B.A.: 3rd Year
MATHEMATICS

Course Code: MATH304TH
Course Credits: 06(DSE)

Numerical Methods

UNITS: 1 to 20

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SYLLABUS

Course Code	MATH304TH
Credits= 6	L-5, T-1, P-0
Name of the Course	Numerical Methods
Type of the Course	Discipline Specific Elective
Continuous Comprehensive Assessment: Based on Assignment	Max. Marks:30
End Semester Examination	Max Marks: 70 Maximum Times: 3 hrs.

Instructions

Instructions for paper setter: The question paper will consist of two Sections A & B of 70 marks. Section A will be Compulsory and will contain 8 questions of 16 marks (each of 2 marks) of short answer type having two questions from each Unit of the syllabus. Section B of the question paper shall have four Units I, II, III, and IV. Two questions will be set from each unit of the syllabus and the candidates are required to attempt one question from each of these units. Each question in Units I, II, III and IV shall be of 13.5 markseach.

Instructions for Candidates: Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

DSE 3 B.1: Numerical Methods

Unit-I

Algorithms, Convergence, Bisection method, False position method, Fixed point iteration method, Newton's method, Secant method, LU decomposition.

Unit-II

Gauss-Jacobi, Gauss-Siedel and SOR iterative methods, Lagrange and Newton interpolation: linear and higher order.

Unit-III

Finite difference operators, Numerical differentiation: Newton's forward difference and backward difference method, Sterling's Central difference method.

Unit-IV

Integration: Trapezoidal rule, Simpson's rule, Euler's method.

Books Recommended

1. B. Bradie, A Friendly Introduction to Numerical Analysis, 2007. 2. M. K. Jain Pearson Education, India,
2. S.R.K. Iyengar and R. K. Jain Numerical Methods for Scientific and Engineering Computation, 5th Ed., New age International Publisher, India, 2007

CONTENTS

Unit - 1

Introduction to Numerical Methods

Structure

1.1 Introduction

Dear student in this unit you will learn, what is a numerical method and why we need numerical method. We will also discuss some important results of calculus which are to be used in this course of numerical method. In this unit we will also learn about the error and error analysis.

1.2 Learning Objectives:

After studying this unit, students will be able to

1. define numerical method.
2. explain the importance of numerical methods
3. define and apply different results of mathematics used in numerical methods.
4. define error and its types

1.3 Numerical Methods and Its Applications

Numerical analysis is a brach of mathematics that deals with devising efficient methods for obtaining numerical solution to different mathematical problems. Most of the mathematical problems that arise in science and engineering are very hard and sometime impossible to solve exactly. Thus an approximation to a difficult mathematical problem is very important to make it easier to solve. Due to immense development in computational technology, numerical approximation has become more popular and a modern tool for scientist and engineers.

Numerical Method:

Numerical methods are techniques that are used to approximate mathematical procedures like integral, differential equations, non-linear equations etc. We need approximation because sometime it is not possible to solve any procedures analytically we are unable to find exact solution. Also analytical methods are intractable, for example solving a set of thousand simultaneous linear equations for thousand unknowns.

Application of Numerical Methods

Numerical methods are used in,

1. Finding the approximate solution or roots of non Linear equations
2. Solving the system of linear equations
3. Finding the approximate solution of ordinary and partial differential equations
4. Interpolation, i.e. finding the value of a given function at a specific moment.
5. M Evaluating derivative
6. Evaluating integral

These are some applications of numerical methods which we will study in this course.

Mathematical Preuminariesfor Numerical Methods

Here we will try to recall some results of calculus that are used frequently in this course.

1. Limit of a Function:

A function f defined on a set x of real number has the limit L at x_0, i.e. $\underset{x \to x_0}{Lim} f(x) = L$, if given any real number $\in$>0, $\exists$ a real number δ> 0 such that $|f(x)-L| < \in$, whenever $x \in X$ and $0 < |x - x_0| < \delta$.

2. Continuity of a Function

A function f : R $\to$R is said to be continuous at a point $x_0 \in R$ if $\underset{x \to x_0}{Lim} f(x) = f(x_0)$.

Or, for any given $\in$>0, $\exists$ a δ> 0 such that

$\left| f(x) - f(x_0) \right| < \in$ Wherever $|x - x_0| < \delta$ (2)

3. Intermediate - Value Theorem

Let f(x) be a continuous function on the interval [a, b]. If $f(x1) < \propto < f(x_2)$ for some number $\propto$ and some x_1 and $x_2 \in$ [a, b] with $x_1 < x_2$ then $\propto = f(\xi)$, for some $\xi \in$ [a, b].

This theorem plays on important basic role in finding initial guess in iterative methods for solving nonlinear equations.

4. Differentiation of a Function

A function f:(a, b)$\to$R is said to be differentiable at a point c$\in$ (a, b) if the limit

$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists. Inthis case the value of the limit is denoted by f'(c) and is called the derivative of f at c. The function f is said to be differentiable in (a, b) if it is differentiable at every point in (a, b).

Using above definition, its equivalent definition are

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

$$f'(c) = \lim_{h\to 0}\frac{f(c)-f(c+h)}{h}$$

$$f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c-h)}{2h}$$

These are useful in deriving forward difference, backward difference and control difference formula in interpolation.

5. Rolle's Theorem

Let f(x) be continuous on the bounded interval [a, b] and differentiable on (a, b).

If f(a) = f(b) then

$f'(\xi)$ = 0for some $\xi \in$(a, b)

6. Mean-Value Theorem for Derivatives

If f(x) is continuous on the bounded interval [a, b] with a ≠ b and differentiable on (a, b) then

Then $\frac{f(b)-f(a)}{b-a} = f^1(\xi)$ for some $\xi \in$(a, b).

7. Mean Value Theorem For Integration:

Let g(x)be a non-negative or non-positive integrable function on [a, b]. If f(x) is continuous on [a, b] then

$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some $\xi \in$ [a, b]

The Prolle's theorem, and mean value theorems for derivative and integration plays crucial role in deriving truncation error for numerical methods.

8. Taylor's Formula with Remainder

If $f(x)$ has n+1 continuous derivative on [a, b] and c is some point in [a, b], then for all x $\in$ [a, b]

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)(x-c)^2}{2!} + \dots\dots + \frac{f^n(c)(x-c)^n}{n!} + R_{n+1}(x)$$

Where

$$R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-t)^n f^{(n+1)}(t)\,dt$$

9. Taylor's Formula in Two Dimensions

If $f(x, y)$ is a continuous function of two independent variables x and y with continuous first and second order partial derivative in a neighborhood D of the point (a, b) then $f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_2(x, y)$, where

$$R_2(x, y) = \frac{f_{xx}(\xi, n)(x-a)^2}{2} + f_{xy}(\xi, n)(x - a)(y - a) + \frac{f_{yy}(\xi, n)(y-b)^2}{2}$$

10. Taylor's Series

The Taylor's series is the foundation of numerical methods. Many of numerical techniques are derived directly from Taylor's Series, as are the estimates of the errors involved in employing these techniques. The value of function $f(x)$ can be expressed in a region of x close to x = a, by the infinite power series given by

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f''(a)$$

$$\dots\dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots\dots \quad (1)$$

Then the function $f(x)$ is said to be analytic in the legion near x = a and the series (1) is unique and is called Taylor's series expansion of $f(x)$ in the neighborhood of x = a. The error in the Taylor series (1) for $f(x)$ when the Series is truncated after the term containing $(x-a)^n$ is not greater than.

$$\left|\frac{d^{n+1}f}{dx^{n+1}}\right|_{\max} \cdot \frac{\left((x-a)\right)^{h+1}}{(n+1)!} \quad \dots . (2)$$

Where the subscript 'max' denotes the maximum magnitude of the derivative on the interval from a to x.

In order to confine ourselves to a Taylor's series for a given function then the following relationship must hold between the error terms of a series truncated at n terms and the same series truanted at n+1 terms.

$$0\ (x-a)^{n+1} < 0\ (x - a)^{n} \qquad ...(3)$$

Let us try to understand above concepts by taking some examples.

Example 1: Find the Taylor Series expansion for sin x near c = a.

Solution: Since $f(x) = \sin x$

$f'(x) = \text{cox } x$

$f''(x) = \sin x$

$f'''(x) = \text{cox } x$

$f^{iv}(x) = \sin x$

Since Taylor Series is

$$f(x) = f(c) + f'(c)\,(x - c) + \frac{f''(c)(x-c)^2}{2!} + ...$$

Here c = 0

$$\sin x = \sin(0)\,(x-0) + \frac{(-\sin 0)(x-0)^2}{2!} + \frac{(cox 0)(x-0)^3}{3!} + \frac{(\sin 0)(x-0)^4}{4!} + ...$$

$$\sin x = 0 + 1.x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ....$$

Example 2: Truncate the Taylor Series for sin x (in example 1) to give a representation of $0(x^4)$. Show that this representation is in fact of $0(x^5)$.

Solution: Since, sin x = sin (0) + x cox (0) - $\frac{x^2}{2!}$ sin (0) + $0(x^4)$

$$\sin x = x - \frac{x^3}{3!} + 0(x^4)$$

But if we carry one max term in the series sin x = sin (0) + x cos (0) - $\frac{x^2}{2!}$ sin (0) - $\frac{x^3}{3!}$ cos (0) + $\frac{x^4}{4!}$ sin (0) + $0(x^5)$we see that additional term is exactly zero since sin (0) = 0.

Therefore

$$\sin x = x - \frac{x^3}{3!} + a(x^5)$$

Thus, this two term representation is actually of $0(x^5)$ rather than $0(x^4)$.

Self Check Exercise

Q.1 Find the Taylor Series expansion about x = a for f(x) = $\log_e$ (1-x).

Q.2 Show that f(x) = $e^{x^{1/2}}$ cannot be expanded in a Taylor's series about x = 0.

for some (ξ_1n)$\in$ D depending or (x, y) and the subscripts of f denotes partial derivative of f.

Taylor's theorem is essential for derivation and error analysis of almost all the numerical method discuss in this course.

Error Analysis of Numerical Method

A real number x can have infinitely many digits. But a digital calculating device can hold only a finite number of digits and therefore, after a finite number of digits (depending on the capacity of the calculating device) The rest should be discussed in same sense. In this way, the representation of the led number x or a computing devices is only approximate. Although, the omitted part of x is very small in its value, this approximation can reads to considerably large error the numerical computation.

Error

The approximate representation of a real number obviously differs from the actual number, and this difference is called an error. Hence

Error = True value - Approximate value

Absolute Error

The absolute value of the error is known as absolute error.

Relative Error

The relative error is a measure of the error in relation to the size of the true value. It is given as

$$\text{Relative error} = \frac{\text{Error}}{\text{True value}}$$

Percentage Error

Percentage error is defined as 100 times the relative error

$$\%\ \text{error} = \frac{\text{Error}}{\text{True value}} \times 100$$

Turncation Error

The term turncation error is used to denote error, which result from approximating a smooth function by turncating its Taylor series representation to a finite number of terms.

If x_A be the approximation of the real number x then

$E(x_A) = \text{Error}\ (x_A) = x - x_A$

$E_a(x_A) = \text{Absolute error}\ (x_A) = |E(x_A)| = |x - x_A]$

$E_r(x_A) = \text{Relative error}\ (x_A) = \frac{E(x_A)}{x}$

1.6 Summary

In this unit we have studied about

1. meaning and application of numerical methods.
2. basic concepts like continuity, differentiability of a function to be used in numerical methods.
3. Discuss the importance of Taylor's series in numerical method using some examples.

4. The error and its types.

1.7 Glossary

- **Approximation :** An approximation is anything that is similar but not exactly equal to something else. A number can be approximated by rounding.

1.8 Answer to self Check exercise

E1 loge(1-x) = $-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+.......\right)$

E2 The function f(x) is bounded at x = 01 but all of the derivatives of f involves negative powers of x which results in those derivatives becoming unbounded at x = 0. Thus f(x) does not satisfy the condition at x = 0 for an expansion in a Taylor series about x = 0.

1.9 References/ Suggested Readings

1. Numerical Analysis, Richard L. Burden, J. Douglas Faires, Cengags, 9th Edition.
2. Numerical Method for Engineer, Steven C. Chapra, Paymond P. Canale McGraw Hill Education 7th Edition.

1.10 Terminal Questions

Study the continuity of f

1. f(x) = x^2 if x < 1

   $\sqrt{x}$ if x $\geq$1

2. f(x) = -x if x < 1

   x if x $\geq$ 1

3. Let f(x) = 1 - $x^{2/3}$ show that f(1) = f(-1) = 0 but f'(x) is never zero in the interval [-1, 1]. use Rolle's theorem.

4. For f(x) = x2, find the point $\xi_1$specified by the mean value theorem for derivative. Verify that this point lies in the interval (a, b).

5. Find the Taylor's expansion for f(x) = $\sqrt{x+1}$ upto n = 2 about c = 0.

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Unit - 2

Solution of Equation by Iteration

Bisection Methods

Structure

2.1 Introduction

Dear student in this unit you will learn to find the solution of equation using numerical methods. Since we know that equation of various kinds arise in a range of physical applications and one of the most common problem of applied mathematics and engineering is to find the solution or roots of an equation. Some equations are simple like single linear equation like ax +b = 0, where a and b are real number and a $\neq$ 0 and its solution is given by x = - $\frac{b}{a}$. In case of non-linear equations like Ax2 + Bx + C = 0 i.e. quadratic equation with real coefficient A, B and C, A $\neq$ 0, the solution of this equation is given by x = $\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$. Similar algebraic formulate are available for finding the roots of a general cubic and biquadratic equation. But for a polynomial equation of degree n$\geq$ 5 with integer coefficient, there is not formula. So in this unit we study the simple numerical method for the approximate solution of the equation f(x) = 0, where f is real-valued function, defined and continuous on a bounded and closed interval of the real line.

2.2 Learning Objectives:

After studying this unit, students will be able to

1. Under the importance of iterative method to solve equation.
2. discuss and fine initial approximation and convergence of a iterative mothod.

3. Apply Bisection Method to solve the equation.

4. Find the order of convergence of Bisection Method.

2.3 Iterative Method

Need and Importance of Iterative Method

Since there is no general formula for the solution of polynomial equations. So for finding the roots of an equation (algebraic or transcendental) with degree higher than two (i.e. non-linear equation) we use numerical methods. The numerical methods for finding the roots of a non-linear equations are known as iterative methods. In iterative method we start with an approximate solution and uses this approximate solution in a recurrence formula to provide another approximate solution. We repeat this process unit we get an approximate solution which converges to the exact solution. Iterative methods are also known as trial and error methods. Generally iterative method gives one root at a time.

Importance of iterative Methods

1. Iterative method make us able to find a root to any specified degree of accuracy.
2. Iterative method are insensitive to the propagation of error.
3. Iterative methods are quite general in technique and have simple operations.
4. These method can easily implemented in computers.
5. It is possible to develop algorithms for iterative method.
6. Through computational algorithm it is very easy to find solution of any non-liner equation.
7. Errors are negligible in iterative methods.

Order of Convergence of an Iterative Method

An iterative method is said to be of order ρ is the largest positive real number such that

$|e_{n+1}| \leq A|e_n|^{\rho}$

Where ρ is a finite positive constant, here

$e_{n+1} = x_{n+1} - r$, where r is root of given equation and x_{n+1} is the approximated root and e_{n+1} is error in $n+1^{th}$ approximation. Similarly $e_n = x_n - r$.

Proof

Let r be the root of equation $y = f\text{f}(x)$ ...(1)

$\therefore \quad f(r) = r$ ...(2)

Let x_n and x_{n+1} be the approximation of 'r' after n and n+1 iteration respectively.

then $x_{n+1} = f(x_n)$ ....(3)

Using (2) and (3) we have

$x_{n+1} - r = f(x_n) - f(r)$. ....(4)

Using mean value theorem of differential calculus,

$$f(x_n) - f(r) = (x_n - r) f'(0), \text{ where } x_n < 0 < r \qquad ...(5)$$

Using (5) in (4), we get

$$x_{n+1} - r = (x_n - r) f'(0) \qquad ...(6)$$

Let k be the minimum value of $|f'(0)|$ in the interval

s.t. $|f'(B)| \leq k$ for all $x \in I$

$$|x_{n+1} - r| \leq k |x_n - r|$$

$$|e_{n+1}| \leq k |e_n| \qquad ...(7)$$

Since here $\rho = 1$, so order of convergence of iterative method is linear. If $\rho = 2$ then the method is of quadrative convergence.

Bisection Method

Bisection Method for finding the root of the equation $f(x) = 0$ is based on the repeated application of intermediate value theorem. This theorem states that if f is a continuous function on the interval [a, b] and $f(a) f(b) < 0$ i.e. $f(a)$ and $f(b)$ are of opposite sine. Then f has at least one root or zero in the interval [a, b].

Steps for Bisection Method

1. Take $c = \frac{a+b}{2}$, point of bisection of the interval (a, b)

2. If $f(a) f(c) < 0$ i.e. root lies in (a, c), so a = a and b = c.

   If $f(c) f(b) < 0$ i.e. root lies in (c, b), so a = c, b = b.

   If $f(c) = 0$, c is the root, so stop.

3. Repeat, the above step 2 until desire accuracy is obtained.

For more simplicity, if $f(a)$ = +ve, $f(b)$ = -ve then the root of equation $f(x) = 0$ lies between a and b.

Then (1) $c = \frac{a+b}{2}$, be the point of bisection of (a, b_

(2) $f(c) < 0$, root lies in (c, b)

$f(c) > 0$ i.e. root lies in (a, c)

$f(c) = 0$, c is the root

(3) Repeat the step 2.

Convergence of Bisection Method

Bisection method is always convergent. To prove this let,

Let a_0 = a and b_0 = b and $[a_n, b_n]$, $n \geq 0$ are the successive intervals in the bisection process.

Clearly, $a_0 \leq a_1 \leq a_2 \leq \ldots\ldots\ldots\ldots \leq b_0 = b$

and $b_0 \geq b_1 \geq b_2 \geq \ldots\ldots\ldots \geq a_0 = a$

So, the sequence $\{a_n\}$ is monotonically increasing and bounded above and the sequence $\{b_n\}$ is monotonically decreasing and bounded below. Hence both sequences converges.

Also,

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} = n \geq 1.$$

because new interval is obtained in each iteration whose length is equal to hay of the length of the interval obtained in previous iteration.

Thus recursively

$$b_n - a_n = \frac{1}{2^{n-1}}\left(b_0 - a_0\right),\ n \geq 1 \qquad (1)$$

Using standard limits law equ. (1) gives us

$$\lim_{n\to\infty}\left(b_n - a_n\right) = \lim_{n\to\infty} bn - \lim_{n\to\infty} an = \left(b_0 - a_0\right) \lim_{n\to\infty} \frac{1}{2^{n-1}} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$$

i.e. the sequences $\{a_n\}$ and $\{b_n\}$ have the same Limits, let

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = r \qquad (2)$$

Now, to show that r is a root of function f. Using bisection algouthum we know that $f(a_n)$ $f(b_n) < 0$.

Taking Limits $\lim\limits_{n\to\infty} f(a_n)\ f(b_n) \leq 0$ (3)

Using (2) in (3)

$f(r)\ f(r) \leq 0$

$\Rightarrow \quad \left[f(r)\right]^2 \leq 0$

$\Rightarrow \quad f(r) = 0$ (4)

Hence both $\{a_n\}$ and $\{b_n)$ converge to a root of $f(x)$ 0 So using (2) and (4) we can say that bisection method is always convergent.

Error in Bisection Method

Theorem: Suppose $f \in c$ [a, b] and f(a) f(b) < 0. Then the bisections method generator a sequence {c_n} approximating a zero r of f with $|c_n - r| \le \frac{b-a}{2^n}$ when n $\geq$ 1

Proof: Since for each n $\geq$ 1, we have

$$b_n - a_n = \frac{1}{2^{n-1}}(b_0 - a_0) = \frac{1}{2^{n-1}}(b - a)$$

Let us apply bisection method to the interval [a_n, b_n] and calculate the mid pointc_n = $\frac{a_n + b_n}{2}$ after iteration. Then the root lies either in [a_n, c_n] or [c_n, b_n]. Let r is the root and c_n is the approximation then

$$|c_n - r| < \frac{1}{2}(b_n - a_n)$$

$$= \frac{1}{2} \cdot \frac{1}{2^{n-1}}(b - a)$$

$$= \frac{1}{2^n}(b - a)$$

$$\Rightarrow \quad |c_n - r| < \frac{1}{2^n}(b - a)$$

This is bound or absolute error, which is independent of the function f.

It can also be seen that, if [a, b] is the initial approximation then maximum error of using either a or b as our approximation is b-a, Because in bisection method we hafe the width of the interval with each iteration, the error is reduced by a factor 2 and Thus, after n iteration the error will be $\left(\frac{b-a}{2^n}\right)$

Order of Convergence of Bisection Method

The speed or fastness of convergence of an iterative method is judged by its order of convergence. Higher the order of converges means the error in the successive approximations obtained decreases more rapidly.

An iterative method is said to be of order þ or has the order of convergence þ if þ is the largest positive real number such that

$$|e_{n+1}| \le A\ |e_n|^{\rho}$$

Here n is iteration idex, A is asymptotic rate e_{n+1} is error at n^{th} stage.

If e_n is error at n^{th} stage then

$$|e_n| = |r - c_n|$$

$$< \frac{1}{2}\left(b_n - a_n\right)$$

and $|e_{n+1}| = |r - c_{n+1}|$ $\left[\because |c_n - r| \le \frac{1}{2}\left(b_n - a_n\right)\right]$

$$< \frac{1}{2}\left(b_{n+1} - a_{n+1}\right) \qquad \left[\because b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}\right]$$

$$= \frac{1}{2}\left(\frac{b_n - a_n}{2}\right)$$

$$\Rightarrow \quad |e_{n+1}| = \frac{1}{2}|e_n|$$

Hence the bisection method converges linearly with asymptotic rate of $\frac{1}{2}$.

Number of Iteration for Bisection Method

To calculate number of iteration 'n' that are needed to achieve a specific accuracy, '$\in$', then the permissible error in the root is $\in$.

Since we know that $|c_n - r| \le \frac{1}{2^n}$ (b - a) gives the bound for error in bisection method. To have specific accuracy $\in$ this bound must be less than or equal to $\in$ i.e.

$$\frac{1}{2^n}(b - a) \le \in$$

$$\Rightarrow \quad 2^n \in \ge b - a$$

$$\Rightarrow \quad 2^n \ge \frac{b-a}{\in}$$

Taking log both side

$$\log 2^n \ge \log\left(\frac{b-a}{\in}\right)$$

$$n \log 2 \ge \log\left(\frac{b-a}{\in}\right)$$

$$n \geq \frac{\log\left(\frac{b-a}{\in}\right)}{\log 2}$$

Above expression gives the minimum numbers of iteration required to achieve an accuracy $\in$.

Advantage And Disadvantage of Bisection Method

Advantages

1. Bisection method is simple to use.
2. If f(x) is a continuous function in a given interval, then bisection method is convergent, mean it always gives a solution of given function.
3. On comparing with other method, this method required less computational effort.
4. This method is suitable for implementation in computer.

Disadvantages

1. Bisection method is slow.
2. This method requires a large number of iterations to achieve a reasonable degree of accuracy.
3. This method is only applicable when the roots of f about which f(x) changes sign, cannot find root of $f(x) = x^2$
4. Its rate of convergence is linear
5. It is incapable of finding complex roots.

To apply what we have studies let us try following examples.

Examples 1: For a given function f(x) = x3 - x - 1, a real root ties in between [1, 2]. Find the minimum number of iteration required to find the root upto the accuracy of two decimal places.

Solution: Given f(x) = x3 - x - 1, and the root lies between [1, 2], so b = 2 and a = 1. Also the required accuracy is two decimal places i.e. $\in = 0.01$.

Using the result,

$$\frac{b-a}{2^n} \leq \in$$

$$\Rightarrow \quad \frac{2-1}{2^n} \leq 0.01$$

$$= \frac{1}{2^n} \leq \frac{1}{100}$$

$$\Rightarrow \quad 2^n \geq 100$$

Since $2^6 = 64$ and $2^7 = 128$

So $2^6 \leq 100 \leq 2^7$

Therefore minimum iteration required = 7.

Example 2: For a given function $f(x) = x^3 - 4x + 3$, a real root lies in between the interval [-3, -2]. Find the minimum number of iteration required to find the root up to the accuracy of three decimal points.

Solution: Given $f(x) = x^3 - 4x + 3$, a = -3, b = -2, $\in$ = .001

Since for accuracy of bisection method we have

$$\frac{b-a}{2^n} \leq c$$

$$= \frac{-2-(-3)}{2^n} \leq 0.001$$

$$\Rightarrow \quad \frac{1}{2^n} \leq \frac{1}{1000}$$

$$\Rightarrow \quad 2^n \geq 1000$$

Since $2^9 = 512$ and $2^{10} = 1030$

$$\Rightarrow \quad 2^9 \leq 1000 \leq 2^{10}$$

Hence minimum number of iteration required to find the root of given function is 10.

Self Check Exercise-1

Q.1 Find out how many iteration are required to get a root of $f(x) = 3x^2 - 5x - 2$ in the interval [1, 2] to get an accuracy of three decimal places.

In order to find the root by Bisection method of given equation following steps are used,

Initial requirement

We have to find the initial bound [a, b] for the root such that f(a) and f(b) have apposite sign.

Iteration Procexy:

For the interval [a, b], we find c = $\frac{a+b}{c}$, then

(1) If f(c) = 0, then c is the root of equation

(2) If f(c) and f(a) have opposite sign, then root lies on [a, c]

(3) If f(c) and f(b) have opposite sign, then root lies on [c, b]

Halting Conditions:

There are three conditions which may cause the iteration process to halt:

(1) If f(c) = 0 then c is the root

(2) We halt if both of the following conditions are met

(i) The width of the interval is sufficiently small i.e. b - a <$\in$ step.

(ii) The function evaluated at one of the end point.

$|f(a)|$ or $|f(b)| < \in$, we choose either a or b, depending on whether $|f(a)| < |f(b)|$ or $|f(a)| > |f(b)|$ respectively.

(3) If we have iterated, say N, number of times, and does not satisfies f(c) = 0, then we halt and indicate that solution is not found.

Let us try following examples:

Example 3: Find the root of $f(x) = x^2 - 3$ upto two decimal places or $\in = 0.01$

Initial requirement

Solution: Given f(x) = x2 - 3

$f(1) = 1 - 3 = -2$ (-ve)

$f(2) = 4 - 3 = 1$ (+ve)

So the root of given equation lies between (1, 2)

Iteration Process:

I[st]Iteration, a = 1, $f(1) = -2$ (-ve)

b = 2 $f(2) = 1$ (+ve)

$$x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

$f(1.5) = -0.75$ (-ve)

Since $f(1.5) = -0.75$ and $f(2) = 1$, so root lies between (1.5, 2)

2[nd] Iteration: a = 1.5, $f(1.5) = -0.75$

b = 2, $f(2) = 1$

$$x_1 = \frac{1.5+2}{2} = \frac{3.5}{2} = 1.75$$

$f(1.75) = 0.0625$ (+ ve)

Since $f(1.5) = -0.75$(-ve) and $f(1.75) = 0.062$ (+ve), so root lies between (1.5, 1.75)

3[rd] Iteration: a = 1.5, $f(1.5) = -0.75$

b = 1.75 $f(1.75) = 0.062$

$x_2 = \frac{1.5+1.75}{2}$ = 1.625 (+ve)

$f(1.625)$ = - 0.359 (-ve)

$f(1.625)$ = - 0.359 (-ve) and $f(1.75)$ = 0.0625 (+ve) so root lies between (1.625,1.75)

4th Iteration: a = 1.625, $f(1.625)$ = -0.359

b = 1.75, $f(1.75)$ = 0.062

$x_3 = \frac{1.625+1.75}{2}$ = 1.6875

$f(1.6875)$ = -0.1523 (-ve)

Since $f(1.6875)$ = - 0.1523, + (1.75) = 0.0625, so root lies between (1.6875, 1.75)

5th Iteration: a = 1.6875, $f(1.6875)$ = - 0.1523

b = 1.75 $f(1.75)$ = 0.0625

$x_4 = \frac{1.6875+1.75}{2}$ = 1.7188

$f(1.7188)$ = -0.0457

Since $f(1.7188)$ = -0.0457 (-ve) and $f(1.75)$ = 0.625 (+ve)

So root lies between (1.7188, 1.75)

6th Iteration: a = 1.7188, $f(1.7188)$ = -0.0457

b = 1.75, $f(1.75)$ = 0.0625

$x_5 = \frac{1.7188+1.75}{2}$ = 1.7344

$f(1.7344)$ = 0.0081 (+ve)

Since $f(1.7188)$ = -0.0457$^{(-ve)}$ and $f(1.7344)$ = 0.0081 (+ve)

So root lies between (1.7188, 1.7344)

7th Iteration: a = 1.7188, b = 1.7344

$f(1.7188)$ = -0.0457, $f(1.7344)$ = 0.0081

$x_6 = \frac{1.7188+1.7344}{2}$ = 1.7266

$f(1.7266)$ = -0.0189.

Since f(1.7266) = -0.0189 (-ve) and f(1.7344) = 0.0081 (+ve)

So root lies between (1.7266, 1.7244)

In 7th iteration, we find that the interval (1.7266, 1.7344) will have the root, also the width of interval is 1.7344 - 1.7266 = 0.0078 < 0.01

Also, $|f(1.7344)|$ = 0.0081 < 0.01

So we choose b = 1.7344 as an approximate root of given equation.

Example 4: Find the root of e^{-x}[3.2 sin (x) -0.5 ces (x)] on the interval [3, 4] for $\in$ = 0.001 or upto 3 decimal places.

Solution: Table of iteration is given as

a	b	f(a)	f(b)	c = $\frac{a+b}{2}$	f(c)	New Interval
3	4	0.047127	-0.038372	3.5	-0.01975	(3,3.5)
3	3.5	0.047127	-0.019757	3.25	0.0058479	(3.25,35)
3.25	3.5	0.0058479	-0.019757	3.375	-0.0086808	(3.25,3.375)
3.25	3.375	0.0058479	-0.0086808	3.3125	-0.0018773	(3.25,3.3125)
3.25	3.3125	0.0058479	-0.0018773	3.2823	0.0018739	(3.2812,3.3125)
3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.000024791	(3.2812,3.2968)
3.2812	3.2968	0.0018739	-0.000024791	3.289	0.00091736	(3.289,3.2968)
3.289	3.2968	0.00091736	-0.00024791	3.2929	.00044352	(3.2929,3.2968)
3.2929	3.2968	0.00044352	-0.000024791	3.2948	0.00021466	(3.2948,3.2968)
3.2948	3.2968	0.00021466	-0.000024791	3.2958	0.000094077	(3.2958,3.2968)
3.2958	3.2968	0.000094077	-0.000024791	3.2963	0.000034799	(3.2963,3.2968)

Since in the interval (3.2958, 3.2968) the width is of length

3.2968 - 3.2958 = 0.001 and $|f(3.2968)|$ < 0.001

So we choose b = 3.2968 as an approximate root of given function.

Example 5: Find the root of $f(x) = 10 - x^2$upto two decimal Haces, starling from [-2, 2]

Solution: The table for bisection method approximation is

Iteration	a	b	c	f(a)	f(b)	f(c)
0	-2	5	1.5	6	-15	7.75
1	1.5	5	3.25	7.75	-15	-0.5625
2	1.5	3.25	2.375	7.75	-0.5625	4.359375
3	2.375	3.25	2.8125	4.359375	-0.5625	2.0898438
4	2.8125	3.25	3.03125	2.0898438	-0.5625	0.8115234
5	3.03125	3.25	3.140625	0.8115234	-0.5625	0.1364746
6	3.140625	3.25	3.1953125	0.1364746	-0.5625	-0.210022
7	3.140625	3.1953125	3.1679688	0.1364746	-0.210022	-0.36026
8	3.140625	3.1679688	3.1542969	0.1364746	-0.36026	0.0504112
9	3.1542969	3.1679688	3.1611328	0.0504112	-0.36026	0.0072393

After 9th iteration, the interval will be [3.1611328, 3.1679688] and the length of the interval is = 0.0068 < 0.01, and $|f(3.1611328)|$ = 0.0072393 < 0.01

So the best approximate root is 3.1611328

Example 6: Find a root of equation $x^3 - x - 1$ using Bisection method, correct to three decimal places.

Solution: Here $f(x)\ x^3 - x - 1$

Firstly we have to find the point a and b such that a < b and $f(a)f(b) < 0$ i.e. function changes its sign.

$f(0) = 0 - 0 - 0 = -1$ (-ve)

$f(1) = 1 - 1 - 1 = 1 - 2 = -1$ (-ve)

$f(2) = (2)^3 - 2 - 1 = 8 - 3 = 5$ (+ve)

So, a = 1 and b = 2 such that $f(a)\ f(2) < 0$

So the root of given equation lies between (1, 2)

I^{st} Iteration: Taking a = 1, b = 2, $f(1) = -1$, $f(2) = 5$, $x_0 = \frac{1+2}{2} = 1.5$

$f(x_0) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875 > 0$

Since $f(1) = -1$ (-ve) and $f(1.5) = 0.875$ (+ve)

So root lies between (1, 1.5)

2^{nd}Iteration: Taking a = 1, b = 1.5, $f(1) = -1$, $f(1.5) = 0.8751$

$x_1 = \frac{1+1.5}{2} = \frac{2.5}{2} = 1.25$

$f(x_1) = f(1.25) = (1.25)^3 - (1.25) - 1 = -0.29688$ (-ve)

Since $f(1) = -1$ (-ve), $f(1.5) = 0.875$ (+ve) and $f(1.25)$

$= - 0.29688$

So the root lies between (1.25, 1.5)

3^{rd} Iteration: Taking a = 1.25, b = 1.5, $f(1.25) = 0.29688$, $f(1.5) = 0.875$

$x_2 = \frac{12.5+1.5}{2} = \frac{2.75}{2} = 1.375$

$f(x_2) = f(1.375) = 0.22461 > 0$ (-ve)

Since $f(1.25)$ = (-ve) and $f(1.375)$ = +ve, so root lies between (1.25, 1.375)

4^{th} Iteration: Taking a = 1.25, b = 1.375, $f(1.25) = -0.29688$

$f(1.375) = 0.2246$

$x^3 = \frac{12.5+1.375}{2} = 1.3125$

$f(x_3) = f(1.3125) = - 0.05151 < 0$ (-ve)

Since $f(1.3125) = -0.05151$ (-ve) and $f(1.375) = 0.2246$ (+ve)

So root lies between (1.3125, 1.375)

5th Iteration: Taking a = 1.3125, b = 1.375

$f(1.3125) = -0.05151$

$f(1.375) = 0.22461$

$x_4 = \frac{1.3125+1.375}{2} = \frac{2.6875}{2} = 1.34375$

$f(1.34375) = 0.08261 > 0$ (+ve)

Since $f(1.3125)$ = -0105151 (-ve) and $f(1.34375)$ = 0.08261 (+ve)

So root lies between (1.3125, 1.34375)

6th Iteration: Taking a = 1.3125, b = 1.34375, $f(1.3125)$ = -0.05151 $f(1.34375)$ = 0.08261

$$x_5 = \frac{1.3125 + 1.34375}{2} = \frac{2.65625}{2} = 1.328125$$

$f(1.32812)$ = 0.01458 > 0 (+ve)

So root lies between (1.3125, 1.32812)

7th Iteration: Taking a = 1.3125, $f(1.3125)$ = -0.05151

b = 1.32812, $f(1.32812)$ = 0.01458

$$x_6 = \frac{1.3125 + 1.32812}{2} = 1.32031$$

$f(1.32031)$ = -0.01871 < 0 (-ve)

Since $f(1.32812)$ is +ve and $f(1.32031)$ is -ve so root lies between (1.32031, 1.32812)

8th Iteration: Taking a = 1.32031, $f(1.32031)$ = - 0.01871

b = 1.32812, $f(1.32812)$ = 0.01458

$$x_7 = \frac{1.32031 + 1.32812}{2} = 1.32422$$

$f(1.32422)$ = -0.00213 < 0 (-ve)

Since $f(1.32812)$ is +ve and $f(1.32422)$ is -ve, so root lies between (1.32422, 1.32812).

9th Iteration: Taking a = 1.32422, b = 1.32812, $f(1.32422)$ = -0.00213 $f(1.32812)$ = 0.01458

$$x_8 = \frac{1.32422 + 1.32812}{2} = 1.32617$$

$f(1.32617)$ = 0.00621 > 0 (+ve)

Since $f(1.32422)$ is -ve and $f(1.32617)$ is +ve, so root lies between (1.32422, 1.32617)

10th Iteration: Taking a = 1.32422, + $f(1.32422)$ = -0.00213

b = 1.32617, $f(1.32617)$ = .00621

$$x_9 = \frac{1.32422 + 1.32617}{2} = 1.3252$$

$f(1.3252)$ = .00204 >0 (+ve)

Since $f(1.32422)$ is -ve and $f(1.3252)$ is +ve so root lies between (1.32422, 1.3252)

11th Iteration: Taking a = 1.32422, $f(1.32422) = -0.00213$

b = 1.3252 $f(1.3252) = 0.00204$

$$x_{10} = \frac{1.32422 + 1.3252}{2} = 1.32471$$

$f(1.32471) = -0.00005 < 0$ (-ve)

Since $f(1.3252)$is +ve = 0.00204 and $f(1.32471)$ is -ve, so root lies between (1.32471, 1.3252)

12th Iteration: Taking a = 1.32471, $f(1.32471) = -0.00005$

b = 1.3252, $f(1.3252) = 0.00204$

$$x_{11} = \frac{1.32471 + 1.3252}{2} = \frac{2.6499}{2} = 1.3249$$

$f(1.3249) = -0.0234$

From 11th and 12th iteration, we find that three is no change in successive approximation upto three decimal places. So the root of $x^3 - x - 1$ is 1.3247 (First three decimal places).

Self Check Exercise - 1

E.2 Find the real root of equation $x^2 - 4x - 10 = 0$ using bisection method.

E.3 Find a root of the equation $x^3 - 2x - 5 = 0$ in the interval [2, 3] correct to three decimal places by using bisection method

E.4 Find a root of equation $x^3 - 4x - 9 = 0$ correct to three decimal places using bisection method.

2.4 Summary:

In this unit, we studies about

1. The iteration method to find the solution/root of given equation.
2. Bisection method, its convergence and its error to lrance

2.5 Glossary:

- **Bounded Function:** A function whose range can be included in a closed interval.
- **Monotonically Increasing Sequence:** A function is monotonically increasing if it is always increasing on its domain and its graph is never horizohal.
- **Monotonically Decreasing Sequence:** A function opposite of monotonically increasing function is monotonically decreasing function.

2.6 Answers to Self Check Exercise

Ans.1 10

Ans.2 $x = 1.7417$

Ans.3 $x = 2.094$

Ans.4 $x = 2.706$

2.7 Reference/Suggested Readings

1. Numerical Analsis, Richard L. Burden, J. Douglas Faires, Cengags, 9th Edition.
2. Numerical Analysis for Engincers, Steven C. Chapra, Raymond P. CaraleMcGraw Hill Education 7th edition.

2.8 Terminal Questions

Use bisection method to find the root, upto 3 decimal places.

1. Find the root of the equation x-cox x = 0 correct to 3 decimal places
2. $x^4 + 2x^2 - 16x + 5 = 0$
3. $x^3 - 4x - 9 = 0$
4. $x^3 - 5x - 3 = 0$
5. $x^3 + x^2 + x + 7 = 0$

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Unit - 3

Iterative Method-Method of False Position

Structure

3.1 Introduction

Dear student in this unit we will learn about one another iterative method which is known as method of false position. This method is also known as method of linear interpolation or Regula-Falxi method. In this unit we will study the algorithm and convergence of this method and will try to use Regula-Falsi method to find the root of given equation.

3.2 Learning Objectives:

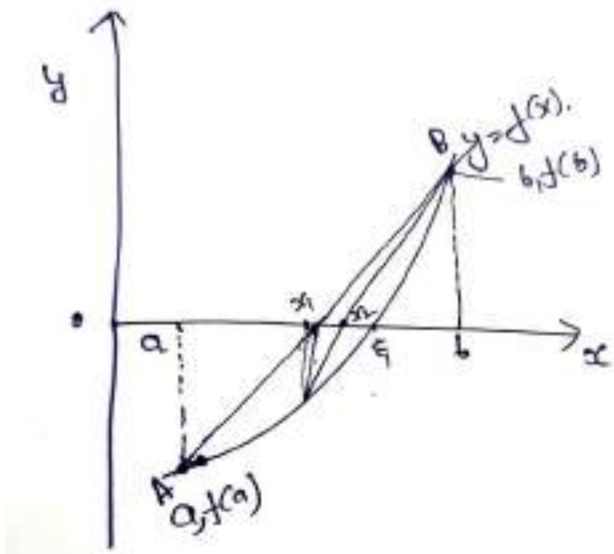
After Studying this unit, students will be able to:

1. Defined mathematically the method of false position.
2. Write the general formula for method of false position
3. Find the order of convergences of this method
4. Apply method of false position to find root of given equation
5. give uses and drawback of this method

3.3 Algorithm of Method of False Position

The method of false position is one of the oldest method. Just like bisection method this method gives an effective tool to solve an equation of the form $f(x) = 0$, for real root. RegulaFalsi method or method of false position closelyresembles with bisection method.

Here we take two points a and b, a < b as initial approximation for the root of equation $f(x) = 0$ such that $f(a)$ and $f(b)$ are of opposite signs i.e. the graph of $y = f(x)$ crosses the x - axis, between these points, as shown in figure 1.



The equation of chord joining the two points $(a, f(a))$ and $(b, f(b))$ is

y - f(a) = $\frac{f(b)-f(a)}{b-a} \times$ (x - a) — By using equation of Line having points (x_1, y_1, and (x_2, y_2) $= y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}$

$$\Rightarrow \quad \frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a} \qquad (1)$$

The point of intersection of this chord with x-axis is given by putting y = 0, in above equation, we get

$$0 - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\Rightarrow \quad -f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

$$\Rightarrow \quad x - a = \frac{-f(b)(b-a)}{f(b)-f(a)}$$

$$\Rightarrow \quad x - a = \frac{-bf(a)+af(a)}{f(b)-f(a)}$$

$$\Rightarrow \quad x = a + \frac{af(a)-bf(a)}{f(b)-f(a)}$$

$$= \frac{af(b) - af(a) + af(a) - bf(a)}{f(b) - f(a)}$$

$$\Rightarrow \qquad x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

This value of x is taken as the approximate value of the root.

So, the first approximate value of the root, say x_1 is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

As in bisection method, Here again there are three possibilities.

1. If $f(x_1) = 0$, then x_1 is the root of given equation.
2. If $f(x_1) < 0$, then the root lies between (x_1, b)
3. If $f(x_1) > 0$, then the root lies between $(a_1\ x_1)$.

Terminating Criterion

If we obtain condition (1), then the process will be stopped but for (2) and (3) condition, we repeat the process untill we get the result of required accuracy.

Geometrical Interpretation

Geometrically, in this method we replace the part of curve $y = f(x)$ between point $(a_1 f(a))$ and $(b_1 f(b))$ of by a straight line birth the end points $(a_1 f(a))$ and $(b_1 f(b))$. The point of intersection of this straight time with x-axis gives us the next approximation x_1 of the root of equation $y = f(x)$,

Order of Convergence of Iterative Method

Let r be root of equation $f(x) = 0$ and e_n be the small quantity by which x_n differ from r. Then

Error at n^{th} approximation is $e_n = x_n - r$ and Error at $n+1^{th}$ approximation is $e_{n+1} = x_{n+1} - r$

Then order of convergence of an iterative method is $þ_1$ if þ is the largest number such that

$|e_{n+1}| < A\ |e_n|^{\rho}$, where A is asymptotic error constant.

Order of Convergence of Method of False Position

Since the formula for I^{st} approximation in Requla-fatsi method is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Replacing $x_1 \to x_{n+1}$, $b \to x_n$ and $a \to x_{n-1}$ we get a general formula i.e.

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})} \qquad (1)$$

If r is the root of equation y = f(x) = 0 and e_{n-1} be the small quantity by which the approximate value of root x_{n-1} differ from r, then errors at n-1, n and n+1 approximation are.

$$\begin{Bmatrix} e_{n-1} = x_{n-1} - r \\ e_n = x_n - r \\ e_{n+1} = x_{n+1} - r \end{Bmatrix}$$

$$\Rightarrow \qquad \begin{Bmatrix} x_{n-1} = e_{n-1} + r \\ x_n = e_n + r \\ x_{n+1} = e_{n+1} + r \end{Bmatrix} \qquad (2)$$

Putting the values of x_{n-1}, x_n and x_{n-1} from (2) in (1) we get

$$e_{n+1} + r = \frac{(e_{n-1} + r) f(e_n + r) - (e_n + r) f(e_{n-1} + r)}{f(e_n + r) - f(e_{n-1} + r)}$$

taking terms containing r in one bracket, we get

$$e_{n+1} + r = \frac{\left[e_{n-1} f(e_n + r) - e_n f(e_{n-1})\right] + r\left[f(e_n + r) - f(e_{n-1} + r)\right]}{f(e_n + r) - f(e_{n-1} + r)}$$

$$= \frac{\left[e_{n-1} f(e_n + r) - e_n f(e_{n-1} + r)\right]}{f(e_n + r) - f(e_{n-1} + r)} + \frac{r\left[f(e_n + r) - f(e_{n-1} + r)\right]}{f(e_n + r) - f(e_{n-1} + r)}$$

$$\Rightarrow \qquad e_{n+1} + r = \frac{\left[e_{n-1} f(e_n + r) - e_n f(e_{n-1} + r)\right]}{f(e_n + r) - f(e_{n-1} + r)} + r$$

$$\Rightarrow \qquad e_{n+1} = \frac{\left[e_{n-1} f(e_n + r) - e_n f(e_{n-1} + r)\right]}{f(e_n + r) - f(e_{n-1} + r)} \qquad (3)$$

Now using Taylor's series expansion to expand $f(e_n + r)$ and $f(e_{n-1} + r)$, in numerator and denominator as well.

Since $f(e_n + r) = f(r) + e_n f'(r) + \frac{e_n^2}{2!} f''(r) + \ldots\ldots\ldots\ldots$.

and $f(e_{n-1} + r) = f(r) + e_{n-1} f'(r) + \frac{e_{n-1}^2}{2!} f''(r) + \ldots\ldots\ldots\ldots$.

So, the numerator of equation (3) is $e_{n-1} f(e_n + r) - e_n f(e_{n-1} + r)$

$$= e_{n-1}\left[f(r)+e_n f'(r)+\frac{e_n^2}{2!} f''(r)+.......... \right] - e_n\left[f(r)+e_{n-1} f'(r)+\frac{e_{n-1}^2}{2!} f''(r)+........ \right]$$

Since r is the root of equation, so $f(r) = 0$

So we get

$e_{n-1}f(e_n + r) - e_n f(e_{n-1} + r)$

$$= e_{n-1}\left[e_n f'(r)+\frac{e_n^2}{2!} f''(r)+.......... \right] - e_n\left[e_{n-1} f'(r)+\frac{e_{n-1}^2}{2!} f''(r)+........ \right]$$

$$= e_{n-1}e_n f'(r) + \frac{e_{n-1}e_n^2}{2!} f''(r) \text{ enen-1} f'(r) - \frac{e_n e_{n-1}^2}{2!} f''(r) + ...$$

$$= \frac{e_{n-1}e_n^2}{2!} f''(r) - \frac{e_n e_{n-1}^2}{2!} f''(r) + ...$$

$$= \frac{e_n e_{n-1}}{2!} f''(r) [e_n - e_{n-1}] + ...$$

Since e_n, e_{n-1}, is small quantities so the term containing higher power can be neglected. So, we get

$$e_{n-1}f(e_n + r) - e_n f(e_{n-1} + r) = \frac{e_n e_{n-1}}{2!} f''(r) [e_n - e_{n-1}] \qquad (4)$$

Now, the denominator of equ. (3) is

$f(e_n + r) - f(e_{n-1} + r)$

Using Taylor's series expansion again, we get

$$= \left[f(r)e_n f'(r)+\frac{e_n^2}{2!} f''(r)+.......... \right] - \left[f(r)e_{n-1} f'(r)+\frac{e_{n-1}^2}{2!} f''(r)+........ \right]$$

as r is root of equation, so $f(r) = 0$, we get

$$e_n f'(r) + \frac{e_n^2}{2!} f''(r) \text{ ....} -e_{n-1} f'(r) - \frac{e_{n-1}^2}{2!} f''(r) + ...$$

$$= f'(r) (e_n - e_{n-1}) + \frac{e_n^2}{2!} f''(r) - \frac{e_{n-1}^2}{2!} f''(r) + ...$$

Neglecting the term containing higher power of e_n and e_{n-1}, we get

$$f(e_n + r) -(e_{n-1} + r) = f'(r) (e_n - e_{n-1}) \qquad ....(5)$$

Using (4) and (5) equation (3) becomes

$$e_{n+1} = \frac{\frac{e_n e_{n-1}}{2!} f''(r)\left[e_n - e_{n-1}\right]}{f'(r)\left(e_n - e_{n-1}\right)}$$

$\Rightarrow$ $$e_{n+1} = \frac{e_n e_{n-1}}{2!} \frac{f''(r)}{f'(r)}$$

$\Rightarrow$ $$e_{n+1} = \frac{1}{2} \frac{f''(r)}{f'(r)} e_n e_{n-1}$$

$\Rightarrow$ $e_{n+1} = e_n e_{n-1}$ (5) where A = $\frac{1}{2} \frac{f''(r)}{f'(r)}$ is a asymptotic error constant.

From (5)

$e_{n+1} \propto e_n e_{n-1}$ ..(6)

Since by definition of order of convergence of iterative method

$$|e_{n+1}| < A|e_n^{\rho}|$$

i.e. $e_{n+1} \propto e_n^{þ}$. ...(7)

also $e_n \propto e^{þ}_{n-1}$ ...(8)

Using (7) and (8), (6) we get

$en+1 \propto e^{þ2}_{n-1}$ ...(9)

Putting (7), (8) and (9) in (6)

$$e_n^{\rho^2} \propto e^{þ}_{n-1} . e_{n-1}$$

$$e_n^{\rho^2} \propto e_{n-1}^{\rho+1}$$

Or $þ^2 = p + 1$

$\Rightarrow$ $þ^2 - þ - 1 = 0$

$\Rightarrow$ $$þ = \frac{-1+\sqrt{5}}{2}$$

as order of convergence cannot be negative, so taking only positive value we get.

$$þ = \frac{1+\sqrt{5}}{2} = \frac{1+2.236}{2} = 1.618$$

þ = 1.618

So, order of convergence of Ragula-Falsi method is 1.618

Advantages and Disadvantages of Method of False Position

Advantages

1. The convergence of this method is guaranteed
2. By Increasing number of iteration, more accurate root can be find
3. This method does not requires any derivation
4. Linear rate of convergence, but faster than bisection method.

Disadvantages

1. Although the convergence of this method is guaranteed, yet it has slow rate of convergence
2. It fails to determine complex root.
3. It cannot be applied if there are discontinuities in the guess interval.
4. It has linear rate of convergence which is steady

To understand more about method of false position Let us try following examples.

Example1: Find the real root of equation $x^3 + x^2 - 3x - 3$ = between 1.5 and 1.75 correct to three decimal places using method of false position.

