M.A. Ist Semester
ECONOMICS

Course Code : ECON 112
Course Code : 06 (DSC)

ELEMENTARY MATHMATICS FOR ECONOMICS

Units: 1 to 20

By: Prof. Sanju Karol
Dr. Dharam Pal

Centre for Distance and online Education,
Himachal Pradesh University
Summer Hill, Shimla, 171005

CONTENTS

COURSE CODE: ECON-112 (DSC)

ELEMENTARY MATHEMATICS FOR ECONOMICS

Block-I: MATRICES AND DETERMINANTS

Their properties, addition, subtraction, and multiplication of matrices. Transpose of a Matrix. Some special forms of square matrices-Trace, Idempotent matrix. Sub-matrix of a matrix. Inverse of a matrix and solution of equations using both the inverse of a matrix and Cramer's rule Rank of a Matrix (Numericals relating to inverse of a matrix and Cramer's rule should to be confined to matrix of order 3x3).

Block-II: DIFFERENTIATION

Derivatives: differentiations of functions of a single variable. Derivative of a composite function, Parametric function, logarithmic function. Exponential. and inverse functions Concave and convex functions. Derivative of higher order Partial Derivatives and total derivative Homogenous functions and Euler's Theorem. Maxima and Minima of functions of single variable. Profit maximization and cost minimization. Constrained optimization of function with two variables. Constrained utility maximization, constrained minimization, and the interpretation of the Lagrange multiplier

Block-III: DIFFERENTIAL AND DIFFERENCE EQUATIONS

Introduction, non-linear and linear differential equations of the first order and first degree. Solutions of differential equations when variables are separable. homogenous equations and non-homogenous equations, exact differential equations and linear equations. Solution of linear differential equations of second with constant coefficient. Finite difference, difference equations. Solutions of homogeneous linear difference equation with constant coefficients. linear first-order difference equations, Linear second order difference equations with constant coefficients.

Application of differential and difference equations in economic models (dynamics of market price, Solow growth model, cob-web model, multiplier- accelerator interaction model. Domar growth model).

Block-IV: ANALYTIC GEOMETRY

Introduction of a Straight Line, section formula, the gradient of a straight in, the equation of a straight line in intercept form, two-point form. Circle: The general equation of a circle. Parabola: equation of a parabola, the points of intersection of line and a parabola. Equation of a rectangular hyperbola. Problems based on applications of analytic geometry in economics.

Integration of function of one variable by parts and substitution. Integration of logarithmic and exponential functions. Definite integral and area between two curves. Simple applications of integration to the relationship between marginal functions and total functions. Consumer's surplus and producer's surplus. Investment and capital formation and the present value of a continuous flow.

Block-V: THE INPUT-OUTPUT MODEL

Its assumptions, technological coefficient matrix, closed and open input -output model, the Hawkins-Simon conditions. Solving the input-output models both open and closed using the inverse matrix.

An Introduction to Linear Programming, Linear equations, slack variables. Feasible and basic solutions. Degeneracy. Solving the primal and Dual with simplex method. Interpretation of the linear programming results.

REFERENCES/SUGGESTED READINGS

- Allen R.G.D. Mathematical Analysis for Economists, MacMillan, India Limited, Delhi
- Baumol, W.J., Economic theory and Operations Analysis, Prentice Hall. New Delhi
- Berchenhal Chris and Paul Grount, Mathematics for Modern
- Budnick, Applied Mathematics for Business economics and social Sciences, 2nd Ed., McGraw Hill.
- Burmeister, E., and R Dobell, Mathematical Theories of Economic Growth.
- Chiang Alpha C. Fundamental Methods of Mathematical Economic Analysis, McGraw-Hill Bank Company London.
- D. Bose, An Introduction to Mathematical Economics, Himkya Publishing House, Delhi.
- Dorfman, R., Linear Programming and Economic Analysis, McGraw
- Dowling, Mathematics for Economists, McGraw Hill Economics, Heritage Publishers, New Delhi.
- Hadley, G. Linear Programming, Narosa Publishing House, New Hill
- Mukherji Badal and V. Pandit. Mathematical Methods for Economic Analysis, Allied Publishers Pvt Ltd., New Delhi.

Unit01

MATRICES-CONCEPTS AND OPERATIONS

STRUCTURE

1.1 INTRODUCTION

In this Unit, we will discuss meaning of matrices and its different types, operation on matrices and trace of a square matrix. This unit ends by giving some properties of matrix and how these properties are used, is explained with the help of some examples.

1.2 LEARNING OBJECTIVES

After studying this Unit, you will able to know

- basic concepts of a matrix
- methods of representing large quantities of data in matrix form
- various operations concerning matrices
- explain the properties of matrix

1.3 MATRIX

A system of *mn* numbers arranged i the form of an ordered set of *m* rows and *n* columns is called an $m \times n$ matrix. In simple words, a matrix is only an arrangement of numbers written in the form of rows columns. For example $m \times n$ matrix as

$$
\begin{array}{l c}
 & \begin{array}{ccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & & \text{Column 4} \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{array} \\
\begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ . \\ . \\ \text{Row m} \end{array} &
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & & a_{mn} \end{bmatrix}_{m \times x}
\end{array}
$$

In the above arrangement of number called a matrix, these are *m* rows and *n* columns and the a matrix is said to be of the order $m \times n$ to be read as *m* by *n*. The number a_{11}, a_{12} etc are called the *elements* of the matrix. It is often convenient to abbreviate the notation. Thus (1) may be written as

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} m\times n$$

or simply A= $[a_{ij}]$ $m \times n$ where $i = 1, 2$ ......*m*

j = 1,2 (2) ....*n* ......(2)

Note:1. In the matrix, (1) there are mu elements

2. In the matrix, the number of row and columns need not be the same.

3. A matrix is only convenient way of representing numbers in row and column form and it has no numerical value as in the case of determinant which has a numerical value.

4. a_{ij} in A means element in the ith row and *j*th column, thus a_{23} means element in the 2nd row and third column.

SELF-CHECK EXERCISE-1.1

Q1. What is meant by Matrix?

1.4 TYPES OF MATRICES

Here we define various types of matrix commonly used in practice.

1.4.1 Square Matrix.A matrix in which the number of rows is equal to the number of columns is called a square matrix. Thus in $m \times n$ matrix A will be called a square matrix if $m = n$ and it will be termed as a square matrix order of n or n rows square matrix.

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$ are the square matrix of order 2 and 3 respectively

Note: Through in a square matrix no. of rows is the same as no. of columns even then, it is not same as determinant. Because a matrix has no value whereas determinant has a value.

$\therefore$The two can never be the same.

Note: In a square matrix the pair of elements $a_{ij}$and a_{ji} are said to be the conjugate elements and the elements a_{11}, a_{22}... a_{nn} are called the diagonal elements.

1.4.2 Diagonal matrix. A square matrix is said to be a diagonal matrix if all its non-diagonal elements are zero. i.e. $a_{ij}=0$ when $i\neq j$.

For example $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{23} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

are all diagonal matrices.

These can be written as diagonal 1[1, 4], diagonal [a_{21},a_{22},a_{23}] and diagonal [4, 5, 6] respectively.

In general we can say that a square matrix A will be a diagonal matrix if all those elements a_{ij} for which $i \neq j$ (i.e. those elements which do not lie on the leading or principal diagonal) are zero. If the diagonal elements are d_1, d_2,.... d_n, then the diagonal matrix is written as

Diagonal (d_1, d_2..... d_n)

1.4.3 Scalar Matrix. A diagonal matrix in which all the diagonal elements are scalar matrix. For Example

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ are all thescalar matrices.

In general, for a scalar matrix

a_{ij}=0 for $i \neq j$

a_{ij}= d for $i \neq j$

1.4.4 Unit (or Identity) Matrix. A square matrix is said to be an identity matrix if all its non-diagonal elements are zero and all its diagonal elements are equal to unity.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are all identity matrix

a_{ij}=0 for $i \neq j$

a_{ij}=1for $i \neq j$

Identity matrices are denoted by 1. Thus I_2, I_3, ...I_n denote identity matrices of order 2, 3.......,*n*.

1.4.5 Zero Matrix or Null Matrix.

Any $m \times n$ matrix in which all the element are zero is called a null matrix of the type *m n* and is denoted by $O_{m \times x}$ A null matrix of the type $n \times n$ is denoted by $O_{n \times n}$or simply by *On*.

1.4.6 Row and Column Matrices.

A matrix in which there is only one row and any number of columns is called a row matrix or a row vector and a matrix in which there is only one column and any number of row is called a column matrix or a column vector. Thus a row matrix is of the type *1* × *n* and a column matrix is of the type *m*× 1.

For example. $[1\ 2\ 3]_{1\times 3}$ is a row matrix whereas

$$\begin{bmatrix}1\\2\\3\end{bmatrix}_{3\times 1}$$ is a column matrix.

Note. Sometimes it is convient to write a column vector as a row vector and enclose the elements bybraces bracket {}

Thus {1 2 4}

1.4.7 Sub Matrices.

If from a given matrix A, we delete any number of row and/or any number of column then the remaining matrix is called the sub-matrix of the given matrix A.

e.g. If $A = \begin{bmatrix}1 & 3 & 5 & 7\\2 & 4 & 6 & 8\\2 & 4 & 5 & 6\end{bmatrix}_{3\times 4}$

then (i) $\begin{pmatrix}1 & 3 & 5\\2 & 4 & 6\end{pmatrix}$ (ii) $\begin{bmatrix}1 & 3\\2 & 4\\3 & 4\end{bmatrix}$

(iii) $\begin{pmatrix}3 & 7\\4 & 6\end{pmatrix}$ (iv) $\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$

are sub-matrix obtained after deleting.

(i) 3rd row 4th column

(ii) 3rd row 4th columns

(iii) 1st and 3rd column and 2nd row.

(iv) 3rd and 4th column 3rd row.

If the resulting sub matrix is a square matrix it iscalled a square sub-matrix.

1.4.8 Determinant of a Square Matrix.

If A is a square matrix of the type $n\times n$ then these numbers also determine a determinant having *n* rows and *n* column and is denoted by [A] or determinant A.

Thus if $A = \begin{bmatrix}1 & 3 & 5\\2 & 4 & 6\\3 & 4 & 5\end{bmatrix}$

Then $|A| = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 4 & 5 \end{bmatrix}$

1.4.9 Minor of a Matrix.

If A be an $m \times n$ then we can have any number of square sub-matrices from it by deleting certain number of rows and certain number of columns. If we delete *m*-4 rows and *n*-4 columns, then we will be left with only 4 rows out of *m* rows 4 columns, out of *n* which will from a square sub matrix of order 4. The determinant of square submatrix is called *minor* of the matrix A or 4 rowed minor of the matrix A in the above case.

1.4.10 Equality of Matrices

Two matrices A=(a) and B = b_{ij} of the sameorder (or type) are defined to be equal if and only if $a_{ij} = b_{ij}$ for each pair of the subscripts. In other words two matrices A and B are equal if and only if

(i) They are of the same order.

(ii) The corresponding elements of the two matrices are the same.

e.g. If A= $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{23} \end{pmatrix}$ and B = $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{23} \end{pmatrix}$

Then A= B if $a_{11} = b_{11}$, $a_{12} = b_{12}$

$a_{21} = b_{21}$, $a_{22} = b_{22}$

SELF-CHECK EXERCISE-1.2

Q1. What is meant by Square Matrix?

Q2. Define Scalar Matrix.

Q3. What is Identity Matrix?

Q. 4 Write orders and types of the following matrices

(i) $\begin{bmatrix} 2 & 9 \\ 3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

(iii) $\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 7 & 6 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 \\ 9 \\ 6 \end{bmatrix}$ (viii) $\begin{bmatrix} 8 & 9 & 1 & 5 \end{bmatrix}$

Q. 5 If $\begin{bmatrix} 3 & x+y \\ xy & 7+z \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 8 & 4 \end{bmatrix}$, find x, y, z

1.5 OPERATION ON MATRICES

1.5.1 Sum of Matrices

Let A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order $m \times n$. Then their sum A+B (or difference A-B) is defined to be another matrix of the same order $m \times n$, say C - (c_{ij}) such that any element of C is the sum (difference) of the corresponding elements of A and B.

C = A+B=$[a_{ij}+b_{ij}]$

Thus, we say that two matrices are conformable for addition if they are of the same order once the matrices are conformable for addition, we add the corresponding elements of the two matrices.

For example

If A = $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 0 \end{pmatrix}_{2\times3}$ and B = $\begin{pmatrix} 4 & 2 & -3 \\ 5 & 0 & 6 \end{pmatrix}_{2\times3}$

Then A + B = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 3 \\ 5 & 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1+4 & 2+2 & 3-3 \\ 2+5 & -3+0 & 0+6 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 & 0 \\ 7 & -3 & 6 \end{bmatrix}$

A-B $= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 & -3 \\ 5 & 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1-4 & 2-2 & 3-(-3) \\ 2-5 & -3-0 & 0-6 \end{bmatrix}$

$= \begin{bmatrix} -3 & 0 & 6 \\ -3 & -3 & -6 \end{bmatrix}$

1.5.1.1 PROPERTIES OF MATRIX ADDITION

Let A = $[a_{ij}]$, B = $[b_{ij}]$ be three matrices conformable for addition, each of order $m \times n$, thenthe following laws hold:

1. Matrix addition is *Commutative*

   i.e. A+B=B+A.

2. Matrix addition is *Associative*

i.e. (A+B)+C=(B+C).

3. Matrix addition is Distributive w.r.t a scalark.

i.e. k(A+B)=kA+kB.

4. Existence of identity.

A+0=0+A-A, 0 being a null matrix.

5. Existence of an inverse

A+(-A)=(-A)+A=0

6. Cancellation law

A+B = A+C$\Rightarrow$ B=C

We shall prove these results.

Proof

(1) A+B $= [a_{ij}]+[b_{ij}]$

$= [a_{ij}]+[b_{ij}]$

$= [b_{ij}]+[a_{ij}]$

$= [b_{ij}]+[a_{ij}]$

=B+A

(2) (A+B) +C = $[a_{ij} + b_{ij}]+[C_{ij}]$

$= [a_{ij}+b_{ij}+ C_{ij}]$

$=[a_{ij}]+ [b_{ij}+ C_{ij}]$

=A+B+C]

(3) k(A+B) $= k[a_{ij} + b_{ij}]$

$= [ka_{ij} + kb_{ij}]$

$= [ka_{ij}] + [kb_{ij},]$

$= k$A + kB.

(4) A + 0 $= [a_{ij}+0]$

$= [0+a_{ij}]$

$= a.0+A$ ($\therefore$ 0+A=0=$(0+a_{ij}) = [a_{ij}]$ = A

$\therefore$ A+0=0+A=0

(5) A+-A$[a_{ij}+(-a_{ij})]$

$= [(-a_{ij}) +a_{ij}]$

=(-A) + A.

Also $[a_{ij}+(-a_{ij})] = [0] = 0$

∴ A+(-A)=(-A) + A = 0.

(6) A+B = A+C.

It implies that the c_{ij} the element on the two sides are equal so that

$a_{ij}+b_{ij}=a_{ij}+ b_{ij}$

Since a_{ij}, $c_{ij}$are scalars, this equality hold if andonly if

$b_{ij} = c_{ij}$

which ⇒B=C

This is known as left cancellation law of addition. In commutative, right cancellation law also holds

i.e. B+A=C+A

⇒B=C

Example 1. If A= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

B= $\begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$ C = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

and $k = 7$.

Verify commnutative associative and distrubutive laws of addition.

Solution 1. A+B= $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1+2 & 2+(-3) \\ 3+(-1) & 4+0 \end{bmatrix}$

$= \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$

B+A $\begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Hence A + B = B + A.

2. A + (B + C) = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2+0 & -3+1 \\ -4+1 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & +(-3)2 \\ 3+(-3) & 4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$(A + B) + C = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix} \right) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \quad 2 & 2+(-3) \\ 3+(-4) & 4+0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0 & -1+1 \\ -1+1 & 4+0 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence A + (B+C)

3. $$7(A + B) = 7 \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1+2 & 2+(-3) \\ 3+(-4) & 4+0 \end{bmatrix}$$

$$= 7 \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & -7 \\ -7 & 28 \end{bmatrix}$$

$$7A + 7\,B = 7 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 7 \begin{bmatrix} 2 & -3 \\ -4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 14 \\ 21 & 28 \end{bmatrix} + 7 \begin{bmatrix} 14 & -21 \\ -28 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 7+14 & 14+(-21) \\ 2+(-28) & 28+0 \end{bmatrix} \qquad = \begin{bmatrix} 21 & -7 \\ -7 & 28 \end{bmatrix}$$

Example 2. If A = $\begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

B = $\begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix}$

C = $\begin{bmatrix} 7 & -6 & 2 \\ 1 & -4 & 11 \end{bmatrix}$

find (i) A + B (ii) A - B (iii) 2A + B - C

(iv) 3A - 4B (v) 4B - 2C

Solution (i) A + B = $\begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2+3 & -5+4 & 1+0 \\ -2+5 & -1-2 & 4+3 \end{bmatrix}$

$= \begin{bmatrix} 5 & -1 & 1 \\ 3 & -3 & 7 \end{bmatrix}$

(ii) A-B = $\begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix} + \begin{bmatrix} -3 & -4 & 0 \\ -5 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 2-3 & -5-4 & 1+0 \\ -2-5 & -1+2 & 4-3 \end{bmatrix}$

$= \begin{bmatrix} -1 & -9 & 1 \\ -7 & 1 & 1 \end{bmatrix}$

(iii) 2A = $2\begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

2A = $2\begin{bmatrix} 4 & -10 & 2 \\ -4 & -2 & 4 \end{bmatrix}$

2A + B – C = $\begin{bmatrix} 4 & -10 & 2 \\ -4 & -2 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 7 & -6 & 2 \\ 1 & -4 & 11 \end{bmatrix}$

$= \begin{bmatrix} 4 & -10 & 2 \\ -4 & -2 & 8 \end{bmatrix} + \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix} - \begin{bmatrix} -7 & 6 & -2 \\ -1 & 4 & -11 \end{bmatrix}$

$$= \begin{bmatrix} 4+3-7 & -10+4+6 & 2+0-2 \\ -4+5-1 & -2-2+4 & 8+3-11 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0 \text{ (matrix)}$$

(iv) $3A = 3\begin{bmatrix} 2 & -5 & 1 \\ -2 & -1 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 6 & -15 & 3 \\ -6 & -3 & 12 \end{bmatrix}$$

$$4B = \begin{bmatrix} 3 & 4 & 0 \\ 5 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 16 & 0 \\ -20 & -8 & 12 \end{bmatrix}$$

$$3A - 4B = \begin{bmatrix} 6 & -15 & 3 \\ -6 & -3 & 12 \end{bmatrix} - \begin{bmatrix} 12 & 6 & 0 \\ 20 & -8 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -15 & 3 \\ -6 & -3 & 12 \end{bmatrix} + \begin{bmatrix} -12 & -16 & 0 \\ -20 & -8 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12 & -5-16 & 3+0 \\ -6-20 & -3+8 & 12-12 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -21 & 3 \\ -26 & 5 & 0 \end{bmatrix}$$

(v) $4B = \begin{bmatrix} 12 & -16 & 0 \\ 20 & -8 & 12 \end{bmatrix}$

$$2C = 2\begin{bmatrix} 7 & -6 & 2 \\ 1 & -4 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -12 & 4 \\ 2 & -8 & 22 \end{bmatrix}$$

$\therefore$ $4B - 2C = \begin{bmatrix} 12 & 16 & 0 \\ 20 & -8 & 12 \end{bmatrix} - \begin{bmatrix} 14 & -12 & 4 \\ 2 & -8 & 22 \end{bmatrix}$

$$= \begin{bmatrix} 12 & 16 & 0 \\ 20 & -8 & 12 \end{bmatrix} - \begin{bmatrix} -14 & -12 & -4 \\ -2 & -8 & -22 \end{bmatrix}$$

$$= \begin{bmatrix} 12-14 & 16+12 & 0-4 \\ 20-2 & -8+8 & 12-22 \end{bmatrix} = \begin{bmatrix} 2 & 28 & -4 \\ 18 & 0 & -10 \end{bmatrix}$$

Example 3. Add the matrices

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 6 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 5 & 6 \\ 9 & 8 \end{bmatrix}$$

Solution: Two matrices are conformable for addition if they are of the same order. Hence the first matrix is of the type 3×3 while the second matrix of the type 3×3. Hence the two matrices are not conformable for addition, i.e. addition of these two matrices is not possible. In other words, adding such matrices do not make any sense.

Example 4. Find a matrix X such that

(i) $3X = \begin{bmatrix} 9 & -12 & 15 \\ -6 & -18 & -21 \\ 1 & 6 & 3 \end{bmatrix}$

(ii) $X + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$

Solution Let $X = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{32} & X_{32} & X_{33} \end{bmatrix}$

$\therefore$ $3X = 3 \begin{bmatrix} 3X_{11} & 3X_{12} & 3X_{13} \\ 3X_{21} & 3X_{22} & 3X_{23} \\ 3X_{32} & 3X_{32} & 3X_{33} \end{bmatrix}$

But $3\ X = \begin{bmatrix} 9 & -12 & 15 \\ -6 & -18 & -21 \\ 1 & 6 & 3 \end{bmatrix}$

$\therefore$Comparing these two, we get

$3x_{11} = 9$, $3x_{12} = 12$, $3x_{13} = 15$

$3x_{21} = -6$, $3x_{22} = -18$, $3x_{23} = -21$

$3x_{31} = 1$, $3x_{32} = 6$, $3x_{33} = 3$

$\therefore x_{11} = 3$, $x_{12} = 4$, $x_{13} = 5$

$x_{21} = -2$, $x_{22} = -6$, $x_{23} = 7$

$x_{31} = 1/3$, $x_{32} = 2$, $x_{13} = 1$

Hence $X = \begin{bmatrix} 3 & 12 & 15 \\ -6 & -18 & -21 \\ 1 & 6 & 3 \end{bmatrix}$

Note. If follow that if

$$3\,X = \begin{bmatrix} 9 & -12 & 15 \\ -6 & -18 & -21 \\ 1 & 6 & 3 \end{bmatrix}$$

$$\text{Then } X = \frac{1}{3}\begin{bmatrix} 9 & -12 & 15 \\ -6 & -18 & -21 \\ 1 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9/3 & 12/3 & 15/3 \\ -6/3 & -18/3 & -21/3 \\ 1/3 & 6/3 & 3/3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 5 \\ -2 & -6 & -7 \\ 1/3 & 2 & 1 \end{bmatrix}$$

(ii) $$X + \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 4 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{Or } X = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -4 & -6 & -2 \\ -6 & -4 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 3 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -2 & -4 & -6 \\ -4 & -6 & -2 \\ -6 & -4 & -2 \end{bmatrix} + \begin{bmatrix} -1 & -2 & -3 \\ 0 & -1 & -5 \\ -3 & -4 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2-1 & 2-4-2 & 3-3-6 \\ 2-4+0 & 3-6-1 & 4-2-5 \\ 3-6-6 & 4-4-4 & 5-2-5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -4 & -6 \\ -2 & -4 & -3 \\ -6 & -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 2 & 4 & 3 \\ 6 & 4 & 2 \end{bmatrix}$$

Example 5. Find a 2×4 matrix X such that

$$A - 2X = 3B$$

given that $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix}$

$$B = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

Solution $\quad A - 2X = 3B$

or $\quad 2X = A - 3B$

$$A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix}$$

$$3B = 3\begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 & 0 & 9 \\ 3 & -3 & 6 & 9 \end{bmatrix}$$

$$-3B = \begin{bmatrix} -6 & -3 & 0 & -9 \\ -3 & 3 & -6 & -9 \end{bmatrix}$$

$$A\text{-}3B = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \end{bmatrix} + \begin{bmatrix} -6 & -3 & 0 & -9 \\ -3 & 3 & -6 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 1-6 & 2-3 & 0+0 & 4-9 \\ 2-3 & 4+3 & -1-6 & 3-9 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{bmatrix}$$

$$\therefore\ 2X = A - 3B = \begin{bmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{bmatrix}$$

$$\text{or } X = \frac{1}{2}\begin{bmatrix} -5 & -1 & 0 & -5 \\ -1 & 7 & -7 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -5/2 & -1/2 & 0 & -5/2 \\ -1/2 & 7/2 & -7/2 & -3 \end{bmatrix}$$

1.5.2 NEGATIVE OF A MATRIX

If A be given matrix and - A is called the negative of the matrix A and all its elements are the corresponding elements of A multiplied by-1.

Thus if A= $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 0 \end{bmatrix}$

Then -A= $\begin{bmatrix} -1 & -2 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

1.5.3 SCALAR MULTIPLE OF A MATRIX

If A be a given matrix and k be any scalar then kA is the matrix all of whose elements are k times the corresponding elements of A.

For Example

if A = $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & -3 \end{bmatrix}$ then

$$3A = 3\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 9 & 12 \\ 3 & 0 & -3 \end{bmatrix}$$

$$-4A = -4\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -12 & -16 \\ -4 & 0 & 12 \end{bmatrix}$$

1.5.4 MULTIPLICATION OR PRODUCT OF MATRICES

Product of a row matrix by a column matrix

If a = (a, a_1 ..... a_n) be a row matrix of order $1 \times n$ and b = {b_1, b_2 ... b_n} be a column matrix or order $n \times 1$, then the product ab is defined as

$$ab = [a_1, a_2 \ldots\ldots a_n]\begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

$$= (a_1b_1+a_2b_2+\ldots\ldots + a_nb_n) = a_ib_i)$$

Product of two matrices in general

Two matrices A and B are said to be conformable for mulplication if the number of colums of A (the first matrix) is equal to the number of B(the second matrix).

Thus If A = (a_{ij}) be $m \times n$ matrix and B=(b_{ij}) be an$n \times p$ matrix, then a product AB is defined as the matrix C = (c_{ik}) of type $m \times p$, where

$$\begin{bmatrix} b_1 k \\ b_2 k \\ \cdots \\ b_n k \end{bmatrix}$$

C_{ik} =(1th row of A) (k the column of B)

= [a_{11} a_{12} .... a_{in}]

= $a_{ij}b_{ik}$+ a_{12} b_2k + ... a_{in} b_{ak}

$$\sum_{j=1} a_{ij} b_{ik}, i = 1, 2, ...m; k = 1, 2......p.$$

= a_{ij}, b_{ij}, where j is the dummy suffix.

Thus if we multiply the Ith of kth column of B,we get the (i, k) the element of AB = C.

The rule of multiplication of matrices is rowcolumn wise. The first row of AB is obtained by multiplying the 1st row of A with 1st, 2nd, 3rd..... columns of B respectively. Similarly second row of AB is obtained by multiplying the 2nd row or A with 1st, 2nd, 3rd..... columns of B respectively and so on. The rule of multiplication is the same for matrices of any order provided the matrices are conformable for multiplication.

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}_{3\times 3}$

Now $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 2 \\ 5 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} [1\ \ 2\ \ 3]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & [1\ \ 2\ \ 3]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} & [1\ \ 2\ \ 3]\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \\ [4\ \ 5\ \ 6]\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} & [4\ \ 5\ \ 6]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} & [4\ \ 5\ \ 6]\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \\ [7\ \ 8\ \ 9]\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} & [7\ \ 8\ \ 9]\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} & [7\ \ 8\ \ 9]\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1.1+2.2+3.3 & 1.0+2.1+3.2 & 1.2+2.2+3.3 \\ 4.1+5.2+6.5 & 4.0+5.1+6.2 & 4.2+5.2+6.3 \\ 7.1+8.2+9.5 & 7.0+8.1+9.2 & 7.2+8.2+9.3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+15 & 0+2+6 & 2+4+9 \\ 4+10+30 & 0+5+12 & 8+10+18 \\ 7+16+45 & 0+8+18 & 14+16+27 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 8 & 15 \\ 44 & 17 & 36 \\ 68 & 26 & 57 \end{bmatrix}$$

Thus Row AB.

$$\begin{bmatrix} (\text{1st row or A}) & (\text{1st col. of B}) & (\text{1st row of A}) & (\text{2nd col. of B}) \\ & & (\text{1st row of A}) & (\text{3rd co. of B}) \\ (\text{2nd col. of A}). & (\text{1st col. of B}) & (\text{2nd row of A}) & (\text{2nd col. of B}) \\ & & (\text{2nd row of A}) & (\text{3rd co. of B}) \\ (\text{3rd row or A}) & (\text{1st col. of B}) & (\text{3rd row of A}) & (\text{2nd col. of B}) \\ & & (\text{3rd row of A}) & (\text{3rd co. of B}) \end{bmatrix}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_3C_3 \\ R_3C_1 & R_3C_2 & R_3C_3 \end{bmatrix}$$

In practice we will follow this rule of multiplication. If we need the element in the 2nd row and 3rdcolumn (i.e. C_{23}) of the product AB, then we neednot find the whole of AB(= C)

$\therefore C_{23}$ = (2nd row of A) (3rd column of A) = R_2C_3

1.5.4.1 PROPERTIES OF MATRIX MULTIPLICATION

If A, B, C are three matrices such that the products AB. BC are well-defined then,

1. Matrix multiplication is *Associative*, i.e.

   A(BC)=(AB)C

2. Matrix multiplication is *Distributive*, i.e.

   A(B+C) = AB+AC

   (B+C)A = BA+CA

3. Martix multiplication is not, in general, *commutative*.

   i.e. AB≠BA, in genral.

(i) It is possible that the matrix AB may exist whereas BA may not exist. For example, if A is of the type m×n and B of the type $n \times p$ but BA will not exist unless $p = m$ (ii) even if AB and BA both exist, it is not necessary that AB=BA. For if the matrix A is of the type $m \times n$ and B is of the type $n \times m$ then both AB and BA exist. But AB is of the $m \times m$ and BA is of the type n x m.

$\therefore$AB and BA cannot be equal.

(iii) If A and B are square matrices of the same order, then both the product matrices AB and BA exist and are also of the same type but not necessarily equal. For example, if we take

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$$

then $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} [1 \quad 2]\begin{bmatrix} 0 \\ 1 \end{bmatrix} & [1 \quad 2]\begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ [3 \quad 4]\begin{bmatrix} 0 \\ 1 \end{bmatrix} & [3 \quad 4]\begin{bmatrix} 2 \\ 3 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1.0 + 2.1 & 1.2 + 2.3 \\ 3.0 + 4.1 & 3.2 + 4.3 \end{bmatrix} = \begin{bmatrix} 0 + 2 & 2 + 6 \\ 0 + 4 & 16 + 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 \\ 4 & 18 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} [0 \quad 2]\begin{bmatrix} 1 \\ 3 \end{bmatrix} & [0 \quad 2]\begin{bmatrix} 2 \\ 4 \end{bmatrix} \\ [1 \quad 3]\begin{bmatrix} 1 \\ 3 \end{bmatrix} & [1 \quad 3]\begin{bmatrix} 2 \\ 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 + 2.3 & 0.2 + 2.4 \\ 1.7 + 3.3 & 1.2 + 3.4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 \\ 10 & 14 \end{bmatrix}$$

Thus AB≠BA.

But if we take $A= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ it can

be verified that AB=BA.

SELF-CHECK EXERCISE 1.3

Q1. If $A = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$, then evaluate the following

(i) 3A + 2B (ii) 2A – 3B (iii) AB

Q. 2 If $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ then find A^4

Q3. Explain the Properties of Matrix Addtion

Q4. Explain the Properties of Matrix Multiplication

1.6 POSITIVE INTEGRAL POWER OF MATRICES

If A is any square matrix, then the product A × A is written as A^2 and we write

$A^2A=(AA)A=A(AA) = AA^2$ as A^3

In general, A A A ...... A(m factors) = A^m and $A^m, A^n= A^{m+n}$

$(A^m)^n = A^{mn}$

AB=0 does not necessary imply that either A = 0or B=0

e.g. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

then AB $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Thus AB=0, even then neither A=0 nor B=0.

SELF-CHECK EXERCISE 1.4

Q1. What is positive integral power of matrices

1.7 TRANSPOSE OF A MATRIX

If A=a_{ij}) be a given matrix of the type $m \times n$ then the matrix obtained by interchanging rows and columns of A is defined as the transpose of A and is written as A' or A^T.

Thus A' = (a_{ij}) and is of the type $n \times m$.

*e.g*if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

1.7.1 PROPERTIES OF THE TRANSPOSE OF A MATRIX

1. (A+B)'=A'+B'
2. (AB)'=B' A' (not A'B')
3. (ABC)' = C'B'A'
4. $(A^k)'=(A')^k$
5. (A')'=A

We shall illustrate the properties of multiplication and transpose by taking example

$$\text{If } A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 1 \\ 4 & 2 & 5 \\ 3 & -2 & 1 \end{bmatrix} \; C = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

Verify the (i) A(BC) = (AB) C

(ii) A(B + C) = AB + AC

(iii) (AB)' = B'A'

$$\text{Solution (i) } B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 4 & 0 & 5 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.3+2.4+3.3 & 1.0+2.2+3.-2 & 1.1+2.5+3.1 \\ 3.3+0.4+2.3 & 3.0+0.2+2.-2 & 3.1+0.5+2.1 \\ 2.3+1.4+1.3 & 2.0+1.2-1.-2 & 2.1+1.5-1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+9 & 0+4-6 & 1+10+3 \\ 9+0+6 & 0+0-4 & 3+0+2 \\ 6+4-3 & 0+2+2 & 2+5-1 \end{bmatrix} = \begin{bmatrix} 20 & -2 & 14 \\ 15 & -4 & 5 \\ 7 & 4 & 6 \end{bmatrix}$$

$$(AB)\,C = \begin{bmatrix} 20 & -2 & 14 \\ 15 & -4 & 5 \\ 7 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20.4+2.0+14.3 & 20.1+2.2+14.-2 & 20.3+2.3+14.1 \\ 15.4+4.0+5.3 & 15.1+4.2+5.-2 & 15.3+4.3+5.1 \\ 7.4+4.0+6.3 & 7.1+4.2-6.-2 & 7.3+4.3-6.1 \end{bmatrix}$$

$$= \begin{bmatrix} 80-0+42 & 20-4-28 & 60-6+14 \\ 60+2+15 & 15-0-10 & 45-12+5 \\ 28+0+18 & 7+8-12 & 21+12+6 \end{bmatrix} = \begin{bmatrix} 122 & -12 & 68 \\ 75 & 3 & 38 \\ 46 & 3 & 39 \end{bmatrix}$$

$$BC = \begin{bmatrix} 3 & 0 & 1 \\ 4 & 2 & 5 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.4+0.0+1.3 & 3.1+0.2+1.-2 & 3.3+0.3+1.1 \\ 4.4+2.0+5.3 & 4.1+2.2+5.-2 & 4.3+2.3+5.1 \\ 3.4+2.0+1.3 & 3.1+2.2-1.-2 & 3.3+2.3-1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+0+3 & 3+0-2 & 9-0+1 \\ 16+0+15 & 4+4-10 & 12+6+5 \\ 12-0+3 & 3-4-2 & 9-6+1 \end{bmatrix} = \begin{bmatrix} 15 & 1 & 10 \\ 31 & -2 & 23 \\ 15 & -3 & 4 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 15 & 1 & 10 \\ 31 & -2 & 23 \\ 15 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1.15+2.31+3.45 & 1.1+2.-2+3.-2 & 1.10+2.23+3.4 \\ 3.15+0.31+2.15 & 3.1+0-2+2.3 & 3.10+0.23+2.4 \\ 2.15+1.31-1.15 & 2.1+1.2-1.-3 & 2.10+1.13-1.4 \end{bmatrix}$$

$$= \begin{bmatrix} 15+62+45 & 1-4-9 & 10+46+12 \\ 45+0+30 & 3-0-6 & 30+0+8 \\ 30+31-15 & 2-2-3 & 20+23-4 \end{bmatrix} = \begin{bmatrix} 122 & -22 & 68 \\ 75 & -3 & 38 \\ 46 & 3 & 39 \end{bmatrix}$$

Hence A (BC) = (AB) C

(ii) $$B+C = \begin{bmatrix} 3 & 0 & 1 \\ 4 & 2 & 5 \\ 3 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+4 & 0+1 & 1+3 \\ 4+0 & 2+2 & 5+3 \\ 3+3 & -2-2 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 1 & 4 \\ 4 & 4 & 8 \\ 6 & -4 & 2 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 7 & 1 & 4 \\ 4 & 4 & 8 \\ 6 & -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1.7+2.4+3.6 & 1.1+2.4+3.-4 & 1.4+2.8+3.2 \\ 3.7+0.4+2.6 & 3.1+0.4+2.-4 & 3.4+0.8+2.2 \\ 2.7+1.4-1.6 & 2.1+1.4-1.-4 & 2.4+1.8+-1.2 \end{bmatrix}$$

$$= \begin{bmatrix} 7+8+18 & 1+8-12 & 4+16+6 \\ 21+0+12 & 3+0-8 & 12+0+4 \\ 14+4-6 & 2+4+4 & 8+8-2 \end{bmatrix} = \begin{bmatrix} 33 & -3 & 26 \\ 33 & -5 & 16 \\ 12 & 10 & 14 \end{bmatrix}$$

$$\text{Now } AC = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 2 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 3 \\ 3 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -1 & 12 \\ 18 & -1 & 11 \\ 12 & 6 & 8 \end{bmatrix} \text{(verify)}$$

$$AB + AC = \begin{bmatrix} 20 & -2 & 14 \\ 15 & -4 & 5 \\ 7 & 4 & 6 \end{bmatrix} + \begin{bmatrix} 13 & -1 & 12 \\ 18 & -1 & 11 \\ 12 & 6 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 20+13 & -2-1 & 14-12 \\ 15+18 & -4-1 & 5+11 \\ 7+5 & 4+6 & 6+8 \end{bmatrix}$$

$$= \begin{bmatrix} 33 & -3 & 26 \\ 33 & -5 & 16 \\ 12 & 10 & 14 \end{bmatrix}$$

Hence A(B + C) = AB + AC

(iii) $AB = \begin{bmatrix} 20 & -2 & 14 \\ 15 & -4 & 5 \\ 7 & 4 & 6 \end{bmatrix}$

$$\therefore (AB)' = \begin{bmatrix} 20 & 15 & 7 \\ -2 & -4 & 4 \\ 7 & 4 & 6 \end{bmatrix}$$

$$\text{Also } B' = \begin{bmatrix} 3 & 4 & 3 \\ 0 & 2 & -2 \\ 1 & 5 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & -1 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 3 & 4 & 3 \\ 0 & 2 & -2 \\ 1 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3+8+9 & 9+0+6 & 6+4-3 \\ 0+4-6 & 0+0-4 & 0+0+2 \\ 1+10+3 & 3+0+2 & 2+5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 15 & 7 \\ -2 & -4 & 4 \\ 14 & 5 & 6 \end{bmatrix}$$

Hence (AB)' = B'A'

Example 2. Given

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \; B = \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$$

find (A + B)', (AB)' and B'A' and show that (AB)' = B'A'

Solution: $A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 1+3 & 2+1 \\ 3+2 & 4+5 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 5 & 9 \end{pmatrix}$$

$\therefore$ $(A + B)' = \begin{pmatrix} 4 & 5 \\ 3 & 9 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4 & 1+10 \\ 9+8 & 3+20 \end{pmatrix} + \begin{pmatrix} 7 & 11 \\ 17 & 23 \end{pmatrix}$$

$\therefore$ $(A + B)' = \begin{pmatrix} 7 & 11 \\ 17 & 23 \end{pmatrix}$

$$B' = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$$

$$A' = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$B'A' = \begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3+4 & 9+8 \\ 1+10 & 3+20 \end{pmatrix} = \begin{pmatrix} 7 & 17 \\ 11 & 23 \end{pmatrix}$$

Hence (AB)' = B'A'

Example 3. Find the value of

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 15 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 15 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 1+0 & 1.5+0 \\ -4+1 & -6+2 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ 0 & -4 \end{pmatrix}$$

$$\therefore \left\{\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1.5 \\ 4 & 2 \end{pmatrix}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1.5 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0 & 0+1.5 \\ 0-0 & 0-4 \end{pmatrix}\begin{pmatrix} 1 & 1.5 \\ 0 & -4 \end{pmatrix}$$

Example 4. Show that for all value of *u*, *v*, *w* and *x* the matrices

$A = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$ and $B = \begin{pmatrix} w & x \\ -x & w \end{pmatrix}$ commute for multiplication.

Solution : $AB = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}\begin{pmatrix} w & x \\ -x & w \end{pmatrix}$

$$= \begin{pmatrix} u.w+v.-x & ux+vw \\ -vw+u.-x & -vx+uw \end{pmatrix}$$

$$= \begin{pmatrix} uw+vx & ux+vw \\ -vw+ux & -vx+uw \end{pmatrix}$$

$$BA = \begin{pmatrix} w & x \\ -x & w \end{pmatrix}\begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

$$= \begin{pmatrix} w.u+x.-v & w.v+x.u \\ -x.u+w.-v & -x.v+w.u \end{pmatrix}$$

$$= \begin{pmatrix} uw+vx & ux+vw \\ -vw+ux & -vx+uw \end{pmatrix}$$

Hence AB = BA for all of u, v, w and x.

Example 5. Show that

$X = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ satisfy the equation $X^3 = 3X^2 + 3X - 1 = 0$

Solution.

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$X2 = X.X = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 2+2+0 & 0+1+0 & 0+0+0 \\ 3+4+3 & 0+2+2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix}$$

$$X^3 = X^2.X = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 4+2+0 & 0+1+0 & 0+0+0 \\ 10+8+3 & 0+4+2 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 21 & 6 & 1 \end{bmatrix}$$

$\therefore X^3 - 3X^2 + 3X - 1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 21 & 6 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 10 & 4 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 21 & 6 & 1 \end{bmatrix} - 3\begin{bmatrix} -3 & 0 & 0 \\ -12 & -3 & 0 \\ -30 & -12 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 6 & 3 & 0 \\ 9 & 6 & 3 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+3-1 & 0+0+0+0 & 0+0+0+0 \\ 6-12+6+0 & 1-3+3-1 & 0+0+0+0 \\ 21-30+9+0 & 6-12+6+0 & 1-3+3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$ Hence the result.

Example 6.

A man buys 8 dozen of mangoes, 10 dozen of apples, and 4 dozen of banana. Mango cost Rs. 18 per dozen, apples Rs 9 per dozen and banana Rs. 6per dozen. Represent the quantities bought by a row matrix and the price by a column matrix and hence obtain the total cost.

Solution:

If A be the row matrix representing the quantities brought i.e. 8 dozen of mangoes, 10 dozen of apples, 4 dozen of bananas, then a is 1 × 3 matrix given by

$$B = \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix}$$

The total cost is given by the elements of the product AB which is a1 × 1 matrix.

$$AB = [8\ 10\ 4] \times \begin{bmatrix} 18 \\ 9 \\ 6 \end{bmatrix} = [8 \times 10 + 10 \times 9 + 4 \times 6]$$

= [144 + 90 + 24] = [258]

Hence the required the total cost in Rs. 258/=

Example 7. A, B, C and X are four matrix given by

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 \\ 11 \\ 5 \end{bmatrix}$$

$$\text{and } X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

(i) Verify: AB=BA = I (is a unit matrix of order 3)

(ii) IfX=BC, find x_1, x_2 and x_3.

Sol.

(i) $$AB = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & -2+2-0 & 7-4-3 \\ 0+0+0 & 0+1+0 & 0-2+2 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1$$

and BA = $\begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +1$

Hence AB = BA =I

(ii) We have X = BC

or $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 11 \\ 5 \end{bmatrix} = \begin{bmatrix} 0-22-35 \\ 0+11-10 \\ 0+0+5 \end{bmatrix} = \begin{bmatrix} 13 \\ 1 \\ 5 \end{bmatrix}$

Hence $X_1 = 13$, $x_2 = 1$, $x_3 = 5$.

SELF-CHECK EXERCISE 1.5

Q1. What is meant by Transpose of a Mtrix?

Q2. If A = $\begin{bmatrix} 2 & 3 & 5 \\ 6 & 8 & 4 \\ 9 & 1 & -3 \end{bmatrix}$ then find tr (A)

Q3. Explain the Properties of Transpose of a Matrix

1.8 SUMMARY

Matrices play an important role in quantitative analysis of managerial decisions. They also provide very convenient and compact methods of writing a system of linear simultaneous equation and methods of solving them. These tools have also become very useful in all functional areas of management. A number of basic matrix operations (such as matrix addition, subtraction, multiplication) were discussed in this unit. This was followed for finding matrix inverse. Numbers of examples were given in support of the above said operations and inverse of a matrix.

1.9 GLOSSARY

1. **Co-factor :** The number C*ij* = $(-1)^{i+j}$ M*ij* is called the co-factor of element a*ij* in A.
2. **Identify Matrix :** A matrix in which diagonal elements are equal to 1 and all other elements are zero.
3. **Matrix :** It is an array number, arranged in rows and columns.
4. **Minor :** The minor of an element is the determinant of the sub-matrix obtained from a given matrix by deleting the row and the column containing that element and is devoted by M*ij*.
5. **Null Matrix :** A matrix in which all elements are zero.

6. **Transpose Matrix :** A matrix obtaining by interchanging rows and column of the original matrix.

1.12 ANSWER TO SELF-CHECK EXERCISES

Self-Check Exercise-1.1

Ans. Q1. Refer to Section 1.3

Self-Check Exercise-1.2

Ans. Q1. Refer to Section 1.4.1

Ans. Q2. Refer to Section 1.4.3

Ans. Q3. Refer to Section 1.4.4

Ans. Q4. Solution

	Order	**Type**
(i)	2×2	Square matrix [$\therefore$ rows and columns are equal in number]
(ii)	2×2	Diagonal matrix [$\therefore$ all the non-diagonal elements are zero]
(iii)	2×2	Scalar matrix [$\therefore$ all the diagonal elements are equal and non-diagonal are zero]
(iv)	2×2	Identify matrix [$\therefore$ all the diagonal elements are unity + non-diagonal element are zero]
(v)	3×3	Upper triangular matrix [$\therefore$ all the elements below the principal diagonal are zero]
(vi)	3×3	Lower triangular matrix [$\therefore$ all the elements above the principal diagonal are zero]
(vii)	3×1	Column matrix [$\therefore$ It has only one column]
(viii)	1×4	Row matrix [$\therefore$ It has only one row]

Ans. Q5: Solution

(i) We know that two matrices A and B are equal if

(a) their orders are same and

(b) the corresponding elements of A and B are equal $\therefore$ On comparing corresponding elements of two matrices, we have

$3 = 3$

$x + y = 6$ ... (1)

$xy = 8$ ... (2)

$7 + 2 = 4 \Rightarrow z = -3$

From (1) $y, = 6 - x$ ... (3)

Putting y from (3) in (2), we get

x(6 – x) = 8

$\Rightarrow\ 6x - x^2 - 8 = 0\ \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x^2 - 4x - 2x + 8 = 0$

$\Rightarrow x(x-4), 2(x-4) = 0 \Rightarrow x(x-4), (x-2) = 0 \Rightarrow x = 4, 2$

when $x = 4$, $y = 6 - 4 = 2$ and when x = 2, $y = 6 - 2 = 4$

$\therefore x = 4, y = 2, z = -3$ or $x = 2$, z = –3

Self-Check Exercise-1.3

Ans. Q1: Solution

(i) $3A + 2B = 3\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} + 2\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & 1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 12 & 15 \\ -9 & 18 & 21 \\ 3 & 24 & 27 \end{bmatrix} + \begin{bmatrix} 6 & 12 & 4 \\ 2 & 5 & 10 \\ 16 & 14 & -2 \end{bmatrix}$

$= \begin{bmatrix} 6+6 & 12+12 & 15+4 \\ -9+2 & 18+8 & 21+10 \\ 3+16 & 24+14 & 27-2 \end{bmatrix} = \begin{bmatrix} 12 & 24 & 19 \\ -7 & 26 & 31 \\ 19 & 38 & 25 \end{bmatrix}$

(ii) $2A - 3B = 2\begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 8 & 10 \\ -6 & 12 & 14 \\ 2 & 16 & 18 \end{bmatrix} - \begin{bmatrix} 9 & 18 & 6 \\ 3 & 12 & 15 \\ 24 & 21 & -3 \end{bmatrix}$

$= \begin{bmatrix} 4-9 & 8-18 & 10-6 \\ -6-3 & 12-12 & 14-15 \\ 2-24 & 16-21 & 18+3 \end{bmatrix} = \begin{bmatrix} -5 & -10 & 4 \\ -9 & 0 & -1 \\ -22 & -5 & 21 \end{bmatrix}$

(ii) $AB = \begin{bmatrix} 2 & 4 & 5 \\ -3 & 6 & 7 \\ 1 & 8 & 9 \end{bmatrix}\begin{bmatrix} 3 & 6 & 2 \\ 1 & 4 & 5 \\ 8 & 7 & -1 \end{bmatrix}$

$= \begin{bmatrix} 6+4+40 & 12+16+35 & 4+20-5 \\ -9+61+56 & -18+24+49 & -6+30-7 \\ 3+8+72 & 6+32+63 & 2+40-9 \end{bmatrix}$

$$= \begin{bmatrix} 50 & 63 & 19 \\ 53 & 55 & 17 \\ 83 & 101 & 33 \end{bmatrix}$$

Ans Q2.: Solution

$$A^2 = AA = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{bmatrix} = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}$$

$$A^4 = A^2A^2 = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} = \begin{bmatrix} 81+128 & 72+136 \\ 144+272 & 128+289 \end{bmatrix}$$

$$= \begin{bmatrix} 209 & 208 \\ 416 & 417 \end{bmatrix}$$

Ans. Q3. Refer to Section 1.5.1.1

Ans. Q4. Refer to Section 1.5.4.1

Self-Check Exercise-1.4

Ans. Q1. Refer to Section 1.6

Self-Check Exercise-1.5

Ans. Q1. Refer to Section 1.7

Ans. Q2: tr(A) = 2 + 8 + (–3) = 7 Ans.

Ans. Q3. Refer to Section 1.7.1

1.11 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.D. (2008). Mathematical Analysis for Economists. Mac Millen, India Limited, Delhi
2. Hughes, A.J.(1983). Applied Mathematics for Business Economics, and Social Sciences, Irwin : Howewood.
3. Raghawachari, M.(1985). Mathematics for Management : An Introduction, Tata Mc Grew Hill (India) Delhi
4. Weber, J.E.(1982). Mathematical Analysis : Business and Economics Application, Harper & Raw : New York.

1.12 TERMINAL QUESTIONS

Q. 1 If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 0 & 3 \\ 4 & 1 & 6 \end{bmatrix}$ then Find

(i) A + B

(ii) B+C

(iii) C+A

(iv) A-2B

(v) 2A-3C

(vi) 3B-5C

(vii) 2A-3B+5C

Q. 2 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 4 & -6 \\ 0 & -1 & -7 \end{bmatrix}$

Find a matrix X such that

(i) 2A+3X=5B

(ii) 2X-3A=4B

(iii)A +2B+3X=0

Q. 3 $A= \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$

Find AB, BA. Is AB = BA

Q.4 Find AB and BA (if defined) where

$A= \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

Q. 5 If $A = \begin{bmatrix} 2 & 2 \\ 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ -2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(i) A (BC)=(AB) C

(ii) (ABC)'=C'B'A'

Is AB BA?

Q. 6 If $A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & -3 \\ 4 & -3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 & -1 & -2 \\ 3 & -2 & -1 & -1 \\ 2 & -5 & -1 & 0 \end{bmatrix}$

Show that AB = AC

Unit2

ADJOINT INVERSE AND RANK OF A MATRIX

STRUCTURE

2.1 INTRODUCTION

In the last unit we studied about the matrices and determinants. In this unit, we will study about the adjoint of the matrices and rank of matrix. We, will also discuss about solving the linear equations by matrix methods.

2.2 LEARNING OBJECTIVES

After studying this unit, you will be able to

- explain adjoint of a square matrix
- differentiate singular and non-singular matrices
- find the inverse or Reciprocal of matrix
- use the inverse of a square of a matrix in solving a system of linear equation.
- find Rank of matrix.

2.3 ADJOINT OF A SQUARE MATRIX

Let A = (a_{ij}) be a square matrix of order n and A_{ij} denote the cofactor of a_{ij} in the determinant A. Then the *adjoint (or adjugate)* of A, to be written as adj A, is defined as the transpose of the matrix of the cofactors (A_{ij})

Thus if $A = \begin{pmatrix} a_{11} & a_{12} & ....a_{1n} \\ a_{21} & a_{22} & ....a_{2n} \\ a_{n1} & a_{n2} & ....a_{nn} \end{pmatrix}_{n\times n}$

and C(A)=Cofactor matrix or matrix of the cofactors of the elements a, s_{ij}'s

$$= \begin{pmatrix} A_{11} & A_{12} & ....A_{1n} \\ A_{21} & A_{22} & ....A_{2n} \\ A_{n1} & A_{n2} & ....A_{nn} \end{pmatrix}_{n\times n}$$

Then adj A= Transpos of the cofactor matrix

= C(A)

$$= \begin{pmatrix} A_{11} & A_{21} & ....A_{n1} \\ A_{12} & A_{22} & ....A_{n2} \\ A_{1n} & A_{2n} & ....A_{nn} \end{pmatrix}_{n\times n}$$

Hence in order to find the adjoint a matrix, repalce each element in the matrix by its corresponding cofactor and then take the transpose.

Example 1. If A=$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find adj A.

Solution. Firstly we shall find the cofactor of theelements of A.

C(1) = Cofactor of 1 = (+) 4 = 4

C(2) = Cofactor of 2 = (-)3 = -3

C(3) = Cofactor of 3 = (-)2=-2

C(4) Cofactor of 4 = (+) 1 = 1

C(A) = Cofactor matrix

$$= \begin{pmatrix} C(1) & C(2) \\ C(3) & C(4) \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ -9 & 1 \end{pmatrix}$$

Example 2. If A= $\begin{pmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{pmatrix}$, find adj A.

Solution. Here we have

A_{11} = Cofactor of a_{11} =+$\begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix}$ =2

A_{12} = Cofactor of a_{12} = - $\begin{vmatrix} 2 & 5 \\ 0 & -7 \end{vmatrix}$ =21

A_{13} = Cofactor of a_{13} =+$\begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix}$ = -18

A_{21} = Cofactor of a_{21} = - $\begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix}$ = 6

A_{22} = Cofactor of a_{22} =+$\begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix}$ = -7

A_{23} = Cofactor of a_{23} = - $\begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix}$ = 6

A_{31} = Cofactor of a_{31} =+$\begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix}$ = 4

A_{32} = Cofactor of a_{32} = - $\begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}$= -8

A_{33} = Cofactor of a_{33} =+$\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix}$ = 4

$$\therefore\ C(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$$

$$\text{and adj } C(A) = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Note:- In all such questions, we first find cofactors of all the elements so as to get cofactor matirx and then transpose to get adjoint.

1.The product of a matrix and its adjoint is commutative. i.e.

A(adj A) = (adj A) A = |A| I

where a is a Square matrix and I is the identitymatrix.

2.If |A|≠0, (i) |adj A|=$|A^{n-1}|$

(ii) $\left(\dfrac{adj\ A}{|A|}\right)$

3.If |A| = 0, A(adj A)=0.

4. Adj (AB)=(adj B), wehre A and B are n-squared matrices.

5. Adj (adj A)= $|A|^{n\ 2}$, A, where A is an n × n matrix.

All these peoperties can be verified by taking a square matrix. Student are advised to verify these statements by taking a 3 × 3 matrix.

SELF-CHECK EXERCISE2.1

Q1. Find the adjoint of each of the following matrices

(i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

2.4 SINGULAR AND NON-SINGULAR MATRICES

A square matrix A is said to be singular if its deteminant is zero i.e. |A| = 0 and said to be non singular if its determinant is zero i.e|A|≠0

SELF-CHECK EXERCISE 2.2

Q1. Distingush between a singular and non-singular matrix

2.5 INVERSE OR RECIPROCAL OF A MATRIX

Let A be a square matrix of order n. Then the matrix B of order n, if it exists, such that AB= I_n=BA, is called the inverse or reciprocal of a and is denoted by A^{-1}

2.5.1 PROPERTIES OF INVERSE OF A MATRIX

1. If *a matrix A has an inverse, then it is unique*.

Let A be an squared matrix where inverse exist. Let us suppose that B and C are the two inverses of A. Then by definition, we have

$AB = BA = I$ ...(1) $\therefore$ B is the inverse of A

$AC = CA = I$ ...(2) $\therefore$ C is the inverse of A

from (1) and (2), it follows that

$AB=I$ Consider $C(AB)\ (CA)B\ C(AB) = CI=C$

and $CA = I$ Consider C(AB) and (CA)B(CA)B=IB=B

But by associated law.

$C(AB) = (CA)B = IB = B$

$\therefore B=C$

Hence the inverse of a matrix is unique.

2. *A squared matrix A can passess an inverse only if A is non singular i.e*. $|A| \neq 0$.

Let A be n-shaped matrix and B be its inverse. Then by definition we have,

$AB = 1$

Taking determinants of both sides, we get.

$|AB|=|I|$

$|A||B|=I$

Since the R.H.S. is non zero, the L.H.S. has to be non-zero which in turn implies that | A | is non zero A is non-singular.

3. If *A non-singular and AB = AC, then B =* C

(Cancellation law)

Since A is non-singular A^{-1} exists.

NowAB = BC

$\therefore$ Pre-multiplying by A^{-1}, we get

$A^{-1}(AB) = A^{-1}(AC)$

or $(A^{-1}A)B = (A^{-1})AC$

or $IB = IC$

$\therefore$ $B = C$

4. Reversal law for the inverse of the productholds i.e. $(AB)^{-1} = B^{-1}A^{-1}$

5. $(A^{1})^{-1}=(A^{-1})$

Remark: Inverse of a matrix exists only if

(i) The given matrix is a square matrix, and

(ii) The determinant of the given matrix $\neq 0$ (i.e.the matrix is non-singular).

In other words,

(i) Every matrix need not have an inverse.

(ii) Every square matrix need not have an inverse.

(iii) Every square non-singular matrix has an inverse.

6. *Inverse of a non-singular diagonal matrix is a diagonal, matrix is a diagonal, matrix*

Let A = diag, (a, b, c) $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

and B = diag $\left(\frac{1}{a} \quad \frac{1}{b} \quad \frac{1}{c}\right) = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

Then AB = BA = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ = I

Hence B is the inverse of A.

In general, If A ding (a_1, a_2, ... a_n)

Then A_{-1}= diag $\left(\frac{1}{a_1} \quad \frac{1}{a_1} \quad \frac{1}{a_n}\right)$

Method to find inverse of a Matrix

Let Abe the given square matrix such that$|A|\neq 0$

B will be the inverse of A if.

AB = BA = 1 ....(1)

So we have to find such a B which satisfies (i). Let us choose B=$\frac{1}{|A|}$adj A.

Since $|A|\neq 0$, our choice of B is justified.

Now AB = A.$\left(\frac{1}{|A|} adj\ A\right)$

Simirly BA = I

$$= \frac{1}{|A|}(\text{Aadj A})$$

$$= \frac{1}{|A|} = |A|\ I = I$$

Hence B $\frac{adjA}{|A|}$=is the inverse of A

i.e. $A^{-1} = \frac{adjA}{|A|}$ ($|A| \neq 0$).

Thus the necessary and the sufficient conditionfor a square matrix *A* to posses an inverse is that it is non singulari.e. $|A| \neq 0$.

For finding the inverse of a square matrix,we shall first find the determinant of Aviz |A|. If |A|=0 inverse does not exist. If |A|=0, we shall find the adjoint matrix and then divide it by to get theinverse matrix.

Example 1. Find the inverse of A=$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Solution: |A|=$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ Cofactor of A = + b

Cofactor of b = -c

Cofactor of c = -b

Cofactor of d=+a

$\therefore$ C(A) = Cofactor matrix = $\begin{pmatrix} d & -c \\ -c & a \end{pmatrix}$

$\therefore$ adj C' (A) = $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$$\therefore A^{-1} = \frac{\text{adj A}}{|A|} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

provided ad - be $\neq$ 0.

Verification. AA^{-1} should be i.

Here $AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & -ab + ab \\ cd - cd & -bc + ad \end{pmatrix}$$

$$= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Hence A^{-1} is correct.

Example 2. Find the inverse of

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$|A| = 0 - 1 (1 - 2) + 3 (3 - 4)$

$\therefore$ We proceed to find *adj* A

Cofactor or the elements of the first row of A

$$+\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}, +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix}$$

or -1, 8, -5

Cofactor of the element of the third row of A

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix}$$

or 1, 2, 3

Cofactor of the element of the third row of A

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

or C(A) = cofactor matrix

$$= \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -5 & 2 & -1 \end{bmatrix}$$

and *adj* A = C' (A)

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\therefore\ A^{-1} = \frac{\text{adj A}}{|A|}$$

$$= \frac{1}{-2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix} \text{or}$$

Verification. AA^{-1} should b e I.

Here $AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & -2 \\ 5 & -3 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0-8+10 & 0+6-6 & 0-2+2 \\ 1+16+15 & -1+12-9 & 1-4+3 \\ 3-8+5 & -3+6-3 & 3-2+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence A^{-1} is correct.

Example 3. Find the inverse of

(i) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

Solution (i)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} \frac{\text{adj } A}{|A|} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

$\therefore$ A has its own inverse.

Actually the given matrix is an identity matrix andwe know that II = I

$\therefore$ Identity matrix has its own inverse.

Which implies that A has its own inverse.

(ii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

|A|=1(15-16)-2(10-12)+3(8-9)

=-1+4-3

=4-4=0

Since |A|=0, $\therefore$ inverse does not exist.

(iii) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$

Since the given matrix is not a square matrix, |A| is not defined and consequently A^{-1} does not exist.

Example 4. Find the adjoint of the matrix.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

and verify that A (*adj* A) = (*adj* A), A = |A|I.

Hence or otherwise find A^{-1}

Solution.

$A= \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$

|A| =1(12-6)-28(8-6)+3(6-9)

=6-4-9=-7≠0.

If A_{ij} donate the cofactor of a_{ij}in A, then A_{ij}= cofactor of]

$a_{11} = + \begin{vmatrix} 3 & 2 \\ 3 & 4 \end{vmatrix} = + (12 - 6) = 6$

A_{12} = cofactor of

$a_{12} = - \begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} = - (8 - 6) = -2$

A_{13} = cofactor of

$$a_{13} = + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = + (6 - 9) = -3$$

A_{21} = cofactor of

$$a_{21} = - \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = - (8 - 9) = 1$$

A_{22} = cofactor of

$$a_{22} = + \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = + (4 - 9) = -5$$

A_{23} = cofactor of

$$a_{23} = - \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = - (3 - 6) = 3$$

A_{31} = cofactor of

$$a_{31} = + \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = + (4 - 9) = -5$$

A_{32} = cofactor of

$$a_{32} = - \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = - (2 - 6) = 4$$

A_{33} = cofactor of

$$a_{33} = + \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = + (3 - 4) = -1$$

$\therefore$ C(A) = cofactor matrix

$$= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 & -3 \\ 1 & -5 & 3 \\ -5 & 4 & -1 \end{bmatrix}$$

adj A = C'(A)

$$= \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$A(adj\ A) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4-9 & 1-10+9 & -5+8-3 \\ -12-6-6 & 2-15+6 & 10+12-2 \\ 18-6-12 & 3-15+12 & -15+12-4 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

$$= -7 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -7I = |A|\, I$$

$$(adj\ A)\, A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+2-15 & 12-3-15 & 18+2-20 \\ -2-1+12 & -4-15+12 & -6-10+16 \\ -3+6-3 & -6+9-3 & -9+6-4 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -7 \end{bmatrix} = -7I = |A|\, I$$

Hence A(adj A) - (adj A) A | A | I

$$\text{Also } A^{1} \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{-7} \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} -6 & -1 & 5 \\ 2 & -5 & -4 \\ 3 & -3 & 1 \end{bmatrix}$$

Note. To verify A^{-1}, we check that $AA^{-1} = 1$.

Example 5. Show that $(AB)^{-1}=B^{-1}A^{-1}$, providedA and B are non-singular matrices of same order.

Solution. $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AIA^{-1}$ $(\therefore BB^{-1}= 1)$

$= AA^{-1}$ $(A^{-1} = 1)$

$=I$

Similarly $(B^{-1}A^{-1})(AB) = I$

$\therefore$ $(AB)(B^{-1}A^{-1})(AB) = I$

Hence by the defination of an inverse.

$(AB)^{-1}=B^{-1}A^{-1}$

Extending this argument, we can show that

$(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$ and so on.

Example 6.

If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ compute $B = I_3 - A(A'A)^{-1} A^{-1}$

Solution. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$, $\therefore$ $A' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$A'A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 1+2+3 \\ 1+2+3 & 1+4+9 \end{bmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$$|A'A| + \begin{vmatrix} 3 & 6 \\ 6 & 14 \end{vmatrix} = 42 - 36 = 6$$

C(A' A) = cofactor matrix of (A' A)

$$= \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$\therefore$ adj (A' A) = C' (A' A)

$$= \begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$$(A'\,A)^{-1} = \frac{\text{adj }(A'\,A)}{|A'\,A|}$$

$$= \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} \frac{14}{6} & -1 \\ -1 & 3/6 \end{pmatrix} = \begin{pmatrix} \frac{7}{3} & -1 \\ -1 & \frac{-1}{2} \end{pmatrix}$$

B = I_3 - A $(A'\,A)^{-1}$ A'

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 7/3 & -1 \\ -1 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{pmatrix} 7/3-1 & 7/3-2 & 7/3-3 \\ -1+1/2 & -1+1 & -1+3/2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \times - \begin{pmatrix} 4/3 & 1/3 & -2/3 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{pmatrix} 4/3-1/2 & 1/3+0 & -2/3+1/2 \\ 4/3-1 & 1/3+0 & -2/3+1 \\ 4/3-3/2 & 1/3+0 & -2/3+3/2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5/6 & 1/3 & -1/6 \\ 1/3 & 1/3 & 1/3 \\ -1/5 & 1/3 & 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} 1-5/6 & 0-1/3 & 0+1/6 \\ 0-1/3 & 1-1/3 & 0-1/3 \\ 0+1/6 & 0-1/3 & 1-5/6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}$$

which is the required result.

SELF-CHECK EXERCISE 2.3

Q1. Explain the proporties of a inverse matrix.

Q2. Find the inverse of A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

Q3. Find the condition under which

A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. Also obtain the inverse of A.

2.6 SOLUTION OF LINEAR EQUATIONS BY MATRIX METHOD

2.6.1 Linear Equation is Two Unknowns

Let us consider two linear equations in x and y.

$$\left.\begin{aligned} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{aligned}\right\} \qquad (1)$$

Let A be the matrix of coefficient= $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

The equation (1) can be written in the matrix notation as

AX = B

Let |A|=0 then A^{-1} exists. Multiplying equation (2) by $A^{-1.}$

A^{-1} (AX)=A^{-1}B

or A^{-1}AX=A^{-1}B

or IX=A^{-1}B

X=A^{-1}B

which gives the required solution.

Example 7. Solve the system of equations x + 2y=4,2x+5y=9 using matrix method.

Solution. The given equations are

x+2y=4

2x+5y=9

Let A = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \end{bmatrix}$ and B = $\begin{bmatrix} 4 \\ 9 \end{bmatrix}$

Then |A| = $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ = 5 - 4 = 1 $\neq$ 0

$\therefore$The given system has a unique solution. The equations (1) can be written in the matrix notationas

AX=B which gives X = $A^{-1}B$

To solve the equation, first we have to calculate A^{-1}.

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 5 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Equation (2) can be written as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 20-18 \\ -8+9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

2.6.2 Linear equations in three unknowns

Let us consider the equations

$$\begin{bmatrix} a_{11}x + a_{12}y + a_{13}Z = b_1 \\ a_{21}x + a_{22}y + a_{23}Z = b_2 \\ a_{31}x + a_{32}y + a_{33}Z = b_3 \end{bmatrix} \qquad (1)$$

Let A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The given equations (1) can be written as AX = B. If |A|≠0, then the equations (1) has a unique solution given by X=A^{-1}B.

Example 8. Solve the following equations by matrix method:

x+y=0, y + z = 1, x+z = 3.

Solution. The given equations are

$$\begin{bmatrix} x + y = 0 \\ y + z = 1 \\ x + z = 3 \end{bmatrix} \text{ or } \begin{bmatrix} x + y + 0.z = 0 \\ 0.x + y + z = 1 \\ x + 0.y + z = 3 \end{bmatrix}$$

Let A = $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Then the system (1) can be written as AX = B.

Now |A| = $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ = z ≠ 0

∴ The system has a unique solution given by

$X = A^{-1} B$

Now A-1 = $\frac{1}{A^{-1}}$ (adj A) = $\frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

From (2)

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0-1+3 \\ 0+1-3 \\ 0+1+3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

⇒x=1,y=-1, z=2

Hence the required solution is

x=1,y=-1, z=2

Before we define the rank of a matrix, we would like to explain the concept elementary trans- formation which will be of much help to us in determining the rank of matrix.

SELF-CHECK EXERCISE 2.4

Q1. Solve the following system of equation by the matrix inverse method :

$x + 2y = 4$, $2x + 5y = 9$

Q2. If A = $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$

are two square matrices, verify that $AB = BA = 6I_3$. Hence solve the system of linear equation : $x - y = 3$, $2x + 3y + 4z = 17$, $y + 2z = 7$

Q3. Solve the following system of homogeneous linear equation by the matrix method

$2x - y + z = 0$, $3x + 2y - z = 0$, $x + 4y + 3 = 3$

2.7 ELEMENTARY TRANSFORMATIONS AND ELEMENTARY MATRICES.

There are three kinds of elementary transformations:

(a) Interchange of any two rows (or columns).

(b) Multiplication of any row (or column) by an non-zero scalar.

(c) Addition to one row (or column), of another row (or column) multiplied by now nonzero scalar.

The operations (a), (b), (c) are called elementary row transformations if applied to row and elementarycolumn transformations if applied to columns.

Square matrices obtained from an identity matrixby any single elementary transformation (a), (b) or (c) are called *Elementary Matrices.*

Notations.

1. R_{ij} (c_{ij}) will denote the interchange of ith and jth rows columns.

2. R_1 (k) [c_1(k)] will stand for the multiplicationof the elements of the ith row (column) by the nonzero scalar K.

3. R_{ij} (k) [C_{ij} (k)] will stand for the addition tothe elements of the ith row (column) K times th corresponding elements of the ith roe (column).

Example. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$ then,

1. R_{12} means inerchanging 1st and 2nd row.

$\therefore$ Applying R_{12} to A, we get

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

Applying C_{13} (e) interchanging 1st and 3rd column we get,

$$C = \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{pmatrix}$$

$R_2(3)$ means the multiplication of the elements of the 2nd by 3. Applying R_2 (3) to A, we get

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 9 & 12 \\ 3 & 4 & 5 \end{bmatrix}$$

3.C_{13} (4) means the addition to the elements of the 1st column, the elements of 3rd column multiplied by 4.

Therefore, applying C_{13} (4) to A, we get

$$B = \begin{bmatrix} 1+12 & 2 & 3 \\ 6+48 & 9 & 12 \\ 3+20 & 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & 2 & 3 \\ 54 & 9 & 12 \\ 23 & 4 & 5 \end{bmatrix}$$

2.7.1 EQUIVALENT MATRICES

Two matrices A and B of the same orderare said to be equivalent, if it is possible to obtain one matrix from the other by the application of elementary transformation. IfB is obtained from A by a series of elementary transformation then we say that A is equivalent to B and write it as A □B.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$

then $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ (Applying R_{12})

$- \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 5 & 4 & 3 \end{bmatrix}$ (Applying R_{13})

2.7.2 INVERSE ELEMENTARY TRANSFORMATION

If by an elementary transformation on a matrix A, we get an equivalent matrix B, then the elementary transformation which when applied onB gives the matrix A will be called the inverse elementary transformation.

1. Inverse Transformation of R_{ij} is R_1

   $R_{ij} = R_{ij} C_{ij}^{-1} = C_{ij}$

2. R_1^{-1} (a) = R_1 (1/α), C_{ij} (α) = C(1/α)

3. R_{ij}^1 (α) = R_{ij}(-α). C_{ij}^1 (α) = C_{ij} (-α)

2.7.3 PROPERTIES OF ELEMENTARY TRANSFORMATIONS AND ELEMENTARY MATRICES

1. Every elementary row (column) transformation of a matrix can be affected by pre (post) multiplication with the corresponding elementary matrix.
2. Two matrices A and B are equivalent if there exist non-singular matrices P and Q such that PAQ= B.
3. Every non-singular square matrix can be expressed as the product of an elementary matrices.
4. Elementary transformations do not alter theorder or rank of a matrix.
5. Equivalent matrices have the same rank.

SELF-CHECK EXERCISE 2.5

Q1. What is equivanlent matrix?

Q2. Explain the properties of Elementary Transformations and Elementary Matrices.

2.8 RANK OF A MATRIX

Let A = $(a_{ij})_{mxn}$be a given matrix of the type $m \times n$. Then the rank of A, to be written as P(A), is defined to be r, where r min $\leq$(m.n) if and only if

(i) Every minor (i.e. determinant of a square submatrix of order (r+1) of A is zero, and

(ii) There exists at least one minor of orderr of Awhich is non-zero.

only (i) $\Rightarrow$ p(A) <r

only (ii)$\Rightarrow$p(A)>r

$\therefore$(i) and (ii) together$\Rightarrow$ P(A)=1.

Note. From the above definition, it clearly follows that

(i) The rank of a null matrix is zero.

(ii) The rank of a non-singular matrix of order n isn.

(iii) The rank of a singular matrix of order n is lessthan n.

(iv) The rank of a non-zero matrix is always$\geq$1.

(v) The rank of an identity matrix of order n is n.

(vi) If A is of order $m \times n$. P(A)$\leq$m and $\leq$n.

(vii) If A' is the transpose of A.P(A) = P(A').

Example 1. Discuss the rank of the following matrices

(i) $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{pmatrix}$ (ii) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Solution. (i) Let A=$\begin{pmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{pmatrix}_{2\times 3}$

Since A is of the type 2 × 3

P(A)≤2

The minors of order 2 are

$\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix}\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix}\begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix}$

which are all zero. Therefore P(A) cannot be equal to 2 but<2 since the matrix is non zero. Therefore P(A)≥1.

P(A)≥ and P(A) <2⇒ P(A)=1.

viz. 1 ≠ 0

∴P(A) = 1

(ii) Let A = $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \\ 2 & 4 & -3 \end{bmatrix}$

|A| = 2(-9+8) + 2(-3+4)

=-2+2=0

Since the matrix is of the type 3 × 3 and it issingular P(A)<3.

Let us find minors of order 2.

One of the minor of order 2 viz $\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} \neq 0$

Hence by definition P(A)=2.

(iii) Let A = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

|A|=1(0-1)=-1

Since the given matrix is a non-singular square matrix of the type 3×3.

∴ By definition, P(A)=3.

(iv) Let $A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$

Clearly |A| = 0. Also all the minors of order twoare zero.

∴ P(A)=1

But it is a non-zero matrix and one minor of order 1 = 1 ≠0.

∴ P(A) = 1

Example 2. Discuss the rank of the matrix.

$$A=\begin{bmatrix} 1 & 3 & 4 & -2 \\ 2 & 6 & 8 & -4 \\ 3 & 0 & 3 & 3 \end{bmatrix}$$

Solution. Since the given matrix is of the type 3 × 4.

P(A)≤ 3.

All the 3 × 3 order minors of A are

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & 6 & -4 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ 2 & 8 & -4 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & -2 \\ 6 & 8 & -4 \\ 0 & 3 & 3 \end{bmatrix}$$

i.e. 0 0 0 0

(verify)

Since each of the 3×3 minor is 0.

P(A) <3.

Now we consider the 2 × 2 minors of A.

There exists at least one minor or order 2 of A viz.

$$\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = -18 \neq 0.$$

Hence p(A)=2.

Note. If all the 2 × 2 minors of A had been zero, then rank of A would have been ≤1. But since the given matrix is a non-zero matrix,... rank would have been 1.

Note. If is very tedious to check all the 3 × 3 order minors. So we devise some method by which we can directly find the rank of a matrix without calculating each minor. We shall state the important theorems and results in this connection.

Important Results

R_1. Elementary transformations do not alter the rank of a matric.

R_2. Equivalent matrices have the same rank.

R_3. Every matrix A of order × n and rank r (>0) can be reduced to one of the following forms:

(i) $\begin{pmatrix} 1_1 & 0 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 1_r \\ 0 \end{pmatrix}$ (iii) (10) (iv) (I_r)

and these are called *normal forms.*

Example 3. Reduce the matrix A to its normal form and hence determine its rank, where

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

Solution. $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 2 \end{bmatrix}$

By the operation R_{21} (-1), we have

$$A \sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 3 & 4 & 5 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad \sim \text{By } R_{31} \text{ (-3)}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad \sim \text{By } C_{21} \text{ (-1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad \sim \text{By } C_{31} \text{ (-1)}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix} \qquad \sim \text{By } C_{41}\ (-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \text{By } R_{32}\ (-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \text{By } C_{31}\ (-2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \sim \text{By } C_{42}\ (-5)$$

Thus $A \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

The rank of f_2=2

Hence p(A)=2.

Note. For finding the rank of the given matrix, itis not necessary to find the normal form. In example (4) above, we would have stopped even at the 5thstep

$$\text{i.e. } A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{bmatrix}$$

$\therefore$ This matrix clearly shows that all the minors of order 3 zero there is a minor of order 2 viz.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \neq 0$$

Hence rank = 2.

Example 4. Find the rank of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4}$$

Solution $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4}$

Since all the four rows (column) are same

$\therefore |A|=0.$

Also all the minors of order 3 and order 2 arezero. But since minor of order 1 is non-zero.. the rank of the given matrix is 1.

In fact, the rank of a matrix of any order each ofwhose element is one is always one.

2.8.1 RANK OF LINEAR INDEPENDENCE

The rank of a matrix is always equal to the number of linearly independent column which also equals to the number of linearly independent rows of the matrix.

If the rank of the matrix A = (a) m × n (m≤n) if r <m, then there are exactly r rows of the matrix which are linearly independent while each of the remaining (m-r) rows can be expressed as a linear combination of these r rows. The same applies to columns.

If $A \sim P = \begin{pmatrix} 1_r & 0 \\ 0 & 0 \end{pmatrix}$, then clearly I, has r independent rows or columns and consequently P, and A also have independent rows or columns.

SELF-CHECK EXERCISE 2.6

Q1. Find the rank of the matrix A, where

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

2.9 SUMMARY

In this unit we have discussed about the adjoint of matrix. We have also studied about the inverse of Reciprocal of a matrix. In the next section we discussed about the properties of inverse of a matrix. We have also discussed about the method of solving linear equation in a variables using matrices giving different and suitable example.

2.10 GLOSSARY

1. **Adjoint** of a square matrix $A = (aij)_{n\times n}$ is defined to be the transpose of the cofactor matrix of A. It is devoted by adj A.

2. **Singular :** A square matrix A is said to be singular it is determinant is zero i.e. $|A| = 0$

3. **Non-singular :** A square matrix A is said to be non-singular if its determinant is zero i.e. $|A| \neq 0$.

4. **Inverse or Reciprocal of matrix :** Let A be a square matrix of order n. Then the matrix B of order n, if it exists, such that AB = In = BA is called the inverse or reciprocal of a and is denoted by A^{-1}.

2.11 ANSWER TO SELF-CHECK EXERCISE

Self-Check Exercise 2.1

Ans. Q1 (i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The Co-factors are

$A_{11} = (-1)^{1+1} \,|\, d \,|\, = d$ $\qquad A_{12} = (-1)^{1+2} \,|\, c \,|\, = -c$

$A_{21} = (-1)^{1+2} \,|\, b \,|\, = -b +$ $\qquad A_{22} = (-1)^{2+2} \,|\, a \,|\, = -a$

$\therefore \text{adj } A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

(ii) Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

The co-factors of the elements of A are

$A_{11} = (-1)^{1+1}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -1$ $\qquad A_{12} = (-1)^{1+2}\begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix} = -2$

$A_{13} = (-1)^{1+3}\begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} = -1$ $\qquad A_{21} = (-1)^{2+1}\begin{bmatrix} -1 & 3 \\ 3 & 5 \end{bmatrix} = 14$

$A_{22} = (-1)^{2+2}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = 13$ $\qquad A_{23} = (-1)^{2+3}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = -5$

$A_{31} = (-1)^{3+1}\begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = -5$ $\qquad A_{32} = (-1)^{3+2}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = -4$

$A_{33} = (-1)^{3+3}\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = 2$

$\therefore \text{ adj } A = \begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix}$

Self-Check Exercise 2.2

Ans. Q1. Refer to Section 2.4

Self-Check Exercise 2.3

Ans. Q1. Refer to Section 2.5.1

Ans. Q2. Here, | A | = (2) (–1) + (–1) (–2) + 3(1) = 3 and adj A = $\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} \text{ adj Ans.}$$

$$= \frac{1}{3}\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 14/3 & 5/3 \\ -2/3 & 13/3 & -4/3 \\ 1/3 & -5/3 & 2/3 \end{bmatrix}$$

Ans. Q3. We have | A | = ad – bc. recall that A is invertible if and only if | A | ≠ 0. That is A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if ad – bc ≠ 0

Also, adj A = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Hence A-1 = $\frac{1}{|A|}$ adj A = $\frac{1}{\text{ad - bc}}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Self-Check Exercise 2.4

Ans. Q1. We can put the given system of equations into matrix mutation as follows :

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}$$

Here the coefficient matrix is given by A = $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

To check if A-1 exists, we not that $A_{11} = (-1)^{1+1}$ | S | = 5 and $A_{12} = (-1)^{1+2}$ | 2 | = –2

Since | A | ≠ 0 A is non-singular (invertible). We also have $A_{21} = (-1)^{2+1}$ | 2 | = –2 : $A_{22} = (-1)^{2+2}$ | 1 | = 1.

Therefore the adjoint of A is

$$\text{adj } A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$\Rightarrow A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$

$\therefore x = A^{-1} B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 20 & -18 \\ -8 & +9 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ or x = 2, y =1

Ans. Q2. AB $\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2+2 & 0-4+10 \end{pmatrix}$

$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$

and BA = $\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 2+4+0 & -2+2-4 & 0+8-8 \\ -4+4+0 & 4+6-4 & 0+8-8 \\ 2-2+0 & -2-3+5 & 0-4+10 \end{pmatrix}$

$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$

Thus AB = BA = 6I3

$\Rightarrow A\left(\frac{1}{6}B\right) = \left(\frac{1}{6}B\right)A = I_3$

This shows that $A^{-1} = \frac{1}{6}B$. Now the given system of equation can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

or Ax = C, where

$$\therefore \text{ x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \text{c} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$\text{X} = \text{A}^{-1}\text{ C} = \frac{1}{6}\text{BC} \qquad \left[\because A^{-1} = \frac{1}{6}B\right]$$

$$= \frac{1}{6}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & 34 & -28 \\ -12 & +34 & -28 \\ 2 & -17 & 38 \end{bmatrix} \qquad = \frac{1}{6}\begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Thus x = 2, y = –1, z = 4 is the required solution.

Ans. Q3. We can write the system of equations as the single matrix equation AX = 0, where

$$\text{A} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & 3 \end{pmatrix}, \text{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The Co factors of | A | are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 10$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10$$

$$\text{and } A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10$$

$\therefore$ | A | = a_{11} A_{12} + a_{12} A_{12} + a_{13} A_{13} = (2) (10) + (-1) (-18) + 1 (10) =4

Since | A | $\neq$ 0, A is non-singular (invertible). This is by known result

X = 0, that x = 0, y = 0 Z = 0.

Self-check Exercise 2.5

Ans. Q1. Refer to Section 2.7.1

Ans. Q2. Refer to Section 2.7.3

Self-Check Exercise 2.6

Ans. Q1. Solution

(i) $$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

we shall find the rank of A by applying elementary transformations.

By performing the operation $R_{31}(-1)$ we have

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2-1 & 6-2 & 5-3 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 1 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

which clearly shows that the determinant of the 3rd order is zero. But a determinant of 2nd order (or minor of 2nd order) viz.

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = 2 \neq 0$$

Hence rank of the transformed matrix is 2.

But equivalent matrices have the same rank.

$\therefore$ P(A) = 2

(ii) $$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}_{4\times 4}$$

By the operation $R_{31}(-1)$, $R_{41}(-1)$, we have

$$A \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \end{bmatrix}$$

Again by the operations $R_{32}(-1)$, $R_{41}(-1)$, we have

$$A \sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 4 & 3 & 9 & -1 \end{bmatrix}$$

By the operation R_{43} (-1), we have

$$\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, all the minors of order 4 and 3 are zero.

But there is one minor of order 2 viz.

$$\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 8 \neq 0$$

Hence P(a)=2.

2.12 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
4. Yamane, T. (2012). Mathematic for Economists : An Elementary Survey. Pretice Hall of India, New Delhi.

2.13 TERMINAL QUESTIONS

Q.1 Find the (i) adjoint and (ii) Inverse of the following matrices.

(i) $\begin{bmatrix} 2 & -2 & 3 \\ 1 & 0 & -3 \\ 1 & 4 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 & 0 \\ 3 & 1 & -2 \\ -1 & 0 & -4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 4 \\ 2 & 1 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 4 \\ 2 & 1 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}$

Q.2 Find the ranks of the following matrices :

(i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 6 \\ 0 & 5 & 10 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -2 & 0 \\ -2 & 3 & 1 \\ -3 & 1 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$

(v) $\begin{bmatrix} 1 & 2 & -4 & 5 \\ 2 & -1 & 3 & 6 \\ 8 & 1 & 9 & 7 \end{bmatrix}$

Q. 3 If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

$B = \begin{bmatrix} -1 & -12 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Find the ranks of A, B, A + B, AB and BA with & by

(i) The help of minors of corresponding matrices.

(ii) Reducing them to canonical forms.

Unit-3

DETERMINANTS

STRUCTURE

3.1 INTRODUCTION

In this Unit, we will study about the determinants. We will also go through the minors and co-factors of the elements of a determinants. In the last section of this unit, we will learn about the properties of determinant.

3.0 LEARNING OBJECTIVES

After completing this unit, you will be able to:

- define Determinant
- find minors and co-factors of square matrices of different orders; and
- Apply properties of determinants

3.3 DETERMINANTS

A determinant is a mathematical tool of a very ordinary kind and involves no new ideas of any description. Briefly, a determinant is a notation that is found convenient in handling certain algebraic processes. Certain expressions of a common form appear in algebraic problems such as that of the solution of linear equation, expressions consisting of sums or differences of a no. of terms each of which is the product of a no. of quantities. Quite apart how other considerations, the labor of writing out the more complicated of these expression is severe and there is every reason to welcome a compact and general notation for them. As some of the characteristics of a vector x can be represented by a scalar, for ex- ample the norm (length) ||x||. Similarly some of the characteristics of a square matrix A can be represented by a scalar, called the determinant, denoted by Al or det, A of the square matrix A. The definition is arbitrary but useful. It should be remembered that the determinant of a square will be scalar eqantity i.e. with a determinant we associate some value," whereas a matrix is essentially an arrangement of numbers and has no value. If the matrix is not square, we cannot associate determinant with it.

3.3.1 DEFINITION

For the square matrix A $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ of second

order the

symbol |A| = $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a determinant of second order or determinant of order 2 and its value is defined by

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The four numbers a_1, b_1, a_2, b_2, are called elements of the determinant.

For the square matrix A = $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

order 3., the symbol

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

consisting of nine number arrangement in three rows and three columns is called determinant of third order or determinant of order 3 and its value is defined by

$$|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

3.3.2 RULE

Write down the elements of the first row (or first column) with alternately positive and negative sign, the first element having always positive sign before it. Multiply each signed element by a determinant of second order after omitting the row and the column in which that element occurs.

Example

Expand the determinant

$$(i)\begin{vmatrix} 15 & -12 \\ -9 & 10 \end{vmatrix} \quad (ii)\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ 10 & 5 & 2 \end{vmatrix}$$

Solution

$$(i)\begin{vmatrix} 15 & -12 \\ -9 & 10 \end{vmatrix} = 15 \text{ x } 10 - (-9)(-12)$$

$$= 150 - 108 = 42$$

$$(ii)\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ 10 & 5 & 2 \end{vmatrix} = 6\begin{vmatrix} 15 & -12 \\ -9 & 10 \end{vmatrix}(-3)$$

$$\begin{vmatrix} 2 & 2 \\ -10 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ -10 & 5 \end{vmatrix}$$

=6(-2-10)+3(4+20)+2 (10 – 10)

=72+72 +2 (0)=0

Remarks:- Determinants are originally connected with the solution of linear equation.

Eliminating x and y from two homogeneous equations.

$a_1x + b_1 y=0$

$a_2x + b_2y=0$

we obtain $a_1b_1 - a_2b_1 = 0$

The expression on the left side of this eliminant issymbolically written as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ which is a determinant of second order.

Similarly, eliminating x, y, z from three equations.

$a_1x+b_1 y+c_1 z=0$

a_2 x+b2y+c2z= 0

a_3x+b3 y+c_3 z=0

we get,

$a_1 (b_2c_3 - b_3 c_2)+b_1 (c_2a_3 - c_3 a_2)+c_1 (a_1b_3 - a_3 b_2)=0$

The expression on the left side of this eliminant issymbolically written as

$$|A|=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

which is a determinant of third order.

3.3.3 MINORS AND COFACTORS OF THE ELEMENTS OF A DETERMINANT

Let us consider a determinant of third order givenby

$$\Delta = \begin{vmatrix} a_{11} & b_{12} & c_{13} \\ a_{21} & b_{22} & c_{23} \\ a_{31} & b_{32} & c_{33} \end{vmatrix}$$

The minor of any element in Δ is a determinant of second order obtained by omitting from Δ the row and the column in which the element occurs.

Thus minor of a_{11}, a_{12}, a_{13}, a_{21}, etc. are respectively.

$$\begin{vmatrix} a_{22} & b_{23} \\ a_{32} & b_{33} \end{vmatrix} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{32} \end{vmatrix} \; etc.$$

Minors of a_{11}, a_{12}, a_{13}, a_{21}, etc. are denoted by M_{11}, M_{12}, M_{13}, M_{21}, etc. respectively.

The cofactors of any element in Δ is the minor of that element in Δ with proper sign depending on the number of the row and the column is which the element occur. If an element occurs in the i th row and j th row columns in Δ, then the cofactor of the element=$(-1)^{i+j}\times$ (minor of the element).

Thus the cofactors of a_{11}, a_{12}, a_{13},a_{21}, etc. in Δ are respectively.

$$(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, (-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$(-1)^{1+3}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, (-1)^{2+1}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

i.e. $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, -\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, +\begin{bmatrix} a_{22} & a_{23} \\ a_{31} & a_{32} \end{bmatrix}, -\begin{bmatrix} a_{12} & a_{13} \\ a_{31} & a_{32} \end{bmatrix}$ etc.

We shall denote the cofactors of a_{11}, a_{12}, a_{13}, a_{21} etc. in Δ by C_{11}, C_{12}, C_{13}, C_{21} etc.

Thus

$$C_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \qquad C_{12} = -\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$C_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad C_{21} = -\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

We have

$$\Delta = a_{11}\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12}\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13}\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

Similarly, we can prove that

$a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} = \Delta$

$a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} = \Delta$

From these results, it follows that we can find the value of the determinant Δ by expanding it along any row or any column.

For quick working, the signs of the different cofactors according to the positions of the corresponding elements in Δ are given by

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Example 2

Write the cofactors of the elements of the second row of the determinant and hence evaluate the determinant.

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

Solution

Let

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$$

Let C_{21}, C_{22}, C_{23} be the cofactors of the element of second row in Δ. Then

C_{21}=cofactor of (-4)=$(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ -7 & 9 \end{vmatrix}$=-(18+21)=-39

C_{22}=cofactor of (3)=$(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 2 & 9 \end{vmatrix}$=+(9-6)=3

C_{23} = cofactor of (6)=$(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 2 & -7 \end{vmatrix}$ = (-7-4)=11

$\therefore \quad \Delta = -4\ C_{21} + 3\ C_{22} + 6\ C_{23}$

$= -4(-39)+3(3)+6\ (11)$

$= 156+9+66$

$= 231$

SELF-CHECK EXERCISE- 3.1

Q1. Define determinant.

Q2. Find the value of the determinant

$$\det A = \begin{vmatrix} 1 & 18 & 72 \\ 2 & 40 & 96 \\ 2 & 45 & 75 \end{vmatrix}$$

3.4 PROPERTIES OF DETERMINANT

Although the following properties of determinants hold good of determinants of any order, we shall verify then for determinants of third only.

(1) The value of a determinant remain unaltered if the rows and columns are interchanged. i.e.

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \end{bmatrix}$$

Proof

$= a_1\ (b_2c_3 - b_3c_2) - b_1\ (a_2c_3 - a_3c_2) + c_1(a_2b_1 - a_3b_2)$

$= a_1\ (b_2c_3 - b_3c_2) - a_2b_1c_3 + a_3b_1\ c_2 + a_2\ b_3\ c_1 - a_3\ b_2\ c_1$

$== a_1\ (b_2c_3 - b_3c_2) - a_2\ (b_1\ c_3 - b_3c_1) + a_3\ (b_1\ c_2 - b_2c_1)$

$$= \begin{bmatrix} a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \\ a_1 & b_2 & c_3 \end{bmatrix} \text{ (by definitions)}$$

(2) If two adjacent rows (or columns) of a determinant are interchanged, the numerical value remains the same, but the sign of the determinant is changed, i.e.

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} - \begin{bmatrix} a_1 & b_2 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Proof:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_2 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1 (b_1 - c_1 - b_3 - c_1) - b_1 (a_1 c_3 - a_3 c_1) + c_1 (a_1 b_3 - a_3 b_1)$

$= a_1 b_1 c_3 - a_1 b_3 c_1 - a_1 b_1 c_3 - a_3 b_1 c_1 + a_1 b_3 c_1 - a_3 b_1 c_1 = 0$

Proceeding as in I, we can expand L. H.S. and R.H.S. and then verify that

L.H.S. = R.H.S.

Instead of the first two rows, we can interchange any two consecutive rows and verify the same result.

The same result can also be verified by interchanging any two adjacent columns.

Cor: The sign of a determinant is either changed or in not changed according as the number of interchanges of two adjacent rows (or columns) is odd or even.

The cor: can be easily proved by using Property (2)

(3) If the two rows (or columns) of a determinantare identical the value of the determinant is zero, i.e

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c1 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0$$

Proof:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c1 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = b_1 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$= a_1 (b_1 - c_1 - b_3 - c_1) - b_1 (a_1 c_3 - a_3 c_1) + c_1 (a_1 b_3 - a_3 b_1)$

$= a_1 b_1 c_3 - a_1 b_3 c_1 - a_1 b_1 c_3 - a_3 b_1 c_1 + a_1 b_3 c_1 - a_3 b_1 c_1 = 0$

Similarly, we can verify the result when two columns are identical.

(4)If all the elements of any one row (or column)are multiplied by the same constant, then the original determinant is multiplied by that constant, i.e.-

Proof:

$$= \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \quad \text{(by definitions)}$$

L.H.S $\begin{bmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = ka_1 \begin{bmatrix} b_2 & c_2 \\ b_3 & c_3 \end{bmatrix} = kb_1 \begin{bmatrix} a_2 & c_2 \\ a_3 & c_3 \end{bmatrix} + kc_1 \begin{bmatrix} a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$

$= k\ a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \begin{vmatrix} a_2 & c_2 \\ a_3 & b_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

$= k \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} =$ R.H.S

Similarly, we can verify the result when all the elements of any one column are multiplied by the same constant *k*.

$$= \begin{bmatrix} ma_1 & mb_1 & mc_1 \\ na_2 & nb_2 & nc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix} = mnk \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(5)If the elements of any row (or column) of a determinant are multiplied in order by the cofactors of the corresponding elements or any other row (or column) then the sum of the products thus obtains is zero.

i.e. a1 $A_2 + b_1 B_2 + c_1 C_2 = 0$. $a_2 A_3 + b_2$

$B_3 + c_2 C_3 = 0$ etc.

and $a_1 c_2 B_2 + a_3 B_3 = 0$ etc.

Proof:

Let $\Delta = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

and $A_1, B_1, C_1, A_2, B_2, C_2$, etc, are the cofactors of $a_1, b_1, c_1, a_2, b_2, c_2$, etc. respectively in Δ.

Then

$A_2 = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} =- (b_1, c_3 - b_3 c_1) = b_1 c_3 + b_3 c_1$

$B_2 = + \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = a_1 c_3 - a_3 c_1$

$C_2 = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} = - (a_1 b_3 - a_3 b_1) = -a_1 b_3 + a_3 b_1$

$\therefore a_1 A_2 + b_1 B_2 + c_1 C_2 = a_1 (-b_1 c_3 + b_1 c_1)$

$+ b_1 (a_1 c_3 – a_3 c_1)$

$c_1 (-a_1 b_3 + a_3 b_1)$

$=-a_1 b_1 c_2+ a_1 b_3 c_1 + a_1 b_1 c_3– a_3 b_1 c_1– a_1 b_3 c_1 + a_3 b_1 c_1 = 0$

Similarly, we can prove the other results.

Cor: If the element of any row (or column) are multiplied in order by the corresponding co-factors of the same elements, then the sum of these products thus obtained is the determinant itself.

We have already proved the results

$a_1 A_1 + a_1 b_1 +c_1 C_1 = \Delta$

$a_2 A_2+ b_2 B_2+ c_2 C_2 = \Delta$

and

$a_3 A_3+a_3 B_3+c_3 C_3 = \Delta$ etc.

(6) If each element of any row (or column) is the sum of two numbers, then the determinant can be expressed as the sum of two determinants whose other rows (or columns) are not altered i.e.

$$= \begin{bmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

Proof:

If A1, B1, C1, be the cofactors of the elements $a_1 +\alpha_1$, $b_1 + \beta_1$, $c_1+\gamma_1$ of the first row of the determinant of the left side, then

$$= \begin{bmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = (a_1 + \alpha_1)A_1 + (b_1+ \beta_1) B_1 +(c_1+\gamma_1) C_1$$

$=(a_1 A_1 +b_1 B_1 + C_1)+(\alpha_1 A_1 + \beta_1 \beta_1 +\gamma_1 C_1)$

$$= \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$$

=R.H.S.

Similarly, we can verify the property when each element of any column is the sum of two numbers.

