B.A. IIIrd Year
Mathematics- DSE IA

Course Code : MATH303TH
New Syllabus- DSE IA

Linear Algebra

Units 1 to 20

Centre for Distance and Online Education (CDOE)
Himachal Pradesh University,
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MATH303TH Linear Algebra

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SYLLABUS
BA 3rd Year Mathematics
MATH303TH Linear Algebra

Himachal Pradesh University
B.A. with Mathematics
Syllabus and Examination Scheme

Course Code	MATH303TH
Credits	6
Name of the Course	Linear Algebra
Type of the Course	Discipline specific Elective
Assignments	Max. Marks:30
Yearly Based Examination	Max Marks: 70 Maximum Times: 3 hrs.

Instructions

Instructions for Candidates: Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

DSE : Linear Algebra

Unit-I

Vector spaces, subspaces, algebra of subspaces, quotient spaces.

Unit-II

Linea combination of vectors, linear span, linear independence, basis and dimension, dimension of subspaces.

Unit-III

Linear transformations, null space, range, rank and nullity of a linear transformation, matrix representation of a linear transformation, algebra of linear transformation.

Unit-IV

Dual Space, Dual Basis, Double Dual, Eigen values and Eigen vectors, Characteristic Polynomial.

Isomorphisms, Isomorphism theorems, invertibility and isomorphisms, change of coordinate matrix.

Books Recommended

1. Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Ed., Prentice-Hall of India Pvt. Ltd., New Delhi, 2004.
2. David C. Lay, Linear Algebra and its Applications, 3rd Ed., Pearson Education Asia, Indian Reprint, 2007.
3. S. Lang, Introduction of Linear Algebra, 2nd Ed., Springer, 2005.
4. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.

Unit - 1

Some Basic Concepts

Structure

1.1 Introduction

1.2 Learning Objectives

1.3 Compositions

1.4 Self Check Exercise-1

1.5 Summary

1.6 Glossary

1.7 Answers to self check exercises

1.8 References/Suggested Readings

1.9 Terminal Questions

1.1 Introduction

Dear students, before proceeding to the concept of vector spaces, Basis and dimensions, Quotient spaces, Linear Transformation etc. it is important to understand some basic concepts like composition (Binary composition, algebraic Structure, Fields etc.)

1.2 Learning Objectives:

The main objective of this unit are

1. to define what are call as a binary composition.
2. to define algebraic structure and types of composition.
3. the concept of fields and its properties.

1.3 Composition

Binary composition (Operation)-

Let A be a non-empty set.

Define a function f :

$A \times A \rightarrow A$. Then f is called a binary composition (or internal composition) or simply a composition on A.

The function f corresponds to each ordered pair $(x, y) \in A$, a unique element $f(x, y)$; $x, y \in A$. We notice the following points.

(i) We can use any convenient notation for a composition but the most commonly used are $*$, 0, $\oplus$, $\odot$, +, ... etc.

Set $*$ be a composition on a non-empty set A. then

$*$ (x, y) or $x * y$, x, y $\in$ A, denotes the image of (x, y) under $*$.

(ii) If $*$ is a composition on A, then we write it as (A, $*$). Here A is a set with operation $*$. If $\oplus$, $\odot$ are two composition on A then we write (A, $\oplus$, $\odot$). Here A is a set with two operation $\oplus$ and $\odot$.

Set us give some examples of Binary Operations

Example 1. Set R be the set of reals and

$f : R \times R \to r$ be defined as

$f(x, y) = xy \ \forall \ (x, y) \in R \times R, x, y \in R.$

then f is a binary composition on R.

2. Set N be the set of naturals and $*$:

$N \times N \to N$ be defined as

$x * y = x + y$, $x, y \in N$.

Since $\forall \ x, y \in N \Rightarrow x + y \in N \Rightarrow x * y \in N$.

3. Define $*$:

$N \times N \to N$ as

$x * y = x - y, x, y \in N$

Set x = 3, y = 5. Then

$3 * 5 = 3 - 5 = -2 \notin N$

$\therefore x * y \notin N$ for all $x, y \in N$

$\therefore$ subtraction is a not a binary composition on N.

Algebraic Structure

A set having one or more binary composition is called Algebraic structure.

Types of Compositions

We shall here discuss some important types of binary compositions which are uesful in defining structure such as Groups, Rings, Fields and Vector spaces.

(I) Commutative Composition

A binary composition $*$ on a set A is called commutative composition iff.

$x * y = y * x \ \forall \ x, y \in A$

For instance, the addition composition in the set of real is commutative, since.

$x + y = y + x \ \forall \ x, y \in r.$

(II) Associative Composition

A binary composition $*$ on a set A is called associative composition iff.

$(x * y) * z = x * (y * z) \ \forall \ x, y \ z \in A.$

For example

(i) The addition composition in the set of reals is associative. Since

$(x + y) + z = x + (y + z) \ \forall \ x, y, z \in R.$

(ii) The composition $*$, defined as

$* : R \times R \to R$

$x * y = x + 2y : x, y \in R$ is not associative

Since

$$
\begin{aligned}
(x * y) * z &= (x + 2y) * z \\
&= (x + 2y) \_ 2z \\
&= x + 2y + 2z
\end{aligned}
$$

and

$$
\begin{aligned}
x * (y * z) &= x * (y + 2z) \\
&= x + 2(y + 2 z) \\
&= x + 2y + 4z
\end{aligned}
$$

Thus $(x * y) * z \neq x * (y * z)$.

(III) Composition with identity element

A binary composition $*$ on a set A is said to be a composition with identity element iff $\exists$ E $\in$ a, such that

$e * x = x * e = e \ \forall \ x \in A$

The element e here is called identity element of A and is always unique.

For example

(i) In R, the set of reals, 0 is the identity element under addition composition since

$x + 0 = x = 0 + x \ \forall \ x \in A.$

(ii) The set of natural numbers N does not have the identity element under the composition of addition, since there is no natural number e such that

$a + e = a \ \forall \ a \in N \qquad (\therefore \ o \notin N).$

(IV) Invertible element

Set e be the identity element of set A under the composition '$*$' on the set A. Set $\alpha \in A$, then $\beta \in A$ is called an inverse element of α iff

$\alpha * \beta = e = \beta * \alpha$.

Then the composition '$*$' is a composition with inverse element, which is always unique.

For example

(i) In the set I of integers, 0 is the identity element under addition compositin and each element $a \in I$ has its additive inverse $(-a) \in I$,

Since

$a + (-a) = 0 = (-a) + a$

Thus every element of I is invertible

(ii) In natural number set N, 1 (one) is the identity element under multiplication composition but there is no element other than I which is invertible.

(V) Distributive Operations

Set $*$ and $\oplus$ be two binary operations on a set A. Then we say that the operation $*$ is distributive with $\oplus$ if

$x * (y \oplus z) = (x * y) \oplus (x * z) \ \forall \ x, y\ z \in A$

(left distributive law)

and

$(y \oplus z) * x = (y * x) \oplus (z * y) \ \forall \ x, y, z \in A$

(Right distributive law)

If the composition $*$ is commutative then

Left Distributive Law = Right Distributive Law

For example

In the set of naturals multiplication composition is distributive over addition composition. Since

$x . (y + z) = x . y + x . z \ \forall \ x, y , z \in N.$

Fields

A non-empty set F having atleast two elements with two binary compositions 't' and '.' (addition and multiplication) is called a field iff the following postulates are satisfied.