Solution: Since given equation is $x^3 + x^2 - 3x - 3 = 0$

So, $f(x) = x^3 + x^2 - 3x - 3$

We have to find the root between 1.5 and 1.75, so taking a = 1.5 and b = 1.75

Now, $f(a) = f(1.5) = -1.875 < 0$ (-ve)

and $f(1.75) = 0.1719 > 0$ (+ve)

So root lies between (1.5, 1.75)

Iteration 1: a = 1.5, b = 1.75, $f(a) = -1.875$, $f(b) = 0.1719$

$$\therefore \quad x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{1.5 \times 0.1719 - 1.75 \times (-1.875)}{0.1719 - (-1.875)}$$

$$x_1 = 1.7290$$

Now, $f(x_1) = f(1.7290) = -0.0288 < 0$ (-ve)

and $f(1.75) = 0.1719 > 0$ (+ve)

So root lies between (1.7290, 1.75)

Iteration 2: a = 1.7290, f(a) = -0.0288

b = 1.75, f(b) = 0.1719

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1.7290(0.1719) - (1.75)(-0.0288)}{0.1719 - (-0.0288)}$$

$\Rightarrow$ x_2 = 1.7320

Now f(x2) = f(1.7320) = -0.0005 < 0 (-ve)

and f(1.75) = 0.1718 > 0 (+ve)

So, root lies between (1.7320, 1.75)

Iteration 3: a = 1.7320, b = 1.75

f(a) = -0.0005, f(b) = 0.1719

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1.7320(0.1719) - (1.75)(-0.0005)}{0.1719 - (-0.0005)}$$

$\Rightarrow$ x_3 = 1.7320

Since in 2nd iteration and 3rd iteration, there is no change in approximated root upto three decimal places So, the approximated root of $x^3 + x^2 - 3x - 3 = 0$ is 1.7320.

Example 2: Use method of false-position to find the real root of $x^3 - 4x - 9 = 0$ correct upto three decimal places.

Solution: Given $f(x) = x^3 - 4x - 9$

First to find the initial approximation

f(0) = 0 - 0 - 9 = -9 (-ve)

f(2) = 8 - 8 - 9 = -9 < 0 (-ve)

f(2.5) = 15.625 - 10 - 9 = -3.375 (-ve)

f(3) = 27 - 12 - 9 = 27 - 21 = 6 > 0 (+ve)

Since f(2.5) is -ve and f(3) is +ve, so root lies between (25, 3).

Iteration 1: a = 2.5, f(2.5) = -3.375

b = 3, f(3)

∴ $x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$= \frac{2.6(6) - 3(-3.375)}{6 - (-3.375)}$

⇒ x1 = 2.6800

Now, f(x1) = f(2.6800) = -0.4712 < 0 (-ve)

and f(3) = 6 > 0 +ve

So, root lies between (2.680, 3)

Iteration 2: a = 2.680, f(2.680) = -0.4712

b = 3, f(3) = 6

$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

⇒ x_2 = 2.7033

Now $f(x_2)$ = (2.7033) = -0.579 < 0 (-ve)

and $f(_3)$ = 6 > 0 (+ve)

So, root lies between (2.70033, 3)

Iteration 3: a = 2.7033, f(2.7033 = -0.579

b = 3, f(3) = 6

$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$= \frac{2.7033(6) - 3(-0.579)}{6 - (-0.579)}$

⇒ x_3 = 2.7061

$f(x_3)$ = f(2.7061) = -0.0077 < 0 (-ve)

and f(3) = 6 > 0 (+ve)

So, root lies between (2.7061, 3)

Iteration 4: a = 2.7061, f(2.7061 = -0.0077

b = 3, f(3) = 6

$x_4 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{2.7061(6) - 3(-0.0077)}{6 - (-0.0077)}$

$\Rightarrow \quad x_4 = 2.7065$

Since from 3rd iteration we get $x_3 = 2.7061$ and in 4th iteration we get $x_4 = 2.7065$ which is same upto three decimal places. So, the real root of equation $x_3 - 4x - 9 = 0$ upto three decimal places is 2.706.

Example 3: Find the fourth root 32 upto three decimal places using method of false position.

Solution: We have to find the fourth root of 32 i.e. $(32)^{\frac{1}{4}}$

Let $\quad x = (32)^{\frac{1}{4}}$

$\Rightarrow \quad x^4 = 32$

$\Rightarrow \quad x^4 - 32 = 0$

So $\quad$ we have to find to root of equation $x^4 - 32 = 0$

first of all, we have to find the initial approximation for the interval in which root of $x^4 - 32$ = lies.

Since $f(1) = 1 - 32 = -31 < 0$ (-ve)

$f(2) = 16 - 32 = -16 < 0$ (-ve)

$f(2.5) = 39.0625 - 32 = 7.0625 > 0$ (+ve)

Since $f(2)$ is -ve and $f(2.5)$ is +ve, so root lies between (2, 2.5)

Iteration 1: a = 2, $f(2) = -16$

b = 2.5 $f(2.5) = 7.0625$

So $\quad x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{2(7.0625) - 2.5(-16)}{7.0625 - (-16)}$

$\Rightarrow \quad x_1 = 2.3469$

Now $f(x_1) = f(2.3469) = -1.6626 < 0$ (-ve)

and $\quad f(2.5) = 7.0625 > 0$ (+ve)

So root lies between (2.3469, 2.5)

Iteration 2: a = 2.3469, f(a) = -1.6626

b = 2.5 f(2.5) = 7.0625

So $x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$= \frac{2.3469(7.0625) - 2.5(-1.6626)}{7.0625 - (-1.6626)}$

$\Rightarrow$ $x_2 = 2.3761$

Now $f(x_2) = f(2.3761) = -0.1244 < 0$ (-ve)

and $f(2.5) = 7.0625 > 0$ (+ve)

So root lies between (2.3761, 2.5)

Iteration 3: a = 2.3761, f(2.3761) = -0.1244

b = 2.5 f(2.5) = 7.0625

$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$= \frac{2.3761(7.0625) - 2.5(-0.1244)}{7.0625 - (-0.1244)}$

$\Rightarrow$ $x_3 = 2.3782$

Now $f(x_3) = f(2.3782) = -0.0115 < 0$ (-ve)

and $f(2.5) = 7.0625 > 0$ (+ve)

So root of given equation lies between (2.3782, 2.5)

Iteration 4: a = 2.3782, f(2.3782) = -0.0115

b = 2.5, f(2.5) = 7.0625

$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$

$= \frac{2.3782(7.0625) - 2.5(-0.0115)}{7.0625 - (-0.0115)}$

$\Rightarrow$ $x_4 = 2.3784$

Since from 3_{rd} iteration we get $x_3 = 2.3782$ and in 4^{th} iteration we get $x_4 = 2.3784$ which is same upto three decimal places, so the fourth root of 32 correct upto three decimal places is 2.3784

Example 4: Find the real root of equation $x\log_{10} x = 1.2$ between 2.7 and 3 using method of False position, correct upto four decimal places.

Solution: Given $x\log_{10} x = 1.2$

$\therefore \quad f(x) = x\log_{10} x - 1.2$. Also we have to find root between 2.7 and 3 so.

Now $f(2.7) = (2.7)\log_{10}(2.7) - 1.2$

$= -0.035318 < 0$ (-ve)

$f(3) = (3)\log_{10}(3) - 1.2$

$= 0.23136 > 0$ (+ve)

So root of given equation lies between (2.7, 3)

Iteration 1: $a = 2.7$, $f(2.7) = -0.035318$

$b = 3$ $f(3) = 0.23136$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2.7(0.23136) - 3(-0.035318)}{0.23136 - (-0.035318)}$$

$\Rightarrow \quad x_1 = 2.73973$

Now $(x_1) = -0.00080 < 0$ (-ve)

and $f(3) = 0.23136 > 0$ (+ve)

So, root of equation lies between (2.73973. 3)

Iteration 2: $a = 2.73973$, $f(2.73973) = -0.00080$

$b = 3$ $f(3) = 0.23136$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2.73973(.23136) - 3(-0.00080)}{0.23136 - (-0.00080)}$$

$\Rightarrow \quad x_2 = 2.74063$

Now $f(2.74063) = -0.00001 < 0$ (-ve)

and $f(3) = 0.23136 > 0$ (+ve)

So, root of equation lies between (2.74063, 3)

Iteration 3: a = 2.74063, $f(2.74063) = -0.00001$

b = 3 $f(3) = 0.23136$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2.74063(.23136) - 3(-0.00001)}{0.23136 - (-0.00001)}$$

$\Rightarrow$ $x_3 = 2.74064$

Since from 2^{nd} we get $x_2 = 2.74063$ and in 3^{rd} iteration we get $x_3 = 2.74064$, which is some upto four decimal places. So the root of given equation $x\log_{10}x = 1.2$ is 2.7406, correct upto 4 decimal places.

Example 5: Using method of False position find the real root of equation $e^x - x^3 = 0$ correct upto four decimal places.

Solution: Given equation is $e^x - x^3 = 0$, to find initial approximation, $f(1) = e^1 - 1 = 2.7182 > 0$ (+ve)

$f(1.5) = e^{1.5} - (1.5)3 = 4.48168 - 3.375 = 1.10668 > 0$ + (ve)

$f(1.75) = 5.75460 - 5.359375 = 0.395225 > 0$ (+ve)

$f(2) = 7.3890 - 8 = -0.61094 < 0$ (-ve)

Since $f(1.75)$ is negative and $f(2)$ is positive, so root of given equation lies between the interval (1.75, 2)

Iteration 1: a = 1.75, $f(1.75) = 0.39523$

b = 2 $f(2) = -0.61094$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{1.75(-0.61094) - 2(0.39523)}{-0.61094 - 0.39523}$$

$\Rightarrow$ $x_1 = 1.84820$

Now $f(1.84820) = 0.03522 > 0$ (+ve)

and $f(2) = -0.61094 < 0$ (-ve)

So, root of equation lies between (1.84820, 2)

Iteration 2: a = 1.84820, $f(1.84820) = 0.03522$

b = 2, $f(2) = -0.61094$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_2 = \frac{1.84820(-0.61094) - 2(0.03522)}{-0.61094 - 0.035522}$$

$\Rightarrow$ $x_2 = 1.85647$

Now $f(x_2) = f(1.85647) = 0.00281 > 0$ (+ve)

and $f(2) = -0.61094 < 0$ (-ve)

So, root of given equation lies between (1.85647, 2)

Iteration 3: a = 1.85647, $f(.85647) = 0.00281$

b = 2, $f(2) = -0.61094$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1.85647(-0.61094) - 2(0.00281)}{-0.61094 - 0.00281}$$

$\Rightarrow$ $x_3 = 1.85713$

Now, $f(x_3) = f(1.85713) = .00021 > 0$ (+ve)

and $f(2) = -0.61094 < 0$ (+ve)

So, root of equation lies between (1.85713, 2)

Iteration 4: a = 1.85713, $f(1.85713) = 0.00021$

b = 2, $f(2) = -0.61094$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1.85713(-0.61094) - 2(0.00021)}{-0.61094 - 0.00021}$$

$\Rightarrow$ $x_4 = 1.85718$

Since from 3rd iteration we get $x_3 = 1.85713$ and form 4th iteration, we get $x_4 = 1.85718$ which is same upto four decimal places. So the approximate root of given equation is 1.8571, correct upto four decimal places.

Self Check Exercises

Find the root of below given equation correct upto three decimal places.

Q.1 $x^6 - x^4 - x^3 - 1 = 0$ using initially a = 1.4 and b = 1.5

Q.2 $x^3 - 2x - 5 = 0$

Q.3 $x^3 + x - 1 = 0$

Q.4 $x^3 + x^2 - 3x - 3 = 0$

3.4 Summary:

Dear students in this unit, we studies about

(1) another iterative method known as method of False position, or method of interpolation or Regulafalsi method.

(2) the algorithm and geometrical interpretation of this method

(3) the convergence of method of false position and find that rate of convergence of this method is 1.68

(4) discuss about the advantages and disadvantages of this method.

(5) numerical procedure by which root of a given equation can be approximated using this method.

3.5 Glossary

- **Chord:** A straight line drawn between two points on a circle
- **Equation of chord:** Chord Length = $2\times\sqrt{r^2 - d^2}$
- **Point of intersection:** The meeting point of two straight line is known as point of interaction

3.6 Answers to Self Check Exercises

Ans. x = 1.4035

Ans. x = 2.0945

Ans. x = 0.682

Ans. x = 1.732

3.7 References/Suggested Readings

1. Numerical methods, M.K. Jain, S.R.K. Iyengar, R.K. Jan New age international Publishes

2. Theory and Problem in numerical methods, TV erarajan, T. Ramachandran, Tata Ms. Graw Hill.

3.8 Terminal Questions

1. Find the real root of $x^3 - 5x + 1 = 0$ in the interval (0, 1), using Regula-Falsi method.
2. Find the root of cos x - $x^e x = 0$, using method of false position
3. Find the negative root of the equation $x^3 + 2x^2 + 2.2x + 0.4 = 0$ that lies between -1 and 0, correct to 4 places of decimal using RegulaFalsi method.
4. Find the root of $x e^x - 3 = 0$, that lies between 1 and 2, correct to 4 places of decimal using method of false position.
5. Find the root of equation sin x = e^x that lies between 0 and 0.5, correct to 4 places of decimal, using Regula-Falsi method.

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Unit - 4

Fixed Point Iteration Method

Structure

4.1 Introduction

Dear student, in this unit we will learn about the iteration method of solving non linear equation. Just like other methods studied in unit 2 and 3 this method is also an approximate method, whose order of convergence depends upon the choice if function we choose (In the basis of given equation) and initial value of variable. In this unit we will learn about the steration method, its order of convergence along with its algorithm. We will also learn to apply iterative method to find approximate solution of non linear problem.

4.2 Learning Objectives:

After studying the unit, students will be able to

1. define iteration method
2. algorithm of iteration method
3. order of convergence of iteration method
4. apply iteration method to solve non linear equation
5. solve non-linear equations using iteration method.

4.3 Fixed Point Iteration Method

Assume we have an equation $f(x) = 0$, for which we have to find the solution. We can write the equation as $x = g(x)$. Select $g(x)$ such that $|g'(x)| < 1$ at $x = x_0$, where x_0 is the initial approximation known as a fixed point iterative plan. Afterward, the iterative approach is used with the consecutive approximation denoted by

$x_n = g(x_{n-1})$ i.e. $x_1 = g(x_0)$, $x_2 = g(x_2)$ and so on.

The Fixed Point Iterative Method Algorithm

Select x_0 as the starting value for the iterative process. Finding the values of x = a and x = b for which $f(a) < 0$ and $f(b) > 0$ is one method to determine x_0. Take x_0 as the average of a and b after reducing the number of options for a and b.

Write the given equation as x = g(x) such that for $x = x_0$, $|g'(x)| < 1$. Select g(x) that has the lowest value of g'(x) at $x = x_0$ if there are multiple possibilities for g(x).

If f is continuous function, we obtain a sequence of $\{x_n\}$ that converges to a point lo, which is the approximate solution of the problem, by applying the consecutive approximations $x_n = g(x_{n-1})$

Important Things to Remember

Several interesting details regarding the fixed point repetition technique are:-

There are numerous method to choose the form of x = g(x). However for $x = x_0$, We select g(x) for which $|g'(x)| < 1$.

We obtain a sequence of x_n by the fixed point iteration approach and this sequence converges to the given equation's root.

The fewer repetitions needed to obtain the approximate answer, the lower the value of g'(x).

If the value of g'(x) is smaller, the rate of convergence increases. The technique is helpful in determining the equation's real rook, which take the form of an infinite series.

There is a linear convergence observed in this method.

Convergence:

The convergence of iterative method depends on the choice of the function g(x) as well as the starting value of x_0.

Let ξ be a simple root of given equation $f(x) = 0$

Since the general iterative formula for iteration method is

$x_n = g(x_{n-1})$ or $x_{n+1} = g(x_n)$, n = 0, 1, 2 .... (1)

If e_n and e_{n+1} are errors in the approximations x_k and x_{k+1} respectively then

$e_n = x_n - \xi$ and $e_{n+1} = x_{n+1} - \xi$

Or $x_n = \xi + e_n$ and $x_{n+1} = e_{n+1} + \xi$.

Putting the value of x_n and x_{n+1} in (1)

$\xi + e_n = g(x_{n-1})$ and $e_{n+1} + \xi = g(\xi + e_n)$

Taking. $\xi + e_{n+1} = g(\xi + e_n)$

expanding $g(\xi + e_n)$ using Taylor's series expansion about point ξ we have

ξ + e_{n+1} = g(ξ) + e_n g'(ξ) + $\frac{e_n^2}{2!}$ g''(ξ) + $\frac{e_n^3}{3!}$ g'''(ξ) + ... on ignoring the higher power of e_n i.e. , e_n^2 , e_n^3 , ... we get ξ + e_{n+1} = g(ξ) + e_n g'(ξ) (2)

As ξ is root of equation f(x) = 0 and x = g(x) integration methods.

So ξ = g(ξ)

So from equation (2) g(ξ) + e_{n+1} = g(ξ) + e_ng'(ξ)

⇒ e_{n+1} = e_n g'(ξ)

⇒ Iteration method has linear order of convergence provided $|g'(\xi)|$ < 1.

Fixed Point Iteration Method:

Let us try to apply fixed point iteration methods on some examples.

Some Examples:

Example 1: Find the first approximation root of the equation $2x^3 - 2x - 5 = 0$ upto 4 decimal places?

Solution: The given equation is $2x^3 - 2x - 5 = 0$

Let $f(x) = 2x^3 - 2x - 5$

As per algorithm, we find the value of x_0, for which we have to find a and b such that f(a) < 0 and f(b) > 0

Now, $f(0) = 2(0)^3 - 2(0) - 5 = -5$

$f(1) = 2(1)^3 - 2(1) - 5 = -5$

$f(2) = 2(2)^3 - 2(2) - 5 = 7$

Thus, a = 1 and b = 2

So, the first approximate root of given equation lies in the interval (1, 2)

Therefore, $x_0 = \left(\frac{1+2}{2}\right) = \left(\frac{3}{2}\right) = 1.5$

Now, we shall find g(x) such that $|g'(x)|$ < 1 at x = x_0 Rewriting the given equation as

$$= 1\ x = \left[\frac{2x+5}{2}\right]^{\frac{-2}{3}}$$

Here $g(x) = \left[\frac{2x+5}{2}\right]^{\frac{-2}{3}}$ which satisfies $|g'(x)|$ < 1 at x = 1.5

as g'(x) = $\frac{1}{3}\left[\frac{2x+5}{2}\right]^{\frac{1}{3}-1}.\left(\frac{x}{x}\right)$ = $\frac{1}{3}\left[\frac{2x+5}{2}\right]^{\frac{-2}{3}}$

$$\left|g'(x)\right|_{x=1.5} = \frac{1}{3}\left[\frac{2(1.5)+5}{2}\right]^{\frac{-2}{3}}$$

$$= \frac{1}{3}\left[\frac{3+5}{2}\right]^{\frac{-2}{3}}$$

$$= \frac{1}{3}[4]^{\frac{-2}{3}} \text{ or } \frac{1}{3(4)^{\frac{-2}{3}}} < 1$$

$\Rightarrow$ $\left|g'(x)\right|$ < 1 at x = 1.5

So, the iterative process defined by $x_n = g(x_{n-1}) = \left[\frac{2x_{n-1}+5}{2}\right]^{\frac{1}{3}}$ will converge to the voot.

Here, the initial approximation is x_0 = 1.5. The successive approximation are calculated as follows:

For n = 1; $x_1 = g(x_0) = \left[\frac{\{2(1.5)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.5874

For n = 2; $x_2 = g(x_1) = \left[\frac{\{2(1.5874)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.5989

For n = 3; $x_3 = g(x_2) = \left[\frac{\{2(1.5989)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.60037

For n = 4; $x_4 = g(x_3) = \left[\frac{\{2(1.60037)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.60057

For n = 5; $x_5 = g(x_4) = \left[\frac{\{2(1.60057)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.60059

For n = 6; x_6= g(x_5) = $\left[\frac{\{2(1.60059)+5\}}{2}\right]^{\frac{1}{3}}$ = 1.600597 ≈ 16006

The approximate root of $2x^3 - 2x - 5 = 0$ by the fixed point iteration method is 1.6006 (correct to 4 decimal places)

Example 2: Find the smallest positive root of the equation $x^2 - 7x + 2 = 0$ using iteration method?

Solution: The given equation is $x^2 - 7x + 2 = 0$

Let $f(x) = x^2 - 7x + 2$

Now, $f(0) = (0)^2 - 7(0) + 2 = 2$

and (1) = $(1)^2 - 7(1) + 2 = -4$

So, a smallest positive root of given equation lies in the interval (0, 1)

Therefore $x_0 = \left(\frac{0+1}{2}\right) = 0.5$

This is because, as per algorithm we find the value of x_0 for which we have to find a and b such that $f(a) < 0$ and $f(b) > 0$

Now, we shall find ϕ (x) such that $|g'(x)| < 1$ at $x = x_0$ Rewriting the given equation as

$$x = \left[\frac{-x^2-2}{7}\right]$$

Here $\phi(x) = \left[\frac{-x^2-2}{7}\right]$ which satisfies $|\phi'(x)| < 1$ at x = 0.5

as $\phi'(x) = \frac{-2x}{7}$

$$|\phi'(x)|_{x=0.5} = \frac{+2(0.5)}{7} = \frac{1}{7} < 1$$

So, the iterative process defined by $x_n = \phi(x_{n-1}) = \left[\frac{-x_{n-1}^2-2}{7}\right]$ will converge to the root.

Here, the initial approximation x = 0.5. The successive approximation are calculated as follows.

For n = 1; $x_1 = \phi(x_0) = \left[\frac{-(0.5)^2-2}{7}\right] = \left[\frac{-0.25-2}{7}\right] = 0.3214$

For n = 2; $x_2 = \phi(x_1) = \left[\frac{-(0.3214)^2 - 2}{7}\right] = \left[\frac{-0.1032 - 2}{7}\right] = 0.30045$

For n = 3; $x_3 = \phi(x_2) = \left[\frac{-(0.3004)^2 - 2}{7}\right] = \left[\frac{-0.0902 - 2}{7}\right] = 0.2986$

For n = 4; $x_4 = \phi(x_3) = \left[\frac{-(0.2986)^2 - 2}{7}\right] = \left[\frac{-0.0891 - 2}{7}\right] = 0.2984$

For n = 5; $x_5 = \phi(x_4) = \left[\frac{-(0.2984)^2 - 2}{7}\right] = \left[\frac{-0.0890 - 2}{7}\right] = 0.2984$

We observe that 4th and 5th iteration, the successive approximation to the root are same. So, a smallest positive root of given equation is given by x = 0.2984. (Correct to 4 decimal.

Example 3: Find a real root of $2x + \log_{10}x - 4 = 0$ correct to three decimal places using iteration method?

Solution: The given equation is $2x + \log_{10} x - 4 = 0$

Let $f(x) = 2x + \log_{10} x - 4$

As per algorithm, we find the value of x_0, for which we have to find a and b such that $f(a) < 0$ and $f(b) > 0$

Now, $f(1) = 2(1) + \log_{10}(1) - 4 = -2$

$f(2) = 2(2) + \log_{10}(2) - 4 = 0.3010$

So, a real root of equation lies in the interval (1, 2)

Therefore $x_0 = \left[\frac{1+2}{2}\right] = 1.5$

Now, we shall find $\phi(x)$ such that $|\phi'(x)| < 1$ at $x = x_0$

Rewriting the given equation as

$$x = \frac{1}{2}(4 - \log_{10}x)$$

Here $\phi(x) = \frac{1}{2}(4 - \log10x)$ which satisfies $|\phi'(x)| < 1$ at x = 1.5

as $\phi'(x) = \frac{1}{2}\left[\frac{-1}{x}\right]\log_{10}e$ $\because \left| \begin{array}{l} pog_e x = \frac{\log_{10} x}{\log_{10} e} \\ \Rightarrow pog_{10} x = log_e x \times log_{10} e \end{array} \right|$

$|\phi'(x)|_{x=1.5} = \frac{1}{2}\left(\frac{0.4343}{1.5}\right) = \frac{1}{3} < 1$

So, the iterative process defined by $x_n = \phi(x_{n-1}) = \frac{1}{2}(4 - \log_{10} x_{n-1})$ will converge to the root.

Here the initial approximation is $x_0 = 1.5$. The successive approximation are calculated as follows:

For n = 1; $x_1 = \phi(x_0) = \frac{1}{2}\left[4 - \log_{10}(1.5)\right] = \frac{1}{2}[4 - 0.1760] = 1.9120$

For n = 2; $x_2 = \phi(x_1) = \frac{1}{2}\left[4 - \log_{10}(1.9120)\right] = \frac{1}{2}[4 - 0.2814] = 1.8593$

For n = 3; $x_3 = \phi(x_2) = \frac{1}{2}\left[4 - \log_{10}(1.8593)\right] = \frac{1}{2}[4 - 0.2693] = 1.8653$

For n = 4; $x_4 = \phi(x_3) = \frac{1}{2}\left[4 - \log_{10}(1.8653)\right] = \frac{1}{2}[4 - 0.2707] = 1.8646$

For n = 5; $x_5 = \phi(x_4) = \frac{1}{2}\left[4 - \log_{10}(1.8646)\right] = \frac{1}{2}[4 - 0.2705] = 1.8647$

From 4^{th} and 5^{th} iteration, it is dear that there is no change in successive approximations to the root upto first three decimal places.

So, a real root of given equation is given by 1.864 (correct to 3 decimal places)

Example 4: Find a positive root of the equation $e^x - \frac{1}{x^2} = 0$ between 0 and 1 using the method of iteration?

Solution: The given equation is $e^x - \frac{1}{x^2} = 0$ or x2ex-1 = 0

Let $f(x) = x^2 e^x - 1$

Now, $f(0) = -1 < 0$

$f(1) = 1e^1 - 1 = 1.7183 > 0$

So, a real root of given equation lies in the interval (0, 1)

Therefore, $x_0 = \left(\frac{0+1}{2}\right) = 0.5$

Now, we shall find g(x) such that $|g'(x)| < 1$ at $x = x_0$

Rewriting the given equation as

$$x = \left[\frac{1}{e^x}\right]^{\frac{1}{2}} \text{ or } e^{-x/2}$$

Here $g(x) = e^{-x/2}$ which satisfies $|g'(x)| < 1$ at x = 0.5 as $g'(x) = \frac{1}{2} e^{-x/2}$

$$|g'(x)|_{x=0.5} = \frac{1}{2} e^{-0.5/2} = \frac{1}{2e^{1/4}} < 1$$

So, the iterative process defined by $x_n = g(x_{n-1}) = e^{-x_{n-1}/2}$ will converge to the root.

Here, the initial approximation is $x_0 = 0.5$. The successive approximation are calculated as follows.

For n = 1; $x_1 = g(x_0) = e^{-0.5/2} = 0.7788$

For n = 2; $x_2 = g(x_1) = e^{-0.7788/2} = 0.6774$

For n = 3; $x_3 = g(x_2) = e^{-6774/2} = 0.7126$

For n = 4; $x_4 = g(x_3) = e^{-7126/2} = 0.7002$

For n = 5; $x_5 = g(x_4) = e^{-7002/2} = 0.7046$

For n = 6; $x_6 = g(x_5) = e^{-7046/2} = 0.7030$

For n = 7; $x_7 = g(x_6) = e^{-7030/2} = 0.7036$

For n = 8; $x_8 = g(x_7) = e^{-7036/2} = 0.7034$

For n = 9; $x_9 = g(x_8) = e^{-7034/2} = 0.7034$

From 8th and 9th iteration, we observe that the successive approximation to the root are approximately same. So, the positive root of the equation is given by x = 0.7034 (correct to 4 decimal places)

Example 5: Find a real root of the equation $-x^2 + 2x + \sin(x) = 0$ correct to three decimal places using iteration method?

Solution: The given equation is $-x^2 + 2x + \sin(x) = 0$

Let $f(x) = -x^2 + 2x + \sin(x)$

As per algorithm, we find the value of x_0, for which we have to find a and b such that $f(a) < 0$ and $f(b) > 0$

Now, $f(0) = -0 + 0 + 0 = 0$

$f(1) = -1 + 2 + \text{Sin}(1) = 1.8414$

$f(2) = -4 + 4 + \text{Sin}(2) = 0.9092$

$f(3) = -9 + 6 + \text{Sin}(3) = -2.8588$

So, a real root of the equation lies in the interval (2, 3)

Therefore, $x_0 = \left(\frac{2+3}{2}\right) = 2.5$

Now, we shall find g(x) such that $|g'(x)| < 1$ at $x = x_0$ Rewriting the equation

$$x = [2x + \sin x]^{\frac{1}{2}}$$

So, the $\phi(x) = [2x + \sin x]^{\frac{1}{2}}$

Which satisfies $|\phi'(x)| < 1$ at $x = 2.5$

as $$\phi'(x) = \frac{1}{2}\left(\frac{2+\cos x}{(2x+\sin x)^{1/2}}\right)$$

$$|\phi'(x)|_{x=2.5} = \frac{1}{2}\left(\frac{2+\cos(2.5)}{(2(2.5)+\sin(2.5))^{1/2}}\right) = \frac{1}{2}\left[\frac{1.1988}{2.3660}\right] < 1$$

So, the iterative process defined by $x_n = \phi(x_{n-1}) = [2x_{n-1} + \sin x_{n-1}]^{\frac{1}{2}}$ will converge to the root.

Here, the initial approximation is $x_0 = 2.5$. The successive approximation are calculated as follows:

For n = 1; $x_1 = \phi(x_0) = [2(2.5) + \sin(2.5)]^{1/2} = 2.3660$

For n = 2; $x_2 = \phi(x_1) = [2(2.3660) + \sin(2.3660)]^{1/2} = 2.3306$

For n = 3; $x_3 = \phi(x_2) = [2(2.3306) + \sin(2.3306)]^{1/2} = 2.3207$

For n = 4; $x_4 = \phi(x_3) = \left[2(2.3207)+\sin(2.3207)\right]^{1/2} = 2.3179$

For n = 5; $x_5 = \phi(x_4) = \left[2(2.3179)+\sin(2.3179)\right]^{1/2} = 2.3171$

For n = 6; $x_6 = \phi(x_5) = \left[2(2.3171)+\sin(2.3171)\right]^{1/2} = 2.3169$

For n = 7; $x_7 = \phi(x_6) = \left[2(2.3169)+\sin(2.3169)\right]^{1/2} = 2.3169$

So, a real root of given equation is x = 2.3169 (correct to 4 decimal places)

Self Check Exercise

Q.1 Find the first approximation root of the equation cos x = 3x - 1 upto 4 decimal places.

Q.2 find a real root of $2x - \log_{10} x = 7$ correct to three decimal places using iteration method.

Q.3 Find the positive root of the equation $xe^x = 1$ between 0 and 1 using the method of iteration.

4.4 Summary:

(i) Fixed point iteration method has linear convergence

(1) The fixed point iteration method is an iterative to find the roots of algebraic and transcendial equation by converting them into a fixed point function.

(2) The given equation $f(x) = 0$ is expressed as x = g(x). By guessing some initial x_0. The iterative method is applied by successive approximations given by $x_n = g(x_{n-1})$. We get a sequence $\{x_n\}$ such that as $x_n \to$ a consequently $f(x) \to 0$ Hence we get x = a as a approximate colution.

4.5 Glossary:

- **Consecutive:** Consecutive meaning in maths represents the unbroken sequence of numbers. It means that in a sequence, the numbers following continuously.
- **Fixed Point:** A fixed point in the fixed point iteration method is a point in the domain of a function g such that g(x) = x.
- **Continuous Function:** A function which
  - (i) Has graph without any breaks or jumps
  - (ii) Exhibits no abrupt changes in values
  - (iii) Maintains continuity at every point within its domain.

4.6 Answers to Self Check Exercise

Ans. 0.6071

Ans. 3.789

Ans. 0.5672

4.7 Reference/ Suggested Reading

1. Numerical Analysis by Richard L. Burden
2. Introductory methods of Numerical Analysis by S.S. Sastry
3. Finite Difference & Numerical Analysis by H.C. Saxena
4. An Introduction to Numerical Analysis by Kendall E. Atkinson.

4.8 Terminal Questions

1. Find a real root of the equation $\sin(x) - x^2 = 0$ correct to three decimal places using iteration method
2. Find the smallest positive root of the equation $x^2 + 8x - 5 = 0$ using iteration method
3. Find a real root of the equation $x^2 + 2x - 2x = 0$ correct to three decimal places using iteration method

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Unit - 5

Newton Raphson Method

Structure

5.1 Introduction

Dear student, in this unit will learn about the mostly used method to find solution of non-linear equation by approximation, which is known as Newton-Rapson method. This method again depends on initial approximation. If the initial approximation is chosen sufficiently close to the root this method gives the final root very fast. In this unit we will learn about Newton-Raphson Method, its algorithm and its convergence. We also learn how to apply this method to non linear equation and find the approximate root of a given equation. This method is also used to find square root, cube root of any any number upto desire number of decimal places.

5.2 Learning Objectives:

After studying this unit, students are able to

1. define Newton-Raphson Method
2. give algorithm of Newton-Raphson Method
3. give order of convergence of Newton-Raphson Method
4. apply Newton-Raphson Method to any non-linear equation
5. solve equation by Newton-Raphson Method
6. find square root and cube root of a given number by Newton-Raphson Method

5.3 Newton Raphson iterative Method

The Newton Method, also known as Newton Raphson Method is an effective method for numerical problem solving. The approximation of the roots of real valued function is its most popular use. The Newton Raphson Method got its name since it was created by Joseph Raphson and Issac Newton.

The Newton Raphson Method entails progressively improving a first estimate in order to converge it on the intended root. For higher degree polynomials or equations, the approach is inefficient in calculating the roots but for lower degree equations, it provides very fast answers. This article will explain the Newton Raphson Method and walk us through the process of utilizing it to find the roots.

Newton Raphson method Formula

Let x0 be the approximation root of $f(x) = 0$and let $x_1 = x_0 + h$ be the correct root.

Then $f(x_1) = 0$

$\Rightarrow \quad f(x_0 + h) = 0 \qquad ..(1)$

By expanding the above equations by Taylor's Theorem, we get

$f(x0) + h\, f1(x0) + ... = 0$

$$\Rightarrow \quad h = \frac{-f'(x_0)}{f''(x_0)}$$

$$\text{Therefore, } x_1 = x_0 - \frac{-f(x_0)}{f'(x_0)}$$

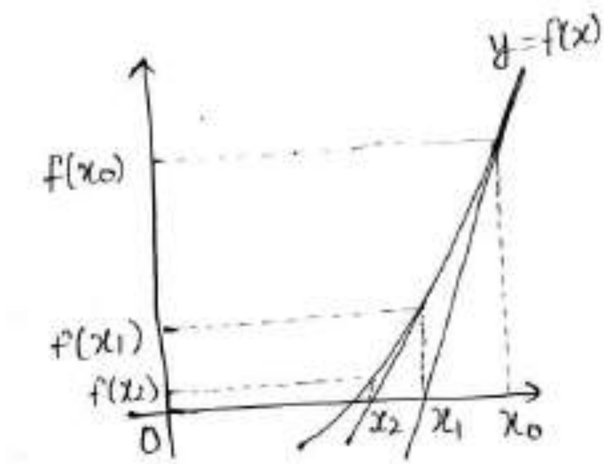
Now, x_1 is better approximation then x_0.

Similarly, the successive approximation x_2, x_3, ..., x_{n+1} are given by this is called Newton Raphson Formula.

$$\text{The general formula is } x_{n+1} = x_n - \frac{-f(x_n)}{f'(x_0)}, \; n = 0, 1, 2, 3, ...$$

Geometrical Interpretation of Newton Raphson Method

Newton Raphson approach can be seen geometrically as drawing a tangent to the curve $y = f(x)$ at the point $[x_0, f(x_0)]$



At x_1, where the x axis is cut, the root is more accurately approximated. The x axis is now cut at x_2 by drawing a second tangent at $[x_1, f(x_1)]$ which is an even better approximation. This method can be repeated until the required accuracy is attained.

Convergence of Newton Raphson Method

In Newton Raphson approach, the order of convergence is two or quadratic. It converges if

$$|f(x).f''(x)| < |f'(x)|^2$$

Also, this method fails if $f'(x) = 0$

Let ξ be the simple root of equation $f(x) = 0$

Since general iteration formula for Newton-Raphson Method is

$$x_{n+1} = x_n \frac{-f(x_n)}{f'(x_n)} \qquad (1)$$

Let e_n and e_{n+1} are the error in the approximation x_n and x_{n+1} respectively then

$e_n = x_n - \xi$ and $e_{n+1} = x_{n+1} - \xi$

or $x_n = \xi + e_n$ and $x_{n+1} = \xi + e_{n+1}$

Putting the value of x_{n+1} and x_n in (1)

$$\xi + e_{n+1} = \xi + e_n - \frac{f(\xi + e_n)}{f'(\xi + e_n)}$$

$$e_{n+1} = e_n - \frac{f(\xi + e_n)}{f'(\xi + e_n)} \qquad (2)$$

Now expanding $f(\xi + e_n)$ and $f'(\xi + e_n)$ using Taylor's series about ξ.

$$e_{n+1} = e_n - \frac{\left\{ f(\xi) + e_n f'(\xi) + \frac{e_n^2}{2!} f''(\xi) + \frac{e_n^3}{3!} f'''(\xi) + \ldots\ldots\ldots\ldots \right\}}{\left\{ f'(\xi) + e_n f''(\xi) + \frac{e_n^2}{2!} f'''(\xi) + \frac{e_n^3}{3!} f^{iv}(\xi) + \ldots\ldots\ldots\ldots \right\}}$$

sin a $f(\xi) = 0$, as ξ is root $f(x) = 0$

$$e_{n+1} = e_n - \frac{\left\{ e_n f'(\xi) + \frac{1}{2} e_n^2 f''(\xi) + \ldots\ldots\ldots\ldots \right\}}{\left\{ f'(\xi) + e_n f''(\xi) + \ldots\ldots\ldots\ldots \right\}}$$

$$= e_{n+1} = e_n - \frac{\left\{e_n + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}}{\left\{1 + e_n \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}}$$

$$= e_{n+1} = e_n - \left\{e_n + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}\left\{1 + e_n \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}^{-1} \quad \text{.....(3)}$$

Using the Binomial theorem for any index, we have

$$= e_{n+1} = e_n - \left\{e_n + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}\left\{1 + e_n \frac{f''(\xi)}{f'(\xi)} + \text{..........}\right\}$$

$$= e_{n+1} = e_n - \left[e_n + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)} - \frac{e_n^2 f''(\xi)}{f'(\xi)} - \frac{1}{2}e_n^3\left(\frac{f''(\xi)}{f'(\xi)}\right)^2 + \text{..........}\right]$$

Neglecting the terms containing higher power of en, we get

$$= e_{n+1} = e_n - e_n + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)}$$

$\Rightarrow$ $\qquad e_{n+1} + \frac{1}{2}e_n^2 \frac{f''(\xi)}{f'(\xi)}$

$\Rightarrow$ $\qquad$ Let c = $e_n + \frac{1}{2}\frac{f''(\xi)}{f'(\xi)}$

So $\qquad$ en+1 = $c\,e_n^2$ $\qquad$ ....(4)

Hence Newton-Raphson Method is second order convergence.

Advantages:

(1) It is one of the best method to find the root of non linear equation

(2) This method has second order convergence, so it is a fast method

(3) This method can easily applied on computer.

(4) By this method root can be find with great speed and little labour

Disadvantages:

(1) This method is very sensitive to initial approximation. If initial approximation is wrong then method never ends.

(2) In this method we have to find $f(x)$ and $f'(x)$ at each step. So having mar computational work.

(3) If derivative of $f(x)$ is not having simple expression so it is difficult to apply.

The Newton Raphson Iterative Method:

Let us try some example in which we find the root by Newton-Raphson Method.

Some Examples:

Example 1: Find a real root of equation $x^3 - x - 1 = 0$ by Newton Raphson Method

Solution: Given equation is $x^3 - x - 1 = 0$

Let $f(x) = x^3 - x - 1$

$\therefore$ $f'(x) = 3x^2 - 1$

Now, $f(1) = (1)^3 - (1) - 1 = -1$

$f(2) = (2)^3 - (2) - 1 = 5$

So, a real root of given equation lies between 1 and 2

Also $f(1)$ is nears to zero than $f(2)$ so take initial approximation x_0 to the root as 1.

Iteration 1: The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{(x_0)^3 - x_0 - 1}{3(x_0)^2 - 1}$$

$$= 1 - \frac{(1)^3 - 1 - 1}{3(1)^2 - 1} = 1 + \frac{1}{2}$$

$$= 1.5$$

Iteration 2: The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{(x_1)^3 - x_1 - 1}{3(x_1)^2 - 1}$$

$$= 1.5 - \frac{(1.5)^3 - (1.5) - 1}{3(1.5)^2 - 1} = 1.5 - \frac{0.875}{5.75}$$

$$= 1.3478$$

Iteration 3: The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{(x_2)^3 - x_2 - 1}{3(x_2)^2 - 1}$$

$$= 1.3478 - \frac{(1.3478)^3 - (1.3478) - 1}{3(1.3478)^2 - 1}$$

$$= 1.3478 - \frac{0.1005}{4.4496}$$

$$= 1.3252$$

Iteration 4: The fourth iteration to the root is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{(x_3)^3 - x_3 - 1}{3(x_3)^2 - 1}$$

$$= 1.3252 - \frac{(1.3252)^3 - 1.3252 - 1}{3(1.3252)^2 - 1}$$

$$= 1.3252 - \frac{0.0020}{4.2684}$$

$$= 1.3247$$

Iteration 5: The fifth iteration to the root is given by

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = x_4 - \frac{(x_4)^3 - x_4 - 1}{3(x_4)^2 - 1}$$

$$= 1.3247 - \frac{(1.3247)^3 - (1.3247) - 1}{3(1.3247)^2 - 1}$$

$$= 1.3247 - \frac{(-0.00007)}{4.2644}$$

$$= 1.3247$$

From 4^{th} and 5^{th} iteration, we observe that the successive approximation to the root are same. So, we stop the iterative procedure. So, a real root of given equation is given by 1.3247.

Example 2: Find a root of equation $2x^3 - 2x - 5 = 0$ between 1 and 2 correct to three decimal places using Newton Raphson method?

Solution: Given equation is $2x^3 - 2x - 5 = 0$

Let $f(x) = 2x^3 - 2x - 5$

So $f'(x) = 6x - 2$

Now, $f(0) = 2(0)^3 - 2(0) - 5 = -5$

$f(1) = 2(1)^3 - 2(1) - 5 = 2 - 2 - 5 = -5$

$f(2) = 2(2)^3 - 2(2) - 5 = 16 - 4 - 5 = 7$

So, root of given equation lies between 1 and 2.

Also, $f(1)$ is nearer to zero than $f(2)$ so take initial approximation

= 1.1904

Iteration 5: Fifth iteration or approximation to the root is given by

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = x_4 = \frac{2(x_4)^3 - 2(x_4) - 1}{6(x_4) - 2}$$

$$= 1.1904 - \frac{2(1.1904)^3 - 2(1.1904) - 1}{3(1.1904) - 2}$$

$$= 1.1904 - \frac{(-0.0071)}{5.1424}$$

= 1.1917

Iteration 6: Sixth approximation to the root is given by

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = x_5 - \frac{2(x_5)^3 - 2(x_5) - 1}{6(x_5) - 2}$$

$$= 1.1917 - \frac{2(1.1917)^3 - 2(1.1917) - 1}{6(1.1917) - 2}$$

$$= 1.1917 - \frac{0.00138}{5.1502}$$

= 1.1914

From 5th and 6th iteration, it is clear that there is no change in the successive approximation upto three decimal places So, a root of given equation is given by 1.191

Example 3: Find a real root of equation 3x = sin x + 1 between 0 and 0.5 correct to three decimal places using Newton-Raphson method?

Solution: Given equation is 3x = sin x + 1

Or 3x - sin x - 1 = 0

Let $f(x) = 3x - \sin x - 1$, $f'(x) = 3 - \cos x$

Now, $f(0) = 3(0) - \sin(0) - 1 = -1$

$f(0.5) = 3(0.5) - 1 = 0.02057$

So, a real root of equation lies between 0 and 0.5.

Also, $f(0.5)$ is nearer to zero than $f(0)$ so take the initial approximation x_0 to the root as 0.5

Iteration 1: The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{3x_0 - \sin x_0 - 1}{3 - \cos x_0}$$

$$= 0.5 - \frac{3(0.5) - \sin(0.5) - 1}{3 - \cos(0.5)}$$

$$= 0.5 - \frac{0.02057}{2.1224}$$

= 0.4903

Iteration 2: The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{3x_1 - \sin x_1 - 1}{3 - \cos x_1}$$

$$= 0.4903 - \frac{3(0.4903) - \sin(0.4903) - 1}{3 - \cos(0.4903)}$$

$$= 0.4903 - \frac{0.00027}{2.1178}$$

= 0.4901

From 1st and 2nd iteration, it is clear that there is no change in the successive approximation upto three decimal places. So, a real root of given equation is given by 0.490

Example 4: Find the real root of x $\log_{10}$ x = 1.2 correct upto to 3 decimal places?

Solution: Given equation is x log10 x = 1.2

$\Rightarrow$ x $\log_{10}$ x - 1.2 = 0

Let f(x) = x $\log_{10}$ x - 1.2

Now, f(1) = 1 $\log_{10}$ (1) - 1.2 = -1.2

f(2) = 2 $\log_{10}$ (2) - 1.2 = -0.59

f(3) = 3 $\log_{10}$ (3) - 1.2 = 0.23

So, a real root of equation lies between 2 and 3

Also, f(3) is nearest to zero than than f(2) so take the initial approximation x_0 to the root as 3.

Here $f'(x) = x.\frac{1}{x}\log_{10}^{e} + \log_{10}^{x}.1 = 0.434 + \log\ \log_{10}^{x}$

Iteration 1: The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 - \log_{10}^{x_0} - 1.2}{0.434 + \log_{10}^{x_0}}$$

$$= 3 - \frac{3\log_{10}^{3} - 1.2}{0.434 + \log_{10}^{3}}$$

$$= 3 - \frac{0.23136}{0.9111}$$

= 2.74606

Iteration 2: The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1 - \log_{10}^{x_1} - 1.2}{0.434 + \log_{10}^{x_1}}$$

$$= 2.74606 - \frac{2.74606\log_{10}(2.746061) - 1.2}{0.434 + \log_{10}(2.746061)}$$

$$= 2.74606 - \frac{0.0047}{0.8727}$$

= 2.74067

Iteration 3: The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2 - \log_{10}^{x_2} - 1.2}{0.434 + \log_{10}^{x_2}}$$

$$= 2.74067 - \frac{2.74067\log_{10}(2.74067) - 1.2}{0.434 + \log_{10}(2.74067)}$$

$$= 2.74067 - \frac{(-0.00027)}{0.87185}$$

= 2.74097

From 2nd and 3rd iteration, it is clear that there is no change in the successive approximation upto three decimal places. So, a real root of given equation is given by 2.740

Example 5: Find a root of cos x = xe^x by Newton Raphson method correct to 3 decimal?

Solution: Rewriting the given equation as

$xe^x - \cos x = 0$

Let $f(x) = xe^x - \cos x$

So $f'(x) = xe^x + 1.e^x + \sin x = e^x (x + 1) + \sin x$

Now $f(0) = 0\, e^0 - \cos 0 = -1 < 0$

and $f(1) = 1.e^1 - \cos 1 = 2.178 > 0$

So, a real root of given equation lies between 0 and 1.

Also, $f(0)$ is nearer to zero than $f'(1)$. So, take initial approximation x0 to the root as 0.

Iteration 1: The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 - e^{x_0} - \cos x_0}{e^{x_0}(x_0+1) + \sin x_0}$$

$$= 0 - \frac{0e^0 - \cos 0}{e^0(0+1) + \sin 0} = 0$$

Iteration 2: The second approximation to the root is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1 e^{x_1} - \cos x_1}{e^{x_1}(x_1+1) + \sin x_1}$$

$$= 1 - \frac{1.e^1 - \cos 1}{e^1(1+1) + \sin 1} = 0.6531$$

Iteration 3: The third approximation to the root is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2 e^{x_2} - \cos x_2}{e^{x_2}(x_2+1) + \sin x_2}$$

$$= 0.6531 - \frac{(0.6531)e^{0.6531} - \cos 0.6531}{e^{0.6531}(0.6531+1) + \sin 0.6531}$$

$$= 0.5313$$

Iteration 4: The fourth approximation to the root is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{x_3 e^{x_3} - \cos x_3}{e^{x_3}(x_3+1) + \sin x_3}$$

$$= 0.5313 - \frac{\left[0.5313e^{0.5313} - \cos 0.5313\right]}{e^{0.5313}(0.5313+1) + \sin(0.5313)}$$

$$= 0.5179$$

Iteration 5: The fifth iteration to the root is given by

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = x_4 - \frac{x_4 e^{x_4} - \cos x_4}{e^{x_4}(x_4+1) + \sin x_4}$$

$$= 0.5179 - \frac{\left[0.5179\, e^{0.5179} - \cos 0.5179\right]}{e^{0.5179}(0.5179+1) + \sin\left(0.5179\right)}$$

= 0.5178

From 4^{th} and 5^{th} iteration, it is clear that there is no change in successive approximation to the root upto 1^{st} three decimal places. So a real root of given equation is given by 0.517

Self Check Exercise

Q.1 Find a root of equation $x^4 + x^3 - 7x^2 - x + 5 = 0$ correct to three decimal places which lies between 2 and 3, using Newton's method.

Q.2 Use Newton-Raphson method to find a root of equation x sin x + cos x = 0

Q.3 Find a root of equation $e^x - x^3 = 0$ correct to four significant digits using Newton-Raphson method.

5.4 Summary

1. Newton Raphson method is an efficient technique to solve the equations numerically. It gives us better approximation in terms of solution.

2. The iterative formula for Newton Raphsonmethos is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

3. The Newton Raphson method is not always convergent. That means it cannot always guarantee that the condition is satisfied. However, this method fails when $f'(x)$ is equal to 0.

5.5 Glossary

- **Real Valued Function:** A real valued function is a function f:S→IR whose codomain is the set of real number R.
- **Convergence:** It is property (exhibited by certain infinite series and function) of approaching a limit more and more closely.
- **Tangent Line:** Tangent line is the line that touches a curve at a single point and does not cross through it.
- **Root:** Root is a solution to an equation.

5.6 Answers to Self Check Exercise

Ans. 2.060

Ans. 2.7984

Ans. 1.857

5.7 References/Suggested Readings

1. Numerical Analysis by Richard L. Burden
2. Introductory Methods of Numerical Analysis by S.s.Sastry.
3. Finite Difference & Numerical Analysis by H.C. Saxena.
4. An Introduction to Numerical Analysis by Kendall E. Atkinson.

5.8 Terminal Questions

1. Consider the function $f(x) = \cos x - x = 0$, approximate a root of f using Newton Raphson method?
2. Find a root of $\sin x = xe^x$ by Newton Raphson Method correct to 3 decimal places.
3. Find a real root of the equation $\log x - 2x + 4 = 0$ correct to 4 decimal places using Newton's method.

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Unit - 6

The Secant Method

Structure

6.1 Introduction

Dear student,in this unit we will learn secant method of solving non linear equation. This is again an approximate method. This method is similar to Regula-Falsi method or method of false position. In method of false position me find the interval in which root of a given equation lie in each step. But in secant method we did not limit the root between an Interval. Here in this method each approximated root is used in next step or approximation. In this unit we will learn secant method, its algorithm and its order of convergence. We will also learn to apply this method to find the approximated root of an non linear equation.

6.2 Learning Objective:

After studying this unit students will be able to

1. define secant method of non-linear equation
2. give algorithm of secant method
3. give order of convergence of secant method
4. apply secant method on non-linear equation to find approximate root
5. solve non linear equation by using secant method

6.3 Secant Method

Another recursive technique for approximating successively the root of the polynomials is the secant method. It is comparable to the RegulaFalsi technique or method, except that after each approximation, we don't have to repeatedly verify that $f(x_1)\ f(x_2) < 0$. This method uses a secant line or chord to the function $f(x)$ to approximate the roots of the neighborhoods. Another benefit of this method is that, unlike the Newton Raphson method, it eliminate the requirement to differentiate the provided function $f(x)$.