(7) The value of a determinant remains unaltered if to all the elements of any row (or column) are added the same multiplies, of the corresponding elements of any numbers of the other rows (or columns) i.e.

$$\begin{bmatrix} a_1 + ma_2 + na_3 & b_1 + mb_2 + nb_3 & c_1 + mc_2 + nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Proof:

L.H.S. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} ma_2 & mb_2 & mc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} +$

$$\begin{vmatrix} na_3 & nb_3 & nc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by property (6)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + n \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

L.H.S. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \times 0 + n \times 0$

=R.H.S.

The same property can be verified by taking columns instead of rows.

(8) If the elements of a determinant are polynomial in x and two rows (or columns) of a determinant become identical when x = a, then (x – a) is a factor of the determinant.

Proof:

Let Δ be the determinant in which the elements are polynomial in x. Then after expansion Δ will also be a polynomial inx.

Let $\Delta = f(x)$

$\therefore \quad \Delta = 0$ when $x=0$

(or columns) are identical]

$\therefore \quad f(a) = 0$

Thus shows that (x – a) is a factor of Δ.

Example 3

Find the value of the determinant Δ without expanding where

$$\Delta = \begin{vmatrix} o & b & -c \\ -b & o & a \\ c & -a & o \end{vmatrix}$$

Solution:

$$\Delta = \begin{vmatrix} o & b & -c \\ -b & o & a \\ c & -a & o \end{vmatrix}$$

Taking out (-1) common, each from R_1, R_2& R_3 we get

$$= (-1)^3 \begin{vmatrix} o & -b & c \\ b & o & -a \\ -c & a & o \end{vmatrix}$$

Interchanging rows & columns, we get

$$= (-1)^3 \begin{vmatrix} o & -b & c \\ b & o & -a \\ -c & a & o \end{vmatrix} = (-1)\,\Delta = -\,\Delta$$

$\therefore \quad = 2\,\Delta = 0 \Rightarrow \Delta = 0$

Example 4

Without expanding the determinant, show that (a +b+c) is a factor of the following determinant:

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution :

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Applying $c_1 + c_2 + C_3$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix}$$

Putting a+b+c = 0 in the determinants

$$\Delta = \begin{vmatrix} 0 & b & c \\ 0 & c & a \\ 0 & a & b \end{vmatrix} = 0$$

$\therefore$ each element of c_1 is zero

$\therefore$ (a+b+c) is a factor of the determinant

Example 5

Show that $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ = (b - c) (c - a) (a - b)

Solution:

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

$\therefore$ R_2 - R_1

R_3 - R_1

= (a - b) (c - a) $\begin{vmatrix} -1 & c \\ 1 & b \end{vmatrix}$ = - (a - b) (c - a) b - c)

= (a - b) (b – c) (c – a)

Example 6

Show that = = a_3 + b_3 + c_3 - 3abc

Solution:

$$\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = \begin{vmatrix} a+b+c & b & a \\ c+a+b & c & b \\ a+b+c & a & c \end{vmatrix}$$

= (a + b + c) $\begin{vmatrix} 1 & b & a \\ 1 & c & b \\ 1 & a & c \end{vmatrix}$

= (a + b + c) $\left[1\begin{vmatrix} c & b \\ a & c \end{vmatrix} - 1\begin{vmatrix} b & a \\ a & c \end{vmatrix} + 1\begin{vmatrix} b & a \\ c & b \end{vmatrix} \right]$

= (a+b+c) ($a^{2-}b^2$+ c² – ab – bc – ca)

=a^3+b^3c^3 – $3abc$

Solution of system of linear equation

Determinants can be usefully employed to solve simultaneous linear equation in two or more unknowns and the method of solving simultaneous linear equation by determination is known as Crammer's rule.

Let us consider two linear equations in two unknowns x any y.

$a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Solving these two equations by ordinary rules, we get

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_1b_2} \qquad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1} \quad (1)$$

where $a_1 b_2 - a_2b_1 \neq 0$

Using determinants of second order, we can write the solutions (1) in the form:

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_1}{\Delta} \textit{ and } y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{\Delta_2}{\Delta}$$

where $\Delta = \begin{vmatrix} a_1 & b1 \\ a_2 & b_2 \end{vmatrix}$ Δ_1 and Δ_2 are

obtained by from by replacing the first and second column by the column of numbers on the right side of the given equation (i.e. by the column of constants c_1, c_2) according as it is the value of x or y.

Example 7

Solve by using determinants

3x – 4y=12x – 7y=3

Solution

The equations are

3x – 4y= 1 and 2x – 7y=3

Here $\Delta = \begin{vmatrix} 3 & -4 \\ 2 & -7 \end{vmatrix} = -21 + 8 = -13 \neq 0$

The solution are

$$x = \frac{\Delta}{\Delta} \begin{vmatrix} \begin{matrix} 1 & -4 \\ 3 & -7 \end{matrix} \\ \hline -13 \end{vmatrix} = \frac{-7+12}{-13} = \frac{5}{13} \textit{ and}$$

$$y = \frac{\Delta}{\Delta}2 - \left| \frac{\begin{matrix} 3 & 1 \\ 2 & 3 \end{matrix}}{-13} \right| = \frac{9-2}{-13} = -\frac{7}{13}$$

Hence the required solutions are

$$x = -\frac{5}{13} \qquad y = -\frac{7}{13}$$

Linear equations is three unknowns

Let us consider the system of linear equation

$a_1x+b_1y+c_1z= d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x+b_3y+c_3z=d_3$

where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

Hence $x = \frac{\Delta_1}{\Delta}$ $y = \frac{\Delta_2}{\Delta}$ $z = \frac{\Delta_3}{\Delta}$

where $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$ $\qquad \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$

and $\Delta_3 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

We can this rule (Crammer Rule) of solving a system of linear equations only when $\Delta \neq 0$.

Example 8

Solve the following equations using determinants:

$2x - y+z=11, x+2y+3z=2, 3x+y - x=6$.

Solution

The given system is

$2x - y+z=11$

$x+2y+3z=2$

$3x+y - z=6$

Here $\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & -1 \end{vmatrix} = 25 \neq 0$

$\Delta_1 = \begin{vmatrix} 11 & -1 & 1 \\ 2 & 2 & 3 \\ 6 & 1 & -1 \end{vmatrix} = -85$

$\Delta_2 = \begin{vmatrix} 2 & 11 & 1 \\ 1 & 2 & 3 \\ 3 & 6 & -1 \end{vmatrix} = 70$

$\Delta_3 = \begin{vmatrix} 2 & -1 & 11 \\ 1 & 2 & 2 \\ 3 & 1 & 6 \end{vmatrix} = -35$

$\therefore \quad x = \frac{\Delta_1}{\Delta} = \frac{-85}{-25} = \frac{17}{5}$

$y = \frac{\Delta_2}{\Delta} = \frac{70}{-25} = -\frac{14}{5}$

$z = \frac{\Delta_3}{\Delta} = \frac{-35}{-25} = \frac{7}{5}$

Hence the required solution are

$x = \frac{17}{5}$ $y \frac{-14}{5}$ $z = \frac{7}{5}$

SELF-CHECK EXERCISE 3.2

Q1. Explain the various properties of determinant.

Q2. Verify the following result

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$$

Q3. Evaluate the following determinants:

(i) $\begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$

(ii) $\begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$

3.5 SUMMARY

In this Unit, we were introduced to the concept of determinants. A determinant is a unique scalar quantity associated with each square matrix. In the last section of this unit we learnt about the different properties of determinants.

3.6 GLOSSARY

1. **Determinant :** A unique scalar quantity associated with each square matrix.
2. **Co-factor :** The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the co-factor of element a*ij* in A.
3. **Minor :** The minor of an element is the determinant of the sub-matrix obtained from a given matrix by deleting the row and the column containing that element in denoted by M*ij*.

3.7 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 3.1

Ans. Q1. Refer to Section 3.3.1

Ans. Q2. Solution

$$\det A = \begin{vmatrix} 1 & 18 & 72 \\ 2 & 40 & 96 \\ 2 & 45 & 75 \end{vmatrix}$$

If you expend the determinant by using the elements of the first column, then you will get

$$\begin{bmatrix} 1 & 18 & 72 \\ 2 & 40 & 96 \\ 2 & 45 & 75 \end{bmatrix} = 1 \begin{vmatrix} 40 & 96 \\ 45 & 75 \end{vmatrix} -2 \begin{vmatrix} 18 & 72 \\ 45 & 72 \end{vmatrix} +2 \begin{vmatrix} 18 & 72 \\ 40 & 96 \end{vmatrix}$$

$= 1(3000 - 4320) - 2(1350 - 3240) + 2\,(1728 - 2880)$

$= 1\times(-1320) - 2 \times (-1890) + 2\,(-1152)$

$= -1320 + 3780 - 2304$

$= -3624 + 3780 = 156$Ans.

Self-check Exercise 3.2

Ans. Q1. Refer to Section 3.4

Ans. Q2. Applying row operation (Property 5)

$$\left| \begin{array}{l} R_2 \to R_2 + (-1)R_1 \\ R_3 \to R_3 + (-1)R_1 \end{array} \right.$$

the given determinant the determinant so obtained

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$

Expanding the new determinant by the elements of first column, you will get

$$\begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}$$

Again performing row operations

$$R_2 \to \frac{1}{(b-a)}R$$

$$R_3 \to \frac{1}{(c-a)}R$$

You will have

(b – a) (c – a) $\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$

= (b – a) (c – a){(c + a) – (b + a)}

= (b – a) (c – a)(c – b)

= (a – b) (b – c)(c – a)

Ans. Q. (i) $\begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$

Let $\Delta = \begin{vmatrix} -3 & 5 & -2 \\ 8 & 9 & -17 \\ 3 & -6 & 3 \end{vmatrix}$

Operating $C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} -3+2-2 & 5 & -2 \\ 8+9-17 & 9 & -17 \\ 3-6+3 & -6 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 5 & -2 \\ 0 & 9 & -17 \\ 0 & -6 & 3 \end{vmatrix} = 0$$

$\therefore$ all the element of C_1 are zero, so using p. 7

(ii) Let $\Delta = \begin{vmatrix} 2 & 3 & 30 \\ 5 & 4 & 54 \\ 6 & 1 & 42 \end{vmatrix}$

Taking 6 common from C3

$$\Delta = 6 \begin{vmatrix} 2 & 3 & 5 \\ 5 & 4 & 9 \\ 6 & 1 & 7 \end{vmatrix}$$

Operating $C_3 \rightarrow C_3 - C_1 - C_2$

$$\Delta = 6 \begin{vmatrix} 2 & 3 & 0 \\ 5 & 4 & 0 \\ 6 & 1 & 0 \end{vmatrix} = 6\,(0) = 0$$

3.8 REFERENCES/SUGGESTED READINGS

1. Budnicks, F. (2017). Applied Mathematics for Business, Economics, and Social Sciences, McGraw Hill : New York.
2. Raghawachari, M. (1985). Mathematics for Management : An Introduction. Tata McGraw Hill (India) : Delhi
3. Weber, J.E. (1976). Mathematical Analysis : Business and Economic Applications. Harper & Raw : New York.

3.9 TERMINAL QUESTIONS

Q.1 Expand the following determinants

(i) $\begin{vmatrix} a+1 & a-2 \\ a+1 & a-2 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & 3 \\ -4 & 3 & 6 \\ 2 & -7 & 9 \end{vmatrix}$

Q.2 Write cofactors of the elements of the second row of the determinant and hence evaluate the determinant.

$$\begin{vmatrix} I & a & bc \\ I & b & ca \\ I & c & ac \end{vmatrix}$$

Q.3 Show that

$$\begin{vmatrix} I & a & a^2 \\ I & b & b \\ I & c & c^2 \end{vmatrix} = (a - b)\ (b - c)\ (c - a)$$

Q.4 Solve the linear equations

x – 2y=4

-3x+5y=-7

Q. 5 Using Crammer's rule solve the following system equations.

2y – 3 z=0,x+3y=-4, 3x+4y=3.

Unit- 4

SIMPLE DIFFERENTIATION

STRUCTURE

4.1 INTRODUCTION

This unit introduces some of the basic techniques of calculus and their application to economic problems. We shall be concerned here with what is known as the differentiation.

4.2 LEARNING OBJECTIVES

The objective of this unit is to make student learn :

- The meaning of Differentiation
- Different theorems on Differentiation
- To explain parametric functions
- To apply the derivates to solve economic problems

4.3 DIFFERENTIATION

Differentiation is a method used to find the slopeof a function at any point. Although this is a useful tool in itself, it also forms the basic for some very powerful techniques for solving optimization problems.The basic technique of differentiation is quite straight forward and easy to apply. Consider a simple function that has only one term

$y=2x^2$

To derive an expression for the slope of this function for any value of x the basic rule of differentiation requires you to:

a) multiply the whole term by the value of the power of x, and

b) deduct 1 from the power of x.

In the above mentioned example, there is a term in x^2 and so the power of x is reduced from 2 to 1. Using the above rule the expression for the slope of this function therefore becomes

$2 \times 2x^{2-1} = 4x$

This is known as the derivative of y with respect to x, and is usually written as dy/dx.

In the study of most economic problems, we are confronted with the issue of finding out the effect of changes in certain economic variables on a certain economic phenomena. We are therefore, interested in knowing the direction and magnitude of change ina particular economic variable as a result of the change in the value of other related variables. It is eventually a problem of finding out the rate of change. It may be the rate of change in the dependent variable say, demand, with respect to the change in the explanatory variable say, price.

Another familiar example is the consumption function.

Let

$C=a+by$

where C is consumption expenditure and y is income. When y is increased by a small increment Δy, C increases by A and we have

C+ΔC = a+b (γ+γ Δ)

=a+bγ+b Δγ

ΔC = –C+ a + bγ == bΔγ

$$\frac{C}{\Delta\gamma} = b$$

i.e. for a small unit change of y (income) C (consumption) increases by the amount b.

$$\frac{\Delta C}{\Delta\gamma} = b$$

b is called the marginal propensity to consume.

We shall now reconsider the derivate more rigorously and show it as a limit and also show it as a slope of a curve.

Instead of using such functions as y = 4x or y = $2x^2$, we may take a more rigorous approach and write it in the abstract form as follows:

Let y be a function of x i.e. y= f(x), then a change in y is due to a change in x and consequently the rate of change in y will depend on the rate of change in x.

Thus $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

If it exists it is called the derivative or differential coefficient of y, w, r, t', x and is denoted by or y' (x) or DY or Y_1 or y'

Thus $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

Note 1: The notation $\frac{dy}{dx}$ is only an operational symbol. It is not ratio of dy to dx. It only stands for derivative of y, w, r, t, x.

2.The derivative of f(x) will exist only if lim of thefunction exists.

∴It follows that the function may have derivative at some points and not at other points where limits do not exist. For example the derivative of y = |x|at x= 1 exists.

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h) - (1)}{h} = \lim_{h \to 0} \frac{1+h-1}{h}$$

$$= \lim_{h \to 0} \left(\frac{h}{h}\right) = \lim_{h \to 0} 1 = 1$$

But the derivative of y=|x| at x=0 does not exist

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} \text{ does not exist.}$$

3. The process of finding the differential coefficient of a function is called differentiation. *Differentiation from abnitio or from first principle*. When derivatives are obtained without making use of the standard theorems on differentiation, the technique of doing it is called differentiation from definition or from first principle or from abnitio. It involves the following five steps:

Step I: Let y = *f*(x) be the given function of x

Step II: Let δ x be increment in x and δ y thecorresponding in δ y.

∴ y+ δ y = *f*(x+ δ x) ...(2)

Step III: Subtract (1) from (2) to get

(y+ δ y) – y = *f*(x+ δ x) –*f*(x)

or δ y = *f*(x + δ x) –*f*(x) ...(3)

Step IV: Divide both sides of (3) by x we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x} \qquad ...(4)$$

Step V: Proceed to the limit x → 0 to get

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} \qquad ...(5$$

Example 1. Differentiate from first principle.

(a) $y = x^2$ (b) $y = \sqrt{x}$

Solution: (a) 1.$y = x^2$ ...(1)

II. Let x be an increment in x and y the corresponding increment in y.

$\therefore y + \delta y = (x + \delta x)^2 = x^2 + 2x\,\delta x + (\delta x)^2$ ....(2)

III. Subtracting (1) from (2), we get

$(y+ \delta y) – y – (x^2 + 2x\delta x + (\delta x)^2 – x^2$

or $\delta y = \delta x\,(2x + \delta x)$ ....(3)

IV. Dividing both sides by x. we get

$$\frac{\delta y}{\delta x} = \frac{\delta x(2x+\delta x)}{\delta x}$$

$= 2x + \delta x$

V. Proceeding to the limit as $\delta x \to 0$, we get

$$\frac{dy}{dx} = \underset{\delta x \to 0}{Lim} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lim} (2x+ \delta x) = 2x$$

Hence $\frac{dy}{dx} = \frac{d(y)}{dx} = \frac{d}{dx}(x^2) = 2x$

(b) 1. y = $\sqrt{x}$ ...(1)

II. Let x be an increment in x and by the corresponding increment in *y*.

$\therefore$ y + δy = $\sqrt{x}$ + δx ...(2)

III. Subtracting (1) from (2), we get

(y + δx) – y = $\sqrt{x}$ + δx - $\sqrt{x}$

or $= \left(\sqrt{x+\delta x} - \sqrt{x}\right) \times \frac{\sqrt{x} + \delta x + \sqrt{x}}{\sqrt{x} + \delta x + \sqrt{x}}$

$$= \frac{(x+\delta x) - x}{\sqrt{x} + \delta x + \sqrt{x}} = \frac{\delta x}{\sqrt{x} + \delta x + \sqrt{x}}$$

IV. Dividing both sides by, δ*x*, we get

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x} + \delta x + \sqrt{x}} \times \frac{1}{\delta x}$$

or $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x} + \delta x + \sqrt{x}}$

V. Proceeding to the limit as δx 0, we → get

$$\frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lim} \frac{\delta y}{\delta x}$$

$$= \underset{\delta x \to 0}{Lim} \frac{1}{\sqrt{x} + \delta x + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} - \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Hence $\frac{\delta y}{\delta x} = \frac{d(y)}{dx} = \frac{d(\sqrt{x})}{dx} = \frac{1}{2\sqrt{x}}$

4.3.1 BASIC THEOREMS ON DIFFERENTIATION

4.3.1.1 Theorem 1. The derivative of a constant is 0

Proof: Let y = c, then

$y + \delta y = c$

$\therefore \delta y = c - c = 0$

and $\frac{\delta y}{\delta x} = \frac{0}{\delta x} = 0$ ...(4)

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (0) = 0$ ...(5)

Examples :(i) $\frac{d}{dx}(20) = 0$ (ii) $\frac{d}{dx}(-36) = 0$

4.3.1.2 Theorem 2.

$\frac{d}{dx}(cu) = c\,\frac{d}{dx}(u)$, u being a function of x.

Proof: Let $y = cu$ ...(1)

then $y + \delta y = c(u + \delta u)$ ...(2)

and $y + \delta y - y = cu + c\delta u - cu$

or $\delta y = c\delta u.$ ...(3)

Dividing both sides by δx, we get

$\frac{\delta y}{\delta x} = c\,\frac{\delta u}{\delta x}$ ...(4)

Taking limits as $\delta x \to 0$, we get

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$

$= c \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$ ...(5)

Hence $\frac{d}{dx}(cu) = c\,\frac{d}{dx}(u)$ ...(6)

Examples : $\frac{d}{dx}(3x^2) = 3\,\frac{d}{dx}(x^2) = 3.2x = 6x.$

4.3.1.3 Theorem 3: $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

where *u* and *v* are (derivable function of x)

Proof: Let $y = u + v$ ...(1)

$\therefore$ $y + \delta y = [(u + \delta u) + (v + \delta v)]$ ...(2)

and $y + \delta y - y = [(u)\ \ \delta u) + (v + \delta v)] - [u + v]$

or $\delta y = \delta u + \delta v$ ...(3)

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x} \quad ...(4)$$

Taking limits as δx → 0, we get

$$\underset{\delta x \to 0}{Lim} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lim} \left[\frac{\delta y}{\delta x} + \frac{\delta v}{\delta x}\right]$$

or $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Hence $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

Similarly $\frac{d}{dx}(u - v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$

Hence $\frac{d}{dx}(u \pm v) = \frac{d}{dx}(u) \pm \frac{d}{dx}(v)$

In general, $\frac{d}{dx}(u_1 + u_2 + u_3 + ...)$

$$= \frac{d}{dx}(u_1) + \frac{d}{dx}(u_2) + \frac{d}{dx}(u_3)$$

also $\frac{d}{dx}(u_1 + u_2 + u_3 + ...)$

$$= \frac{d}{dx}(u_1) - \frac{d}{dx}(u_2) - \frac{d}{dx}(u_3)$$

Note: Combining Theorem 2 and Theorem 3, we get

$\frac{d}{dx}(au+by) + a\frac{d}{dx}(u) + b\frac{d}{dx}(v)$ where *a* and b are constant.

4.3.1.4 Theorem 4. $\frac{d}{dx}(u.v) = u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$

where u and v are functions of x.

Proof: Let y = u. v ...(1)

Then $y + \delta y = (u + \delta u)(v + \delta v)$ ...(2)

and y + δy – y = (u + δu) (v + δv) – uv

or $\delta y = uv + u\,\delta v + v\,\delta u + \delta u \delta v - uv$

or $\delta y = u\delta u + v\delta u + \delta u\delta v.$ ...(3)

Dividing both sides by δx. we get

$$\frac{\delta y}{\delta x} = u\,\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x} + \frac{\delta u}{\delta x}\,.\delta v \qquad ...(4)$$

Taking limits as δx → 0. we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \left[u\frac{\delta u}{\delta x} + v\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\delta v \right]$$

$$+ \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \lim_{\delta x \to 0} .\delta v$$

or $\frac{dv}{dx} = u.\,\frac{dv}{dx} + v.\,\frac{du}{dx} + 0$

[as $\delta x \to 0.\ \delta u \to 0.\ \delta v \to 0.$]

$$= u.\,\frac{dv}{dx}\,(v) + (v).\,\frac{du}{dx} + u \qquad ...(5)$$

Thus $\frac{d}{dx} = (u.\,v) = u.\frac{d}{dx} + (v) + v.\,\frac{d}{dx}u$

i.e. the derivative of the product of two functions = first function × derivative of the second + second functions × derivative of the first.

Similarly. If y = *uvw.*

then $\frac{d}{dx}\,(u) = \frac{d}{dx}\,(uvw) = \frac{d}{dx}[(uv)\,.\,w]$

$$= (uv)\,\frac{d}{dx}\,(w) + w\,\frac{d}{dx}\,(uv)$$

$$= (uv)\,\frac{d}{dx}\,(w) + w\left[u\frac{du}{dx}(v) + v\frac{dv}{dx}\frac{dv}{dx}(u) \right]$$

$$= (uv)\,\frac{d}{dx}\,(w) + w\,u\frac{d}{dx}\,(v) + wv\frac{d}{dx}\,(u)$$

Then result can be generalizd for any number of derivable functions.

4.3.1.5 Theorem 5. If u and v are functions of x, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{d}{dx}\text{(quotient of two functions)} =$$

$$\frac{\text{Denominator} \times \frac{d}{dx}\text{ (Numerator) - Numerator } \times \frac{d}{dx}\text{(denominator)}^2}{\text{Denominator}^2}$$

Proof : Let y = $\frac{u}{v}$ ...(1)

then y + δy = $\frac{u+\delta u}{v+\delta v}$ ...(2)

and y + δy - y = $\frac{u+\delta u}{v+\delta v}$ - $\frac{u}{v}$

δy = $\frac{uv + v\delta - uv - u\delta u}{v(v+\delta v)}$

δy = $\frac{u\delta v - u\delta u}{v(v+\delta v)}$ ...(3)

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{u\delta v - u\delta u}{\delta x.\ v.(v+\delta v)} \quad \frac{\frac{u\delta v - u\delta u}{\delta x}}{v(v+\delta v)}$$

$$\frac{\delta y}{\delta x} = \frac{v\frac{\delta u}{\delta d} - u\frac{\delta v}{\delta x}}{v(v-\delta v)} \quad ...(4)$$

Taking limits as δx→ 0, we get

$$\underset{\delta x \to 0}{Lim}\ \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{Lim}\ \left[\frac{v\frac{\delta u}{\delta d} - u\frac{\delta v}{\delta x}}{v(v-\delta v)}\right]$$

or $\frac{dy}{dx} = \frac{v\frac{\delta u}{\delta d} - u\frac{\delta v}{\delta x}}{v.v} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2}$

Hence $\frac{dy}{dx}\left(\frac{u}{v}\right) = \frac{v.\frac{d}{dx}(u) - u\frac{d}{dx}y}{v^2}$

Now we shall write the derivatives or some most important functions of x. All these results can be derived from the principles discussed earlier. These results along with the thermos discussed above will help to solve problems.

(A) 1. If $y=x^n$

$$\frac{dy}{dx}=\frac{d}{dx}(x^n)=nx^{n-1}$$

2. If y = e^{ax}

$$\frac{dy}{dx}=\frac{d}{dx}(a^x) = a^x \log a$$

3. If y = e^x

$$\frac{dy}{dx}=\frac{d}{dx}\ (e^x)\ e^x$$

4. If y = log x,

$$\frac{dy}{dx}=\frac{d}{dx}\ (\log x) = 1$$

5. (i) If y = sin x,

$$\frac{dy}{dx}=\frac{d}{dx}\ (\sin x)\ \cos x$$

(ii) If y cos x

$$\frac{dy}{dx}=\frac{d}{dx}\ (\cos x\ ) = -\ \sin x \text{ etc.}$$

(B) If instead of x, we have a function of x say a + bx, then

1. If y = $(a + bx)^n$

$$\frac{dy}{dx}=\frac{d}{dx}\ (a + bx)^n = n\ (a + bx)^{n-1}\ (b)$$

2. If y = a^{a+bx}

$$\frac{dy}{dx}=\frac{d}{dx}\ (a^{a.\ bx}) = a^{a\ -\ bx} \log a\ (b)$$

3. If y = e^{a+bx}

$$\frac{dy}{dx}=\frac{d}{dx}\ (e^{a-bx}) = e^{a-bx}\ -\ (b)$$

4. If y = log (a + bx),

$$\frac{dy}{dx} = \frac{d}{dx} \text{ [log } (a + bx)] = \frac{1}{a+bx} \text{ -(b)}$$

5. If y = sin (a + bx)

$$\frac{dy}{dx} = \frac{d}{dx} \text{ [(sin(a + bx)] = cos (a + bx) – (b)}$$

6. If y = cos (a + bx)

$$\frac{dy}{dx} = \frac{d}{dx} \text{ [cos (a + bx] = sin (a + bx) – (b)}$$

Note: In all such questions, we have to multiply by b i.e. coefficient of x.

(C) If instead of x, we get u, which is any function of x i.e. u = u (x) then

1. If y = u, then

$$\frac{dy}{dx} = \frac{d}{dx} (u^n) = n\, u^{n-1} \times \frac{du}{dx}$$

2. If y = a^n, then

$$\frac{dy}{dx} = \frac{d}{dx} (a^u) = a^u.\ \log a \times \frac{du}{dx}$$

3. If y = e^4, then

$$\frac{dy}{dx} = \frac{d}{dx} (u^n) = (e^4) = e^4 \times \frac{du}{dx}$$

4. If y = log u, then

$$\frac{dy}{dx} = \frac{d}{dx} (\log u) = \frac{1}{u} \times \frac{du}{dx}$$

5. If y = sin u. then

$$\frac{dy}{dx} = \frac{d}{dx} (\sin u) = \cos u \times \frac{du}{dx}$$

If y = cos u, then

$$\frac{dy}{dx} = \frac{d}{dx} (\cos u) - \sin u \times \frac{du}{dx}$$

Note : In all such questions, we have to multiply by

$$\frac{du}{dx} = \textit{ i.e. d.c. of u, w.r.t. x}$$

Example 1 : Differentiate w.r.t. x.

(i) $x^5,\ x^{\frac{1}{8}}$, x, $\sqrt[3]{x^5}$, x^e

(ii) $(3x -4)^5$, $(5 - 4x)^7\ \sqrt[5]{2x}$ - 1

(iii) $-\ \frac{1}{2x-1}\ .\ \frac{1}{3-4x}\ .\frac{1}{(a-bx)^{3/2}}$

Solution :

(i) $(x^5) = 5x^{5-1} = 5x^4$

$$\frac{d}{dx}\left(x^{\frac{1}{8}}\right) = \frac{d}{dx}(x^8) = -8x^{-8-1} = -8x^{-9}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{12}) = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\sqrt[3]{x^5}\right) = \frac{d}{dx}(x^{53}) = \frac{5}{3}x^{5/3-1} = \frac{5}{3}x^{2/3}$$

$$\frac{d}{dx}(x^e) = e\ x^{e-1}$$

(ii) $\frac{d}{dx}(3x -4)^5 = 5(3x -4)^{5-1}\ (3) = 15(3x -4)^4$

$$\frac{d}{dx}(5x -4)^7 = 7(5 -4x)^7\ (-4) = -28(5 -4x)^6$$

$$\frac{d}{dx}\left[\sqrt[3]{2x}-1\right] = \frac{d}{dx}\left[(2x-1)^{1/5}\right]$$

$$= \frac{1}{5}(2x - 1)^{1/5-1}\ (2)$$

$$= \frac{2}{5}(2x - 1)^{-4/5}$$

(iii) $\frac{d}{dx}\frac{1}{2x-1} = \frac{d}{dx}\left[(2x-1)^{-1}\right]$

$$= -1(2x - 1)^{-1-1}\ (2)$$

$$= -2(2x - 1)^{-2} = \frac{2}{(2x-1)^2}$$

$$\frac{d}{dx}\left(\frac{1}{3-4x}\right) = \frac{d}{dx}(3 - 4x)^{-1}$$

$= -1(3 - 4x)^{-1-1} = -4$

$= 4(3 - 4x)^{-2} = \frac{4}{(3-4x)^2}$

$\frac{d}{dx}\left[\frac{1}{(a-bx)^{3/2}}\right] = \frac{d}{dx}[(a-bx)^{3-2}]$

$= \frac{-3}{2}(a-bx)^{-3/2}(-b)$

$= \frac{-3}{2}\ (a - bx)^{-5/2} = \frac{3b}{2(a-bx)^{5/2}}$

Example 2. Differentiate w.r.t.x

(i) $y = \frac{2x^2 - 2x^3 + 4}{x^4}$ (ii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution :

(i) $y = \frac{2x^2 - 3x^3 + 4}{x^4} = \frac{2x^2}{x^4} - \frac{3x^3}{x^4} - \frac{4}{x^4}$

$= \frac{2}{x^2} - \frac{3}{x} + \frac{4.}{x^7}$

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left[\frac{2}{x^2} - \frac{3}{x} + \frac{4}{x^7}\right]$

$\frac{d}{dx}(y) = \left(\frac{2}{x^2}\right) - \frac{d}{dx}\left(\frac{3}{x}\right) + \frac{d}{dx}\left(\frac{4}{x^7}\right)$

$= 2\frac{d}{dx}(x^{-2}) - 3\ \frac{d}{dx}(x^{-1}) + 4\ \frac{d}{dx}(x^{-4})$

$= 2(-2)\ x^{-2-1}\ -3.(-1)x^{-1-1}\ +4(-4)\ x^{-4-1}$

$= -\ 4\ x^{-3} + 3x^2\ -16\ x^{-5}$

$= \frac{4}{x^3}\ \frac{3}{x^2}\ \frac{16}{x^5}$

(ii) $y = \sqrt{x} + \frac{1}{x} = x^{1/2} + x^{-1/2}$

$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\left[x^{1/2} + x^{-1/2}\right]$

$$=\frac{d}{dx}\left(x^{1/2}\right)+\frac{d}{dx}\left(x^{-1/2}\right)$$

$$\frac{1}{2}x^{-1/2}-\frac{1}{2}x^{-3/2}$$

$$\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}$$

Example 3. Differentiate w.r.t.x.

(i) $(5 - 2x)(2x^3 + 3)$ (ii) $x(1+x)(1+2x)$

(iii) $\frac{x+1}{\sqrt{x}}$ (iv) $\sqrt{\frac{1+x}{1-x}}$

(v) $\frac{1-\sqrt{x}}{1+\sqrt{x}}$

Solution : (i) Let $y = (5 - 2x)(2x^3 + 3)$

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}[(5-2x)(2x^3+3)]$$

applying u x v formula, we get

$$\frac{d}{dx}=(5-2x)\frac{d}{dx}(2x^3+x)+(2x^3+3)\frac{d}{dx}(5-2x)$$

$$= (5-2x)[2.3x^2+0] + (2x^3+3).[0-2.1]$$

$$= (5-2x) \text{ x } 6x^2 + (2x^3+3) \times (-2)$$

$$= 30x^2 - 12x^3 - x^3 - 6$$

$$=30x^2 - 16x^3 - 6$$

(ii) Let $y=x(1+x)(1+2x)$

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}[\{x(1+x)\}(1+2x)\}$$

Applying u.v. formula, we get

$$\frac{d}{dx}=[x(1+x)\frac{d}{dx}(1+2x)+(1+2x)\frac{d}{dx}[x(1+x)]$$

$$=x(1+x)2+(1+2x)[x\frac{d}{dx}(1+x)+(1+x)(x)]$$

$$= 2x(1+x)+(1+2x)[x.1+(1+x)1]$$

$$= 2x(1+x)+(1+2x)(1+2x)$$

$= 2x\,(1+x) + (1 + 2x)^2$

(iii) Let $y = \dfrac{x+1}{\sqrt{x}}$

$$\therefore \frac{d}{dx}(y) = \frac{d}{dx}\frac{x+1}{\sqrt{x}}$$

Applying $\dfrac{u}{v}$ formula we get

$$\frac{\dfrac{dy}{dx} = \sqrt{x}\dfrac{d}{dx}(x+1) - (x+1)\dfrac{d}{dx}(\sqrt{x})}{\sqrt{x^2}}$$

$$= \frac{\sqrt{x}(1) - (x-1) - \dfrac{1}{2\sqrt{x}}}{\sqrt{x^2}}$$

$$\sqrt{x} - \frac{(x+1)}{2\sqrt{x}}$$

$$= \frac{2.x-(x-1)}{2\sqrt{x}.x} = \frac{2x-x-1}{2x\sqrt{x}}$$

$$= \frac{x-1}{2x\sqrt{x}}$$

(iv) Let $y = \sqrt{\dfrac{1+x}{1-x}} = \dfrac{(1+x)^{1/2}}{(1-x)^{1/2}}$

$$\frac{dy}{dx} = \frac{dy}{dx}\left(\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\right)$$

Applying $\dfrac{u}{v}$ formula, we get

$$\frac{dy}{dx} = \frac{(1-x)^{1/2}\dfrac{d}{dx}\left[(1+x)^{1/2}\right] - (1+x)^{1/2}\dfrac{d}{dx}\left[(1-x)^{1/2}\right]}{(1-x)}$$

$$\frac{(1-x)^{\frac{1}{2}}\dfrac{1}{2}(1-x)^{-\frac{1}{2}} - (1+x)^{\frac{1}{2}}\dfrac{1}{2}(1-x)^{-\frac{1}{2}}}{(1-x)}$$

$$= \frac{\frac{1}{2}\frac{(1-x)^{1/2}}{(1+x)^{1/2}} - \frac{1}{2}\frac{(1+x)^{1/2}}{(1-x)^{1/2}}}{(1-x)}$$

$$= \frac{\frac{1}{2}\left[\frac{(1-x)}{(1+x)^{1/2}} - \frac{(1+x)}{(1-x)^{1/2}}\right]}{(1-x)}$$

$$= \frac{1}{2}\frac{1}{(1+x)^{1/2}(1-x)^{1/2}(1-x)} = \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$$

(v) Let y = $\frac{1-\sqrt{x}}{1+\sqrt{x}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$$

Applying $\frac{u}{v}$ formula, we get

$$\frac{dy}{dx} = \frac{\left(1+\sqrt{x}\right) - \frac{d}{dx}\left(1-\sqrt{x}\right) - \left(1-\sqrt{x}\right)\frac{d}{dx}\left(1+\sqrt{x}\right)}{1+\sqrt{x^2}}$$

$$= \frac{\left(1+\sqrt{x}\right) - \left(\frac{1}{2\sqrt{x}}\right)\left(1-\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}\right)}{1+\sqrt{x^2}}$$

$$\left(\therefore \frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}\right)$$

$$= \frac{\frac{\left(1+\sqrt{x}\right)}{2\sqrt{x}} - \frac{\left(1-\sqrt{x}\right)}{2\sqrt{x}}}{1+\sqrt{x^2}}$$

$$= 1\ \frac{1}{2\sqrt{x}} \ .\ \frac{[1-\sqrt{x}+1-\sqrt{x}]}{1+\sqrt{x^2}} = -\ \frac{1}{2\sqrt{x}}\ .\ \frac{2}{\left(1+\sqrt{x}\right)^2}$$

$$= \frac{1}{\sqrt{x}(1+\sqrt{x})^2}$$

Example 4. Find the derivative w.r.t.x.

(i) $x^a + a^x + a + 2(\sqrt{x})^a - 3(\sqrt{a})^x + \sqrt{e}^x$

(ii) log 2x + sin (3-4x) +4^{e-2x}

Solution:

(i) Let y = x² + a^x + a^a + 2 $(\sqrt{x})^a$ - 3 $(\sqrt{a})^x$ + $\sqrt{e}^{\,x}$

$\frac{d}{dx}$ (y) = $\frac{d}{dx}$[x^a+a^x+a^a+2$x^{a/2}$-3$a^{x/2}$ + $e^{e/2}$]

= $\frac{d}{dx}$ (x^a) + $\frac{d}{dx}$ (a^x) + $\frac{d}{dx}$ (a^a) + $\frac{d}{dx}$ ($x^{a/2}$) -3 $\frac{d}{dx}$($a^{x/2}$) + $\frac{d}{dx}$ (e^{x2})

= ax^{a-1}+a^x log a +0+2. $\frac{a}{2}$ $x^{a/2-1}$ - $3^{ax/2}$

log a $\frac{1}{2}$ + $e^{ex/2}$ $\frac{1}{2}$.

= ax^{a-1} + a^x log a + a$x^{a/2-1}$ - 3/2 log $a^{x/2}$ + $e^{x/2}$

(ii) Let y = log 2x + sin (3-4x) + 4 e^{-2x}]

$\frac{d}{dx}$ (y)= $\frac{d}{dx}$[log 2x + sin (3 - 4x) + e^{-2x}]

= $\frac{d}{dx}$[log 2x]+ $\frac{d}{dx}$ [sin (3-4x)] 4 $\frac{d}{dx}$[e e^{-e2x}]

= $\frac{1}{2x}$ [2] + cost (3-4x). (-4)+i.e^{-2x} (-2)

= $\frac{1}{x}$-4 cos (3-4x) - 8 e^{-2x}

Examples 5: Differentiate w.r.t.x.

(i) x^x+x^{lx} (ii) $x^2\sqrt{\frac{2x-1}{x+1}}$

(iii) log $\left(\frac{ax+b}{cx+d}\right)$

Solution: (i) Let y=x^x+$x^{1/x}$=u+v (say)

so that $\frac{dy}{dx}$ = $\frac{d}{dx}$ (u) + $\frac{d}{dx}$ (v)

Now u = x^x

taking logarithm of both sides, we get

log u= log(x^x)=xlog x

Differentiating w.r.t.x, we get

$$\frac{d}{dx}(\log u) = \frac{d}{dx}(x \log x) = x\frac{d}{dx}(\log x) + \log x \ \frac{d}{dx}(x)$$

$$\text{or } \frac{1}{u}.\frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x\ (1) = 1 + \log x$$

$$\text{or } \frac{du}{dx} = u(1+\log x) = x^x\ (1 + \log x)$$

Also $v = x^{1/x}$

Taking logarithm of both sides, we get

$\log v = \log(x^{1/x}) = x \log x$

differentiating w.r.i.x, we get

$$\frac{d}{dx}(\log v) = \frac{d}{dx}\left(\frac{1}{x}\log x\right) = \frac{1}{x}\frac{d}{dx}(\log x) \log x \ \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\text{or } \frac{1}{v}.\frac{dv}{dx}\left(\frac{1}{x}\right) + \log x\left(\frac{1}{x^2}\right)$$

$$\text{or } \frac{1}{v}\ \frac{dv}{dx} = \frac{1}{x^2}\ \frac{1}{x^2} - \frac{\log x}{x^2} = \frac{1-\log x}{x^2}$$

$$\text{or } \frac{dv}{dx} = v\ \frac{1}{x^2} - \frac{1-\log x}{x^2} = x^{1/x}\left(\frac{1-\log x}{x^2}\right)$$

$$\therefore \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^x\ (1+ \log x) + x^{1/x}\frac{(1-\log x)}{x^2}$$

(ii) Let $y = x^2\sqrt{\frac{2x-1}{x+1}} = = x^2\frac{(2x-1)^{1/2}}{(x+1)^{1/2}}$

taking logarithm of both sides, we get

$$\log y = \log\left[\frac{(2x-1)^{1/2}}{(x+1)^{1/2}}\right]$$

$$= \log x^2 + \log (2x - 1)^{1/2} - \frac{1}{2}\log (x + 1)$$

$$= 2 \log x + \frac{1}{2}\log (2x - 1) - \frac{1}{2}\log (x + 1)$$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}[2 \log x + \frac{d}{dx}\log (2x - 1) - \frac{1}{2}\log (x + 1)$$

$$\frac{1}{y}\ \frac{dy}{dx} = 2\ \frac{d}{dx}\ (\log x) + \frac{1}{2}\ \frac{d}{dx}[\log(2x - 1)] - \frac{1}{2}\ \frac{d}{dx}[\log(x + 1)]$$

$$= 2.\ \frac{1}{x} + \frac{1}{2}.\ \frac{1}{2x-1}\quad (2)\ \frac{1}{2}.\ \frac{1}{x+1}$$

$$= \frac{2}{x} + \frac{1}{2x-1} - \frac{1}{2(x+1)}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{2}{x}+\frac{1}{(2x+1)}-\frac{1}{2(x+1)}\right)$$

$$= \frac{x^2(2x-1)^{1/2}}{(x+1)^{1/2}}\left(\frac{2}{x}+\frac{1}{(2x+1)}-\frac{1}{2(x+1)}\right)$$

(iii) Let y = $\log\left(\frac{ax+b}{cx+d}\right)$

or y=log (ax+b) - log (cx + d)

Differentiating w.r.t.x, we get

$$\frac{dy}{dx} = \frac{d}{dx}[\log (ax + b) - \frac{d}{dx}\ \log (cx + d)]$$

$$= \frac{d}{dx}[\log (ax + b) - \frac{d}{dx}\ \log (cx + d)]$$

$$= \frac{1}{ax+b}\ (a) - \frac{1}{cx+d}.\ (c)$$

$$= \frac{a}{ax+b} - \frac{c}{cx+d}.$$

Derivative Implicit Functions $\frac{dy}{dx}$

If the function is given in the form f(x, y) = A, where

A is a constant and we want to find $\frac{dy}{dx}$(y), then we

differentiate both sides w.r.tx and then solve for $\frac{dy}{dx}$.

Example 6: Find $\frac{dy}{dx}$ if

(i) $x^{3/2}+y^{3/2} = a^{3/2}$

(ii) $ax^2+2hxy+by^2=0$

Solution: $x^{3/2}+y^{3/2} = a^{3/2}$

Differentiating w.r.t. x, we get d

$$\frac{d}{dx}(x^{3/2}+ y^{3/2}) + \frac{d}{dx}(a^{3/2})$$

or $\frac{3}{2}x^{½} + \frac{3}{2}y^{½}\frac{dy}{dx} = 0$

or $\frac{3}{2}y^{½}\frac{dy}{dx} = - \frac{3}{2}x^{½}$

or $\frac{dy}{dx} = - \frac{3/2x^{½}}{3/2y^{½}} = \frac{x^{½}}{y^{½}}$

(ii) $ax^2 + 2hxy + by^2 = 0$

Differentiating w.r.t. x, we get

$$\frac{d}{dx}[ax^2 + 2hxy + by^2] = \frac{d}{dx}(0)$$

$$\frac{d}{dx}[(ax^2) + \frac{d}{dx}(2hxy) + \frac{d}{dx}(by^2) = 0$$

or $a. 2x + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(by)^2 = 0$

or $a. 2x + 2h (x. \frac{d}{dx}x + y.1) + b2y. \frac{dy}{dx} = 0$

or $\frac{dy}{dx} = [2hx + 2by] = - 2 ax - 2hy$

or $\frac{dy}{dx} = \frac{-2[ax+hy]}{2[hy+by]}$

or $\frac{dy}{dx} = \frac{-2[ax+hy]}{2[hy+by]}$

SELF-CHECK EXERCISE 4.1

Q1. Differentiate w.r.t. x

(i) $\frac{d}{dx}(x^3)$ (ii) $\frac{d}{dx}(x^e)$

(iii) $(2x-4)^3$ (iv) $(2-4x)^5$

(v) $(7x-8)^4(5x-1)^3$ (vi) $e^x \log x$

(vii) $\frac{x+e^x}{1+\log x}$

4.4 FUNCTION OF A FUNCTION RULE

(1) If y = f(u) is derivable at u and u= g(x) is derivable at x, then y is derivable at x and

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

(2) If y = f(u), u= g(z), z=h(x) then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dz} \cdot \frac{dz}{dx}$$

Example 8: Find $\frac{dy}{dx}$ if

(i) y= $\sqrt{3x^4 + 5}$ (ii) y = $(ax+b)^n$

Solution: (i) y = $3x^4+5)^{½}$

Put u = $3x^4+5$ so that

y= $u^{½}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now $\frac{dy}{dx} = \frac{d}{du}$ $(u^{½}) = \frac{1}{2}u^{½-1} = \frac{1}{2}u^{½-1} == \frac{1}{2}$ $(3x^4+ 5)^{-½}$

$$\frac{dy}{dx} = \frac{d}{du} [3x^4 + 5] = 3.4x^3 + 0 = 12x^3$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(3x^2+5)^{½}} \times 12x3 = \frac{6x^3}{(3x^4+5)^{½}}$$

OR. We could have directly written as

$$\frac{dy}{dx} = \frac{d}{dx} (3x^4 + 5)^{½}$$

$$= \left[\frac{1}{2}(3x^4+5)^{½-1}\right] \left[\frac{d}{dx}(3x^4+5)\right]\left\{i.e. \frac{dy}{dx} = \frac{du}{dx}\right\}$$

$$= \frac{1}{2}(3x^4 + 5)^{½} [12x^3 + 0]$$

$$= \frac{6x^3}{(3x^4+5)^{½}}$$

(ii) y = $(ax + b)^n$. Put u = ax + b

So that y = u^n

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$\therefore \frac{dy}{dx}$ = n u^{n-1}. and $\frac{du}{dx}$ = a

$\therefore \frac{dy}{dx}$ = n u^{n-1}. a = na. $(ax + b)^{n-1}$

Or Directly we could have written

$\frac{dy}{dx} = \frac{dy}{dx}$ $(ax + b)^n$

= [n $(ax + b)^{n-1}$] $\left[\frac{d}{dx}(ax+b)\right]$

= n $(ax + b)^{n-1}$.a

= na $(ax + b)^{n-1}$

Example 9 : Differentiate w.r.t. x

(i) $\sqrt{\cos x} + \cos\sqrt{x}$ (ii) $\sin(2\sqrt{3x})$

(iii) $\log(1+ e^{2\sqrt{2}})$

Solution : (i) $\sqrt{\cos x} + \cos\sqrt{x}$

= then $\frac{d}{dx}(y) = \frac{d}{dx}\sqrt{\cos x} + \cos(\sqrt{x})$

= $\frac{d}{dx}(\cos x)^{½}\frac{d}{dx}\left(\cos(\sqrt{x})\right)$

= $\frac{1}{2}(\cos x)^{-½} . \frac{d}{dx}(\cos x)$ + - $(\sin\sqrt{x})\ \frac{d}{dx}(\sqrt{x})$

= $\frac{1}{2}(\cos x)^{-½}$ - (sin x) - $(\sin\sqrt{x})$. $\left(\frac{1}{2(\sqrt{x})}\right)$

= $\frac{1}{2}\sin x\,(\cos x)^{½}$ - $\frac{1}{2(\sqrt{x})}\sin\sqrt{x}$

= $\frac{1}{2}$-$\frac{\sin x}{\sqrt{\cos x}} - \frac{1}{2(\sqrt{x})}\sin\sqrt{x}$

(ii) Let y = sin (2 - $\frac{x+1}{\sqrt{x}}$x)

$\frac{d}{dx}$(y) = $\frac{d}{dx}$ sin (2 - $\frac{x+1}{\sqrt{x}}$x)

$= \cos(2 - \frac{u}{v}x).\ \frac{\frac{dy}{dx} = \sqrt{x}\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(\sqrt{x})}{\sqrt{x^2}}$ sin (2 - $\frac{\sqrt{x}(1) - (x-1) - \frac{1}{2\sqrt{x}}}{\sqrt{x^2}}$ x)

$= \cos(2 - \sqrt{x} - \frac{(x+1)}{2\sqrt{x}}x))\ \frac{2.x - (x-1)}{2\sqrt{x}.x}$

$= \cos(2 - \frac{2x - x - 1}{2x\sqrt{x}}x)\ \frac{x-1}{2x\sqrt{x}}$

$= - \sqrt{\frac{1+x}{1-x}}\ \frac{(1+x)^{1/2}}{(1-x)^{1/2}}$ cos (2 - $\frac{dy}{dx}$ x)

(iii) Let y = log $\frac{dy}{dx}$

$\therefore \quad \left(\frac{(1+x)^{1/2}}{(1-x)^{1/2}}\right)$ (y) = $\frac{u}{v}$ log $\frac{dy}{dx}$

$= \frac{(1-x)^{1/2}\frac{d}{dx}\left[(1+x)^{1/2}\right] - (1+x)^{1/2}\frac{d}{dx}\left[(1-x)^{1/2}\right]}{(1-x)} \ . \frac{(1-x)^{\frac{1}{2}}\frac{1}{2}(1-x)^{-\frac{1}{2}} - (1+x)^{\frac{1}{2}}\frac{1}{2}(1-x)^{-\frac{1}{2}}}{(1-x)}$ log

$\frac{\frac{1}{2}\frac{(1-x)^{1/2}}{(1+x)^{1/2}} - \frac{1}{2}\frac{(1+x)^{1/2}}{(1-x)^{1/2}}}{(1-x)}$

$= \frac{\frac{1}{2}\left[\frac{(1-x)}{(1+x)^{1/2}} - \frac{(1+x)}{(1-x)^{1/2}}\right]}{(1-x)} \ . \frac{1}{2}\ \frac{1}{(1+x)^{1/2}(1-x)^{1/2}(1-x)}\ \frac{1}{(1+x)^{1/2}(1-x)^{3/2}}$

$= \frac{1-\sqrt{x}}{1+\sqrt{x}}$

SELF-CHECK EXERCISE 4.2

Q1. Differentiate w.r.t. *x*

(i) y = $(3x - 5)^{-1/3}$

(ii) log [sin $(2x + 5x^2)$]

4.5 THEOREM : PARAMETRIC FUNCTIONS

If x = *f* (t) and y = g (t), then

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

Example 10 : Find $\frac{dy}{dx}$ if x = t e^t and y = 1 + log t

Solution : x = t e^t

$$\frac{d}{dt}(x) = \frac{d}{dt}(t\ e^t)$$

$$= t\ (e^t) + e^t\ (1)$$

$$= e^t\ (t + 1)$$

$$y \quad = 1 + \log t$$

$$\frac{d}{dt}(y) = \frac{d}{dt}(1 + \log t)\ \frac{1}{t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} = \frac{dx}{dt}$$

$$\frac{1}{t} + e^t\ (t + 1)$$

$$= \frac{1}{t(t+1)\ e^t}$$

Miscellaneous Examples

Example 11 : If y = log $\sqrt{x^2 + a^2}$ find $\frac{dy}{dx}$

(ii) y $\sqrt{x^2+1}$ = log (x + $\sqrt{x^2+1}$) show that

$$(x^2 + 1)\ \frac{dy}{dx} + xy - 1 = 0$$

$$\text{and } (x^2 + 1) \ \ + dx\ \frac{dy}{dx} + y = 0$$

Solution : (i) y = log (x + $\sqrt{x^2+1}$)

$$\therefore \frac{d}{dx}\ (y) = \frac{d}{dx}\ (x + \sqrt{x^2+1})$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})}\ \frac{d}{dx}\ (x + \sqrt{x^2+1})$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})}\left[1+\frac{1}{2}(x^2+a^2)^{-1/2}.(x)\right]$$

$$= \frac{1}{(x+\sqrt{x^2+a^2})}\left[\frac{x^2+a^2+x}{(\sqrt{x^2+a^2})}\right]$$

$$= \frac{1}{\sqrt{x^2 + +1}}$$

(ii) y $\sqrt{x^2+1}$ = log (x + $\sqrt{x^2+1}$)

$\therefore \frac{d}{dx}$ [y $\sqrt{x^2+1}$ = $\frac{d}{dx}$log (x + $\sqrt{x^2+1}$)

or y $\frac{d}{dx}$ $[x^2 +1)^{½}$ + $\sqrt{x^2+1}$ = $\frac{d}{dx}$ y.

$$= \frac{1}{(x+\sqrt{x^2+a^2})} \frac{d}{dx} (x + \sqrt{x^2+1})$$

or y $\frac{1}{2}$ $[x^2 +1)^{-½}$ (2x+ $\sqrt{x^2+1}$ = $\frac{d}{dx}$

$$= \frac{1}{(x+\sqrt{x^2+a^2})} \quad [1+ d\ (x^2 +1)^{-½}\ (2x)]$$

or $\frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\ \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \left[1+\frac{1}{x^2+2}\right]$

or $\frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\ \frac{dy}{dx} = \frac{1}{x+\sqrt{x^2+1}} \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$

or $\frac{xy}{\sqrt{x^2+1}} + \sqrt{x^2+1}\ \frac{dy}{dx} - \frac{1}{\sqrt{x^2+1}}$

or xy + $(x^2+ 1)$ $\frac{dy}{dx}$ = 1

Differentiating again w.r.t. x, we get

$$\frac{d}{dx}\left[x^2+1\ \frac{dy}{dx}\right] + \frac{d}{dy}\ (xy) - \frac{d}{dy}\ (1) = \frac{d}{dx}\ (0)$$

or $\left[x^2+1\ \frac{dy}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}.\frac{d}{dx}(x^2+1)\right]$

$$+ \left[x.\frac{dy}{dx}+y.1\right] - 0 = 0$$

or $(x^2 + 1)$ $\frac{d^2y}{dx^2}$ + $\frac{dy}{dx}$. 2x + x $\frac{dy}{dx}$ + y = 0

or $(x^2 + 1)$ $\frac{d^2y}{dx^2}$ +3x + x $\frac{dy}{dx}$ + y = 0

Example 12 : If x y = a + bx show that

$$\text{or} \quad x\frac{d^2y}{dx^2} + 2\ \frac{dy}{dx} = 0$$

Solution : xy = a + bx

$$\therefore \frac{dy}{dx}\ (\text{x y}) = \frac{dy}{dx}\ (\text{a + bx})$$

or $\quad$ $x.\ \frac{dy}{dx} + y.1 = b$

Differentiating again w.r.t. x, we get

$$\frac{d}{dx}\left[x \quad \frac{dy}{dx}\right] + y = \frac{d}{dx}\ (b) = (0)$$

or $\quad$ $\frac{d}{dx}\left[x^2 \quad \frac{dy}{dx}\right] + \frac{d}{dx}\ (y) = (0)$

or $\quad$ $x\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right] + \frac{dy}{dx}\left[\frac{dy}{dx}(x)\right]\frac{dy}{dx} = (0)$

or $\quad$ $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0 \quad \therefore \left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\right]$

or $\quad$ $x\frac{d^2y}{dx^2} + 2\ \frac{dy}{dx} = 0$

Hence the result.

Example 13: Differentiate the following functions

(i) $\quad$ $7x^2 + 2x$

(ii) $\quad$ $\log (x^2)$

Solution: $\quad$ (i) Let $y = 7x^2 + 2x$

Then $\quad$ $= 7^z$ where $Z = x^2 + 2x$

$$\frac{dy}{dx} = 7^z \log e^7 \text{ and } \frac{dz}{dx} = 2x + 2.1 = 2x + 2$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

2. $\quad$ $7x^2 + 2x - (x + 1) \log e^7$

(ii) $\quad$ Let $y = \log (x^2)$

then y = log z

where $z = x^2$

$$\frac{\partial y}{dx} = \frac{oy}{oz}\frac{oz}{ox}$$

$$= \frac{1}{2} \text{ x } 2x$$

$$= \frac{2x}{x^2} = \frac{2}{x}$$

Example 14. Given that

$y=(3x-1)^2 + (2x-1)^3$

Find $\frac{dy}{dx}$ and the points on the curve for which $\frac{dy}{dx} = 0$.

So. We have $y=(3x-1)^2 + (2x-1)^3$

$$\frac{dy}{dx} = 2(3x-1)(3) + 3(2x-1)^2 - (2)$$

$$= 18x-6+6(2x-1)^2$$

if $\frac{dy}{dx} = 0$, then $18x-6+6(2x-1)^2 = 0$

or $3x-1+4x^2-4x+1=0$

or $4x^2 - x = 0$

or $x(4x-1)=0$

$\therefore 0$ or $\frac{1}{4}$.

SELF-CHECK EXERCISE 4.3

Q1. Find $\frac{dy}{dx}$, when

(i) $x = 4t^2 + 3t + 1$, $y = 7t - 1$

(ii) $x = e^t \log t$, $y = t \log t$

4.6 Economic Application of Derivatives

We shall try to express some of the important concepts in economics in terms of derivatives and interpret the derivatives with reference to some economic relations.

1. If $p=f(q)$ is the demand curve then price elasticity of demand (e_d) is given by

$$e_d = \frac{dq|q}{dp|q} = \frac{p}{q}\cdot\frac{dq}{dp}$$

$$|e_d| = \frac{p}{q}\cdot\frac{dq}{dp}$$

Thus by differentiating the demand function, we canget $\frac{dq}{dp}$ and then get e_d

Example 15: A demand function is given by $q=bp^{-n}$ Calculate price elasticity of demand. Hence discuss the case when n=1

$$\frac{d}{dp}(q) = \frac{d}{dp}(b\,p^{-n}) = b\,\frac{d}{dp}(p^{-n}) = b. -n\,p^{-n+1}$$

or $\frac{dq}{dp} = -n\,p^{-n+1}$

$$e_d = -\frac{p}{q}\frac{dq}{dp} = -\frac{p}{q}\;(q = bp^{n+1})$$

$= n.b \quad \because q = bp^{-n}$

$= n\frac{p}{q}$

$= n.$

Thus the demand curve $q=b.p^{-n}$ has elasticity equal to n at all levels of prices.

when n= 1., demand function is

$q = b\,p^{-1}$

and elasticity $e_d = 1$

The curve $q=b\,p^{-n}$ is called the constant outlay curve and price elasticity of demand at any point is equal to unity. Such a demand curve is represented by rectangular hyperbola.

4.6.1 Revenue Functions and Cost Functions

(a) Marginal Revenue M.R. and Average Revenue Functions A.R.

Let R=pq be the total revenue function, then

$$MR = \frac{d}{dp}(R) = \frac{d}{dp}(pq) = p\frac{d}{dp}(q) + q\frac{d}{dp}(p) \qquad ....(1)$$

$$= q + q\,\frac{dq}{dp}$$

$$A\,R = \frac{R}{q} = \frac{dp}{q} - p$$

Since $M\,P = P + \frac{dq}{dp}$. we get

$$M\,R = P\left[\frac{p}{q} + \frac{q}{p}.\frac{dp}{dq}\right]$$

$$= p\left\{1 + \frac{1}{\frac{p}{q}\cdot\frac{dq}{dp}}\right\}$$

$$= p\left\{1 + \frac{1}{e_q}\right\}$$

$$\text{AR}\left\{1 + \frac{1}{e_q}\right\}$$

Thus (4) represent a relation between MR. AR and e_q

From (4) $|e_q| = \frac{AR}{AR - MR}$ ......(5)

which⇒ (i)If MR > 0, $|e_q|>1$

(ii)If MR = 0, $|e_q|= 1$

(iii)If MR < 0.1 $e_q<1$

(b) Cost Function

Let total cost function be taken as

= $aq^2 + bq+c$, a, b, c being constants

Example 16: Given the price equation p=100 – 2Qwhere q is quantity demanded, find

(i) the marginal revenue

(ii) point elasticity of demand when Q = 10

(iii) nature of the commodity.

Solution: (i) Since marginal revenue (MR) isobtained by differentiating the total revenue function with respect to output Q, we find out total revenue first, which is defined as

TR=AR x Q

TR = (100 – 2Q) Q

= $100Q – 2Q^2$

MR = 100 – 4Q

(ii) Point elasticity of demand is obtained from the following relation.

$|e_d| = \frac{AR}{AR - MR}$ when Q = 10

MR=100 – 4 x 10=60

P =AR=100 – 2 x 10= 80

$|e_d| = \dfrac{80}{80-60} = 4$

Example 17: A consumer has a utility function u=u = $\propto(Q)^{\beta}\propto> 0; 0< \beta<1$.

Does the utility function display diminishing marginalutility?

Solution: A utility function will display diminishingmarginal utility if the slope of marginal utility curve isnegative.

Now marginal utility (Mv) is given by the derivative of the utility function

$$\therefore Mv = \frac{du}{dQ} = \propto_{\beta}\ Q^{\beta-1}$$

Now slope of Mv is given by

$$\frac{d}{dQ}(Mu) = \frac{d^2u}{dQ^2}$$

$$= \propto (\beta - 1)\ \beta\ Q\ (\beta - 1)^{-1}$$

$$= \propto (\beta - 1)\ \beta.Q^{\beta-2}$$

Since $|> \beta >0, (\beta -1) <0$

$\therefore \dfrac{d^2U}{dQ^2} < 0$ and the utility function

u=$\propto Q^{\beta}$ displays diminishing marginal utility.

Example 18 : Given the consumption function

$$C=C(y)=1000 - \frac{5000}{3+y}$$

(i) Find marginal propensity to consume when y = 97.

(ii) Find marginal propensity to save when y = 97.

(iii) Determine whether MPC ans MPS move in thesame direction when y changes.

Solution: MPC is given by the differentiation of the function C=1000-$\dfrac{5000}{3+y}$ with respect to y.

Now $C=1000- 5000\ (3 + y)^{-1}$

$$MPC = \frac{dc}{dy} = 0 - 1\ (-1)\ \frac{5000}{(3+y)^2}$$

$$= \frac{5000}{(3+97)^2} = \frac{5000}{(100)^2} = \frac{5000}{10000} = 0.5$$

(ii) Saving function is defined as

$S = y - c$

$S = y - 1000 + 5000\,(3 + y)^{-1}$

MPS = $\frac{dc}{dy} = 1 - 0 + (-1)\ \frac{5000}{(3+y)^2}$

$= 1 - 0.5 = 0.5$

(iii) In order to verify whether MPC and MPS move in the same direction or not, we are to find out the rate of growth of MPC and MPS. That means we are to find out the derivatives of MPC and MPS.

Now $\frac{d}{dy}$ (MPS) = $\frac{d^2S}{dy^2} = -\,(-2)\ \frac{5000}{(3+y)^2}$

since $\frac{d^2C}{dy^2} < 0$ and $\frac{d^2S}{dy^2} > 0$, MPC and

MPS Move in the opposite direction as y changes.

Example 19: (i) Find the total revenue, marginal revenue at q=3. If the demand curve is p= $\sqrt{10-2q}$

(ii) Find the Marginal cost, Average cost and their slopes if the total cost function is $\pi = 0.4q^3 - 0.9q^2 + 10q + 10$.

Solution: (i) Total revenue = px q=q (TR)

Marginal Revenue = $\frac{d}{dq}$(TR)= $\frac{d}{dq}$ $[(10-2q)^{½}]$

(MR)

$$= q\left[\frac{1}{2}(10-2q)^{-½}(-2)\right] + 10\text{-}2q^{½} \qquad (1)$$

$$= \frac{-p}{\sqrt{10-2q}} + \sqrt{10-2q}$$

$$= \frac{-q+\sqrt{10-2q}}{\sqrt{10-2q}}$$

$$= \frac{10-2q}{\sqrt{10-2q}}$$

$\therefore$ TR at q = 3 is equal to 3. $\sqrt{10-2.3}$ = 3. $\sqrt{4}$ = 3.2 = 6

MR at q = 3 is equal to $\frac{10-3.3}{\sqrt{10-2\times 3}} = \frac{10-9}{\sqrt{4}} = \frac{1}{2}$

(ii) Total Cost (TC) is given as

π = .04 q³-0.9q² + 10q = 10

$\therefore$ Marginal Cost (MC)= $\frac{d}{dq}$ (π)

= $\frac{d}{dq}$(0.4q^3-0.9q^2+10q+10)

= .04 × 3q³ -0.9 x 2q + 10

= 12q² - 1.8q+10

Average Cost (AC) = $\frac{\pi}{q}$ = (0.4q² -0.9q² + 10 + $\frac{10}{q}$)

Slope of MC = $\frac{d}{dq}$ (MC)= $\frac{d}{dq}$ (.12q²-1.8q+10) = 024.18

Slope AC = $\frac{d}{dq}$ (AC) = $\frac{d}{dq}$(0.4q² -0.9q² + 10 + $\frac{10}{q}$) = 0.8a + 0.9- $\frac{10}{q^2}$

SELF-CHECK EXERCISE 4.4

Q. 1 A demand function is given by $q = ap^{-n}$ calculate price elasticity of demand.

Q. 2 Given the price equation p = 100 – 2Q where q is quantity demanded, find

(i) marginal revenue

(ii) point elasticity of demand when Q = 10

4.7 SUMMARY

In this unit we studied the concept of differentiation. Then we have discussed various theorems of differentiation. Lastly the use of differentiation to find out the Marginal Revenue, Average Revenue, Average cost and Marginal cost was illustrated.

4.8 GLOSSARY

1. **Differentiation :** Differentiation is a method used to fine the slope of function at any point.
2. **Derivative :** The derivative is the instantaneous rate of change of a function with respect to one of its variables.

4.9 ANSWER TO SELF CHECK EXERCISE

Self-check Exercise 4.1

Ans. Q1. (i) $3x^2$ (ii) ex^{e-1} (iii) $6(2x-4)^2$ (iv) $20(2-4x)^4$

(v) $(7x-8)^3(5x-1)^2(245x-148)$ (vi) $e^x\left[\frac{1}{x}+\log x\right]$ (vii)

$$\frac{\log x(1+e^x)+e^x\left(1-\frac{1.}{x}\right)}{(1+\log x)^2}.$$

Self-check Exercise 4.2

Ans. Q1. (i) $-(3x-5)^{-4/3}$ (ii) $10.x \dfrac{\cos(2+5x^2)}{\sin(2+5x^2)}$

Self-check Exercise 4.3

Ans. Q1. (i) $\frac{7}{8t+3}$ (ii) $\frac{t(c+log\ t)}{et\ (1+tlo\quad t)}$

Self-check Exercise 4.4

Ans **Q1.** Refer to Section 4.6 (Example 15)

Q2. (i) MR = 100 – 4Q (ii) |ed| = 4

4.10 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
4. Yamane, T. (2012). Mathematic for Economists : An Elementary Survey. Pretice Hall of India, New Delhi.

4.11 TERMINAL QUESTIONS

Q.1 Show that the demand curve qq^a =b, where a and b are constants has constant elasticity equal to $-a$.

Q. 2 Find total Revenue (R). Marginal Revenue (R') at $q = 0$, $q = 5$ for the demand curve $p = 100 - e^q$.

Unit-5

PARTIAL DERIVATIVES & HOMOGENEOUS FUNCTIONS

STRUCTURE

5.1 INTRODUCTION

Till now we have considered functions of 4.1 one independent variable only viz. V = $f(x)$. But in economics, we have relations involving more than one independent variables for example, the demand for ghee depends not only on the price of ghee but on the price of other related goods also. Consequently we define functions of more than one variable. Partial Derivatives.

5.2 LEARNING OBJECTIVES

After the completion of Unit, the student will learn

- The meaning if Partial derivatives
- To apply the techniques of obtaining partial derivatives
- To explain higher order partial derivatives
- To apply the derivatives to solve economic problem.

5.3 PARTIAL DERIVATIVES

Definition : Function of Two Variables. Let u be a symbol which has one definite value for every permissible pair of value of the independent variables x and y, then u is called a function of the two variables x and y and we write u = *f*(x, y).

Similarly, we can define a function of the variables and write it is as u = $f(x_1, x_2, ...x_n)$ where $x_1, x_2 ....x_n$ are n independent variables.

Definition: Partial Derivative, Let u= *f*(x, y) be a function of two variables x and y, then the partial derivative of u w.r.t, x is defined to be the ordinary derivative at u w.r.t. x regarding y as constant. Similarly the partial derivative of u w.r.ty is the ordinary derivative of u w.r.t. y regarding x as constant and we write as

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\partial x, y) - f(x,y)}{\partial x}$$

Thus while finding partial derivative of z= *f*(x, y) w.r.t. x at (x, y), we assume that y remains fixed and the change in the function is due to the change in x from x to x+ δx. This renders the function of two variables as the function of a single variable.

Similarly the partial derivative of u= *f*(x, y) w.r.t. y at (x, y) defined as

$$\frac{\partial u}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\partial x, y) - f(x,y)}{\partial x}$$

Notation : Partial derivative of u = *f* (x, y) w.r.t. x is written as

$\frac{\partial u}{\partial x}$ or $\frac{\partial f}{\partial x}$ or u_x or f_x or u_1 or f_1

Partial derivative of u = f (x, y) w.r.t. y is written as $\frac{\partial u}{\partial y}$ or $\frac{\partial f}{\partial y}$ or u_y or f_y or u_2 of f_2

It may be noted that $\frac{\partial u}{\partial y}$ at (x, y) does not depend on x only but depends upon both x and y.

Similarly $\frac{du}{dy}$ depends upon both x &y.

Actually $\frac{\partial u}{\partial x}$ means the relative change in u due to a small unit change in x, regarding y as constant and similarly $\frac{\partial u}{\partial y}$ means the relative change in *u* due to a small unit change in y, regarding x as constant. We shall explain this concept through an example.

Let x be labour, y be land and u be wheat. If we have the functional relationship x, y and u.

then u = *f*(x, y)

i.e. Wheat production depends on land and labour. Our problems is to find the change in wheat (u). When there is a small unit change in the amount of labour (x) holding land (y) constant.

Similarly we want to find the change in wheat (u), when there is a small unit change in the amountof land (y) holding labour (x) constant.

The first problem is equivalent to the partial derivative of u w.r.t. x, regarding y as constant and the second problem is equivalent to finding the partial derivative of u w.r.t. y, regarding x as constant and in notations we would write:

$\frac{\partial u}{\partial y}$ = partial derivative of u w.r.tx.

= change in u due to a small change in x regarding y as constant.

$\frac{\partial u}{\partial y}$ = partial derivative of u w.r.ty.

= change in u due to a small in y regarding x asconstant

Thus partial derivative of a function w.r.t. a variable represents the relative change in the function due to small change in that variable regarding all other variables as constant.

5.3.1 TECHNIQUE OF OBTAINING PARTIAL DERIVATIVES

While obtaining partial derivatives, the variable with which we are not directly concerned is to be regarded as constant. This makes the technique of partial derivative quite similar to that of ordinary partial derivative. Therefore, the rules for theorems used for finding partial derivatives are similar to those applied for finding derivatives. For example If u is a single-valued function of x and y,

ie. u = *f*(x, y), then

1. $\frac{\partial}{\partial x}$ $(u)^n = nu^{-1} \frac{\partial u}{\partial x}$, $\frac{\partial}{\partial y}$ $(u)n = nu^{n-1} \frac{\partial u}{\partial y}$

2. $\frac{\partial}{\partial x}$ $(a^u) = a^u \log a. \frac{\partial u}{\partial x}$, $\frac{\partial}{\partial y}$ $(a^u) = a^u \log a. \frac{\partial u}{\partial y}$

3. $\frac{\partial}{\partial x}$ $(e^u) = a^u \frac{\partial^2 u}{\partial y^2}$, $\frac{\partial}{\partial y}$ $(e^u) = e^u \frac{\partial u}{\partial y}$

4. $\frac{\partial}{\partial x}$ (log u) = $\frac{1}{u}\frac{\partial u}{\partial x}$, $\frac{\partial}{\partial y}$ (log u) = $\frac{1}{u}\frac{\partial u}{\partial y}$.

SELF-CHECK EXERCISE 5.1

Q1. Differentiate Z = $6x^3 + 5x^2 + 10xy$, partially with respect of *x* twice.

Q2. $f(x, y) = x^3 + y^5 + 4x^2y^4$, find *fxy* and *fyx*

5.4 HIGHER ORDER PARTIAL DERIVATIVES

The technique of obtaining higher order partial derivatives is the same as we applied for higher order derivatives.Ifu= *f*(x,y) then we have defined $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ as the first order partial derivatives of u w.r.t x and y respectively. If we find the partial derivative of the firstorder partial derivative get second order partial derivatives.The partial derivative of $\frac{\partial u}{\partial y}$ w.r.t. x is called thesecond order partial derivative of w.r.t. x andis written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x^2} \text{ or } \frac{\partial^2 u}{\partial u^2} \text{ or } u^{xx}f^{xx}$$

Similarly order partial derivative of $\frac{\partial x}{\partial y}$ w.r.t. of y is calledthe second order partial derivative of u w.r.t. y and is written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } u_{yy}f_{yy}$$

The partial derivative of $\frac{\partial u}{\partial x}$ w.r.ty and of $\frac{\partial u}{\partial y}$ w.r.t. x are called the second order cross partial derivatives of u and are written as

$$\frac{\partial}{\partial x}=\frac{\partial^2 u}{\partial y \partial x} \text{or } \frac{\partial^2 f}{\partial yax} \text{ or } u_{xy}f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 f}{\partial yax} \text{or } \frac{\partial^2 f}{\partial xay} \text{ or } u_{yx}f_{yx}$$

Thus we see that a function of two variables u = *f*(x, y) yields

(i) Two first order partial derivatives viz.

f_x and f_y, and

(ii) Four second order partial derivatives, viz

$f_{xx}, f_{yy}, f_{yx}, f_{yx}$,

Example: 1 Find the first order and second order partial derivatives of

$u = 2x^2 + 4xy + 5y^2$

Solution: $u = 2x^2 + 4xy + 5y^2$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x^2 + 4xy + 5x^2)$$

$$= 2\frac{\partial}{\partial x}(x^2) + 4y\frac{\partial}{\partial x}(x) + 0$$

$= 4x + 4y$ ($\because$ y is treated as constant)

$$\frac{dy}{dy} = \frac{\partial}{\partial y}(2x^2 + 4xy + 5x^2)$$

$$= 0 + 4x\frac{\partial}{\partial y}(y) + 5\frac{\partial}{\partial y}(y^2)$$

$= 4x + 10y$

$$\frac{\partial u^2}{\partial x^2}\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(4x + 4y) = 4$$

$$\frac{\partial^2 u}{\partial y^2}\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(4x + 10) = 10$$

$$\frac{\partial^2 u}{\partial y \partial x}\ \frac{\partial}{\partial y} = \left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(4x + 4y) = 4$$

$$\frac{\partial^2 u}{\partial x \partial y}\ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(4x + 10y) = 4$$

Example 2: Find all the first order and second order partial derivatives of the function

$u = \log(x^2 + y^2)$

Solution: $u = \log(x^2 + y^2)$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[\log(x^2 + y^2)]$$

$$= \frac{1}{x^2 + y^2}\frac{\partial}{\partial x}(x^2 + y^2)$$

$$= \frac{1}{x^2 + y^2}2x = \frac{2x}{x^2 + y^2}$$

and $\frac{\partial u}{\partial y}[\log(x^2 + y^2)]$

$$= \frac{1}{x^2 + y^2}\frac{\partial}{dy}(x^2 + y^2)$$

$$= \frac{1}{x^2 + y^2} \text{ 2y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \frac{\partial}{\partial x}\left(\frac{2x}{x^2 + y^2}\right)$$

$$= \frac{\left(x^2 + y^2\right)\frac{\partial}{\partial x}(2x) - 2x\frac{\partial}{\partial x}\left(x^2 + y^2\right)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{\left(x^2 + y^2\right)(2) - 2x\left(2x\right)}{\left(x^2 + y^2\right)^2}$$

$$= \frac{\left(2x^2 + 2y^2\right) - 4x^2}{\left(x^2 + y^2\right)^2}$$

$$= \frac{2y^2 - 2x^2}{\left(x^2 + y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \frac{\partial}{\partial y}\left(\frac{2y}{x^2 + y^2}\right)$$

$$= \frac{(x^2 + y^2)\frac{\partial}{\partial x}(2y) - 2u\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2)2 - 1y(2y)}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$$

Example 3: If u = 2 (ax + by)^2 - (x^2 + y^2) show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = 4\ (a^2 + b^2) - 4$$

Solution : u = 2 (ax = by)^2 - (x^2 + y^2)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ [2\ (ax + by)^2 - (x + y^2)$$

$=2.2\ (ax+b)^1, \frac{\partial}{\partial x}\ (ax + by) - (x^2 + y^2)$

$= 4\ (ax + by)\ (a) – 2x$

$= (ax + by) - 2x$

$= \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\ [4(a^2 + b^2) -4]$

$=4a.a-2=4a^2-2$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\ [2\ (ax + by)^2 - (x^2 + y^2)]$

$= 2.2\ (ax + by) \frac{\partial}{\partial y}(ax + by - \frac{\partial}{\partial y}(x^2 + y^2)$

$=4(ax+by).\ (b) - 2y$

$= 4b\ (ax + by) -2y$

$=2(ax+by)^2 - (x^2 + y^2)$

$= \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\ [4(a^2 + b^2) -4]$

$= 4b.b - 2$

$= 4b^2 - 2$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4a^2 - 2 + 4b2 - 2$

$= 4a^2 + 4b^2 - 4$

$=4(a^2 + b^2+) - 4$

Example 4: Find all the second order cross partial derivations for the function

$u=x^4-5xy^3+6x^2 + 2xz^2 - xyz.$

Solution: Here u = *f*(x, y, z)

$\therefore$ the second order cross partial derivatives are given by

$$\frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial x}, \frac{\partial^2 u}{\partial x \partial z}, \frac{\partial^2 u}{\partial z \partial x}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial z \partial y}$$

Since $u = x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ (x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$

$= 4x^3 - 5y^3 - 1 + 12x + 12z^2 - yz$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$$

$= - 15xy^2 - xy$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial x}(x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$$

$= 4xz - xy$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$$

$= - 15y^2 - z$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial z}(x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$$

$= 4z - y$

$$\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}(x^4 - 5xy^3 + 6x^2 + 2xz^2 - xyz)$$

$= 4z - y$

$$\frac{\partial^2 u}{\partial u \partial u} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial z}(-15xy^4 - xy) = -x$$

$= 4z - y$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}(4xz - xy)$$

$= -x$

Change of order of Differentiation

If $u = f(x, y)$ fx, fy, fxy, fyx, are all continuous at the point (x, y) then

Example 5 : Verify that

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} \text{ if } u \ \frac{xy}{\sqrt{x^2 + y^2}}$$

Solution : $u = \dfrac{xy}{\sqrt{x^2 + y^2}}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{\sqrt{x^2 + y^2}}\right)$$

$$= \frac{\sqrt{x^2} + y^2 \frac{\partial u}{\partial x}(xy) - xy\frac{\partial u}{\partial x}\sqrt{x^2} + y^2}{\left(\sqrt{x^2 + y^2}\right)^2}$$

$$= \frac{\sqrt{x^2 + y^2}\,(y) = xy\left[\frac{1}{2}(x^2 + y^2)^{-1/2}.2x\right]}{\left(x^2 + y^2\right)}$$

$$= \frac{y\sqrt{x^2 + y^2}\,, x^2 y}{\left(x^2 + y^2\right)\left(x^2 + y^2\right)}$$

$$= \frac{y^3}{\left(x^2 + y^2\right)^{3/2}}$$

$$= \frac{\sqrt{x^2 + y^2}\,\frac{\partial}{\partial x}(xy) - xy\frac{\partial u}{\partial y}\sqrt{x^2 + y^2}}{(x^2 + y^2)}$$

$$= \frac{\sqrt{x^2 + y^2}\,(x) - xy\frac{1}{2}(x^2 + y^2)^{-1/2}.(2y)}{(x^2 + y^2)}$$

$$= \frac{\sqrt{x^2 + y^2} - \frac{x^2 y}{\left(\sqrt{x^2} + y^2\right)}}{\left(\sqrt{x^2} + y^2\right)}$$

$$= \frac{y(x^2 + y^2).\ xy^2}{\sqrt{x^2} + y^2(x^2 + y^2)}$$

$$= \frac{y^3}{\left(x^2 + y^2\right)^{3/2}}$$

$$\therefore \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{y^3}{\left(x^2 + y^2\right)^{3/2}}\right)$$

$$= \frac{\left(x^2 + y^2\right)^{3/2}\frac{\partial}{\partial y}(y^3) - y^3\frac{\partial}{\partial y}\left(x^2 + y^2\right)^{3/2}}{\left(x^2 + y^2\right)^2}$$

$$= \frac{3y^2\left(x^2+y^2\right)^{1/2} - 3y^4\left(x^2+y^2\right)^{-1/2}}{\left(x^2+y^2\right)^3}$$

$$= \frac{3y^2\left(x^2+y^2\right)^{1/2}\left[\left(x^2+y^2-y^2\right)\right]}{\left(x^2+y^2\right)^3}$$

$$= \frac{3x^2+y^2}{\left(x^2+y^2\right)^3}\left(x^2+y^2\right)^{1/2} = \frac{3x^2+y^2}{\left(x^2+y^2\right)^{5/2}}$$

$$= \frac{3x^2+y^2}{\left(x^2+y^2\right)^{5/2}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{y^3}{\left(x^2+y\right)^{3/2}}\right)$$

$$= \frac{\left(x^2+y^2\right)^{3/2}\frac{\partial}{\partial x}(x^3) - x^3\frac{\partial}{\partial x}\left(x^2+y^2\right)^{3/2}}{\left(x^2+y^2\right)^3}$$

$$= \frac{\left(x^2+y^2\right)^{3/2}(3x^2) - x^3\left[\frac{3}{2}\left(x^2+y^2\right)^{-1/2}.(2x)\right]}{\left(x^2+y^2\right)^3}$$

$$= \frac{3x^2\left(x^2+y^2\right)^{3/2} - 3x^4[x^2+y^2)^{1/2}}{\left(x^2+y^2\right)^3}$$

$$= \frac{3x^2\left(x^2+y^2\right)^{1/2} - [x^2+y^3-x^2 \quad .}{\left(x^2+y^2\right)^3}$$

$$= \frac{3x^2(x^2+y^2)^{1/2} - y^2}{\left(x^2+y^2\right)^{2/2}}$$

$$\frac{3x^2+y^2 \quad .}{\left(x^2+y^2\right)^{3/2}}$$

Hence $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

SELF-CHECK EXERCISE 5.2

Q1. $f(x, x^2) = \log(x_1^2 + x_2^2)$. Find the second order partial derivatives.

Q2. $f(x, y) = x^2 + xy + y^2$. Find d^2y

Q3. $y = 4x^5 + yx^4 + 3x + 9$, find third order derivative.

5.5 Total Differential and total Derivative

(a) Ifu= f(x, y) be a function of two variables, thendu

= total change in u due to change in x and y

= (change in u due to change in x) +

(change in u due to change in y)

=(change in u due to a unit change in x ×

change in x)+(change in u due to a unit

change in y× change in y).

$$= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} . dy$$

du is called the *total differential of u*.

(b)if u = f(x, y) be a function of two variables,

x= φ(t) and y = Ψ(t)

then $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial t}$

$\frac{du}{dt}$ is called the total derivative of u. Now we shall explain its meaning. We know that $\frac{\partial u}{\partial x}$ is the change in u due to small unitchange in x holding y constant. Furthermore $\frac{dx}{dt}$ is the change in x due to small unit change in t. Thus $\frac{\partial u}{\partial x} . \frac{dx}{dt}$ is the amount of change in u due to a small unit change in t that is transmitted through x.Likewise $\frac{\partial u}{\partial y} . \frac{dy}{dt}$ is the amount of change in u due to a small unit Likewise $\frac{\partial u}{\partial y} . \frac{dy}{dt}$ is the amount of change in udue to a small unit change is t that is transmitted through x.

∴the change in u due toa small unit change in t will be the sum of these two effects, which we write as

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$y = \phi(x)$,

Note: If $u = f(x, y)$ and $y = \Psi(x)$ then

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

and $x = \phi(x)$,

Similarly if $u = f(x, y)$ and $x = \Psi(y)$ then

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\frac{dx}{dy} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y}$$

$$= \frac{\partial u}{\partial x} + \frac{dx}{dy}\frac{\partial u}{\partial y}$$

Example 6: Find the total derivative of u w.r.t. t if $u = x^2+y^2$, $x=t$, $y=2t$. Also find the total differential du.

Solution: $u= x^2+y^2$, $x=1, y=2t$

We know that $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$

Here $\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(x^2 + y^2) = 2x$

$$\frac{dx}{dt} = \frac{\partial}{\partial x}(x)\ \frac{d}{dt}(t) = 1.$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(u) = \frac{\partial}{\partial y}(x^2 + y^2) = 2y$$

$$\frac{dy}{dt} = \frac{d}{dt}(y)\ \frac{d}{dt}(2t) = 2.$$

$$\therefore \frac{dy}{dt} = 2x.1 + 2y.2$$

$$= 2x + 4y$$

Total differential du is give by

$$du = \frac{\partial u}{\partial x}.dx + \frac{\partial u}{\partial y}$$

$$= 2x.\ dx + 2y.\ dy.$$

SELF-CHECK EXERCISE 5.3

Q1. Find the total differential of the function

$y = ax_1^2 + 2hx_1 x_2 + bx_2$

5.6 Application in Economics

Example 7: Let u be utility and x and y be two goods. Then the utility function u = f(x, y) show that the marginal rate of substitution of y for x given by $\frac{dy}{dx}$ is equal in magnitudeto the ratio of the marginal utilities (M.U's) taken in reverse order.

Solution: We assume u is constant because along an indifference curve different combinations of x and y give the same utility.

Let: u= f(x, y)

$$\frac{du}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}$$

$$= f_x + f_y.\ \frac{dy}{dx}$$

Since u is a constant, $\frac{du}{dx} = 0$

$$\therefore f_x + f_y.\ \frac{dy}{dx} = 0$$

$$\text{or } f_y.\ \frac{dy}{dx} = -f\text{ x}$$

$$\text{or } \frac{dy}{dx} = \left[\frac{f_x}{f_y}\right]$$

But $f_x = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x}$ Marginal utility of x = Mu_x

$f_y = \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y}$ Marginal utility of y = Mu_y

$\therefore = \frac{dy}{dx}$ = marginal arate of substitution = $-\frac{MU_x}{MU_y}$

Example 8: If f(x, y)=0 show that

(i) $\frac{dy}{dx} = -\left(\frac{f_x}{f_y}\right)$ ...(1)

(ii) $\frac{d^2y}{dx^2} = \frac{\left[fxxfy - 2\,fxyfxfy + fyyfx^2\right]}{fy^3}$

Solution: We have already shown in the above example that if f(x) = c then

$$\frac{dy}{dx} = -\left(\frac{f_x}{f_y}\right) \qquad (1)$$

Here c = 0 but $\frac{d}{dx}$ (c) = $\frac{d}{dx}$ (0)

$\therefore$ Result is the same.

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \left[-\frac{f_x(x,x)=}{f_y(x,y)=} - \frac{d}{dx}\frac{f_x(x,x)}{f_y(x,y)}\right]$$

$$= \frac{d}{dx}\frac{f_x\left[\frac{d}{dx}(f)\right] - f_x\left[\frac{d}{dx}(f_y)\right]\left[\frac{f_x = f_x(x,y)}{f_y = f_y(x,y)}\right]}{f^2}$$

But $\frac{d}{\partial x}$ (f_x) = $\frac{d}{dx}$ [f_x)x, y)]

$$= \frac{\partial}{\partial x}(f_x)\frac{dx}{dx} + \frac{\partial}{\partial y}(f_x)\frac{dy}{dx}$$

$$= f_{xx} + f_{xy} = \frac{dy}{dx}$$ Applying formula

for total derivative

$$= f_{xx} + f_{xy} = \left(-\frac{f_x}{f_y}\right)$$

$$= \frac{f_y + f_{xx} - f_x + f_{xy}}{f_y} \qquad ...(3)$$

$$\frac{d}{dx}(f_y) = \frac{d}{dx}[f_x(x, y)]$$

$$= \frac{\partial}{\partial x}(f_y)\frac{dx}{dx} + \frac{\partial}{\partial y}(f_y)\frac{dy}{dx}$$

$$= f_{yx} + f_{yy} = \left(-\frac{f_x}{f_y}\right)$$

$$= \frac{f_y f_{xy} - f_x f_{yy}}{f_y} \qquad ...(4)$$

Putting these values in (2) we get

$$\frac{d^2y}{dx^2} = \left[f_y \left\{ \frac{f_y f_{xx} - f_x f_{xy}}{f_y} \right\} - f_y \left\{ \frac{f_y f_{yx} - f_x f_{yy}}{f_y} \right\} + (fy)^2 \right]$$

$$= \frac{(f_y)^2 f_{xx} - f_y f_x f_{xy} - f_x f_{yx} fyx + (fx)^2 fyy}{(f_y)^3}$$

$$= \frac{\left[f_{xx}(f_y)^2 2 f_x f_y fyx + fyx + (fx)^2 \right]}{(f_y)^3}$$

$$(\therefore f_{xy} = (f_{yx})$$

which is the required result.

Example 9: A consumer consumes two commodities x_1 and x_2 and the utility function is given byu = $x^2+3x_1\ x_2+5x_2$, Find out marginal utilities of x_1 and x_2

Solution: The marginal utility is the increase in total utility as a result of consumption of additional unit and is given by the derivative. Since the utility function involves two variables x_1 and x_2, the marginal utility of x_1 and x_2 will be given by the partial derivative of u with respect to x_1 and x_2 respectively.

Marginal utility of x_1 is given by

$\frac{du}{dx_1}$ = $2x_1 + 3x_2 + 0$ (since x_2 is constant)

= $2x_1 + 3x_2$

Similarly, marginal utility of x_2 is given by

$\frac{du}{dx_2}$ = $0 + 3x_1 + 5$ (since x_1 is constant)

= $3x_1 + 5$

Example 10: Given a demand curve of Engel's curvetype

$D=AP^{\propto} N^{\beta}$

where D is demand, P is price, N is income and A,$\propto$, β are parameters. Find the partial derivatives $\frac{\partial D}{\partial P}$ and $\frac{\partial D}{\partial N}$ andalso interpret the value of $\propto$& β.

Solution: In the function $D = A\ P^{\propto} N^{\beta}$, when we differentiate D with respect ot P, N is taken to be constant.

$\therefore \frac{\partial D}{\partial P}$ = $(AP^{\propto-1} N^{\beta}$

or $\frac{\partial D}{\partial P} = \propto . \frac{AP^{\propto} N^{\beta}}{P} = \propto \frac{D}{P} \propto$

Similarly $\frac{\partial D}{\partial P}$ = β. $AP^{\propto}N^{\beta-1}$

$= \beta . \frac{AP^{\propto} N^{\beta}}{P}$ (since P is constant)

$$\frac{\partial D}{\partial N} = \beta \frac{D}{N}$$

From the above partial derivatives

$$\propto = \frac{\partial D}{\partial P} \Big/ \frac{D}{P} = \frac{\partial D}{D} \Big/ \frac{\partial P}{P}$$

= Proportionate change in demand

Proportionate change in price

=Price elasticity of demand

Similarly,

$$\beta = \frac{\partial D}{\partial N} \Big/ \frac{D}{N} = \frac{\partial D}{D} \Big/ \frac{\partial P}{N}$$

= Proportionate change in demand

Proportionate change income

= Income elasticity of demand

∴ ∝ and β represent price elasticity and incomeelasticity of demand respectsvely.

Singns of Partial Derivatives

Ifu= $f(x, y)$, then f_x shows the rate of change of u w.r.t x treating y as constant and f_{xx}shows the rate of change f_x w.r.t x treating y as constant.

∴f_{xx} shows whether the function is increasing at increasing rate, decreasing rate or constant rate, when x varies and y remains constant. Similarly f_{yy} shows the rate of change of f_y w.r.t y when x is treated as constant.

(1)$f_x > 0$ means that the function increases as x increases, treating y as constant. $f_x<0$ means that the function decreases as x increase treating y and constant.

(2)$f_{xx}>0$ means that the rate of change of the function increases as x increases, treating y as constant. $f_{xx}<0$means the function changes at a decreasing rate.

Similarly we can interpret signs of f_y and f_{yy}

(3) $f_{xy} = f_{yx} < 0$ means that f_x decreases as y increases and f_y decreases as x increases.

(4) $fxy=fxy> 0$ means that f_x increase as y increase and f_y increase as x increase.

(5)$f_{xy} = f_{yx} = 0$ means that there is no interaction between the variables.

Marginal Cost and Marginal Products

(a) If the joint-cost function for producing the quantities x and y or two commodities is given by

$c = f(x, y)$

then the partial derivatives of c are the *marginal cost functions.*

$\frac{\partial c}{\partial x}$ is the marginal cost wr.t. y

$\frac{\partial c}{\partial y}$ is the marginal cost wr.t. x

In most economic situations, marginal costs are positive.

For example, If the joint-cost function for producing quantities x and y of two commodities is c = x log (5+y), then

$\frac{\partial c}{\partial y}$ log (5+y) is the marginal cost w.r.t. x,

$\frac{\partial c}{\partial y} = \frac{x}{5+y}$ is the marginal cost w.r.t. y

(b) The production of most commodities requires the use of at least two factors of production, for example, labour, land, capital, machines, or materials. If the quantity u of a commodity is produced using the amounts x and y, respectively of two factors of production, then the production function u= f(x, y) gives the relationship between output u andinputs x and y. The partial derivative $\frac{\partial u}{\partial x}$ ofu w.r.t. x holding y as constant is the marginal productivityof x or the marginal products of x and the partial derivative $\frac{\partial u}{\partial y}$ of u w.r.t. y holding x as constant is the marginal productivity of y or the marginal products of f_y. It may be noted that the marginalproductivity of either input is the rate of increase of the total products as that input is increased, assuming that the amount of other input remains constant. For example, if the production function is

$u = 4xy - x^2 - 3y^2$

then $\frac{\partial u}{\partial x}$ = marginal product of x = 4y - 2x

$\frac{\partial u}{\partial y}$ = marginal product of y = 4x - 6y

It may be noted that

(i) $\frac{\partial u}{\partial x} > 0$ for 2y > x or x <2y

(ii) $\frac{\partial u}{\partial x} = 0$ for x =2y

(iii) $\frac{\partial u}{\partial y} < 0$ for x>2y.

Similarly and $\frac{\partial u}{\partial y} > 0$ for y <3 x, $\frac{\partial u}{\partial y} =< 0$ for y = $\frac{2}{3}$ x.

and $\frac{\partial u}{\partial y} < 0$ for y> $\frac{2}{3}$ x.

Thus the marginal productivities at first increase and then decrease as input increases.

Example 11: Give the production function

$$P = \left(\beta K^{-p} + \alpha L^{-p}\right)^{-1/p}$$

Find the marginal products of Labour and Capital. Also find dP.

Solution: $P = \left(\beta K^{-p} + \alpha L^{-p}\right)^{-1/p} = (u)^{-1/p}$ ...(1)

where $u = \left(\beta K^{-p} + \alpha L^{-p}\right)^{-p}$ ...(2)

$\frac{\partial P}{\partial L}$ = marginal product of labour

$= \frac{\partial}{\partial L} (u)^{-1/p}$

$= -\frac{1}{P} (u)^{1/p-1} \frac{\partial u}{\partial L}$

$= -\frac{1}{P} (u)^{1/p-1} (\alpha.pL^{-p-1})$ [From (2)]

$= \alpha L^{-p-1}(u)^{-1/p-1}$

$= \alpha L^{-p-1}(\beta K^{-p} + \alpha L^{-p})^{-1/p}$ .....(3)

$\frac{\partial P}{\partial K}$ = marginal product of C Capital

$= \frac{\partial}{\partial K} (u)^{-1/p}$

$= -\frac{1}{P} (u)^{-1/p-1} \frac{\partial u}{\partial K}$

$= -\frac{1}{\rho}(u)^{1/p-1}(\beta.pK^{-p-1})$ [From (2)]

$= \beta L^{-p-1}\, u^{-1/p-1}$ .....(4)

$dP = \frac{\partial P}{\partial L}\,\alpha L + \frac{\partial P}{\partial K}.dP$

$= \alpha L^{-p-1}(\beta K^{-p} + \alpha L^{-p})^{-p}\alpha L$

$+ (\beta K^{-p-1} + \alpha L^{-p})^{-1/p}\, dK$

$= (\beta K^{-p} + \alpha L^{-p}[\alpha L^{-1-p} + \beta K^{-p-1} \cdot dK]$

SELF-CHECK EXERCISE 5.4

Q1. The Average Cost (AC) of a firm is AC = $q^2 - 2q + 5$. The maximum capacity of the firm is 30 units. Find the ranges of the output for which AC is decreasing and for which it is increasing.

Q2. The total cost function is given by $C = ae^{bq}$ (a, b are constant). Find the value of q for which marginal and average cost for this function is equal.

5.7 HOMOGENEOUS FUNCTIONS

A function u= $f(x, y)$ of two variables in x and y is said to be a homogeneous function of degree if

(a) $f(tx, ty)=t^n f(x, y)$ ...(1)

where t is any positive real number

or (b) $f(x, y) = x^n\varphi\left(\frac{y}{x}\right)$ or $y^n\varphi\left(\frac{x}{y}\right)$ .....(2)

In other words a function is said to be homogeneous of degree n when each of the independent variables is multiplied by a positive constant t, the whole function gets multiplied by t". We note the following points.

(i) Ifn<1, the function is homogeneous of degree less than one. In this case doubling of x and y will not double the value of function. In other words, the proportionate increase in the function will be less than the proportionate increase in the variables x and y or when x and y are increases by the factor t. the function will increase by less than the value of t.