(I) Properties of Addition

(i) $\forall \ a, b \in F \Rightarrow a + b \in F$ (closure property)

(ii) $\forall \ a, b , c \in F \Rightarrow (a + b) + c = a + (b + c)$ (Associative property)

(iii) $\forall$ a $\in$ F $\exists$ 0 $\in$ F s.t.

a + 0 = 0 + a = 1 (Existence of additive identity)

(iv) $\forall$ a $\in$ F, $\exists$ = a $\in$ F s.t.

a + (-1) = 0 = (-a) +a (Existence of additive inverse)

(v) $\forall$ a, b $\in$ F $\Rightarrow$ a + b = b + a) (Commutative property)

(II) Properties of Multiplication

(i) $\forall$ a, b $\in$ F $\Rightarrow$ a + b $\in$ F (Closure property)

(ii) $\forall$ a, b, c $\in$ F $\Rightarrow$ a (b c) = (a b) c (Associative property)

(iii) $\forall$ a $\in$ F $\exists$ 1 $\in$ F s.t.

1 . a = a . 1 = a, 1 is called unity of F

(iv) $\forall$ a ($\neq$ 0) $\in$ F, $\exists$ = b $\in$ F s.t.

a . b = b . a = 1, (Existence of multiplication inverse)

b is called inverse of a.

(v) $\forall$ a, b $\in$ F $\Rightarrow$ a . b = b . a (Commutative property)

(III) Distributive Laws

$\forall$ a, b, c $\in$ F

a. (b + c) = a . b + a . c

(b + c) . a = b . a + c . a

Some illustrative Examples

Example 1. A binary operation $*$ is defined on I, the set of integers by

$a * b = a + b + 1 \ \forall$ a, b $\in$ I

Is $*$ commutative and associative?

Solution : (i) Set a, b $\in$ I

$a * b = a + b + 1 . \forall$ a, b $\in$ I

and

$b * a = b + a + 1 = a + b + 1$

$\Rightarrow$ $a * b = b * a$

$\Rightarrow$ $*$ is commutative on I

(ii) Set a, b, c $\in$ I

$\therefore$ $a * (b * c) = a * (b + c + 1)$

$= a + (b + c + 1) + 1$

$= a + b + c + 2 \quad \forall\, a, b, c \in 1$

and

$(a * b) * c = (a + b + 1) * c$

$= (a + b + 1) + c + 1$

$= a + b + c + 2 \quad \forall\, a, b, c \in 1$

Hence $*$ is also associative on 1.

Example 2. A binary operation $*$ is defined on R, the set of real by

$a * b = a + 2b \ \forall\, a, b \in R.$

Is $*$ is commutative and associative? Justify.

Solution : (i) a, b $\in$ R then

$a * b = a + 2b$

and $b * a = b + 2a$

$\Rightarrow$ $a * b \neq b * a$

Thus $*$ is not commutative on R.

(ii) a, b, c $\in$ R. Then

$a * (b * c) = a * (b + 2c)$

$= a + 2(2 + 2c)$

$= a + 4 + 4c$

and

$(a * b) * c = (a + 2b) * c$

$= a + 2b + 2\,c$

$\therefore$ $a * (b * c) \neq (a * b) * c$

Thus '$*$' is not associative on R.

Justification

(i) $1, 2, \in R \Rightarrow 1 * 2 = 1 + 2 \times 2 = 5, \ 1, 2 \in R$

$2 * 1 = 2 + 2 \times 1 = 4, \quad 1, 2 \in R$

$\therefore$ $1 * 2 \neq 2 * 1$

(ii) $2, 4, 5 \in R \Rightarrow 2 * (4 * 5) = 2 + 2 \times 14 = 30$

and $(2 \times 4) * 5 = 20$

1.4 Self Check Exercises

Q. 1 A binary operation $*$ is defined on R, the set of reals by

$a * b = |a\ b| \quad \forall\ a, b \in R$

Check $*$ for commutative and associative, Justify your answer.

Q. 2 A binary operation $*$ is defined on Q, the set of rational by

$a * b = a\ b + 1 \qquad \forall\ a, b \in Q$

Is binary operation $*$ is commutative and associative?

1.5 Summary

We have learnt the following concept in this unit.

(i) Binary composition

(ii) Algebraic structure

(iii) Types of composition

(iv) Fields and its properties etc.

1.6 Glossary

1. **Operands :** The binary operations are operations performed on two inputs viz. addition, subtraction, multiplication and division. These inputs are called operands.

2. **Algebraic Structure :** A set having one or more binary compositions is called an algebraic structure.

1.7 Answers to self Check Exercises

Ans. 1 Easy to show commutative and associative

Ans. 2 Try yourself. ($*$ is commutative but $*$ not associative on Q.)

1.8 Reference/ Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.

3. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

1.9 Terminal Questions

1. A binary operation $*$ is defined on

   $I \times I$ b $(a, b) * (c, d) = (a - c, b - d) : \forall (a, b), (c, d) \in I \times I$.

   is the binary operation $*$ is commutative and associative?

2. A binary operation $*$ is defined on M2 (R), the set of 2×2 matrices whose elements are reals by

   $A * B = \frac{1}{2}(AB - BA) \; \forall A, B \in M2\,(R)$.

   Is $*$ commutative and associative? Justify your answer

3. Set $*$ be an associative binary operation on a set S. Set TCS defined by

   $T = \{a \in S : a * x = x * a \; \forall x \in s\}$.

   Prove that T is closed under '$*$'.

Unit - 2

Vector Spaces

Structure

2.1 Introduction

2.2 Learning Objectives

2.3 Binary Compositions

2.4 Self Check Exercise-1

2.5 Vector Spaces

2.6 Summary

2.7 Glossary

2.8 Answers to self check exercises

2.9 References/Suggested Readings

2.10 Terminal Questions

2.1 Introduction

Dear students, we have already learnt about the concepts of algebraic structures, namely, 'Group', 'Rings' and 'Fields' in our previous classes. In the present unit we shall further consider another important algebraic system called 'Linear Vector Space' or simply vector space.

The vector space involves two sets, the set of vectors V and the set of Scalers F (always a Field). We shall use two operations in defining vector space, one is internal composition on the element of V and the other is external composition on the element of V by the element of F.

2.2 Learning Objectives:

The main objective of this unit are

1. to define Binary composition where we shall study internal and external compositions.
2. to define a vector space followed by some important properties like addition multiplication etc.

2.3 Binary composition

Internal Composition : Let A be a set, then the mapping $f : A \times A \rightarrow A$ is called internal composition in it.

For example

(i) Let A = R, the set of all reals

If $f : R \times R \to R$ is defined as

$f(x, y) = x\,y\ \forall\ (x, y) \in R \times R$; x, y are reals.