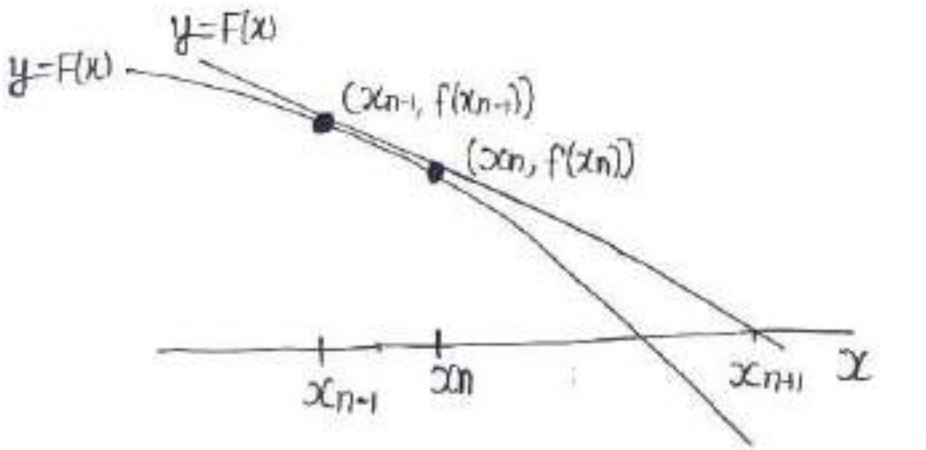
Assume that the function $f(x)$ is continuous. The secant method does not require the use of the derivative of $f(x)$ which can be very useful when handling the derivative difficulty. It approximate $f'(x)$ to avoid using the derivative by $\frac{f(x+h)-f(x)}{h}$ for some h.

In other words, it uses a secant line for f that passes through x to approximate the tangent line to f at x.

In order to reduce the total number of $f(x)$ evaluations needed, it makes use of $x = x_{n-1}$ and $x + h = x_n$.

Functioning of Secant Method

Suppose that we have already found x_n. Next, we denote the equation of secant line that crosses $(x_{n-1}, f(x_{n-1}))$ and $(x_n f(x_n))$ as y = F(x) and we select x_{n+1} as the value of x where F(x) = 0



The equation of the secant line is

$$y = F(x) = f(x_{n-1}) + \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1})$$

So that x_{n+1} is determined by

$$0 = F(x_{n+1}) = f(x_{n-1}) + \frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}(x_{n+1} - x_{n-1})$$

$$\Rightarrow \qquad x_{n+1} = x_{n-1} - \frac{x_n - x_{n-1}}{f(x_n)-f(x_{n-1})} f(x_{n-1})$$

or, Simplifying $x_{n+1} = \dfrac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$

Steps in Secant Method

Step (I) Initialization:- x_0, x_1 of $\propto$ are taken as initial guesses.

Step (II) Iteration:- In the case of n = 1, 2, 3, ... soon, until a specific criterion for termination is satisfied (i.e. the maximum number of iteration has been reached or the desired accuracy of solutions has been attained)

Order of Convergence

The order of convergence of secant method is 1.618 referred to as superliner. (Same as method of false position)

Advantages:

- The secant method converges more quickly than the bisection and regulafalsi method
- It uses two most recent approximations of root to find new approximations, instead of using only those approximations which bounds the interval to enclose root.

Disadvantages:

- In the secant method, convergence is not always assured.
- When using this approach on a computes, we should set a maximum restriction on the number of repetitions because convergence is not guaranteed.

Some Examples: Let us try some examples to have more understanding of secant method.

Example 1: Compute the root of the equation $x^2 e^{\frac{-x}{2}}$ = 1 in the interval [0, 2] using the secant method. The root should be correct to three decimal places. The initial values are 1.42 and 1.43.

Solution: Given equation $x^2 e^{\frac{-x}{2}}$ = 1 or $x^2 e^{\frac{-x}{2}}$ - 1 = 0

Let $f(x) = x^2 e^{\frac{-x}{2}} - 1$

Here a = 1.42, b = 1.43

$\Rightarrow$ $f(a) = -0.0086$, $f(b) = 0.00034$

Iteration 1: The first approximation to the root is given by

$$x_1 = \frac{a\, f(b) - b\, f(a)}{f(b) - f(a)}$$

$$= \frac{1.42(0.00034) - 1.43(-0.0086)}{0.00034 - (-0.0086)}$$

$= 1.4296$

Iteration 2: The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

Taking a = 1.4296, b = 1.43

So that $f(a) = -0.000011$, $f(b) = 0.00034$

$$\text{Thus, } x_2 = \frac{1.4296(0.00034) - 1.43(-0.000011)}{0.00034 - (-0.000011)}$$

$= 1.4293$

From 1st and 2nd iteration, we see that there is no change in the successive approximation to the root upto three decimal places.

So, the root of equation is given by x = 1.429

Example 2: Find real root of the equation $x^3 - 5x + 1 = 0$ between 0 and 1 correct to three decimal places using the secant method.

Solution: The given equation is $x^3 - 5x + 1 = 0$

Let $f(x) = x^3 - 5x + 1 = 0$

Here a = 0, b = 1

So that $f(a) = 1$, $f(b) = -3$

Iteration 1: The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{0(-3) - 1(1)}{-3 - 1} = 0.25$$

Iteration 2: Taking a = 0.25, b = 1 so that $f(a) = -0.2343$ $f(b) = -3$

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{0.25(-3) - 1(-0.2343)}{-3 - (-0.2343)}$$

$= 0.1864$

Iteration 3: Taking a = 0.1864, b = 0.25 So that $f(b) = -0.2343$

$f(a) = (0.1864)^3 - 5(0.1864) + 1 = 0.0744$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{0.1864(-0.2343) - 0.25(-0.0744)}{-0.2343 - 0.0744}$$

= 0.2014

Iteration 4: Taking a = 0.2041, b = 0.25 So that f(b) = - 0.2343

$f(a) = (0.2014)^3 - 5(0.2014) + 1 = 0.0011$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{0.2014(-0.2343) - 0.25(-0.0011)}{-0.2343 - 0.0011}$$

= 0.2013

From 3rd and 4th iteration, we see that there is no change in the successive approximation to the root upto three decimal places.

So, the root of equation is given by 0.201

Example 3: Use secant method to find an approximate value of $3\sqrt{48}$.

Solution: Let x = $3\sqrt{48}$

∴ $x^3 = 48$

⇒ $x^3 - 48 = 0$

Let $f(x) = x^3 - 48$

Now, $f(0) = (0)^3 - 48 = -48$

$f(1) = (1)^3 - 48 = -47$

$f(2) = (2)^3 - 48 = -40$

$f(3) = (3)^3 - 48 = -21$ (-ve)

$f(4) = (4)^3 - 48 = 16$ (+ve)

As per algorithm, we have to find a and b such that $f(a) < 0$ and $f(b) > 0$. So, a root of the given equation lies between a and b.

Here a = 3, b = 4

So that $f(a) = -21$, $f(b) = 16$

Iteration 1: The first approximate to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$x_1 = \frac{3(16) - 4(-21)}{16 - (-21)}$$

= 3.5675

Iteration 2: Taking a = 3.5675, b = 4

So that f(b) = 16 f(a) = $(3.5675)^3$ - 48 = - 2.5962

The second (iteration) approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{3.5675(16) - 4(-2.5962)}{16 - (-2.5962)}$$

= 3.6278

Iteration 3: Taking a = 3.5675, b = 3.6278 so that f(a) = -2.5962

f(b) = $(3.6278)^3$ - 48 = - 0.2547

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{3.5675(-0.2547) - 3.6278(-2.5962)}{-0.2547 - (-2.5962)}$$

= 3.6342

Iteration 4: Taking a = 3.6278, b = 3.6343 so that f(a) = -0.2547

f(b) = $(3.6343)^3$ - 48 = 0.0023

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{3.6278(-0.0023) - 3.6343(-0.2547)}{-0.0023 - (-0.2547)}$$

= 3.6342

∴ f(x4) = $(3.6342)^3$ - 48 = 0

Hence $3\sqrt{48}$ = 3.6342

Example 4: Use secant method to find a real root of equation 2 cos x - x = 0

Solution: Given equation is 2 cos x - x = 0

Let f(x) = 2 cos x - x

Now, f(0) = 2 cos (0) - 0 = 0

f(1) = cos (1) - 1 = 0.0806

f(2) = 2 cos (2) - 2 = -2.8322

So, a real root of equation lies between 1 and 2

Iteration 1: Taking a = 1, b = 2 so that f(1) = 0.0806, f(2) = - 2.8322

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{1(-2.8322) - 2(-0.0806)}{-2.8322 - 0.0806}$$

= 1.0276

Iteration 2: Taking a = 1, b = 1.0276 so that f(a) = 0.0806,

f(b) = 2 cos (1.0276) - 1.0276 = 0.0061

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{1(-0.0061) - 1.0276(-0.0806)}{0.0061 - 0.0806}$$

= 1.0295

Iteration 3: Taking a = 1.0276, b = 1.0295 so that f(a) = 0.0061

f(b) = 2 cos (1.0295) - 1.0295 = 0.0009

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$x_3 = \frac{1.0276(-0.0009) - 1.0295(-0.0061)}{0.0009 - 0.0061}$$

= 1.0292

We observe that in 2nd and 3rd iteration successive approximation o the root are approximately same

So, a real root of given equation is given by x = 1.0292

Example 5: Use secant method to find the real root of equation $x^3 - 2x - 5 = 0$

Solution: The given equation is $x^3 - 2x - 5 = 0$

Let $f(x) = x^3 - 2x - 5$

Now, $f(0) = (0)^3 - 2(0) - 5 = -5$

$f(1) = (1)^3 - 2(1) - 5 = -7$

$f(2) = (2)^3 - 2(2) - 5 = -1$

and $f(3) = (3)^3 - 2(3) - 5 = 16$

So, a real root of equation lies between 1 and 2

Iteration 1: Taking a = 2, b = 3 so that $f(a) = -1$ $f(b) = 16$

The first approximation to the root is given by

$$x_1 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{2(16) - 3(-1)}{16 + 1}$$

= 2.0588

Iteration 2: Taking a = 2, b = 2.0588 so that $f(a) = -1$,

$f(b) = (2.05881)^3 - 2\,(2.0588) - 5$

= -0.3910

The second approximation to the root is given by

$$x_2 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{2(-0.3910) - 2.0588(-1)}{0.3910 - (-1)}$$

= 2.0965

Iteration 3: Taking a = 2.0588, b = 2.0965 so that $f(a) = -0.3910$

$f(b) = (2.0965)^3 - 2\,(2.0965) - 5 = 0.02177 = 0.0218$

The third approximation to the root is given by

$$x_3 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{2.0588(0.0218) - 2.0965(-0.3910)}{0.0218 - (-0.3910)}$$

= 2.09447

= 2.0945

Iteration 4: Taking a = 2.0945, b = 2.0965 so that f(b) = 0.0218

$f(a) = (2.0945)^3 - 2\,(2.0945) - 5 = -0.00057 = 0.0006$

The fourth approximation to the root is given by

$$x_4 = \frac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$$

$$= \frac{2.0945(0.0218) - 2.0965(-0.0006)}{0.0218 - (-0.0006)}$$

= 2.0947

From 3rd and 4th iteration, we see that there is no change in the successive approximation to the root upto three decimal places

So, the real root of the equation is 2.094

Self Check Exercise

Q.1 Find a root of an equation f(x) = x3 + 2x2 + x - 1 using secant method.

Q.2 Use secant method to find the approximate value of $\sqrt{12}$.

Q.3 Use second method to find a real root of equation cos x - xe^x = 0

6.4 Summary

1. Secant method is also known as 2-point method
2. The secant method is a root finding procedure in numerical analysis that uses a series of root of secant lines to better approximate a function's root.
3. The formula used for secant method is

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

4. The order of convergence of Secant Method is 1.618 referred to as superliner.

6.5 Glossary:

- **Function:** Function is a rule that defines a relationship between one variable (the independent variable) and another variable (the defenders variable).

- **Tangent Line:** Tangent line is the line that touches a curve at a single point and does not cross through it.
- **Bounds:**
  - (a) **Lower bound:** A value that is less than or equal to every element of a set of data
  - (b) **Upper bound:** A value that is greater than or equal to every element of a set of data
- **Approximation:** Anything that is similar, but not exactly equal to something else.

6.6 Answers to Self Check Exercise

Ans. 0.4655

Ans. 3.4641

Ans. 0.517

6.7 References/Suggested Readings

1. Numerical Analysis by Richard L. Burden
2. Introductory Methods of Numerical Analysis by S.S. Sastry.
3. Finite difference & Numerical Analysis by H.C. Saxena.

6.8 Terminal Exercise

1. Find a root of an equation $f(x) = x^3 - x - 1$ using secant method.
2. Find a real root of the equation $e^x - x^3 = 0$ using secant method.
3. Use secant method to find a real root of equation $\cos x = 0$.

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Unit - 7

Lu Decomposition or Factorization Method or Triangularization Method

Structure

7.1 Introduction

Dear student, in unit 2 to unit 6. We learn how to solve a non Linear equation using different iterative method. Now, in this unit we will learn to solve a system having three equation in three variable. When linear equations are written together for solution then we know them as system of equation. In this unit we will learn to solve system by using LU decomposition method. We will learn the basic terminology used and algorithm of this method. Also we will learn how to find the solution of a system of linear equations by using LU decomposition method.

7.2 Learning Objectives:

After studying this unit students will be able to

1. define system of linear equation
2. define solution of system of linear equation
3. define direct and iterative method to solve system of linear equation
4. define LU decomposition method of system of linear equation
5. find solution of system of linear equation by LU decomposition method.

7.3 LU Decomposition Method

System of Linear Equations:

A system of linear equations is a collection of two or more linear equations involving the same variables. For example

$$\left.\begin{aligned} 3x+2y-z&=1\\ 2x-2y+4z&=-2\\ -x+\frac{1}{2}y-z&=0 \end{aligned}\right\}$$

is a system of linear equation having three equations in three variables x, y and z.

A general linear system of n equations in n variables can be written as

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$

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$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$.

There are n equations, in n variables, that are x_1, x_2 .... x_n. so represents a system of linear equations in n variables. In matrix form this system can be represented as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ - \\ - \\ b_n \end{bmatrix}$$

$\Rightarrow$ AX = B

Where A = $[a_{ij}]$ is the coefficient matrix

X = $\begin{bmatrix} x_1 \\ x_2 \\ - \\ - \\ x_n \end{bmatrix}$ is the column vector, which gives the variables

and B = $\begin{bmatrix} b_1 \\ b_2 \\ - \\ - \\ b_n \end{bmatrix}$ is the column vector, known as constant vector.

Homogeneous and Nonhomogeneous System

If the values of all b_i i.e. $b_1 = b_2 =$ .... $b_n = 0$ then the system is known as homogeneous system.

and

If the values of all b_i i.e. $b_1 = b_2 =$ .... $b_n \neq 0$ then the system is known as non homogenous system.

Solution of System of Linear Equations

The solution set of the system of linear equations is the set of all possible values of the variables that satisfies the given linear equations i.e. the values of x_1, x_2 .... x_n.

Unique Solution: A system of n linear questions in n variables given by AX = B has a unique solution if $|A| \neq 0$.

Zero or Trivial Solution: A system has a zero solution or trivial solution is B = 0.

Infinite Solution: A system has infinite many solution is |A| = 0.

Direct and Iterative Methods

There are number of methods to save systems of linear equations which are known as direct methods and iterative methods. In this unit we will learn the direct method of solving system of linear equation known as LU Decomposition method.

Iterative methods are the approximation method used to solve system of linear equations.

LU Decomposition or Factorization Method or Triangularization Method

LU Decomposition a square matrix is factorized into two triangular matrices, an upper matrix and a lower matrix, so that the product of these two matrices yields the original matrix. In 1948, Alan Turing - who also invented the Turing machine introduced it.

Finding the inverse of a matrix and the determinant of the matrix are two uses for the LU decomposition method, which factors a matrix as the product of two triangular matrixes. It can also be used to solve a system of equations, which is a necessary step in many other applications, including finding the current in a circuit and solving discrete dynamical system problems.

LU Decomposition Method

The following procedures can be used to divide any square matrix into two triangular matrices one of which is an upper triangular matrix and other is lower triangular matrix.

1. First take a set of linear equations and turn them into a matrix form : AX = B where B is the matrix of integers on the right side of the equations, X is the matrix containing variables and A is the coefficient matrix.

2. Now reduce the coefficient matrix A, which is a matrix derived from the coefficient of variables in each of the given equations, to row echelon form of gauss Elimination method for n variables. The matrix so obtained is U.

3. We can use two methods to find L. The first one is to create equations with the formula A=LU and solve them to get the artificial variables that is the remaining elements. The other method uses the remaining elements as multiplier coefficient, because of which the respective positions become zero in the U matrix.

4. Now, we have X (the $n \times 1$ matrix of variables), L(the $n \times n$ lower triangular matrix, U (the $n \times n$ upper triangular matrix) and B (the $n \times 1$ matrix of integers on the right hand side of equations and A is the $n \times n$ coefficient matrix.

5. The given system of equations is AX = B. We change it to A = LU. As a result, LUX = B. In order to find X, or the values of the variables, as needed, we first solve for LZ = B and then solve for UX = Z, where Z is a matrix or artificial variable.

Thus, by above steps if we AX = B given linear system of equation i.e.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

then by applying Gauss Elimination method on A

L is lower triangular matrix with diagonal elements being equal to 1 i.e.

$$L = \begin{bmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix}$$

and U is upper triangular matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

Let us try this method to find solution of system of linear equations.

Some Examples

Example 1: Solve the system of equations

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 - 3x_3 = 5$

$x_1 - 2x_2 - 5x_3 = 10$

by LU decompositions method.

Solution: Given system of equation are

$x_1 + x_2 + x_3 = 1$

$3x_1 + x_2 - 3x_3 = 5$

$x_1 - 2x_2 - 5x_3 = 10$

These equations are written in the form of AX = B are:

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

Step 1: Let us write the above matrix as LU = A

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

By expanding the left side matrices, we get

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{22} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23}u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -3 \\ 1 & -2 & -5 \end{bmatrix}$$

Thus, by equating the corresponding elements, we get

$u_{11} = 1$, $u_{12} = 1$, $u_{13} = 1$

$l_{21}\, u_{11} = 3$

$l_{21}\, u_{13} + U_{23} = -3$

$l_{31}\, u_{11} = 1$

$l_{31}\, u_{12} + l_{32}\, u_{22} = -2$

$l_{31}\, u_{13} + l_{32}\, u_{23} + u_{33} = -5$

Solving these equations we get

$u_{22} = -2$, $u_{23} = -6$ $u_{33} = 3$

$l_{21} = 3$, $l_{31} = 1$, $l_{32} = \frac{3}{2}$

Step (2): LUX = B

Step (3): Let UX = Y

Step (4): From the previous two steps, we have LY = B

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 3/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \end{bmatrix}$$

So, $y_1 = 1$

$3y_1 + y_2 = 5$

$y_1 + \left(3/2\right) y_2 + y_3 = 10$

Solving these equations, we get

$y_1 = 1,\ y_2 = 2,\ y_3 = 6$

Step (5): Now consider UX = Y. So,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

By expanding these equations we get

$x_1 + x_2 + x_3 = 1$

$-2x_1 - 6x_3 = 2$

$3x_3 = 6$

Solving these equations, we can get

$x_3 = 2$, $x_2 = -7$ and $x_1 = 6$

Therefore, solution of the given system of equation is (6, -7, 2)

Example 2: Solving the following system of equations using the LU Decomposition method.

$x_1 + x_2 + x_3 = 1$

$4x_1 + 3x_3 - x_3 = 6$

$3x_1 + 5x_2 + 3x_3 = 4$

Solution: Here we have A = $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$, X = $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and C = $\begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

Such that AX = C. Now, we 1st consider

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$$

and convert it to row echelon form using Gauss Elimination method. We get

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

UY convert it to column form

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & -5 \\ 3 & 2 & 0 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix} \backsim \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & -10 \end{bmatrix} = 2$$

$\Rightarrow$ $l_{21} = 4$, $l_{31} = 3$, $l_{32} = -2$

Let us assume $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Now, LZ = C

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

So, we have

$z_1 = 1$

$4z_1 + z_2 = 6$

$3z_1 - 2z_2 + z_3 = 4$

Solving we get

$z_1 = 1$, $z_2 = 2$ and $z_3 = 5$

Now, we solve

UX = Z

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Therefore, we get

$x_1 + x_2 + x_3 = 1$

$-x_2 - 5x_3 = 2$

$-10x_3 = 5$

Thus, the solution to the given system of linear equation is

$x_1 = 1$

$x_2 = 0.5$

$x_3 = -0.5$

Hence the matrix x = $\begin{bmatrix} 1 \\ 0.5 \\ -0.5 \end{bmatrix}$

Example 3: Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution: [A] = [L] [U]

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The [U] matrix is the same as found at the end of the forward elimination of Gauss elimination method

$$[U] = \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -15.6 \\ 0 & 0 & 0.7 \end{bmatrix}$$

To find l_{21} and l_{31}, find the multiplier that was used to make a_{21} and a_{31} elements zero in the first step of forward elimination of Gauss elimination method it was

$$l_{21} = \frac{64}{25} = 5.76$$

To find l_{32}, what multiplier was used to make a_{32} element zero? Remember a_{32} element was made zero in the second step of forward elimination. The [A] matrix at the beginning of the second step of forward elimination was

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -15.6 \\ 0 & -16.8 & -4.76 \end{bmatrix}$$

So, $l32 = \dfrac{-16.8}{-4.8} = 3.5$

Hence $L = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}$

Confirm [L] [U] = [A]

$$[L]\,[U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -15.6 \\ 0 & 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Example 4: Use LU decomposition to find the inverse of

$$[A] = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution: We know

[A] = [L] [U]

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

We can solve for the first column of $[B] = [A]^{-1}$ by solving for

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

First Solve [L] [Z] = [C]

that is $\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$z_1 = 1$

$2.56\ z_1 + z_2 = 0$

$5.76\ z_1 + 3.5z_2 + z_3 = 0$

Forward substitution starting from 1st equation gives

$z_1 = 1$

$z_2 = 0 - 2.56\ z_1 = -2.56$

$z_3 = 0 - 5.76\ z_1 - 3.5\ z_2 = 3.2$

Hence $[Z] = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$

Now, solve [U] [X] = [Z]

that is $\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -2.56 \\ 3.2 \end{bmatrix}$

$25b_{11} + 5b_{21} + b_{31} = 1$

$-4.8b_{21} - 1.56b_{31} = -2.56$

$0.7b_{31} = 3.2$

Backward substitutions starting from the third equation give

$b_{31} = \frac{3.2}{0.7} = 4.571$

$b_{21} = \frac{-2.56 + 1.56b_{31}}{25} = \frac{-2.56 + 1.56(4.571)}{-4.8}$

$= 0.9524$

$b_{11} = \frac{1 - 5b_{21} - b_{31}}{25} = \frac{1 - 5(-0.9524) - 4.571}{25}$

$= 0.04762$

Hence, the 1st column of inverse of [A] is

$$\begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0.04762 \\ -0.9524 \\ 4.571 \end{bmatrix}$$

Similarly solving

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & - \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

gives $\begin{bmatrix} b_{12} \\ b_{22} \\ b_{33} \end{bmatrix} = \begin{bmatrix} -0.08333 \\ 1.417 \\ -5.000 \end{bmatrix}$

and solving $\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & - \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

gives $\begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 0.03571 \\ -0.4643 \\ 1.429 \end{bmatrix}$

$$[A]^{-1} = \begin{bmatrix} 0.04762 & -0.08333 & 0.03571 \\ -0.9524 & 1.417 & -0.4643 \\ 4.571 & -5.000 & 1.426 \end{bmatrix}$$

Self Check Exercise

Q.1 Use the LU decomposition method to solve the following system of linear equations.

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Q.2 Use LU decomposition to find [L] and [U]

$4x_1 + x_2 - x_3 = -2$

$5x_1 + x_2 + 2x_3 = 4$

$6x_1 + x_2 + x_3 = 6$

Q.3 The lower triangular matrix [L] in the [L] [U] decomposition of the matrix given below

$$\begin{bmatrix} 25 & 5 & 4 \\ 10 & 8 & 16 \\ 8 & 12 & 22 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

7.4 Summary

LU Decomposition expresses a given square matrix A as the product of two matrices:

(i) L : A lower triangular matrix with ones on the diagonal

(ii) U : An upper triangular matrix

LU Decomposition provides on efficient means to compute the matrix inverse. The inverse has a number of valuable application in engineering practice. It also provides a means for evaluating system conditions.

It is a better way to implement Gauss Elimination method, especially for repeated solving a number of equations

The general formula for LU decomposition is PA = LU where L is an $n \times n$ lower triangular matrix with all diagonal enteries equal to 1. U is an $n \times n$ upper triangular matrix. P is an $n \times n$ permutation matrix.

7.5 Glossary

- **Decomposition:** Decomposition simply means 'to break down' in parts it corresponds to disintegration or rupture of complex things into simpler one.
- **Upper Triangular Matrix:** A square matrix whose all elements below main diagonal are zero.
- **Lower Triangular Matrix:** A square matrix whose all elements above the main diagonal are zero.
- **System of Equations:** Two or more equations to be solved together considered to be system of equations
- **Gauss Elimination Method:** Gauss Elimination Method is a row reduction algorithm for solving linear equations system. It consists of a sequence of operations performed on the corresponding matrix of coefficients.

7.6 Answers to Self Check Exercise

Ans. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.29048 \\ 19.691 \\ 1.0857 \end{bmatrix}$

Ans. L = $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, U = $\begin{bmatrix} 1 & 4 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$

Ans. $\begin{bmatrix} 1 & 0 & 0 \\ 0.4000 & 1 & 0 \\ 0.3200 & 1.7333 & 1 \end{bmatrix}$

7.7 References/Suggested Readings

1. Introductory Methods of Numerical Analysis by S.S. Sastry
2. An Introduction to Numerical Analysis by Kendall Atkinson
3. Numerical Analysis by J. Douglas Eaires
4. Finite difference and Numerical Analysis by H.C. Saxena

7.8 Terminal Questions

1. Solve the following equations by LU decomposition method

   $6x_1 + 18x_2 + 3x_3 = 3$

   $2x_1 + 12x_2 + x_3 = 19$

   $4x_1 + 15_2 + 3x_3 = 0$

2. Solve the below given system of equations by LU decomposition

   x + y + z = 1

   4x + 3y - z = 6

   3x + 5y + 3z = 4

3. Find the solution of the system of equations by LU decomposition

   x + 2y + 3z = 9

   4x + 5y + 6z = 24

   3x + y - 2z = 4

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Unit - 8

Gauss Jacobi Method

Structure

8.1 Introduction

Dear student in this unit we will learn to solve system of linear equations, using iterative method or by approximation. In this unit we learn to solve system of Linear equations by Gauss-Jacobi Method. By solution of a system of linear equation, we have to find the value of the variables x_1, x_2 .... x_n which satisfies the given equations simultaneously. In this unit we learn about Gauss-Jacobi method and which is an iterative or approximation method. In this method the approximate solution find in 1^{st} iteration is used as a whole in second approximation. We will learn Gauss-Jacobi Method to solve system of linear equations.

8.2 Learning Objectives:

Dear students, after studying this unit students will be able to

1. define system of linear equations
2. define iterative method to solve system of linear equation
3. define Gauss-Jacobi Method
4. give algorithm of Gauss-Jacobi Method
5. apply Gauss-Jacobi Method to system of linear equations
6. find solution of system of linear equation by Gauss-Jacobi Method.

8.3 Gauss Jacobi Method

Jacobi method or jacobian method is named after German mathematician corl Gustav Jacob Jacobi (1804-1851). The main idea behind this method is

For a system of linear equations

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$ .... (1)

$a_{21}x_1 + a22x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$ .... (2)

$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots\ldots\ldots + a_{nn}x_n = b_n$ .... (n)

In order to solve the system of equations we assume that the system of equations has a unique solution and that neither the diagonal nor the pivot elements of the coefficient matrix. A contains a zero element

Now, we shall begin to solve equation (1) for x_1, equation (2) for x_2 and so on equation n for x_n, we get

$$x_1 = \frac{1}{a_{11}}[b_1 - a_{12}\, x_2 - a_{13}\, x_3 \ldots\ldots\ldots\ldots\ldots\ldots - a_{1n}x_n]$$

$$x_2 = \frac{1}{a_{22}}[b_2 - a_{21}\, x_2 - a_{23}\, x_3 \ldots\ldots\ldots\ldots\ldots\ldots - a_{2n}x_n]$$

$$x_3 = \frac{1}{a_{33}}[b_3 - a_{31}\, x_1 - a_{32}\, x_2 \ldots\ldots\ldots\ldots\ldots\ldots - a_{3n}x_n]$$

$$x_n = \frac{1}{a_{nn}}[b_n - a_{n1}\, x_1 - a_{n3}\, x_3 \ldots\ldots\ldots\ldots\ldots\ldots - a_{nn-1}\, x_{n-1}]$$

By making an initial guess for solutions $x^{(0)} = \left(x_1^{(0)}, x_2^{(0)}, \ldots\ldots. x_n^{(0)}\right)$ and substitution these values to the right hand side of the above equations we get first approximations. $x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, \ldots\ldots. x_n^{(1)}\right)$. By repeating this method iteratively, we have a sequence of approximations $\left\{x^{(k)}\right\}$ such that as $k\rightarrow\infty$, this sequence converges to exact solution of the system of equation upto a given error tolerance.

Convergence of Approximation

The system of equation must be diagonally dominant is the sufficient condition for the convergence of the approximations obtained by Jacobi method. Here by system of equation we mean the coefficient matrix.

For any i ‡j, the matrix A is considered diagonally dominating if $|aii| > \sum_{j=1}^{n} |aij|$. This indicates that the diagonal elements absolute value is larger than or equal to the total of the corresponding row's elements

How to choose initial approximations?

We must make an initial approximation of the possible solution for the given system of equations in order to start the iterative process. An exact solution can be reached by any initial approximation if the system of equations is diagonally dominant. Even so, we can assume $x^{(0)}$ =

0 where $x_i^{(0)} \forall i$ if appropriate approximations are not available. For every i, the initial approximation thus becomes xi = $\frac{b_i}{a_{ii}}$.

Gauss Jacobi Method:

Let us try to solve system of linear equations by using Gauss-Jacobi method.

Example 1: Solve the following system of equations by Gauss Jacobi's method upto three iteration.

10x - 5y - 2z = 3

4x - 10y + 3z = -3

x + 6y + 10z = -3

Solution: The given System of equations are

10x - 5y - 2z = 3

4x - 10y + 3z = -3

x + 6y + 10z = -3

The given System of equations is a diagonal system so the convergence of Jacobi method is assured

Rewriting the given system of equation Thus, we have

$$x^{(k+1)} = \frac{1}{10}\left[3 + 5y^{(k)} + 2z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{10}\left[3 + 4x^{(k)} + 3z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{10}\left[-3 - k^{(k)} - 6y^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: Putting the initial values in right side of (1), we get

$$x^{(1)} = \frac{1}{10}\left[3 + 0 + 0\right] = 0.3$$

$$y^{(1)} = \frac{1}{10}\left[3 + 4(0) + 3(0)\right] = 0.3$$

$$z^{(1)} = \frac{1}{10}\left[-3 - 0 - 6(0)\right] = -0.3$$

Iteration 2: $x^{(2)} = \frac{1}{10}\left[3 + 5(0.3) + 2(0)\right] = 0.39$

$$y^{(2)} = \frac{1}{10}\left[3+4(0.3)+3(-0.3)\right] = 0.33$$

$$z^{(2)} = \frac{1}{10}\left[-3-0.3-b(0.3)\right] = -0.51$$

Iteration 3: $x^{(3)} = \frac{1}{10}\left[3+5(0.33)+2(-0.51)\right] = 0.363$

$$y^{(3)} = \frac{1}{10}\left[3+4(0.39)+3(-0.51)\right] = 0.303$$

$$z^{(3)} = \frac{1}{10}\left[-3-0.39-6(0.33)\right] = -0.537$$

Hence, the solution to the given system of equation upto three iteration is x = 0.363, y = 0.303, z = -0.537

Example 2: Given system of linear equations is

2x - 6y - z = -38

-3x - y + 7z = -34

-8x + y - 2z = -20 determine the values of x, y and z using Jacobi Iterative Method?

Solution: The coefficient matrix of given system is not diagonally dominant

Hence, we rearrange the equation as follows such that the elements in the coefficient matrix are diagonally dominant.

-8x + y - 2z = -20

2x - 6y - z = -38

-3x - y + 7z = -34

Rewriting the above equations Thus, we have

$$x^{(k+1)} = \frac{1}{8}\left[20+y^{(k)}-2z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{6}\left[38+2x^{(k)}-z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{7}\left[-34+3x^{(k)}+y^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: Putting the initial values in right side of (1) we get

$$x^{(1)} = \frac{1}{8}\left[20+0-2(0)\right] = \frac{20}{8} = 2.50$$

$$y^{(1)} = \frac{1}{6}\left[38 + 2(0) - 0\right] = \frac{38}{6} = 6.333 = 6.334$$

$$z^{(1)} = \frac{1}{7}\left[-34 + 3(0) + 0\right] = \frac{-34}{7} = -4.857$$

Iteration 2: $x^{(2)} = \frac{1}{8}\left[20 + 6.334 - 2(-4.857)\right] = 4.506$

$$y^{(2)} = \frac{1}{6}\left[38 + 2(2.5) - (-4.857)\right] = 7.9761$$

$$z^{(2)} = \frac{1}{7}\left[-34 + 3(2.5) + 6.334\right] = -2.8808$$

Iteration 3: $x^{(3)} = \frac{1}{8}\left[20 + 7.9761 - 2(-2.8808)\right] = 4.2172$

$$y^{(3)} = \frac{1}{6}\left[38 + 2(4.506) - (-2.8808)\right] = 8.7956$$

$$z^{(3)} = \frac{1}{7}\left[-34 + 3(4.506) + 7.9761\right] = -1.7865$$

Iteration 4: $x^{(4)} = \frac{1}{8}\left[20 + 8.7956 - 2(-1.7865)\right] = 4.04607$

$$y^{(4)} = \frac{1}{6}\left[38 + 2(4.2172) - (-1.7865)\right] = 8.0368$$

$$z^{(4)} = \frac{1}{7}\left[-34 + 3(4.2172) + 8.7956\right] = -1.7932$$

Iteration 5: $x^{(5)} = \frac{1}{8}\left[20 + 8.0368 - 2(-1.7932)\right] = 3.9529$

$$y^{(5)} = \frac{1}{6}\left[38 + 2(4.04607) - (-1.7932)\right] = 7.9808$$

$$z^{(5)} = \frac{1}{7}\left[-34 + 3(4.04607) + 8.0368\right] = -1.9749$$

Iteration 6: $x^{(6)} = \frac{1}{8}\left[20 + 7.9808 - 2(-1.9749)\right] = 3.9913$

$$y^{(6)} = \frac{1}{6}\left[38 + 2(3.9529) - (-1.9749)\right] = 7.9801$$

$$z^{(6)} = \frac{1}{7}\left[-34 + 3(3.9529) + 7.9808\right] = -2.0229$$

Iteration 7: $x^{(7)} = \frac{1}{8}\left[20 + 7.9801 - 2(-2.0229)\right] = 4.0032$

$$y^{(7)} = \frac{1}{6}\left[38 + 2(3.9913) - (-2.0229)\right] = 8.0009$$

$$z^{(7)} = \frac{1}{7}\left[-34 + 3(3.9913) + 7.9801\right] = -2.0065$$

Iteration 8: $x^{(8)} = \frac{1}{8}\left[20 + 8.0009 - 2(-2.0065)\right] = 4.0017$

$$y^{(8)} = \frac{1}{6}\left[38 + 2(4.0032) - (-2.0065)\right] = 8.0021$$

$$z^{(8)} = \frac{1}{7}\left[-34 + 3(4.0032) + 8.0009\right] = -1.9985$$

Iteration 9: $x^{(9)} = \frac{1}{8}\left[20 + 8.002 - 2(-1.9985)\right] = 3.9998$

$$y^{(9)} = \frac{1}{6}\left[38 + 2(4.0017) - (-1.9985)\right] = 8.0003$$

$$z^{(9)} = \frac{1}{7}\left[-34 + 3(4.0017) + 8.0021\right] = -1.9989$$

Iteration 10: $x^{(10)} = \frac{1}{8}\left[20 + 8.0003 - 2(-1.9989)\right] = 3.99917$

$$y^{(10)} = \frac{1}{6}\left[38 + 2(3.9998) - (-1.9989)\right] = 7.9997$$

$$z^{(10)} = \frac{1}{7}\left[-34 + 3(3.9998) + 8.0003\right] = -2.0000$$

We observe that in 9th and 10th iteration, the approximation to the unknowns are approximately same, So we stop the iterative procedure.

Hence, the solution of given system of equation is

x = 3.9997, y = 7.9997, z = -2.0000

Example 3: Use Jacobi iterative method to obtain the solution of system of equation

12x + 3y - 5z = 1

3x + 7y + 13z = 76

x + 5y + 3z = 28

With (x^0, y^0, z^0) = (1, 0, 1)

Solution: The coefficient matrix of given system is not diagonally dominant

Hence, we rearrange the equation as follows such that the elements us the coefficient matrix are diagonally dominant

12x + 3y - 5z = 1

x + 5y + 3z = 28

3x + 7y + 13z = 76

Rewriting the above equations Thus,, we have

$$x^{[k+1]} = \frac{1}{12}\left[1 - 3y^{(k)} + 5z^{(k)}\right]$$

$$y^{[k+1]} = \frac{1}{5}\left[28 - x^{(k)} - 3z^{(k)}\right]$$

$$z^{[k+1]} = \frac{1}{13}\left[76 - 3x^{(k)} - 7y^{(k)}\right]$$

Here the initial approximation are (x^0, y^0, z^0) = (1, 0, 1)

Iteration 1: Putting the initial values on right side, we get

$$x^{[1]} = \frac{1}{12}\left[1 - 3(0) + 5(1)\right] = \frac{1}{12}^{[6]} = 0.5$$

$$y^{[1]} = \frac{1}{5}\left[28 - 1 - 3(1)\right] = \frac{1}{5}^{[24]} = 4.8$$

$$z^{[1]} = \frac{1}{13}\left[76 - 3(1) - 7(0)\right] = \frac{1}{13}^{[73]} = 5.6153$$

Iteration 2: $x^{(2)} = \frac{1}{12}\left[1 - 3(4.8) + 5(5.6153)\right] = 1.2230$

$$y^{(2)} = \frac{1}{5}\left[28 - 0.5 - 3(5.6153)\right] = 2.1308$$

$$z^{(2)} = \frac{1}{13}\left[76 - 3(0.5) - 7(4.8)\right] = 3.1462$$

Iteration 3: $x^{(23)} = \frac{1}{12}\left[1 - 3(2.1308) + 5(3.1462)\right] = 0.8615$

$$y^{(3)} = \frac{1}{5}\left[28 - 1.223 - 3(3.1462)\right] = 3.4677$$

$$z^{(3)} = \frac{1}{13}\left[76 - 3(1.223) - 7(3.1462)\right] = 3.8698$$

Iteration 4: $x^{(4)} = \frac{1}{12}\left[1 - 3(3.4677) + 5(3.8698)\right] = 0.8288$

$$y^{(4)} = \frac{1}{5}\left[28 - 0.8615 - 3(3.8698)\right] = 3.1058$$

$$z^{(4)} = \frac{1}{13}\left[76 - 3(0.8615) - 7(3.4677)\right] = 3.7801$$

Iteration 5: $x^{(5)} = \frac{1}{12}\left[1 - 3(3.1058) + 5(3.7801)\right] = 0.8819$

$$y^{(5)} = \frac{1}{5}\left[28 - 0.8288 - 3(3.7801)\right] = 3.1662$$

$$z^{(5)} = \frac{1}{13}\left[76 - 3(0.8288) - 7(3.1058)\right] = 3.9825$$

Iteration 6: $x^{(6)} = \frac{1}{12}\left[1 - 3(3.1662) + 5(3.9825)\right] = 0.9512$

$$y^{(6)} = \frac{1}{5}\left[28 - 0.8819 - 3(3.9825)\right] = 3.0341$$

$$z^{(6)} = \frac{1}{13}\left[76 - 3(0.8819) - 7(3.1662)\right] = 3.9377$$

Iteration 7: $x^{(2)} = \frac{1}{12}\left[1 - 3(3.0341) + 5(3.9377)\right] = 0.9655$

$$y^{(7)} = \frac{1}{5}\left[28 - 0.9512 - 3(3.9377)\right] = 3.0471$$

$$z^{(7)} = \frac{1}{13}\left[76 - 3(0.9512) - 7(3.0341)\right] = 3.9929$$

Iteration 8: $x^{(8)} = \frac{1}{12}\left[1 - 3(3.0471) + 5(3.9929)\right] = 0.9853$

$$y^{(8)} = \frac{1}{5}\left[28 - 0.9655 - 3(3.9929)\right] = 3.0112$$

$$z^{(8)} = \frac{1}{13}\left[76 - 3(0.9655) - 7(3.0471)\right] = 3.9826$$

Iteration 9: $x^{(9)} = \frac{1}{12}\left[1 - 3(3.0112) + 5(3.9826)\right] = 0.9899$

$$y^{(9)} = \frac{1}{5}\left[28 - 0.9853 - 3(3.9826)\right] = 3.01338 \backsim 3.0134$$

$$z^{(9)} = \frac{1}{13}\left[76 - 3(0.0.9853) - 7(3.0112)\right] = 3.9973$$

Iteration 10: $x^{(10)} = \frac{1}{12}\left[1 - 3(3.0134) + 5(3.9973)\right] = 0.9955$

$$y^{(10)} = \frac{1}{5}\left[28 - 0.9899 - 3(3.9973)\right] = 3.0036$$

$$z^{(10)} = \frac{1}{13}\left[76 - 3(0.9899) - 7(3.0134)\right] = 3.9951$$

Iteration 11: $x^{[11]} = \frac{1}{12}\left[1 - 3(3.0036) + 5(3.9951)\right] = 0.9970$

$$y^{[11]} = \frac{1}{5}\left[28 - 0.9955 - 3(3.9951) + 5(3.9951)\right] = 3.0038$$

$$z^{[11]} = \frac{1}{13}\left[76 - 3(0.9955) - 7(3.0036)\right] = 3.9991$$

We observe that in 10th and 11th iteration, the approximation to unknowns are approximately same, so we stop the iteration procedure.

Hence, the solution of given system of equation is

x = 0.9970, y = 3.0038, z = 3.9991

Example 4: Find the solution to the system of equations

2x - 8y + 5z = -5

2x + y - z = 12

-5x + 3y + 9z = -17

Using Jacobi iterative method?

Solution: The coefficient matrix of given system is not diagonally dominant

Hence, we rearrange the equation as follows such that the elements in the coefficient matrix are diagonally dominant

2x + y - z = 12

2x - 8y + 5z = -5

-5x + 3y + 9z = -17

Rewriting the above equations Thus, we have

$$x^{[k+1]} = \frac{1}{2}\left[12 - y^{(k)} + z^{(k)}\right]$$

$$y^{[k+1]} = \frac{1}{8}\left[5 + 2x^{(k)} + 5z^{(k)}\right]$$

$$z^{[k+1]} = \frac{1}{9}\left[-17 + 5x - 3y\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: Putting the initial values in right side of (1), we get

$$x^{(1)} = \frac{1}{2}\left[12 - 0 + 0\right] = 6$$

$$y^{(1)} = \frac{1}{8}\left[5 + 2(0) + 5(0)\right] = \frac{5}{8} = 0.625$$

$$z^{(1)} = \frac{1}{9}\left[-17 + 5(0) - 3(0)\right] = -1.8889$$

Iteration 2: $x^{(2)} = \frac{1}{2}\left[12 - 0.625 + (-1.8889)\right] = 4.74305$

$$y^{(2)} = \frac{1}{8}\left[5 + 2(6) + 5(-1.8889)\right] = 0.9444$$

$$z^{(2)} = \frac{1}{9}\left[-17 + 5(6) - 3(0.625)\right] = -1.2361$$

Iteration 3: $x^{(3)} = \frac{1}{2}\left[12 - 0.9444 + 1.2361\right] = 6.1458$

$$y^{(3)} = \frac{1}{8}\left[5 + 2(4.74305) + 5(1.2361)\right] = 2.5833$$

$$z^{(3)} = \frac{1}{9}\left[-17 + 5(4.74305) - 3(0.9444)\right] = 0.4313$$

Iteration 4: $x^{(4)} = \frac{1}{2}\left[12 - 2.5833 + 0.4313\right] = 4.924$

$$y^{(4)} = \frac{1}{8}\left[5 + 2(6.1458) + 5(0.4313)\right] = 2.4310$$

$$z^{(4)} = \frac{1}{9}\left[-17 + 5(6.1458) - 3(2.5833)\right] = 0.6643$$

Iteration 5: $x^{(5)} = \frac{1}{2}\left[12 - 2.4310 + 0.6643\right] = 5.1166$

$$y^{(5)} = \frac{1}{8}\left[5 + 2(4.924) + 5(0.6643)\right] = 2.2712$$

$$z^{(5)} = \frac{1}{9}\left[-17 + 5(4.924) - 3(2.4310)\right] = 0.0363$$

Iteration 6: $x^{(6)} = \frac{1}{2}\left[12 - 2.2712 + 0.0363\right] = 4.8825$

$$y^{(6)} = \frac{1}{8}\left[5 + 2(5.1166) + 5(0.0363)\right] = 1.9268$$

$$z^{(6)} = \frac{1}{9}\left[-17 + 5(5.1166) - 3(2.2712)\right] = 0.1966$$

Iteration 7: $x^{(7)} = \frac{1}{2}\left[12 - 1.9268 + 0.1966\right] = 5.1349$

$$y^{(7)} = \frac{1}{8}\left[5 + 2(4.8825) + 5(0.1966)\right] = 1.9685$$

$$z^{(7)} = \frac{1}{9}\left[-17 + 5(4.8825) - 3(1.9268)\right] = 1.6321$$

Iteration 8: $x^{(8)} = \frac{1}{2}\left[12 - 1.9685 + 1.6321\right] = 4.1997$

$$y^{(8)} = \frac{1}{8}\left[5 + 2(5.1349) + 5(1.6321)\right] = 2.9288$$

$$z^{(8)} = \frac{1}{9}\left[-17 + 5(5.1349) - 3(1.9685)\right] = 0.30766 \backsimeq 0.3077$$

Iteration 9: $x^{(9)} = \frac{1}{2}\left[12 - 2.9288 + 0.3077\right] = 4.6894$

$$y^{(9)} = \frac{1}{8}\left[5 + 2(4.1997) + 5(0.3077)\right] = 1.8672$$

$$z^{(9)} = \frac{1}{9}\left[-17+5(4.1997)-3(2.9288)\right] = -0.53198 \cong 0.532$$

Iteration 10: $x^{(10)} = \frac{1}{2}\left[12-1.8672+\left(-0.532\right)\right] = 4.8004$

$$y^{(10)} = \frac{1}{8}\left[5+2(4.6894)+5(-0.532)\right] = 1.4648$$

$$z^{(10)} = \frac{1}{9}\left[-17+5(4.6894)-3(1.0672)\right] = 0.0939$$

Iteration 11: $x^{(11)} = \frac{1}{2}\left[12-1.4648+0.0939\right] = 5.3145$

$$y^{(11)} = \frac{1}{8}\left[5+2(4.8004)+5(0.0939)\right] = 1.8837$$

$$z^{(11)} = \frac{1}{9}\left[-17+5(4.8004)-3(1.4648)\right] = 0.2897$$

Iteration 12: $x^{(12)} = \frac{1}{2}\left[12-1.8837+0.2897\right] = 5.203$

$$y^{(12)} = \frac{1}{8}\left[5+2(5.3145)+5(0.2897)\right] = 2.1346$$

$$z^{(12)} = \frac{1}{9}\left[-17+5(5.3145)-3(1.8837)\right] = 0.4357$$

Iteration 13: $x^{(13)} = \frac{1}{2}\left[12-2.1346+0.4357\right] = 5.1505$

$$y^{(13)} = \frac{1}{8}\left[5+2(5.203)+5(0.4357)\right] = 2.1981$$

$$z^{(13)} = \frac{1}{9}\left[-17+5(5.203)-3(2.1346)\right] = 0.2901$$

Iteration 14: $x^{(14)} = \frac{1}{2}\left[12-2.1981+0.2901\right] = 5.046$

$$y^{(14)} = \frac{1}{8}\left[5+2(5.1505)+5(0.2901)\right] = 2.0939$$

$$z^{(14)} = \frac{1}{9}\left[-17+5(5.1505)-3(2.1981)\right] = 0.2398$$

Iteration 15: $x^{(15)} = \frac{1}{2}[12 - 2.0939 + 0.2398] = 5.0729$

$$y^{(15)} = \frac{1}{8}[5 + 2(5.046) + 5(0.2398)] = 2.0363$$

$$z^{(15)} = \frac{1}{9}[-17 + 5(5.046) - 3(2.0939)] = 0.2164$$

We observe that in 14th and 15th iteration, the approximation to unknown are same, So we stop the iteration procedure.

Hence the solution of given system of equation is

x = 5.0729, y = 2.0363, z = 0.2164

Example 5: Apply Jacobi method to given system of linear equations

$3x_1 - x_2 = 2$

$x_1 + 4x_2 = 5$

Solution: Given system of equation are

$3x_1 - x_2 = 2$

$x_1 + 4x_2 = 5$

The given system of equations is a diagonal system so the convergence of Jacobi method is assured.

Rewriting the given system of equation Thus, we have

$$x_1^{(k+1)} = \frac{1}{3}\left[2 + x_2^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{4}\left[5 - x_1^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero. i.e $x^{(0)} = y^{(0)} = z(^{0)} = 0$

Iteration 1: Putting the initial values in right side of (1), we get

$$x_1^{(1)} = \frac{1}{3}\left[2 + x_2^{(0)}\right] = \frac{1}{3}[2 + 0] = \frac{2}{3} = 0.6667$$

$$x_2^{(1)} = \frac{1}{4}[5 - 0] = \frac{5}{4} = 1.25$$

Iteration 2: $x_1^{(2)} = \frac{1}{3}[2 + 1.25] = 1.08333 = 1.0834$

$$x_2^{(2)} = \frac{1}{4}[5 - 0.6667] = 1.0833$$

Iteration 3: $x_1^{(3)} = \frac{1}{3}[2+1.0833] = 1.0278$

$$x_2^{(3)} = \frac{1}{4}[5-1.0834] = 0.9791$$

Iteration 4: $x_1^{(4)} = \frac{1}{3}[2+0.9791] = 0.9930$

$$x_2^{(4)} = \frac{1}{4}[5-1.0278] = 0.9930$$

Iteration 5: $x_1^{(5)} = \frac{1}{3}[2+0.9930] = 0.99766 = 0.9977$

$$x_2^{(5)} = \frac{1}{4}[5-0.9930] = 1.0017$$

Iteration 6: $x_1^{(6)} = \frac{1}{3}[2+1.0017] = 1.0005$

$$x_2^{(6)} = \frac{1}{4}[5-0.9977] = 1.0005$$

We observe that in 5th and 6th iteration, the approximation to unknown are approximately same. So, we stop the iterative procedure here.

Hence the solution of given equations are x = 1.0005, y = 1.0005

Self Check Exercise

Q.1 Solve the following system of equation by Jacobi Method

27x + 6y - z = 85

6x + 15y + 2z = 72

x + y + 54z = 110

Q.2 Solve the following system of equation by Jacobi Method

5x + 2y + z = 12

x + 4y + 2z = 15

x + 2y + 5z = 20

Q.3 Solve equations using Gauss Jacobi Method where equation are

2x + 5y = 16

3x + y = 11

8.4 Summary

Gauss Jacobi or Jacobi method is an iterative method for solving equations of diagonally dominant system of linear equations

It is based on the idea of successive approximations

This method begins with one or two initial approximations of the roots, with a sequence of approximations x_1, x_2, x_3, ..., x_k, ... as K $\rightarrow\infty$, this sequence of roots converges to exact root $\propto$.

The sufficient conditions for the convergence of the approximation obtained by Jacobi method is that the system of equations is diagonally dominant that is the coefficient matrix A is diagonally dominant. i.e. $|a_{ii}| \geq \sum_{J-1}^{n} |a_{ij}|$ for i ‡ j.

8.5 Glossary:

- **System of Linear Equations:** A system of linear equation is the collection of two or more linear equations involving the same variables.
- **Pivot Element:** An element of a matrix which is selected by an algorithm to proceed further calculations.
- **Sequence:** Sequence is a function whose domain is the set of natural number.
- **Convergence:** It is property of approaching a Limit more and more explicitly as an argument of functions increaser or decreases or as the number of terms of series get increased
- **Initial approximation:** Initial approximations are the initial guess that we made to start the iterative procedure.