(ii) As a special we n = 0 so that

$f(tx, ty) = t^0 f(x, y)$

This is a case of *homogenous function of degree zero* when x and y are increased by factor t, the function does not change at all. The most important example is of demand viz, a demand function is homogeneous of degree zero if a fixed proportinate increase in all prices and income leaves the demand unchanged.

(iii) Ifn=1, the function is homogenous of degree. In this case doubling of x and y will exactly double the value of the function. In other words, the proportionate increase in the function will be exactly equal to the proportionate increase in x andy.

This definition of homogeneous function holds good for more than two variables also.

In the general case a function of the n variable $x_1, x_2 ...... x_n$

If (a) $f(tx_1, tx_2.....tx_n)=t^n f(x_1. x_2).... x_n)$

t being a + ve real number

or (b) $f(x_1, x_2 -> x_n) = x1^n \varphi\left(\frac{x_2}{x_1}.\frac{x_3}{x_1} -> \frac{x_n}{x_1}\right)$

Example 12: (i) The function $y = ax^2+2hxy + by^2$ is homogeneour of degree 2.

$\because$ Here $f(x, y) = ax^2 + 2h\,xy + by^2$

$=a\,(tx)^2+2h\,(tx)\,(ty) + b\,(ty)^2$

$= at^2x^2+2h\,t^2xy + bt^2y^2$

$= t^2\,(ax^2 + 2h\,xy + by^2)$

$= t^2\,f(x, y)$

which implies that the function is homogeneous of degree 2.

(ii) The function $y= \frac{x^2+y^2}{xyxy}$ is homogeneous of degree 0.

$\because$ Here $f(x,y) = \frac{x2+y2}{xyxy}$

$= f(x, y) = \frac{(tx)^2}{(tx)(ty)} + \frac{(ty)^2}{(tx)(ty)}$

$= \frac{t^2x^2}{t^2xy} + \frac{t^2y^2}{t^2xy}$

$= \frac{t^2}{t^2}\left[\frac{x^2}{xy} + \frac{y^2}{xy}\right]$

$= t^0 f(x, y)$

which implies that the function is homogenous of degree 0.

(iii) The function y = log (x + y) is not homogenous

$\because$ Here $f(x, y) = \log(x + y)$

$f(tx, ty) = \log tx+ty)$

$= \log\,[t\,(x + y)]$

$\neq t \log (x+y)$

Hence the function is not homogenous.

5.7.1 Euler's Theorem on Homogeneous Function of degree n, then

Statement, If u= f(x, y) is a homogeneous function of degree n, then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n\, u.$$

Proof: Since u= f(x, y) is a homogeneous function of degree n, by definition, we have

u=xn φ(y/x)

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x} [x^n \varphi (y/x)]$$

$$= x^n \frac{\partial}{\partial x} [\varphi (y/x)] + \varphi (y/x)\left[\frac{\partial u}{\partial x}\left(x^n\right)\right]$$

$$= x^n \varphi^1 (y/x) \left[\frac{\partial}{\partial x}\left(y/x\right)\right] + \varphi (y/x) [n\, x^{n-1}]$$

$$= x^n \varphi^1 (y/x) \left[\frac{-y}{x^2}\right] + n\varphi (y/x)\, x^{n-1}]$$

$$= -y.x^{n-2} \varphi^1 (y/x) + n\, x^{n-1} \varphi (y/x).$$

$$\text{or } x \frac{\partial u}{\partial x} = -yx^{n-1} \varphi^1 (y/x) + n\, x^{n-1} \varphi (y/x).$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} [x^n \varphi (y/x)$$

$$= x^n \frac{\partial}{\partial y} [\varphi (y/x)]$$

$$= x^1 \varphi^1 (y/x) \left[\frac{\partial}{\partial y}\left(y/x\right).\right]$$

$$= x^1 \varphi^1 (y/x)\left(\frac{1}{x}\right)$$

$$= x^1 \varphi^1 (y/x)$$

$$\text{or } y \frac{\partial u}{\partial y} = yx^{n-1} \varphi^1 (y/x)$$ Adding (2) and (3) we get

$$= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = -y\, x^{n-1} \varphi^1 (y/x) + n\, x^{n-1} \varphi (y/x)\, y^{n-1} \varphi^1 (y/x)$$

or $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ = n $x^n \varphi^1$ (y/x) = n u. [from (1)]

Hence the result.

Theorem: The partial derivatives of homogeneous function of degree n are homogeneous of degree (n- 1)

Proof: Letu= *f*(x, y) be a homogenous function ofdegee n.

then *f*(x, y)=$x^n \varphi y/x$

(1) Differential partially w.r.t. x we have

$$f_x = x^n\left[\frac{\partial}{\partial x}\varphi(v/n)\right] + \varphi\ (y/x)\left[\frac{\partial}{\partial x}(x)n\right]$$

$$= x^n\varphi\ (y/x) + \left[\frac{\partial}{\partial x}\right] + \varphi\ (y/x).\ n\ x^{n-1}$$

$$= x^n\varphi\ (y/x) + \left[\frac{-y}{x^2}(y/x)\right] + \varphi\ (y/x).\ n\ x^{n-1}$$

$$=-y\ x^{-2}\varphi^1\ (y/x) + nx^{n-1}\varphi(y/x)$$

$$=-x^{n-1}[-y/x\ \varphi^1\ (y/x)+n\varphi\ (y/x)].$$

$\Rightarrow f_x$ is a homogeneous function of degree n-1.

Similarly f_y can be proved to be a homogeneous function of degree n - 1.

Example 13: Verify Euler's Theorem for the following functions.

(i) *f*(L. K)=$AL^{\alpha}K^{-\alpha}$, A.α are constants

(ii) *f*(L.K)=$(\alpha L^{-\alpha}+ \beta K^{-p})^{-1/p}$ α.β.p are constants.

Solution : Here *f* (L.K) = A L α $K^{1-\alpha}$

$$f(LtK) = A(t\ L)^{\alpha}\ (tK)^{1-\alpha}$$

$$= At^{\alpha}\ L^{\alpha}t1\text{-}\ \alpha\ K)1^{-\alpha}$$

$$= t^{\alpha+1-\alpha}\ (AL^{\alpha}\ K^{1-\alpha})$$

$$=t^1 f\ (L.K)$$

which⇒ that the function is homogeneous of degree 1.

∴ By Euler's Theorem

$$= L\frac{\partial u}{\partial L} + K\frac{\partial}{\partial K} = 1.u \qquad ...(1)$$

Since u= A L^{α} $K^{1-\alpha}$ ...(2)

$$\therefore \frac{\partial u}{\partial L} = (A\ K^{1-\alpha})\ L^{\alpha-1} = AK^{\alpha-1}\frac{1}{1} = \frac{\alpha}{1}\ .u$$

or L $\frac{\partial u}{\partial L}$ α.u ...(3)

$\frac{\partial u}{\partial K}$ = (AL$^{\alpha}$) (1 – α) K$^{1-\alpha-1}$

= (1 – α) AL$^{\alpha}\frac{K^{1-\alpha}}{K} = \frac{(1-\alpha)}{K}$ u

or $K\frac{\partial u}{\partial K}$ = (1 − α) u.

Adding (3) and (4), we get

L $\frac{\partial u}{\partial L}$ + K $\frac{\partial u}{\partial K}$ ∝ . u (1 – ∝) u = u ....(1)

Hence the result

(ii) Area f (L, K)= [∝L^{-P}+ βK^{-p}]

f (tL, tK) [∝ (tL)$^{-p}$ + β (tK)$^{-p}$

= [∝ (tL)$^{-p}$ + β (tK)$^{-p}$

= [t^{-p} (∝L^{-p} + ∝K^{-p})]$^{-1/p}$

= t^{1} (∝L^{-p} + βK^{-p})]$^{-1/p}$

= t^{1}f (L, K)

which⇒ that the function is homogeneous of degree 1.

∵By Euler's theorem

$L\frac{\partial u}{\partial L}+K\frac{\partial u}{\partial K}$ = 1.u .....(1)

Since u (αL^{-p} + βK)$^{-1/p}$

∴ $\frac{\partial u}{\partial L}$ = - $\frac{1}{p}$ (α 1.$^{-p}$ + βK^{-p})$^{-1/p-1}$ (-p α L^{-p-1})

= α L^{-p} (αL^{-p} + βK^{-p})$^{-t/p-1}$

or $\frac{\partial u}{\partial L}$ = (αL.$^{-p-1}$ (αL^{-p} + βK^{-p})$^{-1/p-1}$ ....(3)

also $\frac{\partial u}{\partial k}$ = - $\frac{1}{p}$(αL.$^{-p}$+ βK^{-p})$^{-1/p-1}$(-p β K^{-p-1})

= β K^{-p-1} (αL.$^{-p}$ + βK^{-p})$^{-1/p-1}$

or K $\frac{\partial u}{\partial L}$ = β K.$^{-p}$ (αL^{-p} + βK^{-p})$^{-1/p-1}$ ....(4)

Adding (3) and (4) we get

$L\ \frac{\partial u}{\partial L}$ + K $\frac{\partial u}{\partial K}$ = $(\alpha L.^{-p} + \beta K^{-p})^{-1/p-1}$

+ $\beta\ K^{-p}\ (\alpha L^{-p/-1} + \beta K^{-p})^{-1/p-1}$

$=\alpha L^{-p-1} + \beta K^{-p})^{-p}\ (\alpha L^{-p} + \beta K^{-p})$

i.e. L $\frac{\partial u}{\partial L}$ + $\frac{\partial u}{\partial K}$ = $(\alpha L.^{-p} + \beta K^{-p})^{-1/p-1}$= u.

Hence the result

Example 14: Show that the production function

u= $\sqrt{Hab - Aa^2 - Bb^2}$

A. H. B. being constants is linear and homogeneous.

So verify Euler's Theorem.

Solution: u= $\sqrt{2Hab - Aa^2 - Bb^2}$

oru= f(a, b) = $(2Hab - Aa^2 - Bb^2)^{1/2}$

$\therefore f(ta, tb) = [2h(ta)(tb) - A(ta)^2 - B(tb)^2]^{1/2}$

$=[t^2(2Hab - Aa^2 - Bb^2)]^{1/2}$

$= [(2Hab - Aa^2 - Bb^2)^{1/2}$

$=t^1 f(a, b)$

which$\Rightarrow$ that the function is homogeneous or degree 1 i.e. the function is linear and homogenous.

$\therefore$By Euler's Theorem

$a\frac{\partial u}{\partial a} + b\frac{\partial u}{\partial b}$ 1 . u ...(2)

Here $a\frac{\partial u}{\partial a} = b\frac{\partial}{\partial b}(2Hab - Aa^2 - Bb^2)^{1/2}$

$\frac{1}{2}(2Hab - A\alpha^2 - Bb^2)^{1/2}\ (2\ Hb - 2Aa)$

$a\frac{\partial u}{\partial a} = \frac{1}{2\left(2Hab - Aa^2 - Bb^2\right)} 2\ (Hb - Aa)$

$= \frac{\alpha\left(Hb - Aa\right))}{\left(2Hab - Aa^2 - Bb^2\right)}$ ....(3)

$\frac{\partial u}{\partial a} = \frac{\partial}{\partial b}\ (2Hab - Aa^2 - Bb^2)^{1/2}$

$= \frac{1}{2}(2Hab - Aa^2 - Bb^2)^{½} (2\,Hb - 2A\alpha)$

$$= \frac{Hb - Bb}{2(2Hab - Aa^2 - Bb^2)^{1/2}}$$

$$\therefore \frac{\partial u}{\partial u} = \frac{b\left(Hb - Bb\right)}{2(2Hab - Aa^2 - Bb^2)^{1/2}} \qquad ...(4)$$

Adding (3) and (4) we get

$$a\frac{\partial u}{\partial a} + K\frac{\partial u}{\partial b} = \frac{1}{2(2Hab - Aa^2 - Bb^2)^{1/2}} = [Hab - Aa^2 = Bb^2]$$

$$= \frac{2Hab - Aa^2 - Bb^2}{2(2Hab - Aa^2 - Bb^2)^{1/2}}\sqrt{2Hab - Aa^2 - Bb^2} = u$$

Hence the result,

Example 15: u z = f(u) where u is a function of x and y show that

(i) $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ if u = x + y

(ii) x $\frac{\partial z}{\partial x}$ = y $\frac{\partial z}{\partial y}$ if u = xy

(iii) x $\frac{\partial z}{\partial x}$ = y $\frac{\partial z}{\partial y}$ = 0 = x .

Solution: (i) z = f(u) and u = x + y

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(u)] = f(u)\frac{\partial u}{\partial x} = f(u) \text{ [from (1)]}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f'(u)] = f(u)\frac{\partial z}{\partial y} = f(u) \text{ [from (1)]}$$

Hence $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$

(ii) α = f(u) and u = xy

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f'(u) = f'(u)\frac{\partial u}{\partial x} = f'(u) \text{ [from (2)]}$$

$$\text{or } x\frac{\partial z}{\partial x} = f'(u) = xy \qquad ...(3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f'(u)] = f'(u)\frac{\partial u}{\partial y} = f'(u).\,x \text{ [from (2)]}$$

or y $\frac{\partial z}{\partial y} = f'(u) = xy$ ...(4)

Hence from (3) and (4), we get

x $\frac{\partial z}{\partial x}$ = y $\frac{\partial z}{\partial y}$.

(iii) x = f(u) and u = $\frac{x}{y}$. ...(5)

$\therefore \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}$ [$f'(u) = f'(u) \frac{\partial u}{\partial x} f'(u). \frac{1}{y}$ [from (5)]

or x $\frac{\partial z}{\partial x}$ = [$f'(u). = \frac{x}{y}$. ...(6)

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}$ [$f'(u) = f'(u) \frac{\partial u}{\partial y} f'(u).- \frac{x}{y^2}$ [from (5)]

or $\frac{\partial z}{\partial y}$ = [$f'(u). = \frac{x}{y}$. ...(7)

Adding (6) and (7) we get

(1) x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y} = f'(u) = \frac{x}{y} f'(u) \frac{x}{y} = 0$

Hence the result

Example 16 : If U = $\frac{1}{\sqrt{x^2 + y^2 + z^2}}$ $x^2 + y^2 + z^2 \neq 0$

show that $\frac{\partial^2 u}{\partial x^2}$ + y $\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial z^2}$ 0

Solution: We have

$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+ y^2+z^2)^{-3/2.}\ 3x^2\ (x^2 + y^2+z^2)^{-5/2}$

$\frac{\partial^2 u}{\partial x^2} = -1\ (x^2+ y^2+z^2)^{-3/2.}\ 3x^2\ (x^2 + y^2+z^2)^{-5/2}$

Similarly

$\frac{\partial^2 u}{\partial y^2} =-(x^2+ y^2+z^2)^{-3/2}.\ 3x^2\ (x^2 + y^2+z^2)^{-5/2}$

$\frac{\partial^2 u}{\partial z^2} =-(x^2 + y^2+z^2)^{-3/2}.\ 3y^2\ (x^2 + y^2+z^2)^{-5/2}$

Adding, we obtain the result.

Example 17: Find $\frac{dy}{dx}$ of the function

$ax^3 + bx^2y + cxy^2 + d = 0$

Solution: Let $f(x,y) = ax^3 + bx^2 + cxy^2 + d$

$f_x = 3ax^2 + 2bxy + cy^2$

$f_y = bx^2 + 2cxy$

$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-(3ax^2 + 2bxy + cy^2)}{-bx^2 + 2cxy}$

$= \frac{-3ax^2 + 2bxy + cy^2}{-bx^2 + 2cxy}$

SELF-CHECK EXERCISE 5.5

Q1. Find the marginal products of the labour and capital for the production functions:

(i) $q = 2L^2K^3$

(ii) $q = 10L - L^2 + 2LK + 50k - 2K^2$.

(iii) $q = 5L^{0.6}K^{0.4}$.

(iv) $q = 6L^{0.7}K^{0.8}$

Q2. Verify whether the following functions are homogeneous. If so, verify Euler's Theorems.

(i) $u = \frac{x^2 + y^2}{x + y}$ (ii) $u = \sqrt{xy}$ (iii) $u = (u/x)$

(iv) $u = \log\left(\frac{x^2 + y^2}{x + y}\right)$ (v) $u = AL^{3/4}K^{1/4}$

Q3. IfU= $f(q_1. q_2)$ where U is utility and q_1 and q_2 are consumption amounts of two commodities, find dU. IfU is constant, find marginal rate of substitution in terms of marginal utilities.

Q4. A production function is given by $U = AL^{1/3}K^{1/3}$

Show that total product is not exhausted if each factoris paid a price equal to its marginal product.

5.8 SUMMARY

We have learnt the concept and techniques of obtaining partial derivatives. We have also discussed High Order Partial Derivatives. You have learned about the total differential and total derivatives. You have also gone through the concept of homogeneous function. Lastly, you have learned about how the derivative can be applied in economics.

5.9 GLOSSARY

1. **Partial derivative :** Partial derivative of a function w.r.t. a variable represents the relative change in the function due to small change in that variable regarding all other variable as constant.

2. **Higher Order Derivatives :** The derivative is "the rate of change of function at a specific point". The derivative of the function $f(x)$ with respect to x at the point x_0 is the function $f'(x_0)$. The derivative other than the first derivative are called the higher order derivatives.

3. **Total differential :** Consider the function $y = f(x_1, x_2)$. By its total differentia, we measure the total changing due to change in both x_1 and x_2 (where x_1, x_2 are assumed to be independent of each other). Thus

$dy = f_1 dx_1 + f_2 dx_2$ is called the total differential of the function $y = f(x_1, x_2)$.

4. **Total Derivative :** Through total derivative, we measure the rate of change of the dependent variable owing to any change in variable on which it dependents, when now of the variable is assumed to be constant.

Let $y = f(x_1, x_2)$, such that, $x_1 = \text{g(t)}$ and $x_2 = \text{h(t)}$

Then we can write

$$\frac{dy}{dt} = \frac{\partial y}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial y}{\partial x_2} \cdot \frac{dx_2}{dt}$$

$$= f_1 \frac{dx_1}{dt} + f_2 \frac{dx_2}{dt}$$

which is the total derivative of y with respect to t.

5. **Homogeneous function :** The function $f(x_1, x_2)$ is said to be homogeneous of degree n if $f(\text{K}x_1, \text{K}x_2) = \text{K}^n f(x_1, x_2)$. The power of K is called the degree of homogeny.

5.10 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 5.1

Ans. Q1. $Z_x = 18x2 + 10x + 10y$

$Z_{xx} = 36x + 10$

Ans. Q2. $f x = 3x^2 + 8xy^4 + 2y$

$f y = 5y^4 + 16xy^3 + 2x$

$f xy = 32xy^3 + 2$

Self-check Exercise 5.2

Ans. Q1 $f_1 = \frac{2x_1}{x_1^2 + x_2^2}$, $f_2 = \frac{2x_2}{x_1^2 + x_2^2}$

$$f_{11} = \frac{2(x_2^2 - x_1^2)}{\left(x_1^2 + x_2^2\right)^2}$$

$$f_{12} = \frac{4x_1x_2}{\left(x_1^2 + x_2^2\right)^2}$$

$$f_{22} = \frac{2(x_1^2 - x_2^2)}{\left(x_1^2 + x_2^2\right)^2}$$

Ans. Q2 $\frac{d^2y}{dx^2} = \frac{-2x+4}{x+24}$

$$\frac{d^2y}{dx^2} = \left\{(x+2y)\frac{d}{dx}(2x+y) - (2x+y)\frac{d}{dx}(x+2y)\right\}$$

$$= \frac{-6x^2 + 6xy + 6y^2}{(-x+2y)3.}$$

Ans. Q3. $\frac{dy}{dx} = 20x^4 + 28x^3 + 3$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = 80x^3 + 84x^2$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = 240x^2 + 168x$$

Self-check Exercise 5.3

Ans. Q1. $y = ax_1^2 + 2hx_1x_2 + bx_2^2$

$dy = 2ax_1dx_1 + 2h(x_1dx_2 + x_2\, dx_1) + 2bx_2dx_2$

$= 2(ax_1 + hx_2)\, dx_1 + 2(bx_2 + hx_1)\, dx_2.$

Self-check Exercise 5.4

Ans.Q1. AC is decreasing when $\frac{d(AC)}{dq} < 0$

i.e. $2q - 2 < 0$ i.e. $q < 1$

Thus, AC decreases for $0 < q < 1$

and AC increase for $1 < q < 30$

Ans. Q2. MC $= \frac{dc}{dq}$. There derive $\frac{dMc}{dq}$ shows that for continually right q, MC falls, i.e. $\frac{dMc}{dq} < 0$

5.11 SUGGESTED READING

1. Allen, R.G.D. (1998). Mathematical Analysis for Economists, St. Martin's Press, New York.
2. Chiarg, A.C. (1974). Fundamental Methods of Mathematical Economics, 2nd edition, MC Grow-Hill Book Company, New York.
3. Henderson, J.M. and Quandt, R.E. (1980). Microeconomic Theory. MC Grow-Hill Book Company, New York.

5.12 TERMINAL QUESTIONS

Q. 1 Find the first order and second order partial derivatives of the following function:

(i) $u = x^2 + 3xy + y^2$ (ii) $u = ex^2 + y^2$

(iii) $u = e^{xy}$ (iv) $u = y^2/z$.

Q. 2 Verify the Euler's theorem for $u = x^2 \log y/x$.

Q. 3 Find the elasticity of total cost and Average Cost of the function $x = 2x^2 + 4x + 3$

Unit-6

MAXIMA AND MINIMA

STRUCTURE

6.1 INTRODUCTION

Maxima and minima plays a very important role in almost all fields and specially in economics where a rational consumer always thinks in terms of maximum utility and producer always tries to maximise profits and for choosing the least cost combination. We shall develop this important technique and illustrate its application in economics.

6.2 LEARNING OBJECTIONS

After going through this Unit, you will be able to:

- understand the identification process of maximum & minimum points
- prove the necessary conditions for maximum & minimum for functions.
- explain theorems on Maxims & Minima
- apply the concept of maxima and minima to find out minimum cast and maximum profit.

6.3 INCREASING AND DECREASING FUNCTION

y=*f*(x) is said to be an increasing function of x at the point x=a if

$\frac{dy}{dx}$ at x = a > 0 i.e. $\left(\frac{dy}{dx}\right)_{x=a} > 0$

y= *f*(x) is said to be decreasing function of x at the points x=a if

$\frac{dy}{dx}$ at x = a > 0 i.e. $\left(\frac{dy}{dx}\right)_{x=a} > 0$

Note: 1.x is always supposed to increase, y may increase or decrease as x increases.

2. The same function may be an increasingfunction in one interval and a decreasing function in another interval.

e.g. y = sin *x* is an increasing function as x varies from 0 to π/2 and a decreasing function as increases from π/2 to π.

Example. Test $y=20-6x+x^2$ for increasing or decreasing function at the points

(i) x=0 (ii) x= 2 (iii) x=4

Solution: $y=20-6x+x^2$

$\therefore \quad \frac{dy}{dx} = -6+2x = 2x-6$

(i) $\frac{dy}{dx}$ at x=0 = 2.0-6 = 6<0

∴The function or the curve is decreasing at thepoint x=0.

(ii) $\frac{dy}{dx}$ at x=2=2.2 - 6

=4-6=-2<0

∴The function or the curve is decreasing at thepoint x=2

(iii) $\frac{dy}{dx}$ at x=4=2.4-6 = 8-6=2>0

$\therefore$ The function is increasing at the point x = 4

SELF-CHECK EXERCISE 6.1

Q1. Write down the sufficient condition for increasing function and decreasing function.

Q2. Test $y=20 - 6x+x^2$ for increasing or decreasing function at the points

(i)x=0 (ii) x= 2 (iii) x=4

6.4 CONVEXITY OF A CURVE

In order to determine the convexity of the curve y= *f*(x) we consider the derivative of second order.If y-*f*(x) and $\frac{dy}{dx}$ >0 at x=a, then y has been defined as an increasing function of x.

(a) But if $\frac{d^2y}{dx^2} = f''(x) > 0$ we say that the functiony is increasing at an increasing rate i.e. the rate of change of y is increasing. The curve y = *f*(x) lies above the tangent and we say that the curve is *concave upward or convex dowanward*.

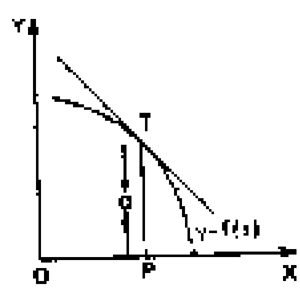
(b) If$f''(x)$ $\frac{d^2y}{dx^2}$= 0 there will be no curvatureand the curve.

y= *f*(x) will be a straight line.

(iii) If $f''(x)$ $\frac{d^2y}{dx^2}$<0. then the curve will be cancave de dowanward or convex upward and it will be below the tagent.

From these we conclude that

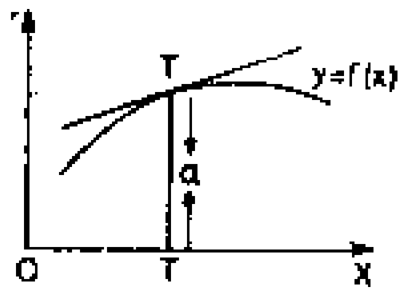
1. If $f''(a) > 0$, the curve y = *f*(x) is concave downward or convex downward at x = a.
2. If $f''(a) < 0$, the curve y = *f* (a) is concave downward or convex upward at x = a.
3. If $f''(a)=0$, the curve is straight line. These cases are illustrated diagrammatically below.



$f''(a)>0.\ f''(a)>0$

[convex from below *f*at x = aor concave from above at x=a]

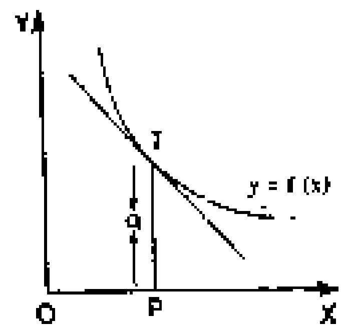
fig (i)



f'' (a)>0. f''(a) <0

[convex from below $f$at x = aor concave from above at x=a]

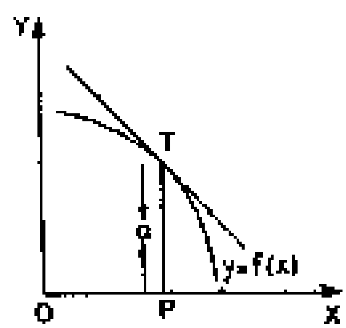
fig (ii)



f''(a) <0. f''(a)>0

[convex from below fat x=a]

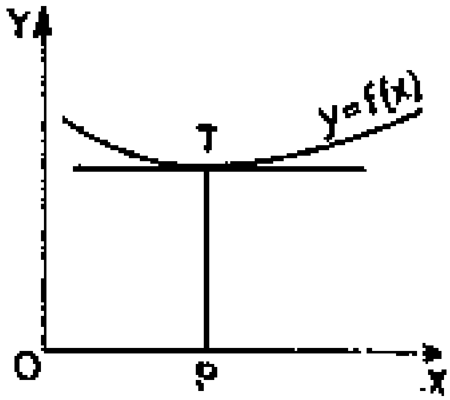
fig (iii)



f''(a) <0. f'' (a) <0

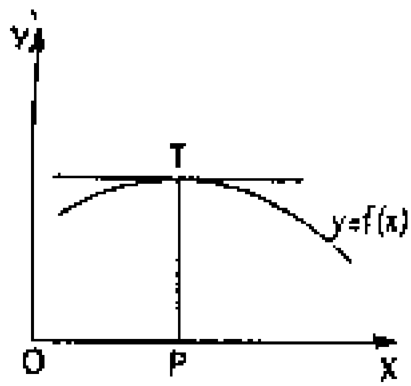
[convex from below fat x=a]

fig (iv)



$[f''(a) = 0.\ f''(a) > 0]$

fig. (v)



$[f''(a) > 0.\ f''(a) < 0]$

fig. (vi)

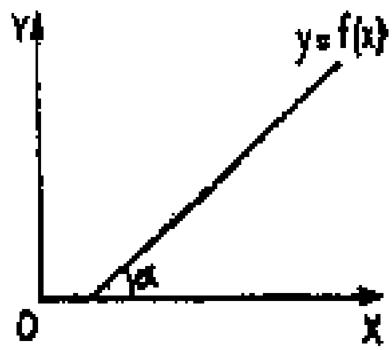


fig (vii)

We shall explain these cases below.

Case I.$f'(x) > 0$ and $f''(x) > 0$

The curve will have shape as given in Fig. (i) above. It is concave from above i.e. concave upward or convex downward.

Since $f'(x) > 0$, the slope of the curve is positive and since f' '(x) > 0, the slope of the curve tends to become steeper and steeper as x increases.

Case II.f'(>0 and $f''(x)$ <0.

The curve will have shape as given in Fig. (ii) above. It is concave from below i.e. concave downward or convex upward.

Since $f'(x) > 0$, the slope of the curve is positive and since $f''(x) <0$, the slope of the curve goes on decreasing as x increases.

Case III.$f'(x) <0$, and $f''(x)>0$

The curve will have shape as given in Fig. (iii) above. It is concave from above i.e. concave upward or convex downward.

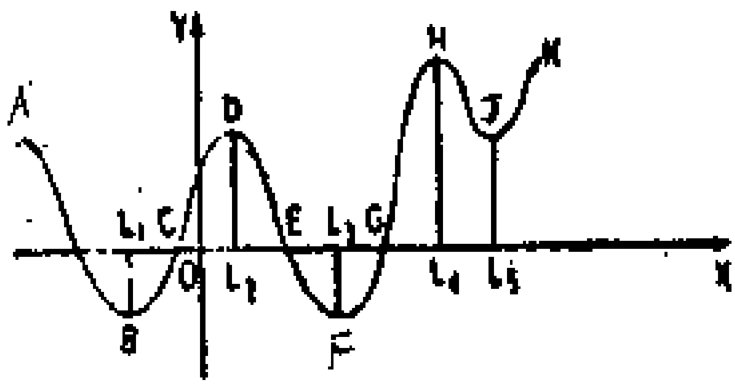
Since $f'(x) <0$ the slope of the curve is negative and since $f''(x)>0$, the slope of the curve goes on increasing as x increases.

Case IV.$f'(x) <0$ and $f''(x) <0$

Since $f'(x) <0$, the slope of the curve is negative and since $f''(x) <0$, the slope of the curve goes on decreasing as x increases.

Thus with the help of second derivative, we have derived the rules to decide about the rising and failing nature of the curve. But what happens when $\frac{dy}{dx} = 0$ as in Fig. (v) and Fig (v) above. Here we have to decide about maximum or minimum point ofthe curve, when = 0

Let us consider the following curve.



1. The curve is failing from A to B, from D to F and from H to J, and the corresponding function is *decreasing.*

2. The curve is rising from B to D, from F to H and from J to K, and the corresponding function is *increasing*.

3. If the curve rises to a certain position and then falls, such a position is called a Maximum Point of the curve. D and H are such points in the abovecurve. The ordinate that is the value of the function at such a point is called a *Maximum value of the Function*.

4. If the curve falls to a certain position and then rises, such a position is called a Minimum point of the curve, B, F and J are such points. The ordinate, that is the value of the function at such points is called a *Minimum Value of the Function*.

Now we can define maximum and minimum values of a function at a point.

SELF-CHECK EXERCISE 6.2

Q1. How can be determined the convexity of curve?

6.5 DEFINITION OF MAXIMUM & MINIMUM VALUE OF FUNCTION

Maximum and Minimum Values of aFunction:

(a) A function y = f(x) is said to have a maximum value f(a) at x = a if f(a) ceases to increase at x = a and begins to decrease as x increases beyond a. Thus, when x is slightly less than $\frac{dy}{dx}$ a, is positive and when x is slightly greater than a, both f(a-h) and f(a + h). In this way, we can also say that a function y = f(x) is maximum at x=aif f(a) >f(x) for all x (x≠a) lying in the interval (a-h, a + h)

(b) A function y = f(x) is said to have a minimum value f(a) x = a and begins to increase as x beyond a. Thus, when x is slightly less than a, $\frac{dy}{dx}$ is negative and when x is slightly greater than a $\frac{dy}{dx}$ is positive. Also for h>0. f(a) is less than both f(a- h) and f(a + h). In this way, we can also say that afunction y = f(x) is minimum at x=a if

f(a) <f(x) for all x (x≠a) lying in the interval (a -h, a + h)

6.5.1 GREATEST AND LEAST VALUES

The greatest and least values of a functionare always considered in a certain finite interval. The greatest value g= $f(d_1)$ means the greatest of all the values of f(x) in the given interval (b.c) whereas the least values $lf(d_2)$ means the least of all the values of f(x) in the interval (b.c).

It may be also be noted the maximum and minimum values are not always equal to the greatest and least values respectively. The distinction between the greatest value $f(d_1)$ and the maximum value f(a) of a function f(x) in an interval (b.c) is that $f(d_1)$ is the greatest of all values of f(x) in the small neighborhood of the point a viz. (a – h.a+h). Similar is the distinction between the least value of f(x) in (a.b) and a minimum value of f(x) at a point in (a.b)

Thus we note that a maximum value of a function f(x) in (a. b) may be less than several other values of f(x) in (a.b) may be greater than several other values of f(x) in a.b) may be greater than several other values of $f(x)$ in (a.b). In fact a function may have several maxims and minima in an interval and a *maximum value may even be less than a minimum value in the interval* (a.b)

If a continuous function has a single maximum or single minimum value in an interval, then that is also the greatest r the least value of the function in that interval. The maximum and minimum values of a function taken together, are called its *extreme values* and the points at which the function attains these extreme values are called the turning points of the function.

6.5.2 CRITERIA FOR A MAXIMA OR MINIMA AT A POINT

$f(x)$ if $y = f(x)$ is maximum at x = a, then

$\frac{dy}{dx}$ is+ve of x <a

and $\frac{dy}{dx}$ is - ve of x > a

Now $\frac{dy}{dx}$ changes sign from +ve to-ve as x passes through the value a. This change of sign can takeplace only when $\frac{dy}{dx}$ = 0 at x=a. Thus

1. y = $f(x)$ is maximum at x = a if

(i) $\frac{dy}{dx}$ = 0 at x = a. and

(ii) $\frac{dy}{dx}$ *changes sign from+ve to = ve as x passes through the valuea.*

Again since $\frac{dy}{dx}$ changes sign from+ve to -ve whilepassing through a, the point of maxima. is a decreasing function of x at $x=a$ and its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is negative, Hence we get anotherrule for maxima as follows.

II. y = $f(x)$ is maximum at x = a if

(i) $\frac{dy}{dx}$ = 0 at x=a.

(ii) $\frac{d^2y}{dx^2}$ is negative at x=a.

(B) If $y = f(x)$ is minimum at x=a, then $\frac{dy}{dx}$ is-ve for x, a and $\frac{dy}{dx}$ is +ve for x> a.

Now $\frac{dy}{dx}$ change sign from -ve to +ve as x passes through the value a. This change of sign can take place only when $\frac{dy}{dx}$ =0 at x=a. Thus

I. **y= *f*(x) is minimum at x=a if**

(i) $\frac{dy}{dx}$ 0 at x=a

(ii) $\frac{dy}{dx}$ *changes sign from-ve to+ve as x passesthrough the value a.*

Again since $\frac{dy}{dx}$ change sign from-ve to +ve while passing through, a the point of minima, therefore $\frac{dy}{dx}$ is an increasing function of x at x=a and its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is positive. Hence we getanother rule for minima as follows.

II. y= *f*(*x*) is maximum at *x*= *a* if

(i) $\frac{dy}{dx}$ =0 at x=a.

(ii) $\frac{d^2y}{dx^2}$ is positive at x = a.

6.5.3 POINTS OF INFLEXION

1. The maximum and minimum values of a function are together called its *extreme values*.
2. The values of y = *f*(*x*) at the points where $\frac{dy}{dx}$ = 0 are called *statianary values of the function.*
3. Points of Inflextion. For y = *f*(x) to have a maximum or minimum value at *x*=a. $\frac{dy}{dx}$ =0

at that point. But if $\frac{dy}{dx}$ = 0 point. But if *dy = 0 at x* = a, it is not necessary that y *f*(x) may have a maximum or minimum value at x=a.

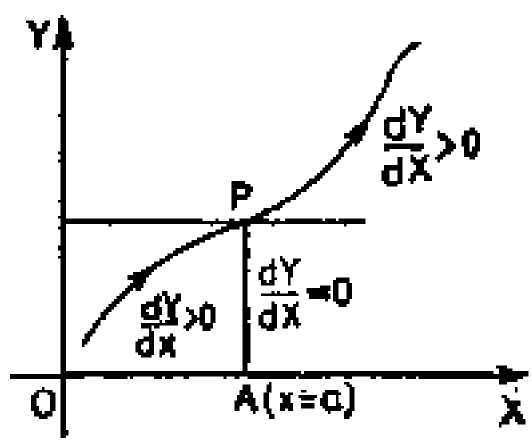
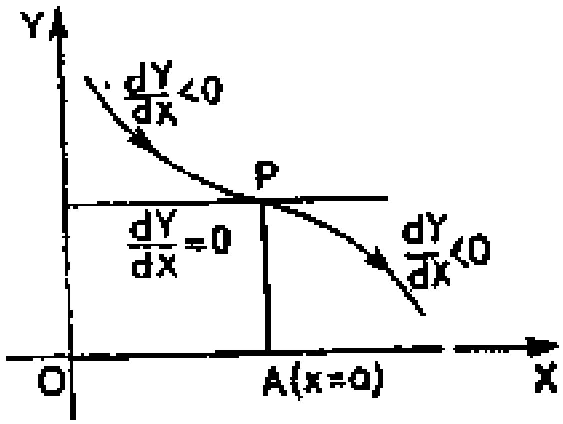


fig. (i)

It may happen that inspect of $\frac{dy}{dx} = 0$ at $x = 0$, the function may go on increasing as in Fig. (i) below or decreasing as in Fig. (ii) below as x passes througha.



The function does not change from an increasing to a decreasing function or vice versa. Thus dy/dx does not change sign while passing through a. Hence at such a point, the function cannot have a maximum or minimum value. Such points are called the points of inflexion of the curve.

SELF-CHECK EXERCISE 6.3

Q1. Find the maxima and minima for the following function

$y = 3x^4 - 10x^3$ +6x2 +5

Q2. Find the point of inflection for the function.

$f(x) = 3x^3 + x^2 + x + 1$

Q3. Find the stationery values and test whether they are maximum or minimum for

$Z = 3x^2 + 6xy + 7y^2$

6.6 THEOREMS ON MAXIMA AND MINIMA

1. If c is a constant, then any value of x which makes $f(x)$ a maximum or a minimum also makes $f(x)$+ca maximum or a minimum and conversely.
2. If c is a positive constant, then any value of x which makes f(x) a maximum or a minimum also makes c f(x) a maximum or a minimum and conversely
3. Ifc is negative constant, then any value of$x$ whichmakes f(x) a maximum makes c f(x) a minimumand any value of x which makes f(x) a minimummakes c f(x) a maximum and conversely.
4. Any value of x which makes f(x) positive and amaximum or a minimum also makes
   (i) $[f(x)]^n$ a maximum or a minimum.
   (ii) Log f(x) a maximum or a minimum and conversely.

5. Any value of x which makes $f(x)$ finite, no-zero and a maximum makes $1/f(x)$ a minimum and any value of x which makes $f(x)$ finite, non-zero and a minimum makes $1/f(x)$ a maximum.

6. If $f(x)$ possesses continuous derivatives up tothen the order in a certain neighborhood of the pointa and if

   (i) $f(a)=0$ $f''(a)$ but $f''(a) \neq 0$, then

   (ii) $f(a)$ *is a maximum* value of $f(x)$ if *n is ever an*f^n (c) < 0.

   (iii) $f(a)$ *is neither a-maximum nor a minimumvalue of f(x) if n is odd working rule finding the Is maximum and minimum values of a function.*

First Method

1. Let$y = f(x)$ be the given function.

1. Find $\frac{dy}{dx}$ and equate it to zero and then solve theequation for real values of x.

   Let these values be x_1, x_2, x_3, ...

2. Consider the value of x slightly less than a and slightly greater that *a*.

3. If $\frac{dy}{dx}$ changes sign from -ve to + ve, then $f(x)$ is maximum at x=a.

   If $\frac{dy}{dx}$ changes sign from - ve to+ve, then $f(x)$ is minimum at x=a.

4. If $\frac{dy}{dx}$ does not change sign, then $x=a$ is a point of inflexion.

Similarly we can discuss maxima or minima at other values or *x*.

Second Method

Ley $y = f(x)$ be the given function.

1. Find $\frac{dy}{dx}$ and equate it to zero and then solve this equation for real values of x. Let these values be x_1, x_2,x_3. ......

2. Find $\frac{d^2y}{dx^2}$ and calculate $\frac{d^2y}{dx^2}$ at these pointsseparately

3. If $\frac{d^2y}{dx^2}$ is-ve when x = x_1, then f (x) is maximum at x=x_1 and the corresponding maximum value of $f(x)$ is $f(x_1)$

   If $\frac{d^2y}{dx^2}$ is+ve when x=x_1, then $f(x)$ is minimum at x = $x_1$and the corresponding maximum value of $f(x)$ is $f(x_1)$

If $\frac{d^2y}{dx^2}$=0 at x=x_1, then $\frac{d^3y}{dx^3}$$f$(x) and calculate its value at x=x_1, If it is not zero, then x=x_1 is a point of inflexion.Similarly we can discuss maxima or minima for other values of x.

Note: First method may be preferred if the process of finding $\frac{d^2y}{dx^2}$ becomes tedious.

Example 1. Find the extreme values, if any, of the functions $y=2x^2-x^3$

Let $y = 2x^2-x^3$

$\therefore \frac{dy}{dx} = 4x-3x^2 = x\,(4 - 3x)$

For maxima or minima.

$\frac{dy}{dx} = 0$

$\therefore$ x (4 - 3x) = 0

which $\Rightarrow$either x = 0 or 4 - 3x = 0

i.e x = 0 or 4-3x= 4/3

So we have to discuss maxima or minima at these points viz.

x = 0 and x = 3/4

(i) Let us take the point x=0

When x is slightly<0.

$\frac{dy}{dx} = (-)\,(+) = -$

When x is slightly >0.

$\frac{dy}{dx} = (+)\,(+) = +$

So $\frac{dy}{dx}$ changes sign from (-) ve to)+) ve as passesthrough the point a. Hence it gives a minimum value and the minimum value is given by

$f(0)=2(0)^2-(0)^3 = 0$

(ii) Take the point x = $\frac{4}{3}$

When x is slightly $<\frac{4}{3}$.

$\frac{dy}{dx} = (+)\,(+) = +$

When x is slightly< $\frac{4}{3}$

$$\frac{dy}{dx} = (+)\ (–) = –$$

So $\frac{dy}{dx}$ changes from (+) ve to (-) ve as x passesthrough the point $\frac{4}{3}$

Hence it gives a maximum value at x= $\frac{4}{3}$ and the maximum value is given by

$$f(4/3) = 2\left(\frac{4}{3}\right)^2 - \left(\frac{4}{3}\right)^3$$

$$= \left(\frac{4}{3}\right)^2 - \left(2 - \frac{4}{3}\right)^3$$

$$= \frac{16}{9} x \frac{2}{3} = \frac{32}{27}$$

Second Method

Let $y = 2x^2 - x^3$

$\therefore$ $\frac{dy}{dx} = 4x - 3x^2 = x\,(4 - 3x)$

For maxima or minima.

$$\frac{dy}{dx} = 0$$

$\therefore$ $x\,(4 - 3x) = 0$

which $\Rightarrow$ either x = 0 or 4-3x=0

i.e. x = 0 or x = 4/3

(1) Take the point x=0

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

At $\frac{dy}{dx} = (4x - 3x^2) = 4 - 6x)$

x=0. $\frac{d^2y}{dx^2}$ -4-6.0 = 4>0

Hence x = 0 gives a minimum value and the minimum value is given by

$f(0) = 0$

(ii) Taking the point x = $\frac{4}{3}$

$$\frac{d^2y}{dx^2} = 4\text{-}6x$$

At x = $\frac{4}{3}$

$$\frac{d^2y}{dx^2} = 4 - 6 \text{ x } \frac{4}{3} = 4 - 8 = -4 < 0$$

Hence x= $\frac{4}{3}$ gives a maximum value and the maximum value is given by $f(4/3)= \frac{32}{27}$

Example 2. Find the maximum and minimum valueof

x^3+2x-4x-8

Solution. Let y=x^3+2x^2-4x-8

$$\therefore \quad \frac{dy}{dx} = 3x^2+4x-4$$

For maxima or minima,

$$\frac{dy}{dx} = 0$$

i.e. $3x^2+4x-4=0$

$$\therefore \quad x = \frac{-4 \pm 16 + 48}{6}$$

$$= \frac{-4 \pm 8}{6} = \frac{2}{3} \text{ or } -2$$

So we have to discuss the maxima or minima at these, two points x=$\frac{2}{3}$ and x-2

$$\frac{d^2y}{dx^2} = 6x + 4$$

At x = $\frac{2}{3}$ $\frac{d^2y}{dx^2}$

$$= 6\frac{2}{3} + 4 = 4 + 4 = 8 > 0$$

$\therefore$ y is minimum t x = $\frac{2}{3}$ and the minimum value is given by

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right) - 8$$

$= \frac{8}{27} + \frac{8}{9} + \frac{8}{3} - \frac{8}{1}$

$= \frac{8+27+72+216}{27}$

$= -\frac{256}{27}$

(ii) At $x = -2$, $\frac{d^2y}{dx^2}$

$=6(-2)+4=-8<0$

$\therefore$ y is maximum at x=-2 and the maximum valueis given by

$f(-2) = (-2)+2(-2)^2 - 4(-2) - 8$

$=-8+8+8-8$

$= 0$

Example 3. Find the maximum and minimum valuesofy = $(x-1)^3 (x+1)^2$

Solution. $y = (x-1)^3 (x+1)^2$

$\frac{dy}{dx} = (x-1)^3 \frac{dy}{dx}(x+1)^2 + (x+1)^2 \frac{dy}{dx}(x+1)^3$

$=(x-1)^3{}_2 (x+1) + (x+1)^2 . 3(x-1)^2$

$=(x-1)^2(x+1)[2(x-1)+3(x+1)]$

$= (x-1)^2(x+1) \phi (5x+1)$

$\frac{dy}{dx} = 0$

$\therefore \quad (x-1)^2 (x+1) \phi (5x+1)=0$

which gives x = 1,-1,-1/5

we now discuss maxima or minima at thesepoints

(i) Atx=1.

When x is slightly <1.

$\frac{dy}{dx} = (+)(+)(+) = +$ ve

When x is slightly> 1

$\frac{dy}{dx} = (+)(+)(+) = +$ ve

$\frac{dy}{dx}$ does not change sign as x passes through 1.

Hence x = 1 is a point of inflexion and gives neither a maximum nor minimum value.

(ii)At x= –1

When x is slightly <-1

$$\frac{dy}{dx} = (+)\ (–)\ (–) = +\text{ ve}$$

When x is slightly>-1

$$\frac{dy}{dx} = (+)\ (+)\ (–) = –\text{ ve}$$

∴ changes sign from + ve to- ve as x passes through 1.

Hence y is maximum at x=-1 and the maximumvalue is given by

$f(-1)=(-1-1)^3\ (-1 + 1)^2 = 0$

(iii) Atx=-1/5

When x is slightly>-1/5

$$\frac{dy}{dx} = (+)\ (+)\ (–) = –\text{ ve}$$

When x is slightly>-1/5

$$\frac{dy}{dx} = (+)\ (+)\ (+) = +\text{ ve}$$

∴ $\frac{dy}{dx}$ change from - ve to+ve as x passesthrough -1/5.

Hence y is minimum at x=-1/5 and the minimum value is given by

$f(-1/5) = (-1/5)^3\ (-1/5 + 1)^2$

$$\left(\frac{-216}{125}\right)\left(\frac{16}{25}\right)$$

$$= \frac{-3456}{3125}$$

Example 4. Find the maximum and minimum values $\left(\frac{1}{x}\right)^x$

Solution. Let y=$\left(\frac{1}{x}\right)^x$

$$\log y = \log \left(\frac{1}{x}\right)^x = \log x \left(\frac{1}{x}\right)^x$$

$=x \log (x^{-1})= -x \log x$

and $\frac{dy}{dx}$ (log y) = $\frac{d}{dx}$ [-x log x]

or $\quad \frac{1}{y}\frac{dy}{dx}$ = - [x. $\frac{1}{x}$ + log x. 1]

or $\quad \frac{1}{y}\frac{dy}{dx}$ = - [1 + log x.]

or $\quad \frac{dy}{dx}$ = -y [1 + log x]

$-\left(\frac{1}{x}\right)$ (1 + log x]

$\frac{dy}{dx} = 0$

$\therefore \quad \left(\frac{1}{x}\right)^{x}$ (1 + log x]

Which gives 1+ logx = 0

or $\quad$ log x=-1-log e-loge^{-1}= log $\frac{1}{e}$

$\therefore \quad x = \frac{1}{e}$

Now we have to discuss maxima or minima at x=$\frac{1}{e}$

When x is slightly< $\frac{1}{e}$ (or logx <- 1)

$\frac{dy}{dx}$ = (–) (+) (–) = + ve

When x is slightly> $\frac{1}{e}$ (or logx>-1)

$\frac{dy}{dx}$ = (–) (+) (+) = + ve

$\therefore \frac{dy}{dx}$ change sign from+ve to -ve as x passes through the point x = *l/e*

Hence y is maximum at x =*l/e* and the maximum value is given by

$f(1/e) = \left(\frac{1}{l/e}\right)^{l/e} = e^{1/e}$

Example 5. Find the maximum and minimum valuesof y=x+ $\frac{1}{x}$

Solution. y=x+ $\frac{1}{x}$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{1}{x^2}$

For maxima or minima.

$$\frac{dy}{dx} = 0$$

$\therefore \quad 1 - \frac{1}{x^2} = 0$

or $\quad x^2 - 1 = 0$

or $\quad x^2 = 1$

or $\quad x = +1.-1$

So we have to discuss maxima and minima at thesetwo points

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}$$

$$= \frac{d}{dx}\left(1 - \frac{1}{x^2}\right) = \frac{2}{x^3}$$

(i) At x = 1

$$\frac{d^2y}{dx^2} = \frac{2}{(1)^2} = 2 > 0$$

$\therefore$ y is minimum at x=1 and the minimum value isgiven by

$$f(1) = 1 + \frac{1}{1} = 1 + 1 = 2$$

(ii) At x = -1

$$\frac{d^2y}{dx^2} = \frac{2}{(-1)^2}$$

$$\frac{2}{-1} = -2 < 0$$

$\therefore$ y is maximum at x=-1 and the maximum value is given by

$f(-1) \quad = -1 + \frac{1}{1}$

$= - 1 - 1 = -2$

Note: In this question we find that maximum value is less than the minimum value. Actually it is 4 less than the minimum value.

Example 6. Find the maximum value of

$$\frac{\log x}{x} \text{ in } 0 < x < \infty$$

Solution Let y= $\frac{\log x}{x}$

$$\therefore \frac{dy}{dx} = \frac{x.\frac{d}{dx}(\log x) - \log x \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x.\frac{1}{x} - \log x.1}{x^2}$$

$$= \frac{1 - \log x}{x^2}$$

For maxima or minima

$$\frac{dy}{dx} = 0$$

i.e. $\frac{1 - \log x}{x^2} = 0$

or 1-log x=0

or log x=1=log e⇒x=e

Now we have to discuss maxima or minima only atthe point viz

When x is slightly<e (i.e. logx<1)

$$\frac{dy}{dx} = \frac{+}{+} = + \text{ ve.}$$

When x is slightly>e (i.e. log cx> 1)

$$\frac{dy}{dx} = \frac{(-)}{+} = + \text{ ve.}$$

$\therefore \frac{dy}{dx}$ changes sign from+ve to -ve as x passesthrough e.

Hence y is maximum at x=e and the maximumvalue is given by

$$f(\text{e})\frac{\log e}{e} = \frac{1}{e}$$

Example 7: Show that the maximum value of$\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$

Solution Lety=$\left(\frac{1}{x}\right)^x$

$$\log y = -x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = -(-1+\log x)$$

$$\frac{dy}{dx}(1+\log x)\left(\frac{1}{x}\right)^x$$

$$\frac{dy}{dx} = 0 \Rightarrow 1 + \log x = 0 \Rightarrow x = e^1$$

Again $\frac{d^2y}{dx^2} = \frac{-1}{x}\left(\frac{1}{x}\right)^x = (1+\log x)^2\left(\frac{1}{x}\right)^x$

At $\quad x = e^1$

$$\frac{d^2y}{dx^2} = e\,(e)^{-1/e} < 0$$

$\Rightarrow$y has maximum for $x = e^{-1}$

and minimum value is $(e)^{1/e}$

SELF-CHECK EXERCISE 6.4

Q1. Find the maximum and minimum valuesofy = $(x-1)^3 (x+1)^2$

Q2. Find the maximum and minimum values $\left(\frac{1}{x}\right)^x$

Q3. Show that the maximum value of$\left(\frac{1}{x}\right)^x$ is $(e)^{1/e}$

6.7 ECONOMIC APPLICATIONS

6.7.1. COST MINIMIZATION:

One of the basic problems of a producer is to find out the level of output at which the average cost of production is minimum or the average variable cost of production is mini- mum. We can apply the conditions of minimization to solve such a problem. Let us consider a total cost function

$$TC = aQ^2 + bQ + C \qquad .....(1)$$

where Q is the quantity and C is the total fixed cost and all parameters are positive

The average cost is given by

$$AC = \frac{TC}{Q} = aQ + b + \frac{C}{Q} \qquad ....(2)$$

To find out the output at which the average cost (AC) will be minimum, we have to satisfy the following first order and second order condition such that

$$\frac{d(AC)}{dQ} = 0 \quad \text{and} \quad \frac{d^2(AC)}{dQ^2} > 0$$

Now $\frac{d(AC)}{dQ} = a + 0 - \frac{C}{Q^2} = 0$

$$Q^2 = \frac{C}{a}$$

$\therefore$ $Q = \pm\sqrt{\frac{c}{a}}$

= either $+\sqrt{\frac{c}{a}}$ or $-\sqrt{\frac{c}{a}}$

Now $\frac{d^2(AC)}{dQ^2} = 0 - (-2)\, CQ^{-2-1} = \frac{2C}{Q^3} \qquad ....(3)$

when $Q = \sqrt{\frac{c}{a}}$, $\frac{d^2(AC)}{dQ^2} = \frac{2C}{Q^3} > 0$

Since $a > 0$ and $C > 0$

when $Q = \sqrt{\frac{c}{a}}$, $\frac{d^2(AC)}{dQ^2} = \frac{2C}{Q^3} > 0$

$\therefore$ the average cost will be minimum at $Q = \sqrt{\frac{c}{a}}$, if theaverage Cost is given by the function.

$AC = aQ^2 + bQ + C \qquad ...(4)$

$(a>0;\ b<0;\ C>0)$

Then the determination of output ot which the average cost (AC) will be minimum requires that

$$\frac{d(AC)}{dQ} = 0 \text{ and } \frac{d^2}{dQ}(AC) > 0$$

Now $\frac{d(AC)}{dQ} = 2aQ + b = 0$

$\therefore$ $Q = -\frac{b}{2a}$

and $\frac{d^2(AC)}{dQ^2}$ = 2a> 0 as a> 0

Thus the average cost will be minimum when theoutput is $\frac{-b}{2a}$

It may be noted that marginal cost curve cuts the average cost curve at the minimum point of AC curveas shown in figure below. We take the total cost function (1). The marginal cost is given by

MC = $\frac{d(TC)}{dQ}$ =2aQ+b .....(5)

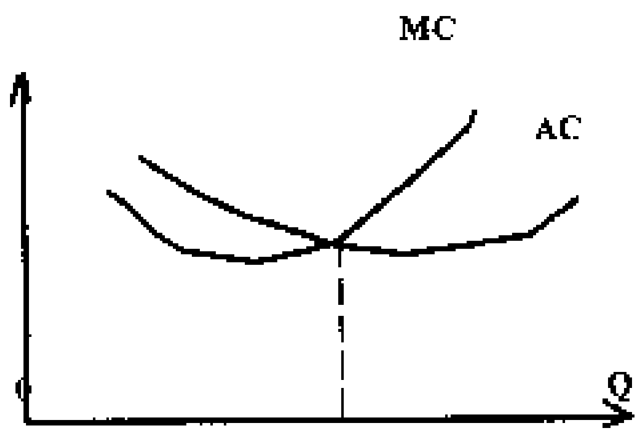


Fig. (i)

Thus at minimum cost, AC=MC

$\therefore a\text{Q} + \text{b} + \frac{C}{q} = 2a\text{Q} + \text{b}$

or $\frac{C}{Q} = a\ \text{Q}$

or $\text{Q}^2 = \frac{C}{a}$

$\text{Q} = + \sqrt{\frac{C}{a}}$

Since output cannot be negative, therefore the average cost will be minimum when Q = $\sqrt{C/a}$. This N is the same value of output we derived using first and second order conditions of minimization.

6.6.2 PROFIT MAXIMIZATION:

In the theory of firm, the basic problem is to choose the combination of price and quantity in order to maximize profits. The optimum level of output which maximizes profit of a firm is arrived at when

a) Marginal revenue equals marginal cost and b(marginal cost curve cuts marginal revenue from below.

Let us now define profit (IT) as the differencebetween total revenue (R) and total cost (C). Since cost of production and revenue vary with the level of output, we can assume that total revenue and total cost are of output (q) such that R= R(q) and C=C(q). So profit can be expressed as

$$\Pi = R - C$$

$$\text{or} \quad II = R(q) - C(q)$$

so final profit (IT) is also a function of quantity (q)

In order to obtain the level of output at which the profit will be maximum, we follow the procedure of maximizing a function in which the first derivative is zero and the second derivative in negative. Thus

Thus $\frac{d\Pi}{dq} = 0$ gives

$$\frac{d\Pi}{dq} = R'(q) = C'(q) = 0$$

$$\text{or} \quad R'(q) = C'(q)$$

$$\text{or} \quad MR = MC$$

The second order condition states

$$\frac{d^2\Pi}{dq^2} = R''(q) - C''(q) < 0$$

$$\text{or} \quad R''(q) < C''(q)$$

or slope of MR < slope of MC

Both these conditions imply that for profit-maximization, MRMC and MC should cut MR from below. The first order and second order conditions of profit maximization under imperfect competition as well as under perfect competition can be more clearly seen from the figures belows.

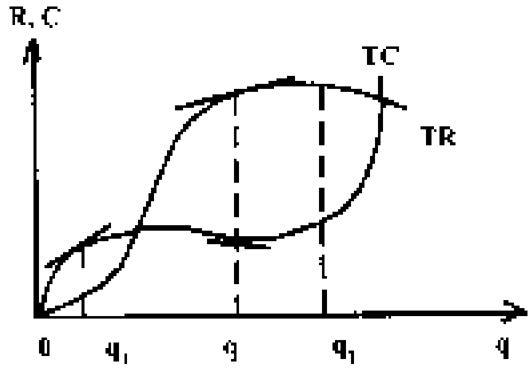


Fig. (ii)

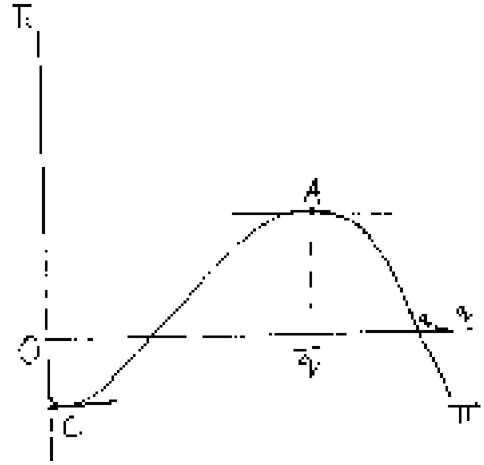


Fig. (iii)

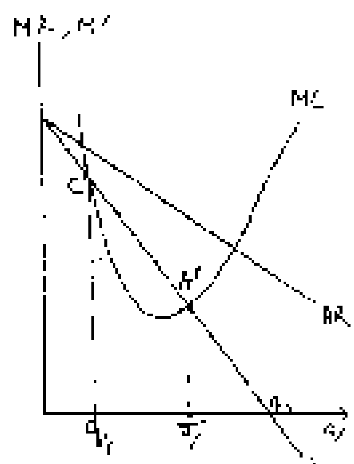


Fig. (iv)

Fig. (v)

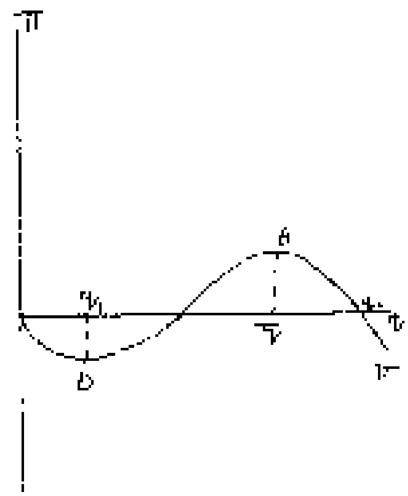


Fig. (vi)

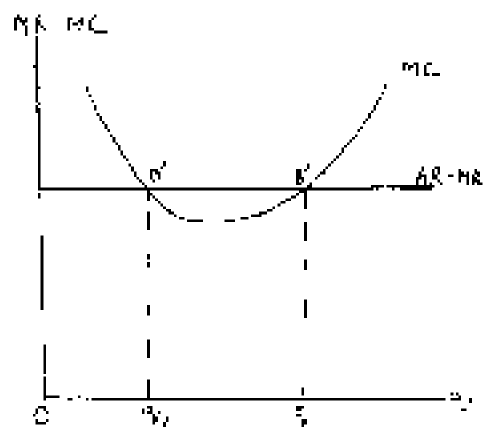


Fig. (vii)

Figures (ii), (iii) and (iv) show that at equilibrium output q, gap between total revenue and total cost in maximum and so the profit function attains the highest point of the profit curve and MC = MR with MC cutting MR from below. At output q_1, total cost over total revenue is maximum and so the profit attains the minimum profit with MR-MC but MC cuts MR from above. The same is the condition under perfect competition as shown in figures (v), (vi) and (vii).

Example 8: Show that the function *f* defined by $f(x)=x^p (1-x)^q \forall$ x ϶ R

Where p, q are positive integers has a maximum

value for x= $\frac{p}{p+q}$, + or all p. q

Solution:

We have

$f'(x) \quad x^p(1 - x)^q$

$f(x) \quad = px^{p-1} (1-x)^q - qx^p(1-x)^{q-1}$

$\quad = x^{q-1} (1-x)^{q-1}[p - x (p + q)]$

$f'(x) = 0 \Rightarrow x=0, 1, \frac{p}{p+q}$

Again

$f''(x) = (p-1)x^{p-2} (1-x)^{q-1}[p-x(p + q)]$

$\quad = (q -1)x^{p-1} (1-x)^{q-2}[p-x(p + q)]$

$\quad - (p = q)x^{p-1} (1 + x)]^{q-1}$

$$f''\left(\frac{p}{p+q}\right) = -(p+q)\left(\frac{p}{p+q}\right)^{p-1}\left(\frac{p}{p+q}\right)^{q-1} < 0$$

where p and q are integers

Thus the function has a max, value at x= $\frac{p}{p+q}$ for all integers p and q and the max value is

$$\frac{p^p q^q}{(p+q)^{p+q}}$$

Example 9: If the demand function is p= $\sqrt{q-x}$ find at what level of output x, the Total Revenue(TR) will be maximum Also find TR.

Solution TR =p ×x

$\quad = \sqrt{9-x} \times x$

$\quad = x (9 – x)^{½}$

TR is maximum when

MR=0

But MR= $\frac{d}{dx}$(TR)

$= \frac{d}{dx} [x (9 - x)]^{½}$

$= x \frac{d}{dx} (9\text{-}x)^{½} + (9 - x)^{½} \frac{d}{dx} (x)$

$= x. \frac{1}{2}(9\text{-}x)^{½} (-1) + (9 - x)^{½}$

$= x \frac{x}{2\sqrt{9-x}} + \sqrt{9-x}$

$\frac{x+2(9-x)}{2\sqrt{9-x}}$

$= \frac{18-3x}{2\sqrt{9-x}}$

But MR = 0 gives 18-3x=0

Maximum TR is given by

$= p^x x at x=6$

At x = 6

$p = \sqrt{9-6} = \sqrt{3}$

∴ TR $= p\ xx = 6x\sqrt{3} = 6\sqrt{3}$

Example 10. (1) The total cost (TC) function for producing a commodity x is TC-60-12x+2x². Findthe level of output at which TC is minimum.

(ii) Find the AC function and the level of output atwhich this function is minimum.

(iii) Then verify that at the low point of the AC curve.

MC = AC.

Solution: (i) Let

y = TC

$= 60\text{-}12x + 2x^2$

∴ $= \frac{d}{dx}$ -12 + 4x

$$\frac{d^2y}{dx^2}=4>0$$

For maxima or minima

$$\frac{dy}{dx} = 0$$

$\therefore$ -12+4x=0

or x=3.

$\therefore$ We discuss maxima or minima at x=3

Since $\frac{d^2y}{dx^2}$ 4>0

$\therefore$ Y is minimum at x=3 and the minimum value isgiven by

$f(3)$ 60-12x3+2(3)^2

= 60-36+18 =

= 42.

$\therefore$ The level of output at which TC is minimum isx=3 and miminum TC is 42.

(ii) let z = AC=$\frac{TC}{x}$

$= \frac{60}{x}$ -12x-2x^2

$= \frac{60}{x}$ -12+2x

$\therefore$ $\frac{dz}{dx} = \frac{60}{x} + 2$

For maxima or minima

$$\frac{dz}{dx -} = 0$$

$$\frac{-60}{x} + 2 = 0$$

or $2x^2 = 60$

or $x^2 = 30$

or $x^2 = \underline{+}\sqrt{30}$

Since output can't be negative, $\therefore$ we reject

$x = -\sqrt{30}$

and consequently $x = -\sqrt{30}$

and we discuss maxima or minima at

$x = \sqrt{30}$

$$\frac{d^2z}{dx^2} = \frac{120}{x^3}$$

At $\quad x = \sqrt{30}$

Hence z gives a minimum at

$x = \sqrt{30}$

and the minimum value is given by

$$f(\sqrt{30}) = \frac{60}{\sqrt{30}} - 12 + 2\sqrt{30}$$

$$= 2\sqrt{30} - 12 + 2\sqrt{30}$$

$$= 4\frac{2x^2 - 3x^3 + 4}{x^4} = 12$$

(iii) $\quad MC = \frac{d}{dx}(TC)$

$$= \frac{d}{dx}(y) = 4x-12$$

At $x = \sqrt{30}$

MC $4\sqrt{30}-12$

At $x = \sqrt{30}$

AC = $4\sqrt{30}-12$

Hence at the minimum point of AC curve

AC = MC $=4\sqrt{30}-12$.

Example 11. The demand function faced by a firmsis p = 500-0.2x and its cost function is C = 25x+ 10000 (p) = price, x = output and C = cost). Find the output at which the prifits of the firm are maximum. Also find the price will charge.

Solution. TC = 25x+10000

TR=$p \times x$

$= (500 - 0.2x)\, x$

$= 500x - 0.2x^2$

Condition for maximum profits is

MR = MC

$MR = \frac{d}{dx}(TR)$

$\frac{d}{dx}(500x - 0.2x^2)$

$= 500 - 0.4x$

$MC = \frac{d}{dx}(TC)$

$= \frac{d}{dx}(25x + 1000) = 25$

$\therefore$ MR = MC gives $500 - 0.4x = 25$

or $0.4x = 475$

$x = \frac{4750}{4} = 1187.50$

$\therefore$ Profit maximising level of output

=1187.50 units

and price at this level of output

$= 500 - 0.2\,(1187.50)$

$= 500 - 337.50$

$= 262.50$

Note: We could also have proceeded as follows:

π =Profits = TR – TC

and make π maximum profits.

$\frac{d\pi}{dx} = 0$

or $\frac{d}{dx}(TR - TC) = 0$

i.e. MR = M. C

Se we get the same result.

Example 12. A monoplist produces x sets per dayat the total cost of Rs. $\left(\frac{x^2}{25}+3x+100\right)$. Show that if the demand curve is x=75-3p is price set, he will produce about 30 sets. What is the monopoly price?

Solution: Let x be the number of sets whichmaximises the net revenue of the monoplist.

$\therefore$ TC for x sets = $\frac{x^2}{25}$ + 3x + 100

MC= $\frac{d}{dx}$(TC)= $\frac{2x}{25}$+3

Demand functions x=75-3p.

$\therefore$ TC for x sets = p x x

= $\frac{2x}{25}$ x x = $\frac{75-x^2}{3}$

$\therefore$ MR = $\frac{d}{dx}$(TC)= $\frac{1}{3}$[75 - 2x]

= 25 - $\frac{2}{3}$ x

Net revenue will be maximum at the level of output where

MC = RC

$\therefore$ $25 - \frac{2}{3}$x= $\frac{2x}{25}$ + 3

or $\frac{2x}{25}$ x + $\frac{2}{3}$ x = 25 - 3

or $\frac{56}{75}$ x = 22

or x = $\frac{75\times 22}{56}$

= $\frac{1650}{56}$ = 30 approx.

Since p = $\frac{75-x}{3}$

$\therefore$ At x = 30

p = $\frac{75-30}{3} = \frac{45}{3}$ = 15 Rs.

Hence net revenue is maximum when about 30 sets are produced per day and the monopoly is Rs. 15 per set.

So far we have applied the techniques of maximum and minimum without any constraints as discussed in this unit, to a variety of economic problems. But when we have an objective function to be maximized or minimized subject to the satisfaction of an equality constraint, Lagrange multiplier method seeks to convert the constrained extremeproblem into a form to which the first order and second order conditions of unconstrained extremism can still be applied. The Lagrange multiplier method would be given in the later unit, after we have discussed the concept of matrices.

SELF-CHECK EXERCISE 6.5

Q1. The Demand function faced by a firm is p = 500 – 0.2x and its cost function is c = 36x + 10000 (p = Price, x = Output and c = Cost). Find the output at which the profits of the firm are maximum. Also find the price at this level of output.

Q2. The Total Cost (TC) function for producing a commodity x is TC = 52 – 10x +2x^2. Find the level of output at which TC is minimum.

6.8 SUMMARY

In this Unit, we have discussed the extreme of a function and the condition under which it attains extreme. We have also discussed about the points of inflexion. Lastly the economic application of maxima and minima were dealt.

6.9 GLOSSARY

1. **Maximum value :** A function $y = f(x)$ is said to have a maximum value $f(a)$ at $x = a$ if f (a) lease to increase at $x = a$ and begins to decrease as x increase beyond a.

2. **Minimum value :** A function $y = f(y)$ is to have a minimum value of $f(a)$ $x = a$ and begins to increase as x beyond a.

3. **Extreme values :** The maximum and minimum value of a function are extreme value.

4. **Stationary points :** The points, at which first order derivatives are zero, are called stationary points.

5. **Points of inflexion :** The point of inflexion is defined as a point at which a curve changes its curvature. The sufficient question for a point of inflexion of $f'(x) = 0$ and $f''(x) \neq 0$

6.10 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 6.1

Ans. Q1. Refer to Section 6.3

Ans. Q2. Refers to Section 6.3

Self-check Exercise 6.2

Ans. Q1. Refer to Section 6.4

Self-check Exercise 6.3

Ans. Q1. First order condition

$12x^3 - 30x^2 + 12x = 0$ or $3x\,(4x - 2)\,(x - 2) = 0$

either, $x = 0$ or $x = 2$ or $x = 1/2$

Second order condition

At $x = 0$, $f''(x) = 12 > 0$

At $x = 2$, $f''(x) = -9 < 0$

Hence the function attains maximum at $x = \frac{1}{2}$ and win at $x = 0$ and $x = 2$

Ans. Q2 $x = -1/3$ is point of inflexion.

Ans. Q3 Here $f(x) = 6x + 6y, f\,y = 6x + 14y$

$f(x)\,(x) = 6, fxy = 6, fyy = 14$

a function requires $fx = fy = 0$ i.e.

$6x + 6y = 0$ ---------- (i)

$6x + 14y = 0$ -------- (ii)

solving (i) and (ii) for x and y we get, x = y = 0.

The given function reaches its minimum value at the stationary point and its minimum value is zero. this is because

$fxx \int_{y-0}^{x-0} = 6 > 0,\ fyy \int_{y-0}^{x-0} = 14 > 0$

Also, $f\,xx\ .\,f\,yy - (fxy)^2 \int_{y=0}^{x=0}$

$= 84 - 36 = 48 > 0$

Self-Check Exercise 6.4

Ans. Q1. Refer to Section 6.6 (Example 3)

Ans. Q2. Refer to Section 6.6 (Example 4)

Ans. Q1. Refer to Section 6.6 (Example 7)

Self-Check Exercise 6.5

Ans. Q1. TC $= 36x + 10000$

TR $= P \times x$

$= (500 - 0.2x)x$

$= 500x - 0.2x^2$

Condition for maximum profits is

MR $= MC \Rightarrow MR = \frac{d}{dx}(TR) = \frac{d}{dx}(500x - 0.2x^2)$

$= 500 - 0.4x$

MC $= \frac{d}{dx}(TC) \Rightarrow \frac{d}{dx}(36 + 10000) = 36$

MR = MC

$500 - 0.4x = 36$

or $0.4x = 464$

$x = \frac{4640}{4} = 1160$

$\therefore$ Profit maximising level of output = 1160 units

and price at this level of output

$= 500 - 0.2\,(1160)$

$= 500 - 232$

$= 268$

Ans. Q2. Let y = TC $= 50 - 10x + 2x^2 \Rightarrow \frac{d}{dx} = -10 + 4x$

$\frac{d^2y}{dx^2} = 4 > 0$

For maxima or minima

$\frac{dy}{dx} = 0 \;\therefore\; -10 + 4x = 0$ or $x = .\ 2/5$

since $\frac{d^2y}{dx^2}$ $4 > 0 \;\therefore$ y is min at x = 2/5 and the min value is given by

$$f\left(\frac{2}{5}\right) = 50 - 10\times\frac{2}{5} + 2\left(\frac{2}{5}\right)^2$$

$$= 50 - 4 + \frac{8}{25} = \frac{1250-100+8}{25}$$

$$= 46.32$$

6.11 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.D. (1998), Mathematical Analysis for Economists, St. Martin's Press, New York.
2. Chiang, A.C. (1974), Fundamental Methods of Mathematical Economics, 2nd edition, MC Grow-Hill Book Company, New York.
3. Henderson, James M and Quudt, Richard E (1980), Microeconomic Theory, MC Grow-Hill Book Company, New York.
4. Varian, Mall (1992), Microeconomic Analysis, W.W. Nortov & Company Inc. New York.

6.12 TERMINAL QUESTIONS.

Q. 1 Find the profit maximizing output given that

Q = 200 – 10p and AC = 10 + Q125

Unit 7

CONSTRAINED OPTIMISATION OF FUNCTIONS

STRUCTURE

7.1 INTRODUCTION

So far we have confined ourselves to the extreme value of function assuming that variable of the given function can take any values.For example, for a hypothetical utility function of two variables U= f(x, y) to get maximized, we took it for granted implicitly that the consumer could purchase an infinite amount of both the goods. But such an assumption has to have relevance in reality because the consumption of two goods also depends on the purchasing power (income of the consumer). As such that we need to find is that how much of x and y should the consumer purchase duly with the given purchasing power to maximize his utility. We also know that with the given purchasing power if the consumer buys more ofx, he will have to buy less of y or vice versa and, therefore, the amount of x and y are not independent of each other. Most of the economic problems concerning maxima and minima are of this nature. There is always a constraint on the variables and as such the variables x and y are not independent.

7.2 LEARNING OBJECTIVES

After studying this unit, you will be able tosolve the basis optimisation problems with equality as well as inequality constraint by using Lagrange method.

7.3 LAGRANGE MULTIPLIER :

This method can be explained in the form of twoconditions:

7.3.1 FIRST ORDER CONDITION:

We combine the given function and the constraint through a new variable in a way such that first order condition can still be applied.

For example Given utility function

$U=4xy - y^2$

and constant: $2x+y - 6=0$

Combining both through new variable λ knownas Langrange's multiplier, we get

$Z= f(x, y) +\lambda (2x+y-6)$ or

$Z=4xy-y^2+\lambda (2x+y-6)$

Treating λ as an additional variable, we have Z as a quadratic in variables x, y and λ.

Applying first order condition which

states:$f_x=f_y=f_\lambda =0$, we get

$$f_x=\frac{\partial z}{\partial x}= 4y+2\lambda=0$$

$$f_y = \frac{\partial z}{\partial y}=4y-2\lambda +\lambda =0$$

$$f_z = \frac{\partial z}{\partial \lambda}= 2x+y-6=0$$

Solving three equations, we get x=2,y=2 and Z=-4. The first order condition gives us the point where the given function has either maximumor minimum values.

7.3.2 SECOND ORDER CONDITION

According to second order condition for minimum value, $d^2z>0$ and maximum value $d^2z<0$.

But d^2z will have positive sign, if all the principal minors (begining from second) of Bordered Hessian determinant.

$$|\overline{H}| = \begin{vmatrix} 0 & \psi_1 & \psi_2 & ....\psi_n \\ \psi_1 & f_{11} & f_{12} & ....f_{1n} \\ \psi_2 & f_{21} & f_{22} & ....f_{2n} \\ ..... & ..... & ..... & .... \\ \psi_n & f_{n1} & f_{n2} & ....f_{nn} \end{vmatrix} \text{ are negative}$$

and d^2z will have negative sign, if the principal minors (begining from second) of Bordered Hessian determineant $|\overline{H}|$ possesses alternative sign, thefirst being negtive and the second being positive.

Example 1. If x and y are positive, show that maximum value of U=xy subject to the constrant $x^2+y^2=a^2$ occures when x=y= $\frac{a}{\sqrt{2}}$. Given U= xy, subject to $\psi(x + y) = x^2+y^2-a^2$ consider fz = U $+\lambda\psi = xy+\lambda (x^2+y^2-a^2)$ where λ is Lagrange's Multiplier.

First Order Condition

$f_1=fx=y+2\lambda x=0$ ... ... (i)

$f_2= fy = x+2\lambda y=0$ ... ... (ii)

$f_3 = x^2 + y^2-a^2 = 0$ ... ... (iii)

Solving (i) and (ii)

$y=-2\lambda x$, $x = -2\lambda y$

further we get

$x = \frac{a}{\sqrt{2}}$, $y = \frac{a}{\sqrt{2}}$ and $\lambda = \frac{-1}{2}$

∴ U can be maximum or minimum

at $\left(\frac{a}{\sqrt{2}}.\frac{a}{\sqrt{2}}\right)$

Second Order condition

In this case Bordered Hessian determinant is

$$|\overline{H}| = \begin{vmatrix} 0 & \psi_1 & \psi_2 \\ \psi_1 & f_{11} & f_{12} \\ \psi_2 & f_{21} & f_{22} \end{vmatrix} = \begin{vmatrix} 0 & 2x & 2y \\ 2x & 2\lambda & 1 \\ 2y & 1 & 2\lambda \end{vmatrix}$$

Now calculating the value of $|\overline{H}|$ at $x = \frac{a}{\sqrt{2}}$

$Y = \frac{a}{\sqrt{2}}$ and $\lambda = \frac{1}{2}$

$$|\overline{H}| = \begin{vmatrix} 0 & \sqrt{2}a & \sqrt{2}a \\ \sqrt{2}a & -1 & 1 \\ \sqrt{2}a & 1 & -1 \end{vmatrix}$$

$= |\overline{H_z}|$

$= 0 - \sqrt{2}\ a\ (-2\sqrt{2}\ a)$

$+ \sqrt{2}\ a\ (2\sqrt{2}\ a)$

$= 8a^2 > 0.$

As $|\overline{H_z}| > 0$

$\therefore$ U will be maximum at $x = y = \dfrac{a}{\sqrt{2}}$

and Max value of $U = xy = \dfrac{a}{\sqrt{2}} .- \dfrac{a}{\sqrt{2}} = \dfrac{a^2}{\sqrt{2}}$

Example 2. Determine the point which maximisesor minimises the function

$U = x^2 + xy + y^2 + 3z^2$

Subject to x+2y+4z=60.

Incorporating Lagranger's multiplier variable λ, we have $z = x^2 + xy + y^2 + 3z^2 + \lambda\ (x+2y+4z-60)$

First Order Condition:

$fx = 2x + y + \lambda = 0$

$fy = x + 2y + 2\lambda = 0$

$fz = 6z + 4\lambda = 0$

$f_\lambda = x + 2y + 4z - 60 = 0$

solving the equations, we get x=0,

$y = \dfrac{90}{7}$, $z = \dfrac{60}{7}$ and $\lambda = -\dfrac{90}{7}$

i.e. these are the point of maxima or minima forthe given function.

Second Order Condition:

$$|\overline{H}| = \begin{vmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 0 & 0 & 6 \end{vmatrix}$$

The principal minors:

$$|\overline{H_2}| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix} < 0 \text{ and}$$

$$|\overline{H_3}| = \begin{vmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 0 & 0 & 6 \end{vmatrix} < 0$$

As all the principal minors are <0, d²z will have positive value, In other words, the given function will have minimum value at the point 0.90/7, 60/7 and the value of the function will be

$$= (0)^2 + 0\left(\frac{90}{7}\right) + \left(\frac{90}{7}\right)^2 + 3\left(\frac{60}{7}\right)^2$$

$$= \frac{8100}{49} + \frac{10800}{49} = \frac{18900}{49}$$

Example 3. A firm production function is $Q = 5L^{0.7}, K^{0.3}$. The price of labour is Re. 1 per unit and the price of capital is Rs. 2 per unit. Find the minimum cost combination of capital and labour for an output of 20.

The cost equation: C=L+2K

Production function $Q = 5L^{0.7}, K^{0.3}$

First order condition gives:

(i) $\frac{\partial z}{\partial L} = 1\text{-}3.5\lambda.L^{-0.3}K^{0.3}=11\text{-}3.5\lambda\left(\frac{K}{L}\right)^{0.3}=0$

(ii) $\frac{\partial z}{\partial k} =2\text{-} 1.5\lambda.L^{0.7}K^{0.7} = 2\text{-}1.5\lambda\left(\frac{K}{L}\right)^{0.7} = 0$

(iii) $\frac{\partial z}{\partial \lambda} =20\text{-}5L^{0.7}, K^{0.3} = 0$

$$3.5\left(\frac{K}{L}\right)^{0.3} = \frac{1.5}{2}\left(\frac{K}{L}\right)^{0.7}$$

i.e. $\frac{L}{K} = \frac{14}{3}$ .... (iv)

Equation (iii) gives : $L^{0.7}, K^{0.3} = 4$

i.e. L. $\left(\frac{3}{14}\right)^{0.3} = 4$ {Substituting from (iv)}

$\therefore$ L = 4 $\left(\frac{14}{3}\right)^{0.3}$ = 4.(4.6)$^{0.3}$

K = $\frac{6}{7}\left(\frac{14}{3}\right)^{0.3}$ = 0.86 (4.6)$^{0.3}$

In other words, the firm should use

$\left[4\left(\frac{14}{3}\right)^{0.3}\right]$ and $\left[\frac{6}{7}\left(\frac{14}{3}\right)^{0.3}\right]$units of labour and

capital respectively for an output rate of 20.This will cause the firm to incur minimum cost (andbring maximum profit)

Example 4. Given a cost function $C = r_1 \times 1 + r_2 + F$

and a production which serves as a constant $q + f(x_1+x_2)$. Find the first and second-order conditions for minimum cost

First order Condition

Set the cost function as$C= r_1x_1+ r_2x_2+F=h\ (x_1, X_2)$

Then by the Lagrange multiplier method

$Z = h(x_1,x_2)+\lambda\{q-f(x_1, x_2)\}$

$$\frac{\partial z}{\partial x_1} = r_1 + \lambda(-f_1) = 0$$

$$\frac{\partial z}{\partial x_2} = r_2 + \lambda(-f_2) = 0$$

$$\frac{\partial z}{\partial \lambda} = q - f\ (x_1, x_2) = 0$$

$\frac{f_1}{r_1}=\frac{f_2}{r_2}=\frac{1}{\lambda}$ expresses the first order.

Conditions, which is the law of equal-marginalproductivity.

Second Order Condition

Using the differential method this is $d^2c < 0$ subject to $d\phi\ (x_1, x_2)=0$

where $\phi\ (x_1,x_2) = q-f(x_1, x_2)=0$

Calculations will show that

$d^2c = r_1d^2x_1$

$d^2\phi = - f_1d^2x_2-f_{11}\ f_{12} - f_{22}\ dx_2^2- 2f_{12}\ dx_1dx_2 = 0$

From $d^2\phi$ we find d^2x_1 and substituting this d^2x_1 in to $d^2\phi$ we find d^2x_1 and substituting this d^2x_1 in to d^2c gives us

$$d^2c = \frac{-r_1}{f_1}[f_{11}\, dx_1^2 + f_{22}\, dx_2^2 + 2f_{12}\, dx_1\, dx_2] > 0$$

Since $r_1 > 0$ and $f_1 > 0$ we need

$f_{11}dx_1^2 + f_{22}\, dx_2^2 + 2f_{12}\, dx_1\, dx_2 < 0$.

for $d^2c < 0$, v.c. for C to be minimum.

SELF-CHECK EXERCISE 7.1

Q1. Use the method of Largeuge multipliers to find the minimum value of the function $f(x, y, z) = x + y + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$

Q2. A consumer's utility function is given by $U = 5q^2_1 + 2q_2^2 + 3q_1,q_2$ and his total budget is Rs. 50. The market price of q_1 and q_2 is Rs. 4 and Rs. 5 per unit respectively.Find the optimum of this consumer.

7.4 LEAST-COST COMBINATION OF INPUTS

As another example of constrained optimization, let us discus the problem of finding the least-cost input combination for the production of a specified level of output Q_0 representing say, a customer'sspecial order.

7.4.1 FIRST ORDER CONDITION

Assuming a production function with two variable inputs, $Q = Q(a, b)$ where $Q_a, Q_b > 0$ in the relevant subset of the domain and assuming both input prices to be exogenous, we may formulate the problem as one of minimizing the cost.

$C = ap_a + bP_b$

Subject to the output constraint

$Q(a, b) = Q_0$

Hence the Lagrangean function is

$Z = aP_a + bP_b + \mu[Q_0 - Q(a, b)]$

To satisfy the first-order condition for a minimumC, the input levels (the choice variables) must satisfy the following simultaneous equations:

$Z\mu = Q_0 - Q(a. b) = 0$

$Z_a = P_a - \mu Q_a = 0$

$Z_b = P_b - \mu\, Q_b = 0$

The first equation in this set is merely the constraint restated, and the last two imply the conditions.

$$\frac{P_a}{Q_a} = \frac{P_b}{Q_b} = \mu \text{ .... (i)}$$

At the point of optimal input combination, the input-price-marginal-product ratios must be the same for input. Since this ratio measures the amount of outlay per unit of marginal product of the input in question, the unit of marginal product of the input in question, the Largrange multiplier u can be given the interpretation of the marginal cost of production in the optimum state.

Equation (i) can be alternatively written in the form

$$\frac{P_a}{P_b} = \frac{Q_a}{Q_b}$$

Presented in this form, this order condition can be explained in terms of isoquants and isocosts. The Q_a/Q_b ratio is the negative of the slope of an isoquant, that is it is a measure of the marginal rate of technical substation of a for b ($MRTS_{ab}$). In the present model, the output level is specified at Q_0, thus only one isopuant is involved, as shown in figure.

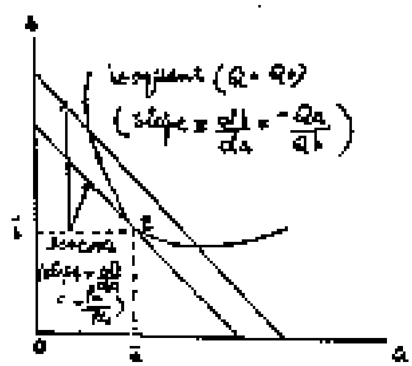


Figure - 1

The P_a/P_b ratio, on the other hand, represents the negative of the slope of isocosts. An isocosts, defined as the locus of the input combinations that entail, the same total cost, is expressionable by the linear equation.

$$C_n = ap_a + bP_b$$

$$\text{or } B = \frac{C_a}{P_b} = \frac{P_a}{P_b} a$$

where C_0 stands for a (parametric) cost figure. When plotted in the ab plane as Fig. 1 therefore it yields a family of straight lines slope P_a/P_b and vertical intercept C_0/P_b. The equality of the two ratios therefore amounts to the equality of the slopes of the isoquant and a selected isocost. Since we are compelled to stay on the given isoquant, this condition leads us to the point of tangency E and the input combination $(\bar{a}. \bar{b})$.

7.4.2 SECOND ORDER CONDITION

To assure a minimum cost, it is sufficient (after the first-order condition is met) to have a negative Bordered Hessian, i.e. to have

$$|\overline{H}| = \begin{vmatrix} 0 & Q_a & Q_b \\ Q & -\mu Q_{aa} & -\mu Q_{ab} \\ Q & -\mu Q_{ba} & -\mu Q_{bb} \end{vmatrix}$$

$= \mu\ (Q_{aa}\ Q^2b - 2Q_{ab}\ Q_a\ Q_b + Q_{bb}\ Q^2a) < 0.$

Since the optimal value of μ (marginal cost) is positive, this reduces to the condition that the expression in parenthesis be negative.

The curvature of an isoquant is represented by the second derivative.

$$\frac{d^2b}{da^2} = \frac{-1}{Qb^3}\ (Q_{ab}Q_b^2 - 2Q_{ab}\ Q_aQ_b + Q_{bb}\ Q_a^2)$$

When the isoquant is strictly converted at the point of tangency, we have the inequality $a^2|da^2>0$, which implies - since Q_b (marginal product of b) is positive that the expression in parentheses is negative. Thusthe strict convexity of the isoquant of fig. 1 at the point of its tangency with an isocost which guarantee the satisfaction of the second order condition stated above. Conversely, if the second-order condition is satisfied, then the isoquant must be strictly convex at the point of tangency.

Example 5. Given a cost function$C = r_1x_1 + r_2x_2 + F$and a production function which serves as a constraint$q=f(x_1,x_2)$find the first and second order conditions for minimum cost.

First order Condition

Set the cost function as

$C= r_1x_1+ r_2x_2+F= h\ (x_1, x_2)$

Then by the Lagrange multiplier method

$Z=h(x_1, x_2)+ \lambda\ [q-f(x_1, x_2)]$

$$\frac{\partial z}{\partial x_1} = r_1 + \lambda\ (-f_1) = 0$$

$$\frac{\partial z}{\partial x_2} = r_2 + \lambda\ (-f_2) = 0$$

$$\frac{\partial z}{\partial \lambda} = q - f\ (x_1 - x_2) = 0$$

$$\therefore \quad \frac{f_1}{r_1} = \frac{f_2}{r_2} = \frac{1}{\lambda}$$

Express the first order conditions, which is the law of equimarginal productivity.

Second Order Conditions

Using the differential method, this is

$d^2c < 0$.

subject to $d\varphi\ (x_1, x_2)=0$

where $\varphi\ (x_1, x_2)=q-f(x_1, x_1)=0$

Calculations will show that

$d^2c=r_1 d^2\ x_1$

$d^2\varphi= -f_1 d^2x_1 - f_1\ f_1^2 - f_{22}\ dx_2^2 - 2f_{12}\ dx_1\ dx_1 = 0$

From $d^2\varphi$, we find d^2x_1, and substituting this d^2x_1 into d^2c give us

$d^2c = -\frac{r_1}{f_1}[f_{11}\ dx_1^2 + f_{22}\ dx_2^2 - 2f_{12}\ dx_1\ dx_2] >$

Since $r_1>0$ and $f_1> 0$, we need

$f_{11}\ dx_1^2+f_{22}dx_2^2+2f_{12}dx_1\ dx_2<0$

for $d^2c<0$, i.e. for C to be minimum.

SELF-CHECK EXERCISE 7.2

Q1. A firm production function is $Q = 5\ L^{0.7}\ K^{0.3}$. The price of labour in Rs. 1 per unit end the price of capital is Rs. 2 per unit. Find the minimum cost combination of capital and labour for an output of 20.

Q2. Given a cost function $C = p_1x_1 + p_2x_2 + OH$ and a production function which serves as a constraint $q = f(x_1, x_2)$. Find the first and second order condition for minimum cost.

7.5 SUMMARY

In this unit, we emphasise the basic theory of constrained optimisation, constrained optimisation in case of quality constraints applied Lagrangean method to solve those problems.

7.6 GLOSSARY

- **Lagrange Multiplier :** The Lagrange multipliers is strategy for finding the local maxima and minima of a function subject to equality constraints.

7.7 ANSWER TO SELF CHECK EXERCISES

Self-Check Exercise 7.1

Ans. Q1. Refer to Section 7.3

Ans. Q2. Solution

The Lagranzian Function,

$L= 5q_1 +2q_2^2 + 3q_1q_2 + \lambda\ (50 - 4q_1 -5q_2)$

Differentiate L w.r.t. q_1, q_2 and setting the derivatives equal to zero.

$$\frac{\partial L}{\partial q_1} = 10q_1 + 3q_2 + 4\lambda = 0$$

$$\frac{\partial L}{\partial q_3} = 4q_2 + 3q_1 + 5\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 50 - 4q_1 - 5q_2 = 0$$

Solving for q_1 from the following

$$\begin{bmatrix} 10 & 3 & -4 \\ 3 & 4 & -5 \\ -4 & -5 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -50 \end{bmatrix}$$

We get $q_1 = \frac{50}{194}$, $q_2 = \frac{1900}{194}$

The 2nd order condition,

$$\begin{bmatrix} U_{11} & U_{12} & -P_1 \\ U_{21} & U_{22} & -P_1 \\ -P_2 & -P_2 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 3 & -4 \\ 3 & 4 & -5 \\ -4 & -5 & 0 \end{bmatrix} > 0$$

for maximum, but it appears $U_{11} > 0$, hence the result is not conclusive. The main assumption of cardinal theory is $U_{11} < 0$, $U_{12} < 0$ which is not fulfilled.

Self-Check Exercise 7.2

Ans. Q1. The cost equation: C=L+2K

Production function $Q = 5L^{0.7} K^{0.3}$

First order condition gives

(i) $\frac{\partial z}{\partial L} = 1\text{-}3.5\lambda.L^{-0.3}K^{0.3} = 11\text{-}3.5\lambda\left(\frac{K}{L}\right)^{0.3} = 0$

(ii) $\frac{\partial z}{\partial k} = 2\text{-} 1.5\lambda.L^{0.7}K^{0.7} = 2\text{-}1.5\lambda\left(\frac{K}{L}\right)^{0.7} = 0$

(iii) $\frac{\partial z}{\partial \lambda} = 20\text{-}5L^{0.7}, K^{0.3} = 0$

$$3.5\left(\frac{K}{L}\right)^{0.3} = \frac{1.5}{2}\left(\frac{K}{L}\right)^{0.7}$$

i.e. $\frac{L}{K} = \frac{14}{3}$ .... (i)

$$L.\left(\frac{3}{14}\right)^{0.3} = 4 \text{ \{Substituting from (i)\}}$$

$$\therefore L = 4\left(\frac{14}{3}\right)^{0.3} = 4.(4.6)^{0.3}$$

$$K = \frac{6}{7}\left(\frac{14}{3}\right)^{0.3} = 0.86\ (4.6)^{0.3}$$

Ans. Q2. Refer to Section 7.4 (Example 5)

7.8 REFERENCES/ SUGGESTED READINGS

1. Henderson, J.M. and Quandt, R. (1980). Microeconomic Theory. Mc Graw-Hill Book Company, New York.
2. Allen, R.G.D. (1938). Mathematical Analysis for Economists. Dt. Martin's Press, New York.
3. Varian, H. (1992). Micro Economic Analysis. W.W. Nortan & Company, Inc. New York.

7.9 TERMINAL QUESTIONS

Q. 1 Use the method of Lagrange multipliers to find the minimum value of

$f(x, y) = x^2 + 4y^2 - 2x + 8y$ subject to constraint $x + 2y = 7$.

Q. 2 Use the method of Lagrange multiplier to find the maximum value of

$f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$ subject to the constraint $3x + 4y = 32$.

Unit 8

DIFFERENCE EQATIONS

STRUCTURE

8.1 INTRODUCTION

The calculus of finite differences, in its broad meaning, deals with the changes that take place in the value of the function, the dependent variable, due to finite changes in the independent variable. It is a study of the relations that exists between the values assumed by the function whenever the independent variable changes by finite jumps whether equal or unequal. In infinitesimal calculus we study, on the other hand, those changes which occur when the independent variable changes continuously in a given interval.

The variable time in various economic data is usually treated discretely, time is divided up into units value of a variable in one period is assumed to be determined by, amongst other things, its value in the previous period, the one before that and so on. This may be because decisions are taken only at discrete intervals, because data is available only at certainties, or for various other reason. So the difference equation frequently express economic relationships more adequately than differential equations. For example, in planning models, the companions in between the initial base year and the terminal year and change in investment over the period is orelated to change in time over a period. Both the changes are said to be discrete.

Consider a function y = $f(x)$ an in Fig. 1. The derivative of f(x) is defined as

$$\lim_{(\Delta \to x0)} \frac{f(x + \Delta x - f(x) =}{(\Delta + \Delta x) - x} \quad \lim_{(\Delta \to x0)} \frac{\Delta y}{\Delta}$$

Instead of taking a limiting process we will now let x be finite quantity and write.

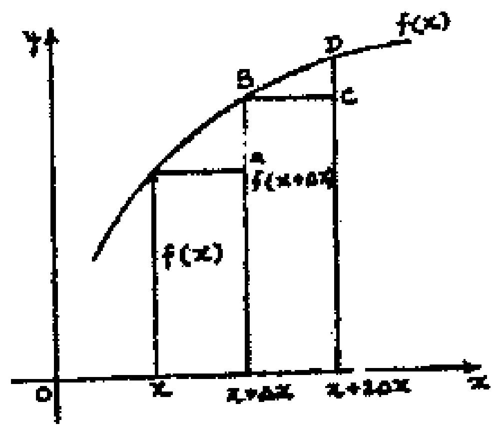


Fig. 1

$f(x+\Delta x) - f(\Delta x) = y(x+\Delta x) - y(x) \;\; \Delta y(x)$

Δ is a symbol denoting that we are operating any in the above fashion and is called a difference operator. The finite quantity is called the difference interval.