Then f is internal composition in R.

(ii) Let A = set of all $n \times n$ matrices over reals.

If $f : A \times A \to A$ is defined as

$f\,(P, Q) = P + Q,\ \forall\ (P, Q) \in A \times A$; P, Q are $n \times n$ matrices over reals.

External Composition.

Let A and F be two non-empty sets. Then a mapping $f : A \times F \to A$ is called an external composition on A by the elements of F.

For example

Let A = set of all $n \times n$ matrices over reals

F = set of all reals

If $f : A \times F \to A$ is defined as

$f(P, k) = k\,P$ for all $P \in A$ and $k \in P$

where k P means the multiplication of matrix P by scalar k.

Then f is an external composition in A over P.

2.4 Vector Spaces

Vector Space. Let (F, +, .) be a given field and V be a non empty set with two compositions, one is internal binary composition on V, called addition of vectors and is denoted by + or $\oplus$ and the other is external binary composition on V by the elements of F, called scalar multiplication and is denoted multiplicatively, then the given set V is called a vector space or linear space over the field F iff the following axioms are satisfied.

I. Properties of Addition :

A-1 Closure Property : $\forall\ x, y \in V$

We have $x + y \in V$

A-2 Associative Property. $\forall\ x, y, z \in V$

we have $(x + y) + z = x + (y + z)$.

A-3 Existence of Addiative identity.

There exists an element $0 \in V$

such that $x + 0 = 0 + x = x \quad \forall\ x \in V$

Here 0 is known as zero vector in V or addiative identity.

A-4 Existence of Addiative Inverse.

For each element $x \in V$, there exists an element $- x \in V$ such that

$$x + (-x) = 0 = (-x) + x$$

The element - x is called addative inverse of x.

A-5 Commutative Property. $\forall$ x, y $\in$ V

We have x + y = y + x.

II. Properties of Scalar Multiplication

M-1 $\forall \alpha \in F, x \in V$ we have $\alpha x \in V$

M-2. $\forall \alpha, \beta \in F, x \in V$ we have $(\alpha + \beta) x = \alpha x + \beta x$

M-3 $\forall \alpha \in F, x, y \in V$ we have $\alpha (x + y) = \alpha x + \alpha y$

M-4 $\forall \alpha, \beta \in F, x \in V$ we have $(\alpha \beta) x = \alpha (\beta x)$

M-5 $\forall x \in V$, we have I. x = x where I is the unity element of F.

Remarks

1. The properties A-1 to A-5 imply that V is an abelian group under +.
2. We may not always explicity mention the operations on V and F. Generally, we shall be writing "V is a vector space over F" or "V (F) is a vector space."
3. The zero element of V is written as 0 in bold type and the zero element of F is written as 0.
4. Generally, we shall use x, y, z .... for elements or vectors of a vector space V and α, β, γ .... for elements (scalars) of the field F.
5. Let V = {0} be a trivial group and F be a filed.

   We define $\alpha\, 0 = 0 \ \forall \alpha \in F$
6. The vector space V (F) is called a rational vector or a real vector space or a complex vector space according as F is the filed of rational numbers or the field R of reals or the field C of complex numbers.

Example 1. Let R be the field of reals and V be the set of vectors in a plane. Show that V (R) is a vector space with vector addition as internal binary composition and scalar multiplication of the elements of R with those of V as external binary composition.

Solution: Given

$V = \{(x, y) | x, y \in R\}$ (The elements of V are ordered pairs as V is a set of vectors in a plane)

Here, we define addition of vectors in V

as (x, y) + (t, z) = (x + t, y + z). for x, y, t, z $\in$ R and the scalar multiplication of $\alpha \in V$ as $\alpha (x, y) = (\alpha x, \alpha y)$.

I. Properties under addition

A-1 Closure Let $(x_1, y_1), (x_2, y_2) \in V$

$\Rightarrow$ $x_1, y_1, x_2, y_2 \in R$

$\Rightarrow$ $x_1 + x_2, y_1 + y_2 \in R$

$\therefore$ $(x_1, y_1) + (x_2, y_2)) = (x_1 + y_2, y_1 + y_2)$

$\in V.$

$\Rightarrow$ V is closed under addition.

A-2 Associative

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in V$

Now $\left[\left(x_1, y_1\right) + \left(x_2, y_2\right)\right] + (x_3, y_3)$

$= \left[\left(x_1, x_1, y_1, y_2\right)\right] + (x_3, y_3)$

$= \left(\left(x_1, x_2\right) + x_3, \left(y_1, y_2\right) + x_3\right)$

$= \left(x_1 + \left(x_2, x_3\right), y_1 + \left(y_2 + y_3\right)\right)$

[$\because$ Associative Property hold in Reals]

$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3)$

$= (x_1, y_1) + \left[\left(x_2, y_2\right) + \left(x_3, y_3\right)\right]$

$\Rightarrow$ addition is associative in V.

A-3 Existence of additive identity

For all $(x_1, y_1) \in V$, there exists $(0, 0) \in V$

Such that $(x_1, y_1) + (0, 0) = (x_1 + 0, y_1 + 0)$

$= (x_1, y_1)$

and $(0, 0) + (x_1, y_1) = (0 + x_1, 0 + y_1)$

$= (x_1, y_1)$

$\Rightarrow$ (0, 0) is addative identity in V.

A-4 Existence of addative inverse.

Let (x, y) be any element of V

$\Rightarrow$ $(-x, -y) \in V$ [Since $x, y \in$ Reals $\Rightarrow$ $-x, -y \in$ Reals]

Now $(x, y) - (-x, -y) = (x + (-x), y + (-y))$

$= (0, 0)$

and $(-x, -y) - (x, y) = (-x + x, -y + y)$

$= (0, 0)$

$\therefore$ $(x, y) + (-x, -y) = (0, 0) = (-x, -y) + (x, y)$

$\Rightarrow$ $(-x, -y)$ is the addative inverse of (x, y) for each $(x, y) \in V$.

A-5 Commutative

Let $(x_1, y_1), (x_1, y_1) \in V$

Now $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_2 + y_1)$

[$\because$ addition is commutative in reals]

$\Rightarrow$ addition is commutative in V.