8.6 Answers to Self Check Exercise

Ans. x = 2.4253, y = 3.5732, z = 1.9259

Ans. x = 0.994, y = 1.993, z = 2.994

Ans. x = 2.9998, y = 1.9999

8.7 References/Suggested Readings

1. Introductory Methods of Numerical Analysis by S.S. Sastry
2. Introduction to Numerical Analysis by Kendall Atkinson
3. Numerical Analysis by J. Douglas Eaires
4. Finite Difference and Numerical Analysis by H.C. Saxena.

8.8 Terminal Questions

1. Solve the following system of Linear equation using Jacobi method

   $4x_1 - 2x_2 + x_3 = 3$

   $2x_1 + 5x_2 - 2x_3 = 0$

$x_1 - x_2 - 3x_3 - 0.1$

2. Find the solution to the system of equations

   $2x + 3y + z = 13$

   $x + y + z = 6$

   $3x + 2y + 2z = 15$

   Using Jacobi Method

3. Use Jacobi Method to solve the system of linear equations

   $7x_1 - 0x_2 + 3x_3 = 2$

   $x_1 - 8x_2 - 4x_3 = 3$

   $x_1 + 2x_2 + 15x_3 = 6$

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Unit - 9

Gauss Seidal Method

Structure

9.1 Introduction

Dear student in this unit we will learn Gauss Seidal method to solve system of linear equation. This is an iterative method to solve system of linear equation. In unit 8 we studied about Gauss Jacobi Method. There is a very small difference between these two methods. Gauss Seidal method is a little modification of Gauss-Jacobi method. In this method we apply the value of variable obtained in one step in the next step. For this reason this method is also known as method of successive displacement. Here in this unit we will learn to apply Gauss Seidal method to system of linear equations and find its solution.

9.2 Learning Objectives:

After studying this unit students will be able to

1. define Gauss Seidal method to solve system of linear equation
2. differentiate between Gauss-Jacobi method and Gauss Seidal method.
3. Solve system of linear equation by Gauss Seidal method.

9.3 Seidal Method

The Gauss Seidal method is a improvisation of the Jacobi method.

Thus method is named after mathematicians Carl Friedrich Gauss (1777-1855) and Philipp L. Seidal (1821-1896). This modification often yields a higher degree accuracy with fewer iterations

The Gauss Seidal method modifies the variables values as soon as a new value is evaluated, whereas the Jacobi approach waits until the following iteration to modify the variable values. For example, in the Gauss Seidal technique, the value of $x_i^{(k)}$ only changes modified until the (k+1)th iteration.

Let us consider the system of equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n = b_n \end{array}\right\rangle \qquad (1)$$

Where the diagonal coefficients are not zero i.e. diagonal elements in the coefficient matrix. Such a system is called diagonally dominant system.

The system of equation (1) may be written as

$$\left.\begin{array}{l} x_1 = \dfrac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 \dots\dots\dots a_{1n}x_n\right] \\ x_2 = \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 \dots\dots\dots a_{2n}x_n\right] \\ \vdots \\ x_n = \dfrac{1}{a_{nn}}\left[bn - a_{n1}x_1 - a_{n2}x_2 \dots\dots\dots an_{n-1}x_{n-1}\right] \end{array}\right\} \qquad (2)$$

Let the initial approximation or approximate solution be

$x_1^{(0)}$, $x_2^{(0)}$, $x_3^{(0)}$, ...

Substitute $x_2^{(0)}$, $x_3^{(0)}$ in the 1st equation of (2) we get

$$\left.\begin{array}{l} x_1^{(1)} = \dfrac{1}{a_{11}}\left[b_1 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} \dots\dots\dots a_{1n}x_n^{(0)}\right] \\ x_2^{(1)} = \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} \dots\dots\dots a_{2n}x_n^{(0)}\right] \\ \vdots \\ x_n^{(1)} = \dfrac{1}{a_{nn}}\left[bn - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} \dots\dots\dots an_{n-1}x_{n-1}^{(1)}\right] \end{array}\right\} \qquad (3)$$

In the same way, beginning with (k-1)th approximations the Kth approximations to the unknowns $x_1^{(k)}$, $x_2^{(k)}$, ... $x_n^{(k)}$ are given by

$$\left.\begin{aligned}
x_1^{(k)} &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)}..........a_{1n}x_n^{(k-1)}\right] \\
x_2^{(k)} &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k-1)}..........a_{2n}x_n^{(k-1)}\right] \\
&\vdots \\
x_n^{(k)} &= \frac{1}{a_{nn}}\left[bn - a_{n1}x_1^{(k)} - a_{n2}x_2^{(k)}..........an,_{n-1}\, x_{n-1}^{(k)}\right]
\end{aligned}\right\} \quad (4)$$

The recurrence formula given by (4) is repeatedly applied (i.e. for k = 1, 2, 3, ...) to obtain a sequence of approximation (approximate solutions) which converge to the exact solutions. (under certain conditions on the coefficients of the system).

Note:- In order to apply Gauss Jacobi and Gauss-Seidal method, the system of equations are to be arranged in diagonal system. In diagonal system, the magnitude of each diagonal element of the coefficient matrix is attest as large as the sum of magnitudes of other elements in its row.

Let us try to find solution of system of linear equations by Gauss-Seidal Method.

Some Examples:

Example 1: Apply Gauss Seidal method to give system of linear equations

$4x_1 + x_2 + x_3 = 7$

$x_1 + 7x_2 + 2x_3 = -2$

$3x_1 + 4x_3 = 11$

Solution: The given system of equations is a diagonal system. So, convergence of Gauss Seldal method is assured.

Rewriting the given system of equation as follows

$$x_1 = \frac{1}{4}\left[7 - x_2 - x_3\right]$$

$$x_2 = \frac{1}{7}\left[2 + x_1 + 2x_3\right]$$

$$x_3 = \frac{1}{4}\left[11 - 3x_1\right]$$

Thus, we have

$$x_1^{(k+1)} = \frac{1}{4}\left[7 - x_2^{(k)} - x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{7}\left[2 + x_1^{(k)} + 2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{4}\left[11 - 3x_1^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: $x_1^{(1)} = \frac{1}{4}\left[7 - 2x_2^{(0)} - x_3^{(0)}\right] = \frac{1}{4}[7 - 0 - 0] = \frac{7}{4} = 1.75$

$$x_2^{(1)} = \frac{1}{7}\left[2 + x_1^{(0)} + 2x_3^{(0)}\right] = \frac{1}{7}[2 + 0 + 2(0)] = \frac{2}{7} = 0.2857$$

$$x_3^{(1)} = \frac{1}{4}\left[11 - 3x_1^{(0)}\right] = \frac{1}{4}[11 - 3(0)] = \frac{11}{4} = 2.75$$

Iteration 2: $x_1^{(2)} = \frac{1}{4}\left[7 - x_2^{(1)} - x_3^{(1)}\right] = \frac{1}{4}[7 - 0.2857 - 2.57] = 0.9912$

$$x_2^{(2)} = \frac{1}{7}\left[2 + x_1^{(1)} + 2x_3^{(1)}\right] = \frac{1}{7}[2 + 1.75 + 2(2.75)] = 1.3214$$

$$x_3^{(2)} = \frac{1}{4}\left[11 - 3x_1^{(1)}\right] = \frac{1}{4}[11 - 3(1.75)] = 1.4375$$

Iteration 3: $x_1^{(3)} = \frac{1}{4}\left[7 - x_2^{(2)} - x_3^{(2)}\right] = \frac{1}{4}[7 - 1.3214 - 1.4375] = 1.0602$

$$x_2^{(3)} = \frac{1}{7}\left[2 + x_1^{(2)} + 2x_3^{(2)}\right] = \frac{1}{7}[2 + 0.9912 + 2(1.4375)] = 0.8380$$

$$x_3^{(2)} = \frac{1}{4}\left[11 - 3x_1^{(2)}\right] = \frac{1}{4}[11 - 3(0.9912)] = 2.0066$$

Iteration 4: $x_1^{(4)} = \frac{1}{4}\left[7 - x_2^{(3)} - x_3^{(3)}\right] = \frac{1}{4}[7 - 0.8380 - 2.0066] = 1.0388$

$$x_2^{(4)} = \frac{1}{7}\left[2 + x_1^{(3)} + 2x_3^{(3)}\right] = \frac{1}{7}[2 + 1.0603 + 2(2.0066)] = 1.0105$$

$$x_3^{(4)} = \frac{1}{4}\left[11 - 3x_1^{(4)}\right] = \frac{1}{4}[11 - 3(1.0603)] = 1.9548$$

Iteration 5: $x_1^{(5)} = \frac{1}{4}\left[7 - x_2^{(4)} - x_3^{(4)}\right] = \frac{1}{4}\left[7 - 1.0105 - 1.9548\right] = 1.0087$

$$x_2^{(5)} = \frac{1}{7}\left[2 + x_1^{(4)} + 2x_3^{(4)}\right] = \frac{1}{7}\left[2 + 1.0388 + 2(1.9548)\right] = 0.9926$$

$$x_3^{(5)} = \frac{1}{4}\left[11 - 3x_1^{(4)}\right] = \frac{1}{4}\left[11 - 3(1.0388)\right] = 1.9709$$

Iteration 6: $x_1^{(6)} = \frac{1}{4}\left[7 - x_2^{(5)} - x_3^{(5)}\right] = \frac{1}{4}\left[7 - 0.9926 - 1.9709\right] = 1.0091$

$$x_2^{(6)} = \frac{1}{7}\left[2 + x_1^{(5)} + 2x_3^{(5)}\right] = \frac{1}{7}\left[2 + 1.0087 + 2(1.9709)\right] = 0.9929$$

$$x_3^{(6)} = \frac{1}{4}\left[11 - 3x_1^{(5)}\right] = \frac{1}{4}\left[11 - 3(1.0087)\right] = 1.9935$$

Iteration 7: $x_1^{(7)} = \frac{1}{4}\left[7 - x_2^{(6)} - x_3^{(6)}\right] = \frac{1}{4}\left[7 - 0.9929 - 1.9935\right] = 1.0034$

$$x_2^{(7)} = \frac{1}{7}\left[2 + x_1^{(6)} + 2x_3^{(6)}\right] = \frac{1}{7}\left[2 + 1.0091 + 2(1.9935)\right] = 0.9994$$

$$x_3^{(7)} = \frac{1}{4}\left[11 - 3x_1^{(7)}\right] = \frac{1}{4}\left[11 - 3(1.0091)\right] = 1.9932$$

Iteration 8: $x_1^{(8)} = \frac{1}{4}\left[7 - x_2^{(7)} - x_3^{(7)}\right] = \frac{1}{4}\left[7 - 0.9994 - 1.9932\right] = 1.0018$

$$x_2^{(8)} = \frac{1}{7}\left[2 + x_1^{(7)} + 2x_3^{(7)}\right] = \frac{1}{7}\left[2 + 1.0034 + 2(1.9932)\right] = 0.9985$$

$$x_3^{(8)} = \frac{1}{4}\left[11 - 3x_1^{(7)}\right] = \frac{1}{4}\left[11 - 3(1.0034)\right] = 1.9947$$

Iteration 9: $x_1^{(9)} = \frac{1}{4}\left[7 - x_2^{(8)} - x_3^{(8)}\right] = \frac{1}{4}\left[7 - 0.9985 - 2.57\right] = 1.0010$

$$x_2^{(9)} = \frac{1}{7}\left[2 + x_1^{(8)} + 2x_3^{(8)}\right] = \frac{1}{7}\left[2 + 1.0018 + 2(1.9974)\right] = 0.9995$$

$$x_3^{(9)} = \frac{1}{4}\left[11 - 3x_1^{(8)}\right] = \frac{1}{4}\left[11 - 3(1.0018)\right] = 1.9986$$

Iteration 10: $x_1^{(10)} = \frac{1}{4}\left[7 - x_2^{(9)} - x_3^{(9)}\right] = \frac{1}{4}\left[7 - 0.9995 - 1.9986\right] = 1.0004 \backsimeq 1.0005$

$$x_2^{(10)} = \frac{1}{7}\left[2 + x_1^{(9)} + 2x_3^{(9)}\right] = \frac{1}{7}\left[2 + 1.0010 + 2(1.9986)\right] = 0.9997$$

$$x_3^{(10)} = \frac{1}{4}\left[11 - 3x_1^{(9)}\right] = \frac{1}{4}\left[11 - 3(1.0010)\right] = 1.9992$$

We observe that 9th and 10th iteration, the approximation to unknowns are approximately same, so we stop the iterative procedure.

Hence, the solution of given system of equation is

x = 1.0005, y = 0.9997, z = 1.9992

Example 2: Solve the following system of equation by Gauss Seidal method

$-4x_1 + 2x_2 = 6$

$3x_1 - 5x_2 = 1$

Solution: The given system of equations is a diagonal system. So, convergence is assured.

Rewriting the above equation Thus, we get

$$x_1^{(k+1)} = \frac{1}{4}\left[-6 + 2x_2^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{5}\left[-1 + 3x_1^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: $x_1^{(1)} = \frac{1}{4}\left[-6 + 2x_2^{(0)}\right] = \frac{1}{4}\left[-6 + 0\right] = \frac{-6}{4} = -1.5$

$$x_2^{(1)} = \frac{1}{5}\left[-1 + 3x_1^{(0)}\right] = \frac{1}{5}\left[-1 + 0\right] = \frac{-1}{5} = -0.2$$

Iteration 2: $x_1^{(2)} = \frac{1}{4}\left[-6 + 2x_2^{(1)}\right] = \frac{1}{4}\left[-6 + 2(-0.2)\right] = -1.6$

$$x_2^{(2)} = \frac{1}{5}\left[-1 + 3x_1^{(1)}\right] = \frac{1}{5}\left[-1 + 3(-1.5)\right] = -1.1$$

Iteration 3: $x_1^{(3)} = \frac{1}{4}\left[-6 + 2x_2^{(2)}\right] = \frac{1}{4}\left[-6 + 2(-1.1)\right] = -2.05$

$$x_2^{(3)} = \frac{1}{5}\left[-1+3x_1^{(2)}\right] = \frac{1}{5}\left[-1+3(-1.6)\right] = -1.16$$

Iteration 4: $x_1^{(4)} = \frac{1}{4}\left[-6+2x_2^{(3)}\right] = \frac{1}{4}\left[-6+2(-1.16)\right] = -2.08$

$$x_2^{(4)} = \frac{1}{5}\left[-1+3x_1^{(3)}\right] = \frac{1}{5}\left[-1+3(-2.05)\right] = -1.43$$

Iteration 5: $x_1^{(5)} = \frac{1}{4}\left[-6+2x_2^{(4)}\right] = \frac{1}{4}\left[-6+2(-1.43)\right] = -2.215$

$$x_2^{(5)} = \frac{1}{5}\left[-1+3x_1^{(4)}\right] = \frac{1}{5}\left[-1+3(-2.08)\right] = -1.448$$

Iteration 6: $x_1^{(6)} = \frac{1}{4}\left[-6+2x_2^{(5)}\right] = \frac{1}{4}\left[-6+2(-1.448)\right] = -2.224$

$$x_2^{(6)} = \frac{1}{5}\left[-1+3x_1^{(5)}\right] = \frac{1}{5}\left[-1+3(-2.215)\right] = -1.529$$

Iteration 7: $x_1^{(2)} = \frac{1}{4}\left[-6+2x_2^{(6)}\right] = \frac{1}{4}\left[-6+2(-1.529)\right] = -2.2645$

$$x_2^{(7)} = \frac{1}{5}\left[-1+3x_1^{(6)}\right] = \frac{1}{5}\left[-1+3(-2.224)\right] = -1.5344$$

Iteration 8: $x_1^{(8)} = \frac{1}{4}\left[-6+2x_2^{(7)}\right] = \frac{1}{4}\left[-6+2(-1.5344)\right] = -2.2672$

$$x_2^{(8)} = \frac{1}{5}\left[-1+3x_1^{(7)}\right] = \frac{1}{5}\left[-1+3(-2.2645)\right] = -1.5587$$

Iteration 9: $x_1^{(9)} = \frac{1}{4}\left[-6+2x_2^{(8)}\right] = \frac{1}{4}\left[-6+2(-1.5587)\right] = -2.27935$

$$x_2^{(9)} = \frac{1}{5}\left[-1+3x_1^{(8)}\right] = \frac{1}{5}\left[-1+3(-2.2672)\right] = -1.56032$$

Iteration 10: $x_1^{(10)} = \frac{1}{4}\left[-6+2x_2^{(9)}\right] = \frac{1}{4}\left[-6+2(-1.56032)\right] = -2.28016$

$$x_2^{(10)} = \frac{1}{5}\left[-1+3x_1^{(9)}\right] = \frac{1}{5}\left[-1+3(-2.27935)\right] = -1.5676$$

Iteration 11: $x_1^{(11)} = \frac{1}{4}\left[-6+2x_2^{(10)}\right] = \frac{1}{4}\left[-6+2(-1.5676)\right] = -2.2838$

$$x_2^{(11)} = \frac{1}{5}\left[-1+3x_1^{(10)}\right] = \frac{1}{5}\left[-1+3(-2.28016)\right] = -1.56809$$

Iteration 12: $x_1^{(12)} = \frac{1}{4}\left[-6+2x_2^{(11)}\right] = \frac{1}{4}\left[-6+2(-1.56809)\right] = -2.28404$

$$x_2^{(12)} = \frac{1}{5}\left[-1+3x_1^{(11)}\right] = \frac{1}{5}\left[-1+3(-2.2838)\right] = -1.57028$$

Iteration 13: $x_1^{(13)} = \frac{1}{4}\left[-6+2x_2^{(12)}\right] = \frac{1}{4}\left[-6+2(-1.57028)\right] = -2.28514$

$$x_2^{(13)} = \frac{1}{5}\left[-1+3x_1^{(12)}\right] = \frac{1}{5}\left[-1+3(-2.28404)\right] = -1.57042$$

Iteration 14: $x_1^{(14)} = \frac{1}{4}\left[-6+2x_2^{(13)}\right] = \frac{1}{4}\left[-6+2(-1.57042)\right] = -2.2852$

$$x_2^{(14)} = \frac{1}{5}\left[-1+3x_1^{(13)}\right] = \frac{1}{5}\left[-1+3(-2.28514)\right] = -1.57108$$

We observe that 13th and 14th iteration, the approximation to unknown are approximately same, so we stop the iterative procedure.

Hence the solution of given system of equation is x = -2.2852, y = 1.57108

Example 3: Solve the following system of equation by Gauss Seidal method

20x + y - 2z = 17

3x + 20y - z = -18

2x - 3y + 20z = 25

Solution: The given system of equations is a diagonal system. So, convergence of Gauss Seidal method is assured.

Rewriting the given system of equation as follows

$$x = \frac{1}{20}\left[17 - y + 2z\right]$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

Thus, we get

$$x^{(k+1)} = \frac{1}{20}\left[17 - y^{(k)} + 2z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{20}\left[-18 - 3x^{(k)} + z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{20}\left[25 - 2x^{(k)} + 3y^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: $x_1^{(1)} = \frac{1}{20}\left[17 - y^{(0)} + 2z^{(0)}\right] = \frac{1}{20}\left[17 - 0 + 0\right] = 0.85$

$$y_1^{(1)} = \frac{1}{20}\left[-18 - 3x^{(0)} + z^{(0)}\right] = \frac{1}{20}\left[-18 - 0 + 0\right] = 0.9$$

$$z_1^{(1)} = \frac{1}{20}\left[25 - 2x^{(0)} + 3y^{(0)}\right] = \frac{1}{20}\left[25 - 0 + 0\right] = 1.25$$

Iteration 2: $x_1^{(2)} = \frac{1}{20}\left[17 - y^{(1)} + 2z^{(1)}\right] = \frac{1}{20}\left[17 - (-0.9) + 2(1.25)\right] = 1.02$

$$y_1^{(2)} = \frac{1}{20}\left[-18 - 3x^{(1)} + z^{(1)}\right] = \frac{1}{20}\left[-18 - 3(0.85) + 1.25\right] = -0.965$$

$$z_1^{(1)} = \frac{1}{20}\left[25 - 2x^{(1)} + 3y^{(1)}\right] = \frac{1}{20}\left[25 - 2(0.85) + 3(-0.9)\right] = 1.03$$

Iteration 3: $x_1^{(3)} = \frac{1}{20}\left[17 - y^{(2)} + 2z^{(2)}\right] = \frac{1}{20}\left[17 - (-0.965) + 2(1.03)\right] = 1.00125$

$$y_1^{(3)} = \frac{1}{20}\left[-18 - 3x^{(2)} + z^{(2)}\right] = \frac{1}{20}\left[-18 - 3(1.02) + 1.03\right] = -1.0015$$

$$z_1^{(3)} = \frac{1}{20}\left[25 - 2x^{(2)} + 3y^{(2)}\right] = \frac{1}{20}\left[25 - 2(1.02) + 3(-0.965)\right] = 1.00325$$

Iteration 4: $x_1^{(4)} = \frac{1}{20}\left[17 - y^{(3)} + 2z^{(3)}\right] = \frac{1}{20}\left[17 - (-1.0015) + 2(1.00325)\right] = 1.00025$

$$y_1^{(4)} = \frac{1}{20}\left[-18-3x^{(3)}+z^{(3)}\right] = \frac{1}{20}\left[-18-3\left(1.00125\right)+1.00325\right] = -1.000025$$

$$z_1^{(4)} = \frac{1}{20}\left[25-2x^{(3)}+3y^{(3)}\right] = \frac{1}{20}\left[25-2\left(1.00125\right)+3\left(-1.0015\right)\right] = 0.99965$$

Iteration 5: $$x_1^{(5)} = \frac{1}{20}\left[17-y^{(4)}+2z^{(4)}\right] = \frac{1}{20}\left[17-\left(-1.000025\right)+2(0.99965)\right] = 0.9999$$

$$y_1^{(5)} = \frac{1}{20}\left[-18-3x^{(4)}+z^{(4)}\right] = \frac{1}{20}\left[-18-3\left(1.0004\right)+\left(0.99965\right)\right] = -1.0000775$$

$$z_1^{(5)} = \frac{1}{20}\left[25-2x^{(4)}+3y^{(4)}\right] = \frac{1}{20}\left[25-2\left(1.0004\right)+3\left(-1.000025\right)\right] = 0.9999$$

We observe that 4th and 5th iteration, the approximation to unknowns are approximately same, so we stop the iterative procedure

Hence, the solution of given system of equation is

x = 0.9999, y = -1.0000, z = 0.9999

Example 4: Solve the following system of equation by Gauss Seidal Method.

$2x_1 - 6x_2 + 2x_3 = 4$

$x_1 - 5x_2 + 8x_3 = 5$

$8x_1 + 4x_2 + 3x_3 = 6$

Solution: The coefficient matrix of given system is not diagonally dominant.

Hence, we rearrange the equation as follows such that the elements in the coefficient matrix are diagonally dominant

$8x_1 + 4x_2 + 3x_3 = 6$

$2x_1 - 6x_2 + 2x_3 = 4$

$x_1 - 5x_2 + 8x_3 = 5$

Rewriting the above equations, we get

$$x_1^{(k+1)} = \frac{1}{8}\left[6-4x_2^{(k)}-3x_3^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{6}\left[-4+2x_1^{(k)}+2x_3^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{8}\left[5 - x_1^{(k)} + 5x_2^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: $x_1^{(1)} = \frac{1}{8}\left[6 - 4x_2^{(0)} - 3x_3^{(0)}\right] = \frac{1}{8}[6-0-0] = 0.75$

$$x_2^{(1)} = \frac{1}{6}\left[-4 + 2x_1^{(0)} + 2x_3^{(0)}\right] = \frac{1}{6}[-4+0+0] = -0.6667$$

$$x_3^{(1)} = \frac{1}{8}\left[5 - x_1^{(0)} + 5x_2^{(0)}\right] = \frac{1}{8}[5-0+0] = 0.625$$

Iteration 2: $x_1^{(2)} = \frac{1}{8}\left[6 - 4x_2^{(1)} - 3x_3^{(1)}\right] = \frac{1}{8}\left[6 - 4(-0.6667) - 3(0.625)\right] = 0.8489$

$$x_2^{(2)} = \frac{1}{6}\left[-4 + 2x_1^{(1)} + 2x_3^{(1)}\right] = \frac{1}{6}\left[-4 + 2(0.75) + 2(0.625)\right] = -0.2083$$

$$x_3^{(2)} = \frac{1}{8}\left[5 - x_1^{(1)} + 5x_2^{(1)}\right] = \frac{1}{8}\left[5 - 0.75 + 5(-0.6667)\right] = 0.1145$$

Iteration 3: $x_1^{(3)} = \frac{1}{8}\left[6 - 4x_2^{(2)} - 3x_3^{(2)}\right] = \frac{1}{8}\left[6 - 4(-0.2083) - 3(0.1145)\right] = 0.8112$

$$x_2^{(3)} = \frac{1}{6}\left[-4 + 2x_1^{(2)} + 2x_3^{(2)}\right] = \frac{1}{6}\left[-4 + 2(0.8489) + 2(0.1145)\right] = -0.3455$$

$$x_3^{(3)} = \frac{1}{8}\left[5 - x_1^{(2)} + 5x_2^{(2)}\right] = \frac{1}{8}\left[5 - 0.8489 + 5(-0.2083)\right] = 0.3887$$

Iteration 4: $x_1^{(4)} = \frac{1}{8}\left[6 - 4x_2^{(3)} - 3x_3^{(3)}\right] = \frac{1}{8}\left[6 - 4(-0.3455) - 3(0.3887)\right] = 0.7769$

$$x_2^{(4)} = \frac{1}{6}\left[-4 + 2x_1^{(3)} + 2x_3^{(3)}\right] = \frac{1}{6}\left[-4 + 2(0.8112) + 2(0.3887)\right] = -0.2667$$

$$x_3^{(4)} = \frac{1}{8}\left[5 - x_1^{(3)} + 5x_2^{(3)}\right] = \frac{1}{8}\left[5 - (0.8152) + 5(-0.3455)\right] = 0.5104$$

Iteration 5: $x_1^{(5)} = \frac{1}{8}\left[6 - 4x_2^{(4)} - 3x_3^{(4)}\right] = \frac{1}{8}\left[6 - 4(-0.2667) - 3(0.5104)\right] = 0.6919$

$$x_2^{(5)} = \frac{1}{6}\left[-4 + 2x_1^{(4)} + 2x_3^{(4)}\right] = \frac{1}{6}\left[-4 + 2(0.7769) + 2(0.5104)\right] = -0.2375$$

$$x_3^{(5)} = \frac{1}{8}\left[5 - x_1^{(4)} + 5x_2^{(4)}\right] = \frac{1}{8}\left[5 - 0.776915 + 5(-0.2667)\right] = 0.3612$$

Iteration 6: $x_1^{(6)} = \frac{1}{8}\left[6 - 4x_2^{(5)} - 3x_3^{(5)}\right] = \frac{1}{8}\left[6 - 4(-0.2375) - 3(0.3612)\right] = 0.7333$

$$x_2^{(6)} = \frac{1}{6}\left[-4 + 2x_1^{(5)} + 2x_3^{(5)}\right] = \frac{1}{6}\left[-4 + 2(0.6919) + 2(0.3612)\right] = -0.3156$$

$$x_3^{(6)} = \frac{1}{8}\left[5 - x_1^{(5)} + 5x_2^{(5)}\right] = \frac{1}{8}\left[5 - 0.6919 + 5(-0.2375)\right] = 0.3901$$

Iteration 7: $x_1^{(7)} = \frac{1}{8}\left[6 - 4x_2^{(6)} - 3x_3^{(6)}\right] = \frac{1}{8}\left[6 - 4(-0.3156) - 3(0.3901)\right] = 0.7615$

$$x_2^{(7)} = \frac{1}{6}\left[-4 + 2x_1^{(6)} + 2x_3^{(6)}\right] = \frac{1}{6}\left[-4 + 2(0.7333) + 2(0.3901)\right] = -0.2922$$

$$x_3^{(7)} = \frac{1}{8}\left[5 - x_1^{(6)} + 5x_2^{(6)}\right] = \frac{1}{8}\left[5 - 0.7333 + 5(-0.3156)\right] = 0.3361$$

Iteration 8: $x_1^{(8)} = \frac{1}{8}\left[6 - 4x_2^{(7)} - 3x_3^{(7)}\right] = \frac{1}{8}\left[6 - 4(-0.2922) - 3(0.3361)\right] = 0.77006$

$$x_2^{(8)} = \frac{1}{6}\left[-4 + 2x_1^{(7)} + 2x_3^{(7)}\right] = \frac{1}{6}\left[-4 + 2(0.7615) + 2(0.3361)\right] = -0.3008$$

$$x_3^{(8)} = \frac{1}{8}\left[5 - x_1^{(7)} + 5x_2^{(7)}\right] = \frac{1}{8}\left[5 - 0.7615 + 5(-0.2922)\right] = 0.3472$$

Iteration 9: $x_1^{(9)} = \frac{1}{8}\left[6 - 4x_2^{(8)} - 3x_3^{(8)}\right] = \frac{1}{8}\left[6 - 4(-0.3008) - 3(0.3472)\right] = 0.7702$

$$x_2^{(9)} = \frac{1}{6}\left[-4 + 2x_1^{(8)} + 2x_3^{(8)}\right] = \frac{1}{6}\left[-4 + 2(0.77006) + 2(0.3472)\right] = -0.2942$$

$$x_3^{(9)} = \frac{1}{8}\left[5 - x_1^{(8)} + 5x_2^{(8)}\right] = \frac{1}{8}\left[5 - 0.77006 + 5(-0.3008)\right] = 0.3407$$

Iteration 10: $x_1^{(10)} = \frac{1}{8}\left[6 - 4x_2^{(9)} - 3x_3^{(9)}\right] = \frac{1}{8}\left[6 - 4(-0.2942) - 3(0.3407)\right] = 0.7693$

$$x_2^{(10)} = \frac{1}{6}\left[-4+2x_1^{(9)}+2x_3^{(9)}\right] = \frac{1}{6}\left[-4+2(0.7702)+2(0.3407)\right] = -0.2964$$

$$x_3^{(10)} = \frac{1}{8}\left[5-x_1^{(9)}+5x_2^{(9)}\right] = \frac{1}{8}\left[5-0.7702+5(-0.2942)\right] = 0.34485$$

Iteration 11: $x_1^{(11)} = \frac{1}{8}\left[6-4x_2^{(10)}-3x_3^{(10)}\right] = \frac{1}{8}\left[6-4(-0.2964)-3(0.34485)\right] = 0.7689$

$$x_2^{(11)} = \frac{1}{6}\left[-4+2x_1^{(10)}+2x_3^{(10)}\right] = \frac{1}{6}\left[-4+2(0.7693)+2(0.34485)\right] = -0.2941$$

$$x_3^{(11)} = \frac{1}{8}\left[5-x_1^{(10)}+5x_2^{(10)}\right] = \frac{1}{8}\left[5-0.7693+5(-0.2964)\right] = 0.3436$$

Iteration 12: $x_1^{(12)} = \frac{1}{8}\left[6-4x_2^{(11)}-3x_3^{(11)}\right] = \frac{1}{8}\left[6-4(-0.2941)-3(0.3436)\right] = 0.7682$

$$x_2^{(2)} = \frac{1}{6}\left[-4+2x_1^{(11)}+2x_3^{(11)}\right] = \frac{1}{6}\left[-4+2(0.7689)+2(0.3436)\right] = -0.2958$$

$$x_3^{(12)} = \frac{1}{8}\left[5-x_1^{(11)}+5x_2^{(11)}\right] = \frac{1}{8}\left[5-0.7689+5(-0.2941)\right] = 0.3451$$

Iteration 13: $x_1^{(13)} = \frac{1}{8}\left[6-4x_2^{(12)}-3x_3^{(12)}\right] = \frac{1}{8}\left[6-4(-0.2958)-3(0.3451)\right] = 0.7684$

$$x_2^{(13)} = \frac{1}{6}\left[-4+2x_1^{(12)}+2x_3^{(12)}\right] = \frac{1}{6}\left[-4+2(0.7682)+2(0.3451)\right] = -0.2955$$

$$x_3^{(13)} = \frac{1}{8}\left[5-x_1^{(12)}+5x_2^{(12)}\right] = \frac{1}{8}\left[5-0.7682+5(-0.2958)\right] = 0.3441$$

We observe that in 12^{th} and 13^{th} iteration, the approximation to unknowns are approximately same, so we stop the iterative procedure.

Hence the solution of the system is

$x_1 = 0.7684$, $x_2 = 0.2955$, $x_3 = 0.3441$

Example 5: Solve the following system of linear equations by Gauss Seidal Method.

6x + 2y - z= 4

x + 5y + z = 3

2x + y + 4z = 27

Solution: Given system of equations is a diagonal system. So, convergence of Gauss Seidal Method is assured.

Rewriting the given system of equations, we get

$$x^{(k+1)} = \frac{1}{6}\left[4 - 2y^{(k)} + z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{5}\left[3 - x^{(k)} - z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{4}\left[27 - 2x^{(k)} - y^{(k)}\right]$$

Let us take the initial approximation to each unknown as zero i.e. $x^{(0)} = y^{(0)} = z^{(0)} = 0$

Iteration 1: $x^{(1)} = \frac{1}{6}\left[4 - 2y^{(0)} + z^{(0)}\right] = \frac{1}{6}\left[4 - 0 + 0\right] = \frac{4}{6} = 0.6667$

$$y^{(1)} = \frac{1}{5}\left[3 - x^{(0)} - z^{(0)}\right] = \frac{1}{5}\left[3 - 0 - 0\right] = \frac{3}{5} = 0.6$$

$$z^{(1)} = \frac{1}{4}\left[27 - 2x^{(0)} - y^{(0)}\right] = \frac{1}{4}\left[24 - 0 - 0\right] = \frac{27}{4} = 6.75$$

Iteration 2: $x^{(2)} = \frac{1}{6}\left[4 - 2y^{(1)} + z^{(1)}\right] = \frac{1}{6}\left[4 - 2(0.6) + 6.75\right] = 1.5916$

$$y^{(2)} = \frac{1}{5}\left[3 - x^{(1)} - z^{(1)}\right] = \frac{1}{5}\left[3 - 0.6667 - 6.75\right] = -0.8833$$

$$z^{(2)} = \frac{1}{4}\left[27 - 2x^{(1)} - y^{(1)}\right] = \frac{1}{4}\left[27 - 2(0.6667) - 0.6\right] = 6.2666$$

Iteration 3: $x^{(3)} = \frac{1}{6}\left[4 - 2y^{(2)} + z^{(2)}\right] = \frac{1}{6}\left[4 - 2(0.8833) + 6.2666\right] = 2.0055$

$$y^{(3)} = \frac{1}{5}\left[3 - x^{(2)} - z^{(2)}\right] = \frac{1}{5}\left[3 - 1.5916 - 6.2666\right] = -0.9716$$

$$z^{(3)} = \frac{1}{4}\left[27 - 2x^{(2)} - y^{(2)}\right] = \frac{1}{4}\left[27 - 2(1.5916) - (-0.8833)\right] = 6.17502$$

Iteration 4: $x^{(4)} = \frac{1}{6}\left[4 - 2y^{(3)} + z^{(3)}\right] = \frac{1}{6}\left[4 - 2(-0.9716) + 6.17502\right] = 2.0197$

$$y^{(4)} = \frac{1}{5}\left[3 - x^{(3)} - z^{(3)}\right] = \frac{1}{5}[3 - 2.0055 - 6.17502] = -1.0361$$

$$z^{(4)} = \frac{1}{4}\left[27 - 2x^{(3)} - y^{(3)}\right] = \frac{1}{4}\left[27 - 2(2.0055) - (-0.9716)\right] = 5.9901$$

Iteration 5: $x^{(5)} = \frac{1}{6}\left[4 - 2y^{(4)} + z^{(4)}\right] = \frac{1}{6}\left[4 - 2(-1.0361) + 5.9901\right] = 2.01038 \cong 2.0104$

$$y^{(5)} = \frac{1}{5}\left[3 - x^{(4)} - z^{(4)}\right] = \frac{1}{5}[3 - 2.0197 - 5.9901] = -1.0019$$

$$z^{(5)} = \frac{1}{4}\left[27 - 2x^{(4)} - y^{(4)}\right] = \frac{1}{4}\left[27 - 2(2.0197) - (-1.0361)\right] = 5.9992$$

Iteration 6: $x^{(6)} = \frac{1}{6}\left[4 - 2y^{(5)} + z^{(5)}\right] = \frac{1}{6}\left[4 - 2(-1.0019) + 5.9992\right] = 2.0005$

$$y^{(6)} = \frac{1}{5}\left[3 - x^{(5)} - z^{(5)}\right] = \frac{1}{5}[3 - 2.0104 - 5.9992] = -1.0019$$

$$z^{(6)} = \frac{1}{4}\left[27 - 2x^{(5)} - y^{(5)}\right] = \frac{1}{4}\left[27 - 2(2.0104) - (-1.0019)\right] = 5.9953$$

Iteration 7: $x^{(7)} = \frac{1}{6}\left[4 - 2y^{(6)} + z^{(6)}\right] = \frac{1}{6}\left[4 - 2(-1.0019) + 5.9953\right] = 1.9998$

$$y^{(7)} = \frac{1}{5}\left[3 - x^{(6)} - z^{(6)}\right] = \frac{1}{5}[3 - 2.0005 - 5.9953] = -0.9992$$

$$z^{(7)} = \frac{1}{4}\left[27 - 2x^{(6)} - y^{(6)}\right] = \frac{1}{4}\left[27 - 2(2.0005) - (-1.0019)\right] = 6.0002$$

Iteration 8: $x^{(8)} = \frac{1}{6}\left[4 - 2y^{(7)} + z^{(7)}\right] = \frac{1}{6}\left[4 - 2(-0.9992) + 6.0002\right] = 1.9997$

$$y^{(8)} = \frac{1}{5}\left[3 - x^{(7)} - z^{(7)}\right] = \frac{1}{5}[3 - 1.9998 - 6.0002] = -1$$

$$z^{(8)} = \frac{1}{4}\left[27 - 2x^{(7)} - y^{(7)}\right] = \frac{1}{4}\left[27 - 2(1.9998) - (-0.9992)\right] = 5.9999$$

Iteration 9: $x^{(9)} = \frac{1}{6}\left[4 - 2y^{(8)} + z^{(8)}\right] = \frac{1}{6}\left[4 - 2(-1) + 5.9999\right] = 1.9999$

$$y^{(9)} = \frac{1}{5}\left[3 - x^{(8)} - z^{(8)}\right] = \frac{1}{5}[3 - 1.9997 - 5.9999] = -0.9999$$

$$z^{(9)} = \frac{1}{4}\left[27 - 2x^{(8)} - y^{(8)}\right] = \frac{1}{4}\left[27 - 2(1.9999) - (-1)\right] = 6.00015$$

Iteration 10: $x^{(10)} = \frac{1}{6}\left[4 - 2y^{(9)} + z^{(9)}\right] = \frac{1}{6}\left[4 - 2(-0.9999) + 6.00015\right] = 1.9999$

$$y^{(10)} = \frac{1}{5}\left[3 - x^{(9)} - z^{(9)}\right] = \frac{1}{5}[3 - 1.9999 - 6.00015] = -1.0000$$

$$z^{(10)} = \frac{1}{4}\left[27 - 2x^{(9)} - y^{(9)}\right] = \frac{1}{4}\left[27 - 2(1.9999) - (-0.9999)\right] = 6.0000$$

We observe that in 9^{th} and 10^{th} iteration, the approximation to unknowns are approximately same, so we stop the iterative procedure.

Hence the solution of the system is

x = 1.9999, y = -1.0000, z = 6.0000

Self Check Exercise

Q. 1 Solve the following system of equation by Gauss Seidal Method

10x + y + z = 12

2x + 10y + z = 13

2x + 2y + 10z = 14

Q.2 Solve the following system of equation by Gauss Seidal Method.

10x + y + 2z = 44

2x + 10y + z = 51

x + 2y + 10z = 61 correct to four significant digits?

Q.3 Solve the following system of equation by Gauss Seidal Method correct to four decimal places.

$20x_1 + x_2 - 2x_3 = 17$

$3x_1 + 20x_2 - x_3 = -18$

$2x_1 - 3x_2 + 20x_3 = 25$

9.4 Summary:

Gauss Seidal method is more efficient than Jacobi method as Gauss Seidal method requires less number of iteration to converge to the actual solution with a certain degree of accuracy.

This method also known as method of successive displacement is an iterative method used to solve a system of linear equations.

Gauss Seidal method converges when the matrix A i.e. coefficient matrix is positive definite.

This method is used to solve both linear and non linear algebraic equations.

9.5 Glossary:

- **Equations:** An equation can be defined as statement that supports the equality of two expressions which are connected by equal (=) sign.

  (i) Linear Equation: Equations of 1st order or when the equation has homogeneous variable of degree e.g. 2x - 3 = 0, 2y = 8

  (ii) Non linear Equation: Equation having degree 2 or more than 2 or the equation which is not linear e.g. $x^2 + y^2 = 1$

- **Positive Definite:** A positive definite matrix is a symmetric matrix with all positive eigenvalues

- **Diagonal Dominant:** Diagonal dominant matrix implies

  $$|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}| \text{ for i ‡ j}$$

  Where A (coefficient matrix) = $\left[a_{ij}\right]_{m \times n}$

9.6 Answers to Self Check Exercise

Ans. x = 2.4255, y = 3.5730, z = 1.9260

Ans. x = 3.000, y = 4.000, z = 5.000

Ans. x_1 = 1.0000, x_2 = -1.0000, x_3 = 1.0000

9.7 References/Suggested Readings:

- Introductory Methods of Numerical Analysis by S.S. Sastry
- Introduction to Numerical Analysis by Kendall Atkinson
- Numerical Analysis by J. Douglas Eaires
- Finite Difference and Numerical Analysis by H.C. Saxena

9.8 Terminal Questions

1. Apply Gauss Seidal Method to given system of linear equations.

   $2x_1 - x_2 = 2$

   $x_1 - 3x_2 + x_3 = -2$

   $-x_1 + x_2 - 3x_3 = -6$

2. Solve the following system of equation by Gauss Seidal Method

   $6x + 2y - z = 4$

   $x + 5y + z = 3$

   $2x + y + 4z = 27$

3. Use Gauss Seidal Method to solve the following system of linear equation

   $8x - 3y + 2z = 20$

   $4x + 11y - z = 33$

   $6x + 3y + 12z = 35$

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Unit - 10

SOR Iterative Method

Structure

10.1 Introduction

Dear student in this unit we will learn SOR Method to find the solution of system of linear equations. The method of successive over relaxation (SOR) is a type of Gauss-Seidal method which has faster convergence. In this unit we will learn the algorithm of SOR method and apply SOR method to find the solution of system of linear equations. We can apply SOR method only if the coefficient matrix of given system of linear equations is of positive definite.

10.2 Learning Objectives:

After studying this unit students are able to

1. define SOR method of solving system of linear equations.
2. give algorithm of SOR method.
3. apply SOR method to system of linear equations
4. find solution of system of linear equations by using SOR method.

10.3 SOR Iterative Method

Gauss Seidal Method extrapolation is used to create the successive over approximation. The extrapolation is presented as a weighted average between the newly computed approximation and the previous approximation using the Gauss Seidal approach.

Let $$\hat{x}_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=i} a_{ij}\,\hat{x}_j^{(k+1)} - \sum_{j>i} a_{ij}\,x_j^{(k)}\right) \quad (1)$$

In stead of taking $\hat{x}_i^{(k+1)}$ as the value of x_i in $(k+1)^{th}$ iteration, let us take

$$x_i^{(k+1)} = \omega \hat{x}_i^{(k+1)} + (1 = \omega) x_i^{(k)} \qquad (2)$$

Where ω is the scalar weight factor or relaxation parameter if ω = 1, formula (2) is Gauss Seidal Method.

If ω< 1, formula (2) becomes under relaxation in which the successive approximation move more slowly towards the solution. If ω> 1, formula (2) is called over relaxation in which the successive approximations move aggressively toward.

SOR Iterative Method

Some Examples

Example 1: Solve the following system of equations by using SOR iterative method.

4x + 3y = 24

3x + 4y - z = 30

-y + 4z = -24

Solution: Given system of equations are

4x + 3y = 24

3x + 4y - z = 30

-y + 4z = -24

Let coefficient matrix be A = $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$

Here A is symmetric matrix as A = AT

Also |4| = 4 > 0, $\begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix}$ = 16 - 9 = 7 > 0

and $\begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ = 4 (16 - 1) -3 (12) = 60 - 36 = 24 > 0

i.e. all the leading principle minors are positive So, matrix A is positive definite. Thus, SOR method can be used with relaxation parameter W from the interval 0 < W < 2.

Using Gauss-Seidal method, the successive approximation are given by

$$\left\{\begin{aligned} \hat{x}^{(k+1)} &= \frac{1}{4}\left(24 - 3y^{(k)} - 0z^{(k)}\right) \\ \hat{y}^{(k+1)} &= \frac{1}{4}\left(30 - 3x^{(k+1)} + z^{(k)}\right) \\ \hat{z}^{(k+1)} &= \frac{1}{4}\left(-24 - 0x^{(k+1)} + y^{(k+1)}\right) \end{aligned}\right\} \qquad \text{(i)}$$

BY SOR Method

$$\left\{\begin{aligned} x^{(k+1)} &= \omega\,\hat{x}^{(k+1)} + (1-\omega)\,x^{(k)} \\ y^{(k+1)} &= \omega\,\hat{y}^{(k+1)} + (1-\omega)\,y^{(k)} \\ z^{(k+1)} &= \omega\,\hat{z}^{(k+1)} + (1-\omega)\,z^{(k)} \end{aligned}\right\} \qquad \text{(ii)}$$

Using (i) and (ii), we have

$$x^{(k+1)} = (1-\omega)\,x^{(k)} + \frac{\omega}{4}\left(24 - 3y^{(k)} - 0z^{(k)}\right)$$

$$y^{(k+1)} = (1-\omega)\,y^{(k)} + \frac{\omega}{4}\left(30 - 3x^{(k+1)} + z^{(k)}\right)$$

$$z^{(k+1)} = (1-\omega)\,z^{(k)} + \frac{\omega}{4}\left(-24 - 0x^{(k+1)} + y^{(k+1)}\right)$$

Taking $x^{(0)} = 0$, $y^{(0)} = 0$, $z^{(0)} = 0$ and $w = 1.25$

1st Approximation

$$x^{(1)} = (1-1.25)(0) + \frac{1.25}{4}\left[24 - 3(0) - 0(0)\right] = 7.5$$

$$y^{(1)} = (1-1.25)(0) + \frac{1.25}{4}\left[30 - 3(7.5) + (0)\right] = 2.34375$$

$$z^{(1)} = (1-1.25)(0) + \frac{1.25}{4}\left[-24 - 0(7.5) + (2.34375)\right] = -6.76758$$

2nd Approximation

$$x^{(2)} = (1-1.25)(7.5) + \frac{1.25}{4}\left[24 - 3(2.34375) - 0(-6.76758)\right] = 3.42773$$

$$y^{(2)} = (1-1.25)(2.34375) + \frac{1.25}{4}\left[30 - 3(3.42773) + (-6.76758)\right] = 3.46069$$

$$z^{(2)} = (1-1.25)(-6.76758) + \frac{1.25}{4}\left[-24 - 0(3.42773) + (3.46069)\right] = -4.72664$$

3rd Approximation

$$x^{(3)} = (1-1.25)(3.42773) + \frac{1.25}{4}\left[24 - 3(3.46069) - 0(-4.72664)\right] = 3.39867$$

$$y^{(3)} = (1-1.25)(3.46069) + \frac{1.25}{4}\left[30 - 3(3.39867) + (-4.72664)\right] = 3.8465$$

$$z^{(3)} = (1-1.25)(-4.72664) + \frac{1.25}{4}\left[-24 - 0(3.39867) + (3.8465)\right] = -5.11631$$

4th Approximation

$$x^{(4)} = (1-1.25)(3.39867) + \frac{1.25}{4}\left[24 - 3(3.8465) - 0(-5.11631)\right] = 3.04424$$

$$y^{(4)} = (1-1.25)(3.8465) + \frac{1.25}{4}\left[30 - 3(3.04424) + (-5.11631)\right] = 3.96056$$

$$z^{(4)} = (1-1.25)(-5.11631) + \frac{1.25}{4}\left[-24 - 0(3.04424) + (3.96056)\right] = -4.98325$$

5th Approximation

$$x^{(5)} = (1-1.25)(3.04424) + \frac{1.25}{4}\left[24 - 3(3.96056) - 0(-4.98325)\right] = 3.02592$$

$$y^{(5)} = (1-1.25)(3.96056) + \frac{1.25}{4}\left[30 - 3(3.02592) + (-4.98325)\right] = 3.9908$$

$$z^{(5)} = (1-1.25)(-4.98325) + \frac{1.25}{4}\left[-24 - 0(3.02592) + (3.9908)\right] = -5.00706$$

6th Approximation

$$x^{(6)} = (1-1.25)(3.02592) + \frac{1.25}{4}\left[24 - 3(3.9908) - 0(-5.00706)\right] = 3.00215$$

$$y^{(6)} = (1-1.25)(3.9908) + \frac{1.25}{4}\left[30 - 3(3.00215) + (-5.00706)\right] = 3.99808$$

$$z^{(6)} = (1-1.25)(-5.00706) + \frac{1.25}{4}\left[-24 - 0(3.00215) + (3.99808)\right] = -4.99883$$

7th Approximation

$$x^{(7)} = (1-1.25)(3.00215) + \frac{1.25}{4}\left[24 - 3(3.99808) - 0(-4.99883)\right] = 3.00126$$

$$y^{(7)} = (1-1.25)(3.99808) + \frac{1.25}{4}\left[30 - 3(3.00226) + (-4.99883)\right] = 3.99966$$

$$z^{(7)} = (1-1.25)(-4.99883) + \frac{1.25}{4}\left[-24 - 0(3.00126) + (3.99966)\right] = -5.0004$$

8th Approximation

$$x^{(8)} = (1-1.25)(3.00126) + \frac{1.25}{4}\left[24 - 3(3.99966) - 0(-5.0004)\right] = 3$$

$$y^{(8)} = (1-1.25)(3.99966) + \frac{1.25}{4}\left[30 - 3(3) + (-5.0004)\right] = 3.99996$$

$$z^{(8)} = (1-1.25)(-5.0004) + \frac{1.25}{4}\left[-24 - 0(3) + (3.99996)\right] = -4.99991$$

9th Approximation

$$x^{(9)} = (1-1.25)(3) + \frac{1.25}{4}\left[24 - 3(3.99996) - 0(-4.99991)\right] = 3.00004$$

$$y^{(9)} = (1-1.25)(3.99996) + \frac{1.25}{4}\left[30 - 3(3.00004) + (-4.99991)\right] = 4$$

$$z^{(9)} = (1-1.25)(-4.99991) + \frac{1.25}{4}\left[-24 - 0(3.00004) + (4)\right] = -5.00002$$

Therefore, solution are

x = 3.00004 ≌ 3

y = 4 ≌ 4

z = -5.00002 ≌ 5

Example 2: Solve equations 5x + 2y = 10, 2x + 3y = 4 using SOR method.

Solution: Given system of equations are

5x + 2y = 10

2x + 3y = 4

∴ coefficient matrix A = $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$

Here A = A^T

$\Rightarrow$ A is symmetric matrix

and $\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ = |5 - 4 = 1|

|5| = 5

So, matrix A is positive definite

Thus, SOR method can be used with relaxation parameter W from the interval 0 < W < 2.