Thus, we have a relationship

$\Delta y(x)=y(x+\Delta x)-y(x)$……………….. (1)

which means that we take a difference interval Δx from the point x and find the difference between the two values of y at the point x and x + Δ x.

In the present case when we are dealing with finite difference, the distance between any two successive points in the domain are a finite distance a part. For our subsequent discussions, not only will two successive points be a finite distance a part, but this will also be a constant. Thus, if we have one point x, we can specify the succeeding points by letting $\Delta x=h$, so that

x, x+h, x+2h, x+3h,....

The points have formed a sequence which will have the characteristics of an arithmetic progression.

Once it is decided that the difference interval $\Delta x = h$ will a constant, we can simplify matters further by changing the scale of the x–axis so that h =1. Then, successive points starting from x will be

x, x +1, x+2, x+3,.....

Δy of equation (1) is called the first difference. By repeating process, we get,

$\Delta [\Delta y (x)]=\Delta [y (x+ h) - y(x)]$

$= \Delta y(x+h) - \Delta y(x0$

which is called the second difference. Thisis written as

$\Delta^2 y(x) = \Delta y(x+h) - \Delta y(x)$.

$= [y(x+2h) – y(x+h)]–\{y(x+h) – y(x)]$

$= y(x+2h) –2y(x+h)+y(x)$

Reputing this process, we have

$\Delta[\Delta^2 y(x)] = \Delta^2[y(x+h) – y(x)]= \Delta^2 y(x+h) - \Delta^2 y(x)$

$= [y(x+3h) – 2y(x+2)+y(x+h)$

$-[y(x+2h) – 2y(x+h)+y(x)]$

or $\Delta^1 y(x)=y(x+3h) – 3y(x+2h) + 3y(x+h) – y(x)$.

By repeating this process,, we can obtainthe general formula.

$\Delta^n y(x) = (-1)^0 C_0^n y(x + xh) + (-1)^1 C_1^n y[x + (x – 1) h] + ...$

$+ \in 1)^{n-1} n_{en-1} y(x+h) + (-1)^n y(x)$

where $c^n{}_m = \frac{<n}{<m<n-m}.nc_0 =1.<0=1$

8.2 LEARNING OBJECTIVES

After going through this Unit, you will be able to :

- Solve Difference Equations
- Find out the order of Difference Equations
- Explain the change of Notation
- Give solution of Difference Equations

8.3 DIFFERENCE EQUATIONS

Def: An equation that relates the independent variable x, the dependent variable y and its finite difference is called a difference equation, i.e.

$I(x, y, \Delta y, \Delta^2 y,......) = 0$

is a difference equation.

8.3.1 ORDER OF THE DIFFERENCE EQUATION

The order of the difference equation is that of the highest difference contained in the equation.For example, consider the following three differenceequation.

(i) $y(x)=5 \Delta y(x)+4 \Delta^2 y(x)+\Delta^3 y(x)=x$ (2)

(ii) $y(x+3)+y(x+2) – y(x – 1)=x$ (3)

(iii) $y(x+x) – y(x+1)=0$ (4)

8.3.2 CHANGE OF NOTATION

For convenience, we shall now change ournotation as follows.

$y(x+2)=y_x +2$

$y (n+x)= y_n+x$

and so forth.

Thus, the equation (3) and (4) above canbe written on

$y_{x-3} + y_{x+2} – y_x = x$ (5)

$y_x + x – y\ x + 1 = 0$ (6)

the first difference equation involves successive differences of the dependent variable y while the second equation involves the successive values of the dependent variable. In practice it will be found more convenient to deal with differences equations involving successive values of the dependent variables andnot successive differences. A differences equation not involving successive value of y_x greater than y_{2+n} is said to be order of x. The order of the equation (2) is 3. It is the difference between the largest and smallest arguements x appearing is an equation. Then equation (5) is a difference equation of order is 3 and the order of equation (6) is n-1.

8.3.3 SOLUTION OF DIFFERENCE EQUATION

A solution of a difference equation over a set S is a relation between the independent variable and the dependent variable which satisfies the equation or is an identity over S. Such a relation on substitution is the equation that makes the left hand and right hand number identically.

An equation over a set S of the form

$Y_{x+n}+ A_1y_{n+1-1} +..... A_ny_n = R(x)$

where A_1 's and R (x) are functions of x or constants defined for all values ofx in the set is called a linear difference equation over S of order x. If R(x) =0, the equation is called linear homogeneous otherwise it is called linear non-homogeneous equation.

SELF-CHECK EXERCISE 8.1

Q1. Find the solution of the equation un = $3u_{n-1} + 4$ given $u_0 = 2$

Q2. Find the general solution of the difference equation un = $u_{n-1} + 4$, N > 1

Q3. Find the first difference of the following function at x=2.

(a) $Y (x)=3x^2+2x$

(b) $y(x)=x(x – 1)$

8.4 HOMOGENEOUS LINEAR DIFFERENCE EQUATION WITH CONSTANT COEFFICIENT

The general equation of a homogeneous linear difference equation with constant coefficients of order n is of the form

$Y_{n+x} + A_1\, Y_{x+n-l} + \ldots\ldots\ldots + A_{x-1}\, y_{x+1}\, A_n y_1 = 0$ (7)

where A^1 's are constants.

The general solution of similar type of differential equation was found by first obtaining an auxiliarly equation. It was done by setting

$y=e^{mx}$

In the case of difference equation, we will be

$y_x = \beta^x$

where β is a constant. Then equation (7) becomes

$(\beta x+A_1 b^{n-1}+ \ldots\ldots+A_n)\ \beta^x=0$

Thus, we have

$\beta^n+ A_1\, b^{n-1} + \ldots.\ A_n = 0$

and we call this equation the auxiliary or characteristic equation. The roots of this equation will be solutions of (7). The general solution is

$Y_x = c_1\beta_1{}^x+\ldots\ldots\ldots + C_n\ \beta_n{}^x$

Case 1

Linear Homogeneous Difference Equation with constant co-efficiency of the First Order

Consider

$y_{n+1} = A_1\, y_n = 0$

Let $y_1 = \beta^n$, then

$\beta^{x+1} - A_1\beta^x = 0$

$\beta^x\, (\beta - A) = 0$

$\therefore\ \beta^x = A_1$

Thus the solution of the difference equation is

$y_x = C_1\, A_1$

Example 1

Find the solution of first order linear homogeneous difference equation

$$y_{n+1} - \frac{3}{2} y_n = 0$$

Solution

$$y_{n+1} - \frac{3}{2} y_x = 0$$

Let $y_n = \beta^x$

On substitution, we obtain the characteristic equation

$$\beta^{x+1} - \frac{3}{2}\beta^x = 0$$

$$\beta\left(b - \frac{3}{2}\right) = 0$$

$$\therefore \beta = \frac{3}{2}$$

Thus, the solution is

$$y_x = C_1\beta^x = C^1 \frac{3^x}{2}$$

Case 2

Linear Homogeneous Difference Equation with constant coefficients of order 2

Consider the equation

$y_{x+2}+A_1y_{x+1}+ A_1\, y_x=0$

which is of order 2.

Let$y_x= \beta^x$ Then the auxiliary equation is

$\beta^{x+2} +A_1\beta^{x+1} + A_2\beta^x = 0$

$\beta^x (\beta^2 + A_1\beta + A_2)=0$

In this case we have three different situations.

(a) When the two roots β_1 and β_2 are real and distinct, the solution is given by

$y_x = C_x\, \beta_1^x + C_2\beta_2^x$

Example 2: Solve

$Y_x - 5y_{x-1} +6y_{x-2}=0$

Let $y_x = \beta^x$ be the solution of the aboveequation. Then the auxiliary equation is

$(\beta^x - 5\beta^{x-1} + y\, \beta^{x-2}) = 0$

or $\beta^{x-2} (\beta^2 -5\, \beta+6)=0$

or $\beta^2 -5\beta +6=0$

$\therefore$ $\beta_1= 2, \beta_2 = 3$

$\therefore$ The general solution is

$y_x =C_1\, 2^x+C_2\, 3^x$

It is given that $y_0 =3$, $y_1 =5$

$\therefore y_0 = C_1 2^0 + C_2 3^0 = 3$ i.e. $C_1 + C_2 = 3$

and $y_0\ C_1 2^1 + C_2 3^0 = 5$ i.e. $2C_1 + 3C_2 = 5$

We find from these two equations

$C_1 = 4$ and $C_2 = -1$

$\therefore$ The solution is

$Y_x = 4.2^x = 3^x$

(b) When the two roots are equal

When the two roots of auxiliary equationsare equal i.e.

$\beta_1 = \beta_2 = \beta$

The general solution is

$y_x = C_1 \beta^x + C_2 x\ \beta^x$

Example 3: Solve and check the solution

$y_{x+2} - 6y_{x+1} + 9y_x = 0$

Solution: The equation

$y_{x+2} - 6y_{x+1} + 9y_x = 0$

is linear homogeneous difference equation of order two with constant coefficients

The auxiliary equation of the above equation is

$\beta_1 - 6\beta + 9 = 0$

or $(\beta - 3)^2 = 0$

i.e. $\beta_1 = \beta_2 = 3$

$\therefore \quad y_x = C_1 3^x + C_2\ x\ 3^x$

For checking the solution, consider left side 2 difference equation

L.H.S. $= y_{x+2} - 6y_{x+1} + 9y_x$

$= C_1 3^{x+2} + C_2 (x+2)\ 3^{x+2} - 6\ [C_1 3^{x+1} + C_2 (x+1)\ 3 + x + 1]^{+9)C}{}_1\ 3^x + C_2 x\ 3^x)$

If $y_x = C_1\ 3_x + C_2\ x\ 3^x$ is a solution, it must satisfy the differential equation.

$= 9\ C_1\ 3^x + 9\ C_2\ x3^x + 18\ C_2\ 3 + x - 18\ C_1\ x^3 - 18\ C_2\ 3^x - 18\ C_2 3^x + 9\ C_1\ 3^x + 9C_2\ x3^x$

$= 0$

= R.H.S.

(c) When the roots are conjugate complex numbers

Let the roots be

$\beta_1 = a + ib = p\ (\text{Cos}\ \theta + i \sin \theta)$ (8)

β_2 =a+ib=p (Cos θ + i sin θ) (9)

On multiplying equation (8) and (9) we get

a_2+b_2 = p² ($Cos^2θ$ - i^2 sin θ) p² ($Cos^2θ$ +$Sin^2θ$) = p²

$$p = \sqrt{a^2 + b^2}$$

and on equating real and imaginary parts in equation(8), we get

a=p Cos θ, b=p Sin θ

$\therefore$ tan θ= $\frac{a}{b}$ or θ = tan^{-1}ba

The solution is

$y_x = d_1\beta_1^x + d_2\beta_2^x$

where y_x need to be real numbers, But if β_2 and β_1 are complex numbers while d_1 and d_2 are not, y_n may be a complex number, To avoid this, we shall assumed, and d_2 are complex conjugates. We can do this because d_1 and d_2 are arbitrary. Thus, let us set

d_1=m+in. d_2 = m – in.

To avoid complex number, let us show one solution in terms of polar coordinates. We have

$d_1\beta_1^x = d_1 p^x$ (Cos θ + i sin θ$)^x$

= $d_1 p^x$(Cos θ + i sin θ$)^x$

$d_2\beta_2^x = d_2 p^x$(Cos θ + i sin θ$)^x$

because of de Moivae's Theorems. Thus

$y_x=p^x$ [(d_1+d_2) Cos θ x+i ($d_1 – d_2$) Sin θ x]

$\therefore y_x=p^x$ (C_1 Cos θ x + C_2 Sin θ x)

where C_1= d_1+d_2=(m+in)+(m – in) = 2m

C_2i ($d_1– d_2$) = i (2 in) = -2n

Thus C_1 and C_2 are real numbers and the y_x we have obtained is a real number.

The solution is sometimes shown in the following form which is easier to interpret when discussing business cycles or economic growth. Let

d_1 = m+in = k (Cos B+i Sin B)

where $K = \sqrt{m^2} + n^2, B \tan^{-1} \frac{n}{m}$

Then

C_1 =d_1 + d_2 = 2 k cos B

C_2i (d_1 – d_2)= – 2k sin B

Substituting these into the solution, we get

$y_n = p^x$ [2k Cos B Cos θ x = 2k Sin B Sine x}

which can be written as

$y_n = 2kp^x$ [Cos B Cos qx – sin B sin qx]

= A ^{x}Cos (θx+B)

(∴ Cos (A+B) = Cos A Cos B – Sin A Sin B)

where A = 2k

Then, for example, if y_n is income, p^n shows the amplitude and θ x shows the period of oscillations of y_x.

Example 4: Solve the differential equation

$y_{n+2} - y_{n+1} + y_n = 0$

The auxiliary equation becomes

$\beta^2 - \beta + 1 = 0$

$$\beta_1 = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$$

$$\beta_1 = \frac{1 \pm \sqrt{1-3i}}{2} \, and \, \beta_2 \frac{1 \pm \sqrt{3}i}{2}$$

Here $a = 1/2 ; b - \sqrt{3}/2$, $thus\ p = a^2 + b2 = \sqrt{1}/4 + 3/4 =$

and $\theta = \tan^{-1} \frac{b}{a} = \tan^{-1} \frac{\sqrt{3} = \pi/3}{1}$

∴ The solution of the given equation is by

$y_x\ p^x\ [C_1 \text{ Cos } \theta\ x + C_2 \text{ Sin } \theta\ x]$

$= 1\ [C_1 \text{ Cos x } \frac{\pi}{3} \text{ x} + C_2 \text{ Sin} \frac{\pi}{3} \text{ x}]$

$= C_1 \text{ Cos} \frac{\pi}{3} \text{ x} + C_2 \text{ Sin } \frac{\pi}{3} \text{x}$

Here amplitude is 1. and period$= \frac{2\pi}{\pi_3} = 6$

SELF-CHECK EXERCISE 8.2

Q1. Find the solution of first order liner homogeneous difference equation.

$y_n + \frac{-5}{3}\ 1\ y_n = 0$

Q2. Solve and check the solution

$$y_{x+2} - 2y_{x+1} + 4yx = 0$$

Q3. Solve the difference equation

$$4x - 3y_{x-1} + 4y_{x-1} = 0$$

8.5 GEOMETRICAL INTERPRETATION OF SOLUTION

The solution when $\beta_1 \neq \beta_2$ and real was

$$y_x - C_1 \beta_1^x + C_2 \beta_2^x$$

Since C_1 and C_2 are constant, the main influence on y_x when $x \rightarrow \infty$ will be values of β_1 and β_2. When $\beta_1 \neq \beta_2$, the larger one will eventually determine the behaviour of y_x. Let us call the larger root in absolute terms the dominant root and assume for the moment it is β_1. We shall form the cases for different value of C_1 and β_1. Letting x = 0, 1, 2, ..., we consider

(i) When $C_1 > 0$, $\beta_1 > 1 : y_x = C_1 \beta_1^x$ would graphic as in Fig. 1.

Fig. 1.

(ii) When C > 0, 1 > β_1,> 0, then we Range the curve shown in Fig. 2.

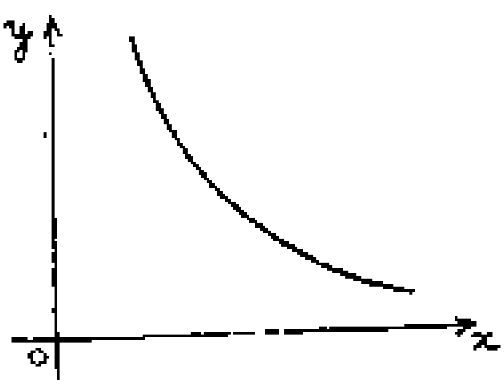


Fig. 2.

(iii) When $C_1 > 0, 0 > \beta > -1$, then wehave the curve show in Fig. 3.

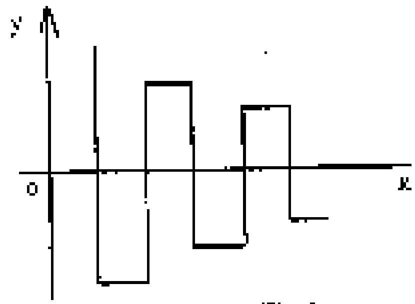


Fig. 3

(iv) When $C_1>0,-1>\beta$, then we have the curve shown in Fig. 4.

Since $y_x=C_1\,\beta^x+C_2\beta^x$, this will be a combination of any of the above situations.

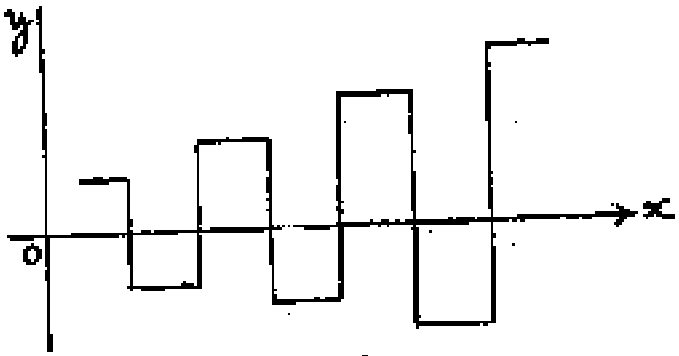


Fig. 4.

$y_x= \rho^x\,(C_1 \text{ Cos } \theta\, x + C_2 \sin \theta\, x)$

$= A\,\rho^x \text{ Cos } (\theta\, x+ \beta)$

where ρ^x will give the magnituder of the oscillation while $\theta\, x$ will determine the periodicity.

(v) When $\rho\ >1$, we get explosive oscillations, curve is shown in Fig. 5

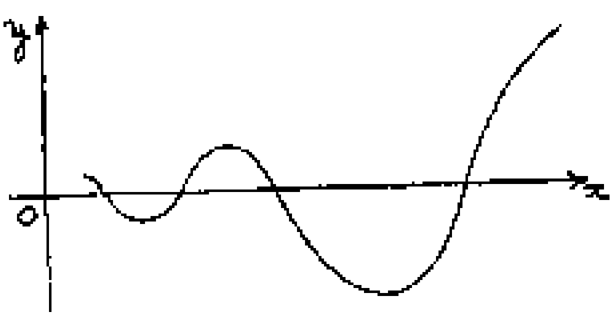


Fig. 5.

(vi) When $\rho\ = 1$, we get simple harmonic,the curve is shown in Fig. 6.

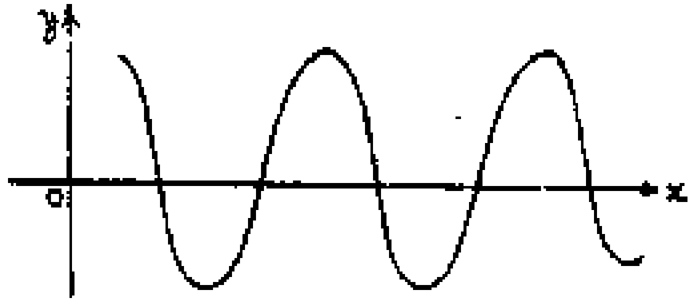


Fig. 6.

$\rho < 1$, we get damped oscillations, curve is shown in Fig. 7.

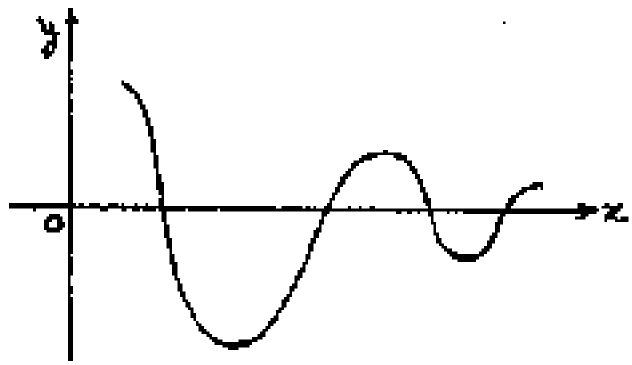


Fig. 7.

8.5.1 Particular Solutions of Non-Homogenous Linear Equations

For showing differential equation, we shall study the method of undetermined coefficient to obtain the particular solution for differential equations. As in the differential equation case, the solution is expressed as general solution = (solution of homogeneous equation)+(particular solution)

The method of undermined coefficients is useful in finding the particular solution of the complete equation when R (x) is of special type. We se up a trial solution which consists of the number of unknown constant coefficients, corresponding to each term present in R(x). The constant coefficients are to be determined by substitution in the difference equation.

Special type of R(x) and its Trial solution

SL No.	Special type of R(x)	Trial solution
1.	$a^x f(x)$	$a^x (A_0 + A_1 x + ... + A_n x^n)$
2.	a^xSin bx or	a^x(A Sin bx+b Cos bx) a^x cos bx
3.	a^x	A. a^x

4. Sin b_x or cos b_x A Sin bx + β cos bx

5. Constant $\begin{Bmatrix} B\frac{1-A^x}{1-A} when\, A\neq 1 \\ Bx \quad when\, A=1 \end{Bmatrix}$

We shall discuss the solution of

(i) linear first-order differential equations with constant coefficients, and

(ii) linear second order differential equations withconstant coefficients.

(i) Linear First-order Differential Equations

Suppose after adjusting the equation it is inthe form

$y_{x+1}=Ay_x+B$

where A and B are constant and the coefficients of $У_{x+1}$ is unity. Then the homogeneous solution can be obtained by letting B=0.

Thus

$Y_{x+1} = Ay_x$

For the difference equation of the present kind, weset

$y_x = B^x$

Substituting this into our equation we obtain,

$\beta^{x-1}=A\beta^x$

∴ β = A

Thus, the homogeneous solution will be

$Y_x = CA^x$

where C is a constant

the particular solution in this case is

$$y_x = \begin{Bmatrix} B\frac{1-A^x}{1-A} when\, A\neq 1 \\ Bx \quad when\, A=1 \end{Bmatrix}$$

Thus, the general solution will be

$$y_x = \begin{Bmatrix} CA + B\frac{1-A^x}{1-A}, when\, A\neq 1 \\ X + Bx \quad A=1 \end{Bmatrix}$$

where x = 0, 1,2,3....

Example 5: Solve $y_{t+1}+3y_1= 4$ when $y_0= 4$

Solution: Here the difference equation is

$y_{t+1} + 3y_t = 4$ or $y_t + -3y_t + 4$

The general solution will be

$y_t = CA^t + B\frac{1-A^t}{1-A}$ when $A \neq 1$

Given that $y_0 = 4$

$\therefore 4 = C(-3)^0 + 1 - (-3)^0$

or C=4

General solution becomes

$y_t = 4(-3)^t + 1 - (-3)^t$

$= 3(-3)^t + 1$

Example 6: Solve the differential equation

$3y_{x+1} = 6y_x + 9x = 0, 1, 2, 3,$ when $y_0 = 7$

Solution

The general solution will be of the form

$y_t = CA^t + B\frac{1-A^t}{1-A}$ when $A \neq 1$ of

the differential equation $y_{x+1} = ay_x + B$

The above differential equation can be written

$Y_{x+1} = 2y_x + 3$

Here A=2 and B=3

$\therefore\ y_1 = C(2)^x + 3\frac{1-2^x}{1-2}$

Given $y_0 = 7$

$7 = C(2)^0 2[1-(2)^0]$

C=7

Thus, the general solution becomes

$y_x = 72^x + 3\frac{1-2^x}{1-2} = 7.2^x - 3 + 32$

$= 10.2^x - 3$

Example 7: Solve $y_x = 7$, given $y_0 = 14$

Solution: Given $\Delta y_x = 7$

or $\quad y_{x+1} - y_x = 7$

$\therefore \quad y_x - y_{x-1} = 7$

$y_{x-1} - y_{x-2} = 7$

$y_2 - y_1 = 7$

$y_1 - y_0 = 7$

Adding, we get

$y_{x+1} - y_0 = (x+1)\ 7$

$\therefore \quad y_{x+1} = y_0 + 7x + 7 + (y_{x+1} - y_x)$

or $\quad y_x = y_0 + 7x$

$y_x\ 14 + 7x$

Example 8: Solve $\Delta y_x = -6y_x$

Solution: Given $\Delta y_x = -6y_x$

$y_{x+1} - y_x = -6y_x$

$y_{x+1} = -5y_x$

$y_{x+1} = -5y_x$ Putting $x = 0$

$y_2 = -5y_1 = -5\ (-5y_0) =)-5)^2\ y_0$

$y_x = (-5)^x y_0$

Hence the required solution is $y_x = (-5)^x\ y_0$

Linear Second-Order Difference Equation with Constant Coefficients

Let the equation be

$y_{x+2} + A_1\ y_{x+2} + A_2 y_{x+2} = R(x)$

here the function R (x) may be constant or a function of x and A_1, A_2 are constant

Several cases of particular interest are considered

Case I

When $R(x) = A^x$ where A is constant, we

try as particular solution

$y_x = CA_x$

Example 9: Solve

$y_{x+2} - 4y_{x+1} + 3y_x = 5^x$

Solution: The second order linear difference equation is

$y_{x+2} – 4y_x+1+3y^x = 5^x$

(10)

The homogeneous equation of the above equationis

$y_{x+2} – 4y_{x+2}+3y_x=0$

Its auxiliary equation is

$\beta^2 – 4\beta +3=0$

or $(\beta – 1)(\beta – 3)=0$

$\therefore$ $\beta_1 = 1, \beta_2 = 3$ are two roots.

Thus the solution of homogeneous equationof complementary function of equation (10) is

$C.F. = C_1 1^x+C_2 3^x$

For the particular solution, let

$y_x = C. 5^x$

Substituting this into the equation (10), we shall get

$C.5^{x+5} – 4C.5^{x+1}+3.C.5^x = 5^x$

or $5^x (C.5^2 – 4C.5+ 3C) = 5^x$

or $(25 – 20 + 3)$

$8C=1$

or $C = 1/8$

Thus the particular solution is $y^x = 1/8\ 5x$

General solution is

$y_x=C_1+C_2 3^x+1/8.\ 5^x$

Example 10. Solve

$y_{x+2} – 4y_{x+2}+3y_x=3^x$

Solution: The homogeneous solution is the same & above, viz.

$y_x = C_1 + C_2\ 3^x$

We notice the part of the homogenous solution is the same as the function Rx i.e. 3x. In such a casewhere the homogenous solution includes a term similar to the function R(x), we multiply the particular solution we are trying by x. Thus we shall try

$y_x = C_x\ 3^x$

On putting this solution into original equation, we get

$C(x+2)\ 3^{x+2} - 4C\ (x+1)\ 3^{x+1} + 3\ Cx\ 3^x = 3x$

$C\ [9x\ 3^x + 18 3^x - 12x.3^x + 3^x 3^x] = 3^x$

or $\quad C[6] = 1$

$\therefore \quad C = \frac{1}{6}$

The particular solution becomes

$$yx = \frac{1}{6} x 3^x$$

$\therefore$ General solution is

$$y_x = C_1 + C_2\ 3x + \frac{1}{6}\ x\ 3^x$$

Example 11.: Solve

$$y_{x+2} - 6Y_{x+1} + 9y_x = 3^x \qquad (11)$$

Solution

The equation $y_{x+2} - 6y_{x+1} 9y_x = 3^x$

is a second order non-homogeneous equation with constant coefficient

In this case, its homogeneous equation is

$$y_{x+2} - 6y_{x+1} + 9y_x = 0$$

The auxiliary equation is

$$\beta^x - 6\beta + 9 = 0$$

$$\beta^x - 6\beta + 9 = 0$$

or $\quad (\beta - 3)2 = 0$

i.e $\quad \beta_1 = \beta_2 = 3$

The homogeneous solution is

$$y_x = C_1\ 3^x + C_1\ x\ 3^x$$

To find the particular solution, we try $y_x = C3^x$

But, the terms in the homogeneous solution include 3^x, we multiply by x and set $y^x = C3^x$. But there is still term in homogeneous solution which is same as the particular solution we propose to try$Cx\ 3^x$, i.e.

$$y_x\ Cx^2\ 3^x$$

Now, there is no term in the homogeneous solution similar to this. On substitution in equation (11) we get

$$C(x+2)^2\ 3^{x+2} - 6\ C(x+1)^2\ 3^{x+1} + 9cx^2\ 3^x = 3^x$$

$C[9(x^2+4x+4) - 18(x^2+2x+1)+9x^2]\, 3^x = 3^x$

$18C=1$

$\therefore C=\frac{1}{18}$

The particular solution is

$$y_x=\frac{x^2}{18}x3^x$$

General solution is

$$y_x = C_1+3^x+C_2\, x3^x+\frac{x^2}{18}3^x$$

Case II

When $R(x)=x^n$, we try, as a particular

$$y_x=A_0 +A_1x+A_2x_2+.... + A_nx^n$$

The method for finding the solution is the same as in CaseI. We first find the homogeneous solution say

$$y_x = C_1\beta_1{}^xC_2\beta_2{}^x$$

if it is a second order equation. Then we check to find if the particular solution has any terms similar to the terms in the homogeneous solution. If has, we multiply with x just as in case.

Example 12: Solve the differential equation

$y_{x+2} - 4y_{x+1}+3y_x= x^2$

Solution: The differential equation is

$$y_{x+2} - 4y_{x+1}+3y_x= x^2 \quad (12)$$

The homogeneous equation of equation (12) is

$$y_{x+2} - 4y_{x+1}+3y_x=0$$

The auxiliary equation is

$\beta_2 - 4+3=0$

or $(\beta- 1)(\beta- 3)=0$

or $\beta_1 = 1\ \beta_2 = 3$

$\therefore \quad y_n=C_1= C_2\, 3^x$

The particular solution we assume is

$$y_n=A_0+A_1\, x+A_2\, x^2$$

This solution has a constant Ao. The homogeneous solution has a constant C. Thus, we have multiply the solution by x to have to different term in the solution and on multiplying the above solutionby x, we get

$y_x = A_0x + A_1x^2 + A_2x^3$

so that there is no similar terms in the homogeneous and particular solution. On substitutingthis in the equation (12), we get

$A_0 (x+2) + A_1 (x+2)^2 + A_2 (x+2)^3$

$-4 [A_0 (x+1) + A_1 (x+1)_2\ A_2 (x+1)^3] + 3 [A_0x + A_1 x^2 + Ax^3] = x^2$

or

$A_0 (x+2) + A_1 (x^2+4x+4) + A_2 (x^38 + 6x^2 + 12x)$

$-4[A_0 (x+1) + A_1 (x_2+2x+1) + A_2 (x^3 + 1 + 3x^2 + 3x)]$

$+3 [A_0x + A_1 x^2 + A_2x^3]$

or

$A_0 (A_2 - 4 A_2 + 3A_2)x^3 + (A_1 + 6A_2 - 4A\ 1 - 3A_2)x^2$

or $-6A2\ x2 + (-4A1)x + (-2Ao + 4A2) = x2$

Equation coefficients, we get

$-6A_2 = 1 \Rightarrow A_2 \frac{-1}{6}$

$-4A_1 = 0 \Rightarrow A_1 = 0$

$-2A_0 + cA_2 = 0$

$\therefore A_0 = 2A_2$

$= 2\left(\frac{1}{6}\right)$

i.e. $A_0 = - -\frac{1}{3}$

So we have $A_0 = \frac{-1}{3}$ $A_1 = 0$, $A_2 = \frac{-1}{16}$

Then the particular solution is

$y_x = \frac{-1}{3}x - \frac{-1}{6}x^3$

General solution is

$y_x = C_1 + C_2\ 3^x - \frac{-1}{3}x - \frac{1}{6}x^3$

Case III

When $R(x)$ = constant, let a particular be gien by $y_x=y$ for all x.

∴Putting $y_x=y$ is

$y_x+A_1y_{x-1}+A_2\ y_{x-2}+ .... + A_xy_x– x = R$

∴ $y + A_1y + A_2y + ... + A_n\ y = R$

or $(1 + A_1 + A_2 + ... + A_n)\ y = R$

But when $1+ A_1+ A_2\ ....+A_n = 0$, then this procedure fails then we take particular solution $y_x = xy$. If this also fails, we then try the particular solution $y_x=x^2y$ and so on.

Example 13: Solve the equation

$y_n – 2y_{n-1}+ y_{n-2}=1,\ y_0 = 2$ and $y_1 = 5.5$

Solution: The difference equation is

$y_n – 2y_{n-1}+ y_{n-2}=1$ (13)

The homogeneous equation to the aboveequation is

$y_x – 2y_x+ y_{x-2}= 0$

The auxiliary equation is

$\beta^2 - 2\beta + 1 = 0$

$(\beta -1)^2=0$

∴ $\beta_1 = \beta_2 = 1$

The complementary function of (13) is

$y_x = (C_1 + C_2\ x)\ \beta_x$

$= (C_1 + C_2\ x)\ 1x$

$= C_1 + C_2\ x$

For particular solution, let y_x =y for all x,

∴and on substituting it is equation (13) we get

$y – 2y+y=1$

i.e. 0=1, which is not possible,

Now substitute, $y_x=x^2y$ is equation (13), we get

$x^2y – 2(2-1)^2y+(x – 2)^2\ y=0$

i.e. $x^2y – 2(x^2+1 – 2x)\ y + (x2 – 4x+4)\ y = 1$

or $x^2y – 2x^2y – 2y+4xy + x^2y – 4xy+4y=1$

$2y=1$ or $y = 1/2$

$\therefore$ The particular solution of equation (13) is

$$y_x = \frac{1}{2}x^2$$

$\therefore$ General solution equation (13) is

$$y_x = C_1 + C_2 x + \frac{1}{2}x^2$$

Note: If R= A^x+ Bx^n

Then in this case, case I and case II are used simultaneously.

Example 14: Solve

$$y_{x+2} - 4\, y_{x+1} + 3y_x = 5x + 2x$$

Solution: The difference equation is

$$y_{x+2} - 4\, y_{x+1} + 3y_x = 5x + 2x$$

The homogeneous equation of the above equationis

$$y_{x+2} - 4y_{x+1} + 3y_x = 0$$

The auxiliary equation is

$$\beta_2 - 4\beta + 3 = 0$$

or $(\beta-3)(\beta-1)=0$

$$\beta_1 = 1,\ \beta_2 = 3$$

The homogeneous solution is

$$y_x = C_1 + C_2\, 3^x$$

The particular solution for R = 5^x is

$$y_x = C.5^x$$

The particular solution for R=2x is

$$yx = A_0 + A_1 x$$

Here constant term A_0 and also a constantterm in the homogeneous solution, viz., C_1 we multiply by x and get

$$y_x = A_0 x + Ax^2$$

Thus, the combined particular solution is

$$y_x = A_0 + A_1 x^2 + C.5^x$$

Substituting this value of y_x in equation (14), we get

$$A_0\,(x+2) + A_1\,(x+2) + C5x+2$$

$$-4\, pA_0\,(x+1) + A_1\,(x+1)\ 2\ c.5x + 1]$$

$+3[A_0x+A_1x_2+C.4^x]=5x+2x$

or $A_0\ (x+2)+A_1\ (x_2+4x-4)+25\ C.5^x$

$-4\ [A_0\ (x+1)+A_1(x_2+2x+1)+4\ C.5^x0$

$+3[A_0\ x+A_1x_2+C.5x]\ =5x+2x$

or $2A_0+4A_1-4A_0-4A_1+(A_0+4A_1-4\ A_0$

$-8A_1+3A_0)x=5x+2x$

$+(A_1-4A_1-3A_1)\ x^2+C\ (25-20+3)\ 5x$

or $-2A_0-4A_1x+9\ C5^x\ 5x+2x$

Equating coefficients, we get

$-2A_0=0 \quad \Rightarrow \quad A_0=0$

$-4A_1=2 \quad \Rightarrow \quad A_1=-1/2$

$8\ C=1 \ \Rightarrow \quad C=\frac{1}{8}$

Thus, the particular solution is

$$y_x=\frac{1}{2}x^2+\frac{1}{8}5x$$

and the general solution is

$$y_x=C_1+C_2\,3^x\,\frac{1}{2}x^2+\frac{1}{8}5x$$

SELF-CHECK EXERCISE 8.3

Q1. Solve $y_{t+1}+3y_1=4$ when $y_0=4$

Q2. Solve the differential equation

$3y_{x+1}=6y_x+9x=0,1,2,3,$ when $y_0=7$

8.6 SUMMARY

In this unit we learnt about the difference equation. An equation that relates the independent variable x_1 the dependent variable y and its finite difference is called a difference equation. In the next section we learnt about the order of the difference equation. Further we discussed about the change of notation. We have also studied about the homogeneous linear difference equation with constant coefficient of order 1 and order 2. In the next section of the unit, we learnt about the geometrical interpretation of the solution.

8.7 GLOSSARY

1. **Difference Equation :** An equation that relates the independent variable x1 the dependent variable y and its finite difference is called a difference equation.

2. **Homogeneous Difference Equation :** A difference equation is homogeneous is the constant term b is zero.

3. **Linear Difference Equation :** A difference equation is linear if (i) the dependent variable y is not raised to any power and (ii) there are no product terms.

4. **Non-homogeneous difference equation :** A difference equation is non-homogeneous is the constant term b is non-zero.

5. **Order of a Difference Equation :** It is determined by the maximum number of periods lagged.

8.8 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 8.1

Ans. 1 Solution

un = $3u_{n-1} + 4$ Given $u_o = 2$

un $= 3^n \times 2 + 2\,(3^n - 1)$

$= 2 \times 3^n + 2 \times 3^n - 2$

$= 4 \times 3^n - 2$

Particular solution of the difference equation has been found.

Ans. 2 $u_0 = u_0 + 4n$.

Ans. Q3. Solution

(a) $y\,(x)=3x^2+ 2x$

$\Delta\, y(x) = y\,(x+1) - y(x)=y(3) - y\,(2)$

$= (3.3^2+2.3) - (3.2^2 +2.2)$

$= 27+6 - 12 - 4 = 17$

(b) $y\,(x)=x(x - 1)$

$= x^2 - x$

$\Delta\, y\,(x) = y\,(x+1) - (x)=(3) - y\,(2)$

$= (3^2-3) - (2^2-2)$

$= 6 - 2 = 4$

Self-check Exercise 8.2

Ans. Q1. Solution

$$y_{n+1} - \frac{5}{2} y_x = 0$$

Let $y_n = \beta^x$

On substitution, we obtain the characteristic equation

$$\beta^{x+1} - \frac{5}{2}\beta^{x} = 0$$

$$\beta\left(b - \frac{5}{2}\right) = 0$$

$$\therefore \beta = \frac{5}{2}$$

Thus, the solution is

$$y_x = C_1\beta^x = C^1 \frac{5^x}{2}$$

Ans. Q2. Solution

The equation $y_{x+2} – 2y_{x+1} + 4y_x = 0$

is linear homogeneous difference equation of order two with constant coefficients

The auxiliary equation of the above equation is

$\beta_1 - 2\beta + 4 = 0$

or $(\beta - 2)^2 = 0$

i.e. $\beta_1 = \beta_2 = 2$

$\therefore \quad y_x = C_1 2^x + C_2 x 2^x$

For checking the solution, consider left side 2 difference equation

L.H.S. $= y_{x+2} – 2y_{x+1} + 4y_x$

$= C_1 2^{x+2} + C_2 (x+2) 2^{x+2} – 4 [C_1 2^{x+1} + C_2 (x + 1) 2 + x + 1]^{+4)C}{}_1 2^x + C_2 \times 3^x)$

If $y_x = C_1 2_x + C_2 x 2^x$ is a solution, it must satisfy the differential equation.

$= 4 C_1 2^x + 4 C_2 \times 2^x + 8 C_2 2 + x – 8 C_1 x^3 – 8 C_2 2^x – 8 C_2 2^x + 4 C_1 2^x + 4C_2 \times 2^x$

$= 0$

= R.H.S.

Ans. Q3. Refer to Section 8.4 (Example 2)

Self-Check Exercise 8.3

Ans. Q1. Refer to Section 8.5 (Example 6)

Ans. Q2. Refer to Section 8.5 (Example 7)

8.9 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.

2. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.

3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.

4. Yamane, T. (2012). Mathematic for Economists : An Elementary Survey. Pretice Hall of India, New Delhi.

8.10 TERMINAL QUESTIONS

Q.1 Find the first difference of

$y(x) = 3x^2$

Q.2 Find the first and second difference of

$y(x)=3x^2+2x$

Q.3 Solve

$$y_{x+1} \frac{-7}{4} y_x = 0$$

Q.4 Solve

$y_x + 2 - 10y_x + 1 + 25y_x = 0$

Q.5 Solve and check the solution of difference equation

$y_x+2 - 10y_x+1+25y_x=0$

Q.6 Solve the difference equation

$2\ y_x + 1 = 6y_x - 4$ when $y_n = 2$

Q.7 Solve

$y_x + 1 = 4y_x + 4,\ y_e = 2$

Q.8 Solve

$y_x + 2 + 5y_x + 1 - 6y_x = 2_x$

DIFFERENTIAL EQUATIONS: INTRODUCTION AND SOLUTION OF FIRST ORDER AND FIRST DEGREE EQUATIONS

STRUCTURE

9.1 INTRODUCTION

A difference equation is used to solve the values of an unknown function y(x) for different discrete value of x. In this Unit, we introduced to the concept of differential equations.

9.2 LEARNING OBJECTIVES

After studying this Unit, you will be able to :

- solve Differential Equation
- know the order of Differential Equation
- identify the degree of Differential Equation
- solve the exact Differential Equation

9.3 DIFFERENTIAL EQUATION

Def: An equation involving derivations of one or more dependent variables with respect to one or more independent variables is called oa differential equation

For Example $\frac{d^2y}{dx^2}+xy\left(\frac{dy}{dx}\right)^2 = 0$ (1)

$\frac{d^4x}{dt^4}+5\left(\frac{d^2y}{dt^2}\right) + 3x = \sin t$ (2)

$\frac{\partial v}{\partial s}+\frac{\partial v}{\partial t} = v$ (3)

$\frac{\partial^2 v}{\partial s^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2} = 0$ (4)

The equations (1) to (4) are differential equations. The differential equations are classified according to whether threr is one or more than one independent variable in the equation.

9.3.1 ORDINARY DIFFERENTIAL EQUATION

Def: A differential equation involving ordinary derivative of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Equation (1) & (2) are ordinary differential equations. In equation (1) the variable x is the single independent variable, and y is a dependent variable & in equation (2) the independent variable is t.

9.3.2 PARTIAL DIFFERENTIAL EQUATION

Def: A differential equation involving partial derivatives of one or more dependent variables with respect to partial differential equation.

Equations (3) and (4) are partial differential equations. In Equation (3) the variables & t are independent variables and v is a dependent variables. In equation (4) there are three independent variables x, y and z, in this equation u is dependent.

We further classify differential equations, both ordinary and partial, according to the order of the highest derivative appearing in the equation. For this purpose we define the order of an equation.

9.3.3 ORDER OF DIFFERENTIAL EQUATION

Def: The order of the highest order derivative involved in a differential equation is called the order of the differential equation.

The ordinary differential equation (1) is of the second order, since the highest derivative involved is a second derivative. Equation (2) is an ordinary differential equation of the fourth order. The partial differential equations (3) and (4) are of the first and second orders, respectively.

9.3.4 DEGREE OF A DIFFERENTIAL EQUATION

The degree of a differential equation is the degree of the highest derivative when the equation has been made free from the redicals and negative inedices as far as the derivatives are concerned.

For example y = $\sqrt{1+\frac{d^2y}{dx^2}}$ on simplifying,

we obtain $y^2 = 1 + \frac{d^2y}{dx^2}$

The highest derivative is $\frac{d^2y}{dx^2}$ The order of equation is 2.

Highest degree of this highest differential is 1, hence the degree of equation is 1.

In equation $\left(\frac{d^3y}{dx^3}\right)^2 + \frac{dy}{dx} + y = 0$ highest derivatives is $\frac{d^3y}{dx^3}$ y is 2. So the degree of the aboveequation is 2.

9.3.5 LINEAR DIFFERENTIAL EQUATION

Def: A differential equation of order in the dependent variable y and the independent variable x, when expressed in the form

$$a_0(x)\ \frac{d^n y^2}{dx^n}+a_1(x)\frac{d^{x-1}y}{dx^{n-1}}+.....+a_{n-1}(x)\frac{dy}{dx}a_n(x)y=b(x)$$

where a, is not indetically zero is said to be linear equation because here (i) the dependet variable y and its various derivatives occur to the first degree only, (ii) that no products of y and or any of its derivatives are present, and (iii) that no transcendental fucntion of y and/or its derivative occur.

For example,

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

$\frac{d^4y}{dx^4}+\frac{d^3y}{dx^3}+x_3\frac{dy}{dx}=xe^x$ one linear differential equation.

9.3.6 NON-LINEAR DIFFERENTIAL EQUATION

A non-linear ordinary differential equationis an ordinary differential equation thath is not linear. For example

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0$$

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

are all non linear differential equations.

Self-check Exercise 9.1

Q1. Define Differential Equation .

Q2. What is meant by Partial Differential Equation.

Q3. What do you understand by Non-Liner Differential Equation.

9.4 SOLUTION OF A DIFFERENTIAL EQUATION

A solution of a differential equation is a function which satisfies the equation and does not involve and derivative or differential.

For example, consider a differential equation

$$\frac{dy}{dx} = 3x^2 \qquad (5)$$

Integrating both sides w.r.t. x, we get y x^3+c (6)

(where C is a constant of integration)

is a solution of the differential equation (5) and this value of y in equation (6) satisfies the differential equation. The definition implies that a differential equation differential and other algebric process of elimination, etc. For this reason, the solution of a differential equation is also called its primitive.

General Solution (or Complete Primitive or Complete Intergal)

The general solutions of a differential equation must contain as many arbitary constants as theorder of the equation.

Particular solution

The solutions deduced from the general solution by giving particular values to the arbitary constants are called particular solutions of the equation.

Singular Solution

A singular solution of a differential equation is that solution which satisfies the equation but cannot be derived from its general solution.

Now, we will classify differential equations which are in the syllabus.

1. Non linear differential equations of the first order and first degree.

(a) Variables are separable

(b) Homogeneous differential equations exact differential equation.

2. Linear differential equation of first order.

3. Linear differential equation of the second order with constant coefficients.

Self-check Exercise 9.2

Q1. Solve $\frac{dy}{dx} = e^{x-y} + x^2e^{-4}$

Q2. Solve $\frac{dy}{dx} = e^{4-x} + 1$

9.5 SOLUTION OF NON-LINEAR DIFFERENTIAL EQUATION OF THE FIRST ORDER AND FIRST DEGREE

(a) When variables are separable:

If the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (7)$$

can be put in the form $f_1(x)\,dx = f_2(y)\,dy$ (8)

where with dx we associate a function $f_1(x)$ which is only a function of x and with dy we associate a function $f_2(y)$ which is only a function of y, we have the variable separable case, Such equations are solved by integrating both sides of (8) and adding an arbitrary constant of integration to any one of the two sides. Thus solution of equation (7) is

$$\int f1(x)\,dx = \int f_2(y)\,dy + C \qquad (9)$$

The constant of C can be selected in any suitable form, for example, logC, SinC, CosC, e^c etc.

Example 1: Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution: $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$= e^{-y}(e^x + x^2)$

or $\frac{dy}{e^{-y}} = (e^x + x^2)\,dx$

or $e^y\,dy = (e^x + x^2)\,dx$

On integrating both sides, we get $e^y = \frac{x^3}{3} + C$

(where c is a constant integration) is the required solution.

(i) Equations reducible to variable separable

:-Equation fo the form

$\frac{dy}{dx}$ =f(ax+by+c) or $\frac{dy}{dx}$ = f(ax+by)

can be reduced to an equation in which variables can be separated. For this purpose we use the substitution ax+by+c=vorax+by=v.

Example 2: Solve (x + y) (dx – dy)= dx + dy.

Solution: (x+y) (dx – dy) = dx + dy (x+y-1) dx = (x+y+1) dy

or $\frac{dy}{dx} = \frac{x+y-1}{x+y-1}$ (1)

Let x = y = v

$1 + \frac{dy}{dx} = \frac{dv}{dx}$ (2)

or $\frac{dy}{dx} = \frac{dv}{dx}$ (3)

Equation (1) with the help of (2) & (3) becomes

$\frac{dy}{dx} - 1 = \frac{v-1}{v+1}$

$\frac{dv}{dx} = \frac{v-1}{v+1} + 1 = \frac{2v}{y+1}$

$\therefore$ 2sz = $\left(1+\frac{1}{v}\right)$dv

On Integrating, we get

2x+c=v+log v

2x+c=x+y+log (x+y)

x-y+c = log (x+y)

Example 3: Solve $\frac{dy}{dx} = e^{x-y}+1$

Solution: $\frac{dy}{dx} = e^{x-y} + 1$

Putx-y=z

$1 - \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Equation (1) can be written as

$$1 - \frac{dy}{dx} = e^z + 1$$

or $\frac{dy}{dx} = e^z$

$$\frac{dz}{e^{-z}} = dx$$

On integrating, we get

$$\frac{e^z}{-1} = x + c$$

or $e^{y-x} = x - c$

or $x = e^y + c$

is the required general solution of the given differential equation.

(b) Homogeneous Differential Equation:- Adifferential equation of first oreder and first degree is said to be homogeneous if it can be put in the

from

$$\frac{dy}{dx} = f\left(\frac{x}{y}\right)$$

To solve such an equation, we put y=vx, where vis a function of x.

Then $\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$

$\therefore$ Equation (1) can be written as v+x+$\left(\frac{dv}{dx}\right) = f(v)$

or x $\frac{dv}{dx} = f(v) - v$

separating the variables, we get

or $\frac{dx}{x} = \frac{dy}{f(v) - u}$

on integration, we get logx+c= $\int \frac{dv}{f(v) - u}$

where C is a constant of integration.

After integration, replacing v by y/x

Example 4

Solve: $(x^2+ y^2)\, dx - 2xy\, dy = 0$

Solution: $(x^2+ y^2)\, dx-2xy\, dy =0$

or $\qquad \frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

Put y = vx, then $\frac{dy}{dx}=v+x\ \frac{dv}{dx}$

Equation (1) can be written as

$$v+x\frac{dv}{dx}=\frac{x^2+u^2x^2}{2x^2v}=\frac{1+v^2}{2v}$$

$$x\frac{dv}{dx}=\frac{1+v^2}{2v}-v$$

$$=\frac{1+v^2-2v^2}{2v}=\frac{1-v^2}{2v}$$

$$\frac{2v}{1+v^2}dv=\frac{dx}{dx}$$

$$-\left(\frac{-2v}{1+v^2}\right)dv=\frac{dx}{dx}$$

On integrating, we get

$-\log(1-v^2) \log|x| -\log C$

(Where c is a constant of integration)

$\log|1-v^2| = \log|x|+ \log c$

$\log(1-v^2)\, x| = \log C$

or $\qquad (1-y^2)\, x=C$

or $\qquad \left(1-\frac{y^2}{x^2}\right)\ x = C$

or $\qquad x^2 - y^2 = Cx$

Example 5

Solve: $x\left(\frac{dy}{dx}\right)= y\,(\log y - \log x + 1)$

Solution: or $\frac{dv}{dx}=\frac{y}{x}\left(log\frac{y}{x}+1\right)$

Putting y=vx, we have $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$From equation (1), we get

v + x $\frac{dv}{dx}$= v (og v + 1)

or $\frac{dx}{x} = \frac{dy}{v \log v} = \frac{1vdv}{\log v}$.

On integrating, we get

log x+log C = log v

or cx=log v

$\therefore$v=e^{ex}

or y_x=e^{ex}

or y=x^{ex}

(ii) Equation reducible to homogeneous form:

The equations of the form

$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c}$ where $\frac{a}{a_1} \neq \frac{b}{b_1}$ can be reduced to homogeneous form.

Let x = X + h&y= Y + k where h and k arecontants

Here dx=dX & dy = dY.

The given equation (1) can be written as

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c}$$

$$= \frac{aX+by+ah+bk+c}{a'X+b'y+a'h+b'k+c} \qquad (2)$$

In order to make equation (2) homogeneous, choosehand k such that

ah+bk+c=0 (3)

and a'h+b'k+c=0 (4)

Solving equation (3) & (4) for h & h, we get

$$h = \frac{dc'-bc'}{ab-ab} \text{ \& } k = \frac{ca'-ca'}{ab'-ab'} \qquad (5)$$

It is given to us that $\frac{a}{a_1} \neq \frac{b}{b}$ ab' –a'b ≠ 0

Hence h and k are given by the equation (5) willexist.

Eqution (2) can be written as

$$\frac{dY}{dX}=\frac{aX+bY}{a'X+b'Y}=\frac{a+b\left(\frac{Y}{X}\right)}{a'+b'\left(\frac{Y}{X}\right)}$$

which being homogeneous in X and Y and can be Y solved by putting $\left(\frac{Y}{X}\right)$ = van usual. After getting solution is terms of X and Y, we remove X and Y by putting X = x- hand Y = y = k.

Example 6

Solve: $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Solution: $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Here $a_1 = 1$, $b_1 = 2$, $a_2 = 2$, & $b_2 = 1$

$\therefore \frac{a_1}{a_2}=\frac{1}{2}$ & $\frac{b_1}{b_2}=\frac{2}{1}=2$

$\therefore \frac{a_1}{a_2}\neq\frac{b_1}{b_2}$

Put x = X+h and y = Y+k

$\therefore$dx=dX and dy = dy

$$\therefore \qquad \frac{dy}{dx}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}$$

$$= \frac{X+2Y+(H+2k-3)}{2X+h+(2H+k-3)}$$

Choose H and k such that h+2k-3=0 and 2h+k=0

$$\frac{h}{-6+3}=\frac{k}{-6+3}=\frac{1}{1-4} \qquad (2)$$

$$\frac{h}{-3}=\frac{k}{-3}=\frac{1}{-3} \qquad (3)$$

$$\therefore \qquad \frac{dY}{dX}=\frac{X+2Y}{2X+Y} \qquad (4)$$

is an homogeneous equation

Put Y=vX, so $\frac{dY}{dX} = v + X\frac{dY}{dX}$

Equation (3) can be written as

$$v + X\frac{dv}{dX} = \frac{1+2v.}{2+v}$$

or $X\frac{dv}{dX} = \frac{1+2v.}{2+v} - v$

$$\frac{1+v^2.}{2+v}$$

(1) $\frac{dv}{dX} = \frac{2+v.}{(2-v)(1+v)}$ dv

(Resolving partial functions)

$$\frac{dx}{X} = \left[\frac{1}{2}\left(\frac{1}{1+v}\right) + \frac{3}{2}\left(\frac{1}{1-v}\right)\right]dv$$

On integrating, we get

log X + logC= $\frac{1}{2}$[log (1 + v) - 3 log (1 - v)]

2log C_x = log $\frac{1+v}{(1-v)^3}$

or C^2X^2 = $\frac{1+v}{(1-v)}$

$$C^2X^2\left(1-\frac{Y}{X}\right)^3 = 1+\frac{Y}{X}$$

C^2 (X-Y)=X+Y

C^2[(x-1)-(y-1)]²=x-1+y-1

or C^2 (x-y)$^{2=}$ x+y-2

Case of failure

In the differential equation

$$\frac{dy}{dx} = \frac{a_1 + b_1 y + c_1}{a_2 x + b_2 y + c_2} \text{ where } \frac{a_1}{a_2} = \frac{Y}{X}$$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{1}{m} \text{ (say)}$$

a_2=ma_1& b_2 = mb_2

The given equation reduces to

$$\frac{dy}{dx}=\frac{a_1+b_1y+c_1}{m(a_1x+b_1y)+c_2}$$

Put $a_1x+b_1y=z$

$$\therefore a_1+b_1\frac{dy}{dx}=\frac{dz_1}{dx} or \frac{dy}{dx}=\frac{\frac{dz}{dx}-a_1}{b_1}$$

$$\therefore \frac{1}{b_1}\left(\frac{dz}{dx}-a_1\right)=\frac{z+c_1}{mz+c_2}$$

or $$\frac{dz}{dx}=\frac{b_1(z+c_1)}{mz+c_2}+a_1$$

$$=\frac{b_1(z+c_1)+a_1(mz+c_2)}{mz+c_2}$$

$$\therefore \frac{mz+c}{z(b_1+a_1m)b_1c_1+a_1c_2}\ dz=dx$$

In the above equation used for variables are separable and can be solved.

Example 7

Solve: (3y+4x+4) dx - (4x+6y+5)=0

Solution: (3y+4x+4)-(4x+6y+5)=0

or $$\frac{dy}{dx}=\frac{4x+3y+4}{4x+6y+5}$$

Here a_1 = 2, b_1 = 3, & a_2 = 4, b_2 = 6

$$\therefore \quad \frac{a_1}{a_2}=\frac{2}{4}=\frac{1}{2}\ \&\ \frac{b_1}{b_2}=\frac{3}{6}=\frac{1}{2}$$

$$\therefore \quad \frac{a_{1-}}{a_2}=\frac{b_1}{b_2}$$

Let 2x + 3y = z

Differentiating the above equation

$$2+3\ \frac{dy}{dx}\ =\ \frac{dz}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{1}{3}\left(\frac{dz}{dx}-2\right)$$

The given equation reduces to

$$\frac{1}{3}\left(\frac{dz}{dx}-2\right)=\frac{z+4}{2z+4}$$

$$\frac{dz}{dx}=-2\frac{3z+12}{2z+5}$$

$$\frac{dz}{dx}=\frac{3z+12}{2z+5}+2\frac{7z+22}{2z+5}$$

or $\frac{2z+5}{7z+22}$ dz = dx (Variables seperable)

On Integrating both sides w.r.t.

$$\text{or } \int\frac{2z+5}{7z+22}dz=x+c$$

$$\frac{\frac{2}{7}(7z+22)+5-\frac{44}{7}}{7z+22}dz=x+c$$

$$\text{or } \frac{2}{7}\int\frac{9}{7}\int\frac{1}{7z+22}dz=x+c$$

$$\text{or } \frac{2}{7}z-\frac{9}{49}\log(72+22)=x+c$$

14z - 9 log (7z + 22) = 49x + 49c

14 (2x + 3y) - 9 log (14x + 21y + 22) = 49x + 49c

21 x - 42y + 9 log (14x + 21y + 22) = -49c

or 7x - 14y + 3 log (14x + 21y + 22) + $\frac{49}{3}$ c = 0

Differentiating Of The Equation

The differential equation of the function *f*(x, y) is

$$df=\frac{\partial t}{\partial x}dx\frac{\partial f}{\partial y}dy=0$$

or df = M(x, y) dx + N(x, y) dy = 0

where M and N have continuous first partial derivatives

$$M = \frac{\partial f}{\partial x}, N = \frac{\partial f}{\partial y}$$

If the differential equation is exact if

$$\frac{\partial M(x,y)}{\partial x}, = \frac{\partial N(x,y)}{\partial y}$$

The solution of equation (1) is given by

$$f(x,y) = \int M(x,y)\,dx + \int \left[N(x,y) - \int \frac{\partial M(x,y)}{\partial y} dx \right] d$$

Example 8: Solve the equation

$(3x^2 + 4xy)$ dx + $(2x^2+2y)$ dy = 0

Solution: First Method

The equation is

$(3x^2 + 4xy)$ dx + $(2x^2+2y)$ dy = 0 …..(1)

First, we want to find whether the above equation is exact or nor. Here

M(x, y) = $3x^2+4xy$, N (x, y) = $2x^2+2y$

$$\frac{\partial M(x,y)}{\partial y} = 4x, \frac{\partial N(x,y)}{\partial y} 4x$$

So the equation (1) is exact equation. Thus we must find *f*(x, y) such that

$$\frac{\partial f(x,y)}{\partial x} = M(x, y) = 3x^2 = 4xy \quad ….. (2)$$

$$\frac{\partial f(x,y)}{\partial x} = N(x, y) = 2x^2 = 2y \quad ……..(3)$$

Integrating equation (2) w.r.t. x

f(x, y) = $\int M(x,y)\,dx + \phi(y)$

[where (y)ϕ is constant integration]

$$\int (3x^2, 2xy)\,dx + \phi(y)$$

$= x^3 + 2x^2 + \phi$ (y)

then $\frac{\partial f(x,y)}{\partial f} = 2x^2 + \frac{d\phi(y)}{dy}$

Substituting the value of $\frac{\partial f(x,y)}{\partial f}$ from equation (3) or $\frac{dp(y)}{dy}$ = 2y

or ϕ (y) = y^2+ C_0 where C_0 is an arbitary conduct

$\therefore f(x, y) = x^2 + 2x^2y + y^2+C_0$

Hence a one-parameter family of solution is $f(x, y)= c_1$

or $x^3+2x^2y+ y^2C_0= C_1$

or $x^3+2x^2y + y^2= C$

(where $C=C_1 -C_0$ is arbitary constant differential equation is exact is given by

$$f(x,y)=\int M(x,y)\,dx+\int\left[N(x,y)-\int\frac{\partial M(x,y)}{\partial y}dx\right]d$$

Here

$$f(x,y)=\int(3x^2,4xy)\,dx+\int\left[(2x^2,y)-\int\frac{\partial}{\partial y}\left(3x^2+4xy\right)dx\right]d$$

$$x^3+2x^2y+\int(2x^2,2y)\int\left[(2x^2,2y)-\int 4xdx\right]d$$

$$x^2+2x^2y+\int\left[(2x^2+2y-2x^2)dy\right]$$

$= x^2+2x^2y + y^2+C$

$= x^2+2x^2y+ y^2+C$ is the sol.

Linear Differential equation of first order

A first order ordinary differential equation is linear in the dependent variable y and the independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

For example

or $\frac{dy}{dx}$ +(x+1) y=x^3

or $\frac{dy}{dx}+\left(1+\frac{1}{x}\right)y=x^2$

is a first orders linear differential equation.

A one-parameter family of solution of this equation is

y =e $\int P(x)\,dx\left[\int e^{\int p(x)dx}\,Q(x)\,dx+c)\right]$

Example 9: Solve

$$\frac{dy}{dx}+\left(\frac{2x+1}{x}\right) y = e^{2x}$$

Solution: $\frac{dy}{dx}+\left(\frac{2x+1}{x}\right) y = e^{2x}$

Here $P(x)=\frac{2x+1}{x}$ and $Q=e^{-2x}$

$$y=e^{-\int p(x)\,dx}\left[\int e^{\int p(x)dx} Q(x)+c)\right]$$

$$=e^{-\int\frac{2x-1}{x}dx}\left[\int e^{\frac{2x-1}{x}dx}(e^{2x})dx+C\right]$$

$$=e^{-\int(2+\frac{-1}{x}dx}\left[\int e^{\left(2+\frac{-1}{x}\right)dx}(e^{2x})dx+C\right]$$

$$=e^{-(2x+\log x)}\left[\int e^{2x+\log x}e^{2x}\,dx+C\right]$$

$$=e^{-(2x+\log x)}\left[\int e^{\log x}\,dx+C\right] \quad \left(\therefore e^{\log x}=X\right)$$

$$=e^{-2x}e^{-\log x}\left[\int\frac{a^2}{2}-+C\right]=e^{-2x}e^{-\log x}\left(\frac{x^2}{2}+C\right)$$

$$=e^{-2x}+\left(+\frac{a^2}{2}-+\frac{c}{x}\right)=\frac{1}{2}xc^{-2x}+\frac{c}{x}e^{-2x}$$

Example 10

Solve:- $(x^2 + 1)\ \frac{dy}{dx} + 2xy = 4x^2$

Solution:- $(x^2 + 1)\ \frac{dy}{dx} + 2xy = 4x^2$

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{4x^2}{x^2+1}$$

Here $P(x) = \frac{2x}{x^2+1}\ and\ Q=\frac{4x^2}{x^2+1}$

$$y = e^{\int p(x)dx}\left[\int e^{\int p(x)dx}Q(X)dx+C\right]$$

$$= e^{-\int \frac{2x-1}{x}dx}\left[\int e^{\frac{2x+1}{x2+1}dx}\left(\frac{4x^2}{x^2+1}\right)dx + C\right]$$

$$= e^{\int \log(x^2+1)^{-1}}\left[\int e^{\log(x^2+1)}\left(\frac{4x^2}{x^2+1}\right)dx + C\right]$$

$$= (x^2+1)^{-1}\left[\int (x^2+1)\left(\frac{4x^2}{x^2+1}\right)dx + C\right]$$

$$= (x^2+1)^{-1}\left[\int 4x^2\,dx + C\right]$$

$$= \frac{1}{x^2+3}\left[\int \frac{4}{3}x^3 + C\right]$$

= y (x^2 +) = $\frac{4}{3}x^3 + C$ is the required solutions.

(b) Equations reducible to linear

We now consider a rather special type of equation that can be reduced to a linear equation by an appropriate transformation. An equation of the form $\frac{dy}{dx}$ + P(x) y = Q(x) y^n is called a Bernulli differential equation. We observe that ifn=0 or 1, then the Bernoulli equation is actually a linear equation and is therefore readily solvable as such. However, in the general case in which n≠0 or, then the transformation $v=y^{1-n}$ reduces the Bernoulli equation to a linear equation in v.

Example 11

Solve the equation $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^6$

Solution:- $\frac{dy}{dx} + \frac{1}{x}y = x^2 y^6$

Dividing by y^6

$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2$

Put $y^{-5} = v$

$-5\, y^{-6}\frac{dy}{dx} = \frac{dv}{dx}$

or $y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dv}{dx}$

$\therefore -\frac{1}{5}\frac{dv}{dx} + \frac{v}{x} = x^2$

or $\frac{dv}{dx} - \frac{5}{x} = -5x^2$

Here P = $-\frac{5}{x}$ and Q = $5x^2$

The solution of equation (2) is given by

$$v = e^{\int p(x)dx}\left[\int e^{\int p(x)dx} Q(X)dx + C\right]$$

$$= e^{-\int \frac{-5}{x}dx}\left[\int e^{\int \frac{-5}{x}dx}(-5x^2)\,dx + C\right]$$

$$= e^{\log x^5}\left[\int e - 5^{\log x}(-5x^2)\,dx + C\right]$$

$$= x^5\left[\int x^{-5}(-5x^2)\,dx + C\right]$$

$$= x^5\left[-5\int x^3\,dx + C\right]$$

$$= x^5\left[-5\left(\frac{x^{-2}}{-2}\right) + C\right]$$

$= y^5 = \frac{5x^3}{2}$ + Cx^5 is the required solutions.

Example 12

Solve $\frac{dy}{dx}$ + $y = xy^3$

Solution:- This is Bernoulli differential equation, where n=3. We first multiply the equation through by y^3, thereby expressing it in the equivalent

form $y^3 \frac{dy}{dx} + y^2 = x$

If we let $v = y^{1-n}$, then

$$\frac{dv}{dx} = -2y^{-3}\frac{dy}{dx}$$

The preceding differential equation is transformedinto the linear equation

$$-\frac{1}{2}\frac{dv}{dx} + v = x$$

or $\frac{dv}{dx}$ $-2v = -2x$ is a linear equation in v where

P=-2, Q=2x.

The between of this equation is

$$v = e^{\int p(x)dx}\left[\int e^{\int p(x)dx} Q(X)dx + C\right]$$

$$= e^{\int 2dx}\left[\int e^{\int 2dx}(-2x) + C\right]$$

$$= e^{2x}\left[\int e^{2x}(-2x) + C\right]$$

$$= e^{2x}\left[-2\int xe^{-2x}\, dx + C\right]$$

$$= e^{2x}\left[-2\int x^{e^{2x}} - -\int e^{-2x}\, dx) + C + C\right]$$

$$= e^{2x}\left[xe^{-2x} + \frac{1}{2}e^{-2x} + C\right]$$

$\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}$ is the required solutions.

Self-check Exercise 9.3

Q1. $(x^2 + y^2)\, dx - 4xy\, dy = 0$

Q2. Solve the equation $(2x^2 + 4xy)\, dx + (2x^2 + 4y)\, dy = 0$

Q3. Solve $\frac{dy}{dx} + \frac{1}{x} = x^2y^6$

9.6 SUMMARY

In this unit, we studied about the differential equations. We also, learnt about the order and the degree of differential equation. In the succession section we studied about the liner and, non-liner differential equation. In the last section we learnt about the exact differential equation.

9.7 GLOSSARY

1. **Differential Equation :** An equation involving derivations of one or more dependent variables with respect to one or more independent variables is called a differential equation.
2. **Partial Differential Equation :** A differential equation involving partial derivatives of one, or more, dependents variable with respect to partial differential equation.
3. **Order or Differential Equation :** The order of the highest order derivative involved in a differential equation is called the order of the differential equation.
4. **Degree of Differential Equation :** The degree of a differential equation is the degree of the highest derivative when the equation has been made free from, the radicals and negative indices us for the derivatives are concerned.

5. **Singular Solution :** A, singular solution of a differential equation is that solution which satisfies the equation but cannot be derived from its general solution.

9.8 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 9.1

Ans. Q1. Refer to Section 9.3

Ans. Q2. Refer to Section 9.3.2

Ans. Q3. Refer to Section 9.3.6

Self-check Exercise 9.2

Ans. Q1. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

$= e^{-y} (e^x + x^2)$

or $\frac{dy}{e^{-y}} = (e^x + x^2)\, dx$

or $e^y\, dy = (e^x + x^2)\, dx$

On integrating both sides, we get $e^y = \frac{x^3}{3} + C$

(where c is a constant integration) is the required solution.

Ans. Q2. Put $y - x = z$

$1 - \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ equation (1) can be written as

$1 - \frac{dy}{dx} = e^z + 1$ or $\frac{dy}{dx} = e^z$ or $\frac{dz}{e^{-z}} = dx$

on integrating, we get x – y

$\frac{e^{z}}{-1} = x - c$ or $e = -x + c$ or $x = e^y + c$ Ans

Self-check Exercise 9.3

Ans. Q1. $(x^2 + y^2)\, dx - 4xy\, dy = 0$

or $\frac{dy}{dx} = \frac{x^2 + y^2}{4xy}$

Put $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx}$

Equation (1) can be written as

$$v+x\frac{dv}{dx}=\frac{x^2+u^2x^2}{4x^2v}=\frac{1+v^2}{4v}$$

$$x\frac{dv}{dx}=\frac{1+v^2}{4v}-v$$

$$=\frac{1+v^2-2v^2}{4v}=\frac{1-v^2}{4v}$$

$$= \frac{4v}{1+v^2}dv=\frac{dx}{dx}$$

$$= -\left(\frac{-4v}{1+v^2}\right)dv=\frac{dx}{dx}$$

On integrating, we get

-log $(1-v^2)$ log |x| -logC

(Where c is a constant of integration)

log $|1-v^2|$ = log|x|+ log c

log $(1-v^2)\,x|$ = log C

or $(1-y^2)\,x=C$

or $\left(1-\frac{y^2}{x^2}\right)x = C$

or $x^2 - y^2 = Cx$

Ans. Q2. Refer to Example 8

Ans. Q3. Refer to Example 11

9.9 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
5. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

9.10 TERMINAL QUESTIONS

1. Solve (1-y) x $\frac{dy}{dx}$ +(1+x)y=0
2. Solve $\frac{dy}{dx}=\sqrt{y-x}$
3. Solve $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$
4. Solve $\frac{dy}{dx}=\frac{2x+9y-20}{6x+2y-10}$
5. Solve (2x + 4y + 3) $\frac{dy}{dx}$ = (x + 2y + 1)
6. Solve (9x + hy + g) dx = (hx + by + f) dy = 0
7. Solve (x^2 - 4xy - 2y^2) dx + (y^2 - 4xy - 2x^2)

Unit-10

LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER WITH CONSTANT COEFFICIENT

STRUCTURE

10.1 INTRODUCTION

In the last unit, we have studied about the first order differential equation. In this unit, we will study about the higher-order differential equations.

10.2 LEARNING OBJECTIVES

After going through this Unit, you will be able to :

- solve higher order differential equation
- solve homogeneous linear equation with constant coefficient
- solve non-homogeneous equation with constant coefficient

10.3 HIGHER-ORDER LINEAR DIFFERENTIAL EQUATION

Higher-order linear differential equation are equations having a great variety of important applications. In particular, second-order linear differential equations with constant coefficients have numerous applications.

Consider the second order (non homogeneous) linear differential equation

$$a_0 \frac{d_2 y}{dx_2} + a_1 \frac{dy}{dx} \text{ a}_2 \text{ y} = 1 \text{ (x)} \qquad (1)$$

and the corresponding homogeneous equation

$$a_0 \frac{d_2 y}{dx_2} + a_2 \frac{dy}{dx} \text{ a}_2 \text{ y} = 0 \qquad (2)$$

where a_0, a_1 and a_2 are contents.

The solution is obtained in two steps.

First Step

The general solution of (2) is called the complementary function of equation (1). We shall denote this by y_0.

Second Step

Any particular solution of (1) involving no arbitrary contents is called a particular integral of y_c. We shall denote this by y_p.

The solution y_c+y_p, where I is the complementary function and y_p is a particular integral of (1), is called the general solution (1)

Thus to find the general solution of (1), we merelyfind:

(i) The complementary function, i.e., a "general linear combination of a linearly independent solutions of the corresponding homogeneous equation (2). The method we will be using dependson the following result which we give without proof. By linearly independent solutions we mean there are two arbitrary constant d, and d, such that $d_1 f_1 + d_2 f_2 = 0$, which implies that $d_1 = d_2 = 0$. Since equation (2) is a second order equation, we expect the solution to have two arbitrary constants.

(ii) A particular integral, i.e., any particular solution of (1) involving no arbitrary constants.

The linearly independence of solutions of second order (or nth order) can also be found from the theorem which's states.

The two solutions f_1 and f_2 of the second order homogeneous linear differential equation are linearly independent on a $\leq$x$\leq$b if the Wronskian off_1 and f_2 is different from zero for some x on the interval a $\leq$ x $\leq$ b.

i.e.

$$W[f_1 \text{ (x)}, f_1 \text{ 2(x0] or } W[f_1 f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1 & f_2 \end{vmatrix} = f_1 f_2 - f_1 f_2 \neq 0$$

In case the non homogeneous member F(x) of the linear differential equation (1) is expressed as a linear combination of two or more functions, then the following theorem may often be used to advantage in finding a particular integral

(i) Let f_1 be a particular integral; of

$$a_0 \frac{d_2 y}{dx_2} + a_1 \frac{dy}{dx} a_2 y = F_1 (x)$$

(ii) Let f be a particular integral of

$$a_0 \frac{d_2 y}{dx_2} + a_1 \frac{dy}{dx} a_2 y = F_2 (x)$$

Then $k_1 f + k_2 f_2$ is a particular integral of

$$a_0 \frac{d_2 y}{dx_2} + a_1 \frac{dy}{dx} a_2 y = F_1 (x) = k_1 F_2 (x) + k_2 F_2 (x)$$

where k_1 and k_2 are constants.

In the remaining section of this unit, we shall proceed to study methods of obtaining the two constituent parts of the general solution.

10.3.1 HOMOGENEOUS LINEAR EQUATION WITH CONSTANT COEFFICIENT

Let us consider the second order homogeneous linear differential equation in which all the coefficients are real constants. That is, we shall be concerned with the equation (2) which is

$$a_0 \frac{d_2 y}{dx_2} + a_1 \frac{dy}{dx} a_2 y = 0 \quad (2)$$

where a_0, a_1 and a_2 are real constants. We shall show that the general solution can be found explicitly.

Thus we seek solutions of above equation of the form $y = e^{mx}$ (because we need a function such that its derivative are constant multiplies of itself), where the constant m will be chosen such that el does satisfy the equation (2) assuming then that

$$y = e^{mx}$$

is a solution for certain m, we have

$$a_0 \frac{dy}{dx} = m e^{mx}$$

$$\frac{d^2 y}{dx^2} = m^2 e^{mx}$$

Substituting in (2) we obtain

$$a_0 m^2 e^{mx} + a_1 m e^{mx} + a^2 e^{mx} = 0$$

or $e^{mx} (a_0 m^2 + a_1 m + a_2) = 0$

Since $e^{mx} \neq 0$, we obtain the polynomial equation is the unknown m:

$a_0 m^2 + a_1 m + a_2 = 0$

This equation is called the auxiliary equation or the characteristic equation of the given differential equation (2). If $y=e^{mx}$ is a solution of (2) then we see that the constant m must satisfy (3). Hence to solve (2), we write the auxiliary equation (3) and solve itform. Three cases arises, according as the roots of (3) are real and distinct, real and repeated, or complex.

First case : Distinct Real Roots

Suppose the roots of (3) are two distinct real numbers m and m Then em^1x, and em^2x are two distinct solutions of (2). Further, using the Wronskian determinant one may show that these two solutions are linearly independent. Thus we have the following result.

It the auxiliary equation (3) has two distinct real root m, and m, then the general solutions of second order homogeneous linear differential equation (2) with constant coefficients is

$Y=C_1\ em^1x+ c_2\ em_2\ x.$

where c_1 and c_2 are arbitrary constant.

Example 1. Find the general solution of

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

Solution:

$$\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=0$$

The auxiliary equation is

$M^2 - 5m+6=0$

Hence (m – 2) (m – 3)=0

or $m_1 = 2$, $m_2 = 3$

The roots are real and distinct. Thus e^{2x} and e^{3x} are solution and the general solution may be written

$Y = c_1e^{2x} + c_2\ e^{3x}$

To verify that the solution e^{2x} and e^{3x} are linearly independent we have to show that their Wronskain is not zero it.

$$w\left[e^{2x},e^{3x}\right]=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}+3xe^{3x} \end{vmatrix}=3e^{5x}-2e^{5x}=e^{5x}\neq 0$$

Thus we are assured of their linear independence.

Example 2. Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$$

Solution: We have

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$$

The auxillary equation is

$M^2 - 3m+2=0$

Hence (m – 1) (m–2) = 0. $m_1= 1$, $m_2= 2$.

The roots are real and distinct. Thus e^x and e^{2x} are solution and the general solution is

$Y = c_1 e^x+c_2 e^{2x}$

To verify that e^x and e^{2x} are linearly independent solution, we shall that their Wronskian is not zero

i.e.

$$w\left[e^{2x},e^{3x}\right]=\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}=e^{3x}\neq 0$$

Hence we conclude that the solutions e^x and e^{2x} are linearly independent solution.

Example 3: Find the general solution of the differential equation.

$$4\frac{d^2y}{dx^2}-12\frac{dy}{dx}+5y=0$$

Solution:

$$4\frac{d^2y}{dx^2}-12\frac{dy}{dx}+5y=0$$

The auxiliary equation is

$4m^2 - 12m+5=0$

or $4m^2 - 10m - 2m+5=0$

or 2m (2m – 5) – (2m – 5)=0

(2m – 1) (2m – 5) = 0, $m_1= ½m_2 = 5/2$

The roots are real and distinct. Thus e1/2x ande5/2x are solution and the general solution is

$Y = C_1 e^{½x} + C_2e^{5/2x}$

The Wronskian of this solution is

$$w\left[e^{x2}, e^{5/2x}\right] = \begin{vmatrix} e^{x}{}_{2} & e^{5}{}_{2}{}^{x} \\ \frac{1}{2} e^{x/2} & \frac{5}{2} e^{5}{}_{2}{}^{x} \end{vmatrix} = \frac{5}{2} e^{3x}$$

$$-\frac{1}{2} e^{3x} = 2e^{3x} \neq 0$$

Hence we conclude that the solutions are linearlyindependent solutions.

Second Case

We consider a simple example first, let

$$\frac{d^2 y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

The auxiliary equation is

m_2–4m+4=0

or $(m-2)^2=0$

The roots of this equations are

$m_1 = 2$, m2 $=2$

(real but not distinct)

Corresponding to the root m_1, we have the solutions e_{2x} and corresponding to m_2 we have the same solution e^{2x}. The linear combination $C_1\ e^{2x}+\ C_2e^{2x}$ of these 'two' solutions is clearly not the general solution the differential equation (4), for it is not a linear combination of two linearly independent solutions. Indeed we may write the combination $C_1e^{2x}C_2e^{2x}$ as simply $c_0\ e^{2x}$, where $C_0 = C_1 + C_2$, and clearly $y = C_0\ e^{2x}$, involving one arbitrary constant, is not the general solution of the given second order equation.

We must find a linearly independent solution, we already know the one solution e^{2x}, we will reduce the order of the equation and let

$y = e^{2x}v$

where v is to be determined. Then we can show

$y = x\ e^{2x}$

is also the solution of equation (4). Thus we find the linearly independent solution e^{2x} and xe^{2x} of equation (4). Thus the general solution of equation (4) may be written

$y=C_1\ e^{2x} + C_2\ xe^{2x}$

$y= (C_1 + C_2x)\ e^{2x}$

Example 4. Find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Solution: The equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

is 2nd order homogeneous equation with constantcoefficient.

The auxiliary equation is

$m^2 - 6m + 9 = 0$

or $(m-3)^2 = 0$

The roots of this equation are real but not distincthere

$m_1 = 3, m_2 = 3$

The general solution of equation is

$y = (C_1 + C_2x)\, e^{3x}$

The solution e^{3x} and xe^{3x} are clearly linearly independent soutions because

$$w\left(e^{3x}, ex^{3x}\right) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{x6} \neq 0$$

Example 5. Solve that equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

Solution: The equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

is 2nd order homogeneous equation with constant coefficient.

The auxiliary equation is

$m^2 - 8m + 16 = 0$

or $(m - 4)2 = 0$

$m_1 = 4, m_2 = 4$

The roots are real, equal. The general solution of the above equation is

$y = (C_1 + C_2x)\, e^{4x}$

Third Case

Let the auxiliary equation has the complex number a+bi (a, b real, 1 =________ b ≠ 0) as a non repeated root. Then, since the coefficient are real the conjugate complex number a – bx is also a non repeated root. The corresponding part of the general solution is

$k_1\, e^{(a+bi)x} + k_2\, e^{(a-b)x}$

where k_1 and k_2 are arbitary constants. The solutions defined by $e^{(a+bi)x}$ and $e^{(a-bi)x}$ are complex functions of the real variable x. It is desirable to replace these by two real linearly independent solutions. This can be accomplished by using Euler's formula

e^{0}=Cos+Sin

which holds for all real. Using this we have

$K_1e^{(a+bi)x}+k_2e^{(a-b)x} = k_1\ e^{ax}\ e^{bix}+k_2\ e^{ax}\ e^{-bix}$

$= e^{ax}\ (k_1\ e^{ibx}+k_2\ e^{-bix}$

$=e^{ax}\ [k_1\ (\text{Cos bx} +i\text{Sin bx})$

$+k_2\ (\text{Cos bx} = i\ \text{Sin bx})]$

$= e^{ax}\ [(k_1 + k_2)\ \text{Cos bx} +i(k_1- k_2)$

Sin bx

$=e^{ax}[C_1\ \text{Cos bx}+C_2\ \text{Sin bx}]$

where $C_1= (k_1+k_2)$, C_2 (I $(k_1 - k_2)$ are two new arbitrary constants. Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots a$\pm$bi is

$e^{ax}[C_1\ \text{Sin bx}+C_2\ \text{Cos bx}]$

Note: Since we are confining our discussion tothe 2^{nd} order homogeneous linear differential equation we shall not uncounted repeated roots.

Example 6. Solve the differential equation

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+10y=0$$

Solution: We have

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+10y=0$$

is a 2^{nd} order homogeneous differential equation.

The auxiliary equation is

$m^2+2m+10=0$

Solving it, we find

$$m=\frac{-2\pm\sqrt{4}-40}{2}=\frac{-2\pm\sqrt{-3}6}{2}=\frac{-2\pm 6i}{2}$$

$= -1 \pm 3i$

Here a = -1, b = 3 the roots are conjugate complex numbers a $\pm$ bi. The general solution is

$Y = e^{x}\ (C_1\ \text{Sin } 3x+C_2\ \text{Cos } 3x)$

Example 7. Solve

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$$

Solution: The equation is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$$

The auxiliary equation is

$$m^2 - 5m + 7 = 0$$

$$m = \frac{5 \pm \sqrt{25-28}}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5}{2} \pm \frac{\sqrt{-3}}{2} i$$

Here $m = \frac{5}{2} + \frac{\sqrt{3}}{2} i, m_2 = \frac{5}{2} = \frac{\sqrt{3}}{2} i$

The general solution may be written

$$y = \left[C^1 \, Cos \frac{\sqrt{3}}{2} + C^2 \, Sin \frac{\sqrt{3}x}{2} \right]$$

Initial and Boundary Value Problem

In the application of both first and higher order differential equations one or more supplementary conditions which the solution of the given differential equation must satisfy. If all the associated supplementary conditions relate to one x value, the problem is called an initialvalue proble, (or one point boundary-value problem). If the conditions relate to two different x values, the problem is called a two point boundary value problem (or simply a boundary value problem)

An Initial-Value Problem

We now apply the results concerning the general solution of a homogeneous linear equation with constant coefficients to an initial value problem involving such an equation

Example 8. Solve the initial value problem,

$$\frac{d^2y}{dx} - \frac{dy}{dx} - 12 = 0, \ y(0) = 3, \ y(0) = 5$$

Solution: The equation

$$\frac{d^2y}{dx} - \frac{dy}{dx} - 12 \ y = 0$$

is the homogeneous linear equation with constantcoefficients.

The auxiliary equation is

$m^2 - m - 12 = 0$

$(m - 4)(m+3) = 0$, $m_1 = 4, m_2 = -3$.

The general solution is

$y = C_1 e^{4x} + C2\, e^{-3x}$. (6)

We shall now find the particular solution of the differential equation that satisfies the two initial conditions y (0) = 3 y'(0) = 5. It is given that at x= 0, y = 3. Substituting these values in equation (6), we get

$3 = C_1 e^0 + C_1 e^0$ or $C_1 + C_2 = 3$ (7)

Now differentiating equation (6) w.r.t. x.

$$\frac{dy}{dx} = 4\, C_1 e^{4x} - 3\, C_2 e^{-3x} \qquad (8)$$

It is given that at x=0, y= 5. On substitutingthese values in equation 8, we get.

$5 = 4C_1 e^0 - 3C_2 e^{-3x}$ or $4C_1 - 3C_2 = 5$ (9)

We have to find the values of C_1 and C_2 from equation (7) and (9). Multiplying equation (7) by 4 and on subtracting equation (9) from it, we get

$7C_2 = 7$, $C_2 = 1$

$C_1 = 3 - C_2$ $C_1 = z$

The general solution (6) can be written as

$y = 2e^{4x} + e^{-3x}$

is the unique solution of the given initial value problem

Self-check Exercise 10.1

Q1. Find the general solution of

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

Q2. Find the general solution of differential equation.

$$6\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 5y = 0$$

10.3.2 NON-HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENT

Consider the non homogeneous differential equation

$$a_0 \frac{d^2y}{dx} + a_1 \frac{dy}{dx} + a_2 y = F(x) \qquad (10)$$

where a_0, a_1, a_2, are constants but where non homogeneous term Fis (in general) a non constant function of x. The general solution of (10) may be written as

$Y = Y_e + Y_p$

where Y_e is the complementary function, that is, the general solution of the corresponding homogeneous.

$$a_0 \frac{d^2 y}{dx} + a_1 \frac{dy}{dx} + a_2 y = 0(x) \qquad (11)$$

and Y_p is a particular integral, that is, any solutions of (10) containing no arbitrary constants. We know how to find the complementary function. Now we consider methods of determining a particular integral. The method of finding particular integral is given in the tabular form where y_p set will be a function of itself and all linearly independent function of which successive derivate of F(x) and either constant multiples or linear combination. Then, if will be a set of

F(x) Yp

1. X^n $\{x^n\ x^{n-1}, x^{n-2}, ...x, 1\}$

2. e^{an} $\{e^{an}\}$

3. Sin (bx + c) or [Sin (bx + c), Cos (bx + c)] Cos (bx + c)] Cos (bx + c)

4. Xn ean $\{X^n e^{an}, X^{n-1} e^{an}, X^{n-2} e^{an}$ .... xe^{en}, e^{ax}

5.x^n Sin (bx + c) or (x^n Sin (bx + c), x^n Cos (bx+ c)

x^n Cos (bx+c) x^{n-1} Sin (bx – c), x^{n-1}Cos (bx + c)

x Sin (bx +c), x Cos (bx + c),

Sin (bx+c), x Cos (bx + c),

Sin (bx + c), Cos (b+c)

6.e^{an} Sin (bx+c) or $\{e^{ax}$ sm (bx + c) e^{ax} as (bx + c)\}

e^{ax} Cos (bx + c)

Note: In case y_p set include one or more members which are solutions of the corresponding homogeneous differential equation. Then we multiply the members of y_p set by the lowest positive integral power of x so that the resulting revised set of y_p contain no members that are solutions of the corresponding homogeneous differential equations.

Now form a linear combination of all the sets of these two categories, with unknown constant coefficients. (Undetermined coefficients.) Determine these unknown coefficients by substituting the linear combination into the differential equation and demanding that it identically satisfy the differential equation (that is, that it be a particular solution).

Example 9. Solve

$$\frac{d^2 y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$$

Solution: The differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x} \qquad (12)$$

is a 2^{nd} order non homogeneous equation with constant coefficient. The solution will consists of y_p and y_c. To findy, consider its homogeneous equation is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y=0 \qquad (13)$$

The auxiliary equation is

$$m^2 - 5m+6=0$$

or $(m-2)(m-3)=0$, $m_1 = 2$, $m_2 = 3$

The auxiliary equation is

The auxiliary equation (13) or complementaryfunction is

$$Y_c = C_1 e^{2X} + C_2 e^{3X} \qquad (14)$$

To find the particular solution y_p let us put

$$Y_p = Ac^{4x} \qquad (15)$$

because here the exponent of e on R.H.S. is 4 which is not a root of auxiliary the equation. From equation (15), we obtain

$$Y'_p = 4\ Ae^{4x}$$

$$Y''_\pi = 16\ A\ e^{4x}$$

These values of y' p and y'' p must satisfy the equation (12). Since we have assumed y_p= Ae^{4x} is a particular solution of equation (12)

$$16\ Ae^{4x} - 20\ Ae^{4x} + 6\ Ae^{4x} = e^{4x}$$

or $2A = 1$

or $1=½$

$$y = ½\ e^{4x} \qquad (16)$$

The solution of equation

$$Y=y_e +Y_p$$

$$= C_1\ e^{2x}+C_2e^{4x}\text{(from equation (14) and (16)}$$

Example 10. Solve

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y=(x-2)\ e^x$$

Solution: The differential equation is

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x - 2)\ e^x \qquad (17)$$

is a non homogeneous second order linear differential equation with constant coefficient. The generalsolution will be of the form.

$Y = Y_e + Y_p$ (18)

where Y_e is the complementary function and Y_p is the particular integral of the equation (17). To find Y_e, we consider, the homogeneous equation of equation (17) which is

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + ey = 0 \qquad (19)$$

The auxiliary equation is

$m^2 - 7m + 6 = 0$

or $(m-6)(m-1) = 0$ or $m_1 = 6$, $m_2 = 1$

$\therefore$ The complementary function is

$Y_e = C_1 e^x + C_2 e^{6x}$ (20)

To find the particular integral, we observe thatthe R.H.S. of equation has a term e which is one ofthe root of the auxiliary equation, i.e., one root isrepeated. So

$Y_p = x(Ax+B)e^x$

$= x^2 e^x A + xe^x B$

$Y'_p = 2xe^x Ax^2 e^x A + e^x B + xe^x B$

$= (Ax^2 + xB)e^x + (2xA + B)e^x$

$Y''_p = [2xA + B + Ax^2 + xB + 2A + 2xA + B]e^x$

The equation (17) becomes

$[2xA+B+Ax^2+xB+2A+2xA+B]e^x - 7(Ax^2+xB)$

$e^x - 7(2xA+B)e^x + 6x^2 e^x A + 6xe^x B = (x-2)e^x$

Cancelling e^x from both sides and on simplifying,we get

$-10xA - 5B + 2A = x - 2$

On comparing the coefficient of x and constant term, we get

$-10A = 1$ and $2A - 5B = -2$

$A = -1/10$ and $2A - 5B = -2$

$2A - 5B = -2$

$2\left(\frac{-1}{10}\right) -5B\ -2$

$-5B = -2 + \frac{-1}{5}$

$$-5B = \frac{-9}{5}$$

or $$B = \frac{+9}{5}$$

$$\therefore\ Y_p = x - \left(\frac{1}{10} \times + \frac{9}{25} \right)$$

Complete genera solution is

$$Y = Y_c + Y_p = C_1e^x + c_2e^{6x} + x\left(\frac{1}{10} \times + \frac{9}{25} \right)$$

Self-check Exercise 10.2

Q1. Solve the equation

$$\frac{d^2y}{dx^2} - \frac{8dy}{dx} + 16y = 0$$

Q2. Solve

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$$

10.4 VARIATION OF PARAMETER

While the process of carrying out the method ofundetermined coefficient is actually quite straight forward, the method applies in general to a rather small class of problems. For example, it would not apply to the apparently simple equation.

$$\frac{d^2y}{dx^2} + y = \tan x$$

We thus seek a method of finding a particular integral that applies in all cases (which incident also applies to variable coefficients) in which the complementary function is known.

Consider second order linear differential equation with constant coefficients

$$a_0\frac{d^2y}{dx^2} + a_0\frac{dy}{dx} + a_2y = F(x) \qquad (21)$$

where a_0, a_1 and a_2 are constants.

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation.

$$a_0\frac{d^2y}{dx^2} + a_0\frac{dy}{dx} + a_2y = F(x) \qquad (22)$$

Then the complementary function of equation (21) is

$C_1 y_1 (x) + C_2 y_2 (x)$.

where y_1 and y_2 are linearly independent solutions of (2) and C_1 and C_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants C_1 and C_2 in complementary function by respective function v_1 and v_2. which will be determined so that the resulting function, which is defined by

$$v_1 (x) y_2 (x) + v_2 (x) y_2 (x). \qquad (23)$$

will be a particular integral of equation 91) (hence the name, variation of parameters).

We have at our disposed the two functions V_1 and v_2 with which to satisfy the one condition that (23) b a solution of (21). Since we have two functions but only one conditions on them, we are thus free to impose a second condition, provided this second conditions does not violate the first one.

We thus assume a solution of the form (23) and write

$$Y_p (x)=v_1(x)y_1(x)+v_2(x) y_2(x) \qquad (24)$$

On differentiating (24), we get

$$Y'_p(x)=v_1(x)y'_1 (x)+v_2(x) y'_2 (x)+v'_1 (x) y_1(x) + v'_2 (x) y_2 (x) \qquad (25)$$

At this point we impose the second condition,we simplify y_p by demanding that

$$v'_1(x) y_1(x)+v'_2 (x)y_2(x)=0 \qquad (26)$$

With this condition (25) reduces to

$$Y'_P (x)=v_1(x) y'_1 (x)+v_2(x)y'_2(x) \qquad (27)$$

On differencing (27), we get

$$Y''_P(x)= v_1(x)y'_1 (x)+v_2(x) y'_2(x)+v'_1 (x) y'_1 (x) - v'_2(x)y'_2(x)$$

We now impose the basic condition that (24) be a solution equation (21) and obtain the identity onsubstituting values y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (21).

a_0 $[v_1(x)y_1 ''(x)+v_2(x) y_2'' (x)+v_1' (x) y_1 '(x)+v_2'(x) y_2'(x)]+a1 [v_1(x)y_1'(x)+v_2(x) y_2'(x)+a_2 [v_1(x) y_1(x)+v_2(x) y_2(x)] = F(x)$

This can be written as

$v_1(x) [a_0 y_1 ''(x)+a_1y_1'(x)+a_2y_1(x)] = v_2(x) \{ay_2''(x)+a1 y_2''(x) + a_2 y_2(x)] + a_0 [v_1'(x) y_1'(x) + v_2'(x) y_2'(x)] = F(x)$

Since y_1 and y_2 are solutions of the corresponding homogeneous differential equation (22), the expressions in the first two brackets in (29) are identically zero. The leaves merely

$$v_1'(x) y_1(x)+v_2'(x) y_2'(x) = \frac{F(x)}{a_0} \qquad (30)$$

This is actually what the basic condition demands. Thus the two imposed conditions require that the function v_1 and v_2 be closed such that the system of equation.

$y_1(x)\ v_1'(x)+y_2(x)\ v_2'(x)=0$

$y_1'(x)\ v_1'(x)+y_2'(x)\ v_2'(x)=\frac{F(x)}{a_0}$

is satisfied. The determined of coefficients of this system is precisely.

$$W[y_1(x), y_2(x)] \begin{vmatrix} y_1(x) & y_2(x) \\ y_1 & y_2(x) \end{vmatrix}$$

Since y_1 and y_2 are linearly independent solution of the corresponding homogeneous differential equation (22), we know that $W[y_1(x).\ y_2(x)] \neq 0$. Hence the system has a unique solution. On solving this system, we obtain

$$v_1(x) \left| \frac{\begin{matrix} 0 & y_2(x) \\ \frac{Fx}{a_0} & y'_2(x) \end{matrix}}{\begin{matrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{matrix}} \right| = \frac{F(x)\, y_2(x)}{a_0\ \ W[y'_1(x), y'_2(x)]}$$

$$v_2(x) \left| \frac{\begin{matrix} 0 & y_2(x) \\ \frac{Fx}{a_0} & y'_2(x) \end{matrix}}{\begin{matrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{matrix}} \right| = \frac{F(x)\, y_2(x)}{a_0\ \ W[y'_1(x), y'_2(x)]}$$

Thus we obtain the function v1 and v2 defined y

$$v_1(x) = -\int^x \frac{F(t) y_2(t)}{a_0 \quad W[y_1(t), y_2(t)]}\, dt$$

$$v_2(x) = -\int^x \frac{F(t) y_2(t)}{a_0 \quad W[y_1(t), y_2(t)]}\, dt$$

Therefore a particular integral y_p of equation (21)is defined by

$Y_p(x)=v_1(x)\ y_1(x)+v_2(x)\ y_2(x).$

where v_1 and v_2 are defined by (31)

Example 11. Solve the differential equation

$$\frac{d^2y}{dx^2} + y = \tan x$$

Solution: The differential equation is

$$\frac{d^2y}{dx^2} + y = \tan x \qquad (32)$$

The auxiliary equation is

$m^2+1=0$

$m^2=-1$, $m = i, -i$

The complementary function is given by

$y_c(x)=C_1 \text{Sin } x + C_2 \text{Cos } x$

We assume

$$Y_p(x)=v_1(x) \text{Sin}x+v_2(x) \text{Cos } x \qquad (33)$$

where $v_1(x)$ and $v_2(x)$ will be determined such that this is a particular integral of the differential equation (32). Thus

$Y_p'(x)=v_1(x)\text{Cos}x-v_2(x) \text{Sin}(x)+v_1'(x) \text{Sin}x+v'_2(x) \text{Cos } x$

We impose the condition

$$v_1'(x) \text{Sin}x+v_2'(x) \text{Cos } x=0 \qquad (34)$$

leaving

$y_p'(x)=v_1(x)\text{Cos}x-y_2(x) \text{Sin } x$

$$y_p''(x)=v_1(x) \text{Sin}x-v_2(x) \text{Cos}x+v_1'(x)\text{Cos}x - v_2'(x) \text{Sin}x \qquad (35)$$

Substituting the values of $y_p''(x)$ and $y_p(x)$ from(35) and (33) into (32), we get

or $\quad v_1'(x) \text{Cos}x - v_1'(x) \text{Sin } x = \tan x$

Thus we have two equation (34) and (36) from which to determine $v_1'(x)$ & $v_2'(x)$. On solving, weget

$$v'_1(x)=\frac{\begin{vmatrix} 0 & \cos x \\ \tan x & -\sin x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}}=\frac{-\cos x \tan x}{-1}=\sin x$$

$$v'_2(x)=\frac{\begin{vmatrix} \sin x & 0 \\ \cos x & -\tan x \end{vmatrix}}{\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}}=\frac{-\sin x \tan x}{-1}=\frac{\sin_2 x}{\cos x}$$

$$=\frac{\cos^2 x-1}{\cos x}=\cos x-\sec x$$

Integrating, we find

$v_1(x)$=cosx+C_3, $v_2(x)$=sinx – log |secx+tanx| + C_4

Substituting (37) in (33)

$y_p(x)$=(-cosx+C_3)sinx+(sinx+(sinx−log|sec x + tan x + C_4) Cos x

= - sin x cos x + C_3 sin x – log | sec x + tax x| cos x

=C_3 sin x + C_4 cos x – cos x (log) secx+tan x|).