II. Properties under scalar multiplication

M-1 Let $\alpha \in r$, $(x, y) \in V$; $x, y \in R$

Then $\alpha (x, y) = (\alpha x, \alpha y)$

$\in V$ [$\because$ $\alpha \in R$ and $x, y \in R$ $\Rightarrow$ $\alpha x, \alpha y \in R$]

M-2 Let $\alpha \in R$ and $(x_1, y_1), (x_2, y_2) \in V$

Now $\alpha \left[(x_1, y_1) + (x_2, y_2)\right]$

$= \alpha \left[(x_1 + x_2, y_1 + y_2)\right]$

$= \left(\alpha(x_1 + x_2), \alpha(y_1 + y_2)\right)$

$= \left(\alpha x_1 + \alpha x_2, \alpha y_1 + \alpha y_2\right)$

$= \left(\alpha x_1, \alpha y_1\right) + \left(\alpha x_2, \alpha y_2\right)$

$= \alpha (x_1, y_1) + \alpha (x_2, y_2)$

M-3 Let $\alpha, \beta \in R$ and $(x_1, y_1) \in V$

Now $(\alpha + \beta)(x_1, y_1) = \left((\alpha + \beta)x_1, (\alpha + \beta)y_1\right)$

$= (\alpha x1 + \beta x1, \alpha y1 + \beta y1)$

$= (\alpha x1, \alpha y1) + (\beta x1, \beta y1)$

$= \alpha (x1, y1) + \beta (x1, y1)$

M-4 Let $\alpha, \beta \in R$ and $(x1, y1) \in V$

Now $(\alpha \beta)(x, y) = \left((\alpha\beta)x, (\alpha\beta)y\right)$

$= \left(\alpha(\beta x), \alpha(\beta y)\right) = \alpha (\beta x, \beta y)$

$= \alpha \left(\beta(x, y)\right)$

M-5 Let $1 \in R$ and $(x_1, y_1) \in V$

Now $1. (x_1, y_1) = (1. x_1, 1. y_1)$

$= (x_1, y_1)$

Hence V is vector space over R.

Note: A plane vector is an ordered pair (x, y) of reals.

The plane vector (x, y) and the directed line segment $\overrightarrow{OP}$ where O is origin and $P \leftrightarrow (x, y)$ are same i.e. $\overrightarrow{OP} = (x, y)$.

Example 2. Let F be an arbitrary field and $M_{m\times n}$ (F) be the set of all $m \times n$ matrices over F. Prove that $M_{m\times n}$ (F) is a vector space over F under the addition of matrices and multiplication of matrix by a scalar as internal and external compositions in $M_{m\times n}$ (F).

Solution: Given $M_{m\times n} (F) = \{A|A = [a_{ij}]_{m\times n}, a_{ij} \in F\}$

Let $A, B, C \in M_{m\times n} (F)$

Then $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$

where $a_{ij}, b_{ij}, c_{ij} \in G$

1. Properties under addition

A-1. Closure.

Let $A, B \in M_{m\times n} (F)$

Now $A + B = [a_{ij}]_{m\times n} + [b_{ij}]_{m\times n}$

$= [a_{ij} + b_{ij})_{m\times n}$

$\in M_{m\times n}$ [$\because$ $a_{ij}, b_{ij} \in F \Rightarrow$ $a_{ij} + b_{ij} \in F$ as F, being field is closed under addition]

$\therefore$ $M_{m\times n} (F)$ is closed under addition.

A-2 Associative.

Let $A, B, C \in M_{m\times n} (F)$

Now $A + (B + C) = [a_{ij}]_{m\times n} + ([b_{ij}]_{m\times n} + [c_{ij}]_{m\times n})$

$= [a_{ij}]_{m\times n} + [[b_{ij} + c_{ij}]_{m\times n}$

$= [a_{ij} + (b_{ij} + c_{ij}]_{m\times n}$

$= [(a_{ij} + b_{ij}) + cij]_{m\times n}$

[$\because$ $aij + (b_{ij} + c_{ij})$

[$= [(a_{ij} + b_{ij}) + cij$ as associative law holds in field F]

$= [a_{ij} + b_{ij})_{m\times n} + [c_{ij}]_{m\times n}$

$= (A + B) + C$

$\Rightarrow$ addition is associative in $M_{m\times n}$ (F).

A-3. Existence of addative identity:

Let $O = [0]_{m\times n}$ where 0 is identity element of F

Now $A + O = [a_{ij}]_{m\times n} + [0]_{m\times n}$ $\forall A \in M_{m\times n}$ (F)

$= [a_{ij} + 0]_{m\times n}$

$= [a_{ij})_{m\times n} = A$

[$\because$ $a_{ij} + 0 = a_{ij}$ since 0 is addative identity of F]

$\therefore$ $A + O = A$

Also $O + A = [0]_{m\times n} + [a_{ij}]_{m\times n}$

$= [0 + a_{ij}]_{m\times n}$

$= [a_{ij}]_{m\times n} = A$

$\therefore$ $O + A = A$

Thus $O + A = A + O = A$

$\Rightarrow$ O is addative identity in $M_{m\times n}$ (F).

A-4. Existence of addative inverse

Let $A = [a_{ij}]_{m\times n} \in M_{m\times n}$ (F)

Then $- A = [-a_{ij}]_{m\times n} \in M_{m\times n}$ (F)

[$\because$ $a_{ij} \in F$ and F is field $\Rightarrow$ $-a_{ij} \in F$.]

Now $A + (- A) = [a_{ij}]_{m\times n} + [- a_{ij}]_{m\times n}$

$= [a_{ij} + (- a_{ij})]_{m\times n}$

$= [0]_{m\times n}$

[$\because$ $a_{ij} + (-a_{ij}) = 0$ as $- a_{ij}$ is addative inverse of a_{ij}]

$= O$

$\therefore$ $A + (- A) = O$

Also $(- A) + A = [- a_{ij}]_{m\times n} + [a_{ij}]_{m\times n}$

$= [(- a_{ij}) + a_{ij}]_{m\times n}$

$= [0]_{m\times n} = O$

$\therefore$ $(- A) + A = O$

Thus $A + (- A) = O = (- A) + A$

$\Rightarrow$ - A is addative inverse in $M_{m\times n}$ (F)

A-5. Commutative.

Let A, B $\in$ Mm$\times$n (F)

$\therefore$ A + B = $[a_{ij}]_{m\times n}$ +$[b_{ij}]_{m\times n}$

= $[a_{ij} + b_{ij}]_{m\times n}$

= $[b_{ij} + a_{ij}]_{m\times n}$ [$\because$ $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ as $a_{ij}, b_{ij} \in$ F and commutative property holds in a field F]

= B + A

$\Rightarrow$ addition is commutative in Mm$\times$n (F)

II. Properties under scalar Multiplication.

M-1. Let α $[\alpha ij]_{m\times n}$

= $[\alpha\alpha ij]_{m\times n}$

$\in M_{m\times n}$ (F) [$\because \alpha, \alpha_{ij} \in$ G $\Rightarrow$ $\alpha\ \alpha_{ij} \in$ F]

M-2. Let $\alpha, \beta \in$ F and A $\in M_{m\times n}$ (F)

Now $(\alpha + \beta)$ A = $(\alpha + \beta)$ $[a_{ij}]_{m\times n}$

= $[(\alpha + \beta)\ a_{ij}]_{m\times n}$

= $[\alpha\ a_{ij} + \beta\ a_{ij}]_{m\times n}$

= $[\alpha\ a_{ij}]_{m\times n}$ $[\beta\ a_{ij}]_{m\times n}$

= α $[a_{ij}]_{m\times n}$ + β $[a_{ij}]_{m\times n}$

= α A + β A

$\therefore$ $(\alpha + \beta)$ A = α A + β A.