Using Gauss-Seidal method, the successive approximation are given by

$$\left\{ \begin{aligned} \hat{x}^{(k+1)} &= \frac{1}{5}\left(10 - 2y^{(k)}\right) \\ \hat{y}^{(k+1)} &= \frac{1}{3}\left(4 - 2x^{(k+1)}\right) \end{aligned} \right\} \qquad \text{(i)}$$

By SOR Method

$$\left\{ \begin{aligned} x^{(k+1)} &= \omega\, \hat{x}^{(k+1)} + (1-\omega)x^{(k)} \\ y^{(k+1)} &= \omega\, \hat{y}^{(k+1)} + (1-\omega)y^{(k)} \end{aligned} \right\} \qquad \text{(ii)}$$

Using (i) and (ii), we have

$$x^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{5}\left(10 - 2y^{(k)}\right)$$

$$y^{(k+1)} = (1-w)y^{(k)} + \frac{w}{3}\left(4 - 2x^{(k+1)}\right)$$

Taking $x^{(0)} = 0$, $y^{(0)} = 0$ and w = 1.25

1st Approximation

$$x^{(1)} = (1-1.25)(0) + \frac{1.25}{5}\left(10 - 2(0)\right) = \frac{1.25}{\cancel{5}}(\cancel{10})2 = 2.5$$

$$y^{(1)} = (1-1.25)(0) + \frac{1.25}{3}\left(4 - 2(2.5)\right) = -0.41666$$

2nd Approximation

$$x^{(2)} = (1-1.25)(2.5) + \frac{1.25}{5}\left(10 - 2(-0.41666)\right) = -0.625 + 2.70833 = 2.08333$$

$$y^{(2)}=(1-1.25)(-0.41666)+\frac{1.25}{3}(4-2(2.08333))=-0.104165-0.06944=0.034723$$

3rd Approximation

$$x^{(3)}=(1-1.25)(2.08333)+\frac{1.25}{5}(10-2(0.034723))=-0.52083+2.48263=1.961808$$

$$y^{(3)}=(1-1.25)(0.034723)+\frac{1.25}{3}(4-2(1.961808))=-0.00868-0.031826=0.023146$$

4th Approximation

$$x^{(4)}=(1-1.25)(1.961808)+\frac{1.25}{5}(10-2(0.0023146))=-0.490452+2.488427=1.997975$$

$$y^{(4)}=(1-1.25)(0.023146)+\frac{1.25}{3}(4-2(1.997975))=-0.0057865-0.0016875=-0.004099$$

5th Approximation

$$x^{(5)}=(1-1.25)(1.997975)+\frac{1.25}{5}(10-2(-0.004099))=-0.49949+2.502049=2.002559$$

$$y^{(5)}=(1-1.25)(-0.004099)+\frac{1.25}{3}(4-2(2.002559))=-0.0010247-0.0021325=-0.001107$$

6th Approximation

$$x^{(6)}=(1-1.25)(2.002559)+\frac{1.25}{5}(10-2(0.001107))=-0.50063975+2.5005535=1.9999137$$

$$y^{(6)}=(1-1.25)(-0.001107)+\frac{1.25}{3}(4-2(1.9999137))=0.00036$$

7th Approximation

$$x^{(7)}=(1-1.25)(1.9999137)+\frac{1.25}{5}(10-2(0.00036))=1.999845$$

$$y^{(7)}=(1-1.25)(0.00036)+\frac{1.25}{3}(4-2(1.999845))=0.000077$$

8th Approximation

$$x^{(8)} = (1-1.25)(1.999845) + \frac{1.25}{5}\left(10 - 2(0.000077)\right) = 2.00000025$$

$$y^{(8)} = (1-1.25)(0.000077) + \frac{1.25}{3}\left(4 - 2(2.00000025)\right) = -0.00001925$$

Hence, the Solution are

x = 2.00000025 ≌ 2

y = -0.00001925 ≌ 0

Example 3: Consider the system of equations

10x + 2y - z = 7

x + 8y + 3z = -4

-2x - y + 10z = 9

Using (0, 0, 0) as initial approximation and ω = 1.25, solve the given system of equations by SOR method.

Solution: Given system of equations are

0x + 2y - z = 7

x + 8y + 3z = -4

2x - y + 10z = 9

Using Gauss Seidal method the successive approximation are given by

$$\left\{\begin{aligned} \hat{x}^{(k+1)} &= \frac{1}{10}\left(7 - 2y^{(k)} + z^{(k)}\right) \\ \hat{y}^{(k+1)} &= \frac{1}{8}\left(-4 - x^{(k+1)} - 3z^{(k)}\right) \\ \hat{z}^{(k+1)} &= \frac{1}{10}\left(9 + 2x^{(k+1)} + y^{(k+1)}\right) \end{aligned}\right\} \qquad \text{(i)}$$

By SOR Method

$$\left\{\begin{aligned} x^{(k+1)} &= w\,\hat{x}^{(k+1)} + (1-\omega)\,x^{(k)} \\ y^{(k+1)} &= w\,\hat{y}^{(k+1)} + (1-w)\,y^{(k)} \\ z^{(k+1)} &= w\,\hat{z}^{(k+1)} + (1-w)\,z^{(k)} \end{aligned}\right\} \qquad \text{(ii)}$$

Given $x^{(0)} = y^{(0)} = z^{(0)} = 0$ and w = 1.25

Using (i) and (ii) we have

$$x^{(k+1)} = (1-\omega)x^{(k)} + \frac{\omega}{10}\left(7 - 2y^{(k)} + z^{(k)}\right)$$

$$y^{(k+1)} = (1-\omega)y^{(k)} + \frac{\omega}{8}\left(-4 - x^{(k+1)} - 3z^{(k)}\right)$$

$$z^{(k+1)} = (1-\omega)z^{(k)} + \frac{\omega}{10}\left(9 + 2x^{(k+1)} + y^{(k+1)}\right)$$

1st Approximation:

$$x^{(1)} = (1-1.25)(0) + \frac{1.25}{10}\left(7 - 2(0) + (0)\right) = 0.875$$

$$y^{(1)} = (1-1.25)(0) + \frac{1.25}{8}\left(-4 - 0 - 3(0)\right) = -0.7617$$

$$z^{(1)} = (1-1.25)(0) + \frac{1.25}{10}\left(9 + 2(0) + (0)\right) = 1.2485$$

2nd Approximation:

$$x^{(2)} = (1-1.25)(0.875) + \frac{1.25}{10}\left(7 - 2(-0.7617) + (1.2485)\right) = 1.0027$$

$$y^{(2)} = (1-1.25)(-0.7617) + \frac{1.25}{8}\left(-4 - (1.0027) - 3(1.2485)\right) = -1.1765$$

$$z^{(2)} = (1-1.25)(1.2485) + \frac{1.25}{10}\left(9 + 2(1.0027) + (1.1765)\right) = 0.9165$$

3rd Approximation:

$$x^{(3)} = (1-1.25)(1.0027) + \frac{1.25}{10}\left(7 - 2(-1.1765) + (0.9165)\right) = 1.033$$

$$y^{(3)} = (1-1.25)(-1.1765) + \frac{1.25}{8}\left(-4 - (1.033) - 3(0.9165)\right) = -0.9219$$

$$z^{(3)} = (1-1.25)(0.9165) + \frac{1.25}{10}\left(9 + 2(1.033) + (-0.9219)\right) = 1.0389$$

4^{th} Approximation:

$$x^{(4)} = (1-1.25)(1.033) + \frac{1.25}{10}\left(7 - 2(-0.9219) + (1.0389)\right) = 0.9771$$

$$y^{(4)} = (1-1.25)(-0.9219) + \frac{1.25}{8}\left(-4 - (0.9771) - 3(1.0389)\right) = -1.0342$$

$$z^{(4)} = (1-1.25)(1.0389) + \frac{1.25}{10}\left(9 + 2(0.9771) + (-1.0342)\right) = 0.9803$$

5^{th} Approximation:

$$x^{(5)} = (1-1.25)(0.977) + \frac{1.25}{10}\left(7 - 2(-1.0342) + (0.9803)\right) = 1.0118$$

$$y^{(5)} = (1-1.25)(-1.0342) + \frac{1.25}{8}\left(-4 - (1.0118) - 3(0.9803)\right) = -0.9841$$

$$z^{(5)} = (1-1.25)(0.9803) + \frac{1.25}{10}\left(9 + 2(1.0118) + (-0.9841)\right) = 1.0099$$

6^{th} Approximation:

$$x^{(6)} = (1-1.25)(1.0118) + \frac{1.25}{10}\left(7 - 2(-0.984) + (1.0099)\right) = 0.9943$$

$$y^{(6)} = (1-1.25)(-0.9841) + \frac{1.25}{8}\left(-4 - (0.9943) - 3(1.0099)\right) = -1.0077$$

$$z^{(6)} = (1-1.25)(1.0099) + \frac{1.25}{10}\left(9 + 2(0.9943) + (-1.0077)\right) = 0.9951$$

7^{th} Approximation:

$$x^{(7)} = (1-1.25)0.9943 + \frac{1.25}{10}\left(7 - 2(-1.0077) + (0.9951)\right) = 1.0027$$

$$y^{(7)} = (1-1.25)(-1.0077) + \frac{1.25}{8}\left(-4 - (1.0027) - 3(0.9951)\right) = -0.9962$$

$$z^{(7)} = (1-1.25)(0.9951) + \frac{1.25}{10}\left(9 + 2(1.0027) + (-0.9962)\right) = 1.0024$$

8th Approximation:

$$x^{(8)} = (1-1.25)(1.0027) + \frac{1.25}{10}\left(7 - 2(-0.9962) + (1.0024)\right) = 0.9987$$

$$y^{(8)} = (1-1.25)(-0.9962) + \frac{1.25}{8}\left(-4 - (0.9987) - 3(1.0024)\right) = -1.0018$$

$$z^{(8)} = (1-1.25)(1.0024) + \frac{1.25}{10}\left(9 + 2(0.9987) + (-1.0018)\right) = 0.9988$$

9th Approximation:

$$x^{(9)} = (1-1.25)(0.9987) + \frac{1.25}{10}\left(7 - 2(-1.0018) + (0.9988)\right) = 1.0007$$

$$y^{(9)} = (1-1.25)(-1.0018) + \frac{1.25}{8}\left(-4 - (1.0007) - 3(0.9988)\right) = -0.9991$$

$$z^{(9)} = (1-1.25)(0.9988) + \frac{1.25}{10}\left(9 + 2(1.0007) + (-0.9991)\right) = 1.0006$$

10th Approximation:

$$x^{(10)} = (1-1.25)(1.0007) + \frac{1.25}{10}\left(7 - 2(-0.9991) + (1.0006)\right) = 0.9997$$

$$y^{(10)} = (1-1.25)(-0.9991) + \frac{1.25}{8}\left(-4 - (0.9997) - 3(1.0006)\right) = -1.0004$$

$$z^{(10)} = (1-1.25)(1.0006) + \frac{1.25}{10}\left(9 + 2(0.9997) + (-1.0004)\right) = 0.9997$$

11th Approximation:

$$x^{(11)} = (1-1.25)(0.9997) + \frac{1.25}{10}\left(7 - 2(-1.0004) + (0.9997)\right) = 1.0002$$

$$y^{(11)} = (1-1.25)(-1.0004) + \frac{1.25}{8}\left(-4 - (1.0002) - 3(0.9997)\right) = -0.9998$$

$$z^{(11)} = (1-1.25)(0.9997) + \frac{1.25}{10}\left(9 + 2(1.0002) + (-0.9998)\right) = 1.0001$$

∴ Solutions by SOR method are

x = 1.0002 ≌ 1

y = -0.998 ≌ 1

z = 1.0001 ≌ 1

Self Check Exercise

Q.1 Consider the system of equations

3x - y + z = -1

-x + 3y - z = 7

x - y + 3z = -7

Using (0, 0, 0) as initial approximations and ω = 1.25, solve the given system of equations by SOR method.

Q.2 Solve using SOR method the following system of equations.

2x + y + z = 5

3x + 5y + 2z = 15

2x + y + 4z = 8

10.4 Summary:

SOR (Successive Over Relaxation) is a variant of the Gauss-Seidal method for solving a linear system of equations, resulting in faster convergence.

It is a method of solving a linear system of equations Ax = b derived by extrapolating the Gauss Seidal Method.

The extrapolation takes the form of a weighted average between the previous iterate and the computed Gauss Seidal iterate successively for each component

$$x_i^k = \omega \hat{x}_i^{(k+1)} + (1-\omega)\, x_i^k$$

10.5 Glossary:

- **Extrapolation:** It is about predicting hypothetical values that fall outside a particular data set.
- **Gauss Seidal Method:** It is a specific iterative method that is always using the latest estimated value for each element in x.

10.6 Answers to Self Check Exercise

1. 1.0002 ≌, 2.00005 ≌ 2, -20001 ≌ -2
2. x ≌ 1, y ≌ 2, z ≌ 1

10.7 References/Suggested Readings

1. Introductory Methods of Numerical Analysis by S.S. Sastry
2. Introduction to Numerical Analysis by Kendall Atkinson

3. Numerical Analysis by J. Douglas Eaires

4. Finite Difference and Numerical Analysis by H.C. Saxena

10.8 Terminal Questions

1. Solve using SOR method the following system of equations

   $45x_1 + 2x_2 + 3x_3 = 58$

   $-3x_1 + 22x_2 + 2x_3 = 47$

   $5x_1 + x_2 + 20x_3 = 67$

2. Consider the system of equation

   $4x - y - 2z = -3$

   $-x + 3y + 2z = 4$

   $-2x + 2y + 6z = -3$

   Using (0, 0, 0) as initial approximation and $\omega = 1.25$, solve the given system of equations by SOR method.

3. Find the 1st two iteration of the SOR method with $\omega = 1.1$ for the following system of linear system using $x^{(0)} = y^{(0)} = z^{(0)} = 0$

   $4x + y - z = 5$

   $-x + 3y + z = -4$

   $2x + 2y + 5z = 1$

*****

Unit - 11

Interpolation

Structure

11.1 Introduction

Dear student, in this unit we will learn about interpolation. Interpolation is a process of finding a unknown value that lie in between the known data point. It is a special case of the general process of curve fitting. In this unit we will learn about interpolation, and different operators which are lesed in interpolation. We will learn the relationship between these operations also. We will learn to write forward and backward difference table of given set of data. We will also learn how to interpolate a unknown value from a given set of data.

11.2 Learning Objectives:

After studying this unit students will be able to

1. define interpolation
2. define forward difference operator and backward difference operator, shift operator, average operator and control operator.
3. define control difference operator and mean operator
4. write forward and backward difference table for a given problem.
5. Interpolate a value from a given data using forward or backward operators.

11.3 Interpolation

Interpolation is a method used to estimate unknown values that fall between known values. It is commonly used in mathematics, statistics and computer graphics to make predictions or generate new data points within the range of a discrete set of known data points often we have a set of data points but need to estimate values at positions not explicitly provided by the data. Interpolation provides a means to do this ensuring a smoother transition between points and a more continuo's representation of data.

Basic Idea:

Interpolation finding a function $f(x)$ that passes through a set of known data points (x_0, y_0), (x_1, y_1),.... (x_n, y_n) and using this function to estimate values of y for any x within the range of known points.

Common methods of Interpolation:

Some methods for interpolation are:

1. Linear Interpolation
2. Polynomial Interpolation
3. Spline Interpolation
4. Nearest Neighbour Interpolation
5. Bilinear and Bicubic Interpolation
6. Kringing

Basic Principle of Interpolation:

Here are the basic principles of interpolation:

1. **Assumption of Continuity:** Interpolation assumes that the function or data set is continuous between the known data points. This continuity implies that there are no abrupt changes or jumps in the data with in the interval.
2. **Use of Known Data Points:** Interpolation uses the values of known data points to estimate unknown values. The accuracy of interpolation depends heavily on the distribution and quality of these known points.
3. **Locality:** Most interpolation methods rely on local information. They use data points in the vicinity of the estimation points to make prediction, ensuring that the estimate reflects local variation and trends.
4. **Smoothness:** Many interpolation methods aim to create a smooth curve or surface through the data points. Smoothness helps in providing a more realistic model of the underlying phenomena being studied.
5. **Piecewise Polynomial:** Some interpolation methods, Like spine interpolation, use piecewise polynomials. These are polynomial function defined on subintervals of the data range, ensuring that they fit smoothly together at the data point.
6. **Weighted Average:** Certain interpolation technique Like Kriging use weighted averages of the known data points. The weight are determined based on the distance from the estimation points with closer points generally given more weight.

7. **Error Minimization:** Effective interpolation methods aim to minimize the error between the estimated values and the actual values if they were known. The goal is to provide the most accurate estimation possible given the known data.

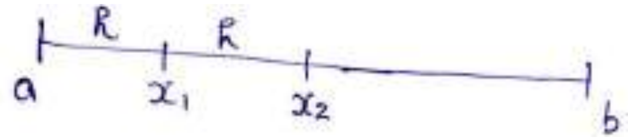
8. **Dimensionality:** Interpolation methods can be one dimensional, two dimensional or even higher dimensional depending on the nature of the data.

Thus, Interpolation is a powerful tool for estimating unknown values within the range of known data. By understanding and applying the basic principles of continuity, locality Smoothness and error minimization one can choose and implement the appropriate interpolation method.

11.4 Finite Difference Operator

The finite difference operator can be classified into three types. Forward Δ, Backward ∇s, Center δ, Shift E, and Average H, in order to understand the difference operators, we divide the interval [a, b], as follows:

x	x_0	x_1	x_2	x_3	x_4
$y=f(x)$	y_0	y_1	y_2	y_3	y_4



Where h is the distance between any two points

1. Forward difference operator Δ:- The forward difference operator is defined as follows

   $\Delta y_i = \Delta f(x_i) = F(x_i + h) - F(x_i) = F(x_{i+1}) - F(x_i) = y_{i+1} - y_i$, i = 0,1,2,......

   $\Delta^2 y_i = \Delta^2 F(x_i) = \Delta(\Delta F(x_i)) = \Delta(F(x_{i-1})) - F(x_i)) = \Delta F(x_{i+1}) - \Delta F(x_i)$

   $= F(x_{i+2}) - F(x_{i+1}) - F(x_{i+1}) + F(x_i)$

   $= y_{i+2} - 2y_{i+1} + y_i$

   By the same way we can find $\Delta^3 y i$, $\Delta^4 y i$. . . . .

   In General, $\Delta^i yr = \Delta^{j-1} yr+1 - \Delta^{j-1} yr$

2. Backward difference operator: The Bad<Ward difference operator is defined as follows:

   $\Delta yi = \nabla F(xi) - F(xi) - F(xi-1) = y_i - y_{i-1}$; i = 1, 2....., n.

   $\nabla^2 yi = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1} = y_i - y_{i-1} - y_{i-1} + y_i$

   $= y_i - 2y_{i-1} + y_{i-2}$

   In general, $\nabla^j yr = \nabla^{j-1} yr - \nabla^{j-1} y^{r-1}$

3. Central Difference operator: The center difference operator is defined as follows:

δyi = δf(xi) = $f\left(xi+\frac{h}{2}\right)$ - $f\left(xi+\frac{h}{2}\right)$ = yi + $\frac{1}{2}$ - yi - $\frac{1}{2}$

δ^2yi = $\delta\left(y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}}\right)$ = $\delta\, y_{i+\frac{1}{2}}$ - $y_{i-\frac{1}{2}}$ = $y_{i+1} - y_i - y_i + y_{i-1}$

= $y_{i+1} - 2y_i + y_{i-1}$

In general, δ2yr = $\delta y_{\frac{r+1}{2}}$ - $\delta y_{\frac{r-1}{2}}$

There are two other useful operators namely shift operator and average operator.

Shift Operator

It is defined as, for any integer

$E^n y_r = y_{r+n}$

For example $E^5 y_0 = y_{0+5} = y_5$

$E^2 y_0 = y_{0+2} = y_2$

$E^{-2} y_3 = y_{3-2} = y_1$

Avepace operator

The average operator is given by μ and is defined as

$\mu y_r = \frac{1}{2}[y_{r-1/2} + y_{r+1/2}]$

For example

$\mu y_1 = \frac{1}{2}[y_{1-1/2} + y_{1+1/2}]$

$= \frac{1}{2}[y_{-1/2} + y_{3/2}]$

$\mu y_2 = \frac{1}{2}[y_{2-1/2} + y_{2+1/2}]$

$= \frac{1}{2}[y_{3/2} + y_{5/2}]$

Relationship between Operators

1. Relation between Δ and E

We know that

$\Delta y_r = y_{r+1} - y_r$

$\Delta y_r = Ey_r - y_r$

$\Delta y_r = (E-1)\, y_r \qquad \because |Ey_r = y_{r+1}|$

$\Delta \equiv E-1$

2. **Relation between ∇ and E**

We know that

$\nabla y_r = y_r - y^{r-1}$

$\nabla y_r = y_r - E^{-1}y_r \quad \because |E^{-1}y_r = y_{r-1}|$

$\nabla y_r = (1-E^{-1})y_r$

$\nabla \equiv 1 - E^{-1}$

3. **Relation between δ and E**

We know that

$\delta y_r = y_{\frac{r+1}{2}} - y_{\frac{r-1}{2}}$

$\delta y_r = E^{1/2} yr - E^{-1/2} y_r$

$\delta y_r = \left(E^{1/2} - E^{-1/2}\right) y_r$

$\delta \equiv E^{1/2} - E^{-1/2}$

4. **Relation between μ and E**

We know that

$$\mu y_r = \frac{1}{2}\left(y\frac{yr+1}{2} + y\frac{r-1}{2}\right)$$

$$\mu y_r = \frac{1}{2}\left(E^{1/2}y_r + E^{-1/2}y_r\right)$$

$$\mu y_r = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) yr$$

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$$

Difference Table or Display:

The difference table corresponding to each type of finite difference are given below:

(i) Forward Difference Table:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

(ii) Backward Difference Table:-

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_1			
x_1	y_1		$\Delta^2 y_2$		
		Δy_2		$\Delta^3 y_3$	
x_2	y_2		$\Delta^2 y_3$		$\Delta^4 y_4$
		Δy_3		$\Delta^3 y_4$	
x_3	y_3		$\Delta^2 y_4$		
		Δy_4			
x_4	y_4				

(iii) Central Difference table:-

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^2 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^2 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Notes:-

(1) In Forward difference table, the subscript remain same each forward diagonal i.e. $y_0,\ \Delta y_0,\ \Delta^2 y_0,\ \Delta^3 y_0,\ \Delta^4 y_0$

(2) In backward difference table, the subscript remain along each backward diagonal i.e.

$y_4, \nabla y_4, \nabla^2 y_4, \nabla^3 y_4, \nabla^4 y_4$

(3) In central difference table, the subscript remain same alone horizontal line i.e.

$y_2, \delta^2 y_2, \delta^4 y_2$

(4) From above tables it is clear that is

$\Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$

Important

(5) If $f(x)$ is a polynomial of degree n then n^{th} order differences are constant and $(n+1)^{th}$ and higher order differences are all zero.

(6) Also if n^{th} differences of a tabulated function are constant and $(n+1)^{th}$ and higher order differences are zero, then the tabulated function represents a poly nominal of degree n.

Some Related Questions

Let us try to understand more about interpolation using forward, backward operation by following examples.

Example 1: Write forward difference table for the following data

x:	0	1	2	3	4
$f(x)$:	1	2	7	13	21

Solution: The forward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^3 f(x)$	$\Delta^4 y$
0 1 2 3 4	1 3 7 13 21	3−1=2 7−3=4 13−7=6 21−13=8	4−2=2 6−4=2	2−2=0

Example 2: Given the function $f(x) = x^2$, use the forward difference operator to approximate the derivative at x = 1 with a step size h = 0.1

Solution: We know forward difference operator is given us:

$$\Delta f(x) = f(x+h) - f(x)$$

$\therefore$ $f(1) = (1)^2 = 1$

$f(1+0.1) = (1.1)^2 = 1.21$

$\Rightarrow$ $\Delta f(1) = f(1.1) - f(1) = 1.21 - 1 = 0.21$

Now, approximating the derivative using the forward difference

$$f'(1) \approx \frac{\Delta f(1)}{h} = \frac{0.21}{0.1}$$

$$= 2.1$$

Example 3: Construct the backward difference table for the data

x:	1	2	3	4	5
$f(x)$:	1	8	27	64	125

Solution: The backward difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	1				
		8−1=7			
2	8		19−7=12		
		27−8=19		18−12=6	
3	27		37−19=18		6−6=0
		62−22=37		24−18=6	
4	64		61−37=24		
		125−64=61			
5	125				

Example 4: By constructing a difference table and using the second order difference as constant, find the sixth term of the series 8, 12, 19, 29, 42,........?

Solution: Let K be the sixth term of the series in the difference table.

The Forward difference table is

x	f(x)	Δ y	Δ^2y
1	8	12−8=4	
2	12	19−12=7	3
3	19	29−19=10	3
4	29	42−29=13	3
5	42	K−42	K−55
6	K		

As second difference are constant

$\Rightarrow$ K - 55 = 3

$\Rightarrow$ $\boxed{K=58}$

$\therefore$ Sixth term of the series is 58.

Example 5: Construct a forward difference table for y = $f(x) = x^3 + 2x + 1$ for x = 1, 2, 3, 4, 5.

Solution: y = $f(x) = x^3 + 2x + 1$ for x = 1, 2, 3, 4, 5.

x	y	Δ y	Δ^2y	Δ^3y	Δ^4y
1	4	9			
2	13	21	12	6	
3	34	39	18	6	0
4	73	63	24		
5	136				

Self Check Exercise

Q.1 Write forward difference table for the following data:

x:	1	2	3	4
f(x):	2	5	9	14

Q.2 Construct a backward difference table for the following data

x:	0	10	20	30
y:	0	0.174	0.347	0.518

Q.3 Find (i) Δe^{ax} (ii) $\Delta^2 ex$ (iii) $\Delta \log x$

11.5 Summary

- Forward Difference $\Delta f(x) = f(x+h) - f(x)$
- Backward Difference $\nabla f(x) = f(x) - f(x+h)$
- Both operators are used to approximate derivatives and solve differential equation numerically
- Thus, these operators are used in numerical analysis and finite difference methods to approximate derivative of function.
- Both forward and backward difference approximations have a truncation error of order 0(h), meaning they are first order accurate.

11.6 Glossary

- **Finite Difference:** A numerical method for estimating derivatives by using values of the function at specific points.
- **Step Size (h):** The interval between points in the finite
- **Truncation Error:** The difference between the true derivative and the finite difference approximation. It is of order o(h) for forward and backward difference.

11.7 Answers to Self Check Exercise

(1)

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	2			
		3		
2	5		1	
		4		0
3	9		1	
		5		
4	14			

(2)

x	f(x)	Δf(x)	$\Delta^2 f$(x)	$\Delta^3 f$(x)
0	0			
		0.174		
10	0.174		−0.001	
		0.173		−0.001
20	0.173		−0.002	
		0.171		
30	0.518			

(3) (i) $e^{ax}\left[e^h - 1\right]$ (ii) $\left[e^h - 1\right]^2$, e^x (iii) $\log 1+\frac{h}{x}$

11.8 References/Suggested Readings

1. Numerical Analysis by Richard L. Burden
2. Introductory Methods of Numerical Analysis by S.S. Sastry.
3. Finite Difference & Numerical Analysis by H.C. Saxena.
4. An Introduction to Numerical Analysis by Kendall E. Atkinson.

11.9 Terminal Question

1. Given the Sequence $\{fn\} = \langle 0,1,4,9,16 \rangle$, compute the first and second forward difference.

2. Construct the backward difference table for the data:

x:	1	2	3	4
f(x):	63	52	43	38

3. Evaluate $\Delta^2\left(\frac{1}{x}\right)$ by taking 1 as the interval of differencing.

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Unit - 12

Newton's Interpolation

Structure

12.1 Introduction

Dear student, in this unit we will study about the Newton's interpolation formula, related to forward differences backward differences and divided differences, using these interpolation formula we can interpolation a given data at starting, ending or central point, using these interpolation formula we can also find the polynomial using the given data. In this unit we will study to evaluate Newton's forward and backward formula to interpolate a value at starting and end point respectively. We will also learn to find to polynomial using these values. Newton's forward and backward interpolation is used when the intervals are equal, whereas Newton's divided difference is used for unequal intervals.

12.2 Learning Objectives:

After studying this unit, students will be able to

1. define Newton's forward difference interpolation formula.
2. Newton's backward difference interpolation formula.
3. apply Newton's forward and backward difference interpolation formula to find polynomial of desire degree, and interpolating a given value.
4. define, Newton's divided difference interpolation formula.
5. apply Newton's divided difference formula whenever required.

12.3 Interpolation

Interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable, while the process of computing the value of the function outside the given range is called extrapolation.

Forward Difference:

The differences $y_1 - y_0$, $y_2 - y$, $y_3 - y_2$ ..... $y_n - y_{n-1}$ when denoted by Δy_0, Δy_1, Δy_2, ... Δy_{n-1} respectively, called the first forward differences.

i.e. $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y$, ... $\Delta y_{n-1} = y_n - y_{n-1}$

in general $\Delta y_r, = y_{r+1} - y_r$

The differences of first order differences are called second order differences and are denoted by $\Delta^2 y_0$, $\Delta^2 y_1$, ... $\Delta^2 y_{n-2}$

i.e. $\Delta^2 y_0 = \Delta y_1 - y_0$, $\Delta^2 y_1 = \Delta y_2 - y_1$, ... $\Delta^2 y_{n-2} = \Delta y_{n-1} - y_{n-2}$

in general $\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r$

Similarly, we can define the third order difference, fourth order difference etc.

Thus, forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0	Δy_0	$\Delta^2 y_0$			
$x_1 (=x_0 + h)$	y_1	Δy_1	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$	
$x_2 (=x_0 + 2h)$	y_2	Δy_2	$\Delta^2 y_2$	$\Delta^3 y_1$	$\Delta^2 y_1$	$\Delta^5 y_0$
$x_3 (=x_0 + 3h)$	y_3	Δy_3	$\Delta^2 y_3$	$\Delta^2 y_2$		
$x_4 (=x_0 + 4h)$	y_4	Δy_4				
$x_5 (=x_0 + 5h)$	y_5					

Newton's Forward Interpolation Formula

$$F(a + hu) = y_0 + y\ \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + ...+ \frac{u(u-1)(u-2)....(u-n-1)}{n!}\Delta^n y_0 \text{ or } f(x)$$

This formula is particularly useful for interpolating the values of $f(x)$ near the beginning of the set of values given, h is called the interval of difference and $u = \frac{x-a}{h}$ here a is the 1st term.

Backward Difference :

In the differences $y_1 - y_0$, $y_2 - y_1$, ... $y_n - y_{n-1}$ are denoted by Δy_1, Δy_2, ...Δy_n respectively where Δ is backward difference operator, then these differences are called first backward differences.

i.e. $\Delta y_1 = y_1 - y_0$, $\Delta y_2 = y_2 - y_1$,..... $\Delta y_n = y_n - y_{n-1}$

in general $\Delta y_r, = y_r - y_{r-1}$

The differences of first order backward difference are called second order backward differences and are denoted by $\Delta^2 y_2$, $\Delta^2 y_3$, ... $\Delta^2 y_n$

i.e. $\Delta^2 y_2 = \Delta y_2 - \Delta y_1$,

$\Delta^2 y_3 = \Delta y_3 - \Delta y_2$

⋮

$\Delta^2 y_n = \Delta y_n - y_{n-1}$

in general $\Delta^2 y_r = \Delta y_r - \Delta y_{r-1}$

Similarly, we can define third order backward differences, fourth order backward differences etc.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_1			
x_1	y_1		$\Delta^2 y_2$		
		Δy_2		$\Delta^3 y_3$	
x_2	y_2		$\Delta^2 y_3$		$\Delta^4 y$
		Δy_3		$\Delta^3 y_4$	
x_3	y_3		$\Delta^2 y_4$		
		Δy_4			
x_4	y_4				

Newton's Backward Interpolation Formula

$$f(x) = y_n + u\Delta y_n + \frac{u(u-1)}{2!}\Delta^2 y_n + ...+ \frac{u(u+1).....(u+(n-1)}{n!}\Delta^n y_n \text{ or } f(an + uh)$$

This formula is useful when the value of $f(x)$ is required near the end of the table, h is called the interval of difference and u = $\frac{x-a_n}{h}$ here an is the last term.

Some examples

Example (1) : Using Newton's Forward interpolation formula find the cubic polynomial.

x :	0	1	2	3
F(x) :	1	2	1	10

Solution: The forward interpolation is

$$y\,(x = x_0 + nh) = y_0 + \frac{n}{1!}\,\Delta y_0 + \frac{n(n-1)}{2!}\,\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\,\Delta^3 y_0 + \ldots\ldots$$

Here $x_0 + nh = x \Rightarrow x_0 = 0$, $h = 1$

$\therefore\ 0 + n = x \Rightarrow n = x$

The difference table is

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		1		
1	2		−2	
		−1		12
2	1		10	
		9		
3	10			

$$\therefore \quad y_{(n-x)} = y_0 + \frac{n}{1!}\,\Delta y_0 + \frac{n(n-1)}{2!}\,\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\,\Delta^3 y_0 + \ldots\ldots$$

$$\Rightarrow \quad y = 1 + \frac{x}{1!}\,(1) + \frac{(x(x-1)(-2)}{2!} + \frac{x(x-1)(x-2)}{6}\,(12)$$

$\Rightarrow \quad y = 1 + x + (x^2 - x)\,(-1) + x\,(x^3 - 3x + 2)\,(2)$

$\Rightarrow \quad y = 1 + x - x^2 + x + 2x^2 - 6x^2 + 4x$

$\Rightarrow \quad y = 1 + 6x - 7x^2 + 2x^3$

Hence the cubic polynomial is

$2x^3 - 7x^2 + 6x + 1$

Example (2) : The population of a city in a censes taken once in 10 years is given below. Estimate the population in the year 1955.

Year	1951	1961	1971	1981
Population in Lakhs	35	42	58	84

Solution : Given

Year (x) :	1951	1961	1971	1981
Population in Lakhs :	35	42	58	84

Since, the population in the year 1955 is asked, which is near the beginning of the table, we have to follow Newton's forward interpolation formula.

$x_0 + nh = 1955 \Rightarrow 1951 + n(10) = 1955$ $\therefore$ h = 10

$\Rightarrow$ 10n = 1955 - 1951 = 4

$\Rightarrow$ $n = \frac{4}{10} = 0.4$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1951	35			
1961	42	7		
1971	58	16	9	
1981	84	26	10	1

Newton's forward interpolation formula is

$$y(x = x_0 + nh) = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0$$

$\Rightarrow$ $y(x = 1955) = 35 + 0.4(7) + \frac{(0.4)(0.4-1)}{2}(9) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(1)$

$\Rightarrow$ $y = 35 + 2.8 + 0.2(-0.6)(9) + \frac{(0.4)(-0.6)(-1.6)}{6}$

$\Rightarrow$ $y = 35 + 2.8 - 1.08 + 0.064$

$\Rightarrow$ $y(x = 1955) = 36.784$

Hence, the population in the year 1955 is 36.784 lakhs.

Example (3) : In an examination the number of candidates who secured marks between certain intervals were as follows :

Marks	0-19	20-39	40-59	60-79	80-99

No. of Candidates	41	62	65	50	17

Estimate the number of candidates whose marks are less than 70.

Solution :

Given

Marks	0-19	20-39	40-59	60-79	80-99
No. of Candidates	41	62	65	50	17

This can be rewritten as

Marks	Below 19	Below 39	Below 59	Below 79	Below 99
No. of candidates	41	41+62=103	41+62+65 = 168	41+62+65 +50 =218	41+62+65 +50+17=235

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
Below 19	41			
Below 39	103	62	3	0
Below 59	168	65	-15	
Below 79	218	50	-33	
Below 99	235	17		

Since, we have to find below 70, use Newton's backward interpolation formula

$\therefore$ x n+hk = 70 $\Rightarrow$ 99 + n(20) = 70

$\Rightarrow$ 20n = 70 - 99 = -29

$\Rightarrow$ $n = \frac{-29}{20} = -1.45$

$$\therefore \quad y_{70} = y_n + \frac{n}{1!}\Delta y_n + \frac{n(n-1)}{2!}\Delta^2 y_n + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_n$$

$$= 235 - 1.45(17) + \frac{(-1.45)(-1.45+1)}{2}(-33) + \frac{(-1.45)(-1.45+1)(-1.45+2)(-18)}{6}$$

$$= 235 - 24.65 + \frac{(-1.45)(1-.45)}{2}(-33) + (-1.45)(-0.45)(0.55)(-3)$$

= 235 - 24.65 - 10.76625 - 1.76625

= 235 - 35.49

= 198.5

Hence the number of students who have scored below 70 are 198.

Example (4) : Find the value of f(x) when x = 32 from the following table

x	30	35	40	45	50
f(x)	15.9	14.9	14.1	13.3	12.5

Solution : Since, 32 is near the beginning of the table, we use the Newton's forward interpolation formula.

$$\therefore y(x = x_0 + nh) = y_0 + \frac{n}{1!} \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + ...$$

$\therefore$ x0 + nh = 32 $\Rightarrow$ 30 + n(5) = 32

$\Rightarrow$ 5n = 2

$\Rightarrow$ n = $\frac{2}{5}$ = 0.4

The difference table is

x	y	Δ y	Δ^2y	Δ^3y	Δ^4y
30	15.9				
35	14.9	-1			
40	14.1	-0.8	0.2		
45	13.3	-0.8	0	-0.2	
50	12.5	-0.8	0	0	0.2

$$\therefore \quad y_{(x=32)} = 15.9 + 0.4\,(-1) + \frac{(0.4)(0.4-1)(0.2)}{2} + \frac{(0.4)(0.4-1)(0.4-2)(-0.2)}{3!}$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.43)(0.2)}{4!}$$

$$= 15.9 - 0.4 + (0.2)\,(-0.6)\,(0.2) + \frac{(0.4)(-0.6)(-1.6)(-0.2)}{6} +$$

$$\frac{(0.4)(-0.6)(-1.6)(-2.6)(0.2)}{24}$$

= 15.9 - 0.4 - 0.24 - 0.0128 - 0.002476

= 15.245

Hence, when x = 32, F(x) = 15.45

Example 5: The following data, gives the meting point of a alloy of lead and zinc where 't' is the temperature in degree C and P is the percentage of lead in the alloy

P	40	50	60	70	80	90
T	180	204	226	250	276	304

Find the melting point of the alloy containing 84 percent lead.

Solution: Let the percentage of lead be x and temperature be y.

Given

P(x)	40	50	60	70	80	90
T(y)	180	204	226	250	276	304

Here we have to find y when x = 84. Since, the temperature required is at the end of the table we apply Newton's Backward interpolation formula.

$\therefore$ x0 + nh = 84

$\Rightarrow$ 90 + n (10) = 84

$\Rightarrow$ 10n = 84 -90 = -6

$\Rightarrow$ n = $\frac{-6}{10}$ = -0.6

The difference table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
40	180					
50	204	24				
60	226	22	-2			
70	250	24	2	4		
80	276	26	2	0	-4	
90	304	28	2	0	0	4

$\therefore$ The Newton's backward interpolation formula is

$$y_i\ (=x_n + nh) = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n +$$

$$\frac{n(n+1)(n+2)(n+3}{4!}\nabla^4 y_n + \frac{n(n+1)(n+2)(n+3(n+4)}{5!}\nabla^5 y_n$$

$\therefore$ $$y(84) = 304 + (-0.6)(28) + \frac{(-0.6)(-0.6+1)}{(2)}(2) + 0 + 0 +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{5!}(4)$$

$\Rightarrow$ $$y(84) = 304 - 16.8 + (0.6)(0.4) + \frac{(-0.6)(0.4)(1.4)(2.4)(3.4)}{120}(4)$$

$\Rightarrow$ y (84) = 304 - 16.8 - 0.24 - 0.09139

$\Rightarrow$ y (84) = 286.86

Hence, the melting point of allow containing 84 percent lead is 286.86^{o}C.

Self Check Exercise - 1

Q.1 Find f(2.8) from the following table.

x	0	1	2	3
f(x)	1	2	11	34

Q.2 In the following table, the values of y are consecutive terms of a series of which 12.5 is the 5th term. Find the 1st and tenth terms.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

Q.3 The following table gives the population of the town during the last six censure. Estimate the increase in population during the period from 1976 to 1978.

Year	1941	1951	1961	1971	1981	1991
Population (In thousand)	12	15	20	27	39	51

12.4 Newton's Divided Difference Formula (Interpolation With Unequally Spaced Points)

Newton's divided difference formula is method for constructing an interpolating polynomial for a given set of data points. This method is particularly useful for numerical interpolating because it provides a straightforward and systematic way to derive the interpolating polynomial.

Divided Difference

Suppose that the function f is tabulated at (n+1) points $x_0, x_1, x_2, ...x_n$ (not necessarily equidistant) and the corresponding values of function f are $y_0, y_1, ..., y_n$. Then $\frac{y_1 - y_0}{x_1 - x_0}$, $\frac{y_2 - y_1}{x_2 - x_1}$, $\frac{y_3 - y_2}{x_3 - x_2}$, ..., $\frac{y_n - y_{n-1}}{x_n - x_{n-1}}$ are called first order divided differences and are denoted by $\Delta_d y_0$, $\Delta_d y_1$, $\Delta_d y_2$, ..., $\Delta_d y_{n-1}$. i.e.

$$\Delta_d y_0 = \frac{y_1 - y_0}{x_1 - x_0},\ \Delta_d y_1 = \frac{y_2 - y_1}{x_2 - x_1},\ \Delta_d y_2 = \frac{y_3 - y_2}{x_3 - x_2}, ...$$

$$\Delta_d y_{n-1} = \frac{y_n - y_{n-1}}{x_n - x_{n-1}}$$ Where Δ_d is called divided difference

In general, $\Delta_d y_r = \frac{y_{r+1} - y_r}{x_{r+1} - x_r}$

The difference $\frac{\Delta_d y_1 - \Delta_d y_0}{x_2 - x_0}$, $\frac{\Delta_d y_2 - \Delta_d y_1}{x_3 - x_2}$, ... , $\frac{\Delta_d y_{n-1} - \Delta_d y_{n-2}}{x_n - x_{n-2}}$ are called second order divided differences and are denoted by $\Delta^2_d y_0$, $\Delta^2_d y_1$, ..., $\Delta^2_d y_{n-2}$

i.e. $$\Delta_d^2 y_0 = \frac{\Delta_d y_1 - \Delta_d y_0}{x_2 - x_0},\ \Delta_d^2 y_1 = \frac{\Delta_d y_2 - \Delta_d y_1}{x_3 - x_1}, ...,$$

$$\Delta_d^2 y_{n-2} = \frac{\Delta_d y_{n-1} - \Delta_d y_{n-2}}{x_n - x_{n-2}}$$

In general, $\Delta_d^2 y_r = \frac{\Delta_d y_{r+1} - \Delta_d y_r}{x_{r+2} - x_r}$

Similarly, higher order divided difference can be defined. In general, the j^{th} order divided difference is given by

$$\Delta_d^j y_r = \frac{\Delta_d^{j-1} y_{r+1} - \Delta_d^{j-1} y_r}{x_{r+j} - x_r}$$

Divided Difference Table:

The table listing all the divided difference is given below:

x	y	$\Delta_d y$	$\Delta^2_d y$	$\Delta^3_d y$	$\Delta^4_d y$
x_0	y_0				
		$\Delta_d y_0$			
x_1	y_1		$\Delta^2_d y_0$		
		$\Delta_d y_1$		$\Delta^3_d y_0$	
x_2	y_2		$\Delta^2_d y_1$		$\Delta^4_d y_0$
		$\Delta_d y_2$		$\Delta^3_d y_1$	
x_3	y_3		$\Delta^2_d y_2$		
		$\Delta_d y_3$			
x_4	y_4				

Here each entry in the difference table is given by the difference between diagonally adjacent eateries to its left divided by the difference between the value of x corresponding to the values of y intercepted by the diagonals passing through the calculated entry.

Procedure for Newton's Divided Difference Formula

Let the function f be tabulated at (n+1) points x_0, x_1, x_2, ... x_n (not necessarily equidistant) and the corresponding values of f be y_0, y_1, y_2, ... y_n.

Let $f(x)$ be approximated by polynomial.

$\phi_n(x) = a_0 + a_1 (x - x_0) + a_2 (x-x_0)(x-x_1) + a_3 (x-x_0)(x-x_1)(x-x_2) + ...$

$+ an (x-x_0)(x-x_1) .... (x-x_{n-1})$ ....(2)

of degree n such that $f(x)$ and ϕn (x) agree at the tabulated

points i.e. $f(x) \approx \phi_n(x)$ ...(3)

and $\phi_n(xi) = y_i$ for i = 0, 1, 2, ...n

Now, imposing conditions (4) on equation (2), we have

$\phi_n(x_0) = y_0 \Rightarrow a_0 = y_0$

$\phi_n(x_1) = y_1 \Rightarrow a_0 + y_1 (x_1-x_0) = y_1$

$\phi_n(x_2) = y_2 \Rightarrow a_0 + y_1 (x_2-x_0) + a_2 (x_2-x_0)(x_2-x_1) = y_2$ and so on.

Solving above equations for a_0, a_1, a_2, ... an, we have $a_0 = y_0$.

$$a_1 = \frac{y_1 - a_0}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

$$a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - a_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{(y_2 - y_0)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_0)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{y_2x_2 - y_2x_0 - y_0x_1 + x_0y_0 - y_1x_2 + y_1x_0 + y_0x_2 - x_0y_0}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{(y_2x_1 - y_2x_0 - x_1y_1 + y_1x_0) - (y_1x_2 - x_2y_0 - x_1y_1 + x_1y_0)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{(x_2 - x_0)}\left[\frac{(y_2(x_1 - x_0) - y_1(x_1 - x_0)) - (x_2(y_1y_0) - x_1(y_1 - y_0))}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{(y_2 - y_1)(x_1 - x_0) - (y_1 - y_0)(x_2 - x_1)}{(x_2 - x_1)(x_1 - x_0)}\right]$$

$$= \frac{1}{x_2 - x_0}\left[\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}\right]$$

$$= \frac{\dfrac{y_2 - y_1}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$$

and so on

Using divided difference, we have

$a_1 = y_0$

$a_1 = \Delta_d y_0$,

$a_2 = \dfrac{\Delta_d y_1 - \Delta_d y_0}{x_2 - x_0} = \Delta_d^2 y_0$ and so on.

Continuing in this way, we have

$a_n = \Delta_d^n y_0$

Substituting the values of a_0, a_1, a_2, ..., a_n in (2), we have

$\phi_n(x) = y_0 + (x-x_0)\,\Delta_d y_0 + (x-x_0)\,(x-x_1)\,\Delta_d^2 y_0 + (x-x_0)$

$(x-x_1)\,(x-x_2)\,\Delta_d^3 y_0 + \ldots\ldots + (x-x_0)\,(x-x_1)\,\ldots\ldots\,(x-x_{n-1})\,\Delta_d^n y_0$ ...(5)

Using relation (3), we can write (5) as

$f(x) = y_0 + (x-x-_0)\,\Delta_d y_0 + (x-x_0)\,(x-x_1)\,\Delta_d^2 y_0 + (x-x_0)\,(x-x_1)\,(x-x_2)$

$\Delta_d^3 y_0$ + ... + (x-x_0) (x-x_1) (x-x_2) .... (x-x_{n-1}) $\Delta_d^n y_0$ ....(6)

Formula given by (5) (or (6)) is called Newton's divided difference formula or Newton's general interpolation formula.

This formula is useful especially when the function is tabulated at unequal intervals.

Some Related Questions:

Let us try following examples to have more understanding of Newton's divided difference.

Example 1: Find the divided difference table for the data in the table below:

x:	-3	-1	0	3	5
f(x):	-30	-22	-12	330	3458

Solution: The divided difference table is

x	y	$\Delta_d f(x)$	$\Delta^2_d f(x)$	$\Delta^3_d f(x)$
0	1			
		$\frac{3-1}{1-0}=2$		
1	3		$\frac{23-2}{3-0}=7$	
		$\frac{49-3}{3-1}=23$		$\frac{19-7}{4-0}=3$
3	49		$\frac{80-23}{4-1}=19$	
		$\frac{129-49}{4-3}=80$		$\frac{37-19}{7-1}=3$
4	129		$\frac{228-80}{7-3}=37$	
		$\frac{813-129}{7-4}=228$		
7	813			

Example 2: Given the values of x and f(x) as follows.

x:	2	2.5	3
f(x):	0.69315	0.91629	1.09861

Find f(2.7) using Newton's Divided Difference formula:

Solution: The divided difference table is

$x f(x)$	$f(x)$	$\Delta_d f(x)$	$\Delta^2_d f(x)$
2 2.5 3	0. 69315 0. 91629 1. 09861	0. 4463 0. 3646	−0. 0816

By Newton Divided Difference formula

$$f(x) = f_0 + (x-x_0)\,\Delta_d f_0 + (x-x_0)\,(x-x_1)\,\Delta_d^2 f_0$$

$= 0.6932 + (2.7-2) \times 0.4463 + (2.7 - 2)\ (2.7 - 2.5)\ (-0.0816)$

$= 0.6932 + 0.7 \times 0.4463 + (0.7)\ (0.2)\ (-0.0816)$

$= 0.6932 + 0.3124 - 0.0114$

$= 0.9941$

$f(2.7) = 0.9941$

Example 3: Find the fourth divided difference with arguments 2, 2.5, 3, 3.5, 4 of the function $f(x) = x^3 - x + 1$.

Solution: The divided difference table is

x	$f(x)$	$\Delta_d f(x)$	$\Delta^2_d f(x)$	$\Delta^3_d f(x)$	$\Delta^4_d f(x)$
2 2.5 3 3.5 4	7 14.125 25 40.375 61	14.25 21.75 30.75 41.25	7.5 9 10.5	1 1	0

Hence the fourth divided difference is 0.

Example 4: Find $f(3.8)$ for an equation $f(x) = 2x^3 - 4x + 1$ using divided difference formula

$s_0 +_0 x_1 = 2$ and $x_2 = 4$ with step size h = 0.5.

Solution: Given $f(x) = 2x^3 - 4x + 1$

$x_1 = 2$ & $x_2 = 4$

Step Size (h) = 0.5

The divided table is

x	f(x)	$\Delta_d f$(x)	$\Delta^2_d f$(x)	$\Delta^3_d f$(x)	$\Delta^4_d f$(x)
2	7				
		26.5			
2.5	22.25		15		
		41. 5		2	
3	43		18		0
		59.5		2	
3.5	72.75		21		
		80.5			
4	113				

By Newton Divided Difference Formula

$$f(x) = f_0 + (x-x_0)\Delta_d f_0 + (x-x_0)(x-x_1)\Delta_d^2 f_0 +$$

$$(x-x_0)(x-x_1)(x-x_2)\Delta_d^3 f_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\ \Delta_d^4 f_0$$

$$f\,(3.8) = 9 + (3.8-2) \times 26.5 + (3.8-2)(3.8-2.5) \times 15 + (3.8-2)(3.8-2.5)(3.8-3)$$

$$\times\, 2 + (3.8 - 2)(3.8 - 2.5)(3.8 - 3)(3.8 - 3.5) \times 0$$

$$= 9 + (1.8) \times (26.5) + (1.8) \times (1.3) + (1.8)(1.3) \times (2) + (1.8)(1.3)(0.8)(0.3) \times 0$$

$$= 9 + 47.7 + 35.1 + 3.744 + 0$$

$$= 95.544$$

$\therefore \quad f(3.8) = 95.544$

Example 5: Construct divided difference table for the data

x:	1	3	4
y:	2	10	22

Solution: The divided difference table is

x	y	Δ dy)	Δ^2dy
1	2		
		$\frac{10-2}{3-1} = 4$	
3	10		$\frac{12-4}{4-1} = \frac{8}{3} = 2.67$
		$\frac{22-10}{4-3} = 12$	
4	22		

Self Check Exercise - 2

Q.1 Prepare a divided difference table for the following data.

x:	0	0.5	1	2
f(x):	1	1.8987	3.7183	11.3891

Q.2 Given values of x and f(x) as follows.

x:	2	4	9	10
f(x):	4	56	711	980

Find f(5) using Newton's Divided Difference Formula.

Q.3 Prepare divided difference table for the following data.

x:	2	5	7	10
f(x):	4	26	58	142

12.5 Summary

1. Newton forward and backward difference interpolation is a numerical method used to estimate the value of a function at a given point based on known discrete data points. This method is particularly useful when data points are equally spaced.

2. Newton Forward Interpolation Formula:

$$f(a+hu) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots + \frac{u(u-1)(u-2).....\big(u-(n-1)\big)}{n!}\Delta^n y_0$$

Newton Backward Interpolation Formula:

$$f(an+uh) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\Delta^2 y_n + \ldots\ldots\ldots + \frac{u(u+1)..........\big(u+(\overline{n-1})\big)}{n!}\Delta^n y_0$$

3. Newton's interpolation is widely used in numerical analysis engineering and computer science to approximate functions and model data where equally spaced intervals are available.