Since a particular integral is a solution free of arbitary constants, we may assign any particular values A and B to C_3 and C_4, respectively, and result will be particular integral

A sin x+B cosx – (cos x) log 1| secx+ tan x |)

Thus y=y_c+y_p

=C_1 sinx+C_2 cosx+ASinx+B cosx+(cosx) (tan)

=C_1' sin x + C_2 cos x – (cos x) (log | secx+tan x|)

where C_1' = C_1+A, C_2 = C_2+B

This is the general solution of the differential, equation (32)

Self-check Exercise 10.3

Q1. Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y - (x-2)e^x$

10.5 SUMMARY

In the first section of this unit, we learnt about the higher order differential equations. In the next section of the unit we learnt about the Homogeneous linear equation with constant function. In the successiding section we discussed Non-homogeneous equation with constant coefficient. In the last section of unit, we studied about variation of parameter.

10.6 GLOSSARY

1. **Higher order liner differential equation :** If contains only one independent variable and one or more of its derivative with respect to the variable.

2.**Complementary function :** Consider the second order linear differential equation (non-homogeneous)

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} a_2 y = 1 (x) \text{ .... } (1)$$

and the corresponding homogeneous equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} a_2 y = 0 \text{ .... } (2)$$

where a_0, a_1 and a_2 are contents

The General solution of (2) is called the complementary function of equation (1). Dentoted by y_0.

3. **Particular Integral :** An particular solution of (1) involving no arbitrary contents is called particular integral of y_c. We shall denote by y_p.

4. **General solution :** The solution $y_c + y_p$, where 1 is the complementary function and y_p is a particular integral of (1), is called the general solution (1).

10.7 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 10.1

AnsQ1. The equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$$

The auxiliary equation is

$M^2 - 6m + 8 = 0$

Hence $(m - 4)(m - 2) = 0$

or $m_1 = 4, m_2 = 2$

The roots are real and distinct. Thus e^{4x} and e^{2x} are solution and the general solution may be written.

$y = C_1 e^{4x} + C_2 e^{2x}$

To verify that the solution e^{4x} and e^{2x} are linearly independent we have to show that their wrouskain is not zero it.

$$W\left[e^{4x}, e^{2x}\right] = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = 3e^{6x} - 4e^{6x} = e^{5x} \neq 0$$

Ans. Q2. Solution

$$6\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 5y = 0$$

The auxiliary equation is

$6m^2 - 13m + 5 = 0$

or $6m^2 - 10m - 3m + 5 = 0$

or $2m(3m - 5) - 1(3m - 5) = 0$

$(2m - 1)(3m - 5) = 0,\ m_1 = \frac{1}{2}\ m_2 = \frac{5}{3}$

The roots are real and distinct. Thus e1/2x ande5/3x are solution and the general solution is

$Y = C_1 e^{½x} + C_2 e^{5/3x}$

The Wronskian of this solution is

$$w\left[e^{x2}, e^{5/3x}\right] = \begin{vmatrix} e^{x}{}_{2} & e^{5}{}_{3}{}^{x} \\ 1\!/\!_2 e^{x/2} & 5\!/\!_3 e^{5}{}_{2}{}^{x} \end{vmatrix} = 5\!/\!_3 e^{4x}$$

$$-\frac{1}{2}e^{4x} = 2e^{4x} \neq 0$$

Self-check Exercise 10.2

Ans.Q1. The equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$$

The auxiliary equation is

$M^2 - 8m + 16 = 0$

or $(m - 4)2 = 0$

or $m_1 = 4, m_2 = 2$

The roots are real, equal. The general solution of the above equation is

$y = (C_1 + C_2 x) e^{4x}$

Ans.Q2. The equation is $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 7y = 0$

The auxiliary equation is

$m^2 - 5m + 7 = 0$

$$m = \frac{5 \pm \sqrt{25} - 28}{2} = \frac{5 \pm \sqrt{-3}}{2} = \frac{5}{2} \pm \frac{\sqrt{-3}}{2} i$$

Here $m = \frac{5}{2} + \frac{\sqrt{3}}{2} i, m_2 = \frac{5}{2} = \frac{\sqrt{3}}{2} i$

The general solution may be written

$$y = \left[C^1 \, Cos\frac{\sqrt{3}}{2} + C^2 \, Sin\frac{\sqrt{3}x}{2}\right]$$

Self-check Exercise 10.3

Ans. Q1. The differential equation is

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = (x - 2)e^x \qquad (1)$$

is a non-homogeneous second order linear differential equation with constant coefficient. The generalsolution will be of the form.

$$Y = Y_e + Y_p \qquad (2)$$

where Y_e is the complementary function and Y_p is the particular integral of the equation (17). To find Y_e, we consider, the homogeneous equation of equation (17) which is

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + ey = 0 \qquad (3)$$

The auxiliary equation is

$$m^2 - 7m + 6 = 0$$

or $(m-6)(m-1) = 0$ or $m_1 = 6$, $m_2 = 1$

$\therefore$ The complementary function is

$$Y_e = C_1 e^x + C_2 e^{6x} \qquad (4)$$

To find the particular integral, we observe thatthe R.H.S. of equation has a term e which is one ofthe root of the auxiliary equation, i.e., one root isrepeated. So

$$Y_p = x(Ax+B)e^x$$

$$= x^2 e^x A + xe^x B$$

$$Y'_p = 2xe^x Ax^2 e^x A + e^x B + xe^x B$$

$$= (Ax^2 + xB) e^x + (2xA + B)e^x$$

The equation (1) becomes

$$[2xA+B+Ax^2+xB+2A+2xA+B]\, e^x - 7\,(Ax^2+xB)$$

$$e^x - 7\,(2x\,A+B)\, e^x + 6x^2 e^x A + 6xe^x B = (x-2)e^x$$

Cancelling e^x from both sides and on simplifying,we get

$$-10xA - 5B + 2A = x - 2$$

On comparing the coefficient of x and constant term, we get

$$-10A = 1 \text{ and } 2A - 5B = -2$$

$$A = -1/10 \text{ and } 2A - 5B = -2$$

$$2A - 5B = -2$$

$$2\left(\frac{-1}{10}\right) -5B\ -2$$

$$-5B = -2 + \frac{-1}{5}$$

$$-5B = \frac{-9}{5} \text{ or } \qquad B = \frac{+9}{5} \qquad \therefore\ Y_p = x - \left(\frac{1}{10}\times + \frac{9}{25}\right)$$

Complete general solution is

$$Y = Y_c + Y_p = C_1 e^x + c_2 e^{6x} + x\left(\frac{1}{10}\times + \frac{9}{25}\right) e^x \text{ Ans.}$$

10.8 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
4. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

10.9 TERMINAL QUESTIONS

Q.1 Find the general solution of each of the following equations.

(i) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}-3y=0$

(ii) $3\frac{d^2y}{dx^2}-14\frac{dy}{dx}-5y=0$

(iii) $4\frac{d^2y}{dx^2}-4\frac{dy}{dx}+y=0$

(iv) $3\frac{d^2y}{dx^2}-6\frac{dy}{dx}+25y=0$

Q.2 Solve the initial value problem

(i) $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+8y=0; y(0)=1, y'(0)=6$

(ii) Solve the initial value problem.

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0; y(0)=3, y'(0)=-1$$

Q.3 Solve the differential equation.

(i) $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=x^2e^x$

(ii) Solve the differential equation.

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0e^{2x}(1+\cos x)$$

Unit - 11

APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS IN ECONOMIC MODELS

STRUCTURE

11.1 INTRODUCTION

In the last units, we have studied the first and second order differential equations and known about differnrent types of differential equations. In this unit, we will learn to solve different economic problem with the help of Difference and Differential equations.

11.2 LEARNING OBJECTIVES

After studying this Unit, you will be able to solve different economic problem with the help of Difference and Differential equation.

11.3 VARIABLE

A variable is something whose magnitude can change i.e. something that can take on different values. Variables frequently used in economics include price, profit, revenue, cost, national income, consumption, investment, imports, exports and so on. Since each variable can assume various values, it must be represented by a symbol instead of a specific number. For example, we represent price by P, profit by π, revenue by R, cost by C, national income by γ, and so forth. Properly constructed, an economic model can be solved to give us the solution values of a certain set of variables. Such variables, whose solution values we seek from the model, are known an endogenous variables (originating from within). However, the model may also contain variables which are assumed to be determined by forces external to model and whose magnitude, are accepted as go data only. Such variables are called exogenous (originating from side). It may so happen that a variable that is endogenous to one model may very well be exogenous to another.

Self-Check Exercise 11.1

Q1. What is meant by the term 'variable'?

11.4 APPLICATIONS OF DIFFERENTIAL AND DIFFERENCE EQUATIONS

Differential and Difference equations find wide applications in all branches of economics. Before we take the application to various economic models, let us first understand what we do mean by economic models. Any economic theory is necessarily an abstraction from the real world. The immense complexity of the real economy makes it impossible to understand all the inter relationships at once, nor, for that matter, all the inter relationships are important. The sensible approach is to pick those primary factors and relationships that are relevant to problem. Such a deliberately simplified analytical framework is called an economic model, An economic model is usually a theoretical and there is no inherent reason why it must be mathematical. If the model is mathematical, however, it will usually consist of a set of equations designed to describe the structure of the model. By relating a number of variables to one another in certain ways, these equation give mathematical form to the set of analytical assumptions adopted. Then, through application of the relevant mathematical operations to these equations, we seek to derive a set of conclusions which logically follow from those assumptions.

11.4.1 MODEL OF PRICE DETERMINATION

Let us consider a "partial equilibrium market model" i.e. a model of price determination in an isolated market. Since only one commodity is being considered. It is necessary to include only three variables in the model: the quantity demanded of the commodity (Q_d) the quantity supplied of the commodity (Q_s) and its price (P). Now we have to make certain assumptions regarding the working of the market. In the equilibrium model, the standard assumption is that equilibrium is obtained in the market if and only if the excess demand is zero ($Q_d - Q_s = 0$), that is, if the market is cleared. We also assume that Q_d is a decreasing linear function of P (as P increases, Q_d decreases). On the other hand, Q_s is postulated to be an increasing linear function of P (as P increases, so does Q_s) with the provision that no quantity is supplied unless the price exceeds aparticular level. In all, then, the model will contain one equilibrium condition plus two behavioral equations which govern the demand and supply sides of the market, respectively.

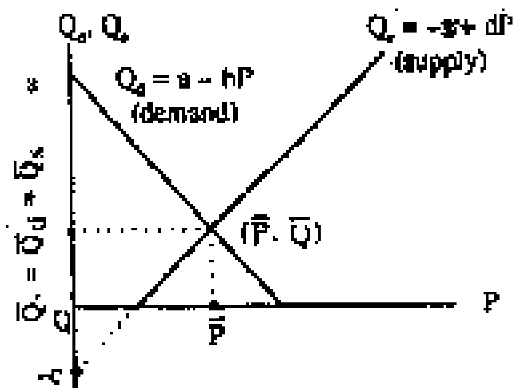
The model in the mathematical form can bewritten as

$$Q_d - Q_s = 0$$

$$Q_d - a - bP \quad (a.b>0)$$

Q_s= - c+dp (c.d>0)

Four parameters, a, b, c and d, appear in the two linear functions and all of them are assumed to be positive. When the demand function is graphed as in figure. Its vertical intercept is at a and its slope is–b. which is negative, as required. The supply function also has the required type of slope, d being positive, but its vertical intercept is negative, at–c. By this way we force the supply curveto have a positive horizontal intercept at P_1 there by satisfying the provision that supply will not be forthcoming unless the price is positive and sufficiently high.



The solution values of the three endogenous variables. $\overline{Q}_d . \overline{Q}_s$ and P. The solution values to be denoted by $\overline{Q}_d . \overline{Q}_s$ and P are those values that satisfy the three equations simultaneously. Since $\overline{Q}_d = \overline{Q}_s$, however, they can be replaced by a single variable Q. An equilibrium solution can be denoted by an ordered for ($\overline{P} . \overline{Q}$). In case the solution is not unique, several ordered pairs may each satisfy the system of simultaneous equations.

By substituting the second and third equation into the first, we get

$$\overline{P} = \frac{a+c}{b+d} \quad (b+d \neq 0)$$

$\overline{P}$ is positive-as a price should be because all the four parameters are positive by model specifications.

The equilibrium quantity $\overline{Q}$ (= $\overline{Q}_d = \overline{Q}_s$) is givenby

$$\overline{Q} = a - \frac{b(a+c)}{b+d} = \frac{ad-bc}{b+d}$$

Since the denominator is positive the positively of Q requires that the numerator (ad–be) be positive as well. Hence, to be economically meaningful, the present model should contain the additional restriction that ad>bc.

The meaning of this restriction will be clear from the figure. The ordered pair ($\overline{P}.\overline{Q}$) of a market model may be determined graphically at the intersection of demand and supply curves. To have $\overline{Q}$>o is to require the intersection point to be located above the horizonted axis in figure, which in turn requires the slope and vertical intercepts of the two curves to fulfill a certain restriction on their relative magnitudes. That restriction, is ad> bc, given that b and d are positive.

11.4.2 DYNAMIC ANALYSIS

In a static equilibrium we confine ourselves to the determination of position and to a comparison of two positions of equilibrium before and after a parameter shift. This is the method of comparativestatic. In using this method we ignore the question of time path that variables may follow as these variables move from one equilibrium position to another, and the associated question whether or not a system that starts out of equilibrium (because, say, of some parameter shift) will ever move back into equilibrium. Dynamic analysis is not to be regarded as just a sophisticated frill added to a fully satisfactory static model. We live in a world in which many magnitudes are changing continuously. Economic growth, trade cycles and inflation are all dynamic phenomena. So are all the processes of adjustment to disequilibrium, whether the adjustment is to be made by the changing of a price or by the migration of people from one part of the world to another. An important idea in dynamic is that, since it is concerned with the behaviour of variables over time, variables must be made functions of time.

11.4.3 DYNAMIC MODEL OF THE MARKET

Suppose for the particular commodity, thedemand and supply functions are as follows:

$Q_d = a - bP$ (a, b>0) …………..(1)

$Q_s = -c + dP$ (C, d>0)..………….(2)

For equilibrium condition, we have

$\overline{P} = \dfrac{a+c}{b+d}$ = some positive constant.

If it happens that the initial price P(o) is precisely at the level of P, the market will clearly be in equilibrium instantly, and no dynamic analysis at all will be needed. In the more likely case of P(o)≠ P, however, P is attainable (if ever) only after a due process of adjustment, during which not only will price change over time but Q_d& Q_s, being functions of P must change over time as well. It is in thiscontext that the price and quantity variables can be taken as functions of time.

Our interest is to find for given sufficient time for the adjustment process to work itself out, does it tend to bring price to the equilibrium level $\overline{P}$ or mathematically does the time path P(t) tend to converge to $\overline{P}$ as $t \to \infty$?

So we must find the time path P(t). But that, in turn, requires a specific pattern of price change to be prescribed In general, price changes governed by the relative strength of the demand and supply forces is the market. Let us assume, for the sake of simplicity, that the rate of price changes (with respect to time) at any moment is always, directly proportional to the excess demand ($Q_d - Q_s$) prevailing at that moment. Such a pattern of change can be expressed symbolically as

$$\frac{dP}{dt} = \alpha(Q_d - Q_s) \ (\alpha > 0) \qquad \ldots\ldots\ldots\ldots\ldots.(3)$$

where a represents a (constant) adjustment coefficient. With this pattern of change, we can have

$$\frac{dP}{dt} = 0 \text{ and only if } Q_d \ Q_s$$

We can write equation (3) by substituting the values of Q_d& Q_s from equation (2) and (3)

$$\frac{dP}{dt} = 0 \text{ and only if } Q_d \ Q_s$$

We can write equation (3) by substituting the values of Q_d& Q_s from equation (2) and (3)

$$\frac{dP}{dt} = \alpha\ (a - bP + c - dP)$$

$$= \alpha\ (a + c) - \alpha\ (b + d)\ P$$

$$\text{or} \qquad \frac{dP}{dt} = \alpha\ (b + d)\ P = \alpha\ (a + c)$$

(complementary form is formed from homogeneous equation)

Complementary if $y_c = e^{\alpha(b+d)t}$

Particular integral sol. $y_p \frac{a+c}{b+d}$

(The particular integral is simply any particular sol. Of the P=some constant

$$\therefore {}^{dp}\!/_{dt} = 0 \Rightarrow p = \frac{\alpha(a+c)}{\alpha(b+d)} = \frac{a+c}{b+d}$$

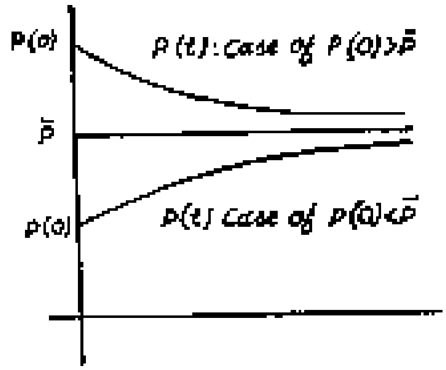
$$\therefore \ t) = A\ e^{-\alpha(+d)t} + \frac{a+c}{b+d}$$

At t = 0, $P(o) = A + \frac{a+c}{b+d} \Rightarrow A = p(o) - \frac{a+c}{b+d}$

$$\therefore \ p(t) = [P(o) - \frac{a+c}{b+d}]e^{-kt} + \frac{a+c}{b+d}$$

$$= [P\ (o) - \overline{P})\ e\text{-}kt + \overline{P} \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

Now the question originally posed, whether P(t)→Past→∞, amount to the question of whether the first term on the right of equation (4) will tend to zero as t→ ∞. Since P(o) and P are both constants, the key factor will be the exponential expression e^{-kt}. In fact k>0, the expression does tend to zero as t→ ∞. Consequently, with the assumptions of our model, the time path will indeed lead the price toward the equilibrium position. In a situation of this sort, where the time path of the relevant variable P(t) converges to the level $\overline{P}$ – interpreted here in its role as the intertemporal (rather that market-cleaning) equilibrium-the equilibrium said to be dynamically stable.



The concept of dynamic stability is an important one. Let us examine it further by a more detailed analysis of equation (4). Depending on the relative magnitudes of P(o) and $\bar{P}$, the solution of equation (4) really encompasses three possible cases. The first is P(0)= $\bar{P}$, which implies P(t) = $\bar{P}$. In that event, the time path of price can be drawn as the horizontal straight line as in adjoining figure. The attainment of equilibrium is in this case immediate. Second, we may have P(o)>$\bar{P}$. In this case, the first term on the right of (4) is positive, but it will decrease as the increase in t lowers the value of e^{-kt}. Thus the time path will approach the equilibrium level $\bar{P}$ from above, as illustrated by the top curve in figure. Third, in the opposite case of P(o)<$\bar{P}$, the equilibrium level $\bar{P}$ will be approached from below, as illustrated by the bottom curve in the same figure. In general, to have dynamic stability, the deviation of the time path from equilibrium must either be identically zero (as in case 1) or steadily decrease with time (as in cases 2 and 3)

The term $\bar{P}$ is nothing but the particular integral y_p, whereas the exponential term is the (definitive) complementary function y_c. Thus, we now have an economic interpretation form y_c, and y_p. $y_p$represents the intertemporal equilibrium level of relevant variable., and y_c is the deviation from equilibrium. Dynamic stability amounts, therefore asymptotic varnishing of the complementary function as t becomes infinite.

In this mode, the particular integral is a constant, so we have a stationary equilibrium is the intertemporal sense, we may interpret it as a moving equilibrium

Example 1. Demand and supply function for tea aregiven by

$$x_d = [120-2p+5\frac{dp}{dt}] \text{ kg. per week,}$$

$$x_s = [3p - 30 + 50 \frac{dp}{dt}] \text{ kg. per week,}$$

where p is the price at time t.

If the initial price is Rs. 36 per kg. find the timepath of price.

Solution :

At equilibrium $x_d = x_s$

$$120-2p+5\frac{dp}{dt} = 3p-30+50\frac{dp}{dt}$$

$$45\frac{dp}{dt}+5p-150 = 0$$

$$\frac{dp}{dt} + p/9 = 30/9 = 10/9$$

$\therefore p(c)=30$, $p = Ae^{1/t}$ where A is constant

$\therefore p(t)=30+ A (e^{-1/9t+})$

At to = o, $p(o)=30+A \Rightarrow A = p(o) - 30$

$\therefore$ $p(t)=30+[p(o) - 30] e^{-1/9}$

$\therefore$ $p(t)=30+(36-30)e^{-1/9}$

$p(t)=30+6 e^{-1/9}$

price after 10 weeks

$p(10)=30+6e^{-10/9}$

11.4.4 DOMAR GROWTH MODEL

It is a well known growth model of Professor E. D. Domar. In this model the idea is to stipulate the type of time path required to prevail if a certain equilibrium condition of the economy is to be satisfied.

The basic premises of the Domar model are asfollows.

(i) Any change in the rate of investment flow per year I(t) will produce a dual effect: it will effect the aggregate demand as well as the productive capacity of the economy.

(ii) The demand effect of a change in I(t) through multiplier process, so that an increase in I(t) will raise the rate of income flow per year Y (t) by a multiple of the investment in I(t). The multiplier is k=where s stands for the given (constant marginal propensity to save. On the assumption that I(t) is the only (para metric) flow that influences the rate of income flow, we can then state that

$$\frac{dY}{dt} = \frac{dI}{dt}\frac{1}{s} \qquad (1)$$

(iii) The capacity effect of investment is to be measured by the change in the rate of potential out-put the economy is capable of producing. Assuming a constant capacity-capital ratio, we can write

$$\frac{\chi}{K} j \text{ (= a constant)}$$

where χ stand for capacity or potential output flow per year, and $\wp$ denoted the given capacity-capital ratio. This implies, of course that with a capital stock K(t) the economy is potentially capable of production an annual product or income amount in to $\chi \equiv \wp k$ dollars. Note that from $\chi \equiv \wp k$ (the production function) It follows that $d\chi = \wp$ dk. &

$$d\chi = \wp \frac{dK}{dt} = \wp 1$$

In Domar's model equilibrium is defined to be a situation in which productive capacity is fully utilized. To have equilibrium is, therefore, to require the aggregate demand to be exactly equal to the potential output producible in a year :that is, $Y=\chi$. If we start initially from an equilibrium situation, however, the requirement will reduce to the balancing of therespective changes

$$\frac{dY}{dt} = \frac{d\chi}{dt}$$

The time path of investment I(t) which satisfies this equilibrium condition at all times can be calculated if we substitute (1) and (2) into the equilibrium condition (3) and we get

$$\frac{dI}{dt} \cdot \frac{1}{s} = \rho I$$

$$\frac{dI}{I} = \wp s \, dt$$

On integrating,

$$|I| = \wp \text{ st} + C$$

$$|I| = e \wp^{st+c} = e \wp^{st} e^{c} = Ae \, st^{\int st} \text{ where } A = e^{c}$$

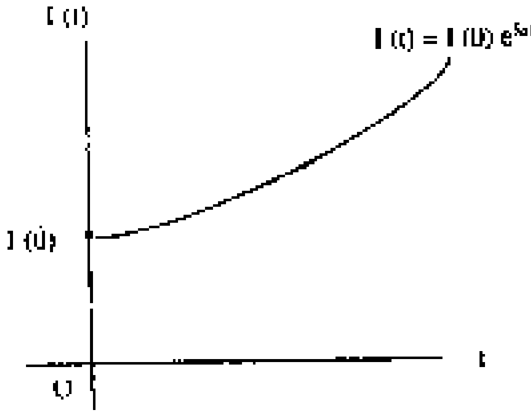
if we take investment to be positive, then |I|= Iand at t=0, we get

$$I(o) = Ae^{0} = A$$

$\therefore$ The required investment path – as

$$I(t) = I(o) \, e \, st$$

where I(o) denotes the initial rate of investment. This result has a some what disquieting economic meaning. In over to maintain the balance between capacity and demand over time, the rate of investment flow must grow precisely at the exponential rate of $\wp s$, along a path as illustrated in figure.



Obviously, larger the required rate of growth investment, the larger will bethe capacity–capital ratio and marginal propensity of save. But at any rate, once the values of $\wp$ & s are known, the required growth path of investment becomes vary rigidly set.

It is now to be seen what will happen if the actual rate of growth of investment-call the rate r= differs from the required rate $\wp$ s.

Domar's approach is to define a coefficient ofutilization.

u= $\underset{t\to\infty}{It} \; \frac{Y(t)}{\chi(t)}$ (u=1 means fullutilizationcapacity)

and show that $u = \frac{r}{\rho s}$ so that u $\underset{<}{\overset{\geq}{-}}$ 1 as r $\underset{<}{\overset{\geq}{-}}$ $\wp$ s.

In other words, if there is a discrepancy between the actual and required rates (r≠ $\wp$ s), then we will find in the end $(as\ t \to \infty)$ either a shortage of capacity (u> 1) or a surplus of capacity (u<1), depending on whether r is greater or less than s.

The capacity shortage and surplus really applies at any time t, not only as $t \to \infty$. For a growth rate of1 implies that

I(t)=I (o) e^{π} and = r I (O) e^{π}

$\therefore$ By (1) & (2), we have

$$\frac{dy}{dt} \cdot \frac{1}{s} = \frac{r}{s} I(o) e^{\pi}$$

$\frac{d\chi}{dt}$ $\wp$ I (t) = $\wp$ I (0) e^{n}

$$\therefore \frac{\frac{dy}{dt}}{\frac{d\chi}{dt}} = \frac{r}{\rho s}$$

the ration make it clear the relative magnitudes of the demand-creating effect and the capacity generating effect of investment at any time t, under the actual growth rate of r. Ifr (the actual rate) exceeds $\wp s$ (the required rate), then, and the demand effect

will out $\frac{dy}{dt} > \frac{d\chi}{dt}$, out strip the capacity

effect, causing a shortage of capacity. Conversely, ifr< $\wp s$, then there will be a deficiency in aggregate demand and, hence, a surplus of capacity.

The curious thing about this conclusion is that if investment actually grows at a faster rate than required (r> $\wp s$), the end result will be a shortage rather than actual growth of investment lags behind the required rate r< $\wp s$), we will encounter a capacity surplus rather than shortage. Indeed, because of such paradoxical results, if we now allow the entrepreneurs to adjust the actual growth rate r (hither to be taken a constant) according to the prevailing capacity situation, they will most certainly make the "wrong" kind of adjustment. In the case of r> $\wp s$, for instance, the emergent capacity shortage will motivate an even faster rate of investment. But this would mean an increase in r, instead of the reduction called for under the circumstances. Consequently, the discrepancy between the two rates of growth would be intensified rather than reduced.

The upshot is that, given the parametric constants $\wp$ and s, the only way to avoid both shortage and surplus of productive capacity is to guide the investment flow ever so carefully along the equilibrium path with a growth rate r= $\wp s$. And, any deviation from such a "razor's edge" time path will bring about a persistent failure to satisfy the norm of full utilization which Domar envisaged in this model. This is perhaps not too joyful a prospect to contemplate. Fortunately, more flexible result become possible when certain assumption of the Domar model are modified, as is done in the growth model of Professor Solow.

11.4.5 SOLOW GROWTH MODEL

In a Domar model, output is explicitly stated as a function of capital alone: $\chi = \wp K$ (the productive capacity, or potential output, is a constant multiple of the stock of capital). The absence of a labor input in the production function carries the implication that labor is always combined with capital in a fixed proportion, so that it is necessary to consider explicitly only one of these factors of production. Solow, in contrast, seeks to analyze the case where capital and labour can be combined in varying proportions.

(The Domar model assumes fixed output-capital ratio and the production function is simple. The Neo Classical Model does away with the assumption of fixed output capital ratio, if the output-capital ratio to vary continuously. In the long run, capital & labour inputs are substitutable & the ratio in which two in- puts are used may change. A purely capitalist economy can choose from these infinitely available ratios, only one of which will ensure a steady state growth which is warranted as well s natural rate of growth. The basic assumptions of Solow model include perfect foresight for all individuals, and smooth adjustment in goods, labour and capital markets.) Thus his production function appears in the form

$Q = f(K.L) \quad (K.L>0)$

where Q is outpur (net of depreciation), K is capital, and L is labor force–all being used in macro sense. It is assumed that f_k and f_L are positive (positive marginal products.) and $f_{kk}$and f_{LL} are negative (diminishing returns to each input). Furthermore, the production $f_n.f$ is taken to be linearly homogeneous (constant returns to scale), consequently, it is possible to write

Q = L $f\left(\frac{K}{L}1\right)$ = Lϕ (K*) where K* $\equiv \frac{K}{L}$ K* is the new variable, to stand for the ratio of capital to labour. (1)

In view of the assumed signs of f_k, and f_{kk}, the newly introduced ϕ function (which, has only a single argument, K*) must be characterized by a positive first derivative and a negative second derivative.

We have Q = 1 ϕ (K*) where K* = $\frac{K}{L}$

$$\frac{\partial k^*}{\partial k}\ \frac{\partial}{\partial k}\left(\frac{K}{L}\right)=\frac{1}{L},\ \frac{\partial k^*}{\partial k}\ \frac{\partial}{\partial L}\left(\frac{K}{L}\right)=-\frac{K}{L^2}$$

$$\frac{\partial Q}{\partial k}\ \frac{\partial}{\partial k}\ [L\phi(K^*)=\frac{\partial\phi(k^*)}{\partial k}=\frac{\partial\phi(k^*)}{dk^*}.\frac{\partial k^*}{\partial k}$$

$$= L\phi(K^*)\left(\frac{1}{L}\right)=\phi(K^*)$$

$$\& \frac{\partial Q}{\partial L}=\frac{\partial}{\partial L}[L\phi(K^*)=\phi(K^*)+L\frac{\partial\phi(k^*)}{\partial L})$$

$$= \phi(K^*)+L\phi'(K^*)\frac{\partial k}{\partial L}$$

$$= \phi(K^*)+L\phi(K^*)\left(-\frac{K^*}{L^2}\right)$$

$$= \phi\ (K^*) - K^*\phi'\ (K^*)$$

which shows that both $\frac{\partial Q}{\partial K}$ & $\frac{\partial Q}{\partial L}$ are functions are K^*alone

So we have

$f_k = \phi'\ (K^*)$

and hence f_k> implies $\phi(k^*)>0$ Then, since

$$f_{kk} = \frac{\partial}{\partial k} = \phi'\ (K^*) = \frac{\partial\phi(k^*)}{dK^*}.\ \frac{\partial k^*}{\partial k} = \phi''(K^*)\ \frac{1}{L}$$

the assumption $f_{kk}<0$ leads directly to the result $\phi''\ (k^*)<0$. Thus the ϕ function is one that increases with k^* at a decreasing.

Given that Q depends on K and L, we shall be finding how the two variables on determined. Solow's assumption are

$$K \equiv \left(\frac{\partial k}{\partial k}\right) = SQ \qquad (2)$$

(constant proportion of Q is invested)

There is a single commodity in the economy, and its annual rate of output is given by Y(t) (here Q). A fractions, of this output is saved and the rest, 1-s is consumed. The society's stock of capital, K, is merely the accumulated stock of single commodity (1), that has been saved in the past. This allows us to say that current saving determines the rate of growth of such society's capital. We write this

K= s Y

or

K = s Q

$L=L_0\ e^{\lambda t}$

($\lambda > 0$) (Labour force grows exponentially)

We now assume that the labour force is growing at a constant rate, λ. Thus labour is a function of time t, & we can write $L=L_0\ e^{\lambda t}$where L(t) is the labour force at time t, L_0 is the initial labour force at time t_0& λ is its rate of growth.

The symbol s represents a (constant) marginal propensity to save, and L_0 and λ are, respectively, the initial labor force & the rate or growth of labor.

$\therefore$ Equation (2) is

$$\begin{aligned} K^* &= s\ Q \\ &= s\ L\phi\ (K^*) \\ &= s\ L_0\ e^{\lambda t}\phi\ (K^*) \text{ from equation…………….. (3)} \end{aligned}$$

We want to find out if the capital labour ratio can always be such as to ensure full employment no matter how fast the labour force may be growing. We also wish to know, if this ratio will approach some stable equilibrium level. To investigate further we assume that the labour force is fully employed. Given this assumption we identify L(t) with the amount of labour input in the production function. This allow us to substitute (3) into (2).

This is a differential equation ……….(4)

Now we have $K = K^*L \neq K^*\ L_0\ e^{\lambda t}$

$$\therefore K = L_0\ e^{\lambda t}\frac{d}{dt}(K^*)+K^*\frac{d}{dt}(L_0\ e^{\lambda t})$$

$$= L_0\ e^{\lambda t}(K^*)+K^*L_0\lambda\ e^{\lambda t} \qquad \text{……………}(5)$$

From = m (4) & (5)

$K^{*} + K^{*}\lambda = s\phi(k^{*})$

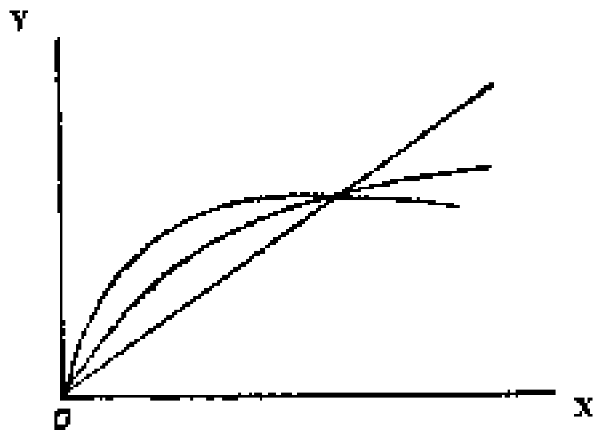
$K^{*} = s\varphi(K^{*}) - \lambda k^{*}$ ……….(6)

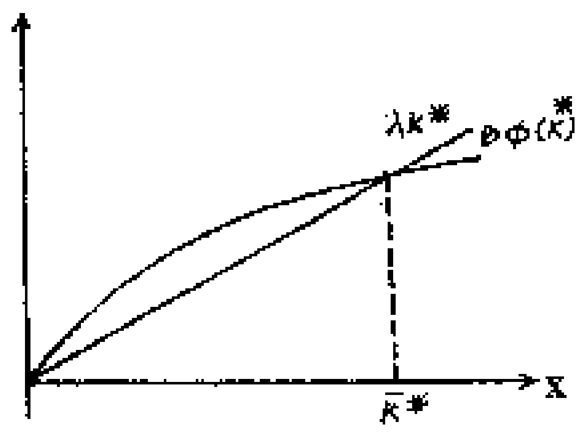
This differential equation, with two parameterss &λ, is the fundamental equation of the Solow model and is a equation with the capital labour ratio K, as its only variable.

$L(t) = L_0 e^{\lambda t}$ where L(t) is of the labour force at time t, L_0 is the initial labour force at time &λ, is its ratio or growth.

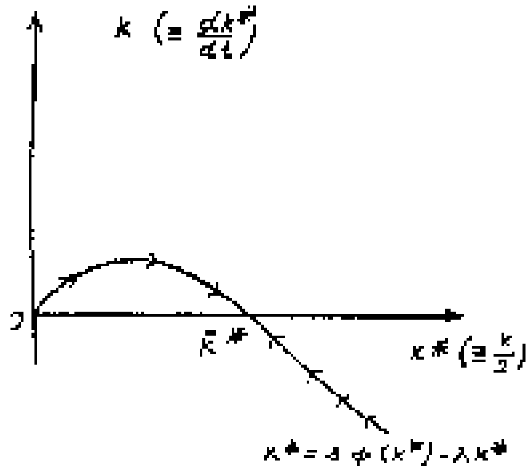
Equation (6) being in a general-function form, no specific quantitative solution is available. Nevertheless, we can analyse it qualitatively. To this end, we should plot a phase line, with k' on the vertical axis and k^{*} on the horizontal.

Since (6) contains two terms on the right, however, let us first plot these as two separate curve.





Obtain the line λK^{*} we set $s\phi(k^{*}) = 0$ and plot the relation between K^{*}& K^{*}, ignoring the negative sign. S line, which has a slope of λ, tells us how fast the capital:



output ratio would be declining for a given rate of growth of the labour force if savings were zero. The term, a linear function of K^*, will obviously show in figure (a) as a straight line, with a zero vertical intercept and a slope equal to λ. To obtain the linesϕ, we let λK^* be zero and plot the relation between & K^* by $K^* = s\phi\ (K^*)$. This line tells us how fast capital: output ratio would be growing as a result of capital accumulation if the labour force were not changing. If both s & λ are non zero, then the actual f K^* will be the difference between λK^* and s $\phi(K^*)$. This difference is represented by the vertical distance between the two lines. The $s\phi\ (K^*)$ term, on other hand, will plot a curve that increases at a decreasing rate, like $\phi(K^*)$, since $s\varphi(K^*)$ is merely a constant fraction of the $\varphi(K^*)$ curve. If we consider K to be an indispensable factor of production, we must start the $s\varphi\ (K^*)$ curve from the point of origin, this is because if K = o and thus K^*=o, Q must also be zero, as will be $\varphi\ (K^*)$ and s $\varphi\ (K^*)$. The way the curve is actually drawn also reflects the implicit assumption that there exists a set of K^* values for which $s\varphi\ (K^*)$ exceeds $\lambda\ K^*$, so that the two curve interact at some positive value of K^* namely K^*.

It remains to consider the shape of the curve $s\varphi\ (K^*)$. The expression $\phi(K^*)$ may be interpreted as the total product curve with labour input held constant at one unit and capital as the variable factor. In this case

K^* equals, K Since $\frac{K}{1}$ = K. The term $s\varphi\ (K^*)$ 1 shows the amount of this total output that is saved and invested per worker. The assumption of diminishing returns to one factor is sufficient to ensure theslope of φ (k) and thus $s\varphi\ (K^*)$ must be declining as $K^*$is increased.

Based upon these two curves, the value of K^* for each value of can be measured by the vertical distance between the two curves. Ploting the value of K^* against k as in fig, b, will then yield the phase line we need. Note that, since the two curves in diagram intersect when the capital labour ratio is K^*, the phase line in diagram b must cross the horizontal axis at K^*. This marks K^* as the (inter temporal) equilibrium capital-labour ratio.

In as much as the phase line has a negative slope at K^*, the equilibrium is readily identified as a stable one, given any (positive) initial value of K^*, the dynamic movement of the model must lead us convergent y to the level of K^*. The significant point is that once this equilibrium is attained and thus the capital-labor ratio is (by definition) unvarying over time-capital must there after simply, in turn that net investment must grow at the rate λ. Note, however, "must" is used here not in sense of requirement, but with the implication of automatcity. Thus, what the Solow model serves to show is that, given a rate of growth of labor λ, the economy by itself, and without the delicate balancing a Domar, can eventually reach a state of steady growth in which investment will grow at the rate λ, the same as K and L. Moreover, in order to satisfy (1), Q must grow at the same rate as well as because φ (K^*) is a constant when the capital labor ratio remains unvarying at the level of K^*. Such a situation in which the relevant variables all grow at the identical rate is called a steady state - a generation of the concept of stationary state, in which the relevant variables all remain constant, or in other words all grow at the zero rate.

11.4.6 THE COBWEB MODEL

A famous illustration of difference equation arises in the case of a single market equilibrium in which supply depends (with a one-period lag) on last periods price. Once the supply is in the market, however, the price depends on current demand.

Usually farmers decide on the basis of this year's price for a particular commodity the acreage they will plant with that crop. Anticipating that the price level will be maintained. If the price is high one year, farmers tend to plant heavily. The following year, when the crop is harvested and brought to the market, the supply exceeds the demand, price fail and farmers cut acreage devoted to this particular commodity. When the next year crop is harvested, supply may be below demand, prices increase, farmers plant more, next years crop exceeds demand, price fall. In this manner this cycle is repeated again and again.

Q = Production is output net of depreciation

Let us assume that the output decision in period t is based on then-prevailing price P_t. Since this output will not be available for sale until period 0 t+1, however, P_t, will determine not Q_{st} but Q_{st+1}. Thus we now have a "lagged" supply function. We are making the implicit assumption here that the entire output of a period will be placed on the market, with no part of it held in storage. Such an assumption is appropriate when the commodity in question is perishable or when no inventory is ever kept.

$Q_{s.t+1} = S(P_t)$ or equivalently $Q_{st} = S\ (P_t)$<i.e. price supply curve relates the supply in any period with the price one period before. When such a sup- ply function interacts with a demand function of the form

$$Q_{dt} = D\ (P_t)\text{i.e.} \begin{bmatrix} \text{price demand is specified in which} \\ \text{quantity demanded is determined by} \\ \text{the price at the time of purchase} \end{bmatrix}$$

interesting dynamic price patterns will result.

To simplify the mathematical analysis of theproblem in hand, we take (suppose) supply (lagged) and demand (unlagged) as a linear functions or in other words, the price-demand and price supply curves are straight lines. Also assuming that in each time period the market is always set at a level which clears the market (i.e. the market price is determined by the available supply, transaction according at which the quantity demanded & the quantity supplied are equal or P_t is determined on the solution of the equation.

$Q_{dt} = Q_{st}$ (1)

$Q_{dt} = a - \beta P_t$ (2) ($\alpha, \beta > 0$)

$Q_{st} = -\Gamma + \delta P_{t-1}$ (3) ($\gamma \in > 0$

where $-\beta$ and a are the slope and D- intercept for demand curve and δ and of Γ are slope and S intercept for the supply curve. The slop of the demand curve is taten to be–ve and that of supply curve positive. The reason for these considerations lies in the fact that an increase of one unit price produces a decrease of β unit is demand but on increase of δ units in supply.

By substituting the last two equations into the first, however, the model can be reduced to a single first-order difference equation as follows.

$\beta P_t + \delta P_{t-1} = \alpha + \Gamma$

$P_t + \frac{\delta}{\beta} P_{t-1} = \frac{\alpha + \Gamma}{\beta}$

In order to solve this equation, it is desirable first or normalize it and shift the time subscripts ahead by on period (after to t+ 1. etc.) the result.

$P_{t-1} + \frac{\delta}{\beta} P_t = \frac{\alpha + \Gamma}{\beta}$

To find solution of diff. = equation

Let $a = \frac{\delta}{\beta}$ and $C = \frac{\alpha + \Gamma}{\beta}$ &y=P

In as much as δ&β are both + ve. it followsthat $a \neq -1$

So we are seeking sol. of equation $y_{t+1} + ay_t = e$ where a & c are two constants.

The solution of this well known difference equations

$$P_t = \left(P_0 - \frac{\alpha + \Gamma}{\beta}\right)\left(\frac{\delta}{\beta}\right) + \frac{\alpha + \Gamma}{\beta + \delta}$$

where P_0 represents the initial price.

Three points may be observed in regard to this time path

(i) In the first place, the expression $\frac{\alpha + r}{\beta + \delta}$ whichconstitutesthe particular integral of the difference = n can be taken as the intermporal equilibrium price of the model.

As far as the market-clearing sense of equilibrium is concerned the price reached in cash period is in equilibrium price, because we have assumed that $Q_{dt} = Q_{st}$ for every t.

$$\bar{P} = \frac{\alpha + r}{\beta + \delta}$$

which is a constant and which is the equilibrium price of the model and this is a stationary equilibrium.

$$P_t\ (P_0) - \ \bar{P})\left(-\frac{\delta}{\beta}\right)^t + P$$

(ii) This leads us to second point namely,the significance of the expression (P_0-$\bar{P}$) which is constant and it depicts the scale effect. Its sign will bear on the question of whether time path will commence above or below the equilibrium (mirror effect), whereas its magnitude will decide how far above or below P_0 the time path starts (scale effect) If (P_0–P)>0, the time path, as said above, will blow up. If (P_0 – P)<0, the time path will start from below the equilibrium price.

$\therefore$ (iii) Lastly, in the expression $\left(-\frac{\delta}{\beta}\right)$ where β, δ > 0.

we have an oscillatory time path where – β and δ are slopes of the demand and supply curve respectively. It is this fact which gives rise to the Cobwebphenomenon.

$\left(-\frac{\delta}{\beta}\right)$ will always be - ve here $\therefore$ β . d>0.

There can β ofcourse, arise there possible varieties of patterns in the model. The oscillations will be.

(i) explosive if δ > β

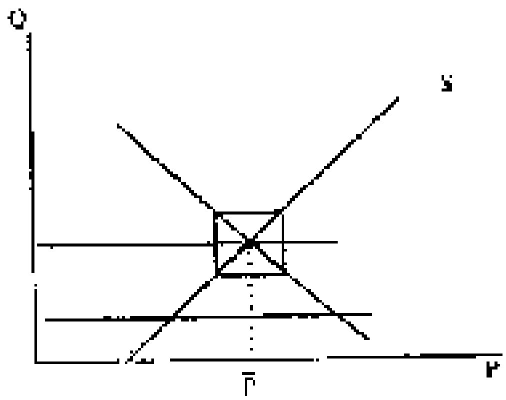
(ii) uniform if δ = β

(iii) damped if δ < β

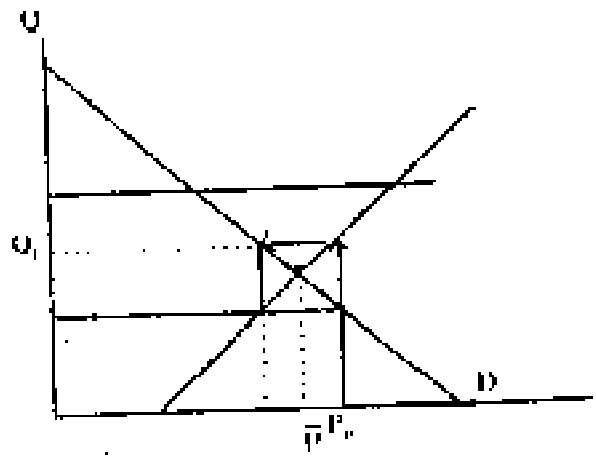
In order to visualize the Cobwebs, let us depict the model (1), (2) and (3) in figures. The equation (2) plots as a downward-sloping linear curve, with its slope numerically equal to β. Similarly, a linear supply curve with a slope equal to β can be drawn from the equation (3). If we let the Q axis represent in this instance a lagged quantity supplied. The intersection of D & S will yield the intertemporal equilibrium price P.

(i) When δ > β (S steeper than D)

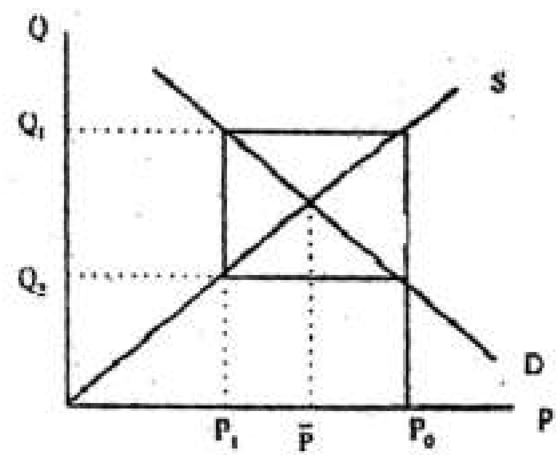
In this case demand and supply will produce an explosive price. Given an initial price P_0 (here assumed above P), we can follow the arrow- head and read off on the S curve that the quantity supplied in the next period (period 1) will be Q_1.



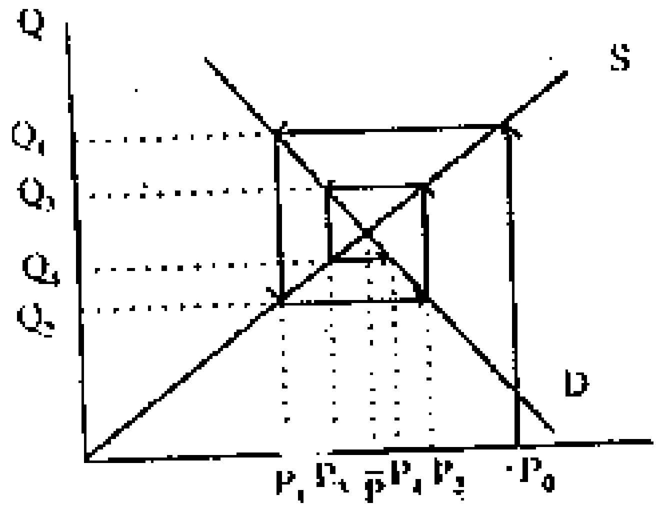
In order to clear the market, the quantity

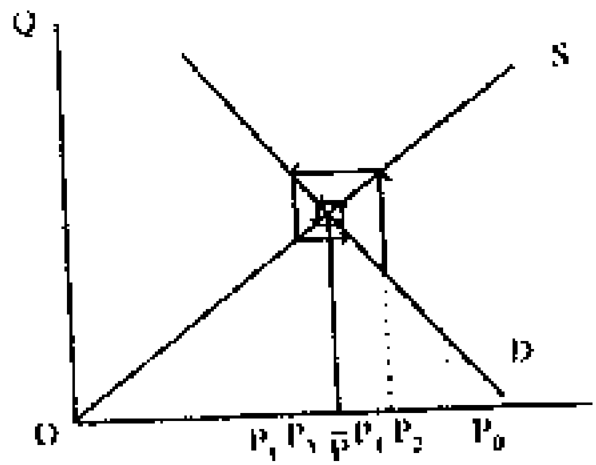


demanded in period, must also be Q_1, which is possible if price is set at the level P_1



(see downwardarrow). Now, via the S curve, the price P_1 will lead to Q_2 as the quantity supplied in period 2, and to clear the market in the latter period, price must be set at the level of P_2 according to the demand curve. Repeating this reasoning, we can trace out the price and quantities in subsequent periods by simply following the arrowheads in the diagram, there by spinning a "cobwed" around the demand and supply curves. By comparing the price levels, P_0, P_1, P_2 ..... we observe in this case not only an oscillatory pattern of change but also a tendency for price to widen its deviation from P as time goes by, with the cobweb being spun form inside out, the time path is divergent and the oscillation explosive.



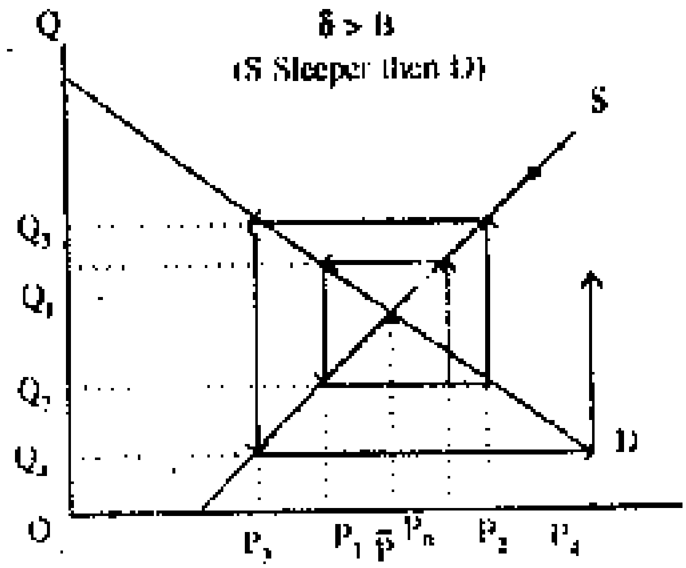


(ii) When δ < β (S flatter than D)

In this case a similar spinning process will create a cobweb which is centre-oriented. From P_0, if we follow the arrowheads, we shall be led ever closer to the intersection of the demand & supply curves, where P is while still oscillatory, this price path is convergent.

(iii) when δ = β

In this case cobweb consists of one square endlessly repeated, price oscillating finitely between just two values and there will be regular oscillations.



Thus the dynamic equilibrium can only be obtained in the (ii) case when $\delta > \beta$ or when demand curve is steeper than the supply curve. The disequilibrium price P, therefore oscillates over successive periods around the equilibrium price P and converge to P if $\delta < \beta$ or if D is steeper than S around the point of intersection.

Example 2. Examine the path represented by$\gamma_1 : 5\left(-\frac{1}{10}\right)+3$

Sol: Here $-\frac{\delta}{\beta} = -\frac{1}{10}$ or $\frac{\delta}{\beta} = \frac{1}{10}$

$\Rightarrow \quad \delta<\beta$

i.e. oscillation is damped. Therefore the time pathconverges to the equilibrium level 3.

Self-check Exercise 11.2

Q1. Demand and supply function, for tea are given by

$x\text{d} = 100 - \text{p} + \frac{dp}{dt}$ million kg. per week

$x_3 = - 50 + 2\text{p} + 10 \frac{dp}{dt}$ million kg. per week

Find the time path of p for dynamic equilibrium if the initial price is given to be Rs. 10 Kg. What will be the price at time t = 10?

Q2. How do you characterize the time path

$y_t = 3^t + 1$?

Q3. Linear demand and supply for the cobweb model as follows, find the inter temporal equilibrium price and determine whether equilibrium is stable

(a) $Q_{dt} = 18 - 3\,P_t\ Q_{st} = 3 + P_{t-1}$

(b) $Q_{dt} = 19 - 6\,P_t\ Q_{st} = -6 + P_{t-1} - 5$

Q4. The demand and supply, when p is the price, Q_d quantity demanded and Q_s, the quantity supplied are given as

$Q_d = a - b_p$ (a, b > 0) - (1)

$Q_s = -c + d_p$ (c, d > 0) - (2)

$\frac{dp}{dt} = x\,(Qd - Q_s)$ (x> 0) - (3)

Find the time path of price.

11.5 SUMMARY

In the last units, we learned about the difference and differential equations. This unit was dedicated to the application of these equation to share economic problems.

11.6 GLOSSARY

(i) **Variable :** A variable is something whose magnitude can change i.e. something that can take on different values.

(ii) **Cobweb Model :** A model where production or supply responds to price with one period lag. This model is after used to analyse the demand supply mechanism for markets of agricultural commodities.

(iii) **Linear Difference Equation :** A difference equation is linear if (i) the dependent variable y is not raised to any power and there are no product terms.

11.7 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 11.1

Ans. Q1. Refer to Section 11.3

Self-check Exercise 11.2

Ans.Q1. Refer to Section 11.4.3 (Example 1)

Ans.Q2. Hint Here $-\frac{\delta}{\beta} = \frac{3}{1} = 3$

$\partial > \beta$

i.e. the time path will explode and will diverge from the equilibrium level

Ans. Q3. Solution

We have $Q_{dt} = x - \beta\,P_t \quad Q_{st} = -\Gamma + P_{t-1}$

$$P_1 = \left(P_0 \frac{\alpha+\Gamma}{\beta}\right)\left(\frac{\delta}{\beta}\right) + \frac{\alpha+\Gamma}{\beta+\delta}$$

$= (P_0 - P \left(\frac{\delta}{\beta}\right)^t + \bar{P})$

Where $\bar{P} = \frac{\alpha + \Gamma}{\beta + \delta}$

Here α = 18 β = 3 Γ = 3 δ = 4

$\bar{P}$ = equilibrium price = $\frac{\alpha + \Gamma}{\beta + \delta} = \frac{18+3}{3+4} = \frac{21}{7} = 3$

and $-\frac{\delta}{\beta} = -\frac{4}{3} \Rightarrow \delta > \beta$

There will be explosive ascillutions and equilibrium will be stable.

(b) Where α = 19 β = 6 Γ = 5 δ = 6

$\bar{P}$ = equilibrium price = $\frac{\alpha + \Gamma}{\beta + \delta} = \frac{19+5}{6+6} = \frac{24}{12} = 2$

and $-\frac{\delta}{\beta} = -\frac{6}{6} = 1$

i.e. δ = β

There will be regular ascillutions and equilibrium will be unstable.

Ans. Q4. Hint equation (3) implies that change in price w.r.t. time (t) is directly proportional to the excess of demand over supply (= Qd – Qs)

= x (3) with held of us (1) & (2) can be written as

$\frac{dp}{dt} = \alpha\ (a - b_p + c - dp)$

$\frac{dp}{dt} = \alpha\ (b + d)\ p = \alpha\ (a + c)$

Hence $y_c = Ae^{\alpha}\ (b + d)\ t$

$y_p = \frac{\alpha(a+c)}{\alpha(b+d)} = \frac{\alpha + c}{b + d} = \bar{P}$ (say)

The complete Sol. Therefore is $y_e + y_p$

i.e. $Pt = \frac{a+c}{b+d} + A^{-\alpha}\ (b + d)\ t$

$= \bar{P} + (P0 - \bar{P})e$ where $P = \frac{a+c}{b+d}$

Now as $t \rightarrow \infty$

So $t \to \infty$ $P = \bar{P} + 0 = \bar{P}$

In other words, in the long run, price will courage to the equilibrium price (P) and in this way the dynamic stability will be obtained.

In the above case, y_p which depicts the particular integral gives the equilibrium price while, Y_c the complementary function, gives the deviation from the equilibrium.

11.8 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Banmal, W.J. (1974). Economic Dynamics (Second Edition) Macmillan, New York. Chapters 9, 10, and 11.
3. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.

11.9 TERMINAL QUESTIONS

Q. 1 Investigate the behavaiam of price in a market, i.e., the stability of a system with demand and supply function :

a) $D_t = 86 - 0.8 P_t$

$S_t = -10 + 0.8 P_{t-1}$

Q. 2 Find the time path represented by the equation $y_t = 2\left(\frac{-4}{5}\right) t + 9$.

Q. 3 Find the solution of the equation $y_{t+1} + \frac{1}{4} y_t = 5$ for $y_0 = 2$

Q. 4 The demand and supply for cobweb model is given as

$Q_{dt} = 19 - 6P_t$ and $Q_{st} = 6P_{t-1} - 5$. Find the intertanporal equilibrium price and comment on the stability of the equilibrium.

Unit - 12

STRAIGHT LINES

STRUCTURE

12.1 INTRODUCTION

The French Philosopher - Mathematician Rene Descartos (1596-1650) was the first realise the geometrical ideas can be translated into algebraic relations. This enabled him to write his book La Geometric (1637) in which geometry was studied systematically by using algebra. The combination of algebra and plane geometry came to known as Co-ordinate Geometry. The name co-ordinate geometry or analytic geometry, was given because of the fact that number (called co-ordinates) which are associated with points of some "plane" or "space" are employed in this study. In this unit, we have introduced co-ordinate system in bath two and three-dimension. Also, the formula for the equation of a straight line passing through two points both in two and three dimension, have been derived.

12.1 LEARNING OBJECTIVES

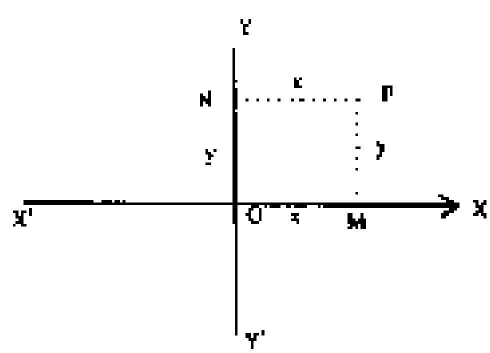
After reading this Unit, you should be able to:

- Locate the position of a point in a plan or in a space;
- Determine the distance between two points;
- Divide a line in any given ratio;
- Find the equation of a straight line;
- Apply the concept of straight line to solve the economic problems.

12.3 TWO DIMENSIONAL COORDINATE SYSTEM

A point is known by its position. A FrenchMathematician and Philosopher Rene Desartes was the first to perceive that a point could be represented in the plane by an ordered pair of real numbers, say (a, b) with the help of two axes and the law of algebra could then the applied to the solution of geometrical problems. We shall now define Cartesian Co-ordinates of a point on the plane with reference to two mutually perpendicular straight lines lying on the plane.

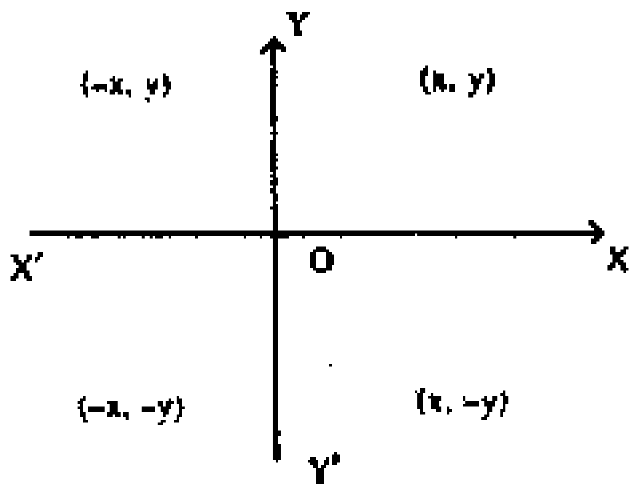
To find the position of a point, say P in a plane, we take two fixed straight lines X' OX' and Y'OY intersecting at right angles at O in the plane. These two lines are called the axes of reference or the axes of co-ordinates. X' O X is called the x-axis, Y' o Y the y-axis and O is termed as the origin.Let PM and PN perpendicular to X'OX and Y' OY respectively and let NP $=x$ and MP = y. Then OM = NP = x and On = MP = y.



When we know the distances OM and MP and the directions in which they are drawn, we know the position of the point P. OM is taken positive when drawn to the right from O and negative when drawn to the left from O, and MP is taken positive or negative when drawn upwards or downwards respectively from M.

The co-ordinates of point P are OM and MP with their proper signs. OM in known as the abscissa or the x co-ordinate and MP the ordinate or the y co-ordinate of the point P. If OM and MP, i.e. if abscissa and ordinate of P are 'x' units of length and 'y' units of length respectively, then x and y are the rectangular cartesian co-ordinates of P which are written as (x, y)

The two axes divide the whole plane into four sections called quadrants. For any point in the first quardrant XOY, both the abscissa x and ordinate y are positive, in the 2^{nd} quardrant YOX'x is negative and y is positive, in the 3^{rd} quadrant X' OY' both x and y are negative, in 4^{th} quadrant Y' O X,x is positive and y is negative. Thus if the position of a point be given, we can determine its co-ordinates and conversely if the co-ordinates (x, y) of a point are given, its position can be determined by measuring 'x' units of length



along the x-axis then measuring 'y' units of length parallel to y-axis, both being measured in the proper directions indicated by the signs of x and y.

12.3.1 Distance between two points

Let P (x_1 y_1) and Q (x_2, y_2) be the two given points. Draw PN and QM perpendicular to OX and then draw PR parallel to OX to meet QM in R.

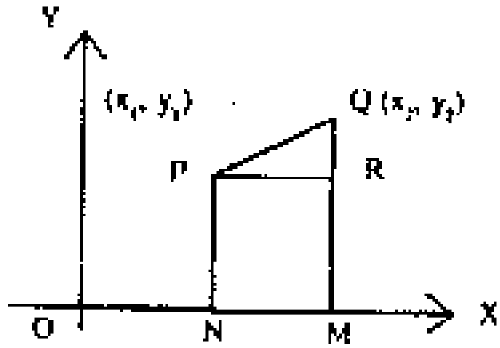
Then PR = NM – ON = X_2 Y_2

and RQ=MQ – MR=MQ – NP = y_2 –y_2

Now from the right-angled trianglePQR

$|PQ|^2|PR|^2+ |RQ|^2$

$|PQ|^2 = (x_2 – x_1)+(y_2 - y_1)^2$



Hence |PQ| = $\sqrt{(x_2-x)^2+(y_2-y_1)^2}$

Cor: The distance of the point P (h, k) from the orgin 0 (o, o) is given by

|OP|= $\sqrt{h^2+k^2}$

Example 1. Find the distance between the points(-5, 3) and (3.1)

Sol. The required distance between the points(-5.3) and (3.1)

$= \sqrt{(x_2-x)^2+(y_2-y_1)^2}$

$= \sqrt{[3-(-5)]^2+(1-3)^2}$

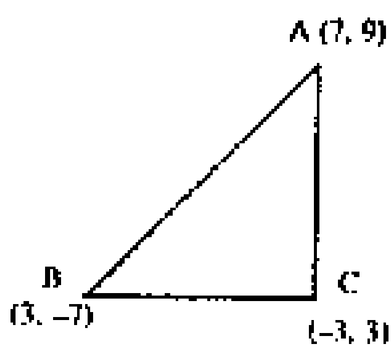
$= \sqrt{(8)^2+(-2)^2}$

$= \sqrt{64+4}$

$= 2\sqrt{17}$ units.

Example 2. Prove that the points (7,9), (3, –7) and (–3, 3) are the vertices of a right angled isosceles triangle.

Sol. Let the vertices of the triangle be A, B, C whose co-ordinates are (7, 9), (3, –7) and (–3,3) respectively.



Then $AB^2 = (3-7)^2 + (-7-9)^2 = 272$

$BC^2=(-3-3)^2+[3-(-7)]^2 = 136$

$CA^2=[7-(-3)]^2+(9-3)^2= 136$

We see that $BC^2+CA^2 = 136+136=272=AB^2$

and $BC^2=CA^2$

or $BC=CA$

Hence ABC is a night-angled isosceles triangle.

12.3.2. Section formula

Division of a finite line in a given ratio

Case I. The co-ordinates of a point R whichdivides the line segment joining (x_1, y_1) and (x_2, y_2) internally in the ratio m:n are

$$\left(\frac{mx_2+nx_1}{m+n}\frac{my_2+ny_1}{m+n}\right)$$

Case II. The co-ordinates of a point R which divides the line segment joining and $(x_1. y_1)$ and $(x_2\ y_2)$ externally in the ratio m:n are

$$\left(\frac{mx_2-nx_1}{m-n}\frac{my_2-ny_1}{m-n}\right)$$

Cor. If m = n in case I. i.e. R becomes the midpoint of PQ, its co-ordinates become

$$\left(\frac{x^1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

Example 3. Find the co-ordinates of the point which divides the join of the points (2, 4) and (6,8) externally in the ratio 5:3.

Sol. The required co-ordinates of the point which divides the join of (2, 4) and (6, 8) externally in the ratio 5:3 are

$$\frac{mx_2-nx_1}{m-n} \qquad \frac{my_2-ny_1}{m-n}$$

$$\frac{5\times6-3\times2}{5-3} \qquad \frac{5\times8-3\times4}{5-3}$$

(ii) (2, 4) and (8, 10) externally in the ratio 7 : 5

12.3.3 Gradient or slope of a line

If a line is not parallel to a co-ordinate axis.It is inclined at an θ angel to the x–axis OX. The angle θ may be acute or obtuse. Let P (x_1, y_1) and $Q(x_2, y_2)$ be two points on the line. Then the quantities $x_2 - x_1$ be two points on the line. Then the quantities $x_2 - x_1 = (PL)$ and $y_2 - y_1 = (LQ)$ are called run and rise respectively.

When $x_2 - x_1 \neq 0$, the number in defined by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$$

is called the gradient for the slope) of the line joining P (x_1, y_1) and Q (x_2, y_2)

Again from figure a, we see that

$$m = \tan\theta = \frac{LQ}{PL} = \frac{y_2 - y_1}{x_2 - x_1}$$

where θ= inclination of the line to the x–axis = < LPQ. Thus the gradient (or slope) of a line which is not parallel to the y–axis is defined by

m=tan θ

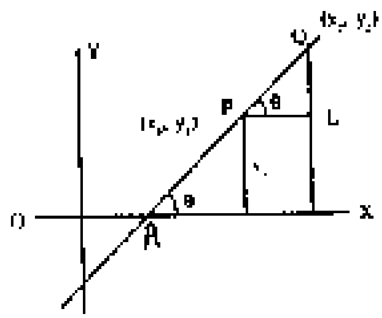
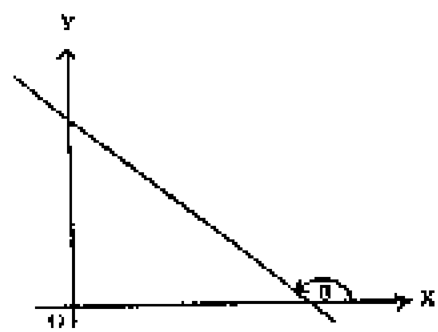


Fig. a



when the inclination of line to the x–axis may be acute or obtuse and hence it may be positive or negative according to the position of the line.

If θis acute (Figure a), the slope of the line is positive ife is obtuse (as in Figure b), the slope is negative.

If the line is parallel to the x–axis, θ = o and hence m = o. But if the lines parallel to the y–axis (or perpendicular to x–axis), $x_2 - x_1 = 0$ and in this case, the slope or gradient of the line is not defined.

Note: This definition cannot be used if the scales on the two axes are not the same. In Co-ordinate Geometry, we shall always assume the same scale on both the axes.

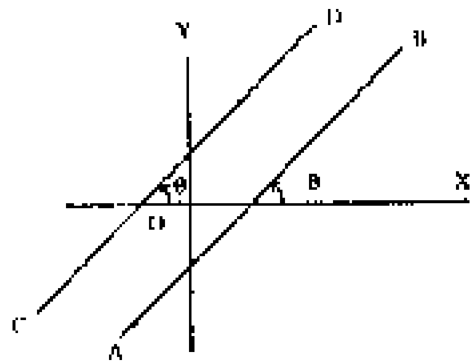
Example 4. Find the slope of the line passing through the points (0,–4) and (–6, 2)

$$\frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{2-(-4)}{-6-0} = -1$$

Condition for parallel and perpendicular lines

Case I If the two lines AB and CD i.e. parallel (none being parallel to y–axis), then their inclinations to the x–axis are the same and hence their slopes m_1 and m_2 are equal i.e. $m_1 = m_2$.



Conversely, if $m_1 = m_2$, then the inclinations of the two straight lines to the *x*–axis are the same and hence the two lines AB and CD are parallel. Hence the condition for two straight lines having slopes m_1 and m_2 to be parallel is $m_1 = m_2$.

Case II Let AB and CD be the two perpendicular straight lines (none being parallel to y–axis). IF AB makes an θ angle with the x–axis OX, then CD will make an angle $\theta + 90^o$ or $\theta - 90^o$ with OX according as θ is acute or obtuse.

∴ The slopes m_1, m_2 of AB and CD aregiven by

$m_1 = \tan\theta$ and $m_2 = \tan(\theta \pm 90) = -$ cote θ

[∴ $\tan(\theta + 90) = -\cot\theta$ and $\tan(\theta - 90) = -\tan(90 - \theta) = -\cot\theta$]

∴ $m_1 m_2 = \tan\theta(-\cot\theta) = -1$

i.e. $m_1 m_2 = -1$.

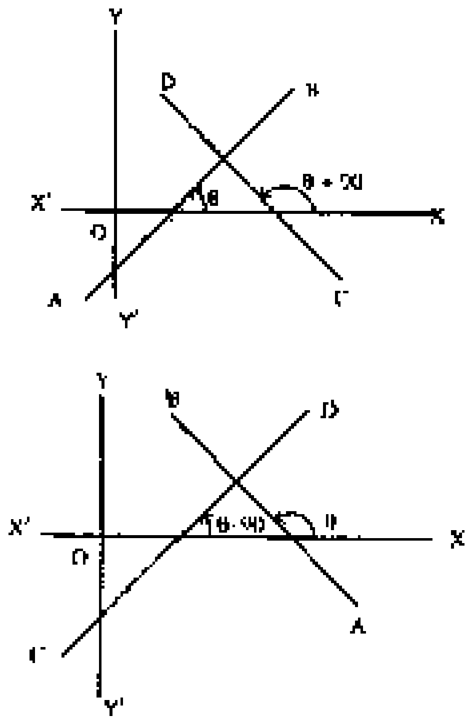


Fig.

Conversely, if m_1, $m_2 = -1$ and $m_1 = \tan\theta$, and $m_2 = \tan\theta_2$, then $\tan\theta_1$, $\tan\theta_2 = -1$

or $\tan\theta_2 = -\frac{1}{\tan} = -\cot\theta_1 = \tan\theta_1\ (\theta_1 \pm 90)$

$\therefore\ \theta_2 = \theta_1 + 90^0$ or $\theta_1 - 90^0$

This shows that the line AB is perpendicular to the line CD.

Hence the condition for two lines having slopes m_1, m_2 to be perpendicular to each other is m_1, $m_2 = -1$.

Example 5:- Show that the points A (6, 6), B (2, 3) and C (4, 7) are the vertices of a right-angled triangle.

Sol. m_1 = slope of AB = $\frac{3-6}{2-6} = 3/4$

m_2 = slope of BC = $\frac{7-6}{4-6} = -2$

and m_3 = slope of AC= $\frac{7-6}{4-6} = -\frac{1}{2}$

$\therefore$ $m_2, m_3 = 2 \text{ x } \left(-\frac{1}{2}\right) = -1$

This show that BC is perpendiculur to AC.

Hence ABC is a right-angled triangle.

Example 6 Show that the points A (1, –2), B (3, 4) and C(4, 7) are collinear.

Sol. m_1 = slope of AB = $\frac{4-(-2)-6}{3-1} = 3$

m_2 = slope of BC = $\frac{7-4}{4-3} = 3$

$\therefore$ $m_1 = m_2 =$

$\therefore$ AB is parallel to BC and B is common to boththe lines AB and BC.

Hence the points A (1, –2), B (3, 4) and C (4,7) are collinear.

12.3.4 Equations of straight lines.

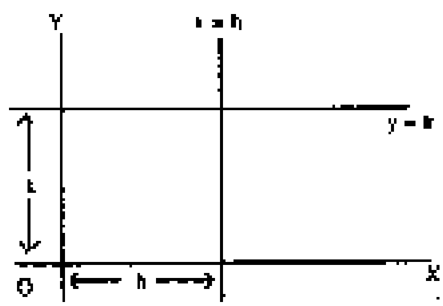
12.3.4.1Straight lines parallel to the co-ordinate axes

(i) The equation of a straight line parallel to the y–axis and at a distance h from it is x = h.

Because all points on the line parallel to the y–axis and at distance h from it have the same x coordinate h. Hence for any point P (x, y) on the line x = h.

Conversely, an equation x=h represents only those points which are at equal distances h from the y–axis.

Hence these points lie on locus x=h which is a lineparallel to the y–axis.



(ii) The equation of a straight line parallel to the x–axis and at a distance k from it is y=k.

Proof is exactly similar to as above.

(iii) Any point on the x–axis has its y co-ordinate equal to zero and hence the equation of the *x*-axis is $y = 0$.

Similarly, the equation of the *y*–axis is x= o.

12.3.4.2Equation of a straight lines: Standard Forms

(i) Point–slope Forms

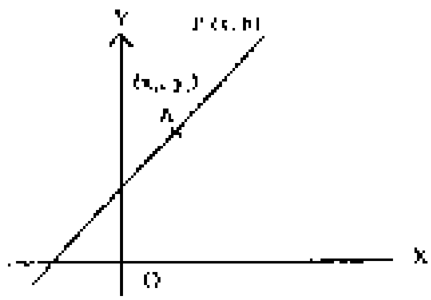
$y - y_1 = m(x - x_1)$

To show that the equation of the straight line passing through a given point (x_1, y_2) having a givenslope m is $y - y_1 = m(x - x_1)$

Proof: Let A be the given point (x_1, y_2) and let Abe any point on the line. Then the slope of line AP.

$$\frac{y - y_1}{x - x_1}$$

But the slope of the line AP is given to be m.



$$\frac{y - y_1}{x - x_1} = m$$

or $\quad y - y_1 = m(x - x_1) \quad (1)$

This is the relation which is satisfied by the co-ordinates of any point on the line and it is not satisfied by the co-ordinates of any point outside the line.

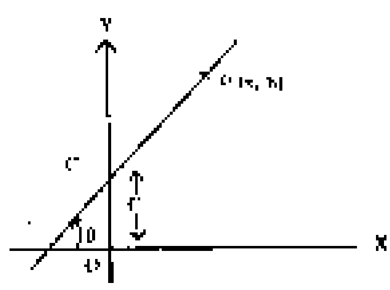
Hence equation (1) is the required equation of the line. **Note:** The slope m is undefined when the line a parallel to the y–axis and hence (1) cannot be used if the line through A (x_1, y_1) is parallel to the y–axis. In this case, the equation of the line through A(x_1, y_1) parallel to y–axis is $x = x_1$.

12.3.4.2Slope-intercept form (or Gradient form)

y – mx + c

To show that the equation of the straight line having a slope m and making a given intercept c on the y–axis is y = mx + c

Proof: Let the line cut the y–axis at C, so that OC= C



The co-ordinates of C are (o, c). Let P(x, y) any point on the line. Then the gradient of line CP is

$$\frac{y-c}{x-0} = \frac{y-c}{x}$$

But the gradient of the line is given to be m

$$\therefore \quad \frac{y-c}{x} = m$$

or $\quad y = mx + c \qquad (2)$

This is the relation with is satisfied by the co ordinates of any point on the line and it is not satisfied by the co-ordinates of any point outside the line. Hence this the required equation on the line.

Cor. The equation of a starlight line having a gradier m and passing through the origin (in this case = 0) is y = mx

12.3.4.2.3. Intercept Form

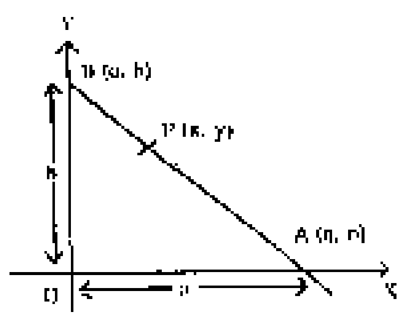
$$\frac{x}{a} + \frac{y}{a} + 1$$

To show that the equation of a straight line which cuts off given intercepts a and b from the axis is

$$\frac{x}{a} + \frac{y}{a} = 1$$

Proof: Let a straight line cut the x–axis at A (a, o) and the y–axis at B (o, b) so that the intercepts on the axes are a and b.

Let P (x, y) be any point on the line.



The slope of AP = $\frac{y-0}{x-0}$

and the slope of AB = $\frac{b-0}{0-a}$

Since AP and AB are on the same line and in the same direction from A to B.

$\therefore \qquad \frac{y-0}{x-a} = \frac{b-0}{0-a}$ or $\frac{y}{x-a} = \frac{b}{-a}$

or $\qquad$ bx – ab = –ay

$\qquad$ bx + ay = ab

Dividing both sides by ab

$\frac{x}{a} + \frac{x}{b} = 1$ which is the required equation of the line.

The slope of this line is $\frac{\frac{-1}{a}}{\frac{1}{b}} = -\frac{b}{a}$

12.3.4.2.4Two points form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

To show that the equation of the straight line passing through two given points A (x_1, y_1) and B (x_2, y_2) is

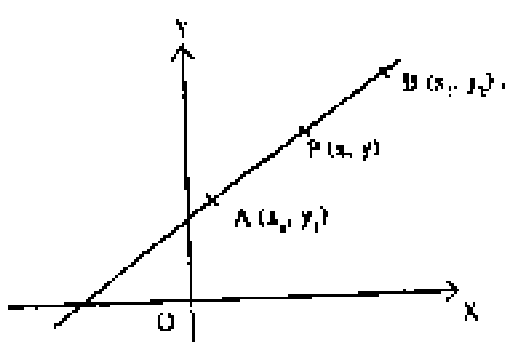
$y - y_1 = \frac{y - y_1}{x - x_1}(x - x_1)$

Proof: Let P (x, y) be any point on the line other than A and B. Clearly, slope of line segment AP= slope of the line segement BA because AP and AB are on the line i.e.

which is the required equation of a line. Condition of collinearity of three points

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $\qquad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$



which is the required equation of a line.

12.3.5 Condition of collinearity of three points

Let the three points be (x_1, y_1), (x_1, y_2) and (x_3, y_3). The equation of the line joining the points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad (1)$$

If the third point $(x_3. y_3)$ also lies on this line, the co-ordinates will satisfy the equation (1)

$$\therefore \quad y_3 - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

or $(y_3 - y_1)(x_2 - x_1) = (y_2 - y_1)(x_3 - x_1)$

or $x_2y_3 - x_2y_1 - x_1y_3 + x_1y_1 = x_3y_2 - x_3y_3 - x_3 y_1$
$-x_1 y_2 + x_1 y_1$

or $x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2) = 0$

which is the required condition of collinearity of threepoints.

Example 7.

Find the equation of a line parallel to Y – axis (or per pendicular to X – axis) at a distance

(i) 4 units to the right (ii) 4 units to the left.

Sol. The equation of any line parallel to Y–axisis $x=h$

(i) Here $h= 4$

$\therefore$ the equation of the line is $x = 4$

or $x - 4=0$

(ii) Here $h= -4$

$\therefore$the equation of the line is$x = -4$

or $x+4=0$.

Example 8

Find the equation of the joining the points (2, 3) and(2, –4).

Sol

Since the x co-ordinates of the points (2,3) and (2, -4) are equal, therefore, the line joining them is vertical i.e. parallel to Y–axis at a distance 2 units from it. Hence the equation of the line joining the points(2, 3) and (2,-4) is $x = 2$.

Example

Show that the three points (1, 4), (3, –2), are collinear. Find also the equation of the line on whichthey lie.

Sol. The equation of the line joining the points (1,4), (3,-2) is

$$y - 4 = \frac{-2-4}{3-1}(x-1) \quad \left[y - y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \right]$$

$= -3(x - 1)$

$= -3x+3$

$\therefore \quad 3x+y - 7=0 \qquad (1)$

Substituting the co-ordinates of third point (4,-5) in(1) we get

$3(4) - 5 - 7 - 0$

$12 - 12 - 0$, which is true.

Thus the third point satisfies the equation (1) of the line joining the first two points.

Hence three given points are collinear and the equation of the line on which they lie is $3x+y - 7=0$.

Example 10

Find the equation of line which passes through the point (-2, 3) and whose intercepts on the axes are equal in magnitude and both positive.

Sol:

Since the line makes equal intercepts on the axes and both are positive.

$\therefore$ let the intercepts be a. a

Then the equation of the line in the intercept form is

$$\frac{x}{a} + \frac{y}{a} = 1$$

or $\quad x+y=a$

$\therefore$ It passes through (-2, 3)

$\therefore -2+3=a$ or $a = 1$

Substituting this value of a in (1); we get

$x + y = 1$

which is the required equation.

SELF-CHECK EXERCISE 12.1

Q1. Find the distance between the points

(i) (–7, 5) and (5, 3)

(ii) (–3, 1) and (2, 1)

Q2. Find the Co-ordinates of the point which divides the join of the points

(i) (4, 6) and (8, 10) externally in the ratio 5 : 3

(ii) (2, 4) and (8, 10) externally in the ratio 7 : 5

Q3. Show that the points A (6, 6), B (2, 3) and C (4, 7) are the vertices of a right-angled triangle.

Q4. Show that the points A (1, –2), B (3, 4) and C(4, 7) are collinear.

Q5. Find the equation of the joining the points (2, 3) and (2, –4).

12.4 ISOPROFIT AND ISOCOST LINES FOR TWO PRODUCTS

An isoprofit line shows different combination of two productsx_1,x_2 which will yield same total profit. Ifx_1 and x_2 are the quantities of the two products, the profit function describing the isoprofit line is given by

$$\pi = a_1x_1 + a_2x_2$$

where Π is profit and a_1a_2 are known values.

The slope of the profit line is found by fixing the value of π say at π_1 thus

$$a_2x_2 = \pi - a_1x_1 \text{ or } x_2 = -\frac{a_1}{a_2}x_1 + \frac{\pi_1}{a_2}$$

The slope is $-\frac{a_1}{a_2}$. The intercept on t the x–axis is$x_1 + \frac{\pi_1}{a_1}$ that on the x-axis is x_2=$x_2 + \frac{x_1}{a_2}$

A family of iso-profit lines Can be drawn by assigning different values to the profit constant.The slopes of all isoprofit lines for a given problem are equal.

An isocost line shows different combinations of twoproducts X_1, X_2 which will involve the same totalcost. The total cost function is given by

$$C=b_1x_1 + b_2x_2$$

where b_1, b_2 are constants. If $C = C_1$ the slope is –

$$-\frac{b_1}{b_2} \text{ since } x_2 = -\frac{b_1}{b_2}x_1 + \frac{C_1}{b_2}$$

A family of isocost lines can be drawn by assigningdifferent values to the cost.

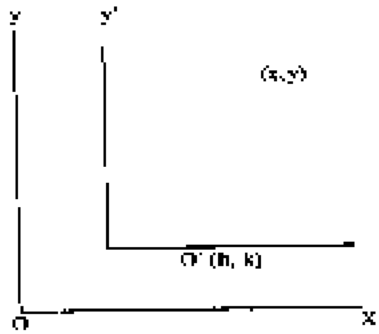
SELF-CHECK EXERCISE 12.2

Q1. What are Iso-profit Lines?

Q2. What are Iso-cost Lines?

12.5 CHANGE OF ORIGIN: TRANSLATION OF AXES

If the coordinates axes are changed, the coordinates of a point would change. The point remains in the same place.Suppose the coordinates of a point P in the old coordinate system (OX, OY) are (x, y). Let the new coordinate system be (O'X', O'Y') with the new origin O' (h, k).



There is a shift or change of origin from O to O'. In other words, there is a translation of axes to a new point O' (h, k)

This means that

x = OA =h+x'

y= OB = k+y'

Thus if the new origin is O' (h, k), the new coordinates of P are given by

x' = x – h

y' = y – k

If in a problem, new coordinates are known we can return to the old coordinate system by using

x = x +h

y = y + k

Example 11 (a). If the origin is shifted to (-5, 1), the coordinates of a point P (-5, 10) with reference to new axes can be found as follows.

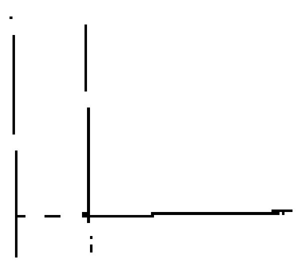
Here (h, k) =(-5, 1);

(x, y) = (–5, 10)

x' =x – h = -5 (–5)= 0

y' =y – k= 1 – 10=9 =

Thus (x', y') = (0,9)



(b) If by a change of origin, p (3, -5) becomes (4, 2), find the new origin.

$x' = x - h \quad h = x - x' = 3 - 4 = -1$

$y' = y - k \quad k = y - y = -5 - 2 = -7$

(c) If there is a change of the coordinate system from O to O' ($\propto$, β) and P(x, y) becomes P(x', y') then

$$x = x' + \propto$$

$$y = y' + \beta$$

(old in terms of new)

or $x' = x - \propto$

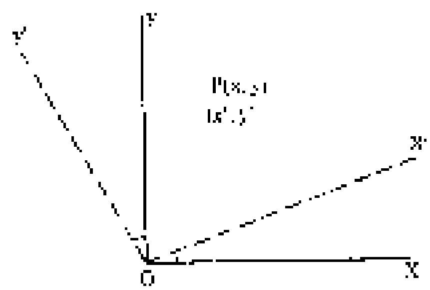
$y' = y - \beta$

(new in terms of old)

The line ax+by+C=0, by shifting the origin to O' becomes

$$a\,(x' + \propto) + b\,(y' + \beta) + C = 0$$

$$ax' + by' + (a\propto + b\beta + C) = 0$$



(d) If the new axes are perpendicular and are throughthe same origin but at an angle θ then

$$x = x' \cos\theta - y' \sin\theta$$

$$y = x' \sin\theta + y' \cos\theta$$

or $x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + \cos\theta$

Example 12.

In some cases it is possible to find an appropriate origin O (h, k) such that the new equation assumes a simple form. If in the equation.

$$(x - 3)^2 + (y+4)^2 = 36$$

the origin is shifted to (3, –4), the new equation becomes

$$x^2 + y^2 = 36$$

Example 13. (a) Show that by shifting the origin suitably the equation $y^2 - 20x - 6y + 149 = 0$ takes the new form $y'^2 = 20x'$.

Sol:- If we factorize the equation, we get $(y-3)^2=20(x-7)$. Now shift the origin to (7,3).

(b) Show that by shifting the origin suitably the equation $x^2-4x-16y=14$ takes the form $x'^2=16'y$

(Do it yourself)

(c) If we shift the origin to (-2, 3) what form does the equation $x^2+y^2+4x-16y=12$ take?

(Ans. $x'^2+y'^2=25$)

The first degree equation in x and y represent a straight line. The graphs of second degree equations are called conic sections. We can easily use the rectangular coordinate system to study the geometry of the conic section: the circle, the parabola, the ellipse, and the hyperbola which we are going to discuss in the nextunit.

Self-check Exercise 12.3

Q1. If the origin is shifted to (-5, 1), the coordinates of a point P (-5, 10) with reference to new axes, find the new origin.

Q2. If by a change of origin, p (3, -5) becomes (4, 2), find the new origin.

12.6 APPLICATION IN ECONOMICS OF STRAIGHT LINE

We consider such special cases where demand and supply curves are linear. The assumption that the functions are linear may look rather restrictive and unlikely to be satisfied in the real world. We see that we can learn a good deal of general nature even on this simple assumption, and besides a straight line may be sufficiently close to a curved one over some range that for small changes at least, the treatment of the curve through it where a straight line leads to acceptable approximation to the correct answer. Our simplest case in that in which both demand and supply curves are straight lines, described by the linear function

$Q_d=a+bP$ ……………(1)

$Q_s=c+dp$ ……………(2)

where Q_d denotes the quantity demanded & Q_s the quantity supplied. These are behavioral equationsm : they state assumptions about market behaviour. Since there is no economic meaning in this modelfor a negative Q, and since there are no subsidies that could create a negative price, we confine both the range and domain of these function to non-negative value of P and Q.

To complete the theory of competitive price determination we add the equilibrium condition.

$Q_d = Q_s$ ………………(3)

Now we can study an important and fascinating topic frequently referred to as qualitative economics. In practice, we frequently do not know parameter values, but only restrictions such as the demand curve slopes down. Hence we are interested in the question of what, if any. We can discover about the solution of the model and its properties on the basis of qualitative restrictions on the parameters. By "qualitative restrictions' we mean (for the moment) such simple and general notations as the demand curves slopes down & supply curves slope up. Evidently if restrictions like these prove to be sufficient to establish some property or result, without need for numbers, we have general results. In the present case we can do quite a lot qualitatively (which is not possible in more complicated models). We now list one qualitative assumption:

(i) $b<0$, i.e. the demand curve slopes down.

(ii) $d>0$, i.e. the supply curve slopes up;

(iii) $a>0$, the demand curve must have a positive intercept.

(iv) $c<a$, because if this were not true, supply would exceed demand at zero price and the good in question would not be an economic good, its price would be zero.

Usually it is assumed that $c<o$ so that the supply curve has a positive intercept on the price axis indicating that nothing is supplied below some minimum positive price. But all that is required for present purposes is $c<a$.

The above linear model can be written as

$Q_d = Qs = Q$

$Q_d = a - bp$ $(a.b>o)$

$Q_s = -c + dP$ $(c.d>o)$

(ii) The cost curve is a linear function of output. The graph of a cost curve is a straight line given by the equation C=a+bq where C = total cost, q=units of output and a, b are positive constant. The slope of this line is marginal cost which remains constant at every level of output. When no output is produced i.e. when q=o, then total cost a, which shows us that a is fixed cost for overhead cost, a is also the y–intercept of cost-line. The variable cost is c = bq.

(iii) In consumer's equilibrium analysis, budget line isa straight line and it expressed

$xPx + y\ Py = M$

where M = level of given income

P_x = price per unit of commodity X,

P_y = price per unit of commodity Y,

x=no. of units produced of commodity X,

y=no. of units produced of commodity Y.

The equation (1) can be written as

$$\frac{x}{M_{P_x}} + \frac{y}{M_{P_y}} = 1$$

Intercept on x–axis= $\frac{M}{P_x}$,which shows number ofunits purchased of X commodity if the consumerspends whole of his income on the commodity X.

Intercept on y–axis = $\frac{M}{P_x}$, which shows number of units purchased of Y commodity if the consumer spends whole of his income on the commodity Y.

Also slope of budged line = $\frac{P_x}{P_y}$

which implyies that slope of budged line is negative and is equal to ratio of prices of X and Y commodities.

(iv) The aggregate consumption in a country may be linearly related to its aggregate disposable income. The consumption function is a straight line given bythe equation.

C=a+bY

where C = aggregate consumption

Y = disposable income

and a, b are positive constants.

Here the slope is b which is the marginal propensity to consume. The intercept on y–axis is a which means that the level of autonomous expenditure is a. So a is the level of consumption when income is zero. The long run consumption function is also a straight line expressed by the equation C=bY. The average and marginal propensity to consume are same.

Example 11

(a) When the price is Rs.80 per watch, 10 watches are sold, 20 watches are sold when the price is Rs.60. Find the linear demand function.

(b) When the price is Rs.100 no watches are sold. (c) When watches are free, 50 are demanded. Find the linear demand function.

(c) When the price is Rs. 50 there are 50 watches of brand XX available for market. When the price is Rs.70 there are 100 watches available for market.

What is the linear supply function.

Sol.

(a) The demand curve passes through points whose co-ordinates are (10, 80) & (20, 60)

where x – coordinate = demand is units and y – coordinate = price in rupees.

$\therefore$ The linear demand curve is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\therefore y - 80 = \frac{60 - 80}{20 - 10}(x - 10)$$

or $2x+y=100$ is the reqd, linear demand curve.

(b) When $x = o$, $y = 100$, where x demand in units and y = price.

So linear demand curve through (0, 100) & (50,0) is

$$y - 100 = \frac{0 - 100}{50 - 0}(x - 0)$$

$2x + y = 100$

(c) Linear supply function through (50, 50) and (100,70) is

$$y - 50 = \frac{70-50}{100-50}(x - 50)$$

where x = no. of watches, y–price rupees

$\therefore 2x - 5y + 150 = o$ is the required supply curve.

Example 12

A firm invests Rs.10,000/- in a business which has a net return of Rs.500/- per years, investment of Rs.20,000/- would yields an income of Rs.2000/- per year. What is the linear relationship between investment and annual income. What would be the annual return on an investment of Rs.12,000/-?

Sol.

Let investment be denoted by x and incomeby y. The income is a linear function of investment.

$y = mx + c$ ……………………(1)

when $x = 10{,}000$, $y = 500$.

when x=20,000, $y = 2000$

$500 = 10{,}000\, m + c$ …………….(2)

$2000 = 20{,}000\, m + c$ ………………(3)

Solving equation (2) and (3), we get

$m = \frac{3}{20}$ and $c = -1000$

$\therefore$ Equation (1) can be written as

$20\, y = 3x - 20{,}000$

Which is the linear relationship between investment and annual income

When x = 12,000 then from equation (4),we have

$20\, y = 3 \times 12{,}000 - 20{,}000$

$= 16{,}000$

$\therefore y = 800.$

Hence when the investment is Rs. 12,000/- = the income is Rs. 800/-.

Self-check Exercise 12.4

Q. 1 When the price is Rs. 75 per watch, 15 watches are sold, 30 watches are sold when the price is Rs. 60. Find the linear demand function.

EXERCISE

1. Show that the point (1, 1), (–3, –1) and(–4, 1) form a right angled triangle.

2. Show that the points (2, 3), (6, 1), (–1, –4) and (–5, –2) are comers of a parallelogram.

3. (i) If the point (9,2) dividesthe segment of a line from P, (6, 8) to P_2 (x_2, y_2) in the ration 3, 7 find the coordinates of P_2,

   (ii) The middle point of a straight lineAB, has co-ordinates (a, b) and the co-ordinates of A are (c, d). Find the co-ordinates of B.

4. Prove that the points (2a, 4a), (2a, 6a)and (2a+$\sqrt{3}$ a. 5a) are the vertices of an equilateral triangle whose side is 2a.

5. What is the slope of the line perpendicular to the line passing through the points (3, 5) and(4,2)

6. A line passes through the points A(2, –3) and B(6,3). Find the slope of the line which are (i) parallel to AB (ii) perpendicular to AB.

7. Find the equation of a straight line parallel to y axis and passing through the point (4, –3).

8. Without using Pythagoras theorem, showthat (4,4), (3, 5) and (–1, –1) are vertices of right triangle

(1) Find the equation of the line joining thepoints $(a\,t_2^2, 2at_1)$ and $(a\,t_2^2, 2at_2)$ $(t_1 \neq t_2)$.

(2) The point (2, 3) is the foot of the perpendicular from the origin on a line. Find the equation of the line.

(3) Find the equation of line which passesthrough the point (–2, 3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

(4) The cost of production of a certain insign.

Production	Total cost
100 units	Rs. 520
150 units	Rs. 670

Assuming a linear cost curve, find the slope.What is the fixed cost?

12.7 SUMMARY

In this unit we have discussed the following points.

1. The position of a point in a plane can be determined by an ordered pair of number (x, y) called its coordinates.

2. The distance between two points P (x, y) and Q ($x_2 y_2$) is $\sqrt{\dfrac{(x_2 - x_1)}{(y_2 + y_1)}}$

3. The coordinates of the point R ($\bar{x}$, $\bar{y}$) dividing PQ in the ratio m : n are

$$\bar{x} = \frac{mx_2 + nx_1}{m+n} \quad \bar{y} = \frac{my_2 + nx_1}{m+n}$$

and if R divides PQ externally, then

$$\bar{x} = \frac{mx_2 + nx_1}{m+n} \quad \bar{y} = \frac{my_2 + nx_1}{m+n}$$

4. An equation of the form ax + by + c = 0 represents a straight line its slope is given by m = b/a

5. The angle between two line having slopes m1 and m2 is tan θ = $\dfrac{m_1 + m_2}{1 + m, m_2}$

12.8 GLOSSARY

1. Gradient or slope of a line : If a line is not parallel to a co-ordinates units. It is inclined at an angle to the x-axis ox. the angle θ may be acute or obtuse. Let P (x_1, y_1) and Q (x_2, y_2) be two points on the line. Then the quantities $x_2 - x_1$ be two points on the lines. Then the quantities $x_2 - x_1$ = (PL) and $y_2 - y_1$ = (LQ) are called run and rise respectively.