M-3. Let $\alpha \in$ F and A, B $\in M_{m\times n}$ (F)

Now α (A + B) = $\alpha([a_{ij}]_{m\times n} + [b_{ij}]_{m\times n})$

= $\alpha([a_{ij} + b_{ij}]_{m\times n})$

= $[\alpha\ (a_{ij} + b_{ij})]_{m\times n}$

= $[\alpha\ a_{ij} + \alpha\ b_{ij}]_{m\times n}$

= $[\alpha\ a_{ij}]_{m\times n}$ + $[\alpha\ b_{ij}]_{m\times n}$

= α $[a_{ij}]_{m\times n}$ + α $[b_{ij}]_{m\times n}$

= α A + α B

$\therefore$ $\alpha\ (A + B) = \alpha$ A + α B $\forall\ \alpha \in$ F, A, B $\in M_{m\times n}$ (F)

M-4 Let $\alpha, \beta \in$ F and A $\in M_{m\times n}$ (F)

Now $(\alpha\ \beta)$ A = $(\alpha\ \beta)$ $[a_{ij}]_{m\times n}$

$= [(\alpha\ \beta)\ a_{ij}]_{m\times n}$

$= [\alpha(\beta\ a_{ij})]_{m\times n}$

[$\because$ multiplication is associative in a field $\therefore$ $(\alpha\ \beta)\ a_{ij} = \alpha\ (\beta\ a_{ij})$]

$= \alpha[\beta\ a_{ij}]_{m\times n}$

$= \alpha(\beta\ [a_{ij}]_{m\times n})$

$= \alpha(\beta\ A)$

$\therefore$ $(\alpha\ \beta)\ A = \alpha\ (\beta\ A)$ $\forall$ $\alpha, \beta \in F, A \in M_{m\times n}$ (F)

M-5. Let 1 be unity element of F

and $A = [a_{ij}]_{m\times n} \in M_{m\times n}$ (F)

Now $1.\ A = 1.\ [a_{ij}]_{m\times n}$

$= [1.\ a_{ij}]_{m\times n}$

$= [a_{ij}]_{m\times n} = A$

$\therefore$ $1.\ A = A$ for all $A \in M_{m\times n}$ (F)

Hence $M_{m\times n}$ (F) is a vector space over F.

Example 3. If P (x) is the set of all polynomials in one indeterminate x over a field F. Then show that P (x) is a vector space over F with addition defined as addition of polynomials and scalar multiplication defined as product of polynomial by an element of F.

Solution: Given P (x) = $\{f(x) | f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + ... + \alpha_n x^n + ...\}$

$$= \left\{ f(x) \mid f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \ for \alpha_k 's \in F \right\}$$

We define addition and scalar multiplication as.

If f (x) = $\sum_{k=0}^{\infty} \alpha_k x^k \in$ P(x) and g(x) = $\sum_{k=0}^{\infty} \beta_k x^k \in$ P (x)

Then f (x) + g(x) = $\sum_{k=0}^{\infty} (\alpha_k + \beta_k) x^k$

and a f (x) = $\sum_{k=0}^{\infty} (\alpha \alpha_k) x^k$ for a $\in$ F.

1. Properties under addition

A-1. Closure. For each f(x), g(x) $\in$ P (x)

$$f(x) + g(x) = \sum_{k=0}^{\infty} (\alpha_k + \beta_k) x^k$$

$\in$ P (x)

[$\because$ α_k, $\beta_k \in F \Rightarrow \alpha_k + \beta_k \in F$ as F is field and field is closed under addition]

Thus P (x) is closed under addition.

A-2. Associative.

For each f (x) = $\sum_{k=0}^{\infty} \alpha_k x^k$, g(x) = $\sum_{k=0}^{\infty} \beta_k x^k$, h(x) = $\sum_{k=0}^{\infty} \gamma_k x^k \in$ P (x)

$$[f(x) + g(x)] + h(x) = \left[\sum_{k=0}^{\infty} \alpha_k x^k + \sum_{k=0}^{\infty} \beta_k x^k\right] + \sum_{k=0}^{\infty} \gamma_k x^k$$

$$= \sum_{k=0}^{\infty} \left(\alpha_k + \beta_k\right) x^k + \sum_{k=0}^{\infty} \gamma_k x^k$$

$$= \sum_{k=0}^{\infty} \left(\left(\alpha_k + \beta_k\right) + \gamma_k\right) x^k$$

$$= \sum_{k=0}^{\infty} \left(\alpha_k + \left(\beta_k + \gamma_k\right)\right) x^k$$

[$\because$ $\left(\alpha_k + \beta_k\right) + \gamma_k = \alpha_k + \left(\beta_k + \gamma_k\right)$ as $\alpha_k, \beta_k, \gamma_k \in$ F (field) and associative law holds in a field]

$$= \sum_{k=0}^{\infty} \alpha_k x^k + \sum_{k=0}^{\infty} \left(\beta_k + \gamma_k\right) x^k$$

$$= \sum_{k=0}^{\infty} \alpha_k x^k + \left[\sum_{k=0}^{\infty} \beta_k x^k + \sum_{k=0}^{\infty} \gamma_k x^k\right]$$

$= f$ (x) + (g(x) + h(x))

$\Rightarrow$ addition is associative in P (x).

A-3 Existence of addative identity

For each f(x) = $\sum_{k=0}^{\infty} \alpha_k x^k$ + $\in$ P (x)

Let O (x) = 0 + 0 x + 0 x^2 + ... + 0 x^n + ... = $\sum_{k=0}^{\infty} 0\, x^k \in$ P (x)

Now f (x) + O (x) = $\sum_{k=0}^{\infty} \alpha_k x^k + \sum_{k=0}^{\infty} 0\, x^k$

$$= \sum_{k=0}^{\infty}(\alpha_k x^k + 0)x^k$$

$$= \sum_{k=0}^{\infty}\alpha_k x^k$$

[$\because$ 0 $\in$ F is addative identity of field F $\Rightarrow$ α_k - 0 = α_k for each $\alpha_k \in$ F]

$= f$ (x)

and O (x) + f (x) = $\sum_{k=0}^{\infty} 0\, x^k + \sum_{k=0}^{\infty}\alpha_k x^k$

$$= \sum_{k=0}^{\infty}(0+\alpha_k)x^k$$

$$= \sum_{k=0}^{\infty}\alpha_k x^k$$

[$\because$ 0 $\in$ F is addative identity of field F $\Rightarrow$ 0 + α_k = α_k for each $\alpha_k \in$ F]

$= f$(x)

$\therefore$ f(x) + O (x) = f(x) = O (x) + f (x) for all f(x) $\in$ P (x)

Thus O (x) is the addative identity of P (x)

A-4. Existence of addative inverse

For each f (x) = $\sum_{k=0}^{\infty}\alpha_k x^k \in$ P (x)

let - f (x) = $\sum_{k=0}^{\infty}(-\alpha_k)x^k$

$\in$ P (x) [$\because$ $\alpha_k \in$ F $\Rightarrow$ - $\alpha_k \in$ F as F is a field]