4. Newton's divided difference is a recursive method used to construct an interpolating polynomial for a given set of data points. The main purpose is to create a polynomial that passes through all the points.

5. Newton's divided difference formula is useful for interpolating data points that are not evenly spaced.

6. It is often used in numerical analysis and scientific computing to approximate functions and model data.

12.6 Glossary:

1. **Interpolation:** Interpolation is the technique of estimating the value of a function for any intermediate value of independent variable, while the process of computing the value of function outside the given range is called extrapolation.

2. **Difference Table:** A tabular arrangement of forward or backward differences for a given set of data points. Each column in the table represents higher order difference.

3. **Degree of Polynomial:** The highest power of x in the polynomial For n+1 data points, the degree of the interpolating polynomial is almost h.

4. **Factorial (n!):** The product of all positive integers upto n. Used in the coefficient of the terms in Newton's forward or backward difference formula.

5. **Divided Difference:** Divided difference are the way to calculate the coefficients of the Newton interpolating polynomial

6. **Interpolating Polynomial"** An interpolating polynomial is a polynomial that passes through a given set of points (x_0, y_0), (x_1, y_1), ... (x_n, y_n). The goal is to find a polynomial P(x) such that $P(x_i) = y_i$ for all !.

7. **Unequally Spaced Points:** Unequal spaced points are the intervals which are not equally spaced.

12.7 Answers to Self Check Exercise - 1

1. 27.992
2. 0.1, 100
3. 25.6

Answers to Self Check Exercise - 2

1.

x	$f(x)$	$\Delta_d f(x)$	$\Delta^2_d f(x)$	$\Delta^3_d f(x)$
0 0.5 1 2	1 1.8987 3.7183 11.3891	1.7974 3.6392 7.6708	1.8418 2.6877	0.4229

2.

x	f(x)	$\Delta_d f(x)$	$\Delta^2_d f(x)$	$\Delta^3_d f(x)$
2 5 7 10	4 26 58 142	7.33 16 28	1.73 2.4	0.084

12.8 References/Suggested Readings

1. Numerical Analysis by Richard L. Burden
2. Introductory Methods of Numerical Analysis by S.S. Sastry.
3. Finite Difference & Numerical Analysis by H.C. Saxena
4. An Introduction to Numerical Analysis by Kendall E. Atkinson.

12.9 Terminal Questions

1. Find Solution using Newton's Forward Difference Formula.

x	1891	1901	1911	1921	1931
f(x)	46	66	81	93	101

2. Find Solution of an equation $2x^3 - 4x + 1$ using Newton's forward difference formula $x_1 = 2$ and $x_2 = 4$

   Step value h = 0.25, x = 2.1

   Find $f(2)$?

3. From the following data estimate the number of students getting marks between 20 and 25.

Marks below	10	20	30	40	50
No. of Students	20	45	115	210	325

4. Prepare a divided difference table for the following data

x:	300	304	305	307
f(x):	2.4771	2.4829	2.4843	2.4871

5. Given values of x and f(x) are

x:	0	1	4	5
f(x):	4	3	24	39

6. Find the divided difference table for the data.

x:	-3	-1	0	3	5
f(x):	-30	-22	-12	330	3458

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Unit - 13

Lagrange Interpolation

Structure

13.1 Introduction

Dear student, in this unit we will study about Lagrange interpolation. Just like Newton's divided difference interpolation, Lagrange interpolation is used unequally spaced points. Using Lagrange's interpolation formula we can interpolate a value of unequal spaced interval as well as polynomial for a given set of data. So, in this unit we will study to interpolate a value or a polynomial using Lagrange interpolation.

13.2 Learning Objectives:

After studying this unit students will be able to

1. define Lagrange interpolation for unequal interval
2. interpolate a given value using Lagrange interpolation formula.
3. Find the polynomial using Lagrange's interpolation formula.

13.3 Lagrange Interpolation

In Lagranges interpolation when the function itself is not known directly, one can use the mathematical tool or technique to approximate the value of the function at any given position within a given range. This approach interpolates values at other places by utilizing known data points from the function.

Lagranges interpolation allow us to get an approximate value of a function. When we don't have the specific formula for the function. It involves using known data points from the function to make these estimates.

Lagrange Interpolation Formula for n^th order

Let's say we have a function y = f(x), where changing the values of x yields varying values of yi.e. for n distinct real values x_1, x_2, x_3, ..., x_n we have n real values y_1, y_2, ..., y_n. Then Lagranges interpolation Formula for nth order is given by

$$f(x) = \frac{(x-x_2)(x-x_3).......(x-x_n)}{(x_1-x_2)(x_1-x_3).........(x_1-x_n)} y_1 + \frac{(x-x_1)(x-x_3).......(x-x_n)}{(x_2-x_1)(x_2-x_3).........(x_2-x_n)}$$

$$y_2 + ... + \frac{(x-x_1)(x-x_2).......(x-x_{n-1})}{(x_n-x_1)(x_n-x_2).........(x_n-x_{n-1})} y_n$$

$$f(x) = \sum_{i=1}^{n} yi \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

Advantages of Lagranges Interpolation:

1. Even in cases when the arguments are not evenly spaces the function's value can be determine using this technique.
2. This formula is used to determine the value of the independent variable x that corresponds to a function for the given value.

Disadvantages of Lagrange Interpolation

1. A change of degree in Lagrangian polynomial involves a completely new computation of all the terms.
2. This formula requires a lot of multiplications for a high degree polynomials, which slows down the process significantly.
3. The degree of polynomial in the Lagrange interpolation is selected at the outset. Therefore, determining the degree of an approximation polynomial that works for a particular set of tabulated points is challenging.

Inverse Interpolation

Inverse interpolation is the process of looking for values of x that correlates to specified values of function f(x). Therefore

$$x = \sum_{i=0}^{n} xi \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(y-y_j)}{(y_i-y_j)}$$

Let us try following examples to have more understanding of Lagrange's Interpolation.

Some Examples:

Example 1: From the following table, interpolate the value of f(x) using Lagrangian polynomial at x = 301.

x:	300	304	305	307
f(x):	2.4771	2.4829	2.4843	2.4871

Solution: Here $x_0 = 300$, $x_1 = 304$, $x_2 = 305$, $x_3 = 307$ and

$f_0 = 2.4771$, $f_1 = 2.4829$, $f_2 = 2.4843$, $f_3 = 2.4871$

We know that Lagrangian Polynomial is given by

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

$$\therefore \quad f(301) = \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)} \times 2.4771 +$$

$$+ \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)} \times 2.4829 + \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)}$$

$$\times 2.4843 + \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)} \times 2.4871$$

$$f(301) = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)} \times 2.4771 + \frac{(1)(-4)(-6)}{(-4)(-1)(-3)} \times 2.4829 + \frac{(1)(-3)(-6)}{(5)(1)(-2)} \times 2.4843$$

$$+ \frac{(1)(-3)(-4)}{(7)(3)(2)} \times 2.4871$$

$$f(301) = \frac{-72}{-140} \times 2.4771 + \frac{24}{12} \times 2.4829 + \frac{18}{-10} \times 2.4843 + \frac{12}{42} \times 2.4871$$

$$f(301) = 2.4786$$

Solution of polynomial at point 301 is $f(301) = 2.4786$

Example 2: Given values of x and f(x) as follows

x:	2	2.5	3
f(x):	0.69315	0.91629	1.09861

Evaluate $f(2.7)$ using Lagrange formula.

Solution: Here $x_0 = 2$, $x_1 = 2.5$, $x_2 = 3$ and

$f_0 = 0.69315$, $f_1 = 0.91629$, $f_2 = 1.09861$

By using Lagrange Formula

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$f(2.7) = \frac{(2.7-2.5)(2.7-3)}{(2-2.5)(2-3)} \times 0.69315 + \frac{(2.7-2)(2.7-3)}{(2.5-2)(2.5-3)} \times 0.91629 +$$

$$\frac{(2.7-2)(2.7-2.5)}{(3-2)(3-2.5)} \times 1.09861$$

$$f(2.7) = \frac{(0.2)(-0.3)}{(-0.5)(-1)} \times 0.69315 + \frac{(0.7)(-0.3)}{(0.5)(-0.5)} \times 0.91629 + \frac{(0.7)(0.2)}{(1)(0.5)} \times 1.09861$$

$$f(2.7) = \frac{-0.06}{0.5} \times 0.69315 + \frac{-0.21}{-0.25} \times 0.91629 + \frac{0.14}{0.5} \times 1.09861$$

$$f(2.7) = 0.994116$$

Example 3: Given Pog_{10} 654 = 2.8156, Pog_{10} 658 = 2.8182, $\log_{10}$ 659 = 2.8189 Pog_{10} 661 = 2.8202. Use Lagrange formula to find the value of $\log_{10}$ 656.

Solution: Let $y = f(x) = \log_{10}x$. Also, $x_0 = 654$, $x_1 = 658$, $x_2 = 659$,

$x_3 = 661$ and $y_0 = 2.8156$, $y_1 = 2.8182$, $y_2 = 2.8189$,

$y_3 = 2.8202$

Using Lagrange Formula

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times 2.8156 +$$

$$\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times 2.8182 +$$

$$\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times 2.8189 + \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}$$

$$\times 2.8202$$

$$f(656) = \frac{(-2)(-3)(-5)}{(-4)(-5)(-7)} \times 2.8156 + \frac{(2)(-3)(-5)}{(4)(-1)(-3)} \times 2.8182 + \frac{(2)(-2)(-5)}{(5)(1)(-2)} \times 2.8189$$

$$+ \frac{(2)(-2)(-3)}{(7)(3)(2)} \times 2.8202$$

= 0.46834 + 7.0455 - 5.6378 + 0.80577

= 2.8168 (rounding off to four decimal places).

Hence $\log_{10} 656 = 2.8168$

Example 4: Using Lagrange interpolation, calculate the profit in the year 2000 from the following data

Year:	1997	1999	2001	2002
Profit of Lakh of Rs:	43	65	159	248

Solution: By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \; y_2 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(x) = \frac{(x-1999)(x-2001)(x-2002)}{(1997-1999)(1997-2001)(1997-2002)} (43) +$$

$$\frac{(x-1999)(x-2001)(x-2002)(65)}{(1999-1997)(1999-2001)(1999-2002)} + \frac{(x-1997)(x-1999)(x-2002)}{(1001-1997)(2001-1999)(2001-2002)}$$

$$(159) + \frac{(x-1997)(x-1999)(x-2001)}{(2002-1997)(2002-1999)(2002-2001)} (248)$$

$$y = \frac{-43}{20} + \frac{65}{2} + \frac{477}{4} - \frac{248}{5}$$

Example 5: Find the third degree polynomial $f(x)$ satisfying the following data:

x:	1	3	5	7
y:	24	120	336	720

Solution:

x:	x_0=1	x_1=3	x_2=5	x_3=7
y:	y_0=24	y_1=120	y_2=336	y_3=720

By Lagrange's interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \ y_2 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-3)(x-5)(x-7)}{(1-3)(1-5)(1-7)}(24) + \frac{(x-1)(x-5)(x-7)}{(3-1)(3-5)(3-7)}(120) + \frac{(x-1)(x-3)(x-7)}{(5-1)(5-3)(5-7)} \ (336)$$

$$+ \frac{(x-1)(x-3)(x-5)}{(7-1)(7-3)(7-5)}(720)$$

$$= \frac{-1}{2}\left[x^3-15x^2+71x-105\right] + \frac{15}{2}\left[x^3-13x^2+47x-35\right] - 21\left[x^3-11x^2+31x-21\right]$$

$$+ 15\left[x^3-9x^2+23x-15\right]$$

$$= \left[\frac{-1}{2}+\frac{15}{2}-21+15\right]x^3 + \left[\frac{15}{2}-\frac{195}{2}+231+135\right]x^2 + \left[\frac{-71}{2}+\frac{705}{2}+605+345\right]x$$

$$+ \left[\frac{105}{2}-\frac{525}{2}+441-225\right]$$

$= x^3 + 6x^2 + 11x + 6$

$f(4) = 4x^3 + 6(4)^2 + 11\ (4) + 6 = 64 + 96 + 44 + 6 = 210$

Example 6: Using Lagrange's interpolation formula, find $f(4)$ given that $f(0) = 2$, $f(1)$, $f(2) = 12$, $f(15) = 3587$

Solution:

x:	x_0 = 0	x_1=1	x_2=2	x_3=15
y:	y_0 = 2	y_1 = 3	y_2 = 12	y_3 = 3587

By Lagranges interpolation formula, we have

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}$$

$$y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \; y_2 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$y = f(4) = \frac{(4-1)(4-2)(4-15)}{(0-1)(0-2)(0-15)} (2) + \frac{(4-0)(4-2)(4-15)}{(1-0)(1-2)(1-15)} (3) +$$

$$\frac{(4-0)(4-1)(4-15)}{(2-0)(2-1)(2-15)} (12) + \frac{(4-0)(4-1)(4-2)}{(15-0)(15-1)(15-2)} (3587)$$

$$= \frac{132}{130} - \frac{264}{14} + \frac{1584}{26} + \frac{86088}{2730}$$

$= 78$

Example 7: The mode of certain frequency curve $y = f(x)$ is very nearer to $x = 9$ and the values of the frequency density $f(x)$ for $x = 8.9, 9, 9.3$ are 0.30, 0.35 and 0.25 respectively. Calculate the approximate value of the mode?

Solution: Given

x:	8.9 (x_0)	9.0 (x_1)	9.3 (x_2)
$f(x)$:	0.30 (y_0)	0.35(y_1)	0.25 (y_2)

By Lagrange's interpolation formula, we have

$$f(x) = \frac{(x-9)(x-9.3)}{(8.9-9)(8.9-9.3)} (0.30) + \frac{(x-8.9)(x-9.3)}{(9-8.9)(9-9.3)} (0.35) + \frac{(x-8.9)(x-9)}{(9.3-8.9)(9.3-9)} \; 10.25$$

$$= \frac{1}{12} (-25x2 + 453.5x - 2052.3)$$

To get the mode, $f'(x) = 0$ and $f''(x) = -ve$

$$\therefore \qquad f'(x) = 0 \Rightarrow \frac{1}{12} (-50x + 453.5) = 0$$

i.e. $(x) = 9.07$

$$f''(x) = \frac{1}{12} (-50) = -ve$$

Hence, $f(x)$ is maximum at $x = 9.07$

Therefore, mode is 9.07

Self Check Exercise

Q.1 Given the table of values as

x:	20	25	30	35
y(x):	0.342	0.423	0.500	0.650

Find x (0.390)

Q.2 Certain corresponding values of x and $Log_{10}x$ are (300, 2.4771), (304, 2.4829), (305, 2.4843), and (307, 2.4871). Find $\log_{10} 301$.

Q.3 Find solution using Lagranges interpolation formula

x:	-1	0	3	6	7
f(x):	3	-6	39	822	1611

X = 1 Finding $f(1)$

13.4 Summary

Lagranges interpolation formula for n^{th} degree polynomial is given as:

$$f(x) = \frac{(x-x_1)(x-x_2)........(x-x_n)}{(x_0-x_1)(x_0-x_2)........(x_0-x_n)} \times y_0 + \frac{(x-x_0)(x-x_2)........(x-x_n)}{(x_1-x_0)(x_1-x_2)........(x_0-x_n)} \times y_1$$

$$+ .... + \frac{(x-x_0)(x-x_1)........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)........(x_n-x_{n-1})} \times y_n$$

This formula is used to find the value of the function at any point even when the function itself is not given.

It is used to find the value of the dependent variable at any particular independent variable if the function itself is not given.

It is used even if the point given are not evenly spaced.

13.5 Glossary:

- **Interpolation:** It is a process of determining the unknown values that lies in between the known data points.
- **Function:** It is an expression, rule or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable)
- **Range:** The set of all outputs of a function is known as the range of the function
- **Degree of Polynomial:** The degree of a polynomial is the highest power of the variable in a polynomial expression
- **Order of Polynomial:** Order of Polynomial is same as degree of Polynomial.

13.6 Answers to Self Check Exercise

1. 22.841
2. 2.4786
3. $f(1) = -3$

13.7 References/Suggested Readings

1. Introductory Methods of Numerical Analysis by S.S. Sasfry.
2. Introduction to Numerical Analysis by Kendall Aticinson
3. Numerical Analysis by J. Douglas Eaires
4. Finite Difference and Numerical Analysis by H.C. Saxena

13.8 Terminal Questions

1. Using Lagrange interpolating polynomial with suitable order, find the mising value of y in the following table:

x:	0	10	20	30	40
y:	1	6	7	-	5

2. Develop a Lagrange interpolation polynomial that passes through the data (3 points) given in the table below:

x:	0	0.64	1.28
y:	5	2	0.75

3. Use the Lagrange interpolating polynomial to estimate the value of $f(x) = 10xe^{-x}$ at x = 3 using the value of $f(x)$ at x = 1, x = 2 and x = 4.

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Unit - 14

Numerical Differentiation By Newton's Formula

Structure

14.1 Introduction

Dear student, in this unit we will study about numerical differentiation. In calculus we have studied about the derivative of a function. But when explicit form of a function is not known, or the function is given in the tabular form or the function is complicated, then to find derivative of a function we use numerical differentiation. So by Numerical differentiation we can calculate the derivative of a function using the given set of values of a function. In this unit we will study about numerical differentiation using Neutron's forward and backward interpolation formula. We will learn how to apply Newton's forward and backward interpolation formula to find derivative of a given functions.

14.2 Learning Objectives:

After studying this unit students will be able to

1. define Numerical differentiation
2. define Numerical differentiation using Newton's forward difference
3. define Numerical differentiation using Newton's backward difference
4. evaluate numerical differentiation of given function using Newton's forward and backward differences.

14.3 Numerical Differentiation

Numerical differentiation is the process of computing the values of the derivatives of an explicitly unknown function, with given discrete set of points (x_i, y_i), i = D, 1, 2, ... , n To differentiate a function numerically. We first determine an interpolating polynomial and then compute the approximate derivative at the given point.

If xi's are equispaced.

(i) Newton's forward interpolation formula is used to find the derivative near the beginning of table.

(ii) Newton's backward interpolating formula is used to compute the derivation near the end of the table.

(iii) Sterling's formula is used to estimate the derivative near the centre of the table

If xi's are not equispaced, we may find f(x) using Newton's divided difference method.

Derivatives using Newton's Forward Interpolation Formula

Newton's forward interpolation formula for the function y = f(x) is given by

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots\dots\dots \quad (1)$$

$$p = \frac{x-x_0}{h}$$

Differentiating (1) w.r.t. 'p'

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \dots\dots\dots\dots\dots\dots \quad (2)$$

Also $\frac{dp}{dx} = \frac{1}{h}$ (3)

Again $\frac{dy}{dx} = \frac{dy}{dp}\frac{dp}{dx}$ (4)

Using (2) and (3) in (4), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \dots\dots\dots\dots\right]$$

Now at x = x_0, p = 0

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \dots\dots\dots\right]$$

Now $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx}$

or $$\frac{d^2y}{dx^2} = \frac{1}{h^2}\frac{d}{dp}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!} + ...\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{6}\Delta^3 y_0 + \frac{12p^2-36p+22}{24}\Delta^4 y_0 + ...\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...\right]$$

Similarly higher order derivatives can be calculated.

Example 1: Using Newton's Forward Difference Formula to Find y'(0.1) and y''(0.1)

x	0.0	0.1	0.2	0.3	0.4
f(x)	1.0000	0.9975	0.9900	0.9776	0.8604

Solution: Newton's forward difference table is

x	y	Δ y	Δ^2y	Δ^3y	Δ^4y
0 0.1 0.2 0.3 0.4	1 0.9975 0.99 0.9776 0.8604	-0.0025 -0.0075 -0.0124 -0.1172	-0.005 -0.0049 -0.1048	0.0001 -0.0999	-0.1

Here $h = x_1 - x_0 = 0.1 - 0 = 0.1$

Newton's forwards difference formula is

$$\left[\frac{dy}{dx}\right]_{x=x_1} = \frac{1}{h}\left(\Delta y_1 - \frac{1}{2}.\Delta^2 y_1 - \frac{1}{3}.\Delta^3 y_1 - \frac{1}{4}\Delta^4 y_1\right)$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=0.1} = \frac{1}{0.1} \cdot \left(-0.0075 - \frac{1}{2}\times(-0.0049) + \frac{1}{3}.(-0.0999) - \frac{1}{4}\times 0\right)$$

$\therefore \quad \left[\frac{dy}{dx}\right]_{x=0.1}$ = -0.3835

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_1} = \frac{1}{h^2} \cdot \left(\Delta^2 y_1 - \Delta^3 y_1 + \frac{11}{12}\Delta^4 y_1\right)$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=0.1} = \frac{1}{0.01} \cdot \left(-0.0049 - (-0.0999) + \frac{11}{12}\times 0\right)$$

$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=0.1}$ - 9.5

$\therefore$ Hence y'(0.1) = -0.3835 and y''(0.1) = 9.5

Example 2: Find solution of an equation $x^3 + x + 2$, x1 = 2 and $x_2 = 4$ at x = 2.25 here step value (h) = 0.25 using Newton's Forward difference formula.

Solution: Given equation is $x^3 + x + 2$

Let $f(x) = x^3 + x + 2$

The value of table for x and y

x	2	2.25	2.5	2.75	3	3.25	3.5
y	12	15.6406	20.125	25.5469	32	39.5781	48.375

Newton's Forward Difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	12				
		3.6406			
2.25	15.6406		0.8438		
		4.4844		0.0938	
2.5	20.125		0.9375		0
		5.4219		0.0938	
2.75	25.5469		1.0312		0
		6.4531		0.0938	
3	32		1.125		0
		7.5781		0.0938	
3.25	39.5781		1.2188		0
		8.7969		0.0938	
3.5	48.375		1.3125		0
		10.1094		0.0938	
3.75	58.4844		1.4062		
		11.5156			
4	70				

Here h = x_1 - x_0 = 2.25 - 2 = 0.25

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}.\Delta^2 y_0 - \frac{1}{3}.\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0\right)$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=2.25} = \frac{1}{0.25}\left(4.4844 - \frac{1}{2}\times 0.9375 + \frac{1}{3}\times 0.0938 - \frac{1}{4}\times 0\right)$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=2.25} = 16.1875$$

and
$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}.\Delta^4 y_0\right)$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=2.25} = \frac{1}{0.0625}.\left(0.9375 - 0.0938 + \frac{11}{12}\times 0\right)$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=2.25} = 13.5$$

Hence y'(2.25) = 16.1875 and y"(2.25) = 13.5

Example 3: Find the first two derivative of $(x)^{\frac{1}{3}}$ at x = 50 The table is

x:	50	51	52	53	54	55	56
y=$(x)^{\frac{1}{3}}$:	3.6840	3.7084	3.7325	3.7563	3.7798	3.8030	3.8259

Solution: Since, we require f'(x) at x = 50, We use Newton's forward difference formula and to get f'(x) at x = 50 we have to plot the difference table first:-

x	y	Δ y	Δ^2y	Δ^3y
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

By Newton's Forward formula

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 \frac{1}{2}\Delta^2 y_0 - \frac{1}{3}\Delta^2 y_0........\right]$$

$$= \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

= 0.02455

and $$\left[\frac{d^2y}{dx^2}\right]_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + ...\right]$$

= 1 [-0.0003]

= - 0.0003

∴ 1st two derivatives of $(x)^{\frac{1}{3}}$ at x = 50 are 0.02455, -0.0003

Example 4: The following data gives the velocity of a particle for twenty seconds at an interval of five seconds. Find the initial acceleration using the entire data

Time t (sec)	0	5	10	15	20
Velocity v (m/sec):	0	3	14	69	228

Solution: The difference table is

t	v	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0 5 10 15 20	0 3 14 69 228	3 11 55 159	8 44 104	36 60	24

At initial acceleration i.e. $\left(\frac{dv}{dt}\right)$ at t = 0 is required, we use Newton's forward formula

$$\left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 \frac{1}{4}\Delta^4 y_0 + ...\right]$$

$$\therefore \qquad \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{5}\left[3 - \frac{1}{1}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right]$$

$$= \frac{1}{5}\ [3 - 4 + 12 - 6]$$

$$= 1$$

Example 5: Find the value of cos (1.74) from the following table.

x:	1.7	1.74	1.78	1.82	1.86
Sin x:	0.9916	0.9857	0.9781	0.9691	0.9584

Solution: Let y = f(x) = Sin x. So that f'(x) = Cos x

The difference table is

t	v	Δ y	Δ ^{2}y	Δ ^{3}y	Δ ^{4}y
1.7 1.74 1.78 1.82 1.84	0.9916 0.9857 0.9781 0.9691 0.9584	-0.0059 -0.0076 -0.0090 -0.0107	-0.0017 -0.0014 -0.0017	0.0003 -0.0003	-0.0006

Since, we require f'(1.74). We use Newton's Forward Difference formula

$$\frac{dy}{dx} = \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right]$$

Here h = 0.04, x0 = 1.7, Δy_0 = - 0.0059, $\Delta^2 y_0$ = - 0.0017 etc.

Substituting these values, we gel.

$$\left[\frac{dy}{dx}\right]_{x=1.74} = \frac{1}{0.04}\left[0.0059 - \frac{1}{2}(-0.0017) + \frac{1}{3}(0.003)\frac{1}{4}(-0.0006)\right]$$

$$= \frac{1}{0.04} (0.007) = 0.175$$

Hence cos (1.74) = 0.175

Self Check Exercise - 1

Q.1 Find solution of an equation $2x^3-4x+1$ at x = 2.25 where x_1 = 2 and x_2 = 4 step value (h) = 0.25 Using Newton's forward difference formula.

Q.2 Using Newton's forward difference method find solution

x	1.96	1.98	2.00	2.02	2.04
f(x)	0.7825	0.7739	0.7651	0.7563	0.7473

Q.3 Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (a) x = 1.1 (b) x = 1.6.

14.4 Numerical Differentiation Using Newton's Backward Difference Formula

Derivatives using Newton's backward interpolation Formula:-

Newton's backward interpolation formula for the function y = f(x) is given by

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots\ldots \qquad (1)$$

$$p = \frac{x - x_n}{h}$$

Differentiating (1) w.r.t. 'p'

$$\frac{dy}{dp} = \nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+1}{3!}\nabla^3 y_n + \ldots\ldots \qquad (2)$$

Also, $$\frac{dp}{dx} = \frac{1}{h} \qquad (3)$$

Again $\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$ (4)

Using (2) and (3) in (4), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+1}{3!}\nabla^3 y_n + \dots\dots\dots\right]$$

Now, at x = x_n, p = 0

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \dots\dots\right]$$

Similarly,

$$\left.\frac{d^2 y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \dots\dots\dots\right]$$

Newton's Backward Difference Method And Differentiation

Some Examples

Example 1: Following table gives the census population of a state for the years 1961 to 2000.

Year	1961	1971	1981	1991	2001
Population (Million)	19.96	36.65	58.81	77.21	94.61

Find the rate of growth of the population in the year 2001:

Solution: Derivative has to be evaluated near the end of the table, thereby constructing backward difference table for the function

y = f(x)

Year x	Population f(x)	Δ y	Δ ^{2}	Δ ^{3}	Δ ^{4}
1961 1971 1981 1991 2001	19.96 36.65 58.81 77.21 94.61	16.69 22.16 18.40 17.40	5.47 -3.76 -1.00	-9.23 2.76	11.99

To find rate of growth (derivatives) in the year 2001, taking x_n = 2001 and apply the relation

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + ...\right]$$

From table h = 10, $\nabla y_n = 17.40$, $\nabla^2 y_n = -1$, $\nabla^3 y_n = 2.76$, $\nabla^4 y_n = 11.99$

Thus, we get

$$\left.\frac{dy}{dx}\right]_{x=2011} = \frac{1}{10}\left[17.40 + \frac{(-1)}{2} + \frac{2.76}{3} + \frac{(11.99)}{4}\right]$$

$$= 2.08175$$

Example 2: The deflection $f(x)$ measured at various distance x from one end of a contilever is given in the following table.

x	0.0	0.2	0.4	0.6	0.8	1.0
$f(x)$	0.0000	0.0456	0.1278	0.3494	0.4027	0.4825

Evaluate $f'(0.85)$ and $f''(1.0)$ based on Newton's backward difference interpolation formula.

Solution : The difference table is as follows :

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0.0 0.2 0.4 0.6 x_0 = 0.8 1.0	0.0000 0.0456 0.1278 0.3494 0.4027 0.4825	455 823 2216 533 798	368 1393 -1683 265	1025 -3076 1948	-4101 5024

(i) $x_0 = 0.8$, $x_p = 0.85$, $h = 0.2$

$$p = \frac{(x_p - x_0)}{h} = \left(\frac{0.85 - 0.8}{0.2}\right) = 0.25$$

Thus, we get

$$f'(0.85) = \frac{1}{0.2}$$

$$\left\{0.0533+\frac{1}{2}(0.25\times 2+1)\times(-0.01683)+\frac{1}{6}\left(3\times\left(0.25\right)^2+6\times 0.25+2\right)\times\right.$$

$$\left.(-0.3076)+\frac{1}{12}\left(2\times\left(0.25\right)^3+9\times\left(0.25\right)^2+11\times 0.25+3\right)\times\left(-0.4101\right)\right\}$$

$= \frac{1}{5}$ {0.0533 - 0.1262 - 0.1891 - 0.2170}

= -2.3950

(ii) $x_0 = 1$; p = 0

Substituting values of p and the required difference we get

$$f''(0.2) = \frac{1}{\left(0.2\right)^2}\left\langle 0.0.65+0.1948+\frac{11}{12}\times 0.5025\right\rangle$$

$= \frac{1}{0.04} \times 0.6819$

= 17.0457

Example 3: Using Newton's Backward Difference formula to find solution at x = 2.2 for the table

x	1.4	1.6	1.8	2.0	2.2
$f'(x)$	4.0552	4.9530	6.0496	7.3891	9.0250

Solution: The difference table is as follows

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.4	4.0552				
		0.8978			
1.6	4.9530		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2.0	7.3891		0.2964		
		1.6359			
2.2	9.0250				

The value of x at we want to find f(x) : x_n = 2.2

$$h = x_1 - x_0 = 1.6 - 1.4 = 0.2$$

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\Delta y_n + \frac{1}{2}.\Delta^2 y_n + \frac{1}{3}.\Delta^3 y_n + \frac{1}{4}.\Delta^4 y_n\right]$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=2.2} = \frac{1}{0.2} \times \left[1.6359 + \frac{1}{2}\times 0.2964 + \frac{1}{3}\times 0.0535 + \frac{1}{4}\times 0.0094\right]$$

$$\therefore \quad \left[\frac{dy}{dx}\right]_{x=2.2} = 9.02142$$

and $$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n\right]$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=2.2} = \frac{1}{0.04}\left(0.2964 + 0.0535 + \frac{11}{12}\times 0.0094\right)$$

$$\therefore \quad \left[\frac{d^2y}{dx^2}\right]_{x=2.2} = 8.96292$$

Hence y'(2.2) = 902142, y''(2.2) = 8.96292

Example 4: Given a polynomial with the following data points

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^3}$ at x = 1.5

Solution: Constructing difference table for the function y = f(x)

x	y=f(x)	Δ y	Δ ^{2}y	Δ ^{3}y	Δ ^{4}y	Δ ^{5}y	Δ ^{6}y
1.0 1.1 1.2 1.3 1.4 1.5 1.6	7.989 8.403 8.781 9.129 9.451 9.750 10.031	0.414 0.378 0.348 0.322 0.299 0.281	-0.036 -0.030 -0.26 -0.023 -0.018	0.006 0.004 0.003 0.005	-0.002 -0.001 0.002	0.001 0.003	0.002

To find the derivative x = 1.5, taking x_n = 1.5 and apply the relation

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \frac{\nabla^5 y_n}{5} + ...\right]$$

From table h = 0.1, ∇y_n = 0.299, $\Delta^2 y_n$ = 0.023, $\Delta^3 y_n$ = 0.003, $\Delta^4 y_n$ = 0.001, $\Delta^5 y_n$ = 0.001

Thus, we get.

$$\left.\frac{dy}{dx}\right]_{x=1.5} = \frac{1}{0.1}\left[0.299 + \frac{(-0.023)}{2} + \frac{0.003}{3} + \frac{(-0.001)}{4} + \frac{0.001}{5}\right]$$

$$= 2.8845$$

Also, $$\left.\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n \frac{5}{6}\Delta^5 y_{n-1}..........\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1.5} = \frac{1}{(0.1)^2}\left(-0.023 + 0.003 + \frac{11}{12}(-0.001) + \frac{5}{6}(0.001)\right)$$

$$= -2.0083$$

Example 5: Find the derivative of f(x) at x = 0.4 from the following table

x:	0.1	0.2	0.3	0.4
y=f(x):	1.10517	1.22140	1.34986	1.49182

Solution: Since, x = 0.4 lies near the end of the table, therefore in this case we shsall use Newton's Backward formula. The difference table is as below.

x	y	Δ y	Δ^2y	Δ^3y
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		0.00127
0.3	1.34986		0.01350	
		0.14196		
0.4	1.49182			

Now, $$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + ...\right]$$

$$f'(0.4) = \frac{1}{0.1}\left(0.14196+\frac{0.01350}{2}+\frac{0.00127}{3}\right)$$

$$f'(0.4) = 1.4913$$

Hence derivative of f(x) at x = 0.4 is 1.4913

Self Check Exercise-2

Q.1 Given that

x:	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y:	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.6

Q.2 Find the first derivative of f(x) at 0.04 for the following table

x:	0.01	0.02	0.03	0.04	0.05	0.06
f(x):	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Q.3 Find the solution by Newton Backward difference method at x = 54 for the following data.

x:	50	51	52	53	54
f(x):	3.6840	3.7083	3.7325	3.7563	3.7796

14.5 Summary

Numerical Differentiation is the process of estimating the derivative of a function using discrete data points. Newton's forward difference interpolation formula can be adopted for this purpose, particularly when the data points are equally spaced.

Formula for 1st and 2nd derivatives using Newton's Forward interpolation is given as:

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\nabla y_0+\frac{\nabla^2 y_0}{2}+\frac{\nabla^3 y_0}{3}-\frac{\nabla^4 y_0}{4}+.........\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0-\Delta^3 y_0-\frac{11}{12}\Delta^4 y_0-\frac{5}{6}\Delta^5 y_0+..........\right]$$

This method is particularly useful when the data points are uniformly spaced and providess a straightforward way to estimate derivative using discrete data.

Newton's Backward difference formula is a method for numerical differentiation that approximates the derivative of a function using backward finite difference.

Newton's Backward difference formula is

$$\left.\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + ...\right]$$

and

$$\left.\frac{d^2 y}{dx^2}\right]_{x=x_n} = \frac{1}{h}\left[\Delta^2 y_n + \Delta^3 y_n + \frac{11}{12}\Delta^4 y_n + \frac{5}{6}\Delta^5 y_n + ...\right]$$

Newton's backward difference formula is particularly useful when the derivative needs to be computed at a specific point and when forward difference is impractical or less accurate.

14.6 Glossary:

- **Numerical Differentiation:** The process of estimating the derivative of a function using discrete data points.
- **Interpolating Polynomial:** A polynomial constructed using the given data points and forward difference, which approximates the unknown function.
- **Polynomial Interpolation:** The process of approximating a function by a polynomial, where the polynomial passes through all given data points.
- **Differences:** Δy_i - First forward difference at x_i

  $\Delta^2 y_i$ - Second forward difference at x_i

  $\Delta^k y_i$ - K^{th} forward difference at x_i
- **Numerical Differentiation:** Approximating derivatives of a function using numerical methods rather than analytic methods.
- **Finite Difference:** Approximation of derivatives based on the difference quotients sof function values at nearby points.
- **Smooth Functions:** Functions that are continuous and have well defined derivatives over their domain, where Newton backward difference formula provides accurate results.
- **Error Analysis:** Examination of difference between the numerical approximation and the exact derivatives, typically discussed in terms of absolute error or relative error.

14.7 Answers to Self Check Exercise-1

1. y'(2.25) = 26.375, y''(2.25) = 27

2. y' (2.03) = -0.4488, y''(2.03) = -1.0418

3. (a) = -3.74, (b) = 2.75

14.7 Answers to Self Check Exercise-2

1. 2.75, -0.715
2. 0.2583
3. 0.0003

14.8 References/Suggested Readings

1. An Introduction to Numerical Analysis by Kendall Atkinson
2. Numerical Analysis by J. Douglas Falres
3. Introductory Methods of Numerical Analysis by S.S Sastry
4. Finite Difference & Numerical Analysis by H.C. Saxena

14.9 Terminal Questions

1. Use the following table of values

x	0.0	0.5	1.0	1.5	2.0
f(x)	2.0286	2.4043	2.7637	3.1072	3.4250

to compute $f'(0.25)$, $f''(0.25)$.

2. Given Sin 0^0 = 0.000, Sin 10^0 = 0.1736, Sin 20^0 = 0.3420, Sin 30^0 = 0.5000, Sin 40^0 = 0.6428

   (a) Find the value of Sin 23^0

   (b) Find the numerical value of cos x at x = 10^0

   (c) Find the numerical value of $\frac{d^3y}{dx^3}$ at x = 20^0 for y = sin x.

3. Find the first and second derivatives of the function tabulated below, at the point x = 1.1

x:	1.0	1.2	1.4	1.6	1.8	2.0
f(x):	0	0.128	0.544	1.296	2.432	4.00

4. Find solution using Newton's Backward Difference formula

x	0	1	2	3
f(x)	1	0	1	10

at x = 4.

5. The population of certain town is given below. Find rate of growth of the population in 1961 from the following table.

Year	1931	1941	1951	1961	1971
Population (in thousands)	40.62	60.80	71.95	103.56	132.68

Estimate the population in the year 1976 and 2003.

6. Find the value of $f'(11)$ from the table given below

x:	6	7	9	12
f(x):	1.556	1.690	1.908	2.158

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Unit - 15

Numerical Differentiation by Newton's Divided Difference

Structure

15.1 Introduction

Dear student, in this unit we will learn to evaluate numerical differentiation of a function using Newton's divided difference formula. We learn numerical differentiation by Newton's forwarded backward method. When the data given is equidistant. But when the given data has not equal distance between from the method we use is Newton's divided difference. Using this method we can calculate the differentiative at a given point. In this unit we will learn how to apply Newton's divided difference formula to find derivative of a given tabular data.

15.2 Learning Objectives:

After studying this unit, students will be able to

1. define numerical differentiation.
2. define Newton's divided difference formula.
3. apply Newton's divided difference formula to evaluate numerical differentiation of given data.

15.3 Numerical differentiation using Newton's Divided Difference Formula

Newton's Divided Difference formula is a method used to construct a polynomial that passes through a given set of points. It is particularly useful for numerical differentiation

Procedure: Suppose that the function f is tabulated at (n+1) points $x_0, x_1, x_2, ..., x_n$ (not necessarily equidistant) and the corresponding values of the function f are $y_0, y_1, y_2, ..., y_n$ respectively.

By using Newton's divided difference formula,

f(x) = y_0 + (x-x_0) $\Delta_d y_0$ + (x-x_0) (x-x_1) $\Delta_d^2 y_0$ + (x-x_0) (x-x_1) x(x−x_2) $\Delta_d^3 y_0$ +. . . . . (1)

So, f'(x) = $\Delta_d y_0$ +

$\left[(x-x_0)+(x-x_1)\right] \Delta_d^2 y_0 + \left[(x-x_1)+(x-x_2)+(x-x_0)+(x-x_2)+(x-x_0)+(x-x_1)\right]$

$\Delta_d^3 y_0$ + ... (2)

Similarly, $f''(x) = 2\ \Delta_d^2 y_0 + 2\left[(x-x_0)+(x-x_1)+(x-x_2)\right] \Delta_d^3 y_0$ . . . . (3)

In the similar manner, we can obtain the derivatives of higher orders.

Some Related Questions:

Let us apply Newton's divided difference formula to find derivative from given data.

Example 1: Find f'(1.5) from the following data

x	y=f(x)	$\Delta_d y$	$\Delta_d^2 y$
0 1 3	0 0.84 0.42	0.84 -0.21	-0.35

By using Newton's divided difference formula:

f(x) = y_0 + (x-x_0) $\Delta_d y_0$ + (x-x_0) (x-x_1) $\Delta_d^2 y_0$

So, f'(x) = $\Delta_d y_0$ + (2x-x_0-x_1) $\Delta_d^2 y_0$

∴ f'(1.5) = 0.84 + (2 × 1.5 - 0 - 1) (-0.35)

= 0.84 + 2 (-0.35)

= 0.84 - 0.70

= 0.14

Example 2: Find F'(x) at x = 1.5 from the following data:

x:	0	1	2	3
F(x):	1	2	0	2

Solution: The divided difference table is:

x	y=f(x)	Δ dy	Δ^2dy	Δ^3dy
0 1 2 3	1 2 0 2	1 -2 2	-15 2	1.1667

The Newton Polynomial p(x) is

$$p(x) = y_0 + (x - x_0)\Delta dy_0 + (x - x_0)(x - x_1)\ \Delta_d^2 y_0 + (x - x_0)(x - x_1)\ x(x - x_2)\ \Delta_d^3 y_0$$

p(x) = 1 + (x - 0) + (x - 0) (x - 1) (-1.5) + (x - 0) (x - 1) (x - 2) 1.1667

p(x) = 1 + x - 1.5x (x - 1) + 1.1667x (x - 1) (x - 2)

$$\therefore \qquad p'(x) = \frac{d}{dx}\left[1 + x - 1.5x(x-1) + 1.1667x(x-1)(x=2)\right]$$

$= 1 + 1.5\ (2x - 1) + 1.1667\ (2x^2 - 5x + 2)$

$= 1 - 3x + 1.5 + 2.3334x^2 - 5.8335x + 2.3334$

$p'(x) = 2.3334x^2 - 8.8335x + 4.8334$

At x = 1.5

$p'(1.5) = 2.3334\ (1.5)^2 - 8.8335\ (1.5) + 4.8004$

$= 2.3334\ (2.25) - 8.8335\ (1.5) + 4.8334$

= 5.25015 - 13.25025 + 4.8334

= -3

p'(1.5) = -3

Example 3: Compute Newton Divided Difference table for the following data

x:	0	1	3	4	7
f'(x):	1	2	49	129	813

Solution: Newton divided difference table is given by

x	$f(x)$	$\Delta d f(x)$	$\Delta^2 d f(x)$	$\Delta^3 d f(x)$
0 1 3 4 7	1 3 49 129 813	2 23 80 228	7 19 37	3 3

Example 4: The distance S covered by a car in a given time 't' is given in following table:

t(in minutes)	10	12	16	17	22
s(in kilometer)	12	15	20	22	32

Find the speed of car at t = 14 minutes.

Self Check Exercise

Q.1 Find $f'(2)$ from the following data

x:	1	2	3
$f(x)$:	1	4	9

Q.2 Compute Newton Divided Difference table for the following data

x:	0	0.5	1	2
$f(x)$:	1	1.8987	3.7183	11.3891

15.4 Summary

1. Newton's divided difference formula is a method used to perform numerical differentiation and interpolation
2. That is we can obtain derivatives of higher order
3. This method is particularly useful for numerical differentiation

15.5 Glossary

1. Numerical differentiation: It involves the computation of a derivatives of a function f from given values of f.
2. Newton's Divided Difference Formula: It is a method used to perform numerical differentiation and interpolation

15.6 Answers to Self Check Exercise

1. $f'(2) = 4$
2.

x	$f(x)$	$\Delta df(x)$	$\Delta^2 df(x)$	$\Delta^3 df(x)$
0 0.5 1 2	1 1.8987 3.7183 11.3891	1.7974 3.6392 7.6708	1.8418 2.6877	0.4229

15.7 References/Suggested Readings

1. Numerical Analysis of Richard L. Burden
2. Introductory Methods of Numerical Analysis by S.S Sastry
3. Finite Difference & Numerical Analysis by H.C. Saxcna
4. An Introduction to Numerical Analysis by Kendall E. Atkinson

15.8 Terminal Questions

1. Find $f'(2)$ from the following data

x:	1	2	3
$f(x)$:	2	3	5

2. Find $f'(x)$ at x = 3 from the following data

x:	0	1	4	5
$f(x)$:	4	3	24	39

3. From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for x = 5.

x:	2	4	9	10
$f(x)$:	4	56	711	980

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Unit - 16

Numerical Differentiation Using Stirling Formula

Structure

16.1 Introduction

Dear student in this unit we will study about sterling control difference method of numerical differentiation. Sterling Central difference method is used when we have to interpolate a value near centre of the given data. After writing sterling central difference table we will apply sterling central difference formula to find the numerical derivative of given data near the given central point of the data.

16.2 Learning Objectives:

After studying this unit students will be able to

1. define central differences
2. define central differences using Sterling's formula.
3. write central difference table using Sterling method.
4. evaluate numerical differentiation at a given point for the given data using sterling formula.

16.3 Numerical Differentiation Using Stirling Central Difference Formula

Stirling's central difference interpolation formula (taking x_0 as the middle value of the table) is given by:

$$y = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \text{..........} \quad (1)$$

Differentiating (1) w.r.t. 'p'

$$\frac{dy}{dp} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\,\Delta^2 y_{-1} + \frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\ldots\ldots\ (2)$$

Also $\frac{dp}{dx} = \frac{1}{h}$ (3)

Again $\frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$ (4)

Using (2), (3) in (4) we get.

$$\frac{dy}{dx} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + p\Delta^2 y_{-1}\,\frac{3p^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \ldots\ldots\ldots\right]$$

Now, at x = x_0, p = 0

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=x_0} \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\ldots\ldots\ldots\right]$$

Similarly $$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} - \ldots\ldots\ldots\right]$$

Sterling's Central Difference Method

Some Examples

Example 1: From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.6 using Stirling's formula.

x:	1.0	1.2	1.4	1.6	1.8	2.0	2.2
y:	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.9530		0.1988		0.0080		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2.0	7.7891		0.2964				
		1.6359					
2.2	9.0250						

Take x0 = 1.6 so that y0 = 4.9530, Δy_0 = 1.0966, Δy_{-1} = 0.8978,

$\Delta^2 y_{-1}$ = 0.1988, $\Delta^3 y_{-2}$ = 0.0361, $\Delta^3 y_{-1}$ = 0.0441, $\Delta^4 y_{-2}$ = 0.0080,

$\Delta^5 y_{-3}$ = 0.0013, $\Delta^5 y_{-2}$ = 0.0014. $\Delta^6 y_{-3}$ = 0.0001

Also, h = 0.2

Now, by Stirling's Formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right)\right]$$

and $$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3}\right]$$

$\therefore$ $$\left(\frac{dy}{dx}\right)_{x=1.6} = \frac{1}{0.2}\left[\frac{1.0966}{2} - \frac{1}{6}\frac{0.0361+0.0441}{2} + \frac{1}{30}\frac{0.0013+0.0014}{2}\right]$$

= 5 [0.9972 - 0.00668 + 0.00004]

= 4.9528

and $$\left(\frac{d^2 y}{dx^2}\right)_{x=1.6} = \frac{1}{(0.2)^2}\left[1.1988 - \frac{1}{12}(0.0080) + \frac{1}{90}(0.0001)\right]$$

= 25 [0.1988 - 0.00067 + 0.000001]

= 4.9533

Example 2: A slider machine moves along a fixed straight rod. Its distance x cm. along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.

t:	0	0.1	0.2	0.3	0.4	0.5	0.6
x:	30.13	31.62	32.87	33.64	33.95	33.81	33.24

Solution: The difference table is

t	x	Δx	$\Delta^2 x$	$\Delta^3 x$	$\Delta^4 x$	$\Delta^5 x$	$\Delta^6 x$
0	30.13						
		1.49					
0.1	31.62		-0.24				
		1.25		-0.24			
0.2	32.87		-0.48		0.26		
		0.77		0.02		-0.27	
0.3	33.64		-0.46		-0.01		0.29
		0.31		0.01		0.02	
0.4	33.95		-0.45		0.01		
		-0.14		0.02			
0.5	33.81		-0.43				
		-0.57					
0.6	33.24						

As the derivatives are required near the middle of the table, We use stirling formula

$$\left(\frac{dx}{dt}\right)_{t_0} = \frac{1}{h}\left(\frac{\Delta x_0 + \Delta x_{-1}}{2}\right) - \frac{1}{6}\left(\frac{\Delta^3 x_{-1} + \Delta^3 x_{-2}}{2}\right) + \frac{1}{30}\left(\frac{\Delta^5 x_{-2} + \Delta^5 x_{-2}}{2}\right) + \text{.............}(i)$$

$$\left(\frac{d^2x}{dt^2}\right) = \frac{1}{h^2}\left[\Delta^2 x_{-1} - \frac{1}{12}\Delta^4 x_{-2} + \frac{1}{90}\Delta^6 x_{-3} - \text{.........}\right] \qquad \text{(ii)}$$

Here h = 0.1, to = 0.3, Δx_0 = 0.31, Δx_{-1} = 0.77, $\Delta^2 x_{-1}$ = -0.46 etc.

Putting these values in above equations (i), (ii) we get

$$\left(\frac{dk}{dt}\right)_{0.3} = \frac{1}{0.1}\left[\frac{0.31+0.77}{2} - \frac{1}{6}\left(\frac{0.01+0.02}{2}\right) + \frac{1}{30}\left(\frac{0.02-0.27}{2}\right)\text{...........}\right]$$

= 5.33

$$\left(\frac{d^2k}{dt^2}\right)_{0.3} = \frac{1}{(0.1)^2}\left[-0.46-\frac{1}{12}(-0.01)+\frac{1}{90}(0.29)\right]$$

$= -45.6$

Hence the required velocity is 5.33 on/sec and acceleration is -45.6 cm/sec^2.

Example 3: The elevation above a datum line of seven points of road are given below

x:	0	300	600	900	1200	1500	1800
y:	135	149	157	183	201	205	193

Find the gradient of road at the middle point

Solution: Here h = 300, $x_0 = 0$, $y_0 = 135$, we require the gradient $\frac{dy}{dx}$ at x = 900.

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	135					
		14				
300	149		-6			
		8		24		
600	157		18		-50	
		26		-26		70
900	183		8		20	
		18		-6		16
1200	201		-14		4	
		4		-2		
1500	205		-16			
		-12				
1800	193					

Using Stirling's Formula for the first derivative, we get

$$y'(x_0) = \frac{1}{h}\left[\left(\frac{\Delta y_0+\Delta y_{-1}}{26}\right)-\frac{1}{6}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{1}{30}\left(\frac{\Delta^5 y_{-2}+\Delta^5 y_{-2}}{2}\right)\right]$$

$$= \frac{1}{300}\left[\frac{1}{2}(18+26)-\frac{1}{12}(-6-26)+\frac{1}{60}(-16+70)\right]$$

$= \frac{1}{300}$ [22 + 2.666 + 0.9]

$= 0.085$

Hence, the gradient of road at the middle point is 0.085

Example 4: Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ using Stirling's formula at x = 15 for the following data.

x:	0	5	10	15	20	25	30
f(x):	0.0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Solution: The value of table for x and y

x:	0	5	10	15	20	25	30
y	0.0	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Stiring method to find solution

h = 5 - 0 = 5

The difference table is

x	y	Δ y	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
0 5 10 15 20 25 30	0 0.0875 0.1763 0.2679 0.3640 0.4663 0.5774	0.0875 0.0888 0.0916 0.0961 0.1023 0.1111	0.0013 0.0028 0.0045 0.0062 0.0088	0.0015 0.0017 0.0017 0.0026	0.0002 0 0.0009	-0.0002 0.0009	0.0011

Now by Stirling formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta x_0+\Delta x_{-1}}{2}\right)-\frac{1}{6}\left(\frac{\Delta^3 x_{-1}+\Delta^3 x_{-2}}{2}\right)+\frac{1}{30}\left(\frac{\Delta^5 x_{-2}+\Delta^5 x_{-3}}{2}\right)\right]$$

$$\left(\frac{dy}{dx}\right)_{x=15} = \frac{1}{5}\left[\frac{0.0963+0.0916}{2}-\frac{1}{6}\frac{0.0017+0.0017}{2}+\frac{1}{30}\frac{-0.0002+0.0009}{2}\right]$$

$$= \frac{1}{5}\ [\ 0.09385 - 0.0002833 + 0.00001166]$$

= 0.018715

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3}\right]$$

$$= \frac{1}{(5)^2}\left[0.0045 - \frac{1}{12}(0) + \frac{1}{90}(0.0011)\right]$$

$$= \frac{1}{25}\left[0.0045 + \frac{1}{90}(0.0011)\right]$$

$$= \frac{1}{25}\ [0.0000122 + 0.0045]$$

= 0.000180488

Example 5: Use Stirling formula to find y_{28} given y_{20} = 49225, y_{25} = 48316, y_{30} = 47236, y_{35} = 45926, y_{40} = 44306?