When $x_2 - x_1 \neq 0$, the number is defined by

$$\text{m} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$$

is called the gradient for the slope.

2. Isoprofit : An isoprofit line shows different combination of two products x_1, x_2 which will yield same total profit.

3. **Isoprofit line :** An isocost line shows different combination of two products x_1, x_2 which will involve the same total cost.

12.9 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 12.1

Ans. Q1. (i) The required distance between the points (–7, 5) and (5, 3)

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[5-(-7)]^2 + (3-5)^2}$$

$$= \sqrt{(12)^2 + (-2)^2} = \sqrt{144+4} = \sqrt{148} = 4\sqrt{37} \text{ Ans.}$$

Q2. (ii) The required distance between the points (–3, 1) and (2, 1)

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{[1-(-3)]^2 + (1-2)^2}$$

$$= \sqrt{(4)^2 + (-1)^2} = \sqrt{16+1} = \sqrt{17} \text{ Ans.}$$

Ans. 2 (i) The required co-ordinates of the point which divides the joint of the points (4, 6) and (8, 10) externally in the ratio 5 : 3 are

$$\Rightarrow \quad \frac{mx_2 - nx_1}{m-n} \qquad \frac{my_2 - ny_1}{m-n}$$

$$= \quad \frac{5\times 8 - 3\times 4}{5-3} \qquad \frac{5\times 10 - 3\times 6}{5-3}$$

= (14, 16) Ans.

(ii) The required co-ordinates of the point which divides the joint of the point (2, 4) and (8, 10) externally in the ratio 7 : 5 are

$$\Rightarrow \quad \frac{mx_2 - nx_1}{m-n} \qquad \frac{my_2 - ny_1}{m-n}$$

$$= \quad \frac{7\times 8 - 5\times 2}{7-5} \qquad \frac{7\times 10 - 5\times 4}{7-5}$$

= (23, 25) Ans.

Ans. Q3. Refer to Section 12.3.3 (Example 5)

Ans. Q4. Refer to Section 12.3.3 (Example 6)

Ans. Q5. Refer to Section 12.3.5 (Example 8)

Self-check Exercise 12.2

Ans. Q1. Refer to Section 12.4

Ans. Q2. Refer to Section 12.4

Self-check Exercise 12.3

Ans. Q1. Refer to Section 12.5 (Example 11)

Ans. Q2. Refer to Section 12.5 (Example 11)

Self-check Exercise 12.4

Ans. Q1. The demand curve passes through points whose co-ordinates are (15, 75) and (30, 60). Where x - coordinate = demand is units and y - coordinates = price in rupees

$\therefore$ The linear demand curve is

$$y - y_1 = \frac{x_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\therefore \quad y - 75 = \frac{60-75}{30-15}(x - 15)$$

$$y = -1\,(x - 15) + 75$$

$$= -x + 15 + 75$$

$\Rightarrow y + x = 90$ is the required linear demand curve.

12.10 READINGD/SUGGESTED READINGS

1. Allen, R.G.D. (1998). Mathematical Analysis for Economists. St. Martin's Press, New York.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Chiang, A.C. (1974). Fundamental Methods of Mathematical Economics, 2nd edition, MC Grow-Hill Book Company, New York.
4. Henderson, J.M. and Quudt, R.E. (1980). Microeconomic Theory. MC Grow-Hill Book Company, New York.

12.11 TERMINAL QUESTIONS.

Q1. Find the equation of the line joining the points $(a\ t_1^2, 2at_1)$ and $(a\ t_2^2, 2at_2)$ $(t_1 \neq t_2)$.

Q2. The point (2, 3) is the foot of the perpendicular from the origin on a line. Find the equation of the line.

Q3. Find the equation of line which passes through the point (–2, 3) and whose intercepts on the axes are equal in magnitude but opposite in sign.

Q4. The cost of production of a certain in sign.

Production	Total Cost
100 units	Rs. 520
150 units	Rs. 670

Assuming a linear cost curve, find the slope. What is the fixed cost?

Unit - 13

CIRCLE, PARABOLA AND HYPERBOLA

STRUCTURE

13.1 INTRODUCTION

In this Unit, we will study about the circles, learn to derive the equation of circle in different form the next section will deal with parabola and we will also go through the different form of equation if Parabola. In the cost section will learn about the hyperbola and its application in the economics.

13.2 LEARNIG OBJECTIVES

After studying this Unit, you should be able to

- Derive the equation of a circle in different forms.
- Derive the points of intersection of a line and a parabola
- Find the equation of the parabola in standard form.
- Explain the Hyperbola
- Apply the concept of hyperbola.

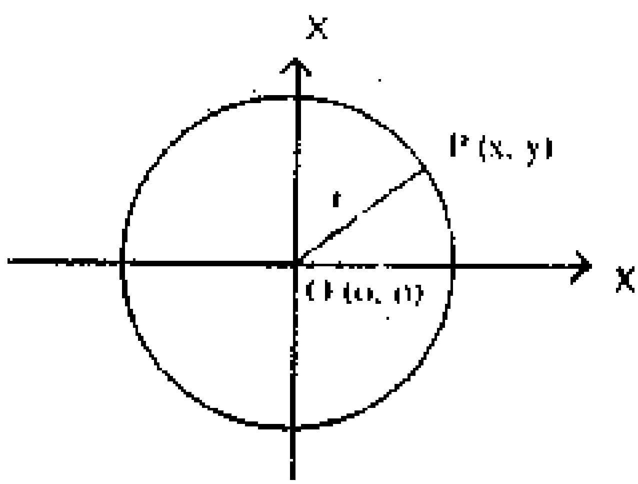
13.3 CIRCLE

Def. A circle is the locus of a point which moves on a plane is such a way that it is always at a constant distance from a fixed point. The fixed point is called the centre and the constant distance the radius of the circle.

13.3.1 EQUATION OF A CIRCLE IN DIFFERENT FORMS

13.3.1.1 Equation of a circle whose centre is at theorigin and radius r.

Let P(x, y) be any point on the circle, Let O be the origin and r be the radius. Then OP=ror $OP^2= r^2$ or $x^2+ y^2=r^2$.



This relation holds for any point P(x, y) on the circle but does not hold for any other point out.

Example 1:

Find the equation of a circle whose centre lies onthe origin and is of radius 4.

Sol. Equation of a circle

$x^2+y^2 = r^2$

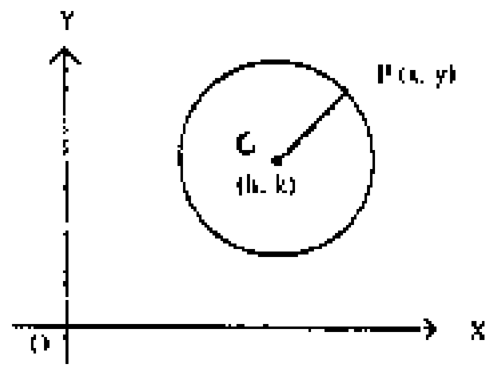
$x^2+ y^2=(4)^2= 16$

$x^2 + y^2 = 16$

13.3.1.2 Equation of a circle with a given centre andradius

Let C(h, k) be the centre and r the radius of thecircle.

Let P (x, y) be any point on the circle. Then CP=r or CP^2 r^2



or $(x - h)^2+(y - k)^2= r^2$

This equation is satisfied by any point P(x, y) or the circle, but by no other point lying outside the circle. Hence this is the equation of the circle having centre at the point C (h, k) and radius = r.

Example 2. Find the equation of the circle whose centre is (-2, 4) and radius 6.

Sol. The general equation of circle is

$(x - h)^2+(y - k)^2= r^2$

Here co-ordinates of centre is (-2, -4) & radius is6.

$[x^2 -(-2)]^2+(y - 4)^2 \quad = 16$

i.e. $x^2+ y^2+ 4x - 8y - 16 = 0$

13.2.1.3 General Equation of a Circle

The equation of a circle can be expressed in the general form.

$x^2+ y^2+2\,gx + 2\,fy+c=0$ (1)

where g, f, c are fixed constants for a particular circle.

The equation of the circle whose centre is (h, k) and radius ris.

$(x - h)^2+(y - k)^2= r^2$

or $x^2+y^2 - 2\,hx - 2\,ky + (h^2 + k^2 - r^2)=0$

which is of the form $x^2+y^2+2\,gx+2\,fy+c=0$

where $g= -h$, $f = -k$ and $c = h^2+k^2 - r^2$

Conversely, given an equation of the form

$x^2+2gx+g^2+y^2 2\,fy+f^2=g^2+f^2 - c$

or $(x + g)^2 + (y + g)^2 = \left(\sqrt{g^2 + f^2 - c}\right)^2$

This represents a circle whose centre is (–g, –1) and radius is $\left(\sqrt{g^2 + f^2 - c}\right)^2$ provided $g^2+f^2 \geq c$.

Note: From the general equation (1), we observe that

(i) the equation of a circle must be a second degree in x and y:

(ii) the coefficient of x^2=the coefficient ofy^2 and

(iii) The equation has no term containing x y.

Example 3.

Find the radius and the co-ordinates of the centre ofthe circle.

$x^2+y^2 - 8x - 16y+78=0$

Sol.

The given equation is $x^2+y^{2\,-}\,8x - 16y+78=0$

(1)

Comparing it with $x^2 + y^2 + 2gx + 2fy + c - 0$ we have

$g = \frac{1}{2}$ (coeff, of x) = $\frac{1}{2}(-8) = -4$

$f = \frac{1}{2}$ (coeff, of y) = $\frac{1}{2}(-16) = -8$

and c = constant term = 78

$\therefore$ the centre is (–g, –f) is [–(–4), –(–8)] i.e. (4,8)

and radius = $\left(\sqrt{g^2 + f^2 - c}\right)^2 = \sqrt{(-4)^2 + (-8)^2 - 78} = \sqrt{2}$

Note: Since the equation $x^2+y^2+2gx+fy+c= 0$ contains three arbitrary constants therefore weneed three conditions to find a circle.

13.3.2 Concentric Circles

Def. Circles having the same centre and different radius called concentric with the circle e.g. equation of any circle with the circle.

$x^2+y^2+2gx+2fy+c = 0$ is$x^2+y^2+2gx+2fy+k + o$ where k is any arbitrary constant.

Example 4

Find the equation of the circle which is concentric to the circle $x^2+y^2 - 6x+12y+15=0$ and radius of double its size.

Sol.

The equation of the given circle is

$$x^2 + y^2 - 6x + 12y + 15 = 0$$

(1)

Its radius = $\sqrt{(3)^2 + (6)^2 - 15} = \sqrt{30}$

Equation of any circle concentric with the circle (1) is

$$x^2 + y^2 - 6x + 12y + k = 0$$

Its radius = $\sqrt{(3)^2 + (6)^2 - k} = \therefore \sqrt{45 - k}$

By the given condition

$$\sqrt{45 - k} = 2\left(\sqrt{30}\right)$$

or $\quad k = -75$

Substituting this value of k in (2), we get

$$x^2 + y_2 - 6x + 12y - 75 = 0$$

which is the required equation of circle

Example 5.

Find the equation of the circle through the points (4, 1) and (6, 5) and 'D' its centralizes on the line 4x+y=16.

Sol.

Let the required equation of the circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ (1)

$\therefore$(1) passes through the points (4, 1) and (6, 5)

$16+1+8g+2f+c=o \Rightarrow 8g+2f+c+17=0$ (2)

and $36+25+12g+10f+c=0 \Rightarrow 12g + 10f + c + 61 = 0$ (3)

Also centre (–g, –f) of the circle (i) lies on

$4x + y = 16$ (4)

$\therefore \quad -4g - f - 16 = 0$

or $\quad 4g+f+16=0$

Subtracting (2) from (3), and get

$$4g+81+44=0$$

$$g+21+11 = 0$$

Solving (4) and (5) by the method of cross-multiplication, we have

$$\frac{q}{11-32}=\frac{f}{16-44}=\frac{1}{8-1}$$

or $\quad \frac{q}{-21}=\frac{f}{-28}=\frac{1}{7} \Rightarrow g= -3 \,\&\, f= -4$

Substituting these values in (2), we get

$$-24-8+c+17=0 \Rightarrow c=15$$

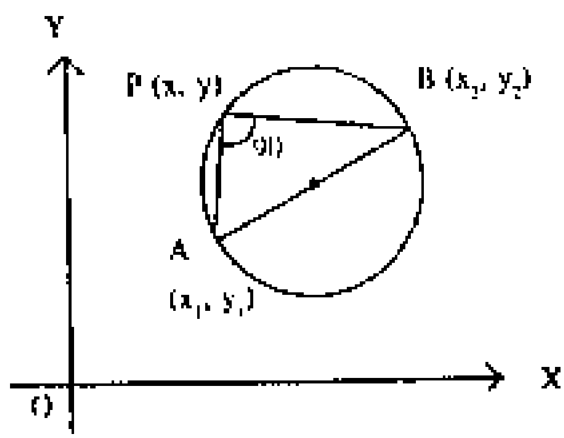
Substituting the value of g, f, c, C in (1), we get

$$x^2+y^2-6x-8y+15=0$$

Which is the required equation of the circle.

Equation of circle with (x_1, y_1) and (x_2, y_2) asthe extremities of a diameter

Let A (x_1, y_1) and B(x_2, y_2) be the extremities of diameter. Let P (x, y) be any point on the circle. Join AP and BP



Then

$$\angle APB = 90^{\circ}$$

$\therefore \quad$ AP is perpendicular to BP

Slope of AP = $\frac{y-y_1}{x-x_1}$

and slope of BP = $\frac{y-y_2}{x-x_2}$

Since AP is perpendicular of BP, we have

$$\therefore \quad \frac{y-y_1}{x-x_1} \times \frac{y-y_2}{x-x_2} = -1$$

or $\quad (x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

which is the required equation of the circle in termsof the co-ordinates of the extremities of a diameter.

Example 6

Find the center, the radius and the equation of the circle drawn on the line joining the points (–1,2) and (3, –4) as diameter.

Sol.

The equation of the circle with A(–1, 2) and B(3, 4) as the ends of a diameter is

$$(x - x_1)\,(x - x_2)+(y - y_1)\,(y - y_2)=0$$

or $(x + 1)\,(x - 3)+(y - 2)\,(y+4)=0$

or $X^2 - 2x - 3+ y^2+2y - 8=0$

or $x^2+y^2 - 2x+2y - 11 =0$

which is the required equation of a circle. The centre of the circle is the mid-point of the diameter AB.

∴ The coordinates of the centre are

$$\left(\frac{-1+3}{2}\ \frac{2-4}{2}\right) = (1, -1)$$

$$AB = \sqrt{(3+1)^2 + (-4-2)^2} = \sqrt{16+36} = 2\sqrt{13}$$

$$\text{Radius of the circle} = \frac{1}{2} \times = \sqrt{13}$$

SELF-CHECK EXERCISE 13.1

Q1. Find the equation of circle whose

(i) Centre is (0, 0) and radius is 3 units

(ii) Centre is (–3, 4) and radius is 6 units

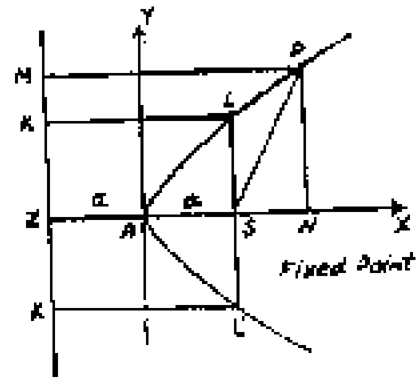
Q2. Find the radius and the co-ordinates of the centre of the circle.

$$x^2 + y^2 - 8x - 16y + 78 = 0$$

Q3. Find the equation of the circle which is concentric to the circle $x^2 + y^2 - 6x + 12y + 15 = 0$ and radius of double its size.

13.4 PARABOLA

Def. A parabola is defined as the locus of a point which moves in a plane is such a way that its distance from a fixed point S (called focus) is always equal to its perpendicular distance from is a fixed straight line (called directrix) in the plane.



The distance from a fixed point on the plane bears a constant ratio to its perpendicular distance from a fixed straight line on the plane, the constant ratio is known as eccentricity e and in case of parabola this constant ratio is 1 i.e.e=1

13.4.1 Equation of the Parabola in Standard Form

Let S be the locus and ZM the directrix of the parabola. Let SZ be perpendicular to the diretrix ZM. Then the mid-point A of the segment SZ lies on the parabola, because AS = AZ. The point A is called the vertex and the line ZAS (produced both ways) is called the axis of the parabola.

Refer to A as origin, ASX as x–axis and the line through A perpendicular to AS as y–axis, let P(x, y) be any point on the parabola. Let AS = a, then AX = a and the co-ordinates of S are (a, o). Draw PN and PM perpendicular to AX and the directrix ZM respectively. Then by definition of parabola. SP= PM or $SP^2 = PM^2 = ZN^3 = (ZA+AN)^2$

i.e. $(x - a)^2+(y - 0)^2 = (a + x)^2$

i.e. $x^2 = 2ax+a^2+ y^2 = a^2+2ax+x^2$

$y^2 = 4 ax$

(1)

This is the standard equation of the parabola.

Some properties of the parabola y^2= 4 ax

(i) The co-ordinates of the vertex A (i.e.the origin) are (0,0):

(ii) The co-ordinate of focus S are (a, o);

(iii) The equation of the directrix is x= –a or x+a=0:

(iv) If y is replaced by –y, the equation remains unchanged. This shows that the parabola is symmetrical about the x–axis:

(v) If LL be the focal chord (i.e. segment of a line through the focus S intercepted by the parabola) perpendicular to the x -axis, then LL is called the latus rectum of the parabola. Clearly,

SL=LK = ZS+2a= SL

Hence the length of the latus rectum = LL

= SL+ SL

= 2a +2a

= 4a:

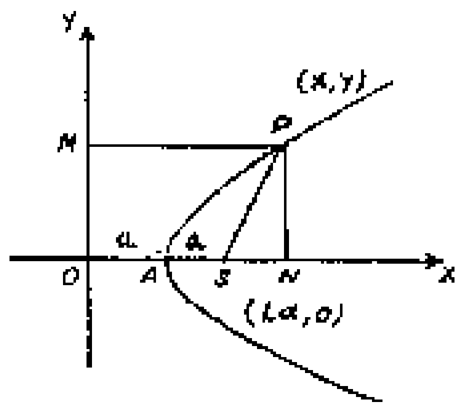
(vi) When x=0, we get two equal values of y as zero, showing that the y–axis is a tangent to the curve at the vertex.

(vii) If x is negative, y is imaginary, hence

there is no point of the parabola to theleft of y–axis.

Equation of parabola with its axis as x–axis the directrix as y–axis–axis

Let S be the focus, OM the directrix and OAS the axis of the parabola. OAS is perpendicular to OM at O.



Referred to O as origin, OX as x–axis and OM as y–axis let P(x, y) be any point on the parabola.

Let AS = a, then OA =a and OS = a + a = 2a.

$\therefore$ co-ordinates of the focus S are (2a, o).

Draw PN and PM perpendicular to OX and OM. Then by definition.

SP = PM or SP2 = $PM^2 = (ON)^2$

$(x - 2a)^2 + (y - o)^2 = (x)^2$

$x^2 - 4ax + 4a^2 + y^2 = x^2$

y^2 4a (x – a).

This is the required equation of the parabola.

Cor. The co-ordinates of the vertex A are (a, o). Ifwe transfer the origin to the vertex A (a, o) then from (1) replacing x by x+a and y by y+o, we get

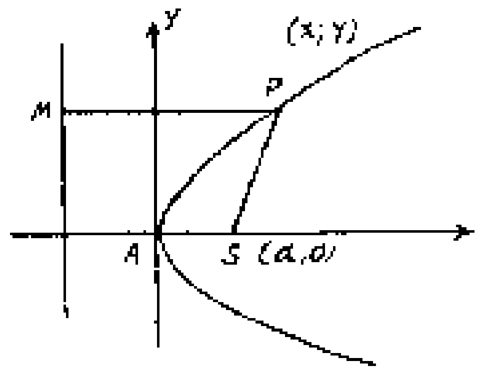
$(y+o)^2 = 4a (x + a - a)$

or $y^2 = 4ax$

which is the standard equation of the parabola

13.4.2 Shape of the Parabola

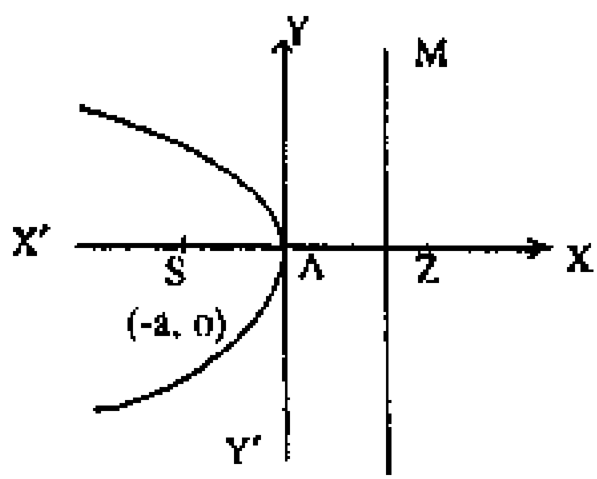
(i) $y^2 = 4ax$



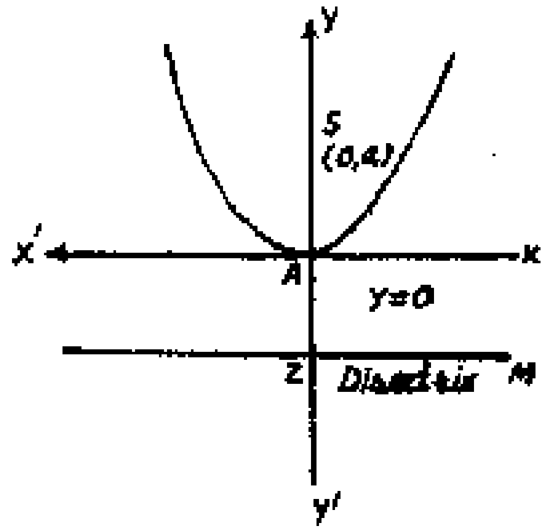
This is the standard equation of a parabola whose vertex is the origin, axis the x–axis and the tangent at the vertex is the y–axis. The co-ordinates of the focus S and (a, o) and the equation of the directrix is x + a= o

(ii) $y^2 = -4ax$

(ii) $y^2 = -4ax$

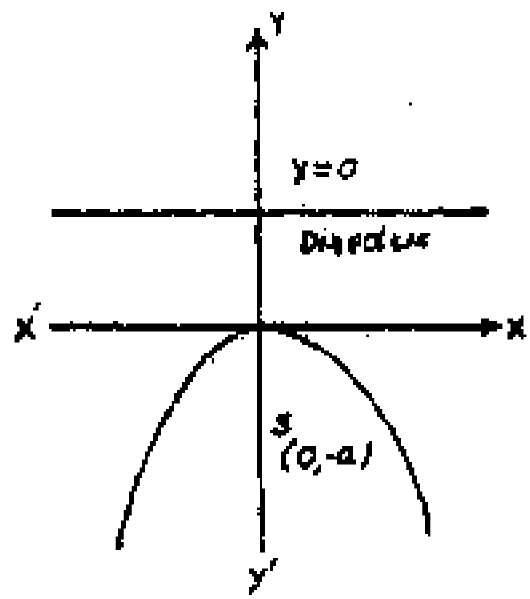


This is the equation of a parabola whose focus S is the point (–a, o) and directrix ZM is the line x – a=o. Here the direction from the vertex A to the focus S is negative. The vertex A is the origin (o, o), the axis of the parabola is y – o, the tangent at the vertex is x = o and the length of the latus rectum is 4a. The concavity of the curve is towards the negative side of the x–axis.



This is the equation of a parabola whose vertex is the origin (o, o) focus is (o, a) the tangent AX at the vertex is the x–axis and the axis AY of the parabola is the y–axis. The equation of the directrix is y= –a, or y+a= o and the length of the latus rectum is 4a. The concavity is towards positive side of the y–axis.

(iv) $x^2= -4ay$



This is the equation of a parabola whose vertex isthe origin (o, o) focus S is the point (o, –a) and thedirectrix is the line y=a or y – a= o.

The tangent at the vertex A is the x–axis and the concavity of the curve is towards the negative side of the y–axis. The length of the latus rectum is 4a.

Equation of the parabola in a parallel translation of co-ordinate axes

(i) When the equation of parabola is $y^2 = 4\,ax$ (1)

If we transfer the origin to the point (h, k) withoutchanging the direction of the axes, the equation istransferred to

$(y+k^2)=4a\,(x + h)$ (replacing x by x + h and y by y + k)

or $y^2+2\,ky+k^2 - 4ax+4ah$

or $x = \frac{y^2 + 2ky + k^2 - 4ah}{4a} = \frac{1}{4a}y^2 + \frac{k}{2a}y + \frac{1}{4a}$

$(k^2 - 4ah)$

which is of the form $x = Ay^2+BC+C$

This is a parabola with its axis parallel to the x–axis.

(ii) When the equation of parabola is $x^2 = 4ay$ (2)

If we transfer the origin from the vertex to the point(h, k) without changing the original direction of the axes, then the equation (2) is transformed to

$(x+h)^2 = 4a\,(Y+k)$

or $x^2+2hx+h^2=4ay + 4\,ak$

or $4ay=x^2 + 2hx + h^2 - 4\,ak$

or $y = \frac{x^2 + 2hx + h^2 - 4ah}{4a} = \frac{1}{4a}x^2 + \frac{h}{2a}x + \frac{1}{4a}$ $(h^2 - 1ak)$

which is of the form $y = Ax^2+ bx+C$.

13.4.3 Point of Intersection of a Line and a Parabola

To find the points of intersection of the line $y=mx+c$ with the parabola $y^2\ 4ax$

$y = mx + c$ (1)

$y^2 = 4ax$ (2)

Substituting the value of the from (1) in equation (2).

$(mx+c)^2=4ax$

$m^2x^2+c^2+2\ mcs = 4ax$

or $m^2x^2+2x\,(mc - 2a)+c^2= 0$ (3)

Which is a quadratic equation in x and it gives two values of x. On substituting the two values of x oneby one in (1), we get the corresponding values of y. These corresponding values x and y are the co-ordinates of the required points of intersection.

Thus the straight line cut the parabola at two points.

13.4.4 Condition of Tangency

If the line (1) touches the parabola (2), then the two values of x given by equation (3) must be equal (ie discriment = o)

ie. $[2(mc - 2a)]^2 - 4m^2c^2 = 0$

or $4(m^2c^2+4a^2 - 4mac) - 4m^2c^2 = 0$

or $a^2 = mac$

or $c = \frac{c}{m}$

Example 7

Find the co-ordinates of the focus, vertices and equation of the directrices of the following parabolas.

(i) $y= -8x$ (ii) $2x^2 = -7y$.

Sol.

(i) The equation of the parabola is $y^2= -8x$.

Clearly it is a left handed parabola and comparing it with $y^2= -4ax$, we have

$4a=8$ or $a=2$

Co-ordinates of focus are $(-a, o)= (-2, o)$

Co-ordinates of the vertex are (o, o)

Equation of its directrix is x–aie.x=2 or x – 2 – o.

(ii) The given equation of the parabola can be written as

$x^2 = \frac{7}{2}y$

Clearly it is a downward parabola and comparing itwith $x^2= -4ay$, we have $4a = \frac{7}{2}$ or $a=\frac{7}{8}$

Co-ordinates of the focus are $(o, -a)= (o, -\frac{7}{8})$

Co-ordinates of the vertex are (o, o) Equation of the directrix is

$Y = a$ ie $y = \frac{7}{8}$ or $8y - 7 = o$.

Example 8

Find the focus, the equation to the directrix and the length of the latus rectum of the parabola

$y^2+12=4x+4y$

Sol.

We have $y^2+12=4x+4y$

or $y^2 - 4y + 4 = 4x - 8$

or $(y - 2)^2 = 4(x - 2)$

which is of the form $y^2 = 4a\times$ where X = x – 2. Y=y – 2 and

$4a = 4, \therefore a = 1$

For the focus X = a, Y=o

i.e. x – 2 = 1 and y – 2 – 0 or x – 3 and y=2

$\therefore$ The co-ordinates of the focus are (3,2)

The equation of the directrix is

X + a = o or x – 1 = 0

The length of the lactus rectum = 4a=4 units.

Example 9

The demand curve is p = a – bx, show that total revenue curve is a parabola with axis vertical and opening downward. At what output is the total revenue maximum.

Sol. The demand curve is p=a – bx

Total revenue is R = p.x.

$= (a - bx)\, x = ax - bx^2$

$\therefore$ The equation of total revenue curve is

$R = ax - bx^2$

$= -bx^2 + ax$

$= -b\ (x^2 - \frac{a}{b}x)$

$= -b\ (x^2 - \frac{a}{2b})^2 + \frac{a^2}{4b^2}$ (completing the square)

or $R - \frac{a^2}{4b^2} = -b\ (x^2 - \frac{a}{2b})^2$

or $(x^2 - \frac{a}{2b})^2 = -\frac{1}{b}\ (R - \frac{a^2}{4b})$

Put $X = x - \frac{a}{2b}$, $Y = R - \frac{a^2}{4b}$

$X^2 = -\frac{1}{b}Y$

which is a downward parabola.

$\therefore$ Total revenue curve is a parabola.

Its axis is X=0 i.e. x – $\frac{a}{2b}$

$\therefore$ axis of the parabola is x = $\frac{a}{2b}$ = 0

Vertex is given by X= o,Y= o $\frac{a}{2b}$ i.e.

$$x - \frac{a}{2b} = 0, R = \frac{a^2}{4b} = 0$$

i.e. $x - \frac{a}{2b}$ =, $R = \frac{a^2}{4b}$

Vertex is $\left(\frac{a}{2b}, \frac{a^2}{4b}\right)$

R is maximum at the vertex of the parabola as the opens downward.

$\therefore$ R is maximum when x = $\frac{a}{2b}$ and max, value of

$R = \frac{a^2}{4b}$

Note:- Te second degree terms, in the equation of a parabola, always form a perfect square.

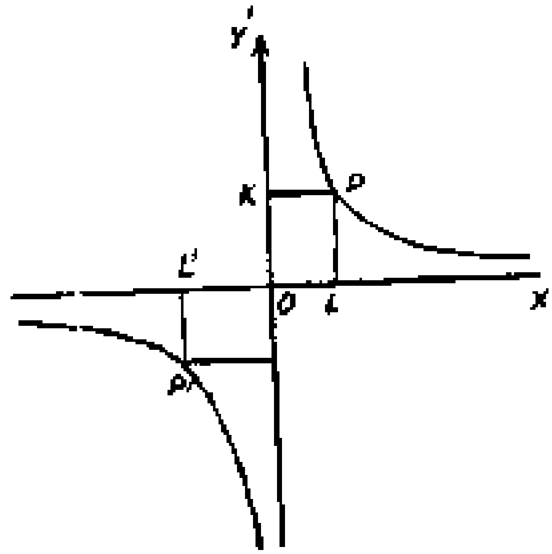
SELF-CHECK EXERCISE 13.2

Q1. Find the focus, the equation to the directrix and the length of the latus rectum of the parabola

$y^2 + 16 = 4x + 4y$

13.5 Rectangular Hyperbola or (Equilateral Hyperbola)

Def.: A rectangular hyperbola is defined as the locus of a point which moves such that the product of its distance from fixed perpendicular line is a positive constant say c^2.The fixed lines perpendicular to each other are called asymptotes and there point of intersection is called the centre of rectangular hyperbola.Take the simplest case when the origin is the centre of the curve, i.e. when the axes are the asymptotes. The one portion of rectangular hyperbola lies in first quadrant and second lies in third quadrant.



Let p be any point on the curve in first quadrant. Draw PL and PK perpendicular to x-axis. As P moves to the right of the curve, the perpendicular distance PL decreases and PK increase in such way that product PL × PK i.e. the area of rectangle OLPK remains constant c^2 (say). In third quadrant the point P moves along the portion of the curve in such a way that product of perpendicular distance from horizontal and vertical asymptotes (i.e. P'L'× P'K') or area OL'P'K' is equal to constant c^2. If the co-ordinates of point P be (x, y). We have PL ×PK c^2 where c^2 is constant

i.e. $xy=c^2$

which is the required equation of rectangular hyperbola.

When the asymptotes of the rectangular hyperbola are parallel to axes and centre be (a, b), then equation of rectangular hyperbola becomes.

$(x - a)(y - b) = c^2$

Example 10

Show that $y = \frac{mx + n}{sx + t}$, for all value of the constant (s≠o), represents a rectangular hyperbola.

Sol. The given equation is $y = \frac{mx + n}{sx + t}$

$sxy+ty = mx + n$

or $xy - \frac{m}{s}x + \frac{t}{s}y = \frac{n}{s}$

or $xy - \frac{m}{s}x + \frac{t}{s}y - \frac{mt}{s^2} = \frac{n}{s} - \frac{mt}{s^2}$

or $\quad x(y - \frac{m}{s}) + \frac{t}{s}\ (y - \frac{mt}{s^2}) = \frac{n}{s} - \frac{mt}{s^2}$

$$(x + \frac{t}{s}(y - \frac{m}{s}) = \frac{ns - mt}{s^2}$$

which is of the form $(x - a)\ (y - b)=c^2$.

Therefore the above equation represents a rectangular hyperbola where $a= -\ \frac{t}{s}$, $b=\frac{m}{s}$

$\therefore$ Centre of rectangular hyperbola is $(-\frac{t}{s}, \frac{m}{s})$

and asymptotes are parallel the axes.

13.5.1 Application of Rectangular Hyperbola

The rectangular hyperbola has many application in economics. Average fixed cost which is defined as the ratios of fixed cost to output is represented by the rectangular hyperbola. In this case, the outputaxis and cost axis are the asymptotes and the product of the distance of any point on average fixed cost curve from the two axes in always equal to fixed cost and hence is a positive constant. Also the demand curve or the average revenue curve has a shape of rectangular hyperbola. Rectangular hyperbola demand curve shows that the total expenditure incurred by a consumer remains constant at all prices. Therefore, the elasticity of demand at any point on such a demand curve is constant and is equal to unity. That is why such a demand curve is also called unitary elastic demand curve. In such a case, the marginal revenue at all level of output is zero, and therefore marginal revenue curve coincides with x–axis. We can also express demand curve for money in the shape of rectangular hyperbola.

The quantity theory of money says that a change in stock of money M implies an proportionate change in the value of money $\frac{ns - mt}{s^2}$ opposite side, wherep represents the price level.

$\therefore \quad M = c^2 \text{ x } P$

or $\quad M \times \frac{1}{p} = c^2$

which is a rectangular hyperbola.

Example 11

Find the centre and asymptotes of rectangular hyperbola $xy - 2x - y - 1=0$

Sol. Given equation is $xy - 2x - y - 1=0$

or $\quad xy - 2x - y+2 - 2 - 1=0$

or $\quad x (y - 2) -(y - 2) - 3=0$

or $\quad (x - 10\ (y - 2)=3$

which is a of a rectangular hyperbola. The centre of rectangular hyperbola is (1,2) The equation of asymptotes are

$x - 1=0, y - 2=0$

Example 12

A point moves in R^2 so that the difference of its distance from the fixed points (α, α) and (–α, –α) is always 2, α. α >o. Derive the equation of the curve described this point.

Sol. Let P(x, y) be the moving point and A(α, α) and B(–α, –α) be given points.

Given PA – PB = 2α

i.e. PA+PB + 2α

or $\sqrt{(x-\alpha)+(y-\alpha)^2} = \sqrt{(x+\alpha)^2+(y+\alpha)^2} + 2\alpha$

or $(x-\alpha)^2+(y-\alpha)^2= (x+\alpha)^2=(y+\alpha)^2 + 4\alpha^2 +$

$4\alpha\sqrt{(x+\alpha)+(y+\alpha)^2}$

or $-\sqrt{(x+\alpha)+(t+\alpha)^2} = x + y + \alpha$

or $x^2+ y^2+2\alpha x +2\alpha y+2\alpha^2=x^2 = x^2+ y^2+\alpha^2+ 2xy + 2\alpha x + 2\alpha$

which is a rectangular hyperbola.

SELF-CHECK EXERCISE 13.3

Q1. Show that $y= \frac{mx+n}{sx+t}$, for all value of the constant (s≠o), represents a rectangular hyperbola.

Q2. Find the centre and asymptotes of rectangular hyperbola $xy - 2x - y - 1=0$

EXERCISE

1. Find the equation of circle whose
   (i) Centre is (0,0) and radius is 3 units.
   (ii) centre is (–3, 4) and radius is 6 units.
2. Find the co-ordinates of the centre and the radius of the circle
   $2(x^2+y^2)=4x+6y+43$
3. Find the equation of the circle concentric with the circle $x^2+ y^2 \quad 6x+4y - 3=0$ of radius 5 units.
4. Find the equation of the circle passing through the points (5, 7) (6, 6) and (2, –2). Find the co-ordinates of its centre and the length of its radius.
5. Find the co-ordinates of the vertex, the focus, the equation of the axis and directrix of the parabola $x^2 +6x+2y=0$.

13.6 SUMMARY

In this unit, we learn about circle, parabola and hyperbola. In the first we studied about circle and the equation of a circle in a different forms. In the next section we studied about parabola equation of a parabola in standard forms we also learnt about the shape of parabola. In the last part of this section we learnt about the condition of Tangency. In the last section we learnt about the Rectangular hyperbola. We also learnt about the application part of rectangular hyperbola to solve economic problem.

13.7 GLOSSARY

1. **Circle :** A circle is the locus of a point which moves on a plane in such a way that it is always at a court distance from a fixed point.
2. **Centre and radius of a circle :** The fixed point is called the centre and constant distance from a fixed point is called the radium of the circle.
3. **General equation of a circle :** The equation of a circle can be expressed in the general form as $x^2 + y^2 + 2yx + 2fy + c = 0$
4. **Concentric circles :** Circles having the same centre and different radium are called concentric with the circle.
5. **Parabola :** A parabola is defined as the locus of a point which moves in a plane in such a way that its distance from a fixed point called focus is always equal to its perpendicular distance from a fixed straight line (called directrix) in the plane.
6. **Rectangular hyperbola :** A rectangular hyperbola is defined as the locus of a point which moves such that the product of its distance from fixed perpendicular line is a positive constant.

13.8 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 13.1

Ans. 1(i) The general equation of circle is

$(x - \text{h})^2 + (\text{y} - \text{R})^2 = r^2$

$[x^2 - (\text{o})]^2 + (y - 0)^2 = (3)^2$

or $x^2 + y^2 - 9 = 0$ Ans.

(ii) The general equation of circle is

$(x - \text{h})^2 + (\text{y} - \text{R})^2 = r^2$

Here co-ordinates of centre is (–3, 4) and radius is 6 level

$[x^2 - (-3)]^2 + (y - 4)^2 = (6)^2$

i.e. $x^2 + y^2 + 6x - 8y - 11 = 0$ Ans.

Ans. Q2. Refer to Section 13.3.1.3 Example 3

Ans. Q2. Refer to Section 13.3.2 Example 4

Self-check Exercise 13.2

Ans. Q1. Refer to Section 13.4 (Example 8)

Self-check Exercise 13.3

Ans. Q1. Refer to Section 13.5 (Example 10)

Ans. Q1. Refer to Section 13.5 (Example 11)

13.9 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
5. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

13.10 TERMINAL QUESTIONS

Q1. Find the Co-ordinates of the Vertese, the focus, the equation of the axis and directrix of the parable $x2 + 6x + 2y = 0$

Q2. Find the equation of the circle passing through the points (5, 7) (6, 6) and (2,– 2). Find the Co-ordinates of its centre and the length of its radius.

Unit - 14

INTEGRATION

STRUCTURE

14.1 INTRODUCTION

In this Unit, a major new concept, the integral of a function and a major new technique, that of integration will be introduced. Only the function of one variable will be considered. The concept of an integral of a function has two distinct aspects. In its first aspect, the integral refers to an area. It measures the area enclosed by the graph of a function $f(x)$ over some range of x values. To obtain, this measure we need to discover the definite integral of the function. In its second aspect, the integral rises from reversing the process of differentiation. Consider an economic example. We know how to derive the marginal cost function if we are given the total cost function.

TC = C(q)

MC = C' (a) = $\frac{d}{dq}$ c (q)

But what if we only know the marginal cost function? Can we use it to derive the total cost function? Evidently this requires that we reverse the process of differentiation whereby the MC function was derived from the TC function. To do this we require what is called the indefinite integral of the function in question.

14.2 LEARNING OBJECTIVES

After going through this Unit, you should be able to :

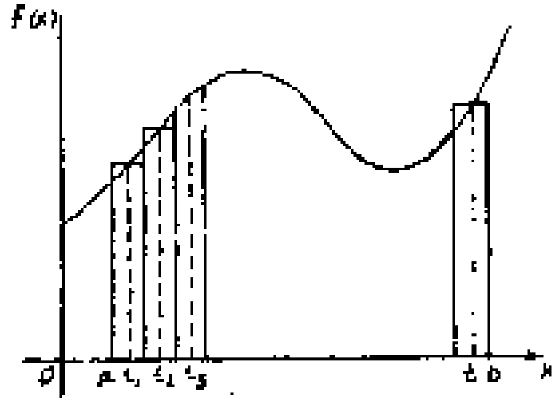
- Define integration
- Define the indefinite integral of function
- Use the method of substitution of simply and evaluates certain integrals
- Integrate by parts a product of two functions.

14.3 DEFINITE AND INDEFINITE INTEGRALS

The definite and the indefinite integrals are closely related concepts. We could start with either and then derive the other. If we start with the concept of the definite integral as an area under a curve and then go on to develop the concept of the indefinite integral as the reverse of the process of differentiation.

Definite Integral as the Limit of a Process of Summation

Suppose we have some function f(x) that is continuous and smooth. We suppose, for the moment, that $f(x)$ is also a positive over the interval with which we are concerned. An example of such a function is shown in figure. Let a and b be particular value of x and suppose that we wish to find the area bounded by $f(x)$, the x–axis and the perpendiculars at x = a and x = b. Fist let us sub-divide the interval between a and b into n equal sub-intervals. Erecting a perpendicular atthe end point of each sub-interval divides the area in which we are interested into n strips of equal width. We pick a single arbitrary value x within each interval and calculate the value of the function $f(x)$ at the arbitrarily chosen value of x. We let $\in_1$ stand for the value of x chosen arbitrarily within the first sub-interval, where $f(\in_1)$ is the value of the function at that point and so on up to $\in_n$ and $f(\in_n)$. This defines xrectangles, each with a width of $\frac{1}{n}$th the interval from a and each with a height of $f(\in_1)$ (1 = 1,....., n)



The sum of the areas of the rectangles $=\sum_{i=1}^{n} f(\in_1)\ \Delta x$ where $\in_i$ stands for the width of each interval. The area of all n rectangles is obtained by summation.

We how define the definite integral of the function $f(x)$ within the interval from a to b as the limit, as n $\to \infty$, of all the sum of areas of n rectangles each of equal with and each of height given by $f(\in_1)$ for an $\in_1$ arbitrarily chosen from written each of the x sub-intervals. We write this

$$\underset{n\to\infty}{LT} \sum_{i=1}^{n} f(\in_1)\ \Delta x = \int_a^b f(x)\,dx$$

where $\Delta x = \frac{b-a}{n}$. The symbol indicates thelimit of the process of summation defined on the left hand side of the identity sign. The symbols a and b attached to the $\int$sign are called the lower and upper limits of integration and tell us the range of x, value from a to b in this case-over which we have integrated the function $f(x)$. The whole expression $\int_a^b f(x)\,dx$ is called the definite integral of the function $f(x)$. The function to be integrated, f(x) in this case, is called the integrand and variable on which it is defined, x in this case, is called the variable of integration. The process of finding the integral is called the integration.

Def. If $f(x)$ is continuous in [a, b] and differentiablein (a, b) and $\frac{d}{dx}f(x)$, then $f(x)$ is said to be indefiniteintegral (or primitive of antidervative of F(x) and iswritten as

F(x) dx = $f(x)$+C

14.3.1 General rules of Integration

(i) $\int \{u(x)+ v(x)\}\, dx = \int u(x)\, dx + \int v(x)\, dx$

(ii) $\int k f(x)\, dx = k \int f(x)\, dx$ where k is a real number.

(iii) $\int f' \{g(x)\}\, g'(x)\, dx = f\{g(x)\}$.

14.3.2 Fundamental Integral

(i) $\int x^n \, dx = \frac{x^{x+1}}{n+1}$+c where n≠ –1 and c is aconstant of integration.

Proof: consider $\frac{d}{dx}\left(\frac{x^{x+1}}{n+1}\right) = \frac{1}{n+1} \frac{d}{dx}$ (x^{n+1})

$= \frac{1}{n+1}$ $(n + 1)x^n$

∴ One of the value of $\int x^n \, dx = \frac{n^{n+1}}{n+1}$ +c

Example 1. Evaluate $(3x^2 - \frac{1}{2}\sqrt{x})\, dx$

Sol. $\int(3x^2 - \frac{1}{2}\sqrt{x})\, dx = 3 \int x^2 \, dx - \frac{1}{2} \int x^{1/2} \, dx$

$= 3 \frac{x^3}{3} - \frac{1}{2} \frac{x^{1/2+1}}{1_2 + 1} + C$

$= x^3 - \frac{1}{3}x^{3/2} + c$

(ii) $\int \frac{1}{x} \, dx = \log |x| + c$

Proof. ifx>0, $\log e_x$ is real and

$\frac{d}{dx}$ (log x) =

∴ $\int \frac{1}{x} \, dx = \log x$ when $x > 0$

If x < 0, then –x > o and log (–x) is real

Also $\frac{d}{dx}$ {log (–x) = $\frac{1}{-x} \times (-1) = \frac{1}{x}$

∴ $\int \frac{1}{x} \, dx = \log |x| + c$

(| x | = x if n > 0 = – x if x < 0)

Example 2

Integrate $\int \frac{a + bx + cx^2}{x^2}$

Sol.

$$\int \frac{a + bx + cx^2}{x^2}\ \text{dx} = \int \left(\frac{a}{x^2} + \frac{bx}{x^2} + \frac{+cx^2}{x^2} \right) dx$$

$$= \int \frac{a}{x^2} dx + \int \frac{bx}{x} dx \int c\ dx$$

$$= \text{a}\ \frac{x^{-2+1}}{-2+1} + \text{b log} \mid x \mid \text{c}x\text{+ c'}$$

$$= -\ \frac{a}{x} + \text{b log} \mid \text{x} \mid + \text{c}x + \text{c'}$$

where c' is the constant of intergration.

(i) $\int e^x\ dx = e^x + c.$

(ii) $\int e^{mn}\ dx = \frac{e^{mn}}{m} + c.$

Proof : (a) $\frac{d}{dx}\ (e^x) = e^x$

$$\int e^x\ dx = e^x + c.$$

(b) $\frac{d}{dx} \left(\frac{e^{mn}}{m} \right) = \frac{1}{m}.\ \frac{d}{dx}\ (e^{mn})$

$$\int e^{mn}\ dx = \frac{e^{mn}}{m} + c$$

Example 3. Evaluate $\int (5x^4 - 3e^{3x} + e^{-x})\ dx$

Sol. $\int (5x^4 - 3e^{3x} + e^{-x})\ dx$

$= 5 \int x^4\ dx - 3 \int e^{3x}\ dx + \int e^{-x}\ dx$

$= 5.\ \frac{x^{4+1}}{4+1} - 3.\frac{e^{3x}}{3} + \frac{e^{-x}}{-1} + c$

$= x^5 - e^{3x} - e^x + c$

(iv) (a) $a^x\ dx = \frac{a^x}{\log_e^a} + c \quad (a > 0)$

(b) a^{mx} dx = $\frac{a^{mx}}{m \log_e^a}$ + c

Proof : We have

$$\frac{d}{dx}\left(\frac{a^x}{\log_e^a}\right) = \frac{1}{\log_e^a} \frac{d}{dx} (a^x) = \frac{1}{\log_e^a} \times a^x \log_e^a = a^x$$

$$\& \frac{d}{dx}\left(\frac{a^{mx}}{m\log_e^a}\right) = \left(\frac{1}{m\log_e^a}\right) \times m\, a^{nm} \log_e^a = a^{mx}$$

$\therefore$ a^x dx = $\frac{a^x}{\log_e^a}$ + c

& a^{mn} dx = $\frac{a^{mx}}{m\log_e^a}$ + c

Example 4. Evaluate

$\int (e^{3a \log x} + e^{3x \log a})$ dx

Sol. $\int (e^{3a \log x} + e^{3x \log a})$ dx

$\int (e^{3a \log x} + e^{3x \log a})$ dx

$$= \left(e^{\log a^{3a}} + e^{\log a^{3x}}\right)$$

$$= \int x^{3a} dx + \int a^{3x} dx$$

$$= \frac{x^{3a+1}}{3\log a} + \frac{a^{3x}}{3\log a} + c$$

Example 5. Evaluate $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3$ dx

Sol. $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3$ dx =

$$= \int\left[\left(\sqrt{x}\right)^3 - \left(\frac{1}{\sqrt{x}}\right)^3 - 3\sqrt{x}\frac{1}{\sqrt{x}}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right] dx$$

$$= \int\left(x^{1/2} - x^{-3/2} - 3x^{1/2} + 3x^{-1/2}\right)dx$$

$$= \frac{x^{5/2}}{5/2} - \frac{x^{1/2}}{-1/2} - 3\ \frac{x^{1/2}}{3/2} + 3\ \frac{x^{1/2}}{1/2}$$

$= \frac{2}{5} x^{½} + 3 x^{½} - 2x\, x^{½} + 6 x^{½} + c$

In the last article we integrated some simple (or standard) functions by inspection and by using the definition of integration as Ant derivative. But often the given function *f*(x) is neither in the simple form nor it can be integrated by mere inspection. In such a case, we use any one or more of the following methods to evaluate the given integral.

(i) Integration by substitution,

(ii) Integration by parts.

SELF-CHECK EXERCISE 14.1

Q1. Evaluate

(i) $\int \left(x + \frac{1}{x} \right)^3$ dx

(ii) $\int$ (2 + 3 sin x + 4e^x)dx

Q2. Evaluate

(a) x^4 (b) $4x^{-2}$ (c) $1 - 2x + x^2$ (d) $\left(x - \frac{1}{x} \right)^2$

Q3. Evaluate the following definite integrals

(a) $\int_5^6 x^4 . dx$

(b) $\int_1^2 \frac{1+x}{x^2} dx$

(c) $\int_2^4 \left(x + \frac{1}{x} \right)^2 dx$

(d) $\int_0^1 (x+1)^3 dx$

14.4 INTEGRATION BY SUBSTITUTION

Consider the integral I = ∫*f*(x) dx

and let us put x = ϕ0 (z)

Then by definition, $\frac{dI}{dx}$ - f (x) & $\frac{dx}{dz}$ ϕ' = (z)

$\therefore \quad \frac{dI}{dz} = \frac{dI}{dx} . \frac{dx}{dz} = f(x)\, \phi' (z)$

∴ By definition

$I = f(x)\,\phi'\ ()\,dz$.....

In the method of substitution, it we put $x = \phi(z)$,

then we get $\frac{dx}{dz} = \phi'(z)$,

which is usually written as $dx = \phi'(z)\,dz$....

Thus, in short. $I = \int f(x)\,dx$ (Put $x = \phi(z)$

$\therefore dx = \phi'(z)\,dz$)

$= \int f\{\phi(z)\}\,\phi'(z)\,dz.$

Some Important Integrals

(a) Prove that $\int f\ (ax+b)\,dx = \frac{1}{a}\int f(z)\,dx$

Proof: $I = \int f(ax+b)\,dx$ Put $ax + b = z$

$= \frac{1}{a} f\int(z)\,dz \quad a\,dx = dz$

$dx = \frac{dz}{a}$

Note: For integrals of the types $\int(ax + b)^n\,dx$:

$\int \sqrt[n]{ax+b}\,dx$ and $\int \frac{dx}{(a+bx)^n}$ suitable

substitution is $ax + b = z$

Example 6.

(a) Evaluate the integrals $\int \sqrt[3]{3x+5}\,dx$

Sol. $\int \sqrt[3]{3x+5}\,dx = \int \sqrt[3]{z}\,\frac{1}{3}\,dz$... (put $3x + 5 = z$)

$= \frac{1}{3}\int z^{1/3}\,dz$... $\therefore 3dx = dz$

$= \frac{1}{3}\frac{z^{1/3+1}}{1/3+1} + c\ dx =)$

$= \frac{1}{4} z^{4/3} + c$

$= \frac{1}{4}(3x + 5)^{4/3} + c$

(b) Prove that $\int \frac{f'(x)}{f(x)}$ dx = log | f (x) | + c

Proof: $\int \frac{f'(x)}{f(x)}$ dx = $\int \frac{dz}{z}$ Put f (x) = z

= log | z | + c f' (x) dx = dz = 1

= log | f (x) | + c

Example 7. Evaluate $\int \frac{x+1}{x2+2x-3}$ dx

Sol. Let I = $\int \frac{x+1}{x2+2x-3}$ dx

Put $x^2+2x-3=t$

i.e. (2x+2) dx = dt

or 2(x + 1) dx = dt

∴ I = $\int \frac{dt}{2t} = \frac{1}{2}$ log | t | + c

= $\frac{1}{2}$ log $|x^2+2x-3|$ + c

(c) Prove that $[f(x)^n]\, f(x)$ dx = $\frac{[f(x)]^{n+1}}{n+1}$ + c

when n≠–1 (Put$f$ (x)=z then

Proof:$\int [f(x)^n]\, f'(x)$ dx = $\int z^n$ dz

f(x) dx = dz)

= $\frac{z^{n+1}}{n+1}$

= $\frac{[f(x)]^{n+1}}{n+1}$ + c, where n ≠ –1 and c is asbitrary constant

Ifn= –1, the integral becomes $\int \{f(x)\}^{-1} f(x)$ dx

= $\int \frac{f'(x)}{f(x)}$ dx

which is the same as the integral discussed in part (b)

Example 8. Evaluate $\int \frac{x-2}{\sqrt[3]{x^2-4x+5}}$ dx

Sol. Let I = $\int \frac{x-2}{\sqrt[3]{x^2-4x+5}}$ dx

Put $x^2 - 4x + 5 = z$

$(2x - 4)\, dx = dz$

$(x - 2)\, dx = \frac{dz}{2}$

$= \int \frac{\frac{1}{2} dz}{(z)^{1/3}} = \frac{1}{2} \int z^{-1/3}$ dz...

$= \frac{1}{2} \frac{z^{-1/3+1}}{-1/3+1}$ z + c

$= \frac{3}{4}$ $z^{2/3 + c}$

$= \frac{3}{4} (x^2 - 4x + 5)^{2/3} + c.$

Some Special Integrals

We shall give below the formulae for some special integrals without profit. These formulae will be used in finding the integrals of many functions.

(a) $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$;

(b) $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$;

(c) $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log \left|x + \sqrt{x^2 \pm a^2}\right| + c$

Example 9. Integrate the following functions

(i) $\frac{1}{\sqrt{1+4x^2}}$ (ii) $\frac{1}{1-9x^2}$

(iii) $\frac{dx}{x^2-16}$

Sol. (i) Let I = $\frac{1}{\sqrt{1+4x^2}}$ dx

Put 2x = t, $\frac{dt}{2}$ = dx

$\therefore$ $1 = \frac{1}{2}\int\frac{1}{1+t^2}\,dt = \frac{1}{2}\log\left|t+\sqrt{t^2+1}\right| + c$

$= \frac{1}{2}\log + \left|2x+\sqrt{1+4x^2}\right|\ c$

(ii) Let I = $\int\frac{dx}{1-9x^2}$

Put 3x = t, $\frac{dt}{3}$ dx

$\therefore$ $1 = \frac{1}{3} = \frac{dt}{1-t^2} = \frac{1}{3}\ \frac{1}{2\times 1}\ \log\ \frac{1+t}{1-t} + c$

$= \frac{1}{6}\ \log\left|\frac{1+3x}{1-3x}\right| + c$

(iii) Let I = $\int\frac{dx}{x^2-16}$

$= \frac{dx}{x^2-4^2} = \frac{1}{2.4}\ \log\left|\frac{x-4}{x+4}\right|$

$= \frac{1}{8}\ \log\left|\frac{x-4}{x+4}\right| + c$

Self-check Exercise 14.2

Q1. Evaluate the integrals $\int\sqrt[3]{3x+5}$ dx

Q2. Evaluate $\int\frac{x+1}{x2+2x-3}$ dx

14.5 INTEGRATION BY PARTS

If u and v be two function x such that u is differentiable and v is in terrible, than u (x) v (x) dx = differentiable and v is integrable, than ∫u (x) v(x) dx=u (x) ∫v (x) dx –∫ [u' (x) ∫v (x) dx] dx

= first function × integral of second function – integral of (derivative of first × integral of second)

Example 10. Evaluate each of the following integrals

(i) $\int x^3 e^x\, dx$ (ii) $x^2 \log_e^x\, dx$

Sol. (i) Let $\int I = x^3 e^x\, dx$

Integrating by parts x^3 as first function

$$I = x^3 e^x - 3\int x^2 e^x\, dx$$
$$= x^3 e^x - 3\,[x^2 e^x - \int 2\, x\, e^x\, dx]$$
$$= x^3 e^x - 3\,[x^2 e^x + 6\int x\, e^x\, dx$$
$$= x^3 e^x - 3x^2 e^x + 6\,[x\, e^x - \int 1.e^x\, dx]$$
$$= x^3 e^x - 3x^2 e^x + 6x\, e^x - 6\, e^x + c$$
$$= (x^3 - 3x^2 + 6x - 6)\, e^x + c$$

(ii) Let $I = \int x^2 \log_e x\, dx$

Taking $\log_e x$ as the first function & x^2 as the second function

$$= \log_e x \int x^2\, dx - \int \left[\frac{d}{dx}(\log x).\int x^2\, dx\right]$$
$$= \log_e x - \int \left(\frac{1}{x}\,\frac{x^3}{3}\right) dx + c$$
$$= \frac{x^3}{3}\log_e x - \frac{1}{3}\int x^2\, dx + c$$
$$= \frac{x^3}{3}\log_e x - \frac{1}{3} + c$$
$$= \frac{x^3}{3}\left(\log_e x - \frac{1}{3}\right) + c$$

Note: If we take 1st function$=x^2$ and 2nd function $=\log_e x$, then it is not possible to find the integral by using the formula for integration by parts.

SELF-CHECK EXERCISE 14.3

Q1. Evaluate each of the following integrals

(i) $\int x^3 e^x\, dx$ (ii) $x^2 \log_e^x\, dx$

14.6 DEFINITE INTEGRAL

We have defined integration as the inverse of differentiation. Now we shall define integration as a process of summation or definite integral as the limit of a sum. We shall also define definite integral as an area

14.6.1 Definite Integral as the limit of a sum

Let f(x) be a single-valued continuous function defined in a chosen interval a≤x≤b, a and be being both finite. Let us divide the interval a ≤x≤b into n equal sub-intervals, each length h, by the points

a+h, a + 2h, ----, a + rh, ---, a + (n – 1) h

so that hn – b – a.

Now we are form the sum

hf(a) +hf (a + h) + h f (a + 2h) +----+ hf (a + rh)

+ ----+h f {a + (n – 1) h}

$= h \sum_{r=0}^{n-1} f$ (a+ rh), where a + nh = b or nh = b – a.

Then the limit $\overset{lt}{h \to 0} \sum_{r=0}^{n-1} f$ (a+rh), if it exists, is called the definite integral of f (x) w.r.t. x between the limits a and b and we denote it by the symbol

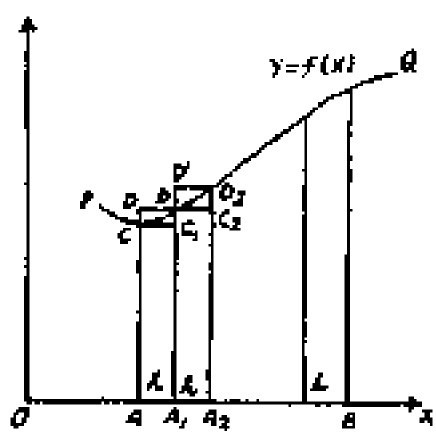
$\int_a^b f(x)\,dx$

Thus f(x) dx = $\overset{lt}{h \to 0} \sum_{r=0}^{n-1} f$ (a + rh), where n h = b – a, a and be being the limits of integration.

= b – a, a and be being the limits of integration.

14.6.2 Definite Integral as Area

Let y=f(x) by a monotonic increasing continuous function of x in the interval a ≤x≤b, and b being both finite and b>a. Let PQ be the continuous curve for y=f(x)



Let AC & BE be the ordinates at the points x = a and x = b respect. Then OA = a and OB = b so that AB = b – a. Let us divide AB (i.e. the interval a ≤x ≤b) into n equal parts each of length h so that n h=b – a, or b=a+nh.

Let us draw ordinates $A_1 D_1$, $A_2 D_2$, - - - at the points x = a+h, a + 2h, - - - -.

Let S denote the area enclosed by the curve y=f(x), the x–axis and the two ordinates at x=a and x= b.

Ifs$_1$ be the sum of the inner rectangles ACC_1A_1,

$A_1D_1C_2A_2$,------ then clearly S_1<S.

(1)

S =hf(a) + hf(a + h) +----- hf(a + n – 1) h

$$= h\sum_{r=0}^{n-1} f\,(a+rh)$$

If S_2 be the sum of the outer rectangles ADD_1,A_1,

$A_1D'D_2A_2$---- than S_2> S,

(2)

and S_2 = hf(a+h)+hf(a+2h) +---+hf(a +nh)

$$= h \sum_{r=1}^{n}(f(a + rh)$$

$$= h \sum_{r=0}^{n-1} f \ (a + rh) - hf(a) \qquad (\because a + nh = b)$$

From (1) and (2), we have S_1<S<S_2.

(3)

Now let n→∞ i.e., the no. of sub-divisions of AB increase indefinitely, then the length h of each subdivision →0, so that h f(a)→0, h f(b) →0 and

$$S_1 \rightarrow \overset{lt}{h \rightarrow 0} \sum_{r=0}^{n-1} f\,(a + rh) = \int_a^b f(x)dx$$

$$S_2 \rightarrow \overset{lt}{h \rightarrow 0} \sum_{r=0}^{n-1} f\,(a + rh) = \int_a^b f(x)dx$$

Hence from (3), we have

S = f(x) dx = dx.

If (x) be a monotone decreasing continuous function of x in a <u>≤</u>x<u>≤</u>b, then we can first prove that S_1>S>S_2 and then as before we can show that

$$S = \int_a^b f(x)\ dx$$

Thus the definite integral $\int_a^b f(x)\, dx$ geometrically a represent the area bounded by the curve y = f (x), x–axis and the two ordinates at x=a and x = b.

Example 11. Evaluate each of the following integrals

(i) $\int_4^5 e^x\, dx$ (ii) $\int_2^3 \frac{dx}{x^2-1}$ (iii) $\int_{-5}^5 x\, dx$

Sol. (i) $\int_4^5 e^x\, dx = e^x \int_4^5 e^5 - x^4$

(ii) $\int_2^3 \frac{dx}{x^2-1} = \int_2^3 \frac{dx}{x^2-1^2} = \left[\frac{1}{2\times 1}\log\left|\frac{x-1}{x+1}\right|\right]_2^3$

$= \frac{1}{2}\left(\log\frac{2}{4} - \log\frac{1}{3}\right)$

$= \frac{1}{2}\left(\log\frac{2}{4} - \log\frac{1}{3}\right)$

$= \frac{1}{2}\,\frac{\frac{1}{2}}{1/3} = \frac{1}{2}\log\frac{3}{2}$

(iii) $\int_{-5}^5 x\, dx = \left|\frac{x^2}{2}\right|_{-5}^{+5} = \frac{1}{2}[25-(+25)] = 0$

14.6.3 Transformation of Definite Integrals by Substitution

When definite integral is to be found by substitution then lower and upper limits of integration is changed. If the substitution is t = ϕ (x) and lower limit of integration is a and upper limit is b then new lower and upper limits will be ϕ (a) and ϕ (b) respectively.

Example 12. Evaluate the following definite integrals,

(i) $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$ (ii) $\int_1^2 3x\sqrt{5-x^2}\, dx$

(iii) $\int_0^1 \frac{x^5}{1+x^6}\, dx$

Sol. (i) Let I = $\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

Let $\sqrt{x} = t, \frac{1}{2}\frac{1}{\sqrt{x}}\, dx = dt$

$\therefore$ $1 = \int_0^1 e^1\ 2\, dt = 2\, e^1 \int_0^1 = 2\,(e^1 - e^0)$

$= 2\ (e - 1)$

(ii) Let I $= \int_1^2 3x\sqrt{5 - x^2}\ dx$

Let $5 - x^2 = t$

$-2x\ dx = dt$

$x\ dx = -\frac{1}{2}\ dt$

when x = 1, t = 5 – 1 = 4 ; when x = 2, t = 5 – 4 = 1

$\therefore\ I = \int_1^2 3x\sqrt{5 - x^2}\ dx = \int_4^1 -\frac{3}{2}\sqrt{t}\ dt$

$= \frac{3}{2}\left.\frac{t^{1/2+1}}{1/2+1}\right]_4^1 = -\frac{3}{2} \times \frac{2}{3}\ t^{3/2}$

$= (1 - 4^{3/2}) = -(1 - 8) = 7$

(iii) Let I $= \int^1 \frac{x^5}{1 + x^6}$

$= \frac{1}{6}\int_0^1 \frac{x^5}{1 + x^6}\ dx$

$= \frac{1}{6}\log(1 + x^6) = \frac{1}{6}\ \log 2 - \log 1)$

$= \frac{1}{6}\ \log 2.$

SELF-CHECK EXERCISE 14.4

Q1. Evaluate $\int_\partial^1 \sqrt{x + x^2}\ dx$

Q2. Evaluate the following integrals

(i) $\int \frac{1+x}{(2+x)^2} e^x\ dx$

(ii) $\int \frac{\sqrt{1+\sin x}}{1+\cos x} e^{-x/2}\ dx$

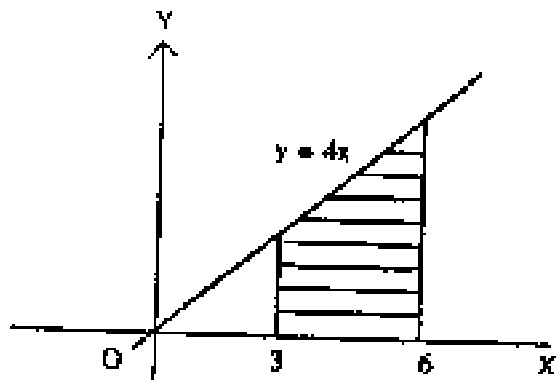
14.7 AREA UNDER THE CURVE

If f (x) be finite and continuous in a≤x≤b. Then area of the region bounded by x–area, y = f (x)and the ordinates at x=a and x = b is equal to $\int_4^1 f\ (x)\ dx$

Example 13:

Find by integration the area bounded by the straight limes: y = 4x, y=0,x=3,x=6

Sol. y = 0 is the x–axis A rough sketch of the graph of the function y = 4x is shown is figure. We have to find the area of the region bounded by the time y = 4x, the x–axis (y = 0) and the two ordinates x=3 and x=3 and x=6 and this is shaded in the figure



Hence the required are of the shaded region.

$$= \int_a^b y\,dx$$

$$= \int_3^6 4x\,dx$$

$$= 4\left|\frac{x^2}{2}\right|_3^6 = 2\,(39 - 9) =$$

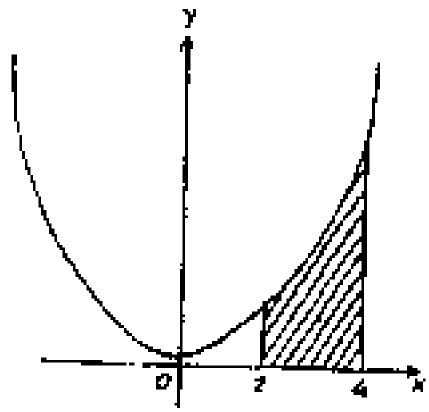
= 2 × 27=54 sq. units.

Example 14

Draw a rough sketch of the curve $y=x^2$ and find the area by the curve, the x–axis and ordinates x = 2 and x = 4.

Sol. The equation of the curve is

$y = x^2$



If x = 0, y = 0, i.e. the curve passing through the origin. If x is replaced by–x, equation (1) remains unaltered. Therefore the curve is symmetrical about the y–axis.

Differentiating (1) w.r.t x, we get = $\frac{dy}{dx}$ 2x and $\frac{d^2y}{dx^2}$=2

$\therefore$ $\frac{dy}{dx}$ > 0 for all x > 0 and $\frac{d^2y}{dx^2}$ and < 0 for all x < 0

$\therefore$The curve is increasing for x>0 & decreasing for x <0

As x=0, $\frac{dy}{dx}$ =0 and $\frac{d^2y}{dx^2}$<0. Therefore x = 0 is a point of local minimum & the minimum value is 0. We find some points on the curve from equation (1)

x	0	1	2	3	4	–1	–2	–3
y	0	1	4	9	16	1	4	9

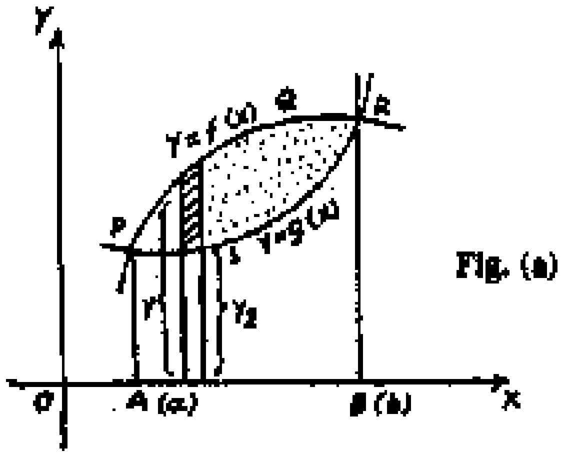
With those ideas we can sketch the curve.The region bounded by the parabola y = x^2, the x–axis and the two ordinates x=2 and x = 4 is shaded Hence the required area = $\int_2^4 y\,dx$ = $\int_2^4 x^2\,dx$

$$= \left|\frac{x^3}{3}\right|_2^4 = \frac{1}{3}\ (64 - 8) = \frac{56}{3} \text{ sq units.}$$

Area between two given curves

Let y = f(x) and y = g(x) be two given curves.

(i) Suppose the two given curves intersect at two points and x = a, x = b are the ordinates of these two points (fig. a) y =f(x) represents, the curve PQR andy=g(x) represents, the curve PSR.



Then the regd. Area between the curves

= Area PSRQP

= are A PQRS – area APSRB

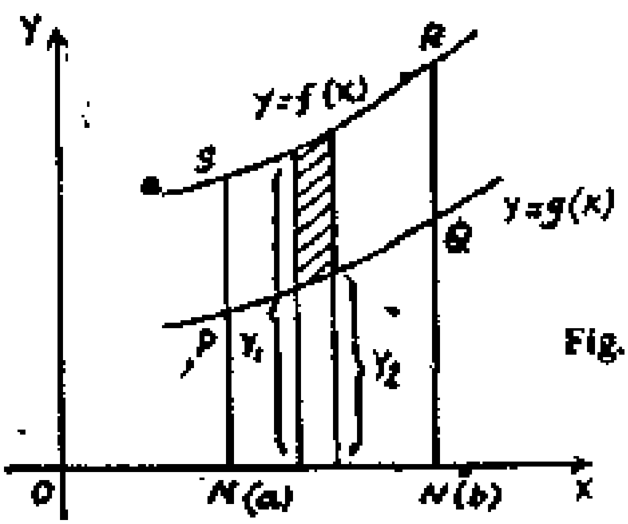
= $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

= $\int_a^b [f(x) - g(x)]\,dx$

= $\int_a^b (y^1 - y^2)\,dx$

where $y_1 = f(x)$ and $y_2 = g(x)$ at the same abscissa x.

(ii) Suppose along with the two curves two co-ordinates, say x=a and x = b are also given (see fig. b). Then the required area bounded by the two given 'corves $y = f(x)$ and $y = g(x)$ and the two given ordinates x = a and x = b



= area PQRSP = area MSRN – area

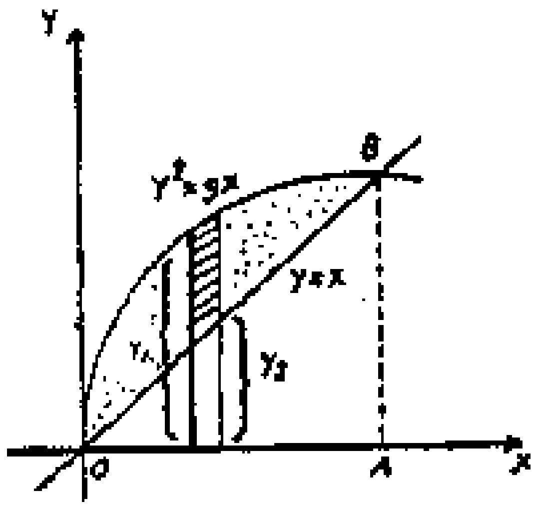
= $\int_a^b f(x)\, dx$ - $\int_a^b g(x)\, dx$

= $\int_a^b (y_1 - y_2)\, dx$

where $y_1 = f(x)$ and y_2 g (x) at the same abscissa x.

Example 15.

Shade the area bounded by $y^2 = 8x$ and y = x along positive direction of x–axis and use integration to find the area of the part.



Sol. We have $y^2 = 8x$ ………(1)

and y = x ………..(2)

Putting y = x in (1), we get

$x^2=8x$ or (x – 8)= 0

i.e. x = 0, 8.

from (2) y = 0, 8.

∴The curves (1) & (2) intersect at (0, 0) and (8, 8). The area bounded by (1) and (2) has been shaded by dots. The dotted region is the required area between (1) and (2) in which x varies – from 0 to 8. the required area between (1) and (2)

= area OABCO – area OAB

= $\int_0^8 y_1\, dx - \int_0^8 y_2\, dx$ where $y^2 = 8x$

= $\int_0^8 \sqrt{8x}\, dx - \int_0^8 x\, dx$ and $y^2 = x$

$$= \sqrt{8}\left|\frac{x^{3/2}}{3_2}\right|_0^8 - \left|\frac{x^2}{2}\right|_0^8$$

$$= \frac{32}{3} \text{ sq. units.}$$

SELF-CHECK EXERCISE 14.5

Q1. Find by integration the area bounded by the straight limes: y = 4x, y=0,x=3,x=6

Q2. Draw a rough sketch of the curve $y=x^2$ and find the area by the curve, the x–axis and ordinates x = 2 and x = 4.

14.8 SUMMARY

In this Unit, we have exposed to varies methods of integration. In the first section we learnt about the Definite and Indefinite Integrals. In the first part of first section we learnt about the general rules of integration. In the second section we learnt about the method of integration by substitution and method of integration by parts to evaluate integration where the given function is not simple. In the last part of the unit, we studied about the Definite Integral as area.

14.9 GLOSSARY

1. General Rules of Integration

(i) $\int \{u(x) + v(x)\}\, dx = \int u(x)dx + \int v(x)\, dx$

(ii) $\int Kf(x)dx = K\int f(x)\, dx$ where R is a real number

2. **Definite Integral :** Integration as a process of summation or definite integral as the limit of a sum.

3. **Transformation of definite integrals by substitution :** When definite integral is to be found by substitution then lower and upper limits of integration is changed.

14.10 ANSWER TO SELF-CHECK EXERCISES

Self-check Exercise 14.1

Ans. Q1. We know that $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

Therefore $\int \left(x + \frac{1}{x}\right)^3 dx = \int \left(x3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) dx$

$= x^3 dx + 3 \int x dx + 3\int \frac{dx}{x} + \frac{dx}{x^3}$ ----------- Rule 2

using integral formulas we have

$$\int\left(x+\frac{1}{x}\right)^3 dx = \left(\frac{x^4}{4}+4\right) + 3\left(\frac{x^2}{2}+c_2\right) + 3\ (|4| \times 1+c_3) + \left(\frac{x^{-2}}{-2}+c_4\right)$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\ln|x| - \frac{1}{2x^2} + (c_1 + 3c_2 + 3c_3 + c_4)$$

$$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + 3\ln|x| - \frac{1}{2x^2} + c$$

(ii) This integral can be written as

$$2\int dx + 3\int \sin x\, dx + 4\int e^x\, dx$$

$$= 2x - 3\cos x + 4e^x + c$$

Ans. Q2. (a) $\frac{x^5}{5} + c$ (b) $-\frac{4}{x} + c$

(c) $x - x^2 + \frac{x^3}{3} + c$ (d) $\frac{x^3}{3} - 2x - \frac{1}{x} + c$

Ans. Q3. (a) $\frac{6^5}{5} - 5^4$ (b) $\frac{1}{2} + \ln 2$

(c) $\frac{2+5}{12}$ (d) $\frac{15}{4}$

Self-check Exercise 14.2

Ans. Q1. Refer to Section 14.4 (Example 6)

Ans. Q2. Refer to Section 14.4 (Example 7)

Self-check Exercise 14.3

Ans. Q1. Refer to Section 14.5 (Example 10)

Self-check Exercise 14.4

Ans. Q1. Now $\int_{\partial}^{1} \sqrt{x+x^2}\ dx$

$$= \int_{\partial}^{1} \sqrt{(x+1/2)^2 - 1/4}\ dx$$

let $x + \frac{1}{2} = u_1$ Then

$$\int_{\partial}^{1} \sqrt{x+x^2}\ dx = \int_{1/2}^{3/2} \sqrt{u2+1/4}$$

$$= \left[\left\{\frac{1}{2}u\sqrt{u^2-1/4}-\frac{1}{8}\ln\frac{u+\sqrt{u^2-1/4}}{1/2}\right\}\right]_{1/2}^{3/2}$$

$$= \frac{3\sqrt{2}}{4} - \frac{1}{8}\ln(3+2\sqrt{2})$$

Ans. 2 (i) $\int \frac{1+x}{(2+x)^2} e^x \, dx$

$$= - \int \frac{(2+x)-1}{(2+x)^2} e^x \, dx$$

$$= \int \left[\frac{1}{2+x} + \frac{-1}{(2+x)^2}\right] e^x \, dx$$

$$= \frac{1}{2+x} e^x \, dx \text{ , since } \frac{-1}{(2+x)^2} = \frac{d}{dx}\left(\frac{1}{2+x}\right)$$

(ii) $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} \, dx$

$$= \int \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{2\cos^2\frac{x}{2}} e^{-x/2} \, dx$$

$$= \frac{1}{2}\int \sec\frac{x}{2} e^{-x/2} \, dx \; - \frac{1}{2}\int \tan\frac{x}{2}\sec\frac{x}{2} e^{-x/2} \, dx$$

Now

$$\int \sec\frac{x}{2} e^{-x/2} \, dx = \left(\sec\frac{x}{2}\right)(-2e^{-x/2}) - \int\left(\frac{1}{2}\sec\frac{x}{2}\tan\frac{x}{2}\right)(-2e^{-x/2}) \, dx$$

$$= -2 \sec\frac{x}{2} e^{-x/2} + \int \sec\frac{x}{2}\tan\frac{x}{2} e^{-x/2} \, dx$$

Thus $\int \frac{\sqrt{1-\sin x}}{1+\cos x} e^{-x/2} \, dx$

$$= - \sec\frac{x}{2} e^{-x/2} + \frac{1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2} dx \frac{-1}{2}\int \sec\frac{x}{2}\tan\frac{x}{2} e^{-x/2} \, dx$$

$$= - \sec\frac{x}{2} e^{-x/2} + c$$

Self-check Exercise 14.5

Ans. Q1. Refers to Section 14.7 (Example 13)

Ans. Q2. Refers to Section 14.7 (Example 14)

14.11 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
5. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

14.12 TERMINAL QUESTIONS

Q1. Integrate the following functions

(i) $\dfrac{1}{1+\dfrac{x^2}{4}}$ (ii) $\dfrac{1}{\sqrt{4+x^2}}$ (iii) $\dfrac{1}{\sqrt{x^2-64}}$

Q2. Evaluate each of the following integrals

(i) $(1 - x^2) \log x \, dx$ (ii) $\int x^4 (\log_e x)^2 \, dx$

Q3. Evaluate each of the following integrals.

(i) $\int_0^4 \left(t^2 + 1\right)$ (ii) $\int_0^2 \dfrac{x+2}{x+1}$ (iii) $\int_a^b \dfrac{1}{x} \, dx$

Q4. Find the integration the area of the circle $x^2 + y^2 = a^2$.

Q5. Find the area of the portion bounded by $y^2 = 4x$ and the latus rectum.

Q6. Shade the area enclosed by the two parabolas $y^2=4x$ and $x^2=4y$ and find the integration, the area of the shaded region.

Unit - 15

ECONOMIC APPLICATIONS OF INTEGRATIONS

STRUCTURE

15.1 INTRODUCTION

In the last unit, we have learnt about the different methods of integration. In this present unit we will study about how integration is used to solve problems related to economic theory.

15.2 LEARNING OBJECTIVES

After going through this unit, you should be able to:

- Indentify the dynamics problem in economics
- Use the mathematical tools of integration to solve problems related to economic theory.

Integrals are used in economic analysis in various ways. Few simple applications are:

15.3 FROM A MARGINAL FUNCTION TO A TOTAL FUNCTION

In non-mathematical economics courses a great deal of time is spent in showing that the area under a marginal curve $f(x)$ between zero and some point x=a>0 is the total cost at the point. Thus the area under the marginal cost curve is total cost and the areas under the marginal revenue curve is total revenue. The analytical reasons for this result is apparent. We know that the total cost is assumed to vary with output, so that total cost (TC) may be written as

$$TC - C(q)$$

where C is total cost and q output then the marginal cost (MC) is given by

$$MC=\frac{d}{dp}(TC) = C'(q) = c(q)$$

IF we begin with the marginal cost function then the equation of the total cost function is obtained from its indefinite integral.

$$TC= \int c(q)\, dq=C(q) + k$$

where the arbitary constant k is of course fixed cost. The total variable cost of producing a particular level of output, a, is given by the definite level of output, a, is given the definite integral of the MC function between 0 and a (the sum of the marginal cost):

$$TC\ (a) = \int_0^a c(q)\, dq = \left|C(q)\right|_0^a$$

We observe that we have just been doing is known by the rather forbidding name of solvingsimple differential equations.

In a nut shell, from a given total function (e.g. a total cost function), the process of differentiation can yield the marginal function (e.g. the marginal cost function). Being the opposite of differentiation, the process g integration enables us, conversely, to infer the total function. We can also determine the average cost (AC) which will be equal to total cost divided by total output i.e. $AC=\frac{c}{q}$

Example I:- If the marginal cost (MC) of a firmis the following function of output, $C'(Q)=2e^{0.2Q}$, and if the fixed cost is $C_F = 90$, find the total-cost function C(Q).

Sol. Marginal cost function $OC'(Q) = 2e^{0.2Q}$ on integrating (MC) w.r.t. Q_1 we get

$$\int C'(Q)\, dQ = \int 2e^{0.2Q}\, dQ$$

$$= 2\frac{e^{0.2Q}}{2} + c$$

where c is constant of integration.

i.e. C(Q) $= 10\ e^{0.2Q} + c$

when Q = 0 total cost C(Q) will consist solely of fixed cost C_F. So

90 = 10 + c i.e. C = 80

Hence the total cost function (TC) is

C(Q) $10\ e^{0.2Q}+80$

Example 2:- The marginal cost and revenue of a firm are given as MC=4+ .08x, MR = 12. Compute the total profit, when x = 100? Given that total cost at zero output is equal to zero.

Sol. Marginal cost function C' (x)=4+0.8x. On integrating MC w.r.tx, we get

C(x)= $\int c'(x)\, dx = \int (4 + .08x)\, dx$

$=4x+.08\frac{x^2}{2}+k$

When x = 0, C= 0, $\therefore$ k = 0

$\therefore$ C=4x+.04x²

Also given MR = 12, Total revenue TR =pq=12x

Profit=12x – (4x+.04)x^2

[$\therefore$ Profit=TR (q) – C(q)]

Atx=100

Profit 12 × 100 – (4 × 100+.04 × 100²)

= 1200 – 400 – 400 = 400

So at x = 100, there is a profit of Rs.400/-

Example 3:- If the marginal revenue function (MR) is 8 – 8p –3q^2, determine the revenue and demandfunctions.

Sol. MR = 8 – 8p – 3q^2

Total revenue (TR) = $\int (8 - 8p - 3p^2)\, dq + C$

$= 8q - \frac{8}{2}q^2 - 3\,{q^3}\!/\!{3} + C$

$8q - 4q^2 - q^3 - q^3 + C$

When q = 0, TR = 0, $\therefore$ C = 0

$\therefore$ TR = $8_q - 4q^2 - q^3$

As demand function p=$\frac{R}{q}$

$$= \frac{8q - 4q^2 - q^3}{q}$$

$8q - 4q - q^2$

Example 4:- If MR = 16 – x^2, find the maxum total revenue. Also find the total and average revenue and demand.

Sol. Given MR = 16 – x^2

We know that TR is maximum when MR =0,

i.e. 16 – x^2=0 i.e. x = $\pm$4.

Hence the total revenue is maximum when output is 4 units. We shall find the maximum total revenue which happens when output is 4 units.

$$\text{TR} = \text{R} = \int_0^4 MR\ dx = \int_0^4 (16 - x^2)\ dx$$

$$= \left|16x - \frac{x^3}{3}\right|_0^4 = \frac{128}{3}$$

(iii) Hence $\frac{128}{3}$ is the maximum total revenues. Total revenues TR is $\int (16 - x^2)\,dx$

$$= 16x - \frac{x^3}{3} + \text{c}$$

When x = 0, revenue must also be zero, $\therefore$ C=0

$\therefore$ TR= $16x - \frac{x^3}{3}$

(iii) Average revenue = $\frac{\text{Total revenue}}{\text{output}} = \frac{R}{x} = \frac{16x - x^3/3}{x}$

$$= 16x - \frac{x^3}{3}$$

Since AR=p, naturally p= = $16x - \frac{x^3}{3}$ is the requireddemand function.

Example 5:- If the marginal propensityto consume function is given as follows.

$$\frac{dc}{dy} = 0.5 - .001y$$

where c is consumption and y is disposable income. Find the total consumption if when income is zero c is 0.2.

Sol.

Consumption function $C = \int \frac{dc}{dy} dy$

$$= \int (0.5 - .001y)$$

$$= .5y - \frac{.001}{2} y^2 + k$$

Aty=0, C=0.2

$\therefore$ k=0.2

$\therefore C=0.2+.5y - .0005y^2$

Note:- C = 0.2 when y = 0 may be termed and subsistence or survival consumption level.

Example 6:- If marginal propersity to save is given to be $0.5+0.2y^2$ (y is income). Find consumption function if consumption is R 50.001 when income is 200.

Sol.

Let s depict total saving, then

$$MPS = \frac{ds}{dy} = 0.5 + 0.2y^2$$

$$\therefore S = \int (0.5 - 0.2y^2) dy$$

$$= 0.5y - 0.2y^1 + k$$

Consumption (=c)=y – S

$$= y - (0.5y - 0.2y^1 + k)$$

$$= 0.5y + \frac{0.2}{y} - A$$

If income (=y)=200, consumption is 50.001

$$\text{ie } 50.001 = 5\times200 + \frac{0.2}{200} - A$$

$$= 100.0 + \frac{1}{1000} - A$$

$$= 100.001 - A$$

$\therefore A = 50$

Hence, the regd. consumption function $C=-50+.5y + \frac{0.2}{y}$

SELF-CHECK EXERCISE 15.1

Q1. The marginal cost and revenue of a firm are given as MC = 2 + .04x, MR = 10. Compute the total profit , when x = 100? Given that total cost at zero output is equal to zero.

Q2. If the marginal propensity to consume function is given as follows.

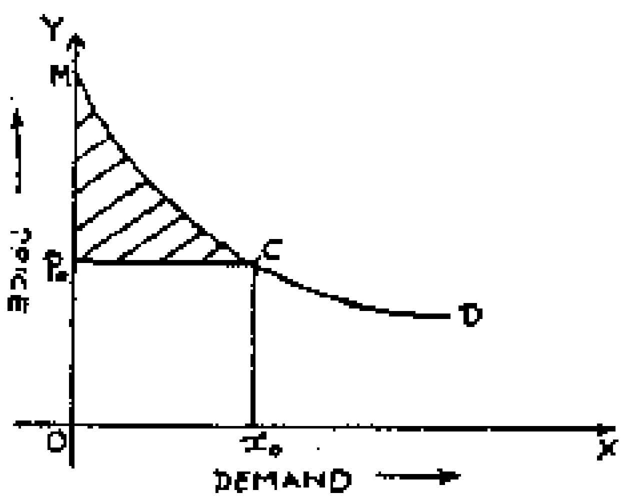
$$\frac{dc}{dy} = 0.5 - .001y$$

where c is consumption and y is disposable income. Find the total consumption if when income is zero c is 0.2.

15.4 CONSUMER'S SURPLUS

The demand curve records for each level of output the maximum price a consumer will pay (rather than go without it). To sum up, any given level of output, thus measures in rupees the total satisfaction he derives from consuming that much of output. Subtracting from this the amount actually paid (in rupees) and the remainder measure the consumer's surplus (C.S)

Consumer surplus = total area of the curve below the demand function from 0 to x_0 minus the area of the rectangle OX_0CP_0.



(i.∴ e) MCD is a demand curve, at price p_0 an amount $0.x = p_0C$ is purchased at a total price of $0x_0\ cp_0$). The area Mp_0 is the consumer's surplus. Its algebraic expression is

$$\text{consumer surplus} = \int_0^x p_d(x)\,dx - p.x$$

where p_d (x) is the demand curve.

Example 7:- If the demand curve is $p=85 - 4x - x^2$, where p and x are respectively the price and the amount demanded of a commodity, what will be the consumers surplus (a: if x_0 =5& (b) if p_0 = 64.

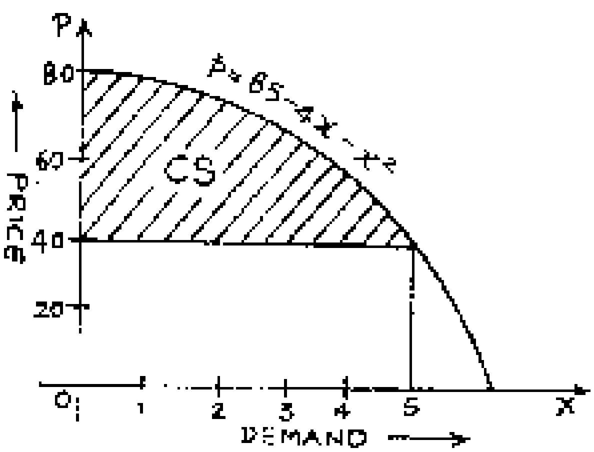
Sol.

(a) If $x_0=5$, $p=85 - 4\times5 - 25=40$

$$\therefore \text{ Consumer Surplus } = \int_0^5 (85-4x-x^2)\,dx-(40\times5)$$

$$=\left|85x - \frac{4x^2}{2} - \frac{x^2}{2}\right|_0^5$$

= 133.33 units



(b) If p_0= 64, then $64 = 85 - 4x - x^2$

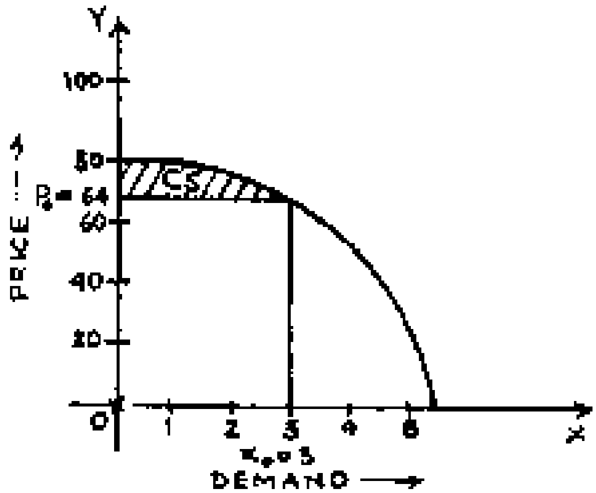
ie$x_0=3$,$x_0= -7$(which has no meaning in demand)

Consumer Surplus

$$= \int_0^3 (85-4x-x^2)\,dx-(64\times3)$$

$$=\left|85x - 2x^2 - \frac{x^3}{3}\right|_0^3 - 192$$

= 36



SELF-CHECK EXERCISE 15.2

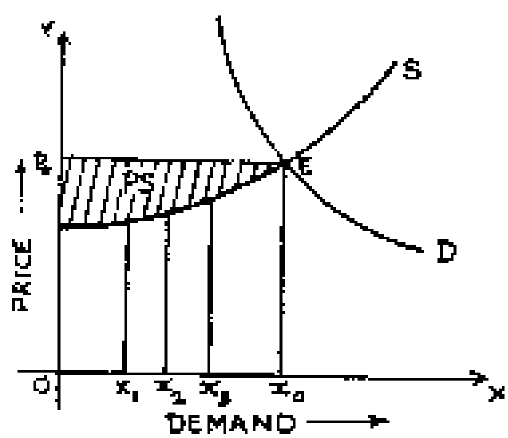
Q1. Support the demand function of a consumer is given by p = 80 – q. If the price offered is p = 60, find the consumer surplus.

15.5 PRODUCER'S SURPLUS

With the given supply function, the producer would have supplied x_1, x_2, x_3.....quantities on different prices less than p_0. At p_0, he supplies all these quantities. Hence the shaded area becomes producer's surplus (P.S) Producer's Surplus = Area of the whole rectangle p_0 Ex_0 0–area of the curve under the supply curve from 0 to x_0.

$$=px_0 - \int_0^{x0} p_s(x)\, dx$$

where p_s (x) is the supply curve and x_0 is the equilibrium output.



Example 8:-Find the producer's surplus when $P_d = 3x^2 - 20+5$

$P_b = 15+9x$ (x is he quantity)

Sol. In equilibrium

Quantity Demand = Quantity Supplied

ie. $3x^2 - 20x+5=15+9x$

or $3x^2 - 29x - 10=0$

or $3x^2 - 30x+x -10=0$

or $3x(x - 10)+(x - 10)=0$

or $(3x+1)(x - 10)=0$

$\therefore$ x 10, or $x = -1/3$ neglected

Atx=10, the equilibrium price is 105.

$\therefore$Producers surplus = total revenues–total supply price

$$=p.x - \int_0^x (15+9x)\,dx$$

$$= 10 \times 105 - \int_0^{10} (15+9x)\,dx$$

$$= 1050 - \left| 15x + \frac{9x^2}{2} \right|_0^{10}$$

= 450

Example 9:- Let p be the price of rice, q the quantity of rice and s the amount of fertilizer used in rice production. Using data for India for 1949-1964: we find for the per capita demand function for rice p =0.964 – 6.773p and for the supply function q = 0.063 +0.0365.

(i) Find the equilibrium in the rice market if

S=0.5

(ii) Find the consumer surplus

Sol.

(i) The demand function for rice in

p = 0.964 0 6.773 q

the supply function for rice is

q=0.063 +0.0368

(ii)

For equilibrium, quantity demanded= quantity supplied

$\therefore$ From the equations (i) & (ii) on eliminating q, wehave

p=0.964 – 6.773 (0.063 +0.0365)

For S=0.5

p=0.964 – 6.773 (0.063 +0.365 × 0.5)= 0.415

$\therefore$ q=0.063+0.036 +0.5= 0.081

$\therefore$ $p_0$0.415, q=0.081 are the equilibrium prices and quantity exchanged.

(ii) Consumer's Surplus = $\int_0^{0.081} pdq - p_0 q_0$

$$\int_0^{0.081} (0.964 - 6.773q)\, dq - 0.415 \times 0.081$$

$$= \left| 0.964q - \frac{6.6773}{2} q^2 \right|_0^{0.081} 0.633615$$

= .0222501635

SELF-CHECK EXERCISE 15.3

Q1. Find the Producer's surplus when $p_d = 3x^2 - 20 + 5$, $P_b = 15 + 9x$

15.6 INVESTMENT AND CAPITAL FORMATION

Capital formation is the process of adding to a given stock of capital. Regarding this process is continuous over time, we may express capital stock as a function time, k(t) and the derivative $\frac{dk}{dt}$ denote the rate of capital formation. But the rate of capital formation at time t is identical with the rate of net-investment flow at time t, denoted by I(t). Thus, capital stock k and net investment I are related by the following two equations.

$$\frac{dk}{dt} = \text{I(t)}$$

and K(t) = $\int I(t)\, dt$

$$= \int \frac{dk}{dt}$$

$$= \int dk$$

The first equation above is an identity, it shows the synonymity between net increment and the increment of capital. Since I(t) is the derivative of k(t), it stands to reason that k(t) will be the integral of (t).

Example 10:- The investment flow is described bythe equation $I(t)=3t^{1/2}$ and that the initial capital stock at time t=0, is k (0). What is the time path of capital k?

Sol.

$I(t)=3t^{1/2}$

$k(t) = \int I(t)\, dt$

$= \int 3t^{1/2}\, dt = 2t^{4/2} + c$

At t = 0, k(t) = k(0)

$\therefore$ K (0) = C

$\therefore$ $k(t) = 3t^{3/2} + k(0)$

$\therefore k(t)=3t^{3/2}+k(0)$ is the time path of capital k. The concept of the definite integral will enter into the picture when one desires to find the amount of capital formation during some interval of time (rather than the time path of k). Since, we may write the definite integral

$$\int_a^b 1(t)\, dt = k(t) \int_a^b = k(b) - k(a)$$

to indicate the total capital accumulation during the time interval [a, b].