Now f(x) + (-f(x)) = $\sum_{k=0}^{\infty}\alpha_k x^k + \sum_{k=0}^{\infty}(-\alpha_k)x^k$

$$= \sum_{k=0}^{\infty}[\alpha_k + (-\alpha_k)]x^k \quad \sum_{k=0}^{\infty} 0\, x^k$$

[$\because$ - α_k is the addative inverse of $\alpha_k \Rightarrow \alpha_k + (-\alpha_k) = 0$]

= O (x)

and (-f (x)) + f(x) = $\sum_{k=0}^{\infty}(-\alpha_k)x^k + \sum_{k=0}^{\infty}\alpha_k x^k$

$$= \sum_{k=0}^{\infty} [(-\alpha_k) + \alpha_k] x^k$$

$$= \sum_{k=0}^{\infty} 0\, x^k$$

[$\because$ - α_k is the addative inverse of $\alpha_k \Rightarrow \alpha_k + (-\alpha_k) = 0$]

$= O$ (x)

$\therefore$ f (x) + (-f(x)) - O (x) = (-f(x)) + f(x)

Thus addative inverse in P (x) exists.

A-5. Commutative.

Let f(x) = $\sum_{k=0}^{\infty} \alpha_k x^k$, g(x) = $\sum_{k=0}^{\infty} \beta_k\, x^k \in$ P (x)

Now f (x) + g(x) = $\sum_{k=0}^{\infty} \alpha_k x^k + \sum_{k=0}^{\infty} \beta_k\, x^k$

$$= \sum_{k=0}^{\infty} (\alpha_k + \beta_k) x^k$$

$$= \sum_{k=0}^{\infty} (\beta_k + \alpha_k) x^k$$

$\alpha_k + \beta_k = \beta_k = \beta_k + \alpha_k$ as $\alpha_k, \beta_k \in$ F commutative law holds in a field]

$$= \sum_{k=0}^{\infty} \beta_k\, x^k + \sum_{k=0}^{\infty} \alpha_k x^k$$

= g (x) + f (x)

Thus addition is commutative in P(x).

II. Properties under multiplication

M-1 For all a $\in$ F and f(x) = $\sum_{k=0}^{\infty} \alpha_k x^k \in$ P (x)

a f(x) = a $\left[\sum_{k=0}^{\infty} \alpha_k x^k \right]$

$$= \sum_{k=0}^{\infty} (a\alpha_k) x^k$$

[$\because$ a $\in$F, $\alpha_k \in$ F $\Rightarrow$ a$\alpha_k \in$ F as F is a field which is closed under scalar multiplication]

M-2. Let a $\in$ F and f (x), g (x) $\in$ P (x)

Now a [f (x) + g (x)] = a $\left[\sum_{k=0}^{\infty}\alpha_k x^k + \sum_{k=0}^{\infty}\beta_k x^k\right]$

= a $\left[\sum_{k=0}^{\infty}(\alpha_k + \beta_k)x^k\right]$

= $\sum_{k=0}^{\infty} \ \left[a(\alpha_k + \beta_k)\right]x^k$

= $\sum_{k=0}^{\infty} \ \left(a\alpha_k + a\beta_k\right)x^k$

[$\because$ a, α_k, β_k $\in$ F and distributive law holds in field F]

= $\sum_{k=0}^{\infty} (a\alpha_k)x^k$ + $\sum_{k=0}^{\infty}(a\beta_k)x^k$

= a $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right]$ + a $\left[\sum_{k=0}^{\infty} p_k x^k\right]$

= a f (x) + a g (x)

M-3. Let a, b $\in$ F and f(x) $\in$ P (x).

Now (a + b) f (x) = (a + b) $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right]$

= $\sum_{k=0}^{\infty}\left[(a+b)\alpha_k\right]x^k$

= $\sum_{k=0}^{\infty} \ \left(a\alpha_k + b\alpha_k\right)x^k$

[$\because$ a, b, αk $\in$ F and F is a field]

= $\sum_{k=0}^{\infty} \ \left(a\alpha_k\right)x^k$ + $\sum_{k=0}^{\infty} \ \left(b\alpha_k\right)x^k$

= a $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right]$+ b $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right]$

= a f (x) + b f (x).

M-4. Let a, b $\in$ F and f (x) $\in$ P (x)

$\therefore$ (ab) f(x) = (ab) $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right] = \sum_{k=0}^{\infty}\left[(ab)\alpha_k\right]x^k$

$= \sum_{k=0}^{\infty}\left[a(b\alpha_k)\right]x^k$ = a $\left[\sum_{k=0}^{\infty}(b\alpha_k)x^k\right]$

$= \left[\left(b\sum_{k=0}^{\infty}a_k x^k\right)\right]$ = a[bf(x)]

$\Rightarrow$ (ab) f(x) = a(bf(x)).

M-5. Let 1 be unity element of field F

and f(x) $\in$ P (x)

Now 1.f(x) = 1. $\left[\sum_{k=0}^{\infty}\alpha_k x^k\right]$

$= \sum_{k=0}^{\infty}(1.\alpha_k)x^k$

$= \sum_{k=0}^{\infty} \alpha_k x^k$ = f(x)

$\Rightarrow$ 1.f (x) = f(x) for all f(x) $\in$ P (x).

Hence P (x) is a vector space over F.

2.4 Self Check Exercises

Q.1 Explain whether the following statements are true or false

(i) A vector space must have attest two elements.

(ii) A vector space has always an infinite number of elements.

Q.2 Which of the following sots are vector spaces over reals

(i) The set of all polynomials with integral coefficients

(ii) All polynomial over R with constant term 2.

2.5 Summary

In this unit we have learnt the following

(i) Binary composition

(ii) Internal composition and external composition

(iii) Vector space or the linear space over the field F.

(iv) Properties of addition and properties of scalar multiplication etc.

2.7 Glossary

1. Rational vector space or a Real vector space or a complex vector space-

   The vector space V(F) is called a rational, real or complex vector space according as F is the field of rational, real or the field of complex numbers.

2. V(F) - V is a vector space over the field F.

2.8 Answers to Self Check Exercises

Ans.1 (i) False

(ii) False

Ans.2 (i) Not a vector space

(ii) Not a vector space

2.9 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.
3. Stephan H:, Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Edition, Prentice Hall of India Pvt. Ltd, New Delhi, 2004.
4. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

2.10 Terminal Questions

1. Prove that C, the set of complex numbers is a vector space over the field C.
2. Show that the set of all elements of the kind a+ $\sqrt{2}$ (b) + C$\left(\sqrt{3}\right)^{\frac{1}{3}}$, c $\in$ Q form a vector space over Q under and addition and scalar multiplication of reals.
3. Show that the set of all matrices of the form

   $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, a, b $\in$ C is a vector space over C under matrix addition and scalar multiplication.