Solution: Here x0 = 30

h = 5

x = 28

$$u = \frac{x - x_0}{h} = \frac{28 - 30}{5} = -0.4$$

By Stirling Formulas

$$y_n = y_0 + \mu\frac{\Delta x_0 + \Delta x_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{6}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \frac{u^2(u^2-1)}{24}\Delta^4 x_{-2}$$

The difference table is:

x	$u = \frac{x-30}{5}$	y	Δ y	Δ^2y	Δ^3y	Δ^4y
20	-2	49225				
			-909			
25	-1	48316		-171		
			-1080		-59	
30	0	47236		-230		-21
			-1310		-80	
35	1	45926		-310		
			-1620			
40	2	44306				

$\therefore$ y(28) = 47236 + (-0.4) $\frac{(-1310-1080)}{2}$ + $\frac{(-0.4)^2}{2}$ (-230) + $\frac{(-0.4)(0.16-1)}{6}$

× $\frac{(-80-59)}{2}$ + (0.16) $\frac{(-0.16-1-59)}{24}$ (-21)

y(28) = 47236 + 478 - 18.4 - 3.8920 + 0.1176

y(28) = 47692

Self Check Exercise

Q.1 From the following table, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 7.5 using Stirling formula

x:	7.47	7.48	7.49	7.50	7.51	7.52	7.53
y:	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Q.2 Compute from the following table the value of the derivation of y = f(x) at x = 1.7489.

x:	1.73	1.74	1.75	1.76	1.77
y:	1.772844100	1.55204006	1.737739435	1.720448638	1.703329888

Q.3 Find f'(93) from the following table

x:	60	75	90	105	120
f(x):	28.2	38.2	43.2	40.9	37.7

16.4 Summary

- Stiring Control Difference method is a technique used for numerical differentiation, specifically for approximating the derivative of a function using equally spaced points.
- Stirling central difference formula is

$$\left.\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\left(\frac{\Delta x_0+\Delta x_{-1}}{2}\right)-\frac{1}{6}\left(\frac{\Delta^3 x_{-1}+\Delta^3 x_{-2}}{2}\right)+\frac{1}{30}\left(\frac{\Delta^5 x_{-2}+\Delta^5 x_{-3}}{2}\right)\text{..........}\right]$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

- Stirling approximation involves the use of a forward difference table
- Thus, it is an enhanced version of the central difference formula that includes higher order correction terms to improve accuracy

16.5 Glossary

- **Numerical Differentiation:** The technique of approximating derivative numerically when an analytic expression for the derivative is not available or impractical to use.
- **Central Difference Formula:** A method for numerically approximating the derivatives of a function using function value at points symmetrically located around x.
- **Stirling Interpolation:** It is a technique which is used to obtain the value of a function at an intermediate point within the range of a discrete set of known data points.

16.6 Answers to Self Check Exercise

1. 0.235, =13.61111
2. -0.17396520185
3. -0.0363

16.7 References/Suggested Readings

1. Introductory Methods of Numerical Analysis by S.S. Sastry
2. Finite Difference & Numerical Analysis by H.C. Saxena.
3. An Introduction to Numerical Analysis by Kendall Atkinson.

16.8 Terminal Questions

1. Given the following table of values of x and y

x:	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y:	1.000	1.025	1.049	1.072	1.095	1.118	1.140

   Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x = 1.15

2. A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t seconds

t:	0	0.2	0.4	0.6	0.8	1.0	1.2
θ:	0	0.12	0.49	0.12	2.02	3.20	4.67

Calculate the angular velocity and the angular acceleration of the rod, when t = 0.6 second.

3. Find $\frac{dy}{dx}$ at x = 1 from the following table by constructing a central difference table:

x:	0.7	0.8	0.9	1.0	1.1	1.2	1.3
y:	0.644218	0.7173560	0.783327	0.841471	0.891207	0.932039	0.963576

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Unit - 17

Numerical Integration-Trapezoidal Rule

Structure

17.1 Introduction

Dear student, the problem of evaluating define integral alises both mathematics and in many areas of Science and engineering. We have already learned to calculate simple integral such as $\int_0^1 e^x dx = e - 1$ and $\int_0^\pi cex\, x\, dx = 1$. But same time it is very hard to evaluate to integral by means of the usual tricks of variable substitution and integration by parts. In this unit we will learn to evaluate the definite integral approximately by using the concept of interpolation, to derive the formula for numerical integration. By using numerical integration technique.

We can determine the integrals which are not possible by symbolic methods. Hence these methods are more useful. So in this unit we will learn to derive such method i.e. Trapezoidal rule using Newton's cotes quadrates formula along with its error analysis.

17.2 Learning Objectives:

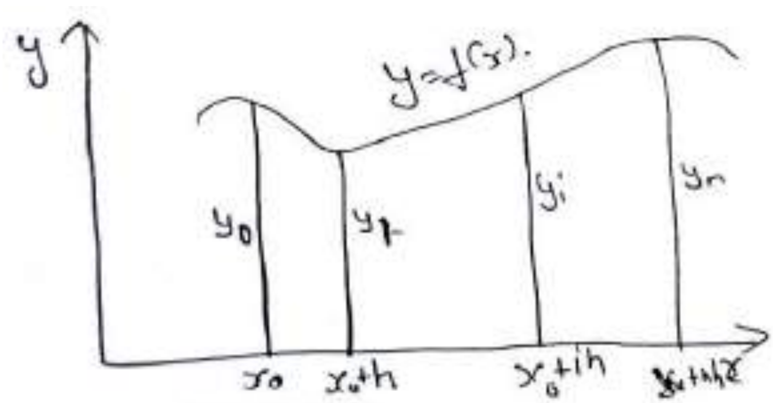
After Studying this unit, you will be able to

1. Know the importance of numerical integration
2. Derive Newton's cotes quadrature formula
3. Derive Trapezaidal rule of Numerical integration
4. Derive the expression of error estimation for T.R. Use Traperzoidal rule of Numerical integration.
5. Trapezoidal rule.

Numerical Integration:

The definite integral $\int_a^b f(x)dx$ can be interpreted as the area bounded by the curve y=f(x), the x-axis and the ordinates x = a and x = b.

A simpler approach for approximating the value of $\int_a^b f(x)dx$ would be evaluating an integral by dividing the interval into several subintervals, applying the rule separately are each of subinterval and then additing them. This is because only, definite integral are additive over subintervals. This strategy is called composite numerical Integration. We can approximate the integrate by a linear curve (y=a+bx) or by a second degree arve (y=a+bx+cx^2) or by a third degree curve.



17.3 Newton-Cote's Quadrature Formula And Trapezoidal Rule

Newton cotes formulas are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with an approximating function that is easy to integrate. Using Newton-Cote's formula we can derive Trapezoidal, Simpson's $3/8$ formula.

Let $\quad I = \int_a^b f(x)dx$

Let us devide the interval (a, b) into n sub intervals of width h so that x0 = a, x_1 = a + h, x_2 = a + 2h ...., x_n = x0 + nh = b

Then $I = \int_a^b f(x)dx = \int_{x^0}^{x_0+nh} f(x)dx$

Taking $x = x_0 + ph \quad \Rightarrow \quad dx = hdp$

Also p = $\frac{x-x_0}{h}$, such that when x = x_0, p = 0 and when x = x_0 + nh, p = n. Using all this

$$I = \int_o^n hf(x_0 + ph)dp$$

$$I = h\int_o^n f(x_0 + ph)dp$$

Since Newton's Forward interpolation formula is

f(x) = y0 + pΔy0 + $\frac{p(p-1)}{2!}\Delta^2 y_0$ + $\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$ +......... using this formula in I, we get

$$I = \left[h\int_o^n y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots.\right]\text{dp.}$$

on integrating term by term we obtain

$$I = h\left[y_0 p + \frac{p^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4 - 4p^3 + 4p^2}{4}\right)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]_0^n$$

$$= h\left[y_0 p + \frac{p^2}{2}\Delta y_0 + \left(\frac{2p^3 - 3p^2}{12}\right)\Delta^2 y_0 + \frac{1}{24}\left(p^4 - 4p^3 + 4p^2\right)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]_0^n$$

$$= h\left[y_0 n + \frac{n^2}{2}\Delta y_0 + \left(\frac{2n^3 - 3n^2}{12}\right)\Delta^2 y_0 + \frac{1}{24}\left(n^4 - 4n^3 + 4n^2\right)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]_0^n$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{2n^3 - 3n}{12}\right)\Delta^2 y_0 + \frac{1}{24}\left(n^4 - 4n^3 + 4n^2\right)\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n^2 - 4n + 4)}{24} + \Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{4!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

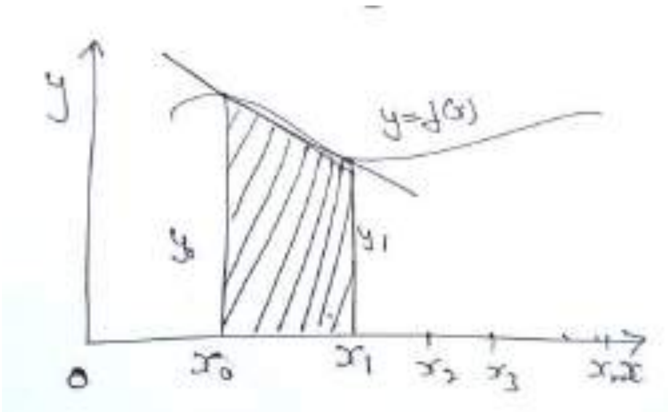
I = $\int_{a}^{b} f(x)dx$ = $\int_{x^0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{4!}\Delta^3 y_0 + \text{..........}\right]$

I = $nh\left[y_0 + \frac{n}{2}\Delta y_0 + \left(\frac{n^2}{3} - \frac{n}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^3}{4} - n^2 + n\right)\frac{\Delta^3 y_0}{3!} + \text{............}\right]$

This is known as the Newton's Cote's quadrature formula. From this formula we can derive Trapezoidal rule, Simpson's $\frac{1}{3}$ rule as well as Simpson's $\frac{3}{8}$ rule, just by changing the value of n = 1, 2, 3 respectively.

Trapezoidal Rule:

Putting n = 1 in Newton-Cote's formula and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e. a polynomial of first order so that differences of order higher than first become zero.



$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{\Delta y_0}{2}\right]$

$= h\left[y_0 + \left(\frac{y_1 - y_0}{2}\right)\right]$

$= h\left[\frac{2y_0 + y_1 - y_0}{2}\right]$

$\int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}$ [y_0 + y_1]

Similarly $\int_{x_0+h}^{x_0+2h} f(x)dx = h\left[y_1+\frac{1}{2}\Delta y_1\right]$

$$= h\left[y_1+\frac{1}{2}(y_2-y_1)\right]$$

$$= \frac{h}{2}\left[y_1+y_2\right]$$

$$\int_{x_0+(n-1)}^{x_0+nh} = \frac{h}{2}\left[y_{n-1}+y_n\right]$$

Using the additive property of definite integral, on adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+n}^{x_0+2h} f(x)dx + \ldots\ldots\ldots\ldots \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}\left[(y_0+y_1)+\frac{h}{2}(y_1+y_2)+\frac{h}{2}(y_2+y_3)+\ldots\ldots\ldots\ldots+\frac{h}{2}(y_{h-1}+y_n)\right]$$

$$= \frac{h}{2}\left[y_0+y_1+y_1+y_2+y_2+y_3+\ldots\ldots\ldots\ldots+y_{h-1}+y_n\right]$$

$$\int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2y_1+y_2+y_3+\ldots\ldots\ldots\ldots+y_{h-1}\right]$$

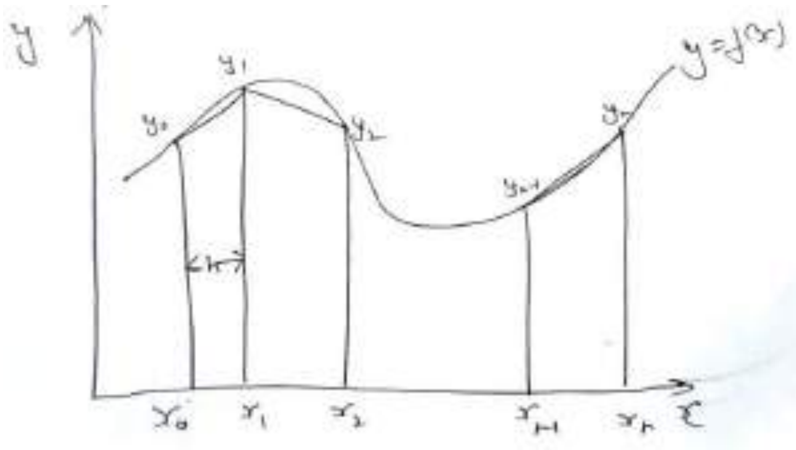
This is known as trapezoidal rule, because when f is a function with positive values, $\int_a^b f(x)dx$ is approximated by the are in a trapezoid.

This rule gives the exact value of the integral if f(x) is a linear function

Geometrical Interpretation:

Geometrically, in trapezoidal rule, we approximate the portion of the curve y = f(x) between x = x_0 and x = x_n by n straight liner joining the points (x_0, y_0) and (x_1, y_1); (x_1, y_1) and (x_2, y_2); .... (x_{n-1}, y_{n-1}) and (x_n, y_n) as shown in the figure.

Then we calculate the area under each of these straight lines i.e. we calculate the area of each of these n trapeziums so formed. The value of the definite integral $\int_{x_0}^{x_n} f(x)\,dx$ i.e. area under the curve y = f(x) between the ordinates x=x_0 and x=x_n and above the x-axis is approximated by the sum of the area of these n trapeziums.



Error In Trapezoidal Rule

Since the error in the quadrature formula is given by

E = $\int_a^b f(x)dx$ - $\int_a^b p(x)dx$, where p(x) is the polynomial presenting the function y = f(x), in the interval [a, b].

Error in trapezoidal rule can be derived by expanding y = f(x) around x = x_0 by Taylor's series i.e.

f(x) = y = y_0 + (x - x_0)y_0' + $\frac{(x-x_0)^2}{2!}$ y_0'' + ... where (1)

Represent the derivative with respect to x.

Therefore, $\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1}\left[y_0 + (x-x_0)y_0^1 + \frac{(x-x_0)^2}{2!}y_0'' + ...\right]dx$

$$= \int_{x_0}^{x_1} y_0 dx + \int_{x_0}^{x_1}(x-x_0)y_0^1 + \int_{x_0}^{x_1}\frac{(x-x_0)^2}{2!}y_0''dx + ...$$

$$= y_0\left[(x)\right]_{x_0}^{x_1} + \left[\frac{(x-x_0)^2}{2!}\right]_{x_0}^{x_1} y_0 + \left[\frac{(x-x_0)^2}{3.2}\right]_{x_0}^{x_1} y_0'' + ...$$

$$\Rightarrow \quad \int_{x_0}^{x_1} f(x)\,dx = y_0(x_1 - x_0) + \frac{(x-x_0)^2}{2}y_0' + \frac{(x_1-x_0)^3}{6}y_0'' + ...$$

Since $x_1 = x_0 + h$

$\Rightarrow \quad x_1 - x_0 = h$

$$\therefore \quad \int_{x_0}^{x_1} f(x)\,dx = h\,y_0 + \frac{h^2}{2!} y0' + \frac{h^3}{3!} y_0'' + \ldots\ldots\ldots \qquad (2)$$

Let A_1 is the area of first trapezium in the interval $[x_0, x_1]$

Which is equal to $A_1 = \frac{h}{2}(y_0 + y_1) \qquad (3)$

Putting $x = x_0 + h$ and $y = y_1$ in (1) we get

$$y_1 = y_0 + (x_0 + h - x_0)\,y_0' + \frac{(x_0 + h - x_0)^2}{2!} y_0'' + \ldots\ldots\ldots$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots \qquad (4)$$

Using the value of y_1 in (3)

$$A_1 = \frac{h}{2}\left(y_0 + y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots\right)$$

$$A_1 = \frac{h}{2}\left(2y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots\right) \qquad (5)$$

Therefore, the error in the interval (x_0, x_1) is given by

$$E_1 = \int_{x_0}^{x_1} f(x)\,dx - \int_{x_0}^{x_1} p(x)\,dx \text{ , Here } \int_{x_0}^{x_1} p(x)\,dx = A_1$$

$$T \quad \Rightarrow \quad E_1 = \int_{x_0}^{x_1} y(x)\,dx - A_1$$

$$= \left(hy_0 + \frac{h^2}{2!} hy_0' + \frac{h^3}{3!} y_0'' + \ldots\ldots\ldots\right) - \left(hy_0 + \frac{h^2}{2!} hy_0' + \frac{h^3}{2.2!} y_0'' + \ldots\ldots\ldots\right)$$

$$= \frac{h^3}{3!} y_0'' - \frac{h^3}{2.2!} y_0'' + \ldots\ldots\ldots$$

$$= \left(\frac{1}{6} - \frac{1}{4}\right) h^3 y_0'' + \ldots\ldots\ldots$$

$$E_1 = \frac{-h^3}{12} y_0'' + \ldots\ldots\ldots$$

Principal part of error in $[x_0, x_1]$ = $\frac{-h^3}{12} y_0''$

Similarly $E_2 = \int_{x_1}^{x_2} f(x)\,dx - A_2 = \frac{-h^3}{12} y_1''$

$E_3 = \int_{x_2}^{x_3} f(x)\,dx - A_3 = \frac{-h^3}{12} y_2''$

$E_n = \int_{x_{n-1}}^{x_n} f(x)\,dx - A_n = \frac{-h^3}{12} y_{n-1}''$

Therefore the total error $E = E_1 + E_2 + \ldots\ldots\ldots\ldots\ldots + E_n$

$\Rightarrow \qquad E = \frac{-h^3}{12}\left[y_0''+y_1''+y_2''+\ldots\ldots\ldots\ldots+y_{n-1}''\right]$

Let maximum $\left\{y_0''+y_1''+y_2''+\ldots\ldots\ldots\ldots+y_{n-1}''\right\}$ = y''(X) i.e. y''(x) is the largest of the n quantifies y_0'', y_1'', y_2'' .... y''_{n-1}, we have

$E < \frac{-nh^3}{12} y''(X)$ using nh = b - a

$E < \frac{-(b-a)^{h^2}}{12} y''(x)$

Hence the error in trapezoidal rule is of order h^2.

Where possible, we can choose h small enough to make this error negligible. In case of hand computation, it may not be possible but in a computer program in which f(x) may be generated anywhere in $x_0 \leq x \leq x_n$, the interval may be subdivided smaller and smaller until there is a sufficient accuracy.

Working of Trade ZOIDAL Rule for $\int_a^b f(x)dx$.

Let y = f(x) take n value f x in interval [a, b]

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	x_2	....	y_n

Then $\int_a^b f(x)dx = \frac{n}{2}\left[(y_0+y_n)+2(y_1+y_{2+}\ldots\ldots\ldots\ldots+y_{n-1})\right]$

Where h = $\frac{b-a}{n}$

For more under of trapezoidal rub let us try following examples:

Example 1: Evaluate $\int_a^b \frac{1}{1+x^3} dx$ taking h = 1 by using trapezoidal rule.

Solution : Here x_0 = 0, x_n = 6 and h = 1

also we know h = $\frac{x_n - x_0}{n}$

= 1 = $\frac{6-0}{n}$

n = 6

So we have to divide (0, 6) into six parts each of width 1. Now the tabulate the function f(x) = $\frac{1}{1+x^2}$ as follow:

x	0	1	2	3	4	5	6
y=f(x)	1	0.5	0.11	0.0357	0.0157	.0079	.0046

By Trapezoidal Rule:

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + ... + y_{n-1}\right)\right]$$

$$\Rightarrow \quad \int_a^b \frac{1}{1+x^3} dx = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5\right)\right]$$

$$= \frac{1}{2}\left[(1+.0046) + 2(0.5 + 0.11 + 0.0357 + 0.0157 + 0.0079)\right]$$

$$= \frac{1}{2}\left[1.0046 + 2(0.669)\right]$$

$$= \frac{1}{2}[1.0046 + 1.338]$$

$$= \frac{1}{2}[2.3426]$$

$$\int_a^b \frac{1}{(1+x^3)}dx = 1.1713$$

Example 2: Evaluate $\int_D^1 \frac{dx}{1+x^2}$ by Trapezoidal Rule, using six coordinates. Also determine the error.

Solution: Here $x_0 = 0$, $x_n = 1$, Here number of coordinator are given six so number of interval swill be :-

So, $h = \frac{x_n - x_0}{n} = \frac{1-0}{5} = \frac{1}{5} = 0.2$

We will divide (0, 1) into 5 parts each of width 0.2 the table value of function $f(x) = \frac{1}{1+x^2}$ is as

x	0	0.2	0.4	0.6	0.8	1
y	1	0.9615	0.8621	0.7353	0.6098	0.5

By Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ...+ y_{n-1})\right]$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[(1+0.5) + 2(0.9615 + 0.8621 + 0.7353 + 0.6098)\right]$$

$$= \frac{0.2}{2}\left[1.5 + 2(3.1687)\right]$$

$$= 0.1\ [1.5 + 6.3374]$$

$$\int_0^1 \frac{1}{1+x^2}dx = .78374$$

To Evaluate the error:

Since $\int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}(x)\right]_0^1 = \left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{\pi}{4}$

$\therefore$ Absolute error = $\frac{\pi}{4}$ - .078374

= 0.00165

Example 3: Evaluate $\int_0^1 \sqrt{\sin x + cox\, x}$ dx taking 5 subintervals using trapezoidal rubs.

Solution: Here given n = 5 and x_0 = 0, x_n = 1

So h = $\frac{x_n - x_0}{n} = \frac{1-0}{5} = \frac{1}{5}$ = 0.2

So, the table of given function is

x	0	0.2	0.4	0.6	0.8	1
y=f(x)	1	1.0857	1.1448	1.1790	1.1891	1.1755

By Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + ...+ y_{n-1}\right)\right]$$

$$\int_0^1 \sqrt{\sin x + cox\, x}dx = \frac{h}{2}\left[\left(y_0 + y_5\right) + 2\left(y_1 + y_2 + y_3 + y_4\right)\right]$$

$$= \frac{0.2}{2}\left[(1+1.7755) + 2(1.0857 + 1.1448 + 1.1790 + 1.1891)\right]$$

$$= 0.1\left[2.1755 + 2(4.5986)\right]$$

= 0.1 [2.1755 + 9.1972]

$$\int_0^1 \sqrt{\sin x + cox\, x}dx = 1.13727$$

Example 4 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by trapezoidal rule using n = 6.

Solution: Here n = 6, x_0 = 0, x_n = 6

h = $\frac{x_n - x_0}{n} = \frac{6-0}{6} = \frac{6}{6}$ = 1

h = 1

The values of x and y = f(x) = $\frac{1}{1+x^2}$ are given by

x	0	1	2	3	4	5	6
y	1	0.5	0.2	0.1	0.0588	0.0385	0.027

So By Trapezoidal Rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0+y_n\right)+2\left(y_1+y_2+y_3+..........+y_{n-1}\right)\right]$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{1}{2}\left[\left(y_0+y_6\right)+2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= \frac{1}{2}\left[1.027+2(6).8973\right]$$

$$= \frac{1}{2}(1.027+1.7946)$$

$$= \frac{1}{2}(2.8216)$$

$$\Rightarrow \quad \int_0^1 \frac{1}{1+x^2}dx = 1.4108$$

Example 5: $\int_0^1 \frac{dx}{1+x}$ using trapezoidal Rule using six interval.

Solution: Here $x_0 = 0$, $x_n = 1$ and n = 6

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{1-0}{6} = \frac{1}{6}$$

The value of x and y = f(x) = $\frac{1}{1+x}$ are given by

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{6}$	1
f(x)	1	0.8511	0.75	0.6667	0.5454	0.5

$\therefore$ By Trapezoidal rule.

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + ... + y_{n-1}\right)\right]$$

$$\int_0^1 \frac{1}{1+x}dx = \frac{1/6}{2}\left[(1+0.5) + 2(0.857 + 0.75 + 0.6667 + 0.5454)\right]$$

$$= \frac{1}{12}(7.1384)$$

$$\Rightarrow \int_0^1 \frac{1}{1+x}dx = 0.5949$$

Example 6: Evaluate $\int_0^4 \sqrt{64 - x^3}\,dx$ by Trapezoidal rule using a ordinates.

Solution: Here $x_n = 1$, $x_0 = 0$, we have to divide (0, 4) into 8 subinterval as the number of ordinals are 9.

$\therefore$ n = 8

Hence $h = \frac{x_n - x_0}{n} = \frac{4-0}{8} = \frac{4}{8} = 0.5$

The value of x and y = f(x) = $\sqrt{64 - x^3}$ = are given by

x	0	0.5	1	1.5	2	2.5	3	3.5	4
f(x)	8	7.9922	7.9372	7.7862	7.4833	6.9552	6.0828	4.5962	0

$\therefore$ By Trapezoidal Rule

$$\int_0^4 \sqrt{64 - x^3}\,dx = \frac{h}{2}\left[\left(y_0 + y_1\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right)\right]$$

$$= \frac{0.5}{2}\left[(8+0) + 2(7.9922 + 7.9372 + 7.7862 + 7.4833 + 6.9552 + 6.0828 + 4.5962)\right]$$

= .025 (105.6662) = 26.4166

$$\int_0^4 \sqrt{64 - x^3}\,dx = 26.4166$$

Example 7: Evaluate $\int_0^{\pi} t\sin t$ dt by using trapezoidal rule using equal subinterval of h = $\frac{\pi}{6}$

Solution: Since $x_0 = 0$, $x_n = \pi$ and h = $\pi/6$

$$\therefore \qquad h = \frac{x_n - x_0}{n} = \pi/6 = \frac{\pi - 0}{n}$$

$$= n = \frac{\pi}{\pi/6}$$

The value of x and y = f(x) = t sin t are given by

x	x_0 0	x_1 $\pi/6$	x_2 $\pi/3$	x_3 $\pi/2$	x_4 $2\pi/3$	x_5 $5\pi/6$	x_6 π
f(x)	0 y_0	0.2618 y_1	0.9069 y_2	1.5708 y_3	1.8138 y_4	1.3090 y_5	0 y_0

By trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + ...+ y_{n-1}\right)\right]$$

$$\int_0^{\pi} t\sin t dx = \frac{h}{2}\left[\left(y_0 + y_6\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 +\right)\right]$$

$$= \frac{\pi/6}{2}\left[(0+0) + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.3090)\right]$$

$$= \pi/12 \ (11.7246)$$

$$\int_0^{\pi} t\sin t dx = 3.0695$$

Example 8: Evaluate the following integral f(x) = $\int_{-2}^{2} \frac{t\sin t}{5+2t} dt$ using h = 1 by trapezoidal rule.

Solution: Here $x_0 = 2$, $x_n = 2$ and h = 1

Since h = $\frac{x_n - x_0}{n}$

$\Rightarrow$ n = $\dfrac{x_n - x_0}{h} = \dfrac{2-(2)}{1}$ = 4.

n = 4.

Therefore the value of x and y = f(x) = $\dfrac{t\sin t}{5+2t}$ is given by

x	-2	-1	0	1	2
f(x)	1.8186	0.2805	0	0.1202	0.2021

$\therefore$ By trapezoidal rule:

$$\int_a^b f(x)dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + ... + y_{n-1}\right)\right]$$

$$\int_{-2}^{2} \frac{t\sin t}{5+2t}dt = \frac{1}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$

$$= \frac{1}{2}\left[(1.8186 + 0.2021) + 2(0.2805 + 0 + 0.1202)\right]$$

$$= \frac{1}{2}(2.8221)$$

$$\int_{-2}^{2} \frac{t\sin t}{5+2t}dt = 1.4110$$

Example 9: Evaluate $\int_0^{\pi/2} \sin x dx$ by trapezoidal releasing n = 6.

Solution: Here $x_0 = 0$, $x_n = \pi/2$, n = 6

$\therefore$ h = $\dfrac{\pi/2 - 0}{6} = \pi/12$

Therefore, dividing the interval $\left(0_1 \pi/12\right)$ in intervals the value of x and y = f(x) = sin x are.

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
y=f(x)	0	0.2588	0.5	0.7071	0.866	0.9659	1

∴ By Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+.........+y_{n-1})\right]$$

$$\int_0^{\pi/2} \sin x dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{\pi}{12.2}\left[(.0+1)+2(0.2588+0.5+0.7071+0.866+0.9659)\right]$$

$$= \frac{\pi}{24}\left[1+2(3.2978)\right] = \frac{\pi}{24}\ (1 + 6.5956)$$

$$= \frac{\pi}{24}\ (4.2978)$$

$$\int_0^{\pi/2} \sin x dx = 0.9943$$

Example 10: Solve $\int_{0.5}^{1.3} \frac{1}{1+\log x}$ by trapezoidal rule taking n = 8.

Solution: Here $x_0 = 0.5$, $x_n = 1.3$, n = 8

∴ $h = \frac{x_n - x_0}{n} = \frac{1.3-0.5}{8} = \frac{0.8}{8} = .1$

Therefore, dividing the interval (0.5, 1.3) into 8 equal sub interval of height 0.1, the value of x and $f(x) = \frac{1}{1+\log x}$ are given as

x	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
f(x)=y	3.26	2.04	1.55	1.29	1.12	1	0.91	0.89	0.79

∴ By trapezoidal rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+.........+y_{n-1})\right]$$

Example 11: Approximate the area under the curve y = f(x) between x = 0 and x = 8 using Trapezoidal Rule with n = 4 subintervals. A. function f(x) is given in the table of values.

x:	0	2	4	6	8
f(x):	3	7	11	9	3

Solution: The Trapezoidal Rule formula for n = 4 subintervals is given as

$$T_4 = \left(\frac{\Delta x}{2}\right)\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)\right)\right]$$

Here the subinterval width $\Delta x = 2$

Now, substitute the values from the table, to find the approximate value of areas under the curve.

$$A = T_4 = \left(\frac{2}{2}\right)[3+2(7+11+9)+3]$$

= [3 + 2 (27) + 3]

= 3 + 54 + 3

= 60

Therefore, the approximate value of area under the curve using Trapezoidal Rule is 60.

Example 12: Find solution using Trapezoidal rule for the following table

x:	1.4	1.6	1.8	2.0	2.2
y:	4.0552	4.9530	6.0436	7.3891	9.0250

Solution: The values of table for x and y are

x:	1.4	1.6	1.8	2.0	2.2
y:	4.0552	4.9530	6.0436	7.3891	9.0250

Here h (subinterval width) = 0.2

Using Trapezoidal Rule

$$\int y\,dx = \frac{h}{2}\left[y_0+y_4+2\left(y_1+y_2+y_3\right)\right]$$

$$= \frac{0.2}{2}\left[4.0552+9.025+2\left(4.953+6.0436+7.3891\right)\right]$$

$$= 0.1\ [4.0552+9.02512\times(18.3857)]$$

= 4.9852

Solution by trapezoidal rule is 4.9852

Example 13: Evaluate $\int_1^2 \frac{1}{x} dx$ by using Trapezoidal rule with h = 0.25

Solution: Here x_0 = 1, x_n = 2 and h = 0.25

So n = $\frac{x_n - x_0}{h} = \frac{2-1}{0.25}$ = 4

Also f(x) = $\frac{1}{x}$

Now, we tabulate the function f(x) = $\frac{1}{x}$ as follows:

x:	1	1.25	1.5	1.75	2
f(x):	1	8	0.6667	0.5714	0.5

By Trapezoidal rule,

$$\int_1^2 \frac{1}{x} dx = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$= \frac{0.25}{2}\left[1 + 2(0.8 + 0.6667 + 0514) + 0.5\right]$$

$$= \frac{0.25}{2}\left[1 + 2(2.0381) + 0.5\right]$$

= 0.697

Solution by trapezoidal rule is 0.697

Example 14: Evaluate $\int_0^\pi t \sin t dt$ using Trapezoidal rule?

Solution: Here to = 0, $t_n = \pi$

Taking h = $\frac{\pi}{6}$

$\therefore$ n = $\frac{t_n - t_0}{h} = \frac{\pi - 0}{\pi/6}$ = 6

Now, we tabulate the function f(x) = t sin t as follows:

x:	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
f(x)=t sin t:	0	0.2618	0.9069	1.5708	1.8128	1.3090	0

By Trapezoidal Rule

$$\int_0^{\pi} t \sin t\, dt = \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6\right]$$

$$= \frac{\pi/6}{2}\left[0 + 2(0.2618 + 0.9069 + 1.5708 + 1.8138 + 1.3090) + 0\right]$$

$$= \frac{\pi}{6}\left[2(5.8623)\right]$$

= 3.0695

Example 15: Find $\int_2^4 (2x^3 - 4x + 1)\,dx$ by using Trapezoidal rule using h = 0.5.

Solution: Here $x_0 = 2$, $x_n = 4$ and h = 0.5

So n = $\frac{x_n - x_0}{h} = \frac{4-2}{0.5} = 4$

Also, $f(x) = 2x^3 - 4x + 1$

Now, we tabulate the function $f(x) = 2x^3 - 4x + 1$ as follows:

x:	2	2.5	3	3.5	4
f(x):	9	22.25	43	72.75	113

By Trapezoidal Rule:

$$\int_2^4 f(x)dx = \int_2^4 (2x^3 - 4x + 1)\,dx$$

$$= \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$= \frac{0.5}{2}\left[9 + 2(22.25 + 43 + 72.75) + 113\right]$$

$= \frac{0.5}{2}\left[9+2(138)+113\right]$

= 99.5

∴ The required solution is 99.5

Example 16: Find the area enclosed by the function f(x) between x = 0 to x = 4 with 4 intervals for f(x) = e^x.

Solution: Here $x_0 = 0$, $x_n = 4$ and n = 4

So, h = $\frac{x_n - x_0}{h} = \frac{4-0}{4} = 1$

Also, $f(x) = e^x$

Now, we tabulate the function $f(x) = e^x$ as follows:

x:	0	1	2	3	4
f(x):	e^0	e^1	e^2	e^3	e^4

The trapezoidal rule for n = 4 is

$T_n = \frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)\right)\right]$

$\Rightarrow \int_0^4 e^x dx = \frac{1}{2}\left[e^0+2\left(e^1+e^2+e^3\right)+e^4\right]$

$= \frac{1}{2} + e+e^2+e^3+\frac{e^4}{2}$

Hence the required solution is $\frac{1}{2} + e+e^2+e^3+\frac{e^4}{2}$

Example 17: Use the Trapezoidal rule to find the given integral with the specified value of n.

$\int_1^5 \frac{2\cos(2x)}{x}$ dx, n = 8.

Solution: Here $x_0 = 1$, $x_n = 5$, and n = 8

So, h = $\frac{x_n - x_0}{h} = \frac{5-1}{8} = 0.5$

Also $f(x) = \frac{2\cos(2x)}{x}$

Now, we tabulate the function f(x) = $\frac{2\cos(2x)}{x}$ as follows:

x:	1	1.5	2	2.5	3	3.5	4	4.5	5
f(x):	2 cos2	$\frac{4}{3}$ cos3	cos4	$\frac{4}{5}$ cos5	$\frac{2}{3}$ cos6	$\frac{4}{7}$ cos7	$\frac{1}{2}$ cos8	$\frac{4}{9}$ cos9	$\frac{2}{5}$ cos10

By trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+..........+f(x_{n-1})+f(x_n)\right)\right]$$

$$\int_1^5 \frac{2\cos(2x)}{2}dx = \frac{\frac{1}{2}}{2}$$

$$\left[2\cos 2+2\begin{pmatrix}\frac{4}{3}\cos(3)+\cos(4)+\frac{4}{5}\cos(5)+\frac{2}{3}\cos(6)+\\ \frac{4}{7}\cos(7)+\frac{1}{2}\cos(8)+\frac{4}{9}\cos(9)+\frac{2}{3}\cos 10\end{pmatrix}\right.$$

$$\int_1^5 \frac{2\cos(2x)}{2}dx = \frac{1}{4}$$

$$\left[2\cos(2)+\begin{pmatrix}\frac{8}{3}\cos(3)+2\cos(4)+\frac{8}{5}\cos(5)+\frac{4}{3}\cos(6)+\\ \frac{8}{7}\cos(7)+\cos(8)+\frac{8}{9}\cos(9)+\frac{2}{5}\cos 10\end{pmatrix}\right.$$

$$\int_1^5 \frac{2\cos(2x)}{2}dx = \frac{1}{4}\begin{bmatrix}-0.83229-2.63998-1.30729+0.45386+1.28023\\ +0.86160-0.14550-0.80989-0.33563\end{bmatrix}$$

$$\int_1^5 \frac{2\cos(2x)}{2}dx = -0.86872$$

$$\therefore \quad \int_{0.5}^{1.3} \frac{1}{1+\log x}dx = \frac{h}{2}\left[\left(y_0+y_8\right)+2\left(y_1+y_2+y_3+y_4+y_5+y_6+y_7\right)\right]$$

$$= \frac{0.1}{2}\left[(3.20+0.79)+2(2.04+1.55+1.29+1.12+1+0.91+0.89)\right]$$

= 0.05 (21.65)

$\Rightarrow \quad \int_{0.5}^{1.3} \frac{1}{1+\log x} dx$ = 1.0825

Self Check Exercises

Use trapezoidal rule to evaluate

Q.1 E_1 $\int_0^1 \frac{x}{x^2+4} dx$, n = 8

Q.2 E_2 $\int_0^2 \frac{2}{x^2+4} dx$, n = 8

Q.3 E_3 $\int_3^5 \frac{1}{\sqrt{x^2-4}} dx$, n = 8

Q.4 E_4 $\int_0^2 \frac{dx}{1+x^2} dx$, h = 0.25

Q.5 E_5 $\int_0^1 \frac{dx}{1+x}$, h = 0.5, 0.25, 0.125

Q.6 E_6 $\int_0^2 \frac{dx}{1+x^2} dx$, h = 0.2

Q.7 Evaluate $\int_0^2 2x\, dx$ by using trapezoidal rule with h = 1.

Q.8 Find area enclosed by the function f(x) between x = 0 to x = 4 with 4 intervals for $f(x) = x^3 + 1$

Q.9 Solve the integral $\int_0^4 \sqrt{5+x^2}\, dx$ using trapezoidal rule with n = 5.

17.7 Summary:

Trapezoidal rule is a rule that evaluates the area under the curves by dividing the total area into smaller trapezoids rather than rectangles.

Trapezoidal Rule Formula

$$I = \int_{x_0}^{x_n} y\, dx = \frac{h}{2}\left[y_0 + 2\left(y_1 + y_2 + \ldots\ldots\ldots\ldots + y_{n-1}\right) + y_n\right]$$

OR

$$I = \left(\frac{1}{2}\right)^{h}\left[\left(Sum of\ 1^{st}\ and\ last\ ordinate\right) + 2\left(Sum of all\ intermediate\ ordinate\right)\right]$$

This rule is used for approximating the definite integrals where it uses the linear approximation of function.

17.8 Glossary

- **Approximate Value:** The value which is nearly correct or exact.
- **Definite integral:** An integral that gives a fixed value for a curve within the two given limits.
- **Trapezoid:** Trapezoid or Trapezium is a four sided shape which has one pair of sides as parallel.
- **Numerical Integration:** It is the method to calculate the approximate value of integral by using numerical technique.

17.9 Answers to Self Check Exercises

Ans. E_1 0.47697

Ans. E_2 0.784241

Ans. E_3 0.605498

Ans. E_4 1.1071

Ans. E_5 0.708, 0.697, 0.694

Ans. E_6 1.1066

Ans. E_7 4

Ans. E_8 72

Ans. E_9 33.5582

17.10 References/Suggested Readings

1. S.S. Shastry, Introductory Method of Numerical Analysis, PhilearningPVt. Ltd, New Delhi.
2. Richard L. Burden, J.D. Faires, Numerical Analysis. Cengage Learning.
3. K.E. Atkinson, Introduction to Numerical Analysis, John Witey.

17.11 Terminal Questions

Use Trapezoidal see to evaluate

1. $\int_{-2}^{2} x^3 e^3 dx$, n = 4

2. $\int_{0}^{\pi} x^2 cox\, x\, dx$, n = 6

3. $\int_{0}^{3\pi/8} \tan x\, dx$, n = 8

4. $\int_{0}^{2} e^{2x} \sin 3x\, dx$, n = 8

5. $\int_{0}^{2} \frac{1}{x+4} dx$, n = 4

6. Determine the area under the curve y = x^2 between x = 0 and x = 4 by utilizing the Trapezoidal Rule Formula with h = 1.

7. By using the trapezoidal formula numerically integrate the following equation from a = 0 to b = 2.

   f(x) = 0.2 + 25x

8. Use the Trapezoidal rule with n = 6 to approximate

   $\int_{0}^{\pi} \sin^2 x\, dx$

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Unit - 18

Numerical Integration-Simpson's $\frac{1}{3}$ Rule

Structure

18.1 Introduction

Dear student, in this unit we will study about one another method used for approximating the value of definite integral just like trapezoidal rule. This rule is also derived by Newton's Cotes formula, taking n = 2 and is known as Simpson's $\frac{1}{3}$ rule of integration. In this unit we will drive the formula for Simpson $\frac{1}{3}$ rule, Study about its geometrical interpretation as well as its error analysis. We will also try to evaluate some definite integral by using Simpson's $\frac{1}{3}$ rule.

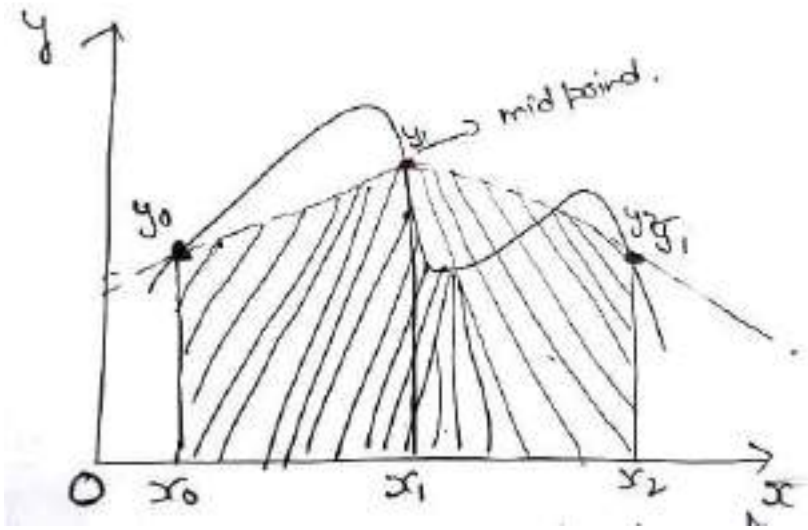
18.2 Learning Objectives:

After studying this unit, students will be able to

1. derive Simpson's $\frac{1}{3}$ rule from Newton Cot's formula.
2. derive expression for error estimation for Simpson's $\frac{1}{3}$ rule.
3. Apply Simpson's $\frac{1}{3}$ rule for approximating definite integral.

18.3 Simpson's $\frac{1}{3}$ Rule

In applying trapezoidal rule we use a polynomial approximation (a straight line) between to successive point (y_0, y_1) and in order to make the error negligible we have to choose h very small. So another way to obtain a more accurate estimate of an integral is to use higher order polynomial is to connect the points. For example, if there is an extract point between (y_0) and (y_2) the three points can be connected with a parabola, as shown in figure.



The formula that results from taking the integral under this polynomial gives Simpson's $\frac{1}{3}$ rule.

In Newton's Cote's Quadiature Formula (unit) taking n = 2 and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola i.e. a polynomial of Second order So that the differences of order higher than the second vanish.

Since Newton's Cote's Quadrature Formula is.

$$I = \int_a^b f(x)\,dx \int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{h}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{4!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

Taking n = 2 in above we get

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0 + \frac{2}{2}\Delta y_0 + 2\left(\frac{2.2-3}{12}\right)\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(\Delta y_1 - \Delta y_0)\right]$$

$$= 2h\left[y_0 + (y_1 - y_0) + \frac{1}{6}(y_2 - y_1) - (y_1 - y_0)\right]$$

$$= \frac{h}{3}\left[6y_1 + y_2 - 2y_1 + y_0\right]$$

$$\Rightarrow \qquad \int_{x_0}^{x_0+2h} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

Similarly $\displaystyle\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$

$$\therefore \qquad \int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right]$$

Using the additive property of definite integrals, or adding all above when n is even

$$\int_a^b f(x)\,dx \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}$$

$$\left[y_0 + 4y_1 + y_2 + y_2 + 4y_3 + y_4 + y_4 + 4y_5 + y_6 + ... + y_{n-1} + 4y_{n-1} + y_n\right]$$

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ...y_{n-1}) + 2(y_2 + y_4 + ...y_{n-2})\right]$$

Here the label "$1/3$" stems from the fact that h is divided by 3.

This rule is known as Simpson's $1/3$ rule, white applying. Simpson's $1/3$ rule, the interval must be divided into an even number of equal subintervals, (Since we find the area of two strips at a time) and odd numbers of point.

Geometrical Interpretation:

Geometrically, in Simpson's $1/3$ rule, we approximate the portion of the curve $y=f(x)$ between $x = x_0$ and $x = x_n$ by $n/2$ parabolas passing through the points (x_0, y_0), (x_1, y_1) and (x_2, y_2); (x_2, y_2), (x_3, y_3) and (x_4, y_4); .... ; (x_{n-2}, y_{n-2}) (x_{n-1}, y_{n-1}) and (x_n, y_n).

Then we calculate the area under each of these $n/2$ parabola. The value of definite integral $\int_{x_0}^{x_n} f(x)\,dx$ i.e. The area under the curve y=f(x), between the ordinater x = x_0, and x = x_n and above x-axis is approximated by the sum of the area under these $n/2$ parabolas.

Error In Simpson's $1/3$ Rule:-

Since the error in Quadrature formula is given by

$\mathrm{E} = \int_a^b y\,dx - \int_a^b p(x)\,dx$, where p(x) is the psdynomial representing the function y=f(x), in the interval [a, b], i.e. a polynomial of second degree ($Ax^2 + Bx + c$).

Error in Simpson' $1/3$ rule can be derived by expanding y = f(x) around x = x_0 by Taylor's Series like.

$f(x) = y = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y0'' + \frac{(x-x_0)^3}{3!}\,y_0''''+........$, where "1" represents the derivative with respect to x.

Therefore over the first strip, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = \int_{x_0}^{x_2}\left[y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0'''' + ...\right]dx$$

$$= \int_{x_0}^{x_2} y_0\,dx + \int_{x_0}^{x_2}(x-x_0)\,y_0'\,dx + \int_{x_0}^{x_2}\frac{(x-x_0)^2}{2!}y_0''\,dx + \int_{x_0}^{x_2}\frac{(x-x_0)^2}{3!}y_0''\,dx + ...$$

$$= (x_2-x_0)\,y_0 + \left[\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_2} y_0' + \left[\frac{(x-x_0)^3}{6}\right]_{x_0}^{x_2} y_0'' + \left[\frac{(x-x_0)^4}{24}\right]_{x_0}^{x_2} y_0'''' + ...$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = (x_2-x_0)\,y_0' + \frac{(x-x_0)^2}{2} + \frac{(x_2-x_0)^3}{6}\,y_0'' + \frac{(x-x_0)^4}{24}\,y_0''''..........$$

Since $x_2 = x_0 + 2h$

$x_0 - x_0 = 2h$

$$\therefore \int_{x_0}^{x_0+2h} f(x)\,dx = 2hy_0 + \frac{(2h)^2}{2}y_0' + \frac{(2h)^3}{3}y_0'' + \frac{(2h)^4}{24}y_0''' + \frac{(2h)^5}{120}y_0^{iv} ....$$

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2hy_0 + \frac{4h^2}{2!} + y_0' + \frac{8h^3}{3!} y_0'' + \frac{26h^4}{4!} y_0''' + \frac{4h^5}{15} y_0^{iv} \ldots\ldots\ldots \qquad (2)$$

Let A1 is the area under the first double strip by Simpson's $1/3$ rule, i.e. parabola passing through the points (x_0, y_0), (x_1, y_1), (x_2, y_2).

Such that $A_1 = \int_{x_0}^{x_2} f(x)\,dx$.

Here $y_0 = f(x_0) = A x_0^2 + Bx_0 + C$

$y_1 = f(x_1) = A x_1^2 + Bx_1 + C$

$y_2 = f(x_2) = A x_2^2 + Bx_2 + C$

Also here, $x_0 = x_0 + 2h \quad \Rightarrow \quad x_2 - x_0 = 2h$

Also the mid point of (x_0, x_2) is x_1 which is

given by $x_1 = \frac{x_2 - x_0}{2}$

$$\therefore \qquad A_1 = \int_{x_0}^{x_2} \left(Ax^2 + Bx + C\right) dx$$

$$= \left[A\frac{x^3}{3} + B\frac{x^2}{2} + Cx \right]_{x_0}^{x_2}$$

$$\Rightarrow \qquad \frac{A}{3}\left(x_2^3 - x_0^3\right) + \frac{B}{2}\left(x_2^2 - x_0^2\right) + C\left(x_2 - x_0\right)$$

$$= \frac{A}{3}\left(x_2 - x_0\right)\left(x_2^2 + x_2x_0 + x_0^2\right) + \frac{B}{2}\left(x_2 + x_0\right)\left(x_2 - x_0\right) + C\left(x_2 - x_0\right)$$

Taking factor $\left(\frac{x_2 - x_0}{6}\right)$ outside we get

$$= \frac{\left(x_2 - x_0\right)}{6}\left[2A\left(x_2^2 + x_2x_0 + x_0^2\right) + 3B\left(x_2 - x_0\right) + 6C\right]$$

$$= \frac{\left(x_2 - x_0\right)}{6}\left[\left(Ax_2^2 + Bx_2 + C\right) + \left(Ax_0^2 + Bx_0 + C\right) + A\left(x_2^2 + 2x_0x_2 + x_0^2\right) + 2B\left(x_2 - x_0\right) + 4C\right]$$

Substituting the values of y_0, y_1, y_2, we get

$$= \frac{(x_2 - x_0)}{6}\left[y_2 + y_0 + A(x_2 - x_0)^2 + 2B(x_2 - x_0) + 4C\right]$$

Putting $2x_1 = x_2 + x_0$, $2h = x_2 - x_0$, we get

$$= \frac{2h}{6}\left[y_2 + y_0 + 4x_1^2 A + 2B.2x_1 + 4C\right]$$

$$= \frac{h}{3}\left[y_2 + y_0 + 4\left(Ax_1^2 + Bx_1 + C\right)\right]$$

$$A1 = \frac{h}{3}\left[y_2 + y_0 + 4y_1\right]$$

Now putting, $x = x_0 + h$, and $y = y_1$ in (1), we have

$$\Rightarrow \quad y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv}..............$$

Again putting $x = x_0 + 2h$, and $y = y_2$ in (1) we have

$$y_2 = y_0 + 2hy_0' + \frac{(2h)^2}{2!}y_0'' + \frac{(2h)^3}{3!}y_0''' + \frac{(2h)^4}{4!}y_0^{iv}..............$$

$$\Rightarrow \quad y_2 = y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + \frac{16h^4}{4!}y_0^{iv}..............$$

Putting the values of y_0 and y_2 in A,

$$A_1 = \frac{h}{3}\left[\left(y_0 + 2hy_0' + \frac{4h^2}{2!}y_0'' + \frac{8h^3}{3!}y_0''' + \frac{2h^4}{3!}y_0^{iv} + ...\right)\right.$$

$$\left. + y_0 + 4\left(y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + ...\right)\right]$$

$$= \frac{h}{3}\left[6y_0 + 6hy_0' + \frac{8}{2!}h^2 y_0'' + \frac{12}{3!}h^3 y_0''' + \frac{15}{18!}h^4 y_0^{iv} + ...\right]$$

$$= 2y_0 + 2h^2 y_0' + \frac{4}{3}h^3 y_0'' + \frac{2}{3}h^4 y_0''' + \frac{5}{18}h^5 y_0^{iv} + ...$$

Therefore the error in the interval (x_0, x_2) is given by

$$E_1 = \int_{x_0}^{x_2} f(x)\,dx - A_1 = \left[\left(2hy_0 + \frac{4}{2!}h^2 y_0' + \frac{8}{3!}h^3 y_0'' + \frac{2}{3!}h^4 y_0''' + \frac{4}{15}h^5 y_0^{iv} + ...\right)\right.$$

$$- \left[2hy_0 - 2h^2 y_0' + \frac{8}{3!} h^3 y_0'' + \frac{2}{3} h^4 y_0''' + \frac{5}{18!} h^5 y_0^{iv} + ... \right]$$

$$E_1 = \left[\left(\frac{4}{15} - \frac{5}{18} \right) h^5 y_0^{iv} + ... \right]$$

$$E_1 = \frac{-1}{90} h^5 y_0^{iv} + \text{term containing higher power of h.}$$

$$E_1 = \frac{-1}{90} h^5 y_0^{iv}, \text{ ignoring the higher powers of h.}$$

Similarly

$E_3 = \int_{x_2}^{x_4} f(x)dx + A_2$, where A_2 is the area under the parabola through the points (x_2, y_2) (x_3, y_2) and (x_4, y_4).