To appreciate the distinction between k (t) and I(to) more fully, let us emphasize that capital k is a stock concept, whereas investment I is a flow concept. Accordingly, while k (1) tells us the amount of K existing at each point of time, I(t) gives us to the information about the rate of (net) investment per year (or per period of time which is prevailing at each point of time. Thus, in order to calculate the amount of net investment undertaken (capital accumulation) we must first specify the length of the interval involved. This fact can also be seen wherewe rewrite the identity $\frac{dk}{dt}$ =1(k) as dk = I(t) dt, which states that dk, the increment ink, is based not only on I(t), the rate of flow, but also on dt, the time elapsed. It is this need to specify the time interval in the picture, and give rise to the area representation under the I(t) curve.

Example 11:- If net investment is a constant flow at I(t)=2000 rupees per year), what will be the total net investment (capital formation) during a year, from t= 0 to t = 1?

Sol.

The answer is Rs.2000/-. This can be found as follows

$$\int_0^1 1(t)\, dt = \int_0^1 2000\, dt = 2000t = 2000.$$

The same answer will be found if, instead the year involved is formt=1 tot = 2.

Example 12:- If $1(t)=3t^{1/2}$ (thousands of rupees per year) a inconstant flow what will be the capital formation during the time interval [1, 4], i.e., during the second, third year and fourth years?

Sol. The answer lies in the definite integral

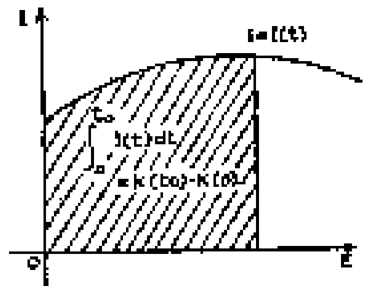
$$\int_1^4 3t^{1/2}dt = \left|2t^{3/2}\right|_1^4 = 16-2=14$$

On the basis of the proceeding examples, we may express the amount of capital accumulation during the time interval [0, 1], for any investment rate I(t), by the definite integral

$$\int_1^1 I(t)\,dt = k(t)\,|_0 = k(t)-k(0)$$

$$k\ (t) = k\ (0) + \int_1^t I(t)\ dt$$

The amount of k at any time to is the initial capital plus the total capital accumulation that has occurred since.



SELF-CHECK EXERCISE 15.4

Q1. Given the rate of net investment I(t) = $9t^{1/2}$, find the level of capital formation in (i) 16 years and (ii) between the 4th and 8th years.

Q2. The investment flow is described bythe equation I(t)=$3t^{1/2}$ and that the initial capital stock at time t=0, is k (0). What is the time path of capital k?

15.7 PRESENT VALUE OF A CASH FLOW

Before we discuss the present value of cash flow, let us define-

15.7.1 Natural Exponential Function e^t

Where e = 2.71828 as the preferred base because the function e^t possesses the remarkable property of being its own derivative (ie. $\frac{d}{dt}e^t=e^t$) fact which will reduce the work of differentiation to practically no work at all. This e may be defined as e = $\underset{m\to\infty}{It}\ f(m)$ = $\underset{m\to\infty}{It}\left(1+\frac{1}{m}\right)^m$

15.7.2 An Economic Interpretation of e.

The compound interest formula is

$$r\ (m) = A\left(1+\frac{r}{m}\right)^{mt}$$

where A is the principal amount the quotient $\frac{r}{m}$(where r interest rate per year and m is thecompounding periods) means that, in each of mt, $\frac{1}{m}$ the nominal rate r will actually be applicable. Finally, the exponent mt tells us that, since interest is to be compounded m times a year, there should be a total of mt compounding m years.

$$\therefore \qquad V_m = \left[A\left(1+\frac{r}{m}\right)^{m_t}\right]^n$$

$$= \left[A\left(1+\frac{1}{w}\right)^{w}\right]^n \qquad \text{where } w = \frac{m}{r}$$

Consequently, the asset value in the generalized continuous-compounding process whenm$\rightarrow\infty$ (i.e. when compounding m is increases) to be V = $\underset{m\rightarrow\infty}{It}$ V(m) = Ae^{rt}.

Applies to some context other them interest compounding, the coefficient r in Ae^{rt} no longer denotes the nominal interest rate. Then r can be reinterpreted as the instantaneous rate of growth of the function Ae^{rt}.

15.7.3 Discrete Growth

Actually growth does not always take place on a continuous bases not even in interest compounding. Fortunately, however, even for cases of discrete growth, where changes occur only once per period rather than from instant to instant, the continuous exponential growth function can be justifiably used.

In case where the frequency of compounding is relatively high, though not infinite, the continuous pattern of growth may be regarded as an approximation to the true growth pattern. But, more importantly, we can show that a problem of discrete or discontinuous growth can always be transformed into an equivalent continuous version.

Suppose that we have a geometric pattern of growth(say, the discrete compounding of interest) as shownby the following sequence:

A. A(1+ C), A $(1+i)^2$, A $(1+i)^3$,... where the effective interest rate per period in denoted by i and where the exponent of the expression (1+i) denotes the number of periods covered in the compounding. It we consider (1+i) to be the base b in an exponential expression then the above sequence may be summarized by the exponential function. Ab^t except that, because of the discrete nature of the problem, t is restricted to integer value only. Moreover, b=1+i is a positive number (positive even if i is a negative interest rate, say, -0.04), so that it can always be expressed as a power if any real number greater than, 1, including e. This means that there must exist a number r such 1+1=b = e^r.

Thus we can transform Ab^t into a natural exponential function.

$$A\,(1+i)^t\ Ab^t = Ae^{rt}$$

For any given value oft in this context, integer values of the function Ae will of course, yield exactly the same value as A $(1+i)^t$, so such as A(1 + i) = Ae^r and $A(1+i)^2=Ae^{2r}$ Consequently, even though a discrete case $A(1+i)^t$ is being considered, we may still work with the continuous natural exponential function Ae^{rt} This explains why natural exponential functions are extensively applied in economic analysis despite the fact that not all growth patterns may actually be continuous.

15.7.4 Discounting and Negative Growth

In a compound-interest problem, we seek to compute the future value V (principle plus interest) from a given present value A (initial principal). The problem of discounting is the opposite one, that of finding thepresent value A of a given sum V which is to be available t years from now.

Let us take the discrete case first, if the amount of principal A will grow into the future value of $A(1+i)^t$ after t years of annual compounding at the interest rate i per annum, i.e. if V = A $(1+i)^t$ then $A=\frac{V}{(1+i)^t}$ = V $(1+i)^{-t}$ V(1+i) which involves the negativeexponent.

Similarly, for the continuous case, if the principal A will grow into Ae^{rt}after t years of continuous compounding out the rate r in accordance with the formula $V=Ae^{rt}$ then $A=\frac{V}{e^{n}}$ = Ve^{-rt}

Here in the above equation the exponential growth function–r being negative, this rate is sometimes referred as a rate of decay, just as interest com- pounding exemplifies the process of growth, discounting illustrates negative growth.Now we are in the position to find the present valueof a cash flow. For single future value V, we havediscounting formulas.

$A=V(1+i)^t$ [discrete case]

&C=Ve^{-rt} [continuous case]

Now suppose that we have a stream or flow of future value–a series of revenues receivable at various times or of cost outlays payable at various times. We are interested in computing the present value of the entire in computing the present value of the entire "cash stream" or cash flow.

In the discrete case, if we assume three future revenue figures R_t (t=1, 2, 3) available at the end of the 1th year and also assume an interest rate of i per annum, the present value of R_t

will be, respectively R_t $(1+i)^{-1}R_2$ $(1+i)^2$, R_3 $(1+i)^3$ it follow that the total present value is the sum

$$\pi = \sum_{t=1}^{3} R_1 (1+i)^t$$

In case of continuous revenue stream at the rate of R(t) rupees per year is discounted at the nominal rate of r of year, its present value should be R(t) e^{-rt} dt. In case one problem is of finding the total present value of a three year stream, that is given by definite integral.

$$\pi \int_0^3 R(t)e^{-rt}\ dt$$

Note:- The upper summation index and the upper limit of integration are identical at 3, the lower summation index, differs from the lower limit of integration 0. This is because the first revenue is the discrete strea, by assumption will be forthcoming until t=1 (end of first year), but in the revenue flow in the continuous case is assumed to commerce immediately after t=0.

Example 13:- What is the present value of a continuous revenue flow lasting for y years at the constant rate of D rupees per year and discounted at the nominal of r per year? Find the present value when D= Rs.3000/- or= 0.06 and y=2.

Sol.

$$\pi = \int_0^y De^{rt}\ dt = \int_0^y De^{rt}\ dt$$

$$= D\left|-\frac{1}{r}e^{rt}\Big|_0^y\right| = -\frac{D}{r}e^{-rt}\Big|_0^y$$

$$= -\frac{D}{r}\left|e^{-ry}-1\right| = \frac{D}{r}(1-e^{ry})$$

When D= Rs.3000/-, r = 0.06, y = 2.

$$\therefore\ \pi = \frac{3000}{.06}(1 - e^{-0.02}) = \text{Rs. } 5655/-$$

15.7.5 Present Value of a Perpetual Flow

If a cash flow were to present forever a situation exemplified by the interest from a perpetual bond or the revenue from an indestructible capital asset such as land the present value of the flow wouldbe

$$\pi = \int_0^\infty R(t)e^{rt}\ dt$$

which is an improper integral

SELF-CHECK EXERCISE 15.5

Q1. What is meant by

(i) Natural Exponential Function
(ii) Discrete Growth
(iii) Discounting and Negative Growth

Q2. What is the present value of a continuous revenue flow lasting for y years at the constant rate of D rupees per year and discounted at the nominal of r per year? Find the present value when D = Rs.3000/- or = 0.06 and y = 2.

15.8 SUMMARY

In this Unit, we have learnt about the use of integration to solve different economic problems.

15.9 GLOSSARY

1. **Consumer surplus :** The notion was introduced by Ayred Marshal to measure the net benefit that a consumer enjoy from his act of purchasing *u* particular, commodity in the market. It is defined in terms of the excess of the consumer's total willingness to pay in units of money over his actual expenditure.
2. **Definite Integral :** Of a the function $f(x)$ over the interval (a, b) is expressed symbolically as $\int_a^b f(x)dx$, read as integral of *f* with respect to x from a to b. the smaller number a is termed as the lower limit and b, the upper limit of integration.
3. **Indefinite Integral :** The Indefinite integral is basically reverse differentiation. To differentiate means to find the rate of change (derivative) of a given function indefinite integration reverse the process and finds the unknown function where rate of change is given.
4. **Capital formation :** Capital formation is the process of adding to given stock of capital.

15.10 ANSWER TO SELF-CHECK EXERCISES

Self-check Exercise 15.1

Ans. Q1. Marginal Cost function C' (x) = 2 + 0.04x. On integrating MC w.r.t. x, we get

$$C(x) = \int c'(x)\,dx = \int (2+0.04x)\,dx$$

$$= 2x + .04\frac{x^2}{2} + k$$

When x = 0, C = 0, $\therefore$ k = 0

$\therefore$ $C = 2x + .02x^2$

Also given MR = 10, Total revenue TR = pq = 10x

Profit = 10x – (2x + .02) x^2

[$\therefore$ Profit = TR (q) – C(q)]

At x = 100

Profit 10 × 100 – (2 × 100 + .02 × 100^2)

= 1000 – 200 – 200 = 600

So at x = 100, there is a profit of Rs. 600/-

Ans. Q2. Refer to Section 15.3 (Example 5)

Self-check Exercise 15.2

Ans. Q1. For p = 60, we get q = 20 from the demand equation. Actual expenditure pq = 1200

Now CS = $\int_0^{20} (80 - q)\, dq - pq$

1400-1200 = 200

Thus the consumer's surplus is Rs. 200/-

Self-check Exercise 15.3

Ans. Q1. Refer to Section 15.5 (Example 8)

Self-check Exercise 15.4

Ans. Q1. (i) k = $\int_0^{16} 9t^{1/2}\, dt = 6(16)^{3/2} - 0 = 384$ Ans.

(ii) k = $\int_4^{8} 9t^{1/2}\, dt = 6(8)^{3/2} - 6(4)^{3/2}$ 135.76– 48 = 87.76 Ans.

Ans. Q2. Refer to Section 15.6 (Example 10)

Self-check Exercise 15.5

Ans. Q1.

(i) Refer to Section 15.7.1
(ii) Refer to Section 15.7.3
(iii) Refer to Section 15.7.4

Ans. Q2. Refer to Section 15.7 (Example 13)

15.11 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
5. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

15.12 TERMINAL QUESTIONS

1. If the marginal cost function F'(q)=3+3+$\sqrt{q}-\frac{5}{\sqrt{q}}$, find total cost function F(q) (1)= 21.

2. Given the marginal cost function $f'(x)$, find thetotal cost function when fixed cost is 50 units and f' (x)=3+x+x^2, x being output produced.

3. If marginal revenue function of a rim is $\frac{ab}{(x-b)^2}$ – C. Find the total revenue function. Give that TR = O when x=0. Prove that the average revenue function AR = $\frac{ab}{b-x}$ – C.

4. The marginal cost function of firm is 2+3e_x where x is the output. Find the total average cost functions if the fixed cost is Rs.500/-.

5. If the marginal propersity of save (MPS) is the following function of income, S' (γ)=0.3 – $1.1\gamma^{-1/2}$ and if the aggregate savings s is nil when income γ is 81, find the saving function S (γ)?

6. If the market demand curve is p=20 – 2x, where p and x are respectively the price and the amount, demanded of a commodity, find the consumer's surplus when p=4&p=4.

7. The supply curve for a commodity is p= $\sqrt{9+x}$ and the quantity sold is 7 units. Find the producer's surplus. Can you find the consumer's surplus. If yes, find it, if not explain with the help of diagram, why not?

Unit – 16

INPUT-OUTPUT ANALYSIS

STRUCTURE

16.1 INTRODUCTION

In this unit, we will study about the Input - Output Analysis. Input-output Analysis is a method of analysing how an industry undertakes production by using the output of other industries in the economy and how the output of the given industries used up in other industries or sectors. I.O. analysis is also known as the inter-industry analysis as it explain the inter dependence and interrelationship among various industries.

16.2 LEARNIG OBJECTIVES

After studying this unit, you will be able to answer :

- with what proportions one sector of the economy are related to other sectors.
- how solution is obtained in a framework of several variables production problems related to input-output.

16.3 INPUT-OUTPUT ANALYSIS

Input-Output analysis is a technique which was invented by W.W. Leontief in the year 1951. The basic idea behind Input-Output analysis is quite simple to understand. Since inputs of one industry are the outputs of another industry and vice-versa, ultimately their mutual relationship must lead to equilibrium between supply and demand in the economy consisting of n industries, and demand in the economy consisting of n industries. For example, the output of industry 1 is needed as an input in many other industries and perhaps for that industry itself, therefore, the total output level of industry 1 must take account of the input requirements of all the industries in the economy. Exactly in the same way since the output of industry n enters into other industries as their "input requirements," the total output of nth industry must be one that is consistent with all input requirements so as to avoid any bottlenecks anywhere in the economy.

Thus the essence of input-output analysis is that, given certain technological coefficients and final demand, each endogenous sector would find its output uniquely determined as a linear combination of multi-sector demand.

Let us suppose that an economic system consists of 4 producing sectors only, and that the production of each sector is being used as an input in all the sectors and is used for final consumption. Suppose (i) X_1, X_2, X_3 and X_4 are the total outputs of the 4 sectors.

(ii) F_1, F_2, F_3 and F_4 are the amounts of final demand, consumption, capital formation and exports

INPUT-OUTPUT TRANSACTION TABLE						
Producing Sector No.	Total Output of the sector	Input requirement of producing sectors				Requirement for final uses
		X_1	X_2	X_3	X_4	
1	2	3	4	5	6	7
1	X_1	X_{11}	X_{12}	X_{13}	X_{14}	F_1
2	X_2	X_{21}	X_{22}	X_{23}	X_{24}	F_2
3	X_3	X_{31}	X_{32}	X_{33}	X_{34}	F_3
4	X_4	X_{41}	X_{42}	X_{43}	X_{44}	F_4
Primary Input (Labour)	Total Primary Input = L $\rightarrow$	L_1	L_2	L_3	L_4	

for output of these sectors.

(iii) X_{11}, X_{12}, X_{13} and X_{14} are the amounts of product of sector I used as an input in 1st, 2nd, 3rd and 4th sectors respectively.

We can now arrange the distribution of total product of 4 producing sectors in the following way.

Two important equations can be derived from theabove table:

(1) Column 2, 4, 5 and 6 of the above table give us total inputs (form all sectors utilized by each sector for its production. In other words, col. 3 gives the production function of sector and col. 6 represents the production function of sector 4.

$X_1 = f_1 (X_{11}, X_{21}, X_{31}, X_{41}, L_1)$

$X_2 = f_2 (X_{12}, X_{22}, X_{32}, X_{42}, L_2)$

$X_3 = f_3 (X_{12}, X_{23}, X_{33}, X_{43}, L_3)$

$X_4 = f_4 (X_{12}, X_{24}, X_{34}, X_{44}, L_4)$

In general terms, if there are 'n' number of producing sectors then the production function ofsector n will be represented by:

$X_n = f_n (X_{1n}, X_{2n}, X_{3n}, \ldots\ldots\ldots X_{4n})$

(2) Rows of the table give us the equality between demand and supply of each product:

$X_1 = X_{11}+X_{12}+ X_{13}+X_{14}+F_1$

$X_2 = X_{21}+X_{22}+ X_{23}+X_{24}+F_2$

$X_3 = X_{31}+X_{32}+ X_{33}+X_{34}+F_3$

$X_4 = X_{41}+X_{42}+ X_{43}+X_{44}+F_4$

$L = L_1+ L_2+ L_3+ L_4$

In general terms, if there are n producing sectors:

$X_1 = X_{11}+X_{12}+ X_{13} + \ldots\ldots\ldots\ldots +X_{1n}+F_1$

$X_2 = X_{21}+X_{22}+ X_{23} + \ldots\ldots\ldots\ldots +X_{2n}+F_2$

....

....

$X_n = X_{n1}+X_{n2}+ X_{n3} + \ldots\ldots\ldots\ldots +X_{nn}+F_n$

INPUT-OUTPUT TRANSACTION TABLE							
Producing Sector	Total Output of the sector	→purchases	Input requirement of producing sectors				Requirement for final uses
			X_1	X_2	X_3	X_4	
	Sales→						
1	$a_{11}X_1$		$a_{12}X_2$	$a_{13}X_3$	$a_{14}X_4$	X_{14}	F_1
2	$a_{21}X_1$		$a_{22}X_2$	$a_{23}X_3$	$a_{24}X_4$	X_{24}	F_2
3	$a_{31}X_1$		$a_{32}X_2$	$a_{33}X_3$	$a_{34}X_4$	X_{34}	F_3
4	$a_{41}X_1$		$a_{42}X_2$	$a_{43}X_3$	$a_{44}X_4$	X_{44}	F_4
Primary Input	I		I_1X_1	I_2X_2	I_3X_3	I_4X_4	

and $L = L_1+ L_2+ L_3+ L_4 + \ldots\ldots\ldots\ldots + L_n$

$$X_1 = \sum_{j=1}^{n} X_{ij} + F_j \text{ and } L = \sum_{j=1}^{n} L_j$$

Where, X→Total output of the sector

X_{ij}→ Output of the ith sector used as input in *j*thsector and F_1→ Final demand for ith sector.

The above identity states that all the output of a particular sector could be utilized either as an input in one of the producing sectors of the economy and/ or as a final demand. Basically, therefore, input-output analysis is nothing more than finding the solution of these simultaneous equations.

16.3.1 Assumptions

The economy can be meaningfully divided into a finite number of sectors (industries):

1. Each industry produces only homogeneous output. Now two produced jointly; but if at all there is such case then it is assumed that products areproduced in fixed proportions.

2. Each producing sector satisfies the properties of linear homogeneous production function-in other words, production of each sector is subject to constant returns to scale so that k-fold change in every input will result in any exactly k-fold change in output.

3. One of the stronger assumption is that each industry uses a fixed input ratio for the production of its output; in other words, input requirements per unit of output in each sector remain fixed and constant. The level of output in each sector (industry) uniquely determines the quantity of each input which is purchased.

16.3.2 The Technological Coefficient Matrix

From the assumption of fixed input requirements we see that in order produce to one unit of the jthcommodity, the input used of jth commodity must be a fixed amount, which we denote by $Q_{ij} = \frac{X_{ij}}{X_1}$.If$X_i$ represents the total output of the jth commodity (on jth producing sector) the input requirement of ith commodity will be equal to Q_{ij} X_j or $X_{ij}=Q_{jj}X_j$.

As such we can now put the input-output transaction table in terms of technical coefficient as follows :

All these coefficients are non-negative (≥ 0). The above table gives us the total output of each sector in terms of technical coefficients, and there are "n" producing sectors:

$X_1 = a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \ldots\ldots\ldots\ldots + a_{in} X_n + F_1$

$X_2 = a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \ldots\ldots\ldots\ldots + a_{in} X_n + F_2$

....

....

$X_n = a_{n1} X_1 + a_{n2} X_2 + a_{n3} X_3 + \ldots\ldots\ldots\ldots + a_{nn} X_n + F_n$

$1 = 1_1X_1 + 1_2X_2 + 1_3X_3 + 1_4X_4$

$$X = \sum_{j=1}^{n} a_{ij} X_i + F_1 \ (i = 1, 2 \ldots\ldots\ldots\ldots n)$$

and $L = \sum I_i X_i$

The equations may be put in matrix notations:

$$\begin{vmatrix} X_1 \\ X_2 \\ X_n \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}a_{23} & \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ a_{n1} & a_{n2}a_{n3} & & a_{nn} \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \\ 1 \\ 1 \\ X_n \end{vmatrix} + \begin{vmatrix} F_1 \\ F_2 \\ 1 \\ 1 \\ 1 \\ F_n \end{vmatrix}$$

$X = AX+F$ and

$L= \sum I_i X_i$

SELF-CHECK EXERCISE 16.1

Q1. What is meant by input-output analysis.

Q2. Write the assumptions of input-output analysis.

16.4 CLOSED AND OPEN INPUT-OUTPUT MODEL

In the above example besides n industries our model contains exogenous sector of final demand which supplies primary input factors (labour services which are not produced by n industries) and consumes the outputs of the n producing industries (no: as input). Such an input, output model is known as open model. It includes, exogenous sectors in terms of "final demand bill"-along with the endogenous sectors in terms of n-producing sectors. Input-output model which has endogenous final demand vector is known as Closed input-output model.

SELF-CHECK EXERCISE 16.2

Q1. Distinguish between Closed and Open Input-Output Model

16.5 SOLUTION OF OPEN MODEL

Let us consider an economy with n-industries. If producing sector is to produce an output just sufficient to meet the input requirements of the n-industries as well as the final demand of the exogenous sector, its output level x_1 must satisfy the following equations.

$$X_1 = a_{11} X_1 + a_{12} X_2 + a_{13} X_3 \ldots\ldots\ldots a_{1n}X_n + F_1$$

or $$(1-a_{11})X_1 - a_{12} X_2 - a_{13} x_3 \ldots\ldots\ldots (-)\ a_{1n} X_n = F_1$$

For the entire set of n-industries, the correct output levels, therefore can be symbolized by following set of n linear equations.

$(1- a_{11})X_1 - a_{12}X_2 - a_{13}X_3 \ldots\ldots\ldots (-)\ a_{1n} X_n = F_1$

$-a_{21} X_1 + (1-a_{22})X_2 - a_{33} X_3 \ldots\ldots\ldots + a_{2n} X_n = F_2$

....

....

$- a_{n1} X_1 - a_{n2} X_2 - a_{n3} X_4..........(1-a_{n4}) X_n = F_n$

In the matrix notation this may be written as:

$$\begin{bmatrix} (a\text{-}a_{11}) & {}^{-a}12 & {}^{-a}13 & .... & {}^{-a}1n \\ {}^{-a}21 & (1\text{-}a_{22}) & {}^{-a}22 & .... & {}^{-a}2n \\ {}^{-a}31 & {}^{-a}22 & (1\text{-}a_{33}) & .... & {}^{-a}3n \\ .... & .... & .... & .... & \\ {}^{-a}n1 & {}^{-a}n2 & {}^{-a}n3 & .... & ({}^{1-a}nn) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ 1 \\ 1 \\ X_n \end{bmatrix} + \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ 1 \\ 1 \\ 1 \\ F_n \end{bmatrix}$$

$[1–A]$ X=For, $X = [1 - A]^{-1} F$

Where A is the given matrix or input coefficients while X and F are the vectors of output and final demand of each producing sector.

$[1\text{-}A]^1 \neq 0$ the $[1\text{-}A]^{-1}$ exists, we can then estimate for either of the 2 matrices X and F by assuming on of them to be given exogenously. In finding the solution $X = [1 - A]^{-1}.F$ only one matrix inversion needs to be performed even if we have to consider thousands of different final demand vectors according to alternative development targets.

16.5.1 The Hawkins-Simon Conditions

Many a time input-output matrix solution may give outputs expressed by negative numbers. If our solution gives negative outputs, it means that more than one tonne (or any unit) of that product is used up in the production of every tonne of that product, which is an unrealistic situation. Such a system is not a viable system. Hawkins - Simon conditions guard against such situations.

Our basic equation is $X=[1 - A]^{-1}.F$, is in such a order that this does not give negative numbers as a solution, the matrix, $[1 – A]$ which in fact is

$$\begin{bmatrix} (1\text{-} a_{11}) & -a_{12} & -a_{13} & .... & -a_{1n} \\ -a_{21} & (1\text{-}a_{22}) & -a_{22} & .... & -a_{2n} \\ -a_{31} & -a_{22} & (1\text{-}a_{33}) & .... & -a_{3n} \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & ..... \\ -a_{n1} & -a_{n2} & -a_{n3} & .... & (1 - a_{nn}) \end{bmatrix}$$

Should be such that:

(i) the determinant of the matrix must always bepositive, and

(ii) the diagonal elements: $(1–a_n)$, $(1–a_{22})$, $(1- a_{33})$... $(1–a_{nn})$ should all be positive or in other words elements: a_{11}, a_{22}, a_{33},.....a_{nb} should all be less than one. One unit of output of any sector should use not more than 1 unit of its own output these are Hawkins-Simon Conditions.

Example 1: The following inter-industry transactions table was constructed for an economy for the year 1978.

Industry	**1**	**2**	**Final Consumption**	**Total**
1	500	1600	400	2509
2	1750	1600	4650	8000
Labour	250	4800	---	5050
Total	2500	8000	5050	50

Construct technology coefficient matrix showing direct requirements. Does a solution exist for this system?

Solution: Technology matrix showing direct requirements per Re. of output is obtained by dividing input by the total output of the sector.

i.e. $a_{11} = \frac{X_{11}}{X_1} = \frac{500}{2500} = 0.20$

$a_{12} = \frac{X_{12}}{X_2} = \frac{1600}{8000} = 0.20$

$a_{21} = \frac{X_{21}}{X_1} = \frac{1750}{2500} = 0.70$

$a_{22} = \frac{X_{22}}{X_2} = \frac{1600}{8000} = 0.20$

	Industry	1	2
$\therefore$ A =	1	0.20	0.20
	2	0.70	0.20
	Labour	0.10	0.60

$$1 = A \begin{pmatrix} 1-0.20 & -0.20 \\ -0.70 & 1-0.20 \end{pmatrix}$$

$$= \begin{pmatrix} 0.80 & -0.20 \\ 0.70 & 0.80 \end{pmatrix}$$

$$[1 - A] = \begin{pmatrix} 0.80 & -0.20 \\ 0.70 & 0.80 \end{pmatrix}$$

$= 0.80 \times 0.80 - 0.20 \times 0.70$

$= 0.50$

Since |1–A| is positive and all elements of principal diagonal of (1–A) are positive, Hawkins-Simon condition are satisfied. Hence the given system has a solution

Example 2: Find out the output by industries 1, 2 and 3 from the following table:

	Inter-Industry Sales			Final demand	Total
	1	2	3		
inter-industry 1	4	8	6	14	32
Purchase 2	10	14	10	14	48
3	6	4	8	22	40
Primary input 4	12	22	16	---	50
Total	32	48	40	50	170

We have the technology matrix and the Leant of Matrix

$$A = \begin{bmatrix} \frac{4}{32} & \frac{8}{48} & \frac{6}{60} \\ \frac{10}{32} & \frac{14}{48} & \frac{10}{40} \\ \frac{6}{40} & \frac{8}{48} & \frac{8}{40} \end{bmatrix} \qquad 1 - A = \begin{bmatrix} 1-\frac{4}{32} & \frac{8}{48} & \frac{6}{60} \\ \frac{10}{32} & 1-\frac{14}{48} & \frac{10}{40} \\ \frac{6}{32} & \frac{8}{48} & 1-\frac{8}{40} \end{bmatrix}$$

$$(I - A)^{-1} = \begin{bmatrix} 27/20 & \frac{18}{50} & \frac{73}{200} \\ 73/100 & \frac{83}{50} & \frac{33}{50} \\ \frac{6}{42} & \frac{13}{50} & \frac{70}{50} \end{bmatrix}$$

We may verify the obvious result

$$\begin{bmatrix} 27/20 & 18/50 & \frac{73}{200} \\ \frac{73}{100} & 83/50 & \frac{33}{50} \\ \frac{39}{100} & 13/50 & \frac{70}{50} \end{bmatrix} \begin{bmatrix} 14 \\ 14 \\ 22 \end{bmatrix} = \begin{bmatrix} 32 \\ 48 \\ 40 \end{bmatrix}$$

If we want to find the effect of a change in one or more final demand levels we can use the above inversion since the technology matrix remains the same.

Suppose the final demand targets are 10.10.20 then the new output will be given by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 27/20 & 18/50 & \frac{73}{200} \\ \frac{73}{100} & 83/50 & \frac{33}{50} \\ \frac{39}{100} & 13/50 & \frac{70}{50} \end{bmatrix} \begin{bmatrix} 10 \\ 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 24.4 \\ 37.1 \\ 34.5 \end{bmatrix}$$

i.e. to satisfy the final demand target of 10, 10, 20 total output worth 24.4, 37, 1, 34.5 must be produced by industries 1.2.3 respectively.

SELF-CHECK EXERCISE 16.3

Q1. Discuss the importance of Hawkins-Simon Conditions of an input-output model.

Q2. Suppose [A] = $\begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$, then check whether any solution will be possible for the system or not.

16.6 THE CLOSED MODEL

If the exogenous sector (final demand level) of the open input-output model is absorbed into the system of endogenous sectors, the model would turn into a closed one. In such a model final demand bill and primary input will not appear any more: rather in their place, we shall have the input requirements and output of this newly conceived industry, the 'household industry' producing the primary input labour. Final demand sector would now be considered as one of endogenous sector. As such now we shall have (n+1) industries in place of n industries and all producing for the sake of satisfying the input requirements.

This newly conceived industry (of demand bill) will also be assumed to have a fixed input ratio as any other industry. In other words, the supply of primary input must now bear a fixed proportion to final demand and consumption of this newly concerned industry. This will mean for example, that household will consume each commodity in fixed proportion to the labour services they supply.

Looking at the problem in this particular way, it appears that the conversion of open model e. into a closed one should not create any significant change in our analyses and solution, because disappearance of final demand means only an addition of one more homogeneous equation.

Let us assume that there are 4 industries only including the new one (of final demand) designated by subscript 0. We shall, therefore, have the following set of equations.

$X_0 = a_{00}a_{01} X_1 + a_{02}X_2 + a_{03}X_3$

$X_1 = a_{10} X_0 + a_{11}X_1 + a_{12}X_2 + a_{13}X_3$

$X_2 = a_{20} X_0 + a_{21}X_1 + a_{22}X_2 + a_{23}X_3$

$X_3 = a_{30} X_0 + a_{31}X_1 + a_{32}X_2 + a_{33}X_3$

This gives us a homogeneous equation system,

$$\begin{vmatrix} (1\text{-}a_{00}) & -a_{01} & -a_{02} & -a_{03} \\ -a_{10} & (1\text{-}a_{11}) & -a_{12} & -a_{13} \\ -a_{20} & -a_{23} & (1\text{-}a_{22}) & -a_{23} \\ -a_{30} & -a_{31} & -a_{32} & (1\text{-}a_{33}) \end{vmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the 4 rows of the input coefficient matrix happen to the linearly dependent, |1-A| will turn out to be zero. Hence the solution is indeterminate.

This means that in a close model no unique output mix of each sector exist. We can at most determine the output levels of endogenous sector in proportion to one another but cannot fix their absolute levels unless additional information is made available exogenously.

Ex Given

$$A = \begin{vmatrix} 0.1 & 0.3 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.3 \end{vmatrix}$$

and final demand are F_1, F_2 and F_3, F in the output levels consistent with the model. What will be the output levels if $F_1 = 20$, $F_2 = 0$ and $F_3 = 100$?

We know that:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = (1\text{-}A)^{-1} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

$$\text{Now } (1 - A) \begin{vmatrix} 0.9 & -0.3 & -0.1 \\ 0 & 0.8 & -0.2 \\ 0 & 0 & 0.7 \end{vmatrix}$$

Co-factors are as follow

$A_{11} = 0.56$ $A_2 = 0.21$ $A_{31} = 0.14$

$A_{12} = 0$ $A_{22} = 0.63$ $A_{32} = 0.18$

$A_{13} = 0$ $A_{23} = 0$ $A_{33} = 0.72$

Hence the value of the determinant developing by first column $0.9 \times 0.56 = 0.504$

$$\text{Hence } (A - A)^1 = \frac{1}{0.504} \begin{vmatrix} 0.56 & 0.21 & 0.14 \\ 0 & 0.63 & 0.18 \\ 0 & 0 & 0.72 \end{vmatrix}$$

$$= \begin{vmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.43 \end{vmatrix}$$

$$\therefore \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{vmatrix} 1.11 & 0.42 & 0.28 \\ 0 & 1.25 & 0.36 \\ 0 & 0 & 1.40 \end{vmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$$

or

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{vmatrix} 1.11 & F_1 + 0.42 & F_2 + 0.28\ F_3 \\ 0 & 1.25 & F_2 + 0.28\ F_4 \\ 0 & 0 & +1.40\ F_4 \end{vmatrix}$$

$\therefore \quad x_1 = 1011F_1 + 0.42F_2 + 0.28F_3$

$1.11 \times 20 + 0 + 0.28 \times 100 = 50.2$

$x_2 = 1.25F_2 + 0.36F_3$

$= 0 + 0.36 \times 100 = 36$

$X_3 = 1.43F_3 = 143.$

SELF-CHECK EXERCISE 16.4

Q1. Describe the features of a closed input-output model.

16.7 SUMMARY

This unit tell us about the interrelationship among different industries in the market. It also shows the way of determining output and price of the product for each industry, which is the most important thing for this final of inter-linkage among the industries.

16.8 GLOSSARY

1. **Closed and open Input - Output Model :** The I-O model that consider 'final demand bill' as exogenous factor is said to be as open I-O model and in closed I-O model "final demand bill" is considered as endogenous factor.
2. **Hawkins-Simon Condition :** It basically states that more than one unit of a product cannot be used up in the production of every unit of that product. If A is the technological coefficient matrix then, according to Hawkins-Simon condition, determinant of |I – A| must be positive and all principal minor of [I – A] must also be positive.
3. **Technological Coefficient Matrix :** The matrix [*aij*], which basically represents input requirement from the industry to produce one output of jth industry, is known as technological coefficient matrix.

16.9 ANSWER TO SELF CHECK EXERCISES

Self-check Exercise 16.1

Ans. Q1. Refer to Section 16.3

Ans. Q2. Refer to Section 16.3.1.

Self-check Exercise 16.2

Ans. Q1. Refer to Section 16.4

Self-check Exercise 16.3

Ans. Q1. Refer to Section 16.5.1

Ans. Q2. Then $[I - A] = \begin{bmatrix} 0.2 & -0.2 \\ -0.9 & 0.3 \end{bmatrix}$ and the value of the determinant $| I - A | = (-)\, 8.12$. Which is less than zero. As the Hawkins-Simon condition are not satisfied no solution will be possible in this case.

Self-check Exercise 16.4

Ans. Q1. Refer to Section 16.6

16.10 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Bose, D. (2018). An Introduction to Mathematical Economical. Himalaya Publishing House, Bombay.
3. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
4. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
5. Mukherji, B. and Pandit, V. (1982). Mathematical Methods for Economic Analysis, Allied Publishers Pvt. Ltd., New Delhi.

16.11 TERMINAL QUESTIONS

Q1. The input-coefficient matrix is of an open input-output system is given as

$A = \begin{bmatrix} 0.2 & 0.3 & 0.2 \\ 0.4 & 0.1 & 0.2 \\ 0.1 & 0.3 & 0.2 \end{bmatrix}$. If the final demand vector in thousand rupees happen to be

$d = \begin{bmatrix} 10 \\ 5 \\ 6 \end{bmatrix}$, solve the system for output production.

Q2. Consider the following inter-industry transaction table. Construct technology coefficient matrix showing direct requirements. Does a solution exist for this system?

Industry	**1**	**2**	**Final Consumption**	**Total**
1	500	1600	400	2500
2	1750	1600	4650	8000
Labour	250	4800	---	5050
Total	2500	8000	5050	15,500

LINEAR PROGRAMMING-SIMPLE METHOD

STRUCTURE

17.1 INTRODUCTION

In this unit, we will learn about the Linear Programming (LP). Linear Programming is a technique used for deriving optimum use of limited resources. We will also learn about the different methods of Linear Programming.

17.2 LEARNING OBJECTIVES

The objectives of this unit is to:

- enable you to grasp the basic idea of linear programming principles.
- enable to apply different methods to solve the LPP

17.3 LINEAR PROGRAMMING

Linear programming is a mathematical technique and is concerned with the optimization of an objective function subject to the availability of limited resources pertaining to different activities or processes.Linear programming problems involve optimization in which all relationships are linear in nature. It deals with deterministic rather than probabilistic situations. Since values attained are constant over time, linear programming problem are of the continues and single stage type.An examination of following simple example should illustrate the basic concepts of linear programming problem abbreviated as (LPP)

Example 1: Industry manufactures two products: x, and x, which are processed in the machine shop and the assembly shop. The times (in hours) required for each product in the profits per unit aregiven along.

Machine		Assembly	Profit Unit
Product X1	2	4	Rs. 3
Product X1	3	2	Rs. 4
Total time available (In a day)	16	16	

Assuming that there is unlimited demand for both the product how many units of each should be produced every day to maximize total profit?

Let x_1 and x_2 be the number of units of x_1 and x_2 be the number of units of each should be. produced every day to maximize may be expressedsymbolically as

$Z=3x+4x_2$

which is subject to

$2x_1 + 3x_2 \leq 16$ Maching Constraint

$4x_1 + 2x_2 \leq 16$ Assembly Constraint

Also, $x_1 \geq 0$, $x_2 \geq 0$, since negative units of any product is meaningless. By analogy the general linear programming problem can be defined by

Maximize (or minimize) $z = c_1x_1 + c_2x_2 + \ldots\ldots c_nx_n$ subject to

$a_1x_1 + a_{12}x_2 + \ldots\ldots + a_{1j}x_j + \ldots. + a_{1n}x_n\ (\leq = \geq)\ b_1$

$a_2x_1 + a_{22}x_2 + \ldots\ldots + a_{2j}x_j + \ldots. + a_{2n}x_n\ (\leq = \geq)\ b_2$

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mj}x_j + \ldots. + a_{mn}x_n\ (\leq = \geq)\ b_m$

and then non-negatively restrictions.

$x_j \geq 0$ where $j=1,2\ldots\ldots\ldots.n$

Also all c's, b's and a_{ij}'s are constants and xj's are variables.

We have used $(\leq = \geq)$, which means any one of the signs could be there.

The linear function that is to be optimized is known as the objective function. Conditions are called the constraints. Solving a linear programming, problem means finding non-negative values of the variables (x_1, x_2....... x_n) which optimize the objective function and satisfy the constraints also.

SELF-CHECK EXERCISE 17.1

Q1. What is a Linear Programming Problem?

17.4 METHOD OF SOLVING LPP'S

17.4.1 Graphical Methods: Students are advised to refer to any book on Basic Mathematics.

17.4.2. Trial and Error Method: graphical Method cannot be used when there are more than 2 variables in an LPP. In that case, we use the simplex Method which is highly efficient and versatile also amenable to further mathematical treatment and offers interesting economic interpretations. Before that we shall understand trial and error method.

Slack Variables

Example I is written below:

Maximize $z=3x_1+4x_2$

Subject to $2x_1 +3x_2 \leq 16$

$4x_1 +2x_2 \leq 16$

$x_1, x_2, \geq 0$

Then$\leq$ type inequalities can be transformed into equalities by the addition of non-negative variables say x_3 and x_4 (Known as slack variables) as below. These variables represent imaginary products with zero profit per unit.

$$\left.\begin{array}{l} 2x_1 + 3x_2 + 1x_3 = 16 \\ 4x_1 + 2x_2 + 1x_4 = 16 \end{array}\right] \text{-A}$$

And the objective function may be rewritten asbelow.

Maximise $z=3x_1 +4x_2 +0x_3 +0x_4$

The trial and error and simple methods are basedon the concept of slak variables and theorems described below.

Extreme Point Theorem: It states that an optimal solution to an LPP occurs at the vertices of the feasible region. The first step of the method is, therefore to convert the inequalities into equalities by the addition (or subtraction) of the slack (or surplus variable) depending on the direction of the inequality. In $\geq$type inequality we subtract a variable (called the surplus variable) to make it an equality.

It is to be noted that the system of equations (A)above has more variables than the number of equations. Such a system of equationshas an infinite number of solutions, yet it has a finite and few vertices the co-ordinates of which can be determined by applying the basis theorem. Basis theorem states that for a system of m equations in n variables (where n >m) a solution in which at least (n-m) of the variables have value of zero is a vertex. This solution is called a basis solution.

Extremes point theorem can be extended to state that the objection function is optimal at least at one of the basic solutions. Some of the vertices may be infeasible in that they have negative co-ordinates and have to the dropped in view of the non-negativity conditions on all variable including the slack and surplus variables.

Consider the LPP of example I

Maximise $x = 3x_1 + 4x_2$

Subject to $2x_1 + 3x_2 \leq 16$

$4x_1 + 2x_2 \leq 16$

$x_1, x_2 \geq 0$

Introducing slack variable x_1 and x_4

Maximise $z = 3x_1 + 4x_2 + 0x_3 + 0x_4$

$$\left.\begin{array}{l} 2x_1 + 3x_2 + 1x_3 + 0x_4 = 16 \\ 4x_1 + 2x_2 + 0x_4 + 1x_4 = 16 \end{array}\right] \text{-B}$$

$x_1, x_2, x_3, x_4 \geq 0$

Heren (number of variables)=4 and m(number of equation) = 2. Thus n-m = 2. According to the basic theorem, we set 2= (n–m) variable in (B) equal to zero at a time, solving resulting system of equations and obtain a basic solution. Thus if we zeroise x_1 and x_2 the resulting system of equations would be

$$\left.\begin{array}{l} 1x_3 + 0x_4 = 16 \\ 0x_4 + 1x_4 = 16 \end{array}\right] - \text{(C) set 1 } (x_1 = x_2)$$

These equations directly yield $x_3 = 16$ and $x_1 = 16$ as the basic solution i.e. the co-ordinates of a vertex.

The other sets of equations, upon zeroising two variables at a time (B) would be as follows :

$$\left.\begin{array}{l} 2x_1 + 3x_2 = 16 \\ 4x_1 + 2x_2 = 16 \end{array}\right] \text{Set 2 } (x_3 = x_4 = 0)$$

$$\left.\begin{array}{l} 2x_1 + 0x_4 = 16 \\ 4x_1 + 1x_4 = 16 \end{array}\right] \text{Set 3 } (x_2 = x_3 = 0)$$

$$\left.\begin{array}{l} 2x_1 + 1x_3 = 16 \\ 4x_1 + 2x_3 = 16 \end{array}\right] \text{Set 4 } (x_3 = x_4 = 0)$$

$$\left.\begin{array}{l} 3x_1 + 1x_3 = 16 \\ 2x_2 + 0x_3 = 16 \end{array}\right] \text{Set 5 } (x_1 = x_4 = 0)$$

$$\left.\begin{aligned} 3x_2 + 0x_4 &= 16 \\ 2x_2 + 1x_4 &= 16 \end{aligned}\right] \text{Set 6 } (x_1 = x_3 = 0)$$

By solving these six sets of simultaneous equations we obtain six basic solution i.e. co-ordinates of the six vertices of the feasible region. The solutions are giver below:

Set	Solution
1	$x_3 = 16, x_1 = 16$
2	$x_1 = 2, x_2 = 4$
3	$x_1 = 8, x_4 = -16$
4	$x_1 = 4, x_3 = 8$
5	$x_1 = 8, x_3 = -8$
6	$x_2 = 16/3, x_2 = 16/3$

Since the solution 3 and 5 yield a negative co-ordinate each, contradicting thereby the non-negativity constraints, these are infeasible and have to be dropped from consideration.

Now according to the basic theorem the optimal solution lies at one of the vertices. By substituting these co-ordinates the values of objective function are derived below:

Set	**Solution**	**Z (Profit)**
1	$x_3 = 16, x_1 = 16$	48
2	$x_1 = 2, x_2 = 4$	22.
3	Infeasible	NA
4	$x_1 = 4, x_3 = 8$	12.
5	Infeasible	NA
6	$x_2 = 16/3, x_4 = 16/3$	$21\frac{1}{3}$

Thus the solution 2 is optimal with a profit of 22.

This is how we can solve an LPP simply by employing the theorems stated above, but the simplex method is a further improvement over the trial and error method.

17.4.3 The Simplex Method

The simplex method is a computation procedure an algorithm for solving linear programming problems. It is an iterative optimizing technique. In the simplex process, we must first find an initial basic solution (extreme point). We then proceed to an adjacent extreme point until we reach an optimal solution. For maximization the simplex method always moves in the direction of steepest ascent, thus ensuring that the value of the objective function improves with each solution

Example Maximise: f=2x+5y

Subject to (1) $x+4y \leq 24$

$3x + y \leq 21$

$x + \leq 9$

and (2) $x, y, \leq 0$

Introducing the slack variables, we obtain following equations:

$x+4y+s_1=24$

$3x+y+s_2=21$

$x+y+s_3 = 9$

which can be written in vector equation from

$$\begin{bmatrix}1\\3\\1\end{bmatrix} x + \begin{bmatrix}4\\1\\1\end{bmatrix} y + \begin{bmatrix}1\\0\\0\end{bmatrix} s_1 +$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$P_4 \qquad P_2 \qquad P_3$$

$P_1x+P_2y+P_3s_1 +P_4s_2 +P_5s_3 = P_0$

Thus the whole problems reduce to:

Max.

$f=2x+5y+0s_1 + 0s_2 + 0s_2+0s_3$ ..... (I)

Subject to:

$P_1x+P_2y+P_3s_1 +P_4s_2 +P_5s_3 = P_0$..... (II)

Simplex Tableau is formed in a particular way as explained below:

(1) All the vectors appear on the top or the table, but their order of appearance is changed.

Simplex table (Example 2)

Cj			0	0	0	0	2	5	
Stage		vector	P_0	P_3	P_4	P_5	P_1	P_2	Ratios
	←0	P_3	24	1	0	0	1	4	$\frac{a_{30}}{a_{32}} = \frac{24}{4} = 6$

Stage I	0	P_4	21	0	1	0	3	1	$\frac{a_{40}}{a_{42}} = \frac{21}{1} = 2$
	0	P_5	9	0	0	1	1	1	$\frac{a_{50}}{a_{52}} = \frac{9}{1} = 9$
	zj		0	0	0	0	0	0	
	zj-cj		0	0	0	0	-2	-5	
	→5	P_2	6	1/4	0	0	1/4	1	$\frac{a_{20}}{a_{21}} = \frac{6}{1/4} = 2$
Iteration (stage 2)	0	P_4	15	-1/4	1	0	11/4	0	$\frac{a_{40}}{a_{41}} = \frac{15}{11/4} = \frac{60}{11}$
←	0	P_5	3	-1/43	0	1	$\frac{3}{4}$	0	$\frac{a_{50}}{a_{51}} = \frac{3}{3/4} = 4$
	zj		30	5/4	0	0	5/4	5	
	zj-cj		30	5/4	0	0	$\frac{3}{4}$	-5	

Iteration II	5	P_2	5	1/3	0	-1/3	0	1
(Stage 3)	0	P_4	4	8/12	1	-11/4	0	0
	2	P_1	4	-1/3	0	1	2	5
	zj		33	1	0	1	2	5
	zj-cj		33	1	0	1	0	0

(1) P_0 vector appears first followed by the Basic (identity) vectors, viz: P_3, P_4 and P_5 followed by structural vectors, viz; P_1 and P_2.

(2) In the first row of the table (*cj*), we write the coefficients of the vector of the objective (1), which is required to be maximized, following the order described in (1) above.

(3) In the first column of table (*cj*) we write the coefficient of the basis vectors at the first stage: but subsequently these coefficients are replaced by the coefficients of the incoming structural vectors.

(4) Formulation of *zj* row; *zj* is the summation of products of element of each column vector with corresponding element of *cj* column.

For example, element of P, column are (0, 0, 1), while corresponding element of *cj* column = (0 × 0) + (1 × 0) + (1 × 0) = 0.

Since *cj* column possesses all zeros in first *zj*, forall vectors will be zero (always)

(5) Formulation of (*zj–cj*) row: Subtract from *zj* value of each column the *cj* value of the vector given in the 1st row of the table. Except in case of columns P_1 and P_2 all the element of (*zj-cj*) row will be zero in the first stage. For the vector P_1 and P_2 the value of *zj-cj* will be (-2) and (-5) respectively; because for the vector P_1*cj*= 2 and *zj*=0 and for vector P_2*cj*= 5 and *zj* = 0.

Before going further to stage II (or iteration I), following test is used to determine whether the solution of the given LP. problem is an optimal feasible solution, or whether it is necessary to make further manipulation (iteration) or whether there can be no finite solution at all of the given problem.

Test

(1) If all *zj-cj*≥0, an optimal solution has been obtained, hence no further iterations are necessary.

(2) *zj*-cj≤0 for some columns, then

(a) If all the element of those columns for which (*zj-cj*I ≤0) possess negative values, the solutions will be infinite.

(b) If some of the elements of those columns 1 for which *zj-cj*≤0 possess positive values, further iterations are necessary to achieve the optimal solution.

If we apply the above test to our problem it is found that in the stage (*zj-cj*) <0 for vectors P_1 and P_2. Also all the, elements of these 2 vectors columns possess positive value hence further iteration is needed to arrive at the optimal solutions.

We proceed as follows for further iteration: A structural vector (P_1 and P_2) is used to replace the basis vectors (P_3, P_4 and P_5) in turn. Replacing vector will be that structural vector which has highest negative *zj–cj* value amongst them. In the first stage of the tableau, for example, we could select P_2 tothe replacing vector since for P_2 we have *zj–cj*=-5.

The replaced vector is determined by means of finding the ratio of each element in P_0, vector to the corresponding elements of the replacing vector P_2. The basis vector associated with the smallest positive ratio would be the vector to be replaced.

Let a_{30} denote the element of the row labeled P_3 and column labeled P_0. It is 24 in our present example. a_{42} denotes the element of the row labeled P_4 and column labeled P_2. It is 1 in out table. A_{51} denote the element of the row labeled P_2 and column labeled P_5. It is 1 out table. In the first stage of the table we have three ratios:

$\frac{a_{30}}{a_{32}}$ (associated with P_3 vector)=6

$\frac{a_{40}}{a_{42}}$ (associated with P_4 vector)=21

$\frac{a_{50}}{a_{52}}$ (associated with P_5 vector)=9

(B) Formulation of Iteration i (or stage II):

(i) We first write new vector (introduced) P_2 in place of the basis vector replaced (P_3) in the 2nd column of the table.

(ii) The element in the row of this new vector P_2(introduced) are obtained by dividing each element of P_3 row by corresponding the element of vector P_2.

Therefore, element in P_2 row will be:

$$\frac{a_{30}}{a_{32}} = \frac{24}{4} = 6.\ \frac{a_{33}}{a_{32}} = \frac{1}{4}$$

$$\frac{a_{34}}{a_{32}} = \frac{0}{4},\ \frac{a_{35}}{a_{32}} = \frac{0}{4} = 0$$

$$\frac{a_{31}}{a_{32}} = \frac{1}{4} \text{ and } \frac{a_{32}}{a_{32}} = \frac{4}{4}$$

(C) Formulation of zj and (zj–cj)

Again we determine zj row by the same procedure given in stage 1, that is multiply each column by the corresponding element in the cj column and then add these products. In stage II, element in the P_0 are 6.15, and 3. Multiplying each of these by the corresponding elements in cj column and then adding them we get.

$(6 \times 5+15 \times 0 \times 3 \times 0)=30$

Value of Cj given in the first row is zero for thiscolumn.

$zj - cj = 30 - 0 = 30$

Value of Cj and ($zj - cj$) for other column are determined in the same way.

As explained under 'test' this iteration process is carried on until all (xj–cj) value are either=0 or more than zero i.e. positive. In our example, we stop after iteration II, when all the solution has been achieved.

This solution is given by P_0 column

$P_0=5P_2+4P_4+4P_1$

But the coeff. Of vector P_2 is y $\therefore$ y=5

Coefficient of vector P_1 is $\therefore$ x=4

Coefficient of vector P_4 is $\therefore$ s = 4

That is, the given function $2x+5y+0s_1 +0s_2+ 0s_3$ will be maximum when x=4 and y=5 and the maximum value of the function will be 2(4)+4(0)= 33, which is also given by the 1st element of zj row.

17.3.3.1 Degeneracy in SimplexMethod

If at any stage in carrying out the simplex operation it is discovered that structural vectors replace more than one basis vector, then LP problem is said to degenerate. In other words this means that in casewe get two or more than two minimum ratio identical, then structural vector would be replacing two or more than two basis vector. This will be the case of degeneration.

Example 3. Maximise $z=x_1 + x_2$

Subject to $8x_1 + x_2 \leq 200$

$x_1 \; 2x_2 \leq 100$

and $x_1 \leq 0$, and $x_2 \leq 0$

Using slack variables x_3, x_4, the inequalities become equalities which should be written in the form

$8x_1 + x_2 + x_3 + 0x_4 = 200$

$x_1 + 2x_2 + 0x_3 + x_4 = 100$

To maximize $z=x_1 + x_2 + 0x_3 + 0x_4$ from the initial toblean with zero profit as the solution corresponding to zero production. This provides us with the initial feasible solution.

			$P_1 = 1$	$P_2 = 1$	$P_3 = 0$	$P_4 = 0$
P*i*	Basis	x_0	x_1	x_2	x_3	x_4
0	x_3	200	8	1	1	0
0	x_4	100	1	2	0	1
	Zi	0	0	0	0	0
	Pi - zi		1	1	0	0

Step 1. First determine the optimal column. The row P*i-Zi* show the net profit when one unit of the variable is added. If there is no positive P*j-Zj* implies the solution can be improved.

Between x_1, x_2 the coming in variable is that which contributed most to the profit. Here since both x_1, x_2 contribute equally we may take, say x_2 as the coming in variable. The x_2 column is the optimal column.

Step 2. Consider the ratios obtained by dividing the quantities of x_3, x_4 rows by the corresponding entries in the optimal column.

$$\frac{200}{1} = 200. \quad \frac{100}{2} = 50$$

The going out variable is the one corresponding to the smaller ratio. Herex_4 is the going out variable to be replaced by x_2 in the new tableau. The largest quantity of x_2 that can be taken 50.

			$P_1 = 1$	$P_2 = 1$	$P_3 = 0$	$P_4 = 0$	Explanation For x
P*i*	Basis	x_0	x_1	x_2	x_3	x_4	row divide old x_4
0	x_3	150	$7\frac{1}{2}$	0	1	$\frac{1}{2}$	row by 2 e.g.
0	x_4	50	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{100}{1} = 50$ etc.
	Zi	50		1	0		
	Pi - zi			0	0		

To find the elements of the rows the following formula is used.

Old row element-old row element correspond- ing element in optimal column in the coming in row.

Thus $200\text{-}1 \times 50 = 150, 8 - 1 \times \frac{1}{2} = 7\frac{1}{2} 1\text{-}1 \times 1 = 0$

Step 3. The positive profit per unit is the P*j*-Z*j* column suggests the need for further improvement with the help of x_1.

We therefore, repeat the steps, between $\frac{150}{71/2} = 20$ and $\frac{50}{1/2} = 100$, the smaller ration corresponds to x_3 which is now the going out variable to be replaced by x_1.

	Cj	0	0	0	3	4	Ratio
		Vectors	P_0	P_3	P_4	P_1	P_2	$a_{31}/a_{32}=6/1 = 6$
Stage I	0	P_3	6	1	0	1	1	$a_{41}/a_{42}=21/4$
	←0	P_4	21	0	1	2	4	21/4 is least, replaced vector P_2.
	zj		0	0	0	0	0	4 is least no is row zj-cj
								∴replacing vector is P_2
	Zj - cj		0	0	0	-3	-4	$a_{30}/a_{31}= 3/4/1/2=3/2$ $a_{20}/a_{21}=21/4/1/2=21/2$
	0	P_3	3/4	1	-1/4	1/2	0	Since 3/2 is least ratio replaced vector is P_3
	→4	P_2	21/4	0	1/4	24/=1/2	4/4=1	Since -I is least no. in row zj-cj
Stage II	zj		21	0	1	2	4	
	zj-cj		21	0	1	-1	0	∴Replacing vector is P_1

	→3	P_1	3/2	2	-1/2	1	0
Stage III	4	P_2	9/2	-1	1/2	0	1
	zj		45/2	2	1/2	3	4
	zj-cj		45/2	2	1/2	0	0

			$P_1 = 1$	$P_2 = 1$	$P_3 = 0$	$P_4 = 0$
P*j*	Basi	x_0	x_1	x_2	x_3	x_4
0	x_1	20	1	0	2/15	-1/15
0	x_2	40	0	1	-1/2	-3/4
zi	.	60	1	1	-11/30	-41/60
pi - zi			0	0	-1/2	-17/60

There is no positive $Pj–Zj$ now so that the optimal solution is $x_1 = 20$, $x_2=40$.

Example 4. Maximize $b = 3x_1 +4x_2$

Subject to $x_1+x_2 \leq 6$

$2x_1+ 4x_2 \leq 21$

Where $x_, \geq 0$, $x_2 \geq 0$

Sol. Introducing the stack variables we have

$x_1+x_2 +s_1 = 6.$

$2x_1+ 4x_2 +s_2 = 21$

which can be written in vector form as

$$\begin{pmatrix}1\\2\end{pmatrix}x_1 + \begin{pmatrix}1\\4\end{pmatrix}x_2 + \begin{pmatrix}1\\0\end{pmatrix}s_1 + \begin{pmatrix}0\\1\end{pmatrix}s_2 = \begin{pmatrix}6\\21\end{pmatrix} \quad ...(1)$$

or $P_1x_1 +P_2x_2 +P_2s_1 +P_4s_2 = P_0$

$\therefore$ our problem becomes

Maximize $f= 3x_1\ 4x_2 +0s_1+ 0s_2$ ...(2)

Subject to $P_1x_1 +P_2x_2 +P_3s_1 +P_4s_2 = P_0=$ ...(3)

Stage 1

Step (i) In stage 1, the elements of *cj* row are values of P_0, P_1, P_2, P_3 and P_4 in equation (2) by comparing equation (2) and (3).

Step (ii) The elements of columns P_0, P_3, P_4, P_2 are written from equation (1) by comparing it with equation (3).

Step (iii) The element of column vector *cj* in stageI are written as coefficient of s_1 and s_2 in equation (2)

Step (iv) The elements of *zj* row are written as sum of product of column vector of *cj* with that of column vector of P_0, P_3, P_4, P_1, P_2.

e.g. first element of *cj* row is

$0\times6+0\times21=0$

Step (v) The elements of row *zj-cj* are written by subtracting corresponding elements of the row of *zj* and *cj*.

Step (vi) The ratios are obtained, the vector corresponding to least ratio (e.g. vector P_4 in this case) is to be replaced by vector P_2 (corresponding to least number in row vector *zj–cj*)

Stage II

Step (i) The elements of row P_2 in stage II are written by dividing each elements of row P_4 in stage1 by number a_{42} (i.e. 4 in this case)

Step (ii) The elements of P_3 row are

$$a_{40}\times\frac{a_{32}}{a_{42}}=6-21\times\frac{1}{4}=\frac{3}{4}$$

$$a_{40}-a_{40}3\times\frac{a_{32}}{a_{42}}=1-0\times\frac{1}{4}=1$$

$$a_{34}-a_{44}\times\frac{a_{32}}{a_{42}}=0-1\times\frac{1}{4}=-\frac{1}{4}$$

$$a_{31}-a_{41}\times\frac{a_{32}}{a_{42}}=1-2\times\frac{1}{4}=\frac{1}{2}$$

$$a_{32}-a_{42}\times\frac{a_{32}}{a_{42}}=1-4\times\frac{1}{4}=0$$

Step (iii) The elements of *zj* row are written as sum of product of corresponding elements of column vector *ci* and P_0, P_3, P_4, P_1, P_2.

e.g. first elements of row *zj* in stage II is

$$0\times\frac{3}{4}+4\times\frac{21}{4}=2$$

Step (iv) The elements of *zj–cj* are written by subtracting corresponding elements or rows of *zj* and *cf*.

Step (v) The ratios are obtained, the vector corresponding to least ration (e.g. vector P_3 in this cases is to be replaced by vector P_1 (corresponding to least number in row vector *zj-cj*)

This process of replacing structural vector (P_2, P_4) by basis vectors (P_1,P_2) will continue till all the elements of row vector *zj-cj* are positive or zero.

Stage III

Step (i) The elements of P_1 row in stage III are written by dividing element of row P_3 in stage II by number a_3 1 be

Step (ii) the elements of P_2 row are

$$a_{20} - a_{30} \times \frac{a_{21}}{a_{31}} = \frac{21}{4} - \frac{3}{4} \times \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{21}{4} - \frac{3}{4} = \frac{18}{4} = \frac{9}{2}$$

$$a_{23} - a_{33} \times \frac{a_{21}}{a_{31}} = 0 - \times \frac{\frac{1}{2}}{\frac{1}{2}} = -$$

$$a_{24} - a_{24} \times \frac{a_{21}}{a_{31}} = \frac{1}{4} - \left(-\frac{1}{4}\right) \times \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2}$$

$$a_{21} - a_{21} \times \frac{a_{21}}{a_{31}} = \frac{1}{2} - \frac{1}{2} \times \frac{\frac{1}{2}}{\frac{1}{2}} = 0$$

$$a_{22} - a_{32} \times \frac{a_{21}}{a_{31}} = 1 - 0 \times \frac{\frac{1}{2}}{\frac{1}{2}} =$$

Step (iii) The elements of row of *zj* are written in similar way e.g. the first element of row *zj* in

$$\frac{3}{2} \times 3 + 4 \times \frac{9}{2} = \frac{9}{2} + 18 = \frac{45}{2}$$

Then the elements of row *zj–cj* are written down. In stage III all the elements of vector row *zj–cj* are positive, hence an optimum has been achieved.

This solution is given by P_0 column in stage III

$$P_0 = \frac{3}{2} P_1 + \frac{9}{2} P_2$$

Compare it with $P_1x_1 + P_2x_2 + P_3s_1 + P_4s_2 = P_0$

$\therefore$ The given function is maximum when

$x_1 = 3/2, x_2 = 9/2/$

and Maximum value of f=$3\left(\frac{3}{2}\right) + 4\left(\frac{9}{2}\right)$

$$= \frac{9}{2} + 18 = \frac{45}{2}$$

SELF-CHECK EXERCISE 17.2

Q1. Define

(a) Slack variable

(b) Extreme Paint Theorem

(c) Degeneracy is Simplex Method

Q2. What are the different methods of solving LPP's?

Q3. Maximise $z = x + y$, subject to $x + y \leq 5$

$x + 3y \leq 12, x \geq 0, y \geq 0$

Q4. Maximise $z = 3x_2 + 7x_2 + 6x_3$

Subject to $2x_1 + 2x_2 + 2x_3 \leq 8$

$x_1 + x_2 \leq 3$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

17.5 SUMMARY

In this unit, we learnt about the Linear Programming. Linear Programming is a mathematical technique and is concerned with the optimization of an objective function subject to the availability of limited resources pertaining to different activities as process. We also studied about the different methods of solving h PP's. In the last two section we leaned about the Graphical method and simples method to solve linear programming.

17.6 GLOSSARY

1. **Basic Feasible Solutions :** These solutions are basic as well as feasible.
2. **Basic solution :** Any set of values of the variables in which the number of non-zero valued variables is equal to the number of constraints is called a Basic solution.
3. **Constraints :** The linear inequalities or the side condition.
4. **Feasible solution :** A set of values of decision variable which satisfies the set of constraints and the non-negativity restrictions.

17.7 ANSWER TO SELF CHECK EXERCISES

Self-Check Exercise 17.1

Ans. Q1. Refer to Section 17.3

Self-Check Exercise 17.2

Ans. Q1. (a) Refer to Section 17.4

(b) Refer to Section 17.4

(c) Refer to Section 17.4.3.1

Ans. Q2. Refer to Section 17.4

Ans. Q3. Refer to Section 17.4

Ans. Q4. Refer to Section 17.4

17.7 REFERENCES/SUGGESTED READINGS

1. Nichason, R.H. (1986). Mathematics for Business and Economics, McGrew Hill.
2. Dorfwan, R. Samuelson P.A. and Solow. R.M. (1987). Linear Programming and Economic Analysis, McGraw Hill.
3. Hadley, G. (2002). Linear Programming, Narosa Publishing House, New Delhi.
4. Bose, D. (2018). An Introduction to Mathematical Economics, Himalaya Publishing House, Bombay.

17.8 TERMINAL QUESTIONS

Q1. Using simple method solve the problem :

Maximise $x = 6x_1 + 2x_2 + 5x_3$

Subject to $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 10 \\ 8 \\ 19 \end{pmatrix}$ and

$x_1, x_2, x_3 \geq 0$

Q2. Maximise $z = 4x + 8y + 2k$

$y_2x + 2y + 4k \geq 4$

$x + y - 2k \geq 6$

$x \geq 0, y \geq 0, k \geq 0$ and minimise the same for z.

Unit - 18

LINEAR PROGRAMMING-PRIMAL AND DUAL

STRUCTURE

18.1 INTRODUCTION

In the last unit, we learnt about the concept of linear programming. In this unit, we will learn about the Primal and Dual, symmetry between then and the correspondence between Primal and Dual optimal solution will be studied is the successiding sections. In this last section of this unit, economic interpretation of primal and dual will be studied.

18.2 LEARNING OBJECTIVES

After going through this unit, you will be able to

- solve the problems based on Duality
- apply the concept of duality to solve the economic problem

18.3 DUALITY

The original problem (whether it is in the form of maximization or minimization function) is referred to as a prime problem. If the prime problem requires maximization, the dual problem is one of minimization and if the prime is a minimization problem, the dual is a maximization problem. In this way minimization are really not so distinct as they appear to be. In fact, since the dual are always identical and also that prime can be translated into its dual and vice versa, we have always an option of picking either of the two to work on nevertheless, the choice will ultimately depend upon:

(1) The formulation that yield more directly thedesired result; and

(2) The formulation that can be more easily solved.

A very good illustration of relationship between optimal problem and its dual is provided in the theory of production and costs. Suppose the prime problem was that of maximization of the total net revenue with given cost out lay. The dual would be that of minimization of cost for the given output.

Suppose a firm produces 2 products with 2 inputs, there are capacity constraints of the inputs, if the prices of two products are p_1 and p_2 then the revenue which the firm will try to maximize will be:

$R = p_1x_1 + p_2x_2$

Suppose a firm produces 2 products with 2 inputs, these inputs which may be written as

$a_{11}x_1 + a_{12}x_2 \leq b_1$

$a_{21}x_1 + a_{22}x_2 \leq b_2$

for input 1 and 2 respectively.

Obviously this presentation can be generated if the number of products are n and number of inputs m. Then the problem will be to maximize.

$$R = \sum_{j=1}^{n} P_i x_i$$

Subject to $\sum_{i=1}^{n} a_{ji}x_i \leq b_1$

.　　.

.　　.

.　　.

.　　or $\sum_{i=1}^{n} a_{ji}x_i \leq b_1$

.　　.

.　　.

.　　.

$\sum_{i=1}^{n} a_{21}x_1 \leq b_2$　　j = 1, 2 ....., m

.　　.

.　　.

.　　.

$\sum_{i=1}^{n} a_{mi}x_i \leq b_m$

And $x_i \geq 0$

Ormaximize $R = px$

Subject to $Ax \leq B$

And $x \geq 0$

Now consider the Dual. Suppose that the firm decides to determine the portion, of total revenue from each of its products it owes to each of the inputs (or capacities spent). This can be done if we consider the imputed prices (or opportunity costs of Shadow prices) of all inputs on each of the products.

We know that one unit of product 1 uses a_{11}, and a_{21} amount of input are c_1 and C_2, the total cost of producting one unit of product 1 will be a $a_{11}c_1 + a_{21}c_2$. This should be at least as much as the price of the product, in the market (p_1). Similarly for the other product, Hence,

$$\left.\begin{array}{l} a_{11}c_1 + a_{21}c_2 \geq p_1 \\ a_{12}c_1 + a_{22}c_2 \geq p_2 \end{array}\right]$$

(1)

The total input cost of input available will be $b_1c_1 + b_2c_2$.

Hence the from will minimize the total input cost (2) subject to the constraints (1). This can be generalised as follows:

Minimise:

$$f = b_1c_1 + b_2c_2 + \ldots + b_mc_m = \sum_{j=1}^{m} bjcj$$

(2) Structural constrains:

$a_{11}c_1 + a_{21}a_2 \ldots\ldots a_{m2}c_m \geq P_2$ or $\sum_{j=1}^{m} a_{j1}c_j \geq P_1$

$a_{12}c_1 + a_{22}a_2 \ldots\ldots a_{m2}c_m \geq P_2$ or $\sum_{j=1}^{m} a_{j2}c_j \geq P_2$

| | |

| | |

| | |

$a_{1n}c_1 + a_{2n}a_2 \ldots\ldots a_{mn}c_m \geq$ or $\sum_{j=1}^{m} a_{jn}c_j \geq P_n$

(3) Non-negatively constraints: $cj \geq 0 (j=1,2 \ldots\ldots\ldots\ldots m)$

or minimise f= $\sum_{j=1}^{m} a_j c_j$

Subject to $\sum_{j=1}^{m} a_{ji} c_j \geq P_i$

i = 1, 2 .... n

and $cj>0$,j=1,2....... m,

or in the matrix notation,

minimise: f = BC

Subject to A C≥P

C≥0

where P is the column vector of prices

X is the column vector of outputs,

A=m×n coefficent matrix

B= capacity constraint vector.

18.3.1 SYMMETRY BETWEEN PRIMAL AND DUAL

Form the above general L.P section we can easily pinpoint following characteristics of the primal and dual programmes which give them remarkable symmetry.

(1) Regarding objective function (i) if the primal involves maximization, the dual involve minimization and vice versa.

(ii) The profit constraints in the primal problemreplace capacity constraints and vice versa.

(2) Regarding Structural Constraints: (i) If the primal involve≥ sign, the dual involve≤ signs and vice versa.

(ii) A new set of variable appear in the dual.

(iii) If in the prime the coefficients in the constraint are found by moving from left to right, coefficients are positioned in the dual form top to bottomand vice versa.

(3) Regarding non-negatively constants: Theconstraints remains unchanged.

(4) Regarding variable: neglecting the number or non-negativity constraints. If there are 'n' variables and 'm' inequalities in the primal problem, in its dual there will be 'm' variables and 'n' inequalities.

These symmetrical characteristics between primal and its dual help us to formulate certain rules fortranslating primal into to dual or vice versa.

Example 1:

Primal	Dual
Minimize f=4x+5y Subject to: x $\geq$4 x $\geq$ 3 x + y $\geq$ 8 and A≥0. y≥0.	Maximize f=4A+3B+8C Subject to: A+C$\leq$4 B +C$\leq$ 5 and A $\geq$ 0. B $\geq$ 0, c $\geq$ 0.

structural constraints may be put in matrix form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

The basic rules to transformation are as below:

(1) The row vector of the coefficients in the primal objective function gives us the column vector of constrains in the dual constraints. Similarly the column vector of constraints in the primal constraints becomes the row vector of the coefficients in the dual objective function.

(2) Transpose of the coefficient matrix of the primal constraints gives us the coefficients of the primal constraints gives us the coefficients of the constraints in the dual and vice versa.

(3) The inequality sign in the dual constraints is reversed, but inequalities of non-negativity conditions retain their direction

Example 2. Write the dual of programme

Minimize $f=x_1 + x_2 + 3x_2 + 2x_5$

Subject to $x_1 + 3x_2 - x_2 + 2x_5 \geq 7$

$-2x_2 + 4x_3 + x_4 \geq 12$

$-4x_2 + 3x_3 + 8x_5 + x_6 \geq 10$

and $x_j \geq$ (j = 1 ....6)

(1) The row vector of objective function is = [1, 1,3,0,2,0]. This become the column vector of the constraints

$$\begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

(2) The column vector of the constraints is = $\begin{bmatrix} 7 \\ 12 \\ 10 \end{bmatrix}$

This become row vector of the coefficients of the objective function with new set of variables (x,y, z).

$\therefore$ Objective function

f=7x+12y+10z.

(3) The coefficient of constraints of primal are given by matrix.

$$A = \begin{bmatrix} 1 & 3 & -1 & 0 & 2 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & -4 & 3 & 0 & 8 & 1 \end{bmatrix}$$

$$\text{Transpose } A = A' = \begin{bmatrix} 1 & 0 & 0 \\ 3 & -2 & -4 \\ -1 & 4 & 3 \\ 0 & 1 & 0 \\ 2 & 0 & 8 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A' \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 3 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

Since we have introduced a new set of variable (x, y, z) therefore, the required constraints in the dual will now be:

$x \leq 1$ ...(1)

$3x - 2x - 4z \leq 1$ ...(2)

$-x + 2y + 3z \leq 3$ ...(3)

$y \leq 0$ ...(4)

$2x+8z\leq 2$ ...(5)

$z \leq 0$ ...(6)

and $x, y, z \leq 0$ ...(7)

with the objective function: f=7x+ 12y + 10z of course, constraints 4, 6 and 7 imply that y and z must be zero.

We always select the problem in the form which involves lesser number of constraints. But in case the primal and its dual have an equal or nearly number of constraints, preference should be given to the problem in its maximization form because there is no need to introduce artificial variables along with the slack variables as would be in the minimization form.

18.3.2 CORRESPONDENCE BETWEEN PRIMAL AND DUAL OPTIMAL SOLUTIONS

Example 3. Write the Dual of the following problem and solve it.

Primal	Dual
Maximizez=$3x_1+4x_2$	Maximize z=$16y_1 +16y_2$
Subject to $2x_1 +3x_2 \leq 16$	Subject to $2y+4y_2 \geq 3$
$4x_1 +2x_2 \leq 16$	$3y_1+2y_2 \geq 4$
$x_1, x_2 \geq 0$	$y_1, y_2 \geq 0$

Introducing surplus and artificial varietals.

Minimize $z = 16y_1 +16y_2+MA_1 +MA_2$

Subject to $2y_1 +4y_2 - s_1 +A_1 = 3$

$3y_1+2y_2 - s_2 + A_2 = 4$

$y_1, y_2, s_1, s_2, A_1, A_2 \geq 0$

Fixed Prog.		Cost	Qty.	16	16	0	M	M	Replacement
Ratio			y1	y2	s1	s2	A1	A2	Ratio
A_1	M	3	2	4	-1	0	1	0	3/4←
1/2 A_2	M	4	3	2	0	-1	0	1	2
		16-5 M		16-2M		MM	0	0	
				↑					
1/2 Y_2	16	3/4	1/2	1	-1/40	1/4	0	3/2	
A2	M	5/2	2	0	-1/2	-1	-1/2	5/4	
	3-2	M	0	4	$-\frac{1}{2}$MM	$-4+\frac{1}{2}$	MM	0	
	↑								
Y_2	16	1/8	0	1	-1/8	1/4	3/2	-1/4	
Y_1	16	5/4	1	0	1/4	-1/2	-1/4	1/2	
			0	0	2	4	M-2	M-4	

We put the optimal table of the primal and the dual below to bring out the correspondence between them.Except for sign reversal the value in the primal and the dual are the same. In other words the dual problems gives the solution in term of marginal value of resources for the primal problem.There is exact correspondence between the primal and the dual.Thus we can extract the primal optimal solution from the dual optimal table vice versa.

Primal Optimal Table

Prog.	Profit	Qty	x_1	x_2	x_3	x_4
x_2	4	4	0	1	1/2	-1/4
x_3	3	2	1	0	-1/4	3/8
NER			0	0	-5/4	-1/8

Dual Optimal Table

Prog.	Cost.	Qty	y_1	y_2	s_1	s_2	A_1	A_2
y_2	16	1/8	0	1	-3/8	1/4	1/8	-1/2
y_3	16	5/4	1	0	1/4	1/2	-1/4	1/2
			0	0	2	4	M-4	M-4

Marginal value of resources is synonymous with opportunity cost or shadow price.

18.3.3 ECONOMIC INTERPRETATION OF PRIMAL AND DUAL

Example 4:

Wordsworth Ltd. has three departments (Assembly, Finishing And Packing) with capability to make three products Table (T) at Rs.2/ unit profit, Chairs (C) at Rs. 4/unit profit and Book Case (B) at Rs. 3/unit profit. One table requires 3 hrs of assembly, 2hrs of finishing and 1 hrs of packing time. One Chair requires 4 hrs, 1 hrs and 3 hrs of assembly, finishing and packing time respectively. One book case require 2 hrs each of assembly, finishing and packaging time. Total time available for assembly, finishing, and packing are 60 hrs, 40 hrs and 80 hrs, respectively. Find the number of each product that should be produced in order to maximize the profit.

Solution: The primal for the problem is

Maximize $2T+4C+3B$

Such that $3T + 4C + 2B \leq 60$ Assembly constraint

$2T+1C+2B\leq 40$ Finishing constraint

$1T+3C+2B\leq 80$ Packing constraint

All variable≥ 0

The final table of primal is

			T 2	C 4	B 3	S_1 0	S_2 0	S_3 0
4	C	$6\frac{2}{3}$	1/3	1	0	1/3	-1/3	0
3	B	$16\frac{2}{3}$	5/6	0	1	-1/6	2/3	0
0	s_2	$26\frac{2}{3}$	-5/3	0	0	-2/3	-1/3	0
	cj-zj		-11/6	0	0	-5/6	-2/3	1

The optimal solution is to produce $6\frac{2}{3}$ chairs, $16\frac{2}{3}$ book cases and no tables. The total contribution for the product mix is Rs.76.67. The value under the s_1, s_2, s_3 columns in the c_j-z_j row indicate that to remove one productive hour form each of the three departments would reduce the total contribution, respectively, by Rs.5/6, Rs.2/3 and Rs.0.

Now the manager or the company recognizes that the productive capacity of the three departments is a valuable resource to the firm. He soon comes to think in terms of how much he would receive from another furniture producer, a renter who wanted to rent all the capacity in Woodworth company's three departments. He reasons along the following lines, suppose the rental charge were Rs.y_1 per hour of assembly time, Rs.y_2 per hour of finishing time Rs.y_3 per pacing time. The cost to the renter of all the time would be Rs.$60y_1 + 40y_2 + 80y_3$= total rent paid of course, the rented would want to set the rental pricein such a way as to minimize the total rent to be minimize. Hence objective function

Minimize $60y_1 + 40y_2 + 80y_3$

One table requires 3 assembly hours, 2 finishing hours and packing hour. The time that goes making a table would be rented out for Rs. ($3y_1 + 2y_2 + ly_3$) if the manager used that time to make a table, he would earn Rs.2 in contribution to profit, and so he will not rent out the time unless

$3y_1 + y_2 + ly_3 \geq 4$

Similar reasons give the other two dual constraints.

$4y_1 + 1y_2 + 3y_3 \geq 4$

$2y_1 + 2y_2 + 2y_3 \geq 0$and or course, the rental must be non-negative.

Thus the dual problem which determines the value of the productive resources is

Minimise $60y_1 + 40y_2 + 80y_1$ = total rent paid.

Subject to $3y_1 + 1y_1 + 1y_3 = \leq 2$

$4y_1 + 1y_2 + 3y_3 = \leq 4$

$2y_1 + 2y_2 + 1y_3 = \leq 3$

$y_1, y_2, y_3 \leq 3$

The final table of the dual problem is

			60 y_1	40 y_2	80 y_3	0 s_1	0 s_2	0 s_3	M A_1	M A_2	M A_3
60	y_1	5/6	1	0	2/3	0	-1/3	1/6	0	1/3	-6
0	s_1	11/6	0	0	5/6	1	-1/3	-5/6	-1	1/3	5/6
40	y_2	2/3	0	1	1/3	0	1/3	-2/3	0	-1/3	2/3
			0	0	$26\frac{2}{3}$	0	$6\frac{2}{3}$	$16\frac{2}{3}$	M	M	$-6\frac{2}{3}$
										M	$-16\frac{2}{3}$

The optimal solution indicates that the worth of the company of a productive hour in assembly isRs.5/6 in finishing department Rs.2/3 and in pack- aging department Rs.0. Of course, these are the same values we get by looking at the *cj-zj* in the final table of the primal. Thus if we solve primal, we can get solution to dual. Similarly, if we solve primal, we get solution to primal which can be obtained from *cj-zj* row of dual corresponding to the slack variables. In this case cj-zj corresponding to s_1, S_2 and s_3 0, and which is the solution to the primal problem.

Example 5: Find the dual of the following problem.

Maximize X $Z = x_1 + 2x_2$

Subject to $x_1 + x_2 \leq 3$

$2x_1 + x_2 \leq 10$

$x_1 \geq x_2 \geq 0$

Solution. The 1st constraint must be brought to ≥type of changing signs before we can derive the dual. This is done below.

Maximize X $Z = x_1 + 2x_2$

Subject to $x_1 + x_2 \leq -3$

$2x_1 + x_2 \leq 10$

$x_1, x_2 \geq 0$

Dual is now formulated below.

Minimize $z = -3y_1 + 10y_2$

Subject to $-y_1 + 2y_2 \geq 1$

$y_1 + y_2 \geq 2$

$y_1, y_2 \geq 0$

Example 6. Formulate the dual for the followingproblem.

Maximize $3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution. Since this is a minimization problem firstof all we make the <type inequalities of the $\geq$ typeas below.

Maximize $3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$-x_1 - 3x_2 \geq -3$

$-x_2 \geq -4$

$-x_1, x_2 \geq 0$

The dual can be written as below

Maximize $2y_1 - 3y_2 - 4y_3$

Subject to $2y_1 - y_2 \leq 3$

$y_1, 3y_2, y_3 \leq -1$

$y_1, y_2, y_3 \geq 0$

Example 7. Find the dual of the following problem.

Minimize $z = 30x_1 + 20x_2$

Subject to $x_1 + 4x_2 \leq 8$

$6x_1 + 4x_2 \geq 12$

$5x_1 + 8x_2 = 20$................. (iii)

$x_1 + 8x_2 \geq 0$

Solution. The equality (iii) can be restated as two inequalities as below

$$\left.\begin{array}{l} 5x_1 + 8x_2 \geq 20 \\ 5x_1 + 8x_2 \leq 20 \end{array}\right]$$

or $$\left.\begin{array}{r} 5x_1 + 8x_2 \geq 20 \\ -5x_1 - 8x_2 \geq -20 \end{array}\right]$$

The entire problem is now restated as below

Minimize $z = 30x_1 + 20x_2$

Subject to $-x+_1 - x_2 \geq -8$

$6x_1+4x_2 \geq 12$

$5x_1 + 8x_2 \geq 20$

$-5x_1 - 8x_2 \geq -20$

This dual is formulated below:

Maximize $-8y_1 + 12y_2 + 20y_3 - 20y_4$

Subject to $-y_1 + 6y_2 + 5y_3 - 5y_4 \leq 30$

$-y_1 + 4y_2 + 8y_3 - ly_4 \leq 20$

$y_1, y_2, y_3, y_4 \geq 0$

Example 8. To maintain his health a person mustfulfill certain minimum daily requirements for following three nutrients: Calcium. Protein and Calories, His diet consists of only two items I and II whose prices and nutrient are shown below.

	Food I(per Ib)	Food III(Per Ib)	Mini.DailyRequirement
Price	0.60	1.00	
Calcium (unit)	10	4	20
Protein (..)	5	5	20
Colaries (..)	2	6	12

Set up linear programming problem mathematically and solve it by simplex method. The objective being the minimization of the cost for the combination of food items.

Solution. Let x and y be the units of Food I and Food II respectively, then given linear programming problem becomes.

Minimize $z=0.60x+1.00y$

subject $10x+4y \geq 20$

$5x+5y \geq 20$

$2x+6y \geq 12$

where $x \geq 0$, $y \geq 0$.

The matrix of primal problem is

$$\begin{bmatrix} 10 & 4 & 20 \\ 5 & 5 & 20 \\ 2 & 6 & 12 \\ 0.60 & 1.00 & 0 \end{bmatrix}$$

Its Transpose is $\begin{bmatrix} 10 & 5 & 2 & 0.60 \\ 4 & 5 & 6 & 1.00 \\ 20 & 20 & 12 & 0 \end{bmatrix}$

$\therefore$ The dual problem is

Max. $z=20x_1+20x_2+12x_3$ subject to

Constraints $10x_1+5x_2+2x_3 \leq 0.60$

$4x_1+5x_2+6x_3+\leq 1.00$

To solve Dual problem by simplex method.

Introducing slack variables $x_4 \geq 0$, $x_5 \geq 0$, we obtain

$10x_1+5x_2+2x_3+1x_4+0x_5=0.40$

$5x_1+5x_2+6x_3+0x_4+1x_5=1000$

and obvious initial basic feasible solution is

$X_B=[0.60, 1.00\}$, (x_4, x_5, basic), with B_1 I_2 as basic sub matrix.

Starting Table

C_b	Y_b	*cf* X_b	20 y_1	20 y_2	12 y_3	0 y_4	0 y_5	
$R_1 \leftarrow 0$	y_4	0.60	10	5	2	1	0	0.60/10 = 0.60
R_2 0	y_5	1.00	4	5	6	0	1	$\frac{1}{4}=0.25$
zf		0	0	0	0	0	0	As = 20 is most negative element
zj-cj		-20	-20	-12	0	0	0	in now *cj-zj*, we choose arbitrary y_1 column as key element.
$R^1_1 \leftarrow 20$	y_1	0.60	1	1/2	1/5	1/10	0	0/60 = 0.12
R^1_2 0	y_5	0.76	0	3	26/5	-2/25	1	0.76/3 = 0.25
zj *zj-cj*			20	10	4	2	0	
$R''_1 \rightarrow 20$	y_2	0.12	2	1	2/5	1/5	0	0.12/2/5 = 0.3
$R''_2 \leftarrow 0$		0.40	-6	0	4	-1	1	0.40/4 = 0.10
zj			40	20	8	4	0	
zj-cj		0.08	20	0	-4	4	0	
20 y_2		0.10	13/5	1	0	3/10	-1/10	
$12y_3$			-3/2	0	1	-1/4	1/4	
zj			34	20	12	3	1	
zj-cj			14	0	0	03	1	

As all element of row $zj-cj$ are + ve

$\therefore$ An optimum solution is obtained at

$$\left(0\frac{13}{5}, -\frac{3}{2}\right)$$

and so minimum is obtained at (3, 1)

Atx_1= 3, x_2=1, Mini. Z= 0.60 × 3+1.00× 1

= 1.8+1=2.8

Example 9. Minimizez=6x+30y

Subject $x + 2y \geq 3$

$x + 4y \geq 4$

and $x \geq 0, y \geq 0.$

Solution. The dual of given problem is

Maximize $T=3x_1 +4x_2$

Subject to $x_1 + x_2 \leq 6$

$2x_1 + 4x_2 \leq 30$

and $x_1 \leq 0, x_2 \leq 0$

Introducing the slack variable.

$x_1 + x_2 + s_1 = 6$

$2x_1 + 3x_2 + s_2 = 30$

which can be written in vector form as

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 4 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} s_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} s_2 = \begin{pmatrix} 6 \\ 30 \end{pmatrix} \quad ...(1)$$

or $p_1x_1+p_2x_2+p_3s_1 + p_4s_2=P_0$

our problem becomes

Maximize $f= 3x_1 +4x_2+0s_1 + 0s_2$ ...(2)

Subject to $p_1x_1+p_2x_2+p_3s_1 + p_4s_2=P_0$ ...(3)

Simplex Table

cf vectors		0 P_0	0 P_3	0 P_4	0 P_1	0 P_1	Ratio
stage I ← 0	P_3	6	1	0	1	1	$a_{31}/a_{32} = \frac{1}{6} = 6$
0	P_3	30	0	1	2	4	$a_{41}/a_{42} = \frac{30}{4} = 7.5$
zj		0	0	0	0	0	6 is lest replaced vector
is							P3 and as
zj - cj		0	0	0	0	0	6 is last num ∴replacing vector is P_2.
← 4.	P_3	6	1	0	1	1	$a_{20}/a_{21} = \frac{1}{6} = 6$
0	P_4	6	-4	1	-2	0	$a_{42}/a_{31} = 6/(-2) = -3$
Stage II *zj*		24	4	0	4	4	
zj-cj		24	4	0	1	0	

Since all the elements of row $zj - cj$ are + ve or zero in stage optimal solution is obtained.The solution of maximization problem is (0.6) and of dual is (4.0) minimized value of given function is$6x+30y=6\times4 +30 \times0 = 24$

Exercise 18.1

Q1. Construct the dual of follow L.P. problem and solve the primal and the dual.

Maximise $Z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \leq 12$

$2x_1 + 3x_2 \leq 21$

$x_1 \leq 8$, $x_2 \leq 6$, and $x_1, x \geq 0$

Q2. Formulate the dual for the followingproblem.

Maximize $3x_1 - x_2$

Subject to $2x_1 + x_2 \geq 2$

$x_1 + 3x_2 \leq 3$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

18.4 SUMMARY

This unit was in continuation with the last unit. In this unit we have leaned about the Duality in Linear Programming. We also studied about the economic interpretation of Primal and Dual.

18.5 GLOSSARY

1. **Dual Problem :** Associated with every linear programming there is a linear programming problem. Which is called its dual problem.
2. **Primal :** The original LPP is called the primal problem.

18.6 ANSWER TO SELF CHECK EXERCISES

Exercise 18.1

Ans. Q1. Refer to Section 18.3.2 (Example 3)

Ans. Q2. Refer to Section 18.3.3 (Example 6)

18.7 REFERENCES/SUGGESTED READINGS

1. Nichason, R.H. (1986). Mathematics for Business and Economics, McGrew Hill.
2. Dorfwan, R. Samuelson P.A. and Solow. R.M. (1987). Linear Programming and Economic Analysis, McGraw Hill.
3. Hadley, G. (2002). Linear Programming, Narosa Publishing House, New Delhi.
4. Bose, D. (2018). An Introduction to Mathematical Economics, Himalaya Publishing House, Bombay.

18.8 TERMINAL QUESTIONS

Q1. Solve the following problem

Maximise $10x_1 + 10x_2 + 20x_3 + 20x_4$

Subject to $12x_1 + 8x_2 + 6x_3 + 4x_4 \leq 210$

$3x_1 + 6x_2 + 12x_3 + 24x_4 \leq 210$

$x_1 , x_2 , x_3 , x_4 \leq 0$

Q2. How will you state the problem of linear programming.

Unit-19

SETS

STRUCTURE

19.1 INTRODUCTION

In the present unit, we will study about a concise overview of some fundamental will team concepts of sets. In the first part we will learn about the basic elements of set theory and the relationships between sets. Finally we will learn about some operations on sets that are most frequency encountered in economics.

19.2 LEARNING OBJECTIVES

After going through this Unit, you will be able to-

- Define set and object
- Identify the elements of a given set
- Describe contentious used to list sets
- List the elements of a set using natatias
- Apply basic set concepts to economic Analyses.

19.3 CONCEPT OF SETS

A set is defined as a collection of distinct objects. These objects may be a group of students or a deck of cards or a group of numerical numbers. The objects of a set are called the elements.

19.3.1 SET NOTATION

The sets are usually denoted by capital letters like A, B, C, D, X, Y, Z etc. If a is an element of a set A, then we write $a \in A$ and Say a belongs to A. If a does not belong to A, then we write $\alpha \notin A$. It is assumed here that if A is any set and a is any element, then either $a \in A$ or $A \notin A$ and the two possible............ inclusive. The following are some sets/

a. The collection of first five natural numbers is a set containing the elements 1, 2, 3, 4, 5.

b. The collection of all twelve districts of Himachal Pradesh is a set.

c. The collection of brilliant students is a class in not a set, since the term "brilliant" is vague and is not well defined. However, the collection of all students in a class is a set. In the following paragraphs, we will use some sets frequently which are listed below:

N: for the set of natural numbers.

Z: for the set of integers.

Z^+: for the set of all positive integers.

Q: for the set of all rational numbers.

Q^+: for the set of all positive rational numbers.

R: for the set of all real numbers.

R^+: for the set of all positive real numbers.

C: for the set of all complex numbers.

19.3.2 DESCRIPTION OF A SET

A set is often described in the following two ways

19.3.2.1Roster Method

One way of defining a particular set is by enumeration. We simply list the items included in set the elements of the set.

Example 1. The set of even numbers between 1 and 13 may be described as

S = {2, 4, 6, 8, 10, 12}

Example 2. The set of first five prime natural numbers can be written as

A = {2, 3, 5, 7, 11}

Example 3. The set of even natural numbers can be described as A = {2, 4, 6...}

Here the dots stand for 6 and so on.

NOTE: The order in which the elements are written in a set makes no difference. Also repetition of an element has no effect.

19.3.2.2 Set-Builder Method

Alternatively, we can describe a set by stating a specific property P(x) of the elements x. If an item possesses that property, it is an element of that set, but if it does not, then it is excluded from the set. In such a case the set is described by

{x: P(x) holds} or {x | P(x) holds},

Which is read as 'the set of all x such that P(x) holds'. The symbol ":" or "l' is read as 'such that.'

Example The set X = {1, 2, 3, 4, 5,} can be written as X = {x ∈ N : x ≤ 5).

Example The set of all real numbers greater than-1 and less than 1 can be described as {x ∈ R: - 1 < x < 1}

Self-Check Exercise 19.1

Q1. Define Set.

Q2. Are all empty set equal?

19.4 TYPES OF SETS

19.4.1 Empty Set

A set is said to be empty or void or null set if it does not have any element and it is denoted by ϕ. In roster method, ϕ is denoted by{}. The null set is unique in the sense there is only one set in the whole world that can be considered a subset of any conceivable set. It from the above definition that a set A is an empty set if the statement x $\in$A is not true for any x.

Example: A = {x $\in$N: 8<x<9)= ϕ

Example: A= {x:x is an even prime number greater than 2} is an empty set because 2 is the only even prime number.

19.4.2 Singleton Set

A set consisting of a single element is termed as unit of singleton set.

Example: A= {10} is a singleton set.

Example: The set {x:x ε N and x^2=9} is α singleton set equal to {3}

19.4.3 Finite and Infinite Sets

A set is finite if it contains finite number of elements. In other words if the elements of a set can be listed by natural numbers 1, 2, 3,.... and the process of listing goes on till a certain natural number say n, then the set is finite set.

On the other hand, a set whose elements cannot be listed by the natural numbers n is called an infinite set. In other words if the number of elements of a set is very large and infinite, then the set is infinite set.

Example: Each one of the following sets is afinite set:

(i) Set of all persons on the Earth.

(ii) Set of even natural numbers less than 1000.

Example: Each one of the following sets is aninfinite set.

(i) Set of all in a plane.

(ii) A= {x:x is a natural number}

Relationship between Sets

When two sets are compared with each other, on can observe several possible relationship between them.

Equality of Two Sets

Two sets are said to be equal if every element of A is a member of B, and every element of B is a member of A.

If sets A and B are equal, we write A =B and A ≠ B when A and B are not equal.

Example If A = {2, 4, 7, 8} andB = {7, 4, 2, 8}

Then A = B, because each element of A is an element of B and vice-versa.

Note that the elements of a ------------------------ any order. However, even if one element ------------------- ent, two sets are not equal.

Example

A = {1, 5, 7}

B={1, 5, 8}

A ≠ B

19.4.5 Equivalent Sets

Two finite sets A and B are equivalent if their cardinal numbers are same i.e. n(A)= n(B). In other words, two sets are equivalent if there is one to one correspondence between the elements of the two sets. Equivalent sets have same number of distinct elements but notthe same elements.

Example:

A= {a, b, c)

B = {9, 10, 11}

Then A & B are equivalent sets and are writtenas A ≡ B or A ↔ B.

19.4.5 Subsets

Let A and B two set. Ifevery element of A is an element of B, then A is called a subset of B. If A is subset of B, we write A ⊆B, which is read as "A is a subset of B" or "A is contained in B." Thus, A⊆B ifa∈e A⇒a∈B.

The symbol"⇒" stands for "implies."

If A is a subset of B, we say that B contains A or

B is a super set of A and we write B⊇A.

If A is not a subset of B,

We write A ⊄ B.

Every set is a subset of itself and the empty set is subset of every set. These two subsets are calledimproper subsets.

19.4.6 Proper Subset

A subset A of a subset B is called proper subset of B if A≠ B and we write A ⊂B. In such a case, we also say that B is a super set of A. Thus, if A is a proper subset of B, then there exists an element X∈ B such that x ∈A.

It follows immediately from this definition and the definition of equal sets that two sets A and B are equal if A⊆B and B⊆ A. Thus whenever we want to prove that two sets are equal, we must prove that A⊆B and B⊆ A.

Example: A = {1, 9, 20}

B={1,9}

Then B⊂A, and is a proper subset of A.

19.4.7 Universal Set

In any discussion in set theory, there always happens to be a set that contains all sets under consideration i.e. it is a super set of each of the given sets. Such a set is called theuniversal set and is denoted by U.

Thus, a set that contains all sets in a given context is called the universal set.

Example: If A = {1, 2, 3}, B = {2, 4, 5, 6} and C = {1, 3, 5, 7} then

U= {1, 2, 3, 4, 5, 6, 7} can be taken as universal set.

SELF-CHECK EXERCISE 19.2

Q1. Define finite and Infinite Sets

Q2. What is meant by Equivalent Set?

Q3. What is Universal Set?