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Unit - 3

Vector Subspaces

Structure

3.1 Introduction

3.2 Learning Objectives

3.3 Vector Subspaces

3.4 Self Check Exercise

3.5 Summary

3.6 Glossary

3.7 Answers to self check exercises

3.8 References/Suggested Readings

3.9 Terminal Questions

3.1 Introduction

Dear students, in unit-2 we have learnt the concept of vector space and its properties under addition and multiplication. There in this unit we shall extend this knowledge to study the concept of vector subspaces. In Linear algebra, a linear vector subspace is a vector space that is a subset of some larger vector space. A linear subspace is simply called a subspace when the context serves is distinguish it from other types of subspaces.

3.2 Learning Objectives

The main objectives of this unit are

(i) to study the concept of vector subspace

(ii) to prove some important theorems to show how a non-empty subset of vector space become a vector space.

(iii) to prove how the intersection of two subspaces of a vector space become a vector subspace etc.

3.3 Vector Subspaces

Definition (Vector subspace)

Let V be a vector space over a given field F and W is a subset of V. The n W is called a subspace of V iff W itself is a vector space under the operation of addition and multiplication defined for V.

Two main things to be noted here are as

(i) The two binary operation in W, i.e., vector addition and scalar multiplication are same as those in V.

(ii) For any V(F), the set { 0 } and the set V, both are subset of V. Also both are vector spaces under addition and scalar multiplication operation of V(F). Both { 0 } and V are subspaces of V, known as trivial (Improper) subspaces and the subspaces other than { 0 } and V are called proper or non-trivial subspaces of V(F).

Theorem : Prove that a non-empty subset W of a vector space V(F) is a subspace of V iff W is closed under addition and scalar multiplication.

Proof : Given, W is a subspace of V(V).

$\therefore$ by definition of vector subspace, W is closed under addition and multiplication.

Hence the result holds.

Conversely.

It is given that W is closed under addition and scalar multiplication.

$\Rightarrow$ For all $x, y \in W$, $\alpha \in F$

we have $x + y \in W$ ....(1)

$\alpha \in W$ ....(2)

We have to prove W is a subspace of V (F).

Now $-1 \in F$ and $x \in W$

$\Rightarrow$ $(-1)\, x \in W$ (Using 2)

$\Rightarrow$ $-x \in W$

[Since $x \in W$ implies $x \in V$ and $(-1)\, x = -x$ holds in V]

so that addative inverse of every element in W exists.

And $\forall\, x \in W$, $-x \in W$

$\Rightarrow$ $x + (-x) \in W$ (Using 1)

$\Rightarrow$ $0 \in W$

so that addative identity exists in W.

$\therefore$ $x + 0 = x = 0 + x \;\forall \in W$

and $\forall\, x \in W$, there exists $-x \in W$ such that

$x + (-x) = 0 = (-x) + x.$

Now, since $W \subset V$ i.e., all the elements of W are also the elements of V, therefore, we have

$x + y = y + x.\ \forall\, x, y \in W$

$(x + y) + z = x + (y + z)\ \forall\, x, y, z \in W$

$\alpha\, (x + y) = \alpha\, x + \alpha\, y\ \forall\, \alpha \in F,\ x, y \in W$

$(\alpha + \beta)\, x = \alpha x + \beta x \;\forall\, \alpha, \beta \in F, x \in W$

$1. x = x \;\forall\, x \in W, 1 \in F.$

Thus W satisfies all the axioms for a vector space

$\Rightarrow$ W is a vector space over F

Hence W is a subspace of V (F).

Theorem : Prove that a non empty subset W of a vector space V(F) is a subspace of V iff (i) $\forall$ x, y $\in$ W, we have x - y $\in$ W.

(ii) $\forall\, \alpha \in F, x \in W$ we have $x\alpha \in W$.

Proof : Given W is a subspace of V(F).

(i) $\forall$ x, y $\in$ W $\Rightarrow$ x - y $\in$ W. [$\because$ y $\in$ W $\Rightarrow$ - y $\in$ W as W is vector space over F]

$\Rightarrow x + (-y) \in W$ [$\because$ W is closed under addition]

$\Rightarrow x - y \in W$.

(ii) $\forall\, \alpha \in F, x \in W$

$\Rightarrow \alpha \in F$ [$\because$ W is a vector space and by def. of a vector space W is closed under scalar multiplication]

Hence (i) and (ii) hold.

Conversely.

Given the properties (i) and (ii) hold

We have to prove, W is a vector subspace of V(F)

Now $-1 \in F$ and $x \in W$

$\Rightarrow$ $(-1)\, x \in W$ (Using ii)

$\Rightarrow$ $-x \in W$ [since $x \in W \Rightarrow x \in V$ and $(-1)\, x = -x$ holds in V]

So that addative inverse of every element in W exists.

And $\forall\, x \in W$ and $y \in W$

$\Rightarrow$ $x \in W$ and $-y \in W$

$\Rightarrow$ $x - (-y) \in W$ [Using property (i)]

$\Rightarrow$ $x + y \in W$

So that W is closed under addition.

Now $\forall\, x \in W$ and $-x \in W$

$\Rightarrow$ $x + (-x) \in W$

$\Rightarrow$ $0 \in W$

So that addative identity exists in W.

Now, since $W \subset V$ i.e., all the elements of W are also the elements of V. Therefore, we have

$x + y = y + x.\ \forall x, y \in W$

$(x + y) + z = x + (y + z)\ \forall x, y, z \in W$

$\alpha (x + y) = \alpha x + \alpha y\ \forall \alpha \in F, x, y \in w$

$(\alpha + \beta) x = \alpha x + \beta x\ \forall \alpha, \beta \in F, x \in W$

$1 . x - x\ \forall x \in W, 1 \in F.$

Thus W satisfies all the axioms for a vector space

$\Rightarrow$ W is a vector space over F

Hence W is a subspace of V(F).

Theorem : The necessary and sufficient condition for non-empty subset W of a vector space V (F) to be a subspace of V is that $\alpha x + \beta y \in W$ for all $\alpha, \beta \in F$ and $x, y \in W$.

Proof : The condition is necessary

Let W be a subspace of V (F)

Let $\alpha , \beta \in F$ and $x, y, \in W$

Now $\alpha \in F, x \in W \Rightarrow \alpha x \in W$

[Since W is a subspace so closed under scalar multiplication]

and $\beta \in F, y \in W \quad \Rightarrow \beta y \in W$

since $\alpha x, \beta y \in W$

$\Rightarrow \quad \alpha x + \beta y \in W$ [$\because$ W is closed under addition]

So that for all $\alpha, \beta \in F, x, y \in W$

$\Rightarrow \quad \alpha x + \beta y \in W.$

The condition is sufficient

It is given that $\forall \alpha, \beta \in F$ and $x, y \in W \Rightarrow \alpha x + \beta y \in W$ ...(1)

To prove that W is a subspace of V

(i) Putting $\alpha = 1$ and $\beta = -1$ in, (1) we get,

$1. x + (-1) y \in W$

i.e., $x - y \in W$

(ii) Again putting $\beta = 0$ in (1), we get,

$\alpha x + - y \in W$ [$\because$ $0 y = 0$]

$\Rightarrow \quad \alpha x + 0 \in W$

$\Rightarrow \quad \alpha x \in W$

Hence W is a subspace of V(F). [Using Th. 2]

Theorem : Prove that the intersection of two subspaces W_1 and W_2 of a vector space V(F) is also a subspace.

Proof : Let $\alpha, \beta \in F$ and $x, y \in W_1 \cap W_2$

Now $x \in W_1 \cap W_2 \quad \Rightarrow \quad x \in W_1$ and $x \in W_2$

and $y \in W_1 \cap W_2 \quad \Rightarrow \quad y \in W_1$ and $y \in W_2$

Since W_1 is a subspace of V

$\therefore \quad \forall\, x, y \in W_1; \alpha, \beta \in F \quad \Rightarrow \quad \alpha x + \beta y \in W_1$

Also W_2 is a subspace of V

$\therefore \quad \forall\, x, y \in W_2; \alpha, \beta \in F \quad \Rightarrow \quad \alpha x + \beta y \in W_2$

So that $\alpha x + \beta y \in W_1$ and $\alpha x + \beta y \in W_2$

$\Rightarrow \quad \alpha x + \beta y \in W_1 \cap W_2$

Hence $W_1 \cap W_2$ is a subspace of V (F).

Remark. (i) $W_1 \cap W_2$ is the largest subspace contained in W_1 and W_2 both.

(ii) The union of two subspaces is not necessarily a subspace.

For example,

Let the vector space V_3 (r) = {(x, y, z) | x, y, z $\in$ R}

Let $W_1 = \{(x, 0, 0) \mid x \in R\}$

and $W_2 = \{(0, y, 0) \mid y \in R\}$

Firstly, we prove that W_1 and W_2 are subspaces of V_3 (R)

Let $u = (x_1, 0, 0)$ and $v = (x_2, 0, 0) \in W_1$ for same $x_1, x_2 \in R$

and $\alpha, \beta \in R$

Then $\alpha u + \beta v = \alpha (x_1, 0, 0) + \beta (x_2, 0, 0)$

$= (\alpha x_1 + \beta x_2, 0, 0)$

$\in W_1 \quad [\because \alpha, \beta, x_1, x_2 \in R \Rightarrow \alpha x_1 + \beta x_2 \in R]$

$\therefore \alpha u + \beta v \in W_1$ for all $u, v \in W_1$ and $\alpha, \beta \in R$

$\Rightarrow \quad W_1$ is a subspace of V_3 (R).

Similarly we can show W_2 is a subspace of V_3 (R)

Now, we shall prove here W1 U W2 is not a subspace of V3 (R)

Let α β be non zero reals

and $w_1 = (x, 0, 0) \in W_1$, $w_2 = (0, y, 0) \in W_2$

$\Rightarrow$ $w_1 = (x, 0, 0) \in W_1 \cup W_2$, $w_2 = (0, y, 0) \in W_1 \cup W_2$

$\therefore$ $\alpha w_1 + \beta w_2 = \alpha (x, 0, 0) + \beta (0, y, 0)$

[$\because$ W_1 and W_2 both are subsets of $W_1 \cup W_2$]

$= (\alpha x, \beta y, 0)$

$\Rightarrow$ $\alpha w_1 + \beta w_2 \notin W_1$ as well as $\alpha w_1 + \beta w_2 \notin W_2$

$\Rightarrow$ $\alpha w_1 + \beta w_2 \notin W_1 \cup W_2$

Hence W1 U W2 is not a subspace.

Theorem : Prove that the union of two subspaces is a subspace iff one of them is a subset of the other.

Or

If W_1 and W_2 are subspaces of V (F). Then prove that $W_1 \cup W_2$ is a subspace of V (F) iff either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Or

Prove that the union of two subspaces W_1 and W_2 of a vector space V(F) is a subspace of V iff they are comparable.

Proof : Let W_1 and W_2 be subspaces of a vector space V(F)

Firstly suppose $W - W_1 \cup W_2$ be a subspace of V(F)

Now, we wish to prove that either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If possible, suppose neither $W_1 \subseteq W_2$ nor $W_2 \subseteq W_1$

$\Rightarrow$ there exists $x \in W_1$ such that $x \notin W_2$ ...(1)

and $y \in W_2$ such that $y \notin W_1$ ...(2)

$\therefore$ $x \in W_1$ $\Rightarrow x \in W_1 \cup W_2$ [$\because$ $W_1 \subseteq W_1 \cup W_2$]

$\Rightarrow x \in W$

and $y \in W_2$ $\Rightarrow y \in W_1 \cup W_2$ [$\because$ $W_2 \subseteq W_1 \cup W_2$]

$\Rightarrow y \in W$

So that $x, y \in W$

$\Rightarrow x + y \in W$

[$\because$ W is a subspace of V (by supposition)]

$\Rightarrow x + y \in W_1 \cup W_2$

$\Rightarrow$ Either $x + y \in W_1$ or $x + y \in W_2$ ...(3)

If $x + y \in W_1$

Then $x + y, x \in W_1$ $\Rightarrow (x + y) - x \in W_1$

[$\because$ W_1 is a subspace of V (F)]

$\Rightarrow y \in W_1$

which contradicts (2)

$\therefore$ $x + y \notin W_1$

If $x + y \in W_2$

Then $x + y, y \in W_2$ $\Rightarrow (x - y) - y \in W_2$

[$\because$ W_2 is a subspace of V (F)]

$\Rightarrow x \in W_2$

which contradicts (1)

$\therefore$ $x + y \notin W+$

Thus $x + y \notin W_1$ and $x + y \notin W_2$

which contradicts (3)

$\therefore$ our supposition is wrong

Hence either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Conversely

Now suppose either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

We have to prove that $W_1 \cup W_2$ is a subspace of V (F)

Since $W_1 \subseteq W_2$ $\Rightarrow$ $W_1 \cup W_2 = W_2$

and $W_2 \subseteq W_1$ $\Rightarrow$ $W_1 \cup W_2 = W_1$

$\therefore$ $W_1 \cup W_2 = W_2$ or W_1

$\Rightarrow$ $W_1 \cup W_2$ is a subspace if V (F)

[$\because$ W_1 and W_2 both are subspaces of V (F)]

Hence $W_1 \cup W_2$ is a subspace of V(F).

Example 1. Let a, b , c be fixed elements of a field F. Show that

$W = \{(x, y, z) \mid a x + b y + c z = 0 ; x, y, z \in F\}$

is a subspace of V_3 (F).

Solution : Since $(0, 0, 0) \in W$ as $a . 0 + b . 0 + c . 0 = 0$; $0 \in F$

$\Rightarrow$ $W \neq \phi$

Clearly $W \subset V_3$

Let $\alpha, \beta \in F$ and $u, v \in W$

The $u = (x_1, y_1, z_2)$ and $v = (x_2, y_2, z_2)$

where $x_1, y_1, z_2, x_2, y_2, z_2 \in F$

$$\left.\begin{array}{ll}\text{such that} & ax_1 + by_1 + cz_1 = 0 \\ \text{and} & ax_2 + by_2 + cz_2 = 0\end{array}\right] \qquad \ldots(1)$$

Now $\alpha\, u + \beta\