So, $E_3 = \frac{-h^5}{90} y_2^{iv}$

Similarly $E_{n-1} = \int_{x_{n-2}}^{x_n} f(x)dx + A_{n-2} = \frac{-h^5}{90} \left(y_{(n-2)} \right)$

Therefore, total Error $E = \frac{-h^5}{90} \left[y_0^{iv} + y_0^{iv} + ...y_{2(n-1)}^{iv} \right]$

Assuming that $y^{iv}(x)$ is the largest of y_0^{iv}, y_2^{iv}, ... $y_{2(n-1)}^{iv}$, we get

$$E < - \frac{-nh^5}{90} y_0^{iv} (X)$$

using $n = \frac{b-a}{2h}$, hence

$$E = \frac{(b-a)h^5}{180h} y_0^{iv}(X)$$

$$E = \frac{(b-a)h^4}{180} y_0^{iv}(X)$$

Therefore, error in Simpon's $1/3$ rule is of the order h^4.

Working of Simpon's $\frac{1}{3}$ Rule for $\int_a^b f(x)\,dx$

Let y = f(x) takes n value of x in [a, b] i.e.

x	x_0	x_1	x_2	....	x_n
y	y_0	y_1	y_2	....	y_n

Then $\int_a^b f(x)\,dx \; \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.......y_{n-1})+2(y_2+y_4+........y_{n-2})\right]$

Where h = $\frac{b-a}{n}$, here (a, b) is divided into an even number of equal subintervals.

To have more understanding of Simpon' s $\frac{1}{3}$ rule Let us try following Examples:

Example 1: Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Simpson's $\frac{1}{3}$ rule taking h = $\frac{1}{4}$.

Solution: Here x_0 = 0, x_n = 1, h = $\frac{1}{4}$ = 0.25

∴ n = $\frac{x_n - x_0}{h} = \frac{1-0}{\frac{1}{4}}$ = 4

∴ We have to subdivide the given interval into 4 equal parts. On tabulating given function f(x) = $\frac{1}{1+x^2}$ we get

x	0	0.25	0.50	0.75	1
y=f(x)=$\frac{1}{1+x^2}$	1	0.94	0.8	0.64	0.5

Since Simpson's $\frac{1}{3}$ rule is

$$\int_a^b f(x)\,dx \; \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.......y_{n-1})+2(y_2+y_4+........y_{n-2})\right]$$

$$= \int_0^1 \frac{1}{1+x^2}dx = \frac{0.25}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \frac{0.25}{3}\left[(1+0.5)+4(0.94+0.64)+2(0.8)\right]$$

$$\Rightarrow \quad \int_0^1 \frac{1}{1+x^2}\,dx = 0.7850$$

Example 2: Evaluate $\int_0^{\frac{\pi}{2}} e^{\sin x}dx$ by Simpson's $1/3$ rule taking n = 4.

Solution: Here $x_0 = 0$, $x_n = \frac{\pi}{2}$, n = 4

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$$

on tabulating the given function $y = e^{\sin x}$, we get

x	0	$\frac{\pi}{8}$	$\frac{2\pi}{8}=\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{4\pi}{8}=\frac{\pi}{2}$
$y=f(x)=e^{\sin x}$	1	1.4662	2.0281	2.5191	2.7183

Since Simpon's $1/3$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.........y_{n-1})+2(y_2+y_6+...........y_{n-2})\right]$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} e^{\sin x}dx = \frac{\pi/8}{3}\left[(y_0+y_4)+4(y_1+y_3)+2(y_2)\right]$$

$$= \pi/24\left[(1+2.7183)+4(1.4662+2.5191)+2(2.0281)\right]$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} e^{\sin x}dx = 3.1044$$

Example 3: Evaluate $\int_0^{\pi/2} \sin x\,dx$ using Simpson's $1/3$ rule taking n = 6

Solution: Here $x_0 = 0$, $x_n = \pi/2$ and n = 6

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{\pi/2 - 0}{6} = \pi/12$$

Therefore dividing the interval (0, $\pi/2$) into six equal intervals and on tabulating given function y = sin x on these intervals we get

x	0	$\pi/12$	$\frac{2\pi}{12} = \frac{\pi}{6}$	$\frac{3\pi}{12} = \frac{\pi}{4}$	$\frac{4\pi}{12} = \frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{6\pi}{12} = \frac{\pi}{2}$
y=f(x)=sinx	0 y_0	0.2588 y_1	0.5 y_2	0.7071 y_3	0.866 y_4	0.9659 y_5	1 y_6

Since Simpon's $1/3$ rules is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ...y_{n-1}) + 2(y_2 + y_4 + y_{n-2})\right]$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\pi/12}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{36}\left[(1+4) + 4(0.2588 + 0.7071 + 0.9659) + 2(0.5 + 0.866)\right]$$

= 1.000004

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} \sin x\,dx = 1.000004$$

Example 4: Evaluate $\int_{0.5}^{1.3} \frac{1}{1+\log x}dx$ using Simpson's $1/3$ rule taking n = 8.

Solution: Here $x_0 = 0.5$, $x_n = 1.3$, n = 8

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{1.3 - 0.5}{8} = \frac{0.8}{8} = 0.1$$

Therefore on dividing the interval (0.5, 1.3) into 8 subintervals the tabulated value of given function $f(x) = \frac{1}{1+\log x}$, is as

x	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$y=f(x)=\frac{1}{1+\log x}$	3.26 y_0	2.04 y_1	1.55 y_2	1.29 y_3	1.12 y_4	1 y_5	0.91 y_6	0.89 y_7	0.79 y_8

Since Simpon' $\frac{1}{3}$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.........y_{n-1})+2(y_2+y_4+y_{n-2})\right]$$

$$\int_{0.5}^{1.3} \frac{1}{1+\log x}dx = \frac{0.1}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$= \frac{0.1}{3}\left[(3.26+0.79)+4(2.04+1.29+1+0.89)+2(0.7+1.12+0.91)\right]$$

$$= \frac{0.1}{3}[32.09]$$

$$\Rightarrow \quad \int_{0.5}^{1.3} \frac{1}{1+\log x}dx = 1.070$$

Example 5: Estimate the integral $\int_1^3 \frac{dx}{x}$ using n = 8 by Simpson's $\frac{1}{3}$ ruel.

Solution: Here $x_0 = 1$, $x_n = 3$, n = 8

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{3-1}{8} = \frac{2}{8} = 0.25$$

On tabulating the function $f(x) = \frac{1}{x}$ we get following:

x	x^1 1	x^1 1.25	x^2 1.50	x^3 1.75	x^4 3.0	x^5 2.25	x^6 2.50	x^7 2.75	x^8 3
y=f(x)=$\frac{1}{x}$	1 y_0	0.8 y_1	0.6667 y_2	0.5714 y_3	0.5 y_4	0.4444 y_5	0.4 y_6	0.3636 y_7	0.3333 y_8

Since Simpon's $\frac{1}{3}$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + ...y_{n-1}\right) + 2(y_2 + y_4 + y_{n-2})\right]$$

$$\therefore \qquad \int_1^2 \frac{dx}{x} = \frac{0.25}{3}\left[\left(y_0 + y_8\right) + 4\left(y_1 + y_3 + y_5 + y_7\right) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{0.25}{3}\left[\left(1 + 0.3333\right) + 4\left(0.8 + 0.5714 + 0.4444 + 0.3636\right) + 2\left(0.6667 + 0.5 + 0.4\right)\right]$$

$$\int_1^2 \frac{dx}{x} = 1.0987$$

Example 6: Compute $\int_0^1 \frac{x}{x^3 + 10}\,dx$ with 9 ordinates by Simpon's $\frac{1}{3}$ Rule.

Solution: Here x_0 = 0, x_n = 1 and given that we have to use a ordinates, hence number of subinterval will be 8.

$$\therefore \qquad h = \frac{x_n - x_0}{n} = \frac{1-0}{8} = \frac{1}{8} = 0.125$$

On tabulating f(x) = $\frac{x}{x^3 + 10}$, we get

x	x^1 1	x^1 1.25	x^2 2.50	x^3 0.375	x^4 0.5	x^5 0.625	x^6 0.750	x^7 0.875	x^8 1
y=f(x)= $\frac{x}{x^3 + 10}$	0 y_0	0.0125 y_1	0.0250 y_2	0.0375 y_3	0.0494 y_4	0.0010 y_5	0.0720 y_6	0.0820 y_7	0.0909 y_8

Since Simpon's $\frac{1}{3}$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+.........y_{n-1})+2(y_2+y_4+y_{n-2})\right]$$

$$\therefore \quad \int_0^1 \frac{x}{x^3+10} = \frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$\Rightarrow \quad \int_0^1 \frac{x}{x^3+10} =$$

$$\frac{1/8}{3}\left[(0+0.0909)+4(0.0125+0.0375+0.0610+0.0820)+2(0.0250+0.0494+0.0720)\right]$$

$$\int_0^1 \frac{x}{x^3+10}\,dx = 0.0481$$

Example 7: Evaluate numerically $\int_3^5 \frac{4}{2+x^2}$ taking 8 intervals.

Solution: Here $x_0 = 3$, $x_n = 5$ and $n = 8$

$$\therefore \quad h = \frac{x_n - x_0}{n} = \frac{5-3}{8} = \frac{2}{8} = 0.25$$

on tabulating the function $f(x) = \frac{4}{2+x^2}$, we get

x	x^1 3	x^1 3.25	x^2 3.50	x^3 3.75	x^4 4.0	x^5 4.25	x^6 4.50	x^7 4.75	x^8 5
$y=f(x)=\frac{4}{2+x^2}$	0.3636 y_0	0.3184 y_1	0.2807 y_2	0.2490 y_3	0.2222 y_4	0.2118 y_5	0.1798 y_6	0.1628 y_7	0.1481 y_8

Since Simpon's $\frac{1}{3}$ rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+.........y_{n-1})+2(y_2+y_4+y_{n-2})\right]$$

$\therefore \quad \int_3^5 \frac{4}{2+x^2}$ dx = $\frac{0.25}{3}\left[\left(y_0+y_8\right)+4\left(y_1+y_3+y_5+y_7\right)+2(y_2+y_4+y_6)\right]$

=

$\frac{0.25}{3}\left[\left(03636+0.1481\right)+4\left(0.3184+0.2490+0.2118+0.1628\right)+2\left(0.2807+0.2222+0.1798\right)\right]$

$\Rightarrow \quad \int_3^5 \frac{4}{2+x^2}$ dx = 0.4704

Applications of Simpson's $\frac{1}{3}$ Rule

Numerical integration has numerous practical applications in the field of calculus. Simpson's $\frac{1}{3}$ rule due to its case in application and higher accuracy is a preferred method in various application areas as given below:

1. Area bounded by a curve y = f(x) between the ordinates x = a and x = b, above x-axis is given by A = $\int_a^b y\,dx$
2. Volume of Solid formed by revolving the curve y = f(x) between the ordinates x = a and x = b along x-axis, is given by V = $\int_a^b \pi y^2 dx$.
3. Length of an are of the curve y = f(x) between the ordinates x = a and x = b and x-axis is given by $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
4. To find velocity when acceleration at different times is given in tabular form.
5. To find displacement when velocity is given as a function of time in digresses form.

To understand these application of numerical integration Let us try following examples.

Example 8: From the following table, find the area bounded by the curve and x-axis, between ordinates x = 7.47 to x = 7.5

x	7.47	7.48	7.49	7.50	7.51
y=f(x)	1.93	1.95	1.98	2.01	2.03

Solution: Here n = 4, and h = .01, therefore, the area bounded by the curve y = f(x), x axis and x = 47 and x = 7.51 is since Simpson's $\frac{1}{3}$ rule is $\int_{7.47}^{7.51} f(x)dx$.

$$\int_a^b f(x)dx = \frac{h}{3}\left[\left(y_0+y_n\right)+4\left(y_1+y_3+.........y_{n-1}\right)+2(y_2+y_4+y_{n-2})\right]$$

$$\therefore \quad \int_{7.47}^{7.51} f(x)dx = \frac{0.01}{3}\left[\left(y_0+y_4\right)+4\left(y_1+y_3\right)+2(y_2)\right]$$

$$= \frac{0.01}{3}\left[\left(1.93+20.3\right)+4\left(1.95+2.01\right)+2\left(1.98\right)\right]$$

$$= 0.0033\left[\left(3.96\right)+4\left(3.96\right)+\left(3.96\right)\right]$$

= 0.0033 [3.96 + 15.84 + 3.96]

= 0.0033 [23.76]

$$\int_{7.47}^{7.51} f(x)dx = 0.07840 \text{ square units}$$

Example 9: The velocity v of an airplane which starts from rest i given at fixed interval of time t as shown

t (minutes)	2	4	6	8	10	12	14	16	18	20
V=f(t) Km/min.	8 y_1	17 y_2	24 y_3	28 y_4	30 y_5	20 y_6	12 y_7	6 y_8	2 y_9	0 y_{10}

Estimate the approximate distance covered in 20 minutes

Solution: Since the airplane starts from rest, so its initial velocity is zero. So at t = 0 v = 0.

Let S be the distance covered at any instant of time t,

then v = $\frac{ds}{dt}$ or ds = vdt

Therefore distance covered in 20 minutes is given by

S = $\int_0^{20} ds = \int_0^{20} v\,dt$,

Here we will use Simpson's $\frac{1}{3}$ rule with h = 2 and n = 10.

Here $v_0 = 0$ at t = 0.

So by Simpson's $\frac{1}{3}$ rule

= $\int_0^{20} v\,dt\ \frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$

= $\left[(0+0)+4(8+24+30+12+2)+2(17+28+20+6)\right]$

S = 297.33 Km.

Example 10: A solid of revolution is formed by rotating about x-axis, the area between x-axis, the line x = 0 and as curves through the points with the following coordinates.

x	0	0.25	0.50	0.75	1
y=f(x)	1	0.5846	0.5586	0.5085	0.7328

Estimate the volume of solid formed,

Solution: Volume of solid formed by revolving the curve y = f(x) between the ordinates x = a, y = b along x -s axis is given by V = $\int_a^b \pi y^2 dx$. Here a = 0, b = 1 and h = 0.025, n = 4.

$\therefore$ Volume = $\pi\int_0^1 y^2 dx$.

Using Simpson's $\frac{1}{3}$ rule we get

Volume = $\pi\int_0^1 y^2 dx = \frac{h}{3}\left[(y_0^2+y_4^2)+4(y_1^2+y_3^2)+2(y_2^2)\right]$

= $\frac{0.25\times\pi}{3}\left[1^2+(0.7328)^2+4(0.5846)^2+(0.5085)^2+2(0.5586)^2\right]$

= $\frac{0.25\times\pi}{3}\left[1.5370+4(0.60033)+2(0.31203)\right]$

Volume = 1.944 meter cube.

Self Check Exercises

Q.1 E_1 Evaluate $\int_0^1 \frac{x}{1+x^2}dx$ using Simpson's $\frac{1}{3}$ rule by taking n = 4.

Q.2 E2 $\int_{-3}^{3} x^4 dx$ taking 7 ordinates by using Simpson's $\frac{1}{3}$ rule

Q.3 Calculate $\int_{0}^{\pi/2} \sin x\, dx$ by Simpson's $\frac{1}{3}$ rule taking n = 10

Q.4 $\int_{0}^{6} \frac{dx}{1+x^2}$ n = 6 using Simpson's $\frac{1}{3}$ rule

Q.5 $\int_{0}^{5} \frac{dx}{4x+5}$ n = 10, using Simpson's $\frac{1}{3}$ rule.

Q.6 The following table gives the velocity v of a particle at time t:

t (sec)	0	2	4	6	8	10	12
v (m/s)	4	6	16	34	60	94	136

Find the distance moved by the particle in 12 seconds.

Q.7 The velocity v(km/min) of a body which starts from ret is given at fixed interval of time t (min) as follows:

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

Estimate approximately, the distance covered by the body in 20 minutes.

Q.8 The velocity v of a particle at distance s from a point or its path is given by the table:

s(met)	0	10	20	30	40	50	60
v m/s	47	58	64	65	61	52	38

Estimate the time taken to travel 60 miter.

Q.9 A solid of revolution i formed by rotating about the x axis, the area between x-axis, the line x = 0 and x = 1 and the curve through the points with following coordinates:

x	0	0.25	0.50	0.75	1.00
y	1.0	.9896	.9589	.9089	.8415

Q.10 The body is in the form of solid of revolution. The diameter D in cm of it sections at distances x cm from one end are given below. Estimate the volume of the solid.

x	0	2.5	50.	7.5	10.0	12.5	15.0
D	5	5.5	6.0	6.75	6.25	5.5	4.0

18.7 Summary

In this unit we subject about the

1. Simpson's $\frac{1}{3}$ rule of Numerical integration
2. Error estimation of Simpson's $\frac{1}{3}$ rule
3. Evaluating the definite integral using sismposns' $\frac{1}{3}$ rule
4. Different application of Simpson's $\frac{1}{3}$ rule related to area of under the curve, velocity of a particle in a given time and volume of solid.

18.8 Glossary

- **Numerical Integration:** It is the approximate computation of an integral using numerical technique.
- **Trapezoid:** A quadrilateral with no side parallel.

18.9 Answers to Self Check Exercises

Ans. 0.3196

Ans. 97.2

Ans. 1.0000

Ans. 1.3662

Ans. 0.4025

Ans. Distance s = 552m.

Ans. S = Distance = 309.33

Ans. Time = 1.063 sec

Ans. Volume V = 2.819 cubic unit

Ans. volume v = 402.7625 cm^3.

18.10 References/Suggested Readings

1. S.S. Shastry, Introductory methods of Numerical Analysis, PHI Learning Pvt Ltd, New Delhi.
2. Richard L. Burden, J.D. Fairess, Numerical Analysis engage Learning.
3. K.E. Atkinson, Introduction to Numrical Analysis, John Witey.

18.11 Terminal Questions

1. Evaluate using Simpson's $\frac{1}{3}$ rule $\int_0^{10} \frac{1}{1+x^2}$, n = 10

2. A rocket is launched from the ground. Its acceleration is registered during the first so second and is given in the table below. Find the velocity of the rocket at t = 80 sec.

t (sec)	0	10	20	30	40	50	60	70	80
f m/sec^2)	3.0	31.63	33.34	35.47	37.75	40.33	43.25	43.69	50.67

3. A curve is drawn by the table s

x	0	1	2	3	4	5	6
y	0	2	2.5	2.3	2	1.7	1.5

Calculate area bounded by the curve using Simpson's $\frac{1}{3}$ rule.

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Unit - 19

Numerical Integration (Simpson $\frac{3}{8}$ Rule)

Structure

19.1 Introduction

Dear student, in this unit will study about Simpson's $\frac{3}{8}$ rule of numerical integration. Just like Simpson's $\frac{1}{3}$ rule, Simpson's $\frac{3}{8}$ rule can be derive easily from Newton's cotes formula by putting n = 3. In this unit we will study about the derivation of Simpson's $\frac{3}{8}$ rule, about its geometrical interpolation as well as about it error estimation. We will also learn how to evaluate definite integral by using Simpson' $\frac{3}{8}$ rule.

19.2 Learning Objectives:-

After studying this unit, students will be able to

1. define Simpson's $\frac{3}{8}$ rule from Newton-cotes formula.
2. derive expression for error estimation for Simpson's $\frac{3}{8}$ rule.

3. Apply Simpson's $\frac{3}{8}$ rule for approximating definite integral.

Simpson's $\frac{3}{8}$ Rule

In Simpson's $\frac{1}{3}$ Rule for integration was derived by approximating the integrand f(x) with 2nd order (quadratic) polynomial function. In Similar fashion, Simpson's $\frac{3}{8}$ rule for integration can be derived by approximating the given function f(x) with the 3rd order (cubic) polynomial $Ax^3 + Bx^2 + (x + D$, passing through the point (x_0, y_0), (x_1, y_1) (x_2, y_2) and (x_3, y_3)

In Newton's Cote's Quadrature Formula (unit -) taking n = 3 and taking the cubic curve passing through four points (x_0, y_0), (x_1, y_1), (x_2, y_2) and (x_3, y_3) i.e. a polynomial of third order, so that the differences of order higher than the third Vanishes.

Since Newton-Cote's quadrature Formula is

$$I = \int_{a}^{x_0+xb} f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx = nh\left[y_0 + \frac{h}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)}{4!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

taking n = 3, in above we get

$$I = \int_{a}^{b} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx = 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3(3)}{12}\Delta^2 y_0 + \frac{3}{4!}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

$$= 3h\left[y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{12}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0 + \ldots\ldots\ldots\ldots\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(\Delta y_1 - \Delta y_0) + \frac{1}{8}(\Delta^2 y_1 - \Delta^2 y_0) + \ldots\ldots\ldots\ldots\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - y_1 - y_1 - y_0) + \frac{1}{8}[(\Delta y_2 - \Delta y_1) - (\Delta y_1 - \Delta y_0)]\ldots\ldots\ldots\ldots\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 - y_0) + \frac{1}{8}[y_3 - y_2 - (y_2 - y_1) - (y_2 - 2y_1 + y_0)]\ldots\ldots\ldots\ldots\right]$$

$$= 3h\left[y_0 + \frac{3}{2}(y_1 - y_0) + \frac{3}{4}(y_2 - 2y_1 - y_0) + \frac{1}{8}[y_3 - 3y_2 + 3y_1 - y_0]\ldots\ldots\ldots\ldots\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6\left(y_2 - 2y_1 + y_0\right) + \left(y_3 - 3y_2 + 3y_1 - y_0\right)\right]$$

$$= \frac{3h}{8}[8y_0 + 12y_1 - 12y_0 + 6y_2 - 12y_1 + 6y_0 + y_3 - 3y_2 + 3y_1 - y_0]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

$$I = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly $\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$

So on.

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + y_n\right]$$

Using the additive property of definite integral, on adding all above, we gets

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_a^b f(x)dx = \frac{3h}{8}$$

$$\left[y_0 + 3y_1 + 3y_2 + y_3 + y_3 + 3y_4 + 3y_5 + y_6 + \ldots\ldots\ldots\ldots + y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

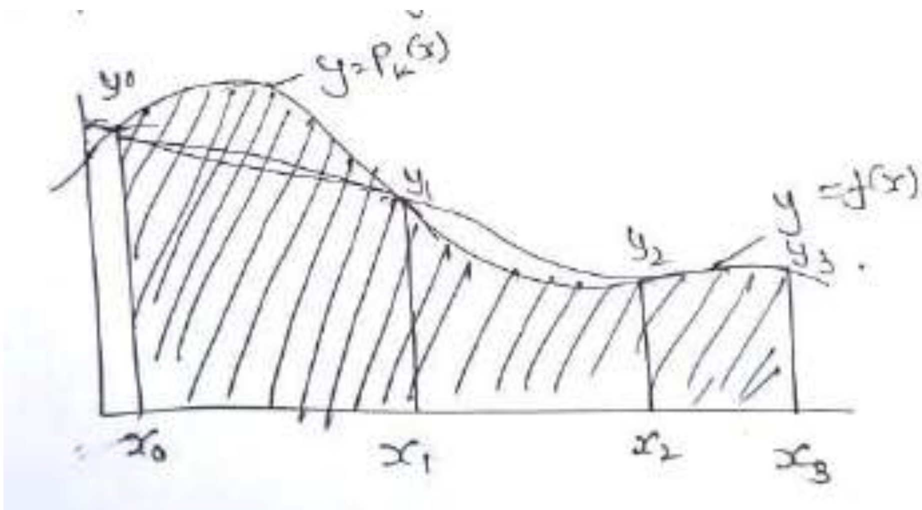
$$\Rightarrow \qquad \int_a^b f(x)dx = \frac{3h}{8}$$

$$\left[\left(y_0 + y_n\right) + 3y_1 + y_2 + y_4 + y_5 + \ldots\ldots\ldots\ldots + y_{n-2} + y_{n-1} + 2\left(y_3 + y_6 + \ldots\ldots\ldots\ldots y_{n-3}\right)\right]$$

This equation is called Simpson's $\frac{3}{8}$ rule because h is multiplied by $\frac{3}{8}$.

Geometric Interpretation:

The Simpson's $\frac{3}{8}$ rule is similar to the $\frac{1}{3}$ rule except that curve y = f(x) is replaced by are of 3rd degree polynomial curve, as shown in figure. It is used when it is required to take 3 segments at a time. Thus number of interval must be multiple of 3.



Error In Simpson's $\frac{3}{8}$ Rule.

Since the error ins quadrature formula is given by E = $\int_a^b y\,dx - \int_a^b p(x)dx$, where p(x) is the polynomial representing the function y = f(x) in the interval [a, b], here the polynomial is of 3rd degree i.e. $Ax^3 + Bx^2 + Cx + D$. Error in Simpon's rule can be derived by expanding y = f(x) around x = x_0 by Taylor's curies like.

f(x) = y = $y_0 + (x - x_0)\, y_0' + \frac{(x-x_0)^2}{3} y_0'' + \frac{(x-x_0)^3}{3} y_0''' + ...$ (1), where (1) represents the derivative with respect to x.

Therefore over the first sub interval, we get

$$\int_{x_0}^{x+3h} f(x)dx = \int_{x_0}^{x_3}\left[y_0 + (x-x_0)\, y_0\,' + \frac{(x-x_0)^2}{2!} y_0\,'' + \frac{(x-x_0)^3}{3!} y_0\,''' + \frac{(x-x_0)^4}{4!} y_0^{iv} + \frac{(x-x_0)^5}{5!} y^{v} + ...dx \right]$$

$$= \int_{x_0}^{x_3}\left[y_0 dx + \int_{x_0}^{x_3} (x-x_0)\, y_0\,'dx + \int_{x_0}^{x_3} \frac{(x-x_0)^2}{2!} y_0\,''dx + \int_{x_0}^{x_3} \frac{(x-x_0)^3}{3!} y_0\,'''dx + \int_{x_0}^{x_3} \frac{(x-x_0)^4}{4!} y_0^{iv} dx + \int_{x_0}^{x_3} \frac{(x-x_0)^5}{5!} y^{v} \right] dx$$

$$\left[= y_0 [x_3 - x_0] + \left[\frac{(x-x_0)^2}{2!} \right]_x^{x_3} y_0 ' + \left[\frac{(x-x_0)^3}{2!3} \right]_x^{x_3} y_0 '' + \right.$$

$$\left. \left[\frac{(x-x_0)^4}{3! \times 4} \right]_x^{x_3} y_0 ''' + \left[\frac{(x-x_0)^5}{5 \times 4!} \right]_x^{x_3} y_0^{iv} + \left[\frac{(x-x_0)^6}{5!6} \right]_x^{x_3} y^v + ... \right]$$

$$\int_{x_0}^{x+3h} f(x)dx = (x_3 - x_0) + \frac{(x_3-x_0)^2}{2!} y_0 ' + \frac{(x_3-x_0)^3}{3!} y_0 '' + \frac{(x_3-x_0)^4}{4!}$$

$$y_0 ''' + \frac{(x-x_0)^5}{5!} y_0^{iv} + \frac{(x_3-x_0)^6}{6!} y^v + ...$$

Since $x_3 = x_0 + 3h$

$x_3 - x_0 = 3h$

$$\Rightarrow \quad \int_{x_0}^{x+3h} f(x)dx = 3hy_0 + \frac{(3h)^2}{2!} y_0 ' + \frac{(3h)^3}{3!} y_0 '' + \frac{(3h)^4}{4!} y_0 ''' +$$

$$\frac{(3h)^5}{5!} y_0^{iv} + \frac{(3h)^5}{5!} y^v + \qquad ...... (2)$$

Let A_1 be the area under the first subinterval by Simpson's $\frac{3}{8}$ rule, here it is a cubic polynomial passing through the points

(x_0, y_0), (x_1, y_1) (x_2, y_2) and (x_3, y_3) such that $A_1 = \int_{x_0}^{x_3} p(x)\, dx$

Here $y_0 = f(x_0) = A_0^3 + Bx_0^2 + Cx_0 + D$

$y_1 = f(x_1) = A_1^3 + Bx_1^2 + Cx_1 + D$

$y_3 = f(x_2) = A_2^3 + Bx_2^2 + Cx_2 + D$

$y_4 = f(x_3) = A_3^3 + Bx_3^2 + Cx_3 + D$

Also here $x_3 = x_0 + 3h \Rightarrow \quad x_3 - x_0 = 3h$

Also $x_0 = a$

$$A_1 = \int_{x_0}^{x_3} \left(Ax^3 + Bx^2 + Cx + D \right) dx$$

$$= \left[\frac{Ax^4}{4} + \frac{Bx^3}{3} + \frac{Cx^2}{2} + Dx \right]_{x_0}^{x_3}$$

$$= \frac{A}{4}\left(x_3^4 - x_0^4 \right) + \frac{B}{3}\left(x_3^3 - x_0^3 \right) + \frac{C}{2}\left(x_3^2 - x_0^2 \right) + D\left(x_3 - x_0 \right)$$

$$= \frac{A}{4}\left\{ \left(x_3^2 + x_0^2 \right)\left(x_3^2 + x_0^2 \right) \right\} + \frac{B}{3}\left(x_3 - x_0 \right)\left(x_3^2 + x_3 x_6 x_0^2 \right) + \frac{C}{2}\left(x_3^2 - x_0^2 \right) + D\left(x_3 - x_0 \right)$$

Taking factor $\frac{\left(x_3 - x_0 \right)}{12}$ out side

$$= \frac{\left(x_3 - x_0 \right)}{12} \left[3A\left(x_3 - x_0 \right)\left(x_3^2 + x_0^2 \right) + 4\left(x_3^2 + x_3 x_0 x_0^2 \right) + 6\left(x_3 - x_0 \right) + 12D \right]$$

Let us try some examples to evaluate define integral using Simpson' $\frac{3}{8}$ rule.

Some Examples:

Example 1: Find integral of f(x) using Simpson's $\frac{3}{8}$ rule for the following table

x:	0.0	0.1	0.2	0.3	0.4
f(x):	1.0000	0.9975	0.9900	0.9776	0.8604

Solution: Here x_0 = 0.0 and x_n = 0.4

h = 0.1

By Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\, dx = \frac{3h}{8} \left[\left(y_0 + y_n \right) + 3\left(y_1 + y_2 + y_4 + ... \right) + 2\left(y_3 + y_6 + ... \right) \right]$$

$$= \frac{3(0.1)}{8} \left[(1.0000 + 0.8604) + 3(0.9975 + 0.99) + 2(0.9776) \right]$$

$= \frac{0.3}{8}\left[1.8604+3(1.9875)+2(0.9776)\right]$

= 0.36668

$\therefore \quad \int_{x_0}^{x_n} f(x)\,dx$ = 0.36668

Example 2: Evaluate $\int_0^3 \frac{1}{1+x}dx$ with n = 6 by using Simpson's $\frac{3}{8}$ rule.

Solution: Here $x_0 = 0$, $x_n = 3$ and n = 6

So, h = $\frac{x_n - x_0}{6} = \frac{3-0}{6} = \frac{3}{6}$ = 0.5

Also, f(x) = $\frac{1}{1+x}$

Now, we tabulate the function f(x) = $\frac{1}{1+x}$ as follow:

x:	0	0.5	1	1.5	2	2.5	3
f(x):	1	0.6667	0.5	0.4	0.3333	0.2857	0.25

By Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+.....)+2(y_3+y_6+.....)\right]$$

$$\int_0^3 \frac{1}{1+x}dx = \frac{3(0.5)}{8}\left[(1+0.25)+2(0.4)+3(0.6667+0.5+0.3333+0.2857)\right]$$

$$\int_0^3 \frac{1}{1+x}dx = 1.3888$$

Example 3: Evaluate $\int_{0.2}^{1.4}\left(\sin x - \log_e x + e^x\right)dx$ by using Simpson's $\frac{3}{8}$ rule.

Solution: Here $x_0 = 0.2$, $x_n = 1.4$

Let us take n = 6 so that h = $\frac{x_n - x_0}{n} = \frac{1.4-0.2}{6}$ = 0.2

Now, we tabulate the function $f(x) = (\sin x - \log_e x + e^x)$ as follows:

x:	0.2	0.4	0.6	0.8	1.0	1.2	1.4
f(x):	3.0295	2.7975	2.8976	3.1660	3.5598	4.0698	4.7042

By Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+.....)+2(y_3+y_6+.....)\right]$$

$$\int_{0.2}^{1.4}\left(\sin x-\log_e x+e^x\right)dx = \frac{3(0.2)}{8}$$

$$\left[(3.0295+4.7042)+3(2.7975+2.8976+3.5598+4.0698)+2(3.1660)\right]$$

$$= \frac{0.6}{8}\left[7.7337+3(13.3247)+2(3.1660)\right]$$

$= 0.6 \times 6.7544975$

$= 4.0530$

$$\therefore \quad \int_{0.2}^{1.4}\left(\sin x-\log_e x+e^x\right)dx = 4.0530$$

Example 4: approximate value of $\int_{1}^{4}\left(e^{-2x}+4x^2-8\right)dx$ by application of Simpson's $\frac{3}{8}$ rule with

n = 3

Solution : Here $x_0 = 1$, $x_n = 4$ and n =3

So, $h = \frac{x_n - x_0}{n} = \frac{4-1}{3} = 1$

Also $f(x) = e^{-2x}+4x^2-8$

Now, we tabulate the function $f(x) = e^{-2x}+4x^2-8$ as follows:

x:	1	2	3	4
f(x):	-3.864665	8.018316	28.002479	56.000335

By Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + ...) + 2(y_3 + y_6 + ...)\right]$$

$$\int_1^4 \left(e^{-2x} + 4x^2 - 8\right)dx = \frac{3(1)}{8}\left[(-3.864665 + 56.000335) + 3(28.002479 + 56.000335)\right]$$

= 60.074271

$$\therefore \quad \int_1^4 \left(e^{-2x} + 4x^2 - 8\right)dx = 60.074271$$

Example 5: Find the vertical distance covered by a rocket from x = 8 and x = 30 Second is given by

$$S = \int_8^{30} \left(2000 \ln\left(\frac{14000}{14000 - 2100t}\right) - 9.8x\right)dx$$

Use Simpson's $\frac{3}{8}$ rule to find the approximate value of integral:

Solution: Here $x_0 = 8$, $x_n = 30$ and n = 3

So, $h = \frac{30-8}{3} = 7.3333$

Now, we tabulate the function for $f(x) = 2000 \ln\left(\frac{14000}{14000 - 2100t}\right)$dx.

x:	8	15.3333	22.6666	30
f(x):	177.2667	372.4629	608.8976	901.6740

By Simpson's $\frac{3}{8}$ rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + ...) + 2(y_3 + y_6 + ...)\right]$$

$$\int_8^{30} f(x)dx = \frac{3}{8} \times 7.3333\ \left[1.77.2667 + 3(372.4629 + 608.8976) + 901.6740\right]$$

$$\int_8^{30} f(x)dx = 11063.3104$$

Hence $\int_{8}^{30}\left(2000\ln\left(\frac{14000}{14000-2100t}\right)-9.8x\right)dx$ = 11063.3104

Self Check Exercise

Q.1 Evaluate $\int_{0}^{6}\frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$ rule upto four decimal places.

Q.2 Evaluate the integral $\int_{0}^{4}xe^{2x}dx$ by Simpson's $\frac{3}{8}$ rule with h = $\frac{4}{3}$

Q.3 Find $\int_{0}^{1}\frac{2}{x^2+1}$dx by Simpson's $\frac{3}{8}$ rule with n = 10

19.4 Summary:

Simpson Rule is one of the numerical Method which is used to evaluate the definite integral.

Simpson $\frac{3}{8}$ Rule is completely based on cubic interpolation rather than quadratic interpolation.

The Formula for Simpson $\frac{3}{8}$ rule is

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[y_0+y_n+3\left(y_1+y_2+y_4+y_5+.....+y_{n-1}\right)+2\left(y_3+y_6+.....+y_{n-3}\right)\right]$$

19.5 Glossary:

- **Numerical Method:** A numerical method is a complete and definite Set of procedures for the solution of a problem together with computable error estimate.
- **Definite Integral:** Definite Integral is an integral that gives a fixed value for a curve within the two given Limits.
- **Ant derivative:** It simply implies inverse derivative, primitive function primitive integral or indefinite integral of a function whose derative is equal to original function.

19.6 Answers to Self Check Exercise

1. 1.3571
2. 6819.209
3. 1.5446

19.7 References/Suggested Readings:

1. Finite Difference & Numerical Analysis by H.C. Saxena.
2. Introductory Methods of Numerical Analysis by S.S. Sastry.
3. Numerical Analysis by J. Douglas Faires
4. An introduction to Numerical Analysis by Kendall Atkinson.

19.8 Terminal Questions

1. Find the value of $\int_0^8 In\left(3+x^2\right)dx$ by using Simpson's $\frac{3}{8}$ rule with n = 4.

2. Evaluate $\int_{-3}^{3} x^4 dx$ by using Simpson's $\frac{3}{8}$ rule with n = 6

3. Use Simson's $\frac{3}{8}$ rule to numerically integrate

   $f(x) = x^5 - 4x^3 - x + 1$

   from $-2 < x < 4$. Given the following step sizes

   (a) h = 1 (b) h = 0.5

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Unit - 20

Euler's Method

Structure

20.1 Introduction

Dear student, in this unit we learn about the differential equation and numerical method to solve these differential equation. Here in this unit we learn the numerical solution of the differential equation using Euler's Method. Here in this unit we learn to solve first order differential equation $\frac{dy}{dx} = f(x, y)$ with given initial conditions. We will drive the Euler's formula of solving first order differential equation and learn how to apply this formula to solve the given differential equation.

20.2 Learning Objectives:

After studying this unit, students will be able to

1. define first order differential equation
2. define Euler's method of solving first order differential equation
3. solve first order differential equation by using Euler's method.

20.3 Euler's Method

Euler's method is based on the idea of approximating a curve using tangent lines. The tangent line to the curve at a point is the line that touches the curve at that point and has the same slope as the curve at that point. We can use the slope of the tangent line to estimate the slope of the curve at that point. Then, we can use this estimate to find the value of the function at the next point by extrapolating along the tangent line. In this way, we can approximate the value of the function at any point.

Consider an initial value problem

$y' = f(x, y), y(x_0) = y_0$ (1)

We are interested in computing approximate values of the solution of equation at equally spaced points $x_0, x_1, x_2, x_n = b$ in an interval $[x_0, b]$. Thus

$x_i = x_0 + ih, i = 0, 1, 2, ... n$

and $h = \frac{b - x_0}{n}$

We will denote the approximate value of the solution at these point by $y_0, y_1, ... y_n$.

Integrating (1) in $[x_{i-1}, x_i]$, we get

$$\int_{y_{i-1}}^{y_i} dy = \int_{x_{i-1}}^{x_i} f(x, y)dx$$

$$y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} f(x, y)dx$$

$$y_i - y_{i-1} = \int_{x_{i-1}}^{x_i} f(x, y)dx \qquad (2)$$

Suppose $f(x_{i-1}, y_{i-1})$ is the approximate value of $f(x, y)$ in the sub interval $[x_{i-1}, x_i]$

$\therefore$ (2) $\Rightarrow$ $y_i = y_{i-1} + f(x_{i-1}, y_{i-1}) \int_{x_{i-1}}^{x_i} dx$

$y_i - y_{i-1} = f(x_{i-1}, y_{i-1}) (x_i, x_{i-1})$

$y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$ (3)

where i = 1, 2, 3, ... n

Equ. (3) is the required Euler's iteration formula

Euler's Method

Some Examples

Example 1: Use Euler's method to find the solution to the differential equation = $\frac{dy}{dx}$ = 3x + 4y at x = 1 with the initial condition y(0) = 0 and step size h = 0.25

Solution: Given $f(x, y) = 3x + 4y$

To find y(1), we shall carryout calculations in the following steps with h = 0.25

Step (I) : Taking x0 = 0, y0 = 0 and h = 0.25, we have

y1 = y0 + h f(x0, y0)

= $0+(0.25)(3(0)+4(0))$

$\Rightarrow$ y$(0.25)=0$

Step II: Taking $x_1 = 0.25$, $y_1 = 0$ and h = 0.25, we have

$y_2 = y_1 + h\,f(x_1, y_1)$

= $0+(0.25)(3(0.25)+4(0))$

= 0.25×0.75

$\Rightarrow$ y(0.50) = 0.1875

Step III: Taking x2 = 0.50, y2 = 0.1875 and h = 0.25, we have

y3 = y2 + hf(x2, y2)

= $0.1875+(0.25)(3(0.5)+4(0.1875))$

= 0.1875 + 0.5625

$\Rightarrow$ y(0.75) = 0.75

Step IV: Taking $x_3 = 0.75$, $y_3 = 0.75$ and h = 0.25, we have

$y_3 = y_3 + hf(x_3, y_3)$

= $0.75+(0.25)(3(0.75)+4(0.75))$

= 0.75 + 1.3125

$\Rightarrow$ y(2) = 2.0625

Hence y(1) = 2.0625

Example 2: Solve the equation $\frac{dy}{dx}=\frac{y}{x}$ with initial conditions x=1, y=1, using Euler's algorithm and tabulate the solutions x=1.25, 1.50, 1.75, 2.

Solution: Given $f(x, y) = \frac{y}{x}$

To find y(1.25), y(1.50), y(1.75) and y(2), we shall carryout calculations in the following four steps with h=0.25.

Step I: Taking $x_0 = 1$, $y_0 = 1$ and h = 0.25

$y_1 = y_0 + h\,f(x_0, y_0)$

= 1 + (0.25) $\left(\frac{1}{1}\right)$

= 1 + 0.25 = 1.25

i.e y(1.25) = 1.25

Step II: Taking $x_1 = 1.25$, $y_1 = 1.25$ and h = 0.25

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 1.25 + (0.25)\left(\frac{1.25}{1.25}\right)$

$= 1.25 + 0.25$

$= 1.50$

$\Rightarrow$ y(1.50) = 1.50

Step III: Taking $x_2 = 1.50$, $y_2 = 1.50$ and h = 0.25

y3 = y2 + h f(x2, y2)

$= 1.50 + (0.25)\left(\frac{1.50}{1.50}\right)$

$= 1.50 + 0.25$

$\Rightarrow$ y(1.75) = 1.75

Step IV: Taking $x_3 = 1.75$, $y_3 = 1.75$ and h = 0.25

y4 = y3 + h f(x3, y3)

$= 1.75 + (0.25)\left(\frac{1.75}{1.75}\right)$

$= 1.75 + 0.25$

$= 2.00$

$\Rightarrow$ y(2) = 2

Hence, y(1.25) = 1.25, y(1.5) = 1.5), y(1.75) = 1.75 and y(2) = 2

Example 3: Given $\frac{dy}{dx}$ = cos x, y(0) = 0, Use Euler's method to solve given differential equation taking h = $\frac{\pi}{10}$ at x = $\frac{\pi}{2}$.

Solution: Given $f(x, y) = \cos x$

To find $y\left(\frac{\pi}{2}\right)$ We shall carryout calculations in the following steps with h=$\frac{\pi}{10}$.

Step I: Taking $x_0 = 0$, $y_0 = 0$ and h=$\frac{\pi}{10}$, we have

$y_1 = y_0 + f(x_0, y_0)$

$= 0 + (0.314)\ (\cos 0)$

$= 0.3146$

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 0 + \left(\frac{\pi}{10}\right) \cos 0$

$= \frac{\pi}{10} = 0.31459$

$\Rightarrow \quad y\left(\frac{\pi}{10}\right) = \frac{\pi}{10} = 0.3146$

Step II: Taking $x_1 = \frac{\pi}{10}$, $y_1 = \frac{\pi}{10}$ and $h = \frac{\pi}{10}$, we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= \frac{\pi}{10} + \left(\frac{\pi}{10}\right) \cos \frac{\pi}{10}$

$= \frac{\pi}{5} = 0.6$

$\Rightarrow \quad y\left(\frac{\pi}{5}\right) = \frac{\pi}{5} = 0.61294$

Step III: Taking $x_2 = \frac{\pi}{5}$, $y_2 = \frac{\pi}{5}$ and $h = \frac{\pi}{10}$, we have

$y_3 = y_2 + h\, f(x_2, y_2)$

$= \frac{\pi}{5} + \frac{\pi}{10} \cos\left(\frac{\pi}{5}\right)$

$= 0.86710$

$\Rightarrow \quad y(\frac{3\pi}{10}$ or $0.94241) = 0.86710$

Step IV: Taking $x_3 = 0.94247$, $y_3 = 0.86710$ and $h = \frac{\pi}{10}$, we have or $\frac{3\pi}{10}$

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 0.8671 + \frac{\pi}{10} \cos\left(\frac{3\pi}{10}\right)$

$= 1.05176$

$\Rightarrow \quad y\left(\frac{2\pi}{5}\right) = 1.05176$

Step V: Taking $x_4 = \frac{2\pi}{5}$, $y_4 = 1.05176$ and $h = \frac{\pi}{10}$, we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= 1.05176 + \frac{\pi}{10}\cos\left(\frac{2\pi}{5}\right)$

$= 1.14884$

$\Rightarrow \quad y\left(\frac{\pi}{2}\right) = 1.14884$

Hence $y\left(\frac{\pi}{2}\right) = 1.14884$

Example 4: Find y(0.2) for $y_1 = \frac{x-y}{2}$, y(0) = 1, with step length 0.1 using Euler method.

Solution : Given $y_1 = \frac{x-y}{2}$, y(0) = 1, h = 0.1, y(0.2) = ?

Using Euler's method

Step I: Taking $x_0 = 0$, $y_0 = 1$ and h = 0.1, we have

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 1 + (0.1)\, f(0, 1)$

$= 1 + (0.1)\, \frac{(0-1)}{2}$

$= 1 - 0.05$

$= 0.95$

$\Rightarrow \quad y(0.1) = 0.95$

Step II: Taking $x_1 = 0.1$, $y_1 = 0.95$ and h = 0.1, we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= 0.95 + (0.1)\, f(0.1, 0.95)$

$= 0.95 + (0.1)\left(\frac{0.1-0.95}{2}\right)$

$= 0.95 + (0.1)(-0.425)$

$= 0.95 - 0.0425$

$= 0.9075$

$\Rightarrow \quad y(0.2) = 0.9075$

Hence $y(0.2) = 0.9075$

Example 5: Find $y(0.5)$ for $\frac{dy}{dx} = -2x - y$, $y(0) = -1$, with step length 0.1 using Euler method.

Solution: Given $\frac{dy}{dx} = -2x - y = f(x, y)$, $y(0) = -1$, $h = 0.1$

To find $y(0.5)$, we shall carryout the calculations in the following steps with $h = 0.1$

Step 1 : Taking $x_0 = 0$, $y_0 = -1$ and $h = 0.1$, we have

$y_1 = y_0 + h\, f(x_0, y_0)$

$= -1 + (0.1)\, f(0, -1)$

$= -1 + 0.1\,(1)$

$= -0.9$

$\Rightarrow \quad y(0.1) = -0.9$

Step 2: Taking $x_1 = 0.1$, $y_1 = -0.9$ and $h = 0.1$, we have

$y_2 = y_1 + h\, f(x_1, y_1)$

$= -0.9 + (0.1)\, f(0.1, -0.9)$

$= -0.9 + (0.1)(0.7)$

$= -0.83$

$\Rightarrow \quad y(0.2) = -0.83$

Step 3: Taking $x_2 = 0.2$, $y_2 = -0.83$ and $h = 0.1$, we have

$y_3 = y_2 + h\, f(x_2, y_2)$

$= -0.83 + 0.1\, f(0.2, -0.83)$

$= -0.83 + 0.1\,(0.43)$

$= -0.787$

$\Rightarrow \quad y(0.3) = -0.787$

Step 4: Taking $x_3 = 0.3$, $y_3 = -0.787$ and $h = 0.1$, we have

$y_4 = y_3 + h\, f(x_3, y_3)$

$= 0.787 + (0.1)\, f(0.3, -0.787)$

$= 0.787 + (0.1)\,(0.187)$

$= 0.7683$

$\Rightarrow \quad y(0.4) = -0.7683$

Step 5: Taking $x_4 = 0.4$, $y_4 = -0.7683$ and $h = 0.1$, we have

$y_5 = y_4 + h\, f(x_4, y_4)$

$= -0.7683 + 0.1\, f(0.4, -0.7683)$

$= -0.7683 + 0.1\,(-0.0317)$

$= -0.77147$

$\Rightarrow \quad y(0.5) = -0.77147$

Hence $y(0.5) = -0.77147$

Self Check Exercise

Q.1 Suppose we have the following differential equation with the initial condition

$$\frac{dy}{dx} = 0.5x\,(1 - x),\ p(0) = 2$$

Use Euler's method to approximate p(2), using a step size of 1.

Q.2 Use Euler's method to find the solution to the differential equation $\frac{dy}{dx} = y^2$ ex at $x = 6$ with the initial conditions $y(0) = 0.01$ and step size $h = 1$.

Q.3 Find y(0.2) for $y' = -y$, $x_0 = 0$, $y_0 = 1$ with step length 0.1 using Euler method.

20.4 Summary:

Euler's method is a numerical method for approximating solutions of ordinary differential equation.

It is based on the idea of approximating a curve using tangent lines.

The formula for Euler's method is

$y_i = y_{i-1} + h\, f(x_{i-1}, y_{i-1})$ where $i = 1, 2, 3, \ldots\ldots, n$

$$h = \frac{b - x_0}{n}$$

Euler's method may not be appropriate when dealing with differented equation that have high curvature or rapidly changing behaviour.

20.5 Glossary:

- **Ordinary Differential Equation:** Ordinary differential equation is a differential equation that contains only one independent variables and its derivation
- **Tangent Line:** Tangent line to the curve at a point is the line that touches the curve at that point and has the same slope as the curve at that point.
- **Step Size:** Step size in Euler's method determines the interval between the computed approximations of the solution.

20.6 Answers to Self Check Exercise

1. p(2) = 2
2. 0.1089
3. 0.81

20.7 References/Suggested Readings

1. An introduction to Numerical Analysis by Kendall Atkinson.
2. Numerical Analysis by J. Douglas Faires
3. Introductory Methods of Numerical Analysis by S.S. Sastry.
4. Finite Difference & Numerical Analysis by H.C. Saxena.

20.8 Terminal Questions

1. Use Euler's method with step sizes h = 0.1 to find the approximate values of the solution of the initial value problem $y' + 3y = 7e^{4x}$, y(0) = 2.
2. Solve the equation $y' + \frac{2}{x}y = \frac{3}{x^3} + 1$ with the initial condition x = 1, y = 1, using Euler's algorithm and tabulate the solutions x = 0.1, 0.2, 0.3 with h = 0.1
3. Use Euler's method solve the differential equation $\frac{dy}{dx} = \frac{2x+1}{5y^4+1}$ = y(2) = 1 taking h = 0.05 at x = 2.0

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