Q4. Let U = {u, v, w, x, y, z}

(i) Find the number of subsets of ∪

(ii) Find the number of proper non-empty subsets of ∪.

19.5 VENN DIAGRAM

Operations on sets or any property or theorem relating to sets can be well understood with the help of a diagram known as Venn-Euler diagram or simply Venn diagram. In Venn diagram the universal set U is denoted by rectangular region of U by a region enclosed by a closed curve (or a circle) lying within the rectangular region. These closed curves (or circles) representing the subsets of U will intersect each other if they have some common elements among them.

19.5.1 Union of Sets

Def: The union of two sets A and B, written as A∪B, is the set of all elements which belongs either to A or to B or to the both A and B.

Symbolically, AUB = {x: xεAU x εB}.

Here U means and/or' A∪B is read as 'A unionB'.

For example

If A ={1, 3, 4, 5}

and B = {2, 3, 4, 6, 8}

then A∪B = {1, 2, 3, 4, 5, 6, 8).

Note that no element is to be repeated even if it belongs to both the sets.

Union can be extended to more than two sets. We can construct a set A as the union of three sets A, B and C, i.e.

X=A∪B∪C.

Set Z consists of all the elements belonging to A,B and C without duplication, and no more elements other then the elements of the sets A, B & C.

Set Z consists of all the elements belonging to A,B and C without duplication and no more elements other then the elements of the sets A, B & C.

Suppose A = {1,3}, B = {1, 6, 9} and C = {2,3, 5, 6, 7}

Then

X=A∪B∪C= {1,3} ∪{2, 3, 5, 6, 7}

= {1, 2, 3, 5, 6,,9}

We can extend the notation of union to any number of sets.

Venn diagram for A∪B

In Venn diagram we have shaded A∪B, i.e. the area of A and the area of B.

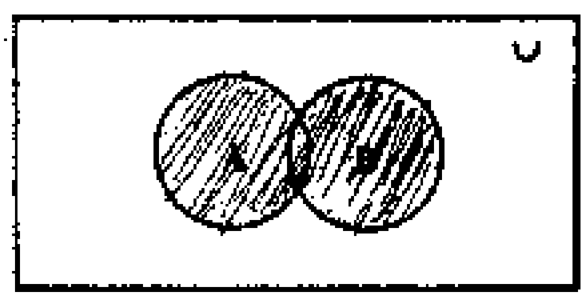


Fig. A∪B is shaded

It follows from definition that A∪B=B∪A and both A and B are always subsets of A∪B, i.e.. A ⊂A∪B and B⊂ A∪B.

19.5.2 INTERSECTION OF SETS

Def. The intersection of two sets A and B, written as $A \cap B$ is the set of all elements which are common to both A and B.

Symbolically, $A \cap B = \{x : x \in A \cap B x \in B\}$.

Here $\cap$ means intersection and $A \cap B$ is read as 'A intersection B'.

For example

If A = {2, 3, 5, 7}

and B = {1, 3, 4, 6, 7, 9}

then $A \cap B$ = {3, 7}

Like the set union the operation of intersectioncan be extended to the more than two sets. For any three sets X, Y and Z we may define $W = X \cap Y \cap Z$. Clearly, W consists of elements which are common to all the three sets X, Y and Z. Thus if

X = {a, b, c, d, e}

Y = {b, d, f}

Z= {a, b, d, g, h},

Then

$W = X \cap Y \cap Z = (b, d)$.

Venn diagram for $A \cap B$

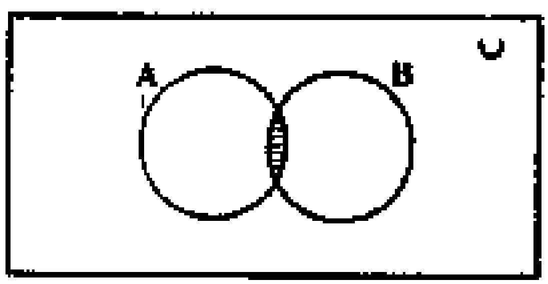


Fig $A \cap B$ is shaded

In Venn diagram we have shaded $A \cap B$, i.e. the area common to A and B.

It follows from definition that $A \cap B = B \cap A$ and each of A and B contains $A \cap B$i.e. $A \cap B$isa subset of both A and B i.e.

$A \cap B \subset C$ A and $A \cap B \subset B$.

19.5.3 DISJOINT SETS

Def: If two sets A and B have no elements in common, i.e. if no element of A is in B and no element of B is in A, then A and B are said to be disjoint or mutually exclusive sets.

Clearly $A \cap B = \phi$ when A and B are disjoint

For example,

If A = {2, 5, 7} and

B = {1,3,6,8}

Then two sets A and B are disjoint sets since they have no common elements.

Venn diagram for disjoint sets

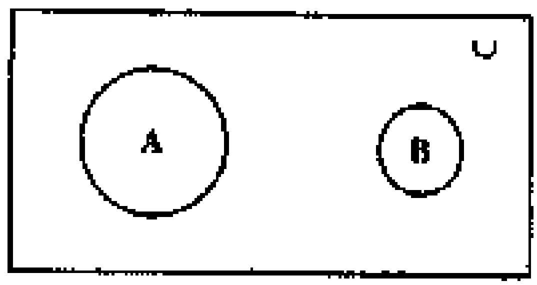


Fig A ∩ B= ϕ

Two disjoint sets A and B having no common elements among them are shown in the Venn diagram.

19.5.4 DIFFERENCE OF TWO SETS

Def: The difference of two sets A and B is the set of elements which belongs to A but which does not belong to B.

We denote the difference of A and B by A-B

Symbolically, A – B= {x:x∈A∩x∉ B}

Similarly B – A= {x:x∈ B ∩x∉A}/

For Example,

If A = {1, 2, 3, 5, 7}

and B = {2, 3, 4, 5, 6}

then A – B={1,7}

and B – A = {4,6}

Venn diagram for difference of two sets

In Venn diagram we have shaded A – B, i.e. the area in A which does not include any part of B.

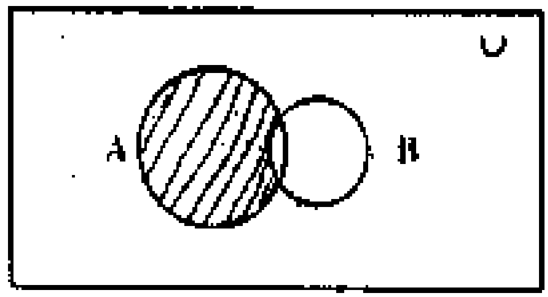


Fig A-B is shaded

It follows from definition that

A – B⊂A and B – A⊂B.

19.5.5 Complement of a Set (or Negation of a Set)

Def: The complement of a Set A is the set ofall the elements of the Universal set U which do notbelong to A.

The complement of a Set is the difference of the universal set U and the set A. the complement of the set A is denoted by A' of A^c.

Symbolically A' = {x:x∈ U∩x ∉A}

Clearly. A'∩ A = ϕ

A∪A'=U. U' = ϕ . ϕ = U

For example.

Let U= {a, e, i, o, u}

and A = {e, o}

Then A' = U – A = {a, i, u}

The complement of the complement of a set A is the set A itself.

(A')'= A

Venn diagram for the complement of a set

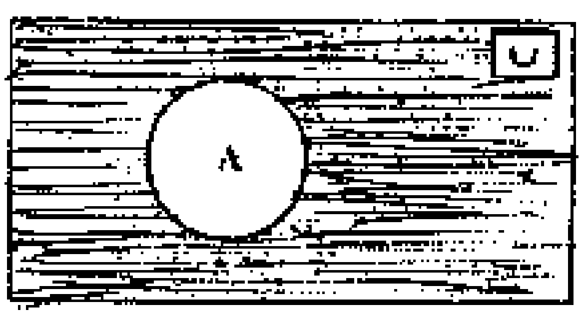


Fig A is Shaded

In the Venn diagram, we have shaded the complement of A i.e. the area outside A.

Example 1. Write down the following in settheoretic notations:

(i) 4 is an element of A

(ii) 8 does not belong to set B

(iii) X is a subset of Y

(iv) S & T are disjoint sets.

Sol. (i) $4 \in A$ (ii) $8 \notin B$

(iii) $X \subset Y$ (iv) $S \cap T = \phi$

Example 2. State which of the following are nullsets.

(i) $\{x: 3x^2 - 4 = 0, x \text{ is an integer}\}$

(ii) $\{x: (x+3)(x+3) = 9, x \text{ is a real number}\}$

(iii) $(A \cap B) - A$

Sol. (i) We have $3x^2 - 4 = 0$, or $3x^2 = 4$,

or $x^2 = 4/3$

or $\pm\sqrt{\frac{4}{3}}$ which is not integer

$\therefore$the given set has no element in it. i.e., it is a nullset

(ii) We have

$(x+3)(x+3) = 3$

or $x^2 + 6x = 0$

$x(x+6) = 0$

i.e. $x = 0, x = -6$

The given set contains two elements 0 and -6.

Hence it is not a null set.

(iii) Clearly, $A \cap B < A$.

Hence $A \cap B - A$ is a null set.

Hence the first and the third sets are null sets.

Example 3. If A = {1, 3, 5), B = {2, 4, 6, 8},C = {2, 5, 10} and

U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10), verify byactually writing the sets that

(i) $(A \cap B)^c = A^c \cap B^c$

(ii) $A \cap (B \cap C) = (A \cap B) \cap (A \cap C)$

Sol.

(i) A = {1, 3, 5}, B = {2, 4, 6, 8}, C= {2, 5,10}

$\therefore$ $A \cap B = \{1,3,5\} \cap \{2, 4, 6, 8\} = \phi$.

$\therefore \quad (A\cap B)=U-(A\cap B)-=U=\phi\ \{1,2,3,4,5,6,7,8,9,10\}$…… (1)

Again $A^c=U-A=(1,2,3,4,5,6,7,8,9,10\}-\{1,3,5\}$

$=\{2,4,6,7,8,9,10\}$

and $B^c=U-B=\{1,3,5,7,9,10)$

$\therefore A^c\cup B^c=\{2,4,6,7,8,9,10\}\cup\{1,5,7,9,10\}$

$=\{1,2,3,4,5,6,7,8,9,10\}$………….. (2)

Hence from (1) and (2), we get

$(A\cap B)^c=A^c\cup B^c$

(ii) $B\cup C\{2,4,6,8\}\cup\{2,5,10\}=\{2,4,5,6,8,10\}$

$\therefore A\cap(B\cup C)=\{1,3,5\}\cap\{2,4,5,6,8,10\}=5$ …………(3)

Again $A\cap B=\phi$ and $A\cap C=\{1,3,5\}\ \{2,5,10\}=\{5\}$

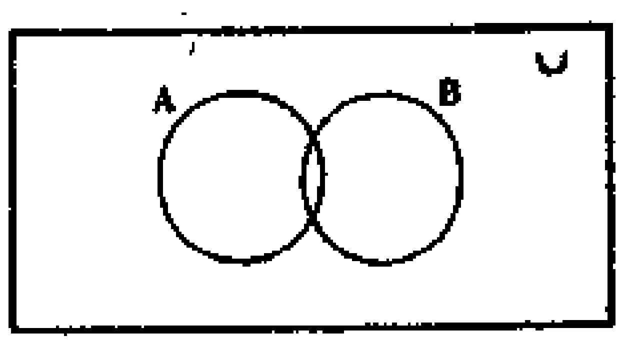
$\therefore(A\cap B)\cup(A\cap C)=\phi\cup\{5\}=\{5\}$ …………(4)

Hence from (3) and (4) we get

$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$

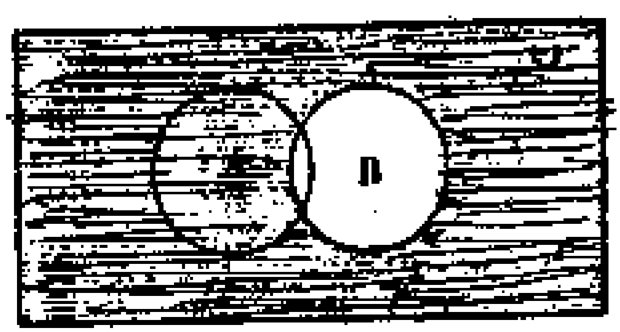
Example 4.

In the Venn diagram below shade



(i) B' (ii) (B – A)' (iii) A' ∩ B'.

Sol.



Fig

(i) B' is the complement of B and therefore, B' consists of elements which do not belong to B. Hence we shade the area outside B.

(ii) First we shade the area B – A with upward slanted strokes. (iii) then (B-A)' is the area outside B – A which is shaded with horizontal lines and is shown in fig. 2.

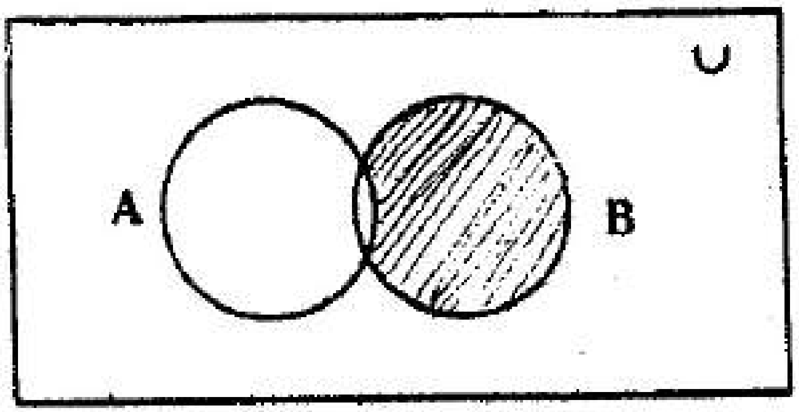


Fig. 1. B–A is shaded

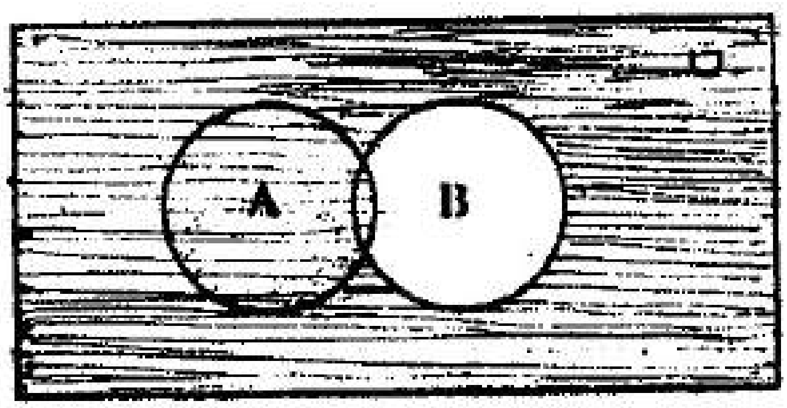


Fig. 2. B – A' is shaded

(iii) We first shade A', the area outside A, with upward slanted strokes (iii) and then shade B' with downward slanted strokes (iii) A'∩B' is the cross shaded (or cross-hatched) area i.e. the area common to A' and B' which is shaded with horizontal lines and is shown in Fig. 4

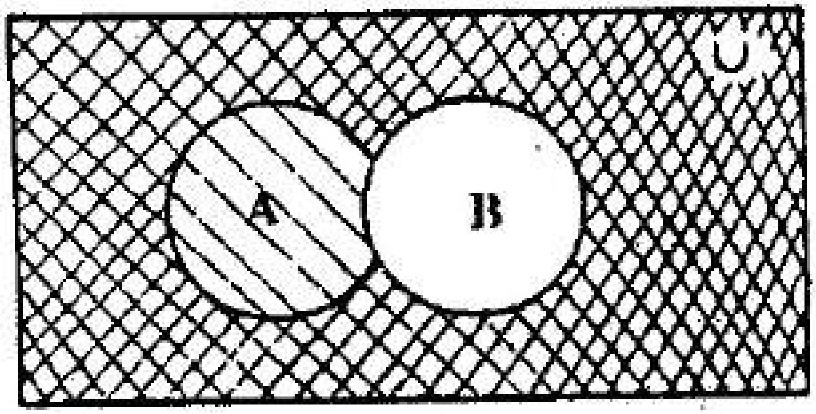


Fig. 3

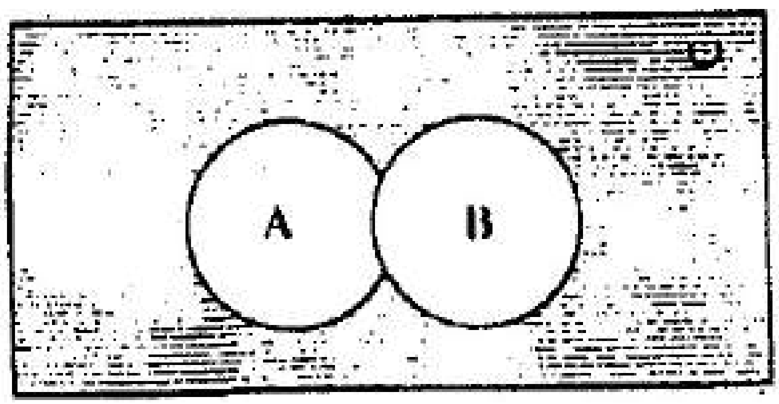


Fig. 4A'∩B' is shades

SELF-CHECK EXERCISE 19.3

Q1. What is Ven Diagram

Q2. What is meant by Complement of a Set?

Q3. If A ={1, 3, 4, 5}and B = {2, 3, 4, 6, 8}

then Find (i) A∪B and (ii) A∩ B

19.6 LAWS OF ALGEBRA OF SETS

Three main operations of sets, viz. intersection (∩), union (∪) and complement (') satisfy the certain laws of Algebra. These laws are stated below.

19.6.1. Idempotent Law

For any set A, we have

(i) A∪A = A and (ii) A∩ A = A

19.6.2 Associative Law:

For any three sets A, B and C we have

$(A\cup (B\cup C)=A\cup B\cup C$.

(ii) $A\cap (B\cap C)=(A\cap B)\cap C$

19.6.3 Commutative Law:

For a pair of sets A and B, we have

(i) $A\cup B = B\cup A$ and

(ii) $A\cap B = B\cap A$.

19.6.4 Distributive Law:

For any three sets A, B and C, we have

(i) $A\cup (B\cap C)=(A\cup B)\cap(A\cup C)$ and

(ii) $A\cap (B\cup C)=(A\cap B)\cup (A\cap C)$

19.6.5 De Morgan's Law:

For any two sets A and B, we have

(i) $(A\cup B)'= A'\cap B'$ and

(ii) $(A\cap B)'=A'\cup B'$.

19.6.6 Identity Law:

(i) $A\cup\phi = A$, (ii) $A\cap U=A$

(iii) $A\cap A\ \phi = A$ (iv) $A\cup U=U$

19.6.7 Complement Law:

(i) $A\cup A'=U$ (ii) $A\cap A' = \phi$

(iii) $(A')'= A'$ and (iv) $U' = \phi$, $\phi'=U$

Let us verify the Associative Law and de Morgan's Law by using Venn diagrams and analytical proofs using first definitions. The proof of idem- potent Law, Commutative Law, Distributive Law, Identity law, Complement Law are left as exercises to the reader.

Proof: Associative Law

(i) With the help of Venn diagram

We have to shows that

(i) $A\cup (B\cup C)=(A\cup B)\cup C$

(ii) $A\cap (B\cap C)=(A\cap B)\cap C$

(iii) L.H.S = $A\cup (B\cup C)$

In Fig. 5, we first shade A with upward slantedstrokes (II) and shade B∪C with downward slanted strokes (III). A∪ (B∪C) is the total area which is shaded with horizontal limits is shown in Fig. 6.

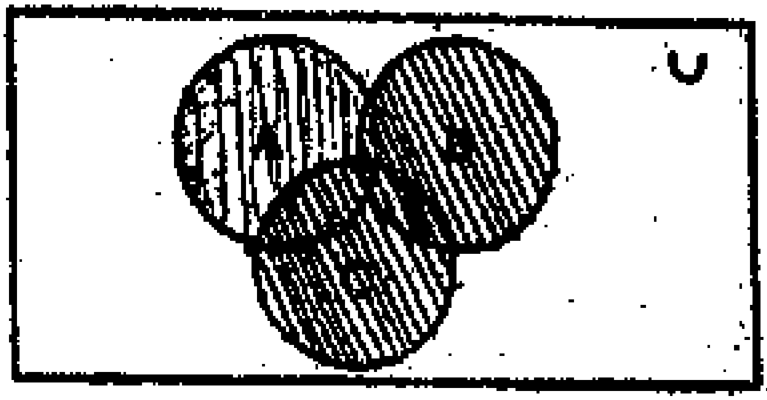


Fig. 5

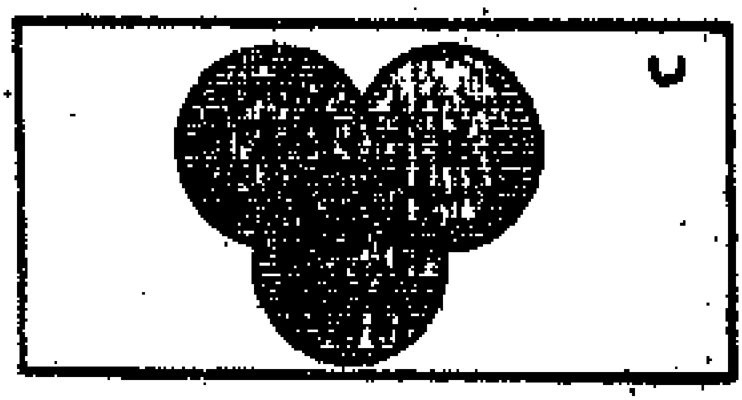


Fig. 6

R.H.S. (A∪B) ∪C

In Fig 7. We first shade (A∪ B) with upward strokes and then shade C with downward slanted strokes.

Fig. 7

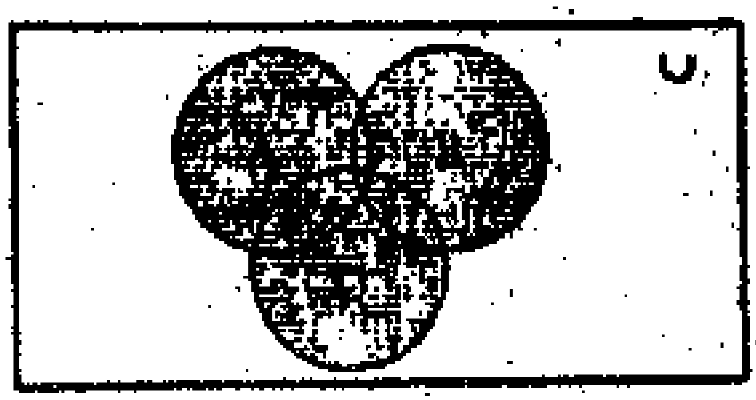


Fig. 8

(III) (A∪B)∪C is the total area which is shadedwith horizontal lines and is shown in fig 8.

(ii) L.H.S. A∪ (B∪C).

In fig 9. first we shade A with upward strokes

(II)and then shade B∩C with downward strokes (III)

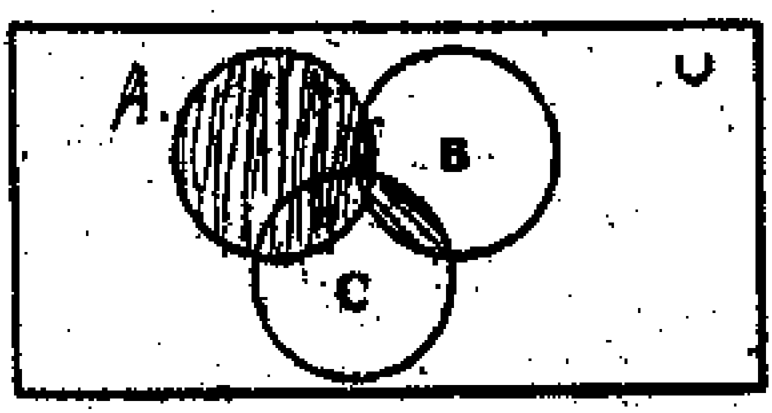


Fig. 9

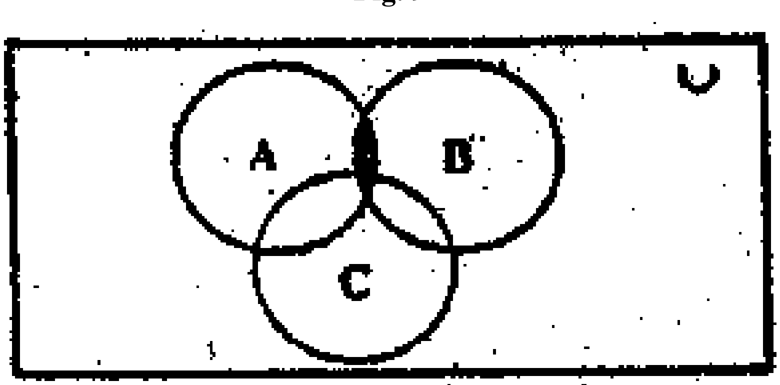


Fig. 10 A ∩ (B ∩ C) is shades

A∩ (B∩C) is the cross-shaded area which is shown in Fig. 10 by shaded it with horizontal lines.

In Fig. 7A first we shade A∩B with upwardslanted strokes (II) and shade C with downward strokes (III) (A∩B) ∩Cis the cross-shade area which is shown in Fig. 8A. by shading it with horizontal lines

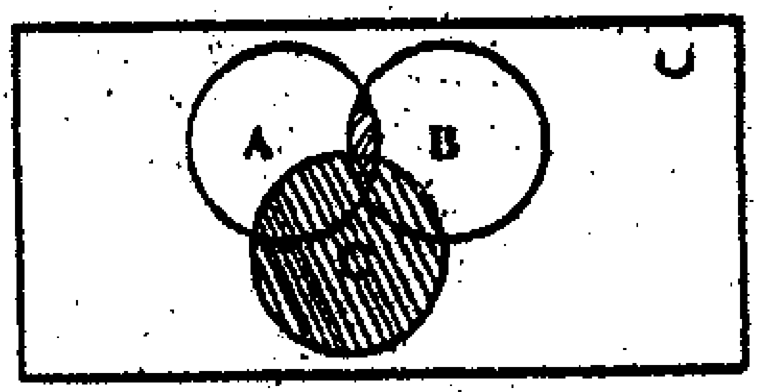


Fig. 7. A

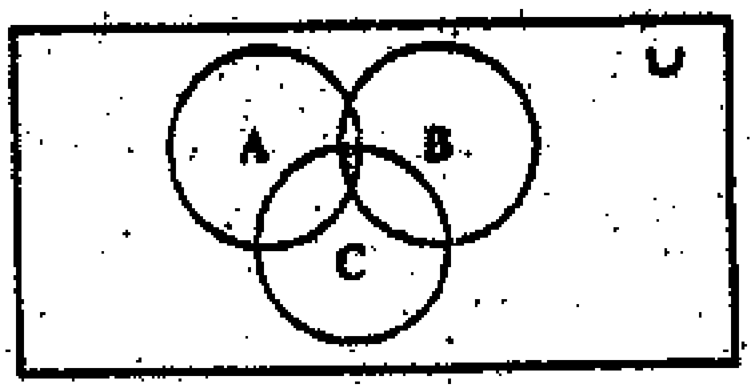


Fig. 8A (A ∩ B) ∩ C is shaded

Hence from Fig. 7A and Fig. 8A we obtain

A∩(B∩C)=(A∩B) ∩C

(b) Analytical Proof

To Prove

(i) A∪(B∪C)=(A∪B) ∪C and

(ii) A∩ (B∩C)=(A∩B)∩ C.

Sol. (i) Let *xε* A∪ (B∪C). Then

x∈ A ∪ (B∪C)

$\Rightarrow x \in$ A or/and x$\in$ (B$\cup$C)

$\Rightarrow x \in$ A or/and $x \in$ (B$\cup$C)

$\Rightarrow$($x \in$ A or/and $x \in$ B) or/ and $x \in$ c

$\Rightarrow x \in$ (A$\cup$B) or/and $x \in$ C

$\Rightarrow x \in$ (A$\cup$B) $\cup$C

Thus $x \in$ A$\cup$ (B$\cup$C)

$x \in$ (A $\cup$ B) $\cup$ C

$\therefore$ A$\cup$(B$\cup$C)$\leq$(A$\cup$B) $\cup$C

(1)

Now, let y $\in$ (A$\cup$B) $\cup$C. Then by definition,

$y \in$ A $\cup$ (B$\cup$C)

$\Rightarrow y \in$ (A$\cup$B) or/and $y \varepsilon$ C

$\Rightarrow y \in$ A or/and y $\in$B or/and $y \in$ C

$\Rightarrow y \in$ A or/and $y \in$ (B$\cup$C)

$\Rightarrow y \in$ A or/and $y \in$ B $\cup$C

(2)

$\therefore$(A$\cup$B) $\cup$C$\subseteq$ A $\cup$ (B $\cup$ C)

Hence from (1) and (2), we get

A$\cup$ (B$\cup$C)=(A$\cup$B) $\cup$C.

(ii) Using the definition of intersection and proceeding and above we can also prove the result.

A$\cap$ (B$\cap$C)=(A$\cap$B)$\cap$ C

This is left as an exercise to the reader.

Proof of De Morgan's Law

(a) With the help of Venn diagram

We have to show that

(i) (A$\cup$B)'=A'$\cap$B'

(ii) (A$\cap$B)'=A'$\cup$B'

(i) L.H.S. = (A$\cup$B)'

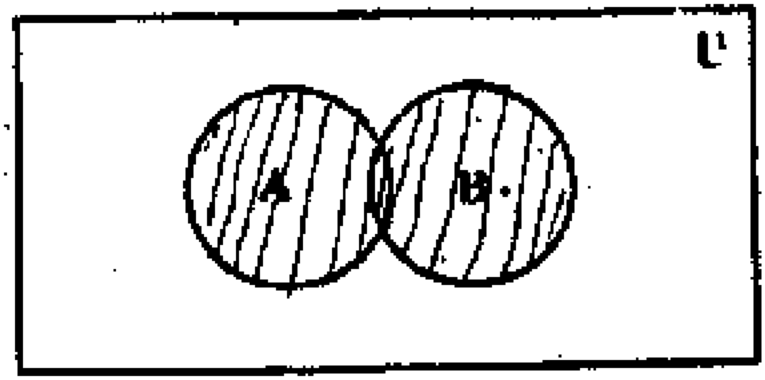


Fig. 11 (A∪) B is shaded

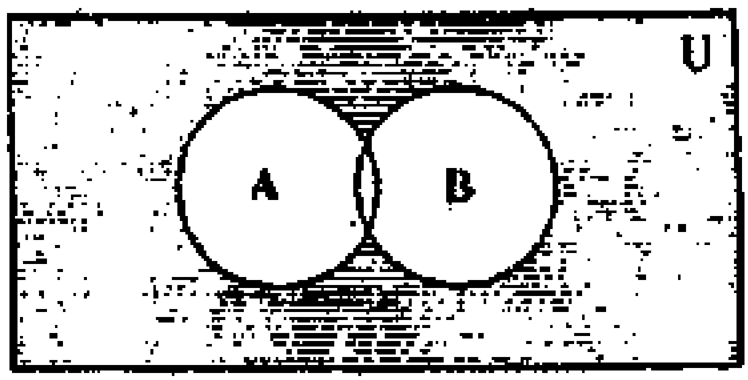


Fig. 12 (A∪B)' is shaded

In Fug, 11. A∪B is shaded with upward slanted strokes (III). (A∪B)' is the area outside A ∪B which is shaded with horizontal line and shown inFig. 12.

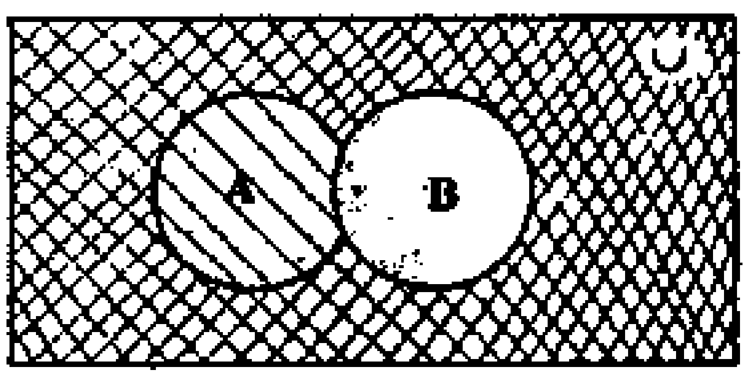


Fig. 13

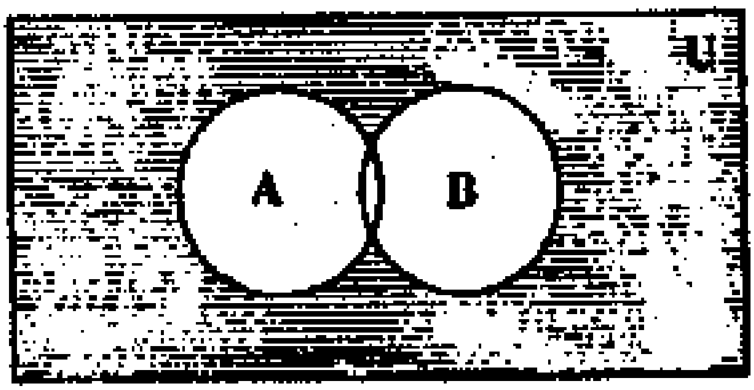


Fig. 14 (A' ∩ B')

R.H.S. A'∩ B'

We first shade A' i.e. the area outside A with upward slanted strokes (III) and then shade B', the area outside B, with downward strokes (III). A'∩B' is the cross-hatched area, i.e. the area common to both A' & B' is shaded with horizontal lines and is shown is Fig. 14. Hence from fig. 12 and fig. 14, wehave -

(A∪B) A'∩B'

L.H.S. = (A∩B)'

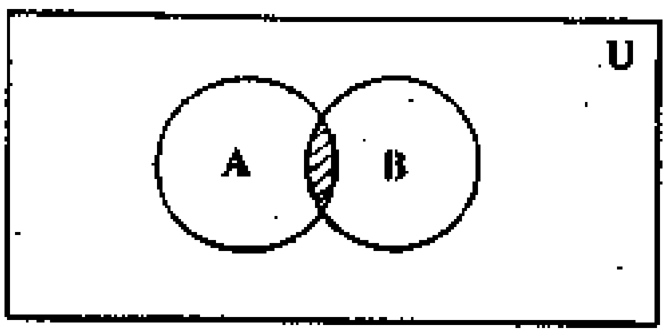


Fig. 15

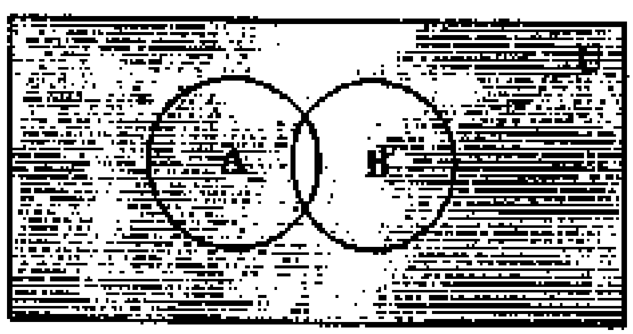


Fig. 16

In fig. 15. we have shaded A∩B i.e., the area common to A & B. (A∩B)' is the area outside A ∩B which is shaded with horizontal lines and is shown in fig. 16.

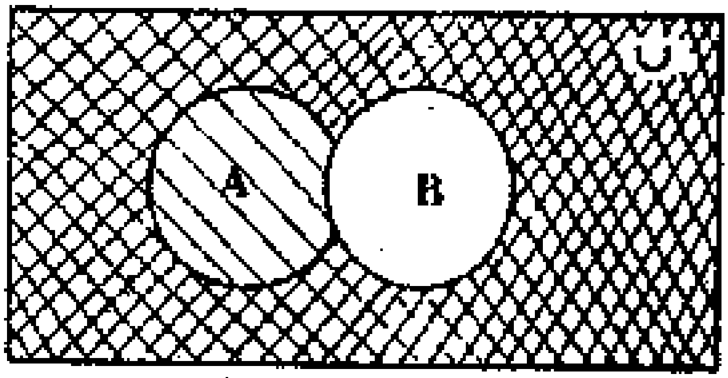


Fig. 17

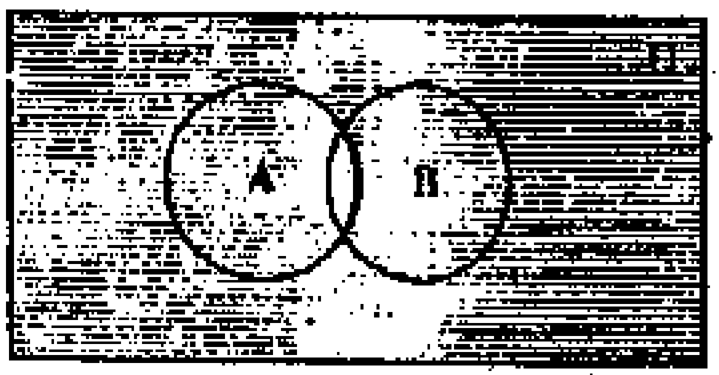


Fig. 18 A' ∪ B' is shaded

R.H.S. A'∪B'

First we shade A', the area outside A with up- ward slanted strokes (III) and then B' with downward slanted strokes (III). A'∪ B' is the total shaded area with is shaded with horizontal lines andshown in fig. 18. Hence from Fig. 16 and Fig. 18, we have (A∩B)=A' ∪ B'.

Analytical Proof :

(i) (A∪B)'= (A' ∩B)'

(i) (A∩ B)'= A' ∪B'

(i) Let*x* ε (A∪B)' Then by definition of complement

*x*ε (A∪B)' ⇒x∉ (A∪B)

⇒x ∉A and x ∉ B

⇒x∈ A' and x∈ B'

Thus x 8 (AUB) ⇒x∈ (A' ∪B')

$\therefore (A \cup B)' \qquad (A \cap B)'$

(1)

Next, let $y \in (A' \cap B')$. Then by definition

$y \varepsilon (A' \cap B') \quad \Rightarrow y \in A'$ and $y \varepsilon B'$

$\Rightarrow y \notin$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

$A' \cap B'. \qquad \subseteq (A \cup B)' \qquad (2)$

Hence form (1) and (2), we get

$(A \cup B)' = A' \cup B'$

(ii) Using definition of complement and proceeding and above, we can also prove the result

$(A \cup B)' = A' \cup B'$

Example 5. If A and B are two given sets, thenshow that

$A \cap (B - A) = \phi$

Solution : If possible, let $A \cap (B - A) \neq \phi$ where ϕ is the null set and A, B are not null sets. Then there is at least one element, say x. such that $x \in A \cap (B - A)$

$x \in A \cap (B - A) \Rightarrow x \in A$ and $x \in (B - A)$

$\Rightarrow x \in A$ and ($x \in B$ and $x \notin A$)

$\Rightarrow x \in A$ And $x \in B$ and $x \notin A$

which is absurd, since $x \in A$ and $x \notin A$ cannot holds simultaneously.

Hence $A \cap (B - A) = \phi$

If A, B are null sets, the result is obvious.

Example 6 : Let S = {1, 2, 3, 4, 5, 6} be the universal set.

Let $A \cup B = \{2, 3, 4\}$ find A^c A B^c, where A^c, $B^c$are the complements of A and B respectively.

Solution By De Morgan's Law, we have

$A^c \cap B^c \ (A \cup B)^c$

Again $\qquad (AB)^c = S - (A \cup B)$

$= \{1, 2, 3, 4, 5, 6\} - \{2, 3, 4,\}$

$= \{1, 5, 6\}$

Hence from (1), we get

$A^c \cap B^c = \{1, 5, 6\}$

SELF-CHECK EXERCISE 19.4

Q1. Let $\in = \{1, 2, 3, 4, 5, 6, 7\}$ and $A = \{1, 2, 3, 4, 5\}$

$B = \{2, 5, 7\}$ show that

(a) $(A \cup B)' = A' \cap B'$

(b) $(A \cup B) = B \cup A$

Q2. Let $P = \{a, b, c, d\}$ $Q = \{b, d, f),$ $R = \{a, c, e\}$

verify that $(P \cup Q) \cup R = P \cup (Q \cup R)$

19.7 SUMMARY

In this unit we have discussed notations used in set theory, operation of sets, building blocks of relations and functions. Starting with meaning of a set as one of collection of distinct objects, called elements, which are normally endeared within brockets and separated by commas, we went an to learn different ways of forming of sets. The operation an difference, i.e. all elements of one that are not elements of the other and compliment set viz all elements in the universal set that are not in a given set were covered.

19.8 GLOSSARY

1. **Complement set :** Set containing one set's elements that are not members of the other set.
2. **Disjoint set :** Sets having no members in common, having an intersection equal to the empty set.
3. **Element :** An object in a set.
4. **Power set :** The set of all subsets of a set.
5. **Set :** collection of objects, disregarding their order and repetition.
6. **Subset :** with respect to another set, a set such that each of the elements is also an element of the other set.
7. **Venn Diagram :** Diagram representing sets by circles or ellipses.

19.9 ANSWER TO SELF CHECK EXERCISE

Self-Check Exercise 19.1

Ans. Q1. Refer to Section 19.3

Ans. Q2. Yes, all empty sets are equal

Self-Check Exercise 19.2

Ans. Q1. Refer to Section 19.4.3

Ans. Q2. Refer to Section 19.4.4

Ans. Q3. Refer to Section 19.4.7

Ans.Q4. (i) $2^6 = 64$

(ii) 62

Self-Check Exercise 19.3

Ans. Q1. Refer to Section 19.5

Ans. Q2. Refer to Section 19.5.5

Ans. Q3. (i) {1, 2, 3, 4, 5, 6, 8)

(ii) {3, 4}

Self-Check Exercise 19.4

Ans. Q2. (a) L.H.S. = R.H.S. = {6}

(b) {1, 2, 3, 4, 5, 7}

Ans. Q3. {a, b, c, d, e, f}

19.10 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
4. Yamane, T. (2012). Mathematic for Economists : An Elementary Survey. Pretice Hall of India, New Delhi.
5. Mukherji, B. andPandit, V. (1982). Mathematical Methods for Economic Analyses, Allied Publishers Pvt. Ltd., New Delhi.

19.11 TERMINAL QUESTIONS

Q1. Prove that $A - (B \cup C) = (A - B) \cap (A - C)$

Q2. If u = {a, b, c, d, e, f} be the universal set and A, B, C, and three subsets of U_1 where A = {a, b, c, d, f}, $B \cap C$ = {a, b, f}, final $(A \cup B)$ $(A \cup C)$ and $B' \cup C'$.

Unit - 20

FUNCTIONS, LIMITS AND CONTINUITY

STRUCTURE

20.1 INTRODUCTION

Sets, relations and functions, inter alia are basic ingredients of mathematics and they have immense use in economics. In the study of economics, we come across situations where a certain relation exists between two or more economic variables. In order to examine the mathematical representation of such economic relationships, such as the relation- ship between cost of production and quantity produced, or between quantity demanded and price etc., we need to know how such relationship are handled in mathematics. The first step in doing this involves defining a distinct collection of entities as a set. The next step will be the examinations of the concept of "ordered pairs" followed by the final step of defining the concepts of "relations and functions."

20.2 LEARNING OBJECTIVES

After studying this unit, students will be able to -

- Define Functions
- Explain Limits
- Elucidate continuity of Function

20.3 ORDERED PAIRS

In writing a set of two numbers (x, y), we do not care about the order in which the elements x and y appear since by definition $(x, y) = \{y, x\}$. In such a case, the elements x and y are said to be "unordered pair." But when x and y have distinct meaning denoting, say, height and weight of students or price and demand of a commodity, the ordering of the pair of elements will have a particular significance. In such a case we write two distinctly different ordered pairs given by (x, y) and (y, x) such that $(x, y) \neq (y, x)$ unless $x = y$.

In general, a set consisting of two elements with the order of the elements specified say price and demand or height and weight, is called an "ordered pair." Ordered pairs are normally written in ordinary brackets as we have shown above.If we include another element Z, say age of the students or income of the consumers, then we can write ordered quintuples, etc. having the location of the elements in the specific order.

The ordered pair can be represented graphically in rectangular co-ordinate plane as shown in figure I dividing the plane into four quadrants. The xy plane is an infinite set of points, with each point representing an ordered pair whose first element is the value of x and the second element is the value of y. If we have two sets $x = \{2, 3\}$ and $y = \{4, 5\}$, we can generate all possible ordered pairs with (x, y) = (2, 4), (2, 5), (3, 4), (3, 5)

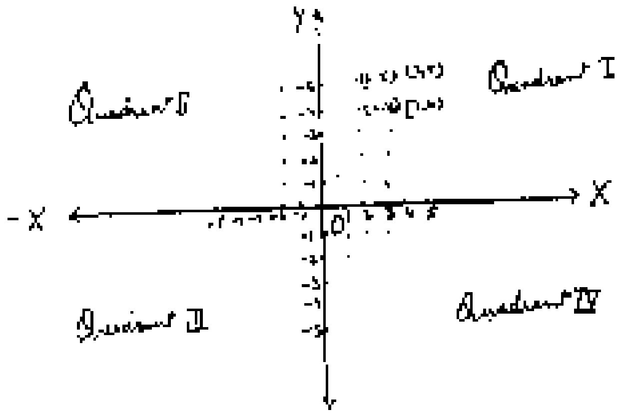


Fig. 1

Since an ordered pair indicates the value of y associated with a given value of x, the collection of ordered pairs will constitute a relation between y and x. The relation will give the value of y for a particular value of x. For example, there can be a relation between cost of production (y) and quantity produced (x) or between total revenue (y) and quantity sold (x) indicating that value of y depends on x.

In a given set {(x, y/y=2x), we can have various ordered pairs having the value of y double thevalue of x such as (-2, -1), (0, 0), (2, 1), (4, 2), ... which satisfy the equation y = 2x. This set constitutes a relation and is represented in the graph be- low (Fig. 2) by a straight line given by the set of points. In this particular relation, the equation y=2x provides the value of y associated with the value of x.

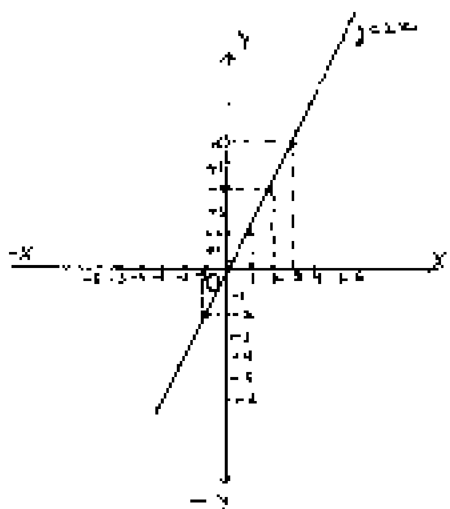


Fig. 2

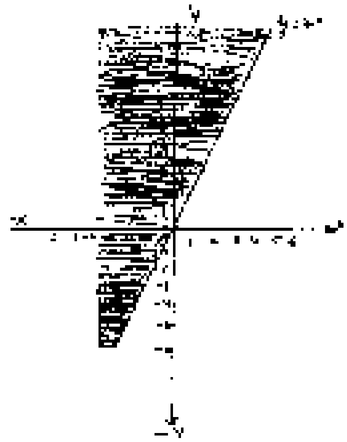


Fig. 3

Similarly another set {(x, y)/ Y≥2x)} provides a relation between y and x such that the ordered pairs satisfy the inequality y ≥2x. We can have the ordered pairs satisfying the above inequality as (-1, 2), (0, 2), (1, 2), (2, 5), (3, 11), etc. implying that y value will be equal to or greater than two times of the value of x. Such a relation is graphically represented by the set of all the points in the shaded area including the straight ling y=2x as shown in the Fig. 3. from the above two examples of sets

{(x, y)| y=2x} (1)

and {(x, y)| y≥2x} (2)

It appears that in the set (2), the relation between x and y is given by the inequality y≥2x. This means that each value of y associated with the value of x is an ordered pair must satisfy the inequality condition y≥2x. But in case of the set (1), we have a relation between x and y such that for each value of x there exists only one corresponding value of y.

This type of relation between y and x consisting of a set of ordered pairs with the property that the value of x determines a 'unique' value of y. is called a 'function'. In such a situation y is said to be a function of x and it is symbolically expressed as y= *f*(x). Here y is dependent variable and x is called independent variable or explanatory variable. It may be noted that *f* is a symbol implying a particular function. We can also use other symbols like g, h, 4, etc. to symbolize a particular function.Normally two different symbols should be used to indicate two different functions even of the same variable (s).

For example, if we write y = *f*(x) and y = g(x), it means that there exists relation between y and x in both the functions, but the nature of functional relations are different. It may be noted thatthe relation between y and x represented by the straight line y=2x qualities as a function where as the relation given in the inequality y ≥ 2x does not qualify as a function since there exists more than one value of y for a particular value of x.

But the relation between y and x given by the curve in Fig. 4 qualifies as a function. This ex- ample indicates that while the definition of a function requires a unique y for each x, but there may be cases where we can have a single value of y for more than one value of x.

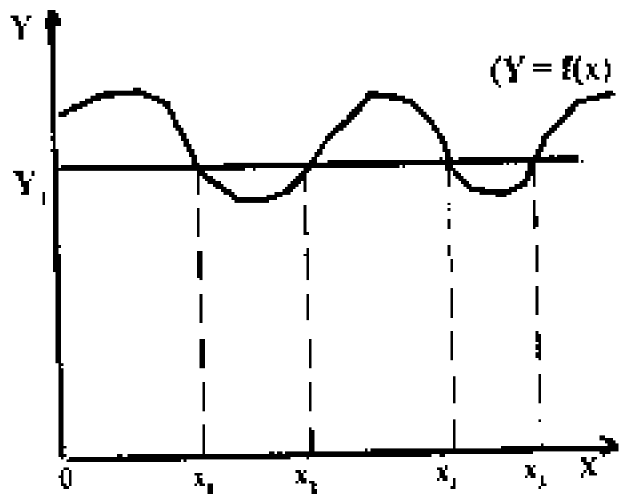


Fig. 4

The above fig clearly shows that in the function y = f(x), the value of $y=y_1$ is associated with four values of x viz,, x_1, x_2, x_3, and x_4.Here, it is appropriate to note that a function is also called a "mapping" or "transformation."

It is also relevant to distinguish between "do- main" and "range" of a function. In a function y= = f(x), the set of values that x can take in a certain context is called the "domain" of the function. But the set of values of y into which the set of values of x is mapped, is called the "range" of the function. Suppose, for example, the value of x is restricted to a set {|x| 2≥x≥-2}, then in the function y = 2x,the value of y will be restricted to the set {|y|-4≥y ≥4). So the value of x between -2 and +2 is called the domain and that of y between -4 and +4 is called the range of the function y = f(x)=2x.

The functional notation y=f(x) only states that there exists some functional relationship between x and y but does not tell us the exact way in which y depends on x. If we assume that our function y = f(x) is given as y=2x+3, then this equation states the exact functional relationship viz linear relation- ship between the two variables x and y. From the given equation, given the values of x variable, we can find the corresponding values of the dependent variable y. Here we say that y is an explicit function ofx.

20.4 FUNCTION

20.4.1 EXPLICIT AND IMPLICIT FUNCTIONS:

y is said to be an Explicit Function of x ify is expressed directly in terms of x in the form y = f(x) e.g. y=2x+3, $y=x^2+x-1$, y=log x + 4.

But if y and x are mixed up in the functional relation of the from f(x, y) = 0, y is said to be an Implicit Function of x.

e.g. 2xy + 3x+4y+5=0, $x^2 + y^2 = a^2$ are examples of implicit function.

If it is to solve the equation $f(x) = 0$ fory, the implicit function may be changed into an explicit one.

e.g. the implicit form $x^2+y^2=a^2$ can be written inthe explicit form as

$y = \pm \sqrt{a^2 - x^2}$

In such cases, each of the two functions is called the inverse function of the other.

Single-Valued and Many - Valued Function

y is said to the single - valued function of x if a value of x gives rise to only one value of y e.g.$y = x^2+4$, $y = \log x$, $y = e^x$ are all single valued function of x, y is said to be Many Valued Function of x if a value of x gives rise to more than one value of y e.g...$y^2=x(x\geq 10)$. Y $\tan^{-1} x$ are examples of many valued functions.

20.4.2 EVEN AND ODD FUNCTIONS

$y=f(x)$ is said to be an Even function of x if

$f(-x) = f(x)$ (1)

e.g. $y=x^2$, $y = \cos x$, $y = x^2$ are all examples or even functions

$\therefore$in all these examples $f(-x) = f(x)$ (2)

Similarly a function $y = f(-x)$ is said to be oddfunction of x if

$f(-x) = f(x)$

e.g., $y=x^3$, $y = \sin x$ are examples of odd function.

$\therefore$in these examples $f(-x) = -f(x)$.

20.4.3 INVERSE FUNCTION:

Ify be a function of x given by the relation $y = f(x)$. then the relation which ex- presses x as a function of y (if such a function is possible) is called the Inverse function of y and is symbolically written as

$x = f^1 (y)$

For example, if $y=x^2$, the inverse function is $x = \pm \sqrt{y}$

ify=sin x, the inverse function if $x=\sin^{-1}y$

if $y=e^x$, the inverse function is x=logy.

20.4.4 INCREASING AND DECREASING FUNCTION

y is said to be an increasing function of x if the value of y always increases and x increases, y is said to be a decreasing function of x if the value of y always decreases as x increases. The class of in- creasing and decreasing function together is known as monotonic functions. The function will be called increasing function if the curve of thefunction rises from left to right without interruption and called monotonically decreasing function if the curve of the function falls from left to right without interruption. Demand functions are monotonically de- creasing functions and total cost functions are monotonically increasing function. For example

$$f(x) = \frac{1}{x} \quad X \neq 0$$

is a monotonically decreasing function ofx and

$f(x) = x^2+2$

is a monotonically increasing function of x. There are some function which may increase as x increases for some value of x and may decrease over values of x. Such functions are not monotonic.

20.4.5 TYPES OF FUNCTION

Functions are divided into two broad groups algebraic and non-algebraic. Algebraic functions include basically polynomial function and rational function. But the non-algebraic functions broadly com- 3 prise exponential, function, logarithmic functions, s trigonometric functions etc.

20.4.5.1 Constant Functions:

A Function whose range consists of only one specific value, is called a constant function. Or in other words, when the value of y in a function y = f(x) does not change or remains the same irrespective of the values of x, the said function is called a constant function. So a constant function is expressed as $y = f(x)=C=$(constant).

For example, the average revenue (AR) function under perfect competition is a constant function. Since total revenue is a function of quantity sold, AR is also a function of quantity (Q). But the Aver- age Revenue under perfect competition is fixed and so the AR curve is horizontal to the X-axis. The following figure 5 shows that when the output (0)increase from Q_1 to Q_2 and from Q_2 to Q_3 etc. the AR or Price remains the same at the level OP.

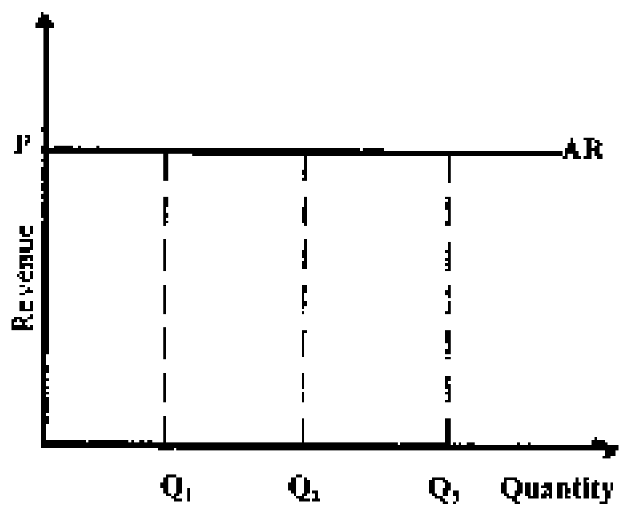


Fig. 5

Another example of constant function may be cited from national Income models, where investment is determined exogenously. In such a case, we may have the investment function of the form $I = I_0$ = Rs.100 crores where I_0 is a fixed value of investment.

20.4.5.2 Polynomial Function

A constant function referred above is also a formof polynomial function. The general form of a poly nominal function of a single variable x is given by

$$y = \propto_0 + \propto_1 a + \propto_2 x^2 + \propto_3, x^3 + —\propto_n x^n$$

Where $\propto_0$, $\propto_1$, $\propto x_2$——$_n$ are the parameters. The above function has the largest power of x equal to n and, therefore, is called a polynomial function of degree n. Depending on the value of n, we can have several sub-classes of polynomial functions. For instance, when

n=0,	$y = \propto_0$, it is a constant function
n = 1,	$y=\propto_0+\propto_1 x$,it is a linear function,
n - 2,	$y = \propto_0 + \propto_1 x + \propto_2 x^2$, it is a quadratic function.
N = 3,	$y = \propto_0+\propto_1 x + \propto_2 x_2+\propto_3 x_3$, it is acubic function.

Similarly, we can have polynomial of fourth degree or fifth degree or sixth degree etc. depending on n = 4 or 5 or 6.

The parameter $\propto_0$ in a polynomial function represents the intercept of the curve on y-axis. The general form of a linear function is a straight line as shown below in Figure 6. where $\propto_0$ is the intercept of the curve and $\propto_1$ is the slope of the curve.

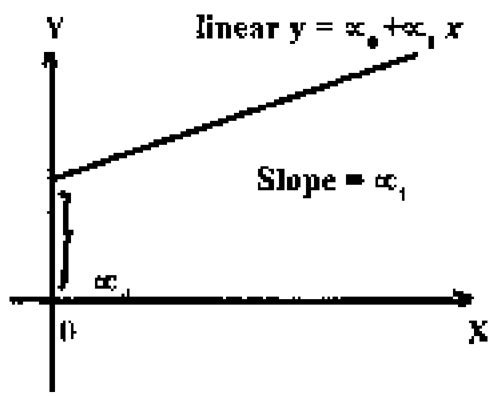


Fig. 6

Quadratic $y = \propto_0+\propto_1 x +\propto_2 x^2 \propto_1 >0, \propto_2< 0$

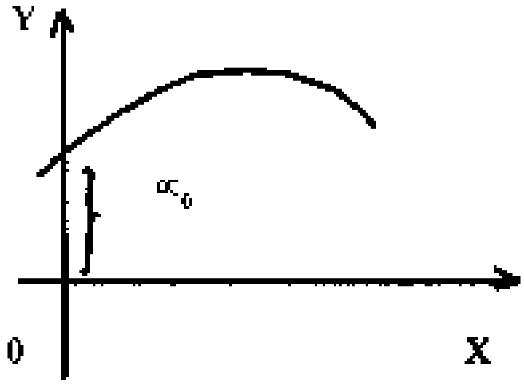


Fig. 7

Cubic $y = \propto_0 + \propto_1 x + \propto_2 x^2 + \propto_3 x^3 \propto_1 > 0, \propto_2 > 0, \propto_3 > 0$

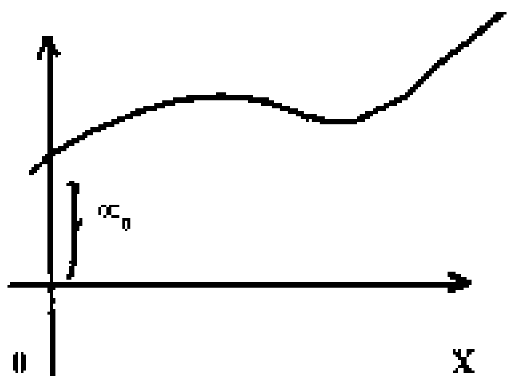


Fig. 8

The general form of the quadratic and cubic functions are shown in Figures 7 and 8 respectively. The shape of the parabolic and cubic functions will be different if $\propto_2>0$ or $\propto_1<0$ or $\propto_3<0$. So the exact shape of the polynomial functions will depend on the sign and value of the parameters.

20.4.5.3 Rational Function

A function which is expressed as the ratio of two polynomial functions in the same variable x is called a 'rational function'. For example, if a function y = *f*(x) is defined as

$y = f(x) = \quad \propto_3 + \propto_3 x$

$\beta_0 + \beta_1 x + \beta_2 x^2$

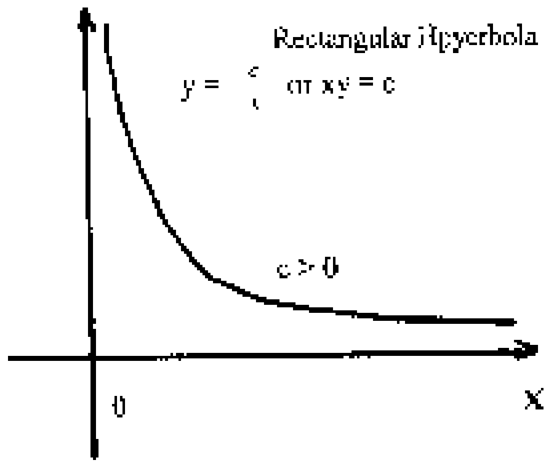
where $\propto_0$, $\propto_1$, β_1,β_2 are parameters, it is rational.

There can be special form of rational function in the form

$y = f(x) = \frac{c}{x}$ (c is a constant)

or $\quad xy=c$

which has an interesting application in economics. It is a rational function as it is a ratio of a constant function to a linear function having zero intercept and unitary slope. The shape of such a rational function is shown in figure 9.



The curve is convex to the origin and is asymptotic to both x-axis and y-axis implying that the curve will never touch the x-axis or y-axis, for all positive values of x. Such a rational function is popularly known as 'rectangular hyperbola.' The indifference curve in consumer's behaviour or the isoquant curve in production behaviour are examples of rectangular hyperbola.

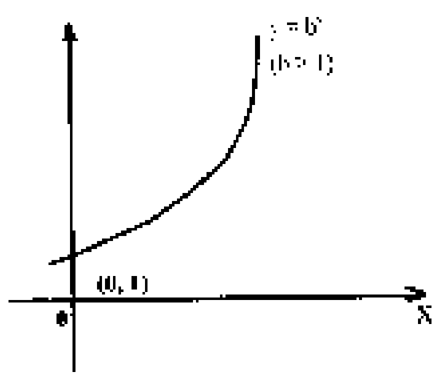
20.4.5.4 Exponential Function

In an algebraic function the exponent of a variable happens to be a constant such as x^2 or x^3 or any power ofx.

But it is also possible to have a function where the independent variable is the exponent of a constant such as 5^x or 2^x etc. So a function whose independent variable appears as the exponent of a constant is called an exponential function. The simplest form of exponential function may be represented in the form

$y = f(x) = b^x \; (b>1)$

The standard shape of an exponential function is given in figure 10



The curve passes through the (0, 1) intersecting y–axis and the slope of the curve dependson the value of b.

But in the exponential function the base value 'b' may be replaced by a certain irrational number denoted by e = 2.71828. When e is taken as base value in such function, it is termed as 'natural exponential function', and is defined as $y = e^x$. The generalized form of natural exponential function is defined as $y = f(x) = Ae^{rx}$ where A and r are constants.

20.4.5.5 Logarithmic Function

In a function where the dependent variable (y) is a function of the logarithm of the independent variable x such that $y=\log_{10}x$ for common logarithm

or $y=\log_e x$ for natural logarithm.

The function is known as logarithmic function. The standard shape of a logarithmic function is shown in figure below.

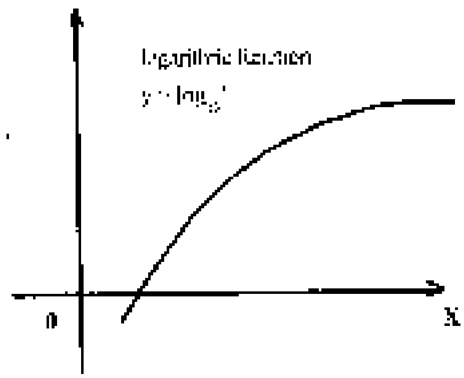


Fig. 11

20.4.6 Functions in Economics

Functions are very important in economics as economics is concerned with functional relation- ships between measurable quantities. We shall discuss some important function is economics.

1. Demand function: $q_d=a - bp$ with constant negative slope = -b.
2. Supply function: $q_s = -c + dp$, with constant positive slope = d.
3. Consumption function: C=a+cY, where c=marginal propensity to consume.

We have assumed that all these functions are linear but sometimes these relations or certain other relations suggested by economic theory can be adequately represented only by non-linear form e.g. constant elasticity demand functions. Cobb-Douglas production functions, U-shaped marginal cost functions.

Non-Linear Functions in Economics

(i) If initial income is y, and income grows at g percent per year, then income after t years is

$Y_t = y_o(1+g)^t = y_o r^t$, where r = 1 + g

Here income is said to be an exponential function of time. In other words, income is growing at an exponential rate.

(ii) If consumption is taken as a logarithmic function of income then

$y = \alpha + \beta \log X$ ($\propto > \beta > 0$).

This is termed as semi-log transformation.

Here when y = 0, $\log X = \frac{-\alpha}{\beta}$ or $x = e^{-\alpha\beta}$

(iii) Production function is generally of the type

$y = Ax_1^{\alpha} x_1^{\beta}$

where x_1 and x_2 are labour and capital respectively. If we take only one factor labour ($\therefore$ in the short run capital remains constant and hence can be combined with $x_1^{\propto}$ then the Cobb-Douglas production will be

$y = Ax^{\alpha}$

where x is labour. Obviously when x = 0, y = 0 and y increases with x if $\beta > 0$. If$\beta > 1$, output changes at increasing rate with change in labour (x) i.e. there are increasing returns. If $0 < \beta < 1$, there are decreasing returns.

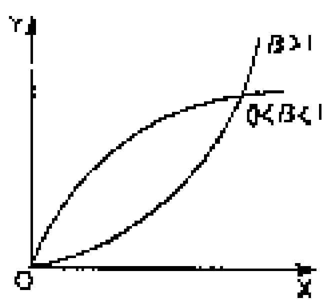
Since $y = Ax^{\alpha}$

Taking logarithms, we get

$\log y = \log A + \alpha \log x$

or $\log y = A' + \alpha \log x$

(where A'=log A)



This is termed as double logarithmic transformation. (iv) Demand is an inverse function of price i.e. if price increases, demand decreases by such an amount that total expenditure remains constant, price elasticity of demand is 1.

Hence p. q.= constant, = a^2 (say)

or $P = \frac{1}{3x}$

This represents a rectangular hyperabola. Here neither price nor demand can be zero for finite value of the other variable.

(v) U-shaped marginal and average cost function can be represented by parabola, the general equation being

$C = a + bq + cq^2$

where q is outpur and c is M.C.

We have to choose a, b, c in such a manner that c and q always lie in positive quadrant.

(vi) Total Revenue function expressed as

$R = p \times q$

Let $p = a - bq$

then $R = (a - bq) \times q = aq - bq^2$

If we take $p = 20 - 10q$, then

$R = 20q - 10q^2$

and $AR = \frac{R}{q}$ where AR = average revenue functions.

Thus average revenue and price imply one and the same thing,

We have discussed only a few important functions in economics. There are many other functions e. g. utility function, supply function, investment function etc. which can be similarly described.

Example 1: (i) if $f(x) = x^6 - 2x^4 + 5$.

show that $f(-x) = f(x)$

(ii) if $f(x) = 2x^2 + 3x + 4$.

find $f(0)$. $f(1)$ and $f(½)$

(iii) if $(x) = \frac{x}{x+1}$

Solution: (i) $f(x) = x^6 - 2x^4 + 5$

$f(-x) = (-x)_6 - 2(-x)_4 + 5$

Hence $(-x) = f(x)$

$\therefore$ the function is even.

(ii) $f(x) = 2x^2 + 3x + 4$

$f(x) = 2(0)^2+2(0)+4$

$= 0+0+4$

$= 4$

$f(x) = 2(1)^2+3(1)+4$

$= 2+3+4$

$= 9$

$f(x) = 2(-1)^2+3(-1)+4$

$= 2.1+3(-1)+4$

$= 2-3+4$

$= 3$

$f(x) = 2.(½)^2+3.½+4$

$= 2½ + 3½ + 4$

$= 2 + 4$

$= 6$

(iii) $f(x) = \frac{x}{x+1}$

$\therefore f(2) = \frac{2}{2+1} = \frac{2.}{3}$

$\therefore f[(2)] = f\left(\frac{2}{3}\right) = \frac{2/3}{2/3+1} = \frac{2}{3} \times \frac{3}{5} \times \frac{2}{5}$

Example 2: Find the domain of definition of the following functions:

(i) $\sqrt{2x+1}$ (ii) $\frac{1}{\sqrt{x-3}}$ (iii) $\frac{1}{\sqrt{(x-2)(x-4)}}$

Solution: (i) Let $f(x) = \sqrt{2}x+1$

$f(x)$ is defined only for those values of x for which $2x + 1$ is ≥ 0 otherwise $f(x)$ becomes imaginary.

$\therefore 2x+1 \geq 0$ or $2x \geq -1$ or $x \geq -½$.

Hence $f(x)$ is defined only for those $x \geq -½$ which is its domain of definition.

(ii) Let $f(x) = \frac{1}{\sqrt{x-3}}$

$f(x)$ is defined only for those values of x for which $x - 3 > 0$

∵If x-3=0, $f(x)=\frac{1}{0}$ which is not defined.

if x - 3<0, f(x) is imaginary

∴ x -3> 0 which gives x > 3

Hence f(x) is defined only for values of x>3, which is its domain to definition.

(iii) Let $f(x) = \frac{1}{\sqrt{(x-2)(x-4)}}$

f(x) is defined for only those values of x for which(x-2)

(x-4) is non-negative, for f(x) is imaginary when

(x – 2) (x – 4)≥0 then two cases arise.

Case I x –2≥ and x – 4≥0

⇒ x≥2 and x≥4

⇒ x ≥4

Case II x – 2 ≤ 0 and x – 4≤0

⇒ x≤2 and x ≤4

⇒ x ≤ 2

Hence f(x) is defined only for values of x≥4 orx≤ 2 which is its domain of definition.

SELF-CHECK EXERCISE 20.1

Q1. Point out the domain of definition of the following functions :

(i) $\sqrt{2x+1}$

(ii) $\frac{1}{\sqrt{x-2}}$

Q2. Find the inverse of y = 3x – 2 ⇒

20.5 LIMITS

Before defining limits, we would explain the concept of Absolute Value of Numbers.

Modulus (or Absolute value) of a Real Number:

We define the modulus (or absolute value) of a real number (x) as follows:

x=x if x≥0

= x if x <0.

i.e. modulus of a number is always positive. It is in fact the numerical value of the number regardless of its sign. For example, absolute value of – 4 as well as +4 is 4. i.e. |-4|= 4 and |+ 4| = 4

If x and y are assumed to be two real numbers, then the following properties hold good:

(i) $|xy| = |x|\,|y|$

(ii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

(iii) $|x + y| \le |x| + |y|$

(iv) $|x - y| \ge |x| - |y|$

Now we will introduce the concept of limit of a function.

Let $y = \frac{1}{3x}$ where

y is single–valued function of x. The each value ofx, there corresponds one and only one value of y. We want to see the behaviour of the function as a sequence of values are allotted to x. From the above function, we get

x :	1	2	3	4	.... 100	
y:	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{12}$	.... $\frac{1}{300}$	

To the x sequence, there corresponds a y sequence. y sequence has been obtained according to some rule and not as arbitrary numbers from the sequence. It is obvious that as x becomes larger and larger, y or f(x) becomes smaller and smaller. Continuing this argument, we say that as x tends to infinity (i.e. very-very large) y tends to zero. It may be noted that y can never be equal to zero by making x larger and larger, but it can be very close to zero. Let us consider the function

$y= f(x) = \frac{x^2 - 1}{x - 1}$

Let x approach 1 through values < 1, then the corresponding values of y or $f(x)$ are shown asbelow.

x:	.9	.99	.999	...	...	...	...
y:	1.9	1.99	1.999	...	...	...	...

Let x approach 1 through values > 1, then we get

x :	1.1	1.01	1.001	...	...	...	...
y :	2.1	2.01	2.001	...	...	...	...

We observe that x takes values nearer and nearer to 1 remaining always < 1 as in the first case > 1 asin the second case, y or $f(x)$ takes values nearer and near to 2. Thus, the difference between $f(x)$ and 2 can be made as small as we please by giving x a value sufficiently close to 1. In other words as x approaches 1 (written as $x \rightarrow 1$), $f(x)$ tends to the limit 2 as x tends to 1.

i.e. $f(x) \to 2$ as $\lim_{x \to 1}$ or Lim $f(x)=2$

Now we define limit of a function at a point x = a.

Definition: If x approaches a (through values < a or>a) and thereby $f(x)$ approaches a real number ℓ, then $f(x)$ can be brought as near to ℓ as we please by bringing x close enough to a but $x \neq a$. In this case we say that $f(x)$ tends limit ℓ as x tends to a and write it as

$f(x) \to \ell$ as $x \to a$ or $\lim_{x \to a} f(x) = \ell$

Note:-

1. When $x \to a$ through values which are greaterthan a, we say that x approaches a from the right(i.e. $x \to a+0$)
2. When $x \to a$ through values which are less than a, we say that x approaches a from the left (i.e. $\to a - 0$)
3. In the first case $\lim_{x \to a} f(x)$ is called the right-hand limit of $f(x)$ and in the second case $\lim_{x \to a} f(x)$is called the left hand limit of $f(x)$.

Existence of the limit of a function

$\lim_{x \to a} f(x)$ is said to exist if

(i) left hand Limit and right hand exist

(ii) left hand Limit=right hand limit.

Thus if $\lim_{x \to a} f(x) = \lim_{x \to a} f(x) = \ell$, only then we say that:

$\lim_{x \to a} f(x) = \ell$

Formal Definition of Limit

The function $f(x)$ tends to a limit ℓ as x tends to a if the numerical difference between $f(x)$ and ℓ can made as small as we like by making the positive difference between x and a small enough. In symbols we write

$\lim_{x \to a} f(x) = \ell$

To be more rigorous, we state that $f(x)$ tends to limit ℓ as x tends to a, if for each given $\in>0$ however small, there exists a positive number δ (that depends on $\in$) such that

$| f(x) - \ell / < \in$

for all values of x for which 0 <1x – a≤ δ. The condition (f(x)-) ℓ <∈ is equivalent to the condition ℓ– ∈<f(x) < ℓ +∈. Hence it is clear that the limit exists if f(x) can be confined to any arbitrary small interval (ℓ – ∈, ℓ +∈)

20.5.1 Distinction between the Value and Limit of a Function

The value of a function f(x) as x = a is obtained by putting x = a. Limit of a function as x → a is obtained by considering the values of x in the Lim neighbourhood of a. Thus $\lim_{x \to a}$ f(x) may exist even if the function is not defined at x = a. For example, we have seen above that

$$\lim_{x \to a} \frac{x^2 - 1}{x - 1} = 2$$

where as the value of the function is not defined at 1.

$\because f(1) = \frac{1-1}{1-1} = \frac{0}{0}$ which is indeterminate.

Infinite Limits and Variable tending to Infinity

1. A function f(x) is said to tend to infinity (+ ∞ or -∞) as x tends to a. If for each arbitrarily assigned positive value G, no matter how large, we can find a positive number δ such that

f(x)>G (or < – (i)

for all values of x for which 0<(x – a) < δ.

2. A function f(x) is said to tend to a limit ℓ as x tends to + ∞ (or -∞), if to each arbitrarily assigned positive number δ no matter how small, wecan find a positive number G such that

| f(x)| ≠ ℓ<δ

for every value of x> (or < – G).

In less rigorous language, the function f(x) tends to limit ℓ as x ends to + ∞ (or – ∞) if it is possible to make the positive difference between f(x) and as small as one likes by making x large (or small enough)

Now we shall state (without proofs) important theorems on limits which help us in solving problems.

20.5.2 Theorems on Limits:

Let f(x) and g(x) are functionsof x, then

1. Lim [f(x)+g(x)] = Lim f(x) + Lim g(x)

i.e. the limit of the sum of two functions is equal to the sum of their limits.

2. Lim [f(x) – g(x)] = Lim f(x) – Lim g(x)

3. Lim [*f*(x). g(x)= Lim *f*(x). Lim (x)

i.e. the Limit of the product of two functions is equalto the product of their limits.

4. $\left[\frac{f(x)}{g(x)}\right] = \frac{Lim\ f(x)}{Lim\ g(x)}$

For example, if g (x) = x^2 and *f*(x)

Then as x→2, g(x) →4 and *f*(x)→12.

Then Lim | *f*(x)+g(x) = Lim *f*(x) + Lim g(x) = 12 +4=16

Lim *f*(x) – g(x) = Lim *f*(x) – Lim g(x)=12 – 4= 8

Lim | *f*(x). g(x) = Lim *f*(x) Lim g(x) = 12.4=48.

Lim [*f*(x)/g(x)]= $\frac{Lim\ f(x)}{Lim\ g(x)} = \frac{12}{4} = 3$

Illustrative Examples

Example 3: Evaluate

(i) $\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1}$ (ii) $\lim_{x \to 1} \frac{x^2 - 9}{x - 3}$ (iii) $\lim_{x \to 1} \frac{x^n - a^n}{x - a}$

(a being +ve and x is any real number different from

Solution : (i) Here y = *f* (x) $= \frac{x^3 - 1}{x^2 - 1}$

Put x = 1 + h so that as x → 1. h → 0.

$$\lim_{x \to 1} \left(\frac{x^3 - 1}{x^2 - 1}\right) = \lim_{h \to 0} \frac{(1+h)^3 - 1}{(1+h)^2 - 1}$$

$$= \lim_{h \to 0} \frac{(1+3h+3h^2+h^3)-1}{(1+2h+h)^2 - 1}$$

$$= \lim_{h \to 0} \frac{3h+3h^2+h^3}{2h+h^2}$$

$$= \lim_{h \to 0} \frac{(3+3h+h^2)}{h(2+h)}$$

$$= \lim_{h \to 0} \frac{3+3h+h^2}{2+h}$$

$$\left(\begin{array}{l}\text{as h } \to \text{ 0 and h} \neq 0 \\ \therefore \text{h can be cancelled}\end{array}\right)$$

$$= \frac{3 + 3.0 + 0^2}{2 + h} = \frac{3}{2}$$

Second Method

$$\lim_{x \to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x+1)(x+1)}$$

$$= \lim_{x \to 1} \frac{(x^2+x+1)}{(x+1)}$$

$$\left[\begin{array}{l}\text{Since } x \to 0 \text{ and } x \neq 1 \\ \therefore x - 1 \text{ can be cancelled}\end{array}\right]$$

$$= \frac{(1)+1+1}{1+1} = \frac{3}{2}$$

$$\lim_{x \to 3} \left(\frac{x^2 - 9}{x - 3}\right)$$

$$\lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)}$$

$$\lim_{x \to 3} (x-3)$$

$$\left[\begin{array}{l}\text{Since } x \to 3 \text{ and } x \neq 3 \\ \therefore x - 3 \text{ can be cancelled}\end{array}\right]$$

$= 3 + 3 = 6$

$$\lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right)$$

Let $x = a$ + h so that as $x \to a.\ h \to 0$

$$\lim_{x \to a} \left(\frac{x^n - a^n}{x - a}\right) = \lim_{h \to 0} \left[\frac{(a+h)^n - a^n}{(a+h)a}\right]$$

$$= \lim_{h \to 0} \left[\frac{(a+h)^n - a^n}{h}\right]$$

$$= \lim_{h \to 0} \left(\frac{a\left(1 + \frac{h}{a}\right)^n - a^n}{h}\right)$$

$$= \lim_{h \to 0} \left[\frac{a^n \left(1+\frac{h}{a}\right)^n - a^n}{h} \right]$$

$$= \lim_{h \to 3} a^n \left[\frac{1+\frac{h}{a} + \text{2nd higher power of } h}{h} \right] -1$$

(By Binomial Theorem. $\left|\frac{h}{a}\right| < 1$)

$$= \lim_{h \to 0} a^n \left[\frac{1+n\ \frac{h}{a} + \text{2nd higher power of } h}{h} \right] -1$$

$$= \lim_{h \to 0} a^n\ \frac{h}{a} \left[\frac{n+\frac{n(n-1)}{2}\frac{h}{a} + \text{higher power of } h}{h} \right] -1$$

$$= \lim_{h \to 0} a^{n-1} a \quad n+\frac{n(n-1)}{2}\frac{h}{a} + \text{higher power of } h \quad -1$$

$= a^{n-1}$ [n+0+0+ ... ...|

$= n\ a^{n-1}$

Hence $\lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right)$ n a ($a^{n-1} > 0$)

Example 4: Evaluate (i) $\lim_{\lambda \to 0} \frac{\sqrt{(x+a)}}{x} - 1$

(ii) $\lim_{h \to 0} \frac{1}{h} \frac{1}{\sqrt{x=h}} + \frac{1}{\sqrt{c}}.$

Solution : (i) If we directly calculate by put yx = 0 we get 0/0 which is indeterminate. Hence to seek the limits of the given function, we must divide out x from the denominator.

$$\therefore \lim_{h \to 0} \frac{\sqrt{(x-a)}-1}{x} = \frac{\left(\sqrt{x+1}-1\right)\left(\sqrt{x+1}-1\right)}{\left(x\sqrt{x+1}+1\right)}$$

By Rationalisation)

$$= \frac{(x+1)-(1)}{x[\sqrt{x+1}+1]}$$

$$= \frac{x}{x[\sqrt{x+1}+1]}$$

$\because x \to 0$ and $x \neq 0$

$\therefore x$ can be cancelled)

$$= \frac{x}{\sqrt{x+1+1}}$$

$$= \frac{x}{\sqrt{0+1+1}}$$

$$= \frac{1}{1+1}$$

$$= \lim_{h \to 0} \frac{1}{h} \left(\frac{1}{\sqrt{x=h}} + \frac{1}{\sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sqrt{x}}{\sqrt{x}} - \frac{\sqrt{x+h}}{\sqrt{x+h}} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \text{ x } \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{(x) - (x+h)}{h \ \sqrt{x+h}\,[\sqrt{x} + \sqrt{x+h}]}$$

$$= \lim_{h \to 0} \frac{-h^1}{h \ \sqrt{x+h}\,[\sqrt{x} + \sqrt{x+h}]}$$

$$= \lim_{h \to 0} \frac{-1}{h \ \sqrt{x+h}\,[\sqrt{x} + \sqrt{x+h}]}$$

$\because h$ 0 and $h \neq 0$

$\therefore h$ can be cancelled)

$$= \frac{-1}{\sqrt{x}\sqrt{x} + [\sqrt{x} + \sqrt{x}]}$$

$$= \frac{-1}{x.2\sqrt{x}} \qquad = \frac{-1}{2x^{3/2}}$$

Example5: Evaluate

$$\lim_{x \to \infty} \frac{4x^2 + 5x + 6}{3x^2 + 4x + 5}$$

Solution : Note : To evaluate $\underset{x \to \infty}{Lim} \frac{f(x)}{g(x)}$ where f (x) and g (x) are polynomials in x, divide f(x) and g(x) by the highest power of x in the fraction $\frac{f(x)}{g(x)}$ As x → ∞, $\frac{a}{x}$, b etc. all → 0.

$$\therefore \underset{x \to \infty}{Lim} \frac{4x^2 + 5x + 6}{3x^2 + 4x + 5}$$

$$4 + \frac{5}{x} + \frac{6}{x^2}$$ (Dividing the numerator and denominator by x^2)

$$3 + \frac{4}{x} + \frac{5}{x^2}$$

$$= \frac{4+1-0}{3+0+0} = \frac{4}{3} \qquad \left(\frac{5}{x}\ \frac{6}{x},\ \frac{4}{x},\ \frac{5}{x^2} all \to 0\right)$$

Example 6. Prove that

(i) $\underset{x \to \infty}{Lim} \left(1 + \frac{1}{n}\right)$ = e

(ii) $\underset{x \to 0}{Lim} (1+x)^1 = e$

(iii) $\underset{x \to 0}{Lim} \frac{a^a - 1}{x}$ = log a

(iv) $\underset{x \to 0}{Lim} \frac{a^a - 1}{x}$ = 1

Solution : (i) As n→ ∞, $\frac{1}{n}$ is positive and less than unity and therefore expansion of $\left(1 + \frac{1}{n}\right)^n$ by Binomial. Theorem for any index is possible.

$$\therefore \left(1 + \frac{1}{n}\right)^n = 1 + n.\ \frac{1}{2} + \frac{n(n-1)}{n}$$

$$\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3} \frac{1}{n^3} + ...$$

$$= 1 + 1 = \frac{1}{2}\left(1 + \frac{1}{n}\right) + \frac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + ...$$

As n→ ∞, $\frac{1}{n}$, $\frac{1}{n^2}$.......... all → 0

$$\lim_{n \to 0} \left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2}+\frac{1}{3}+..........$$

(By Def.)

(ii) Putting $x = \frac{1}{n}$ so that as $\to \infty$, $n = \frac{1}{x} = 0$

$$\therefore \lim_{n \to 0} \left(1+\frac{1}{n}\right)^n \lim_{n \to 0} = (1 + x) = e....$$

$$= \lim_{n \to 0} \frac{a^a - 1}{x} \lim_{n \to 0} \left[\frac{1}{x}(a^a - 1)\right]$$

$$= \lim_{n \to 0} \frac{1}{x} \left[\left\{1 + x\log a - \frac{x^2(\log a)^2}{2} + ...\right\} - 1\right]$$

$$= \lim_{n \to 0} \frac{1}{x} \left[1 + x\log a - \frac{x^2(\log a)^2}{2} + ...\right]$$

$$= \lim_{n \to 0} \frac{1}{x} \left[1 + x\log a - \frac{0(\log a)^2}{2} + ...\right]$$

= log a + 0 + 0 + ...

= log a.

$$\lim_{x \to 0} \frac{a^a - 1}{x} = \log e. \qquad \text{(from (iii) above)}$$

Example 7. Evaluate (i) $\lim_{x \to 0} \frac{a^a - b^a}{x}$

(ii) $\lim_{x \to 0} (1 + ax)^{lx}$

Solution : (i) $\lim_{x \to 0} \left(\frac{a^a - b^a}{x}\right)$

$$= \lim_{x \to 0} \left[\frac{(a^a - 1) - (b^x - 1)}{x}\right]$$

$$= \lim_{x \to 0} \left[\frac{(a^a - 1)}{x} - \frac{(b^x - 1)}{x}\right]$$

$$= \lim_{x \to 0} \left(\frac{(a^a - 1)}{x}\right) - \lim_{x \to 0} \left(\frac{(b^x - 1)}{x}\right)$$

$= \text{Log a - log b} = \log \frac{a}{b}$

(ii) $\lim_{x \to 0} (1+ax)^{lx}$

$= \lim_{x \to 0} \left[(1+ax)^{l/ax}\right]$

$= \lim_{y \to 0} \left[(1+y)\right]$

$= \lim_{y \to 0} \left[(1+y)\right] [\because \lim_{y \to 0}(1+y)^{ly} = e$

$= e^a.$

Example 8. Evaluate $\lim_{n \to 0} \frac{e^x - e^{-a}}{x}$

Solution :

$= \lim_{n \to 0} \frac{e^x - e^{-1}}{x} \lim_{n \to 0} \frac{e^a - e^{-x}}{x} \frac{e^x(e^x - e^{-x})}{e^x.x}$

$= \lim_{n \to 0} \frac{e^{-a} - 1}{x}$

$= \lim_{n \to 0} \left(\frac{e^{-a} - 1}{x} \times \frac{1}{e^a}\right)$

$= \lim_{n \to 0} \frac{e^2 - 1}{x} \times \lim_{n \to 0} \frac{1}{e^a}$

$= \lim_{z \to 0} \frac{e^z - 1}{z/2} \times \lim_{n \to 0} \frac{1}{e^a}$

$= \lim_{z \to 0} \frac{e^z - 1}{z} \times 1$ where 2x = z or n = z/2 cm n→0. z→0.

$= 2 \times 1 \times 1 = 2$

SELF-CHECK EXERCISE 20.2

Evaluate the following limits

(i) $\lim_{x \to 0} \frac{\sqrt{a+x}+\sqrt{a-x}}{x}$

(ii) $\lim_{x \to 0}$ Sin x = 0

(iii) $\lim_{x \to 0}$ sin x = sin θ

(iv) $\lim_{x \to 1} \frac{x^2-4}{x-3}$

20.6 CONTINUITY OF FUNCTIONS

1. A function y = f(x) is said to be continuous atx=c if for any positive number∈, however small, there exists a positive number (depending on ∈), such that

| f(x) – f(c) | <∈for |x – c| ≤ δ

It can be defined in other way as follows:

2. A function f(x) is said to be continuous at x=c, if for any positive number ∈, however small, there exists a positive number δ (depending on ∈) such that

f(c) – ∈ <f(x)<f(c) + ∈ for

c – δ ≤ x ≤ c + .

3. In simple language. A function f(x) is continuous at

x – cif Lim f(x) – f(c).

x → c

For continuity of functions, the following three conditions must be fulfilled.

1. Lim f(x) exists. i.e. right hand limit and left hand x → c.
2. The value of a function f(x) at x=c exists i.e.f(c) exists.
3. Lim f(x) = f(c) i.e. Limit of function and value ofthe x → c.

function are same at that point

Note: A function f(x) is continuous is an interval (a, b) if it is continuous at every point of the interval.

Note: If any of the three conditions is not fulfilled,the functions is discontinuous.

Note: Continuity represents the agreement between limit and value where both exists, i.e. the assumes a definite value of f(a) at the point and that f(x) tends to the same value f(x) as x approaches a from either side. Hence the curve has no gaps or jumps at x = a.

Example 9. Examine whether the function is continuous or discontinuous.

(i) f (x) = $x^z - 1$ at x = 1

(ii) f (x) = $\frac{x^z - 1}{x - 1}$ at x^z - 1

(iii) f (x) = $\frac{1}{x-1}$ at $x^z - 1$

Solution : (i) f (x) = x – 1

Lim f (x) = $(1)^z - 1 = 1 = 0$

x – 1

f (1) = $(1)^z - 1 = 0$

Since Lim f (x) = f (1)

x – 1

∴ the function is continuous as x = 1.

(ii) f (x) = $\frac{x^z - 1}{x - 1}$

$\therefore f(x) = \frac{1-1}{1-1} = \frac{0}{0}$ which is meaningless

In the question $\underset{x-1}{Lim} f(x)$ exists and is equal to 2, which can be verified. But since the value of the function does not exist, there is no point in finding the limit of the function. Hence we say that function is discontinuous.

(ii) f (x) = $\frac{1}{x - a}$

Clearly f(x) is not defined at x = a. Hence it is discontinuous.

Example 10: Show that the function

$$y = \left.\begin{matrix} e^{1/x} - 1 & \text{when } x \neq 0 \\ e + 1 & \text{when } x = 0 \end{matrix}\right\}$$ is discontinuous at zero.

Solution : As x→ 0. $\frac{1}{x} \to \infty$ or $-\frac{1}{x} - \to \infty$

$$\therefore e^{1/x} \quad \frac{1}{e^{-1/x}} \Rightarrow \frac{1}{\infty} \Rightarrow 0 \text{ e1/x}$$

Thus left hand Limit = $\underset{x-0}{Lim} \left[\frac{e^{1/x} - 1}{e^{1/x} + 1}\right]$

$$= \left[\frac{0-1}{0+1}\right] = \left(\frac{-1}{1}\right) = -1$$

Again let x → 0. + then $\frac{1}{x} \to 0$

$e^{1/x} \to \infty$ and $\frac{1}{e^{1/x}} \to 0$

∴ right hand limit = $\lim_{x-0} \left[\frac{e^{1/x}-1}{e^{1/x}+1}\right]$

$$= \lim_{x-0} = \left\{\frac{1-\frac{1}{e^{1/x}}}{1+\frac{1}{e^{1/x}}}\right\}$$

Thus left hand limit right hand limit.

Hence $\lim_{x \to 0}$ of the given does not exist.

∴ The function is discontinuous at x = 0

Example 11: The function f is discontinuous at x=0

Solution: Atx=0, f(x)=x – 3.∴f(o)= – 3

$\lim_{x \to 0}$ +f(x)= $\lim_{x \to 0}$ +x^2=0

and $\lim_{x \to 0}$ +f(x)= $\lim_{x \to 0}$ +x - 3 = - 3

$\lim_{x \to 0} f(x) \neq \lim_{x \to 0} f(x)$

i.e. f(x) does not exist.

Hence f is discontinuous at x = 0.

Example 12. The function $f(x) = \frac{x^2-1}{x^3-1}$ is undefined at thepoint x = 1: what should be the value of f(1) such that f(x) may be continuous at x = 1? Give arguments.

Solution: For the function f(x) to be continuous at x =, we must have $\lim_{x \to 1} f(x) = f(1)$.

Now $\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x^3 - 1}$

$= \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x^2+x+1)}$

$= \lim_{x \to 1} \frac{(x+1)}{x^2+x+1}$ $\quad (\therefore \ x \neq 1)$

SELF-CHECK EXERCISE 20.3

Q1. Find the paint of discontinuity of the following function

$$f(x) = \frac{x^2 - 2x + 4}{x^2 - 5x + 6}$$

Q2. $f(x) = \frac{x^2 - 4}{x - 2}$. What should the value of f (2) be, so that f (x) is continuous at x = 2.

20.7 SUMMARY

In this unit, you were introduced to the three important basic concept of calculates namely, function, limit and continuity. Important types of function the limiting value of a function and its substance. Important cut property of function, continuity and when the limit of function existed and when it is continuous was discussed in detail.

20.8 GLOSSARY

1. **Explicit and Implic Function :** y is said to be an explicit function of x if y is expressed directly in terms of x in terms of x in the form y = *f* (x). But if y and x are mixed up in the functional relation of the from *f* (x, y) = 0 y is said to be an implic function of x.
2. **Constant Function :** A function whose range consists of only one specific value, is called a constant function.
3. **Polynomial Function :** A polynomial function is a function that can be defined by evaluating a polynomial.
4. **Rational Function :** A function which is expressed as the ratio of two polynomial function in the same variable x is called a 'rational function'.
5. **Exponential Function :** In an algebraic function the exponent of a variable happens to be a constant such as x^2 or x^3 or an power of x.

20.9 ANSWER TO THE SELF-CHECK EXERCISE

Self-check Exercise 2.1

Ans.Q1 (i) $\sqrt{2x+1}$

Let f (x) = $\sqrt{2x+1}$

f (x) is defined only for those value of x for those value of x for which 2x + 1 is ≥ 0 otherwise f (x) become imaginary.

$\therefore$ $2x + 1 \geq 0$ or $2x \geq -1$ or $x \geq -\frac{1}{2}$

Hence f (x) is defined only fare there $x \geq -\frac{1}{2}$ which is its domain of definition.

Ans. Q1 (ii) Let f (x) = $\frac{1}{\sqrt{x-2}}$

f (x) is defined only for those value of x for which $x - 2 > 0$

$\therefore$ if $x - 2 = 0, f(x) = \frac{1}{0}$ which is not defined

if $x - 2 < 0$, f (x) is imaginary

$\therefore$ $x - 2 > 0$ which gives $x > 3$

Hence f (x) is defined only for value x > 2, which is its domain to definition.

Ans. Q2 First get $\frac{y+2}{3}$ = x

Switch the location y to x and x to y have y = $\frac{x+2}{3}$ which is required inverse.

Self-check Exercise 2.2

Ans. (i) $\lim_{x \to 0} \frac{\sqrt{a+x}-\sqrt{ax+x}}{x}$

$= \lim_{x \to 0} \frac{a+x-a-x}{x\left(\sqrt{a+x}-\sqrt{a-x}\right)}$

$= \lim_{x \to 0} \frac{2x}{x\left(\sqrt{a+x}-\sqrt{a-x}\right)} = \lim_{x \to 0} \frac{2}{\sqrt{a+x}+\sqrt{a-x}}$

(since $x \to 0$ and $x \neq 0$ we can cancel x in the ratio)

Now as x → 0, $\sqrt{a \pm x} - \sqrt{a}$

Thus, the limit = $\frac{2}{2\sqrt{a}} = \frac{1}{\sqrt{a}}$

Ans. (ii) |Sin x – 01 = 1 Sin x| can be made arbitrarily small by making 1 × 1 arbitrarily small.

Thus $\lim_{x \to 0}$ sin x = 0

Ans. (iii) Sin x = Sin θ = 2 Sin $\frac{1}{2}$ (x – θ) Cos $\frac{1}{2}$ (x + θ)

As x → θ, Sin $\frac{1}{2}$(x – θ) → θ

Also |Cos $\frac{1}{2}$ (x + 0)|

Thus, $\lim_{x \to \theta}$ (Sin x – Sin θ) = 0

i.e. $\lim_{x \to \theta}$ Sin x – Sin θ

Ans. (iv) $\lim_{x \to 1} \frac{x^2 - 4}{x - 2}$

$= \lim_{x \to 1} \frac{(x-2)(x+2)}{x-2}$

$= \lim_{x \to 1}$ x + 2

= 1 + 2 = 3

Self-check Exercise 2.3

Ans. Q1. f (x) = $\frac{x^2 - 2x + 4}{x^2 - 5x + 6}$ is the ratio of two continuous function (Polynomials are continuous can be verified lastly). There by the property III of the continuous function f (x) will be continuous at all value of x except when x^2 – 5x +6 equal zero i.e., the point of discontinuity of f (9x) are x = 2, 3.

Ans. Q2 $\lim_{x \to 2} f$ (x) = $\lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)}$ = 4

Thus, f (2) = 4 is the requirement for f (x) to be continuous at x = 2

20.10 REFERENCES/SUGGESTED READINGS

1. Allen, R.G.C. (2015). Mathematical Analysis for Economists. MacMillan, India Limited, Delhi.
2. Budrick, F. (2017). Applied Mathematics for Business, Economics and Social Sciences, MC Grew-Hill Book Company, London.
3. Chiang. A.C. and Wainwright, K. (2017). Fundamental Methods of Mathematical Economics. MCGraw-Hill Book Company, London.
4. Henderson, J.M. and Quandt, R.E. (1980). Microeconomic Theory. McGraw Hill Book Company, New York.
5. Yamane, T. (2012). Mathematic for Economists : An Elementary Survey. Pretice Hall of India, New Delhi.

20.11 TERMINAL QUESTIONS

Q1. Examine the continuity of the function $\frac{x^2+1}{x^2-1}$ at x = 2.

Q2. Evaluate :

(i) $\lim\limits_{x^2 \to a} \frac{2x^2-5x+6}{x+x-3}$ (ii) $\lim\limits_{x \to a} \frac{2x^3+3}{3x-2x-10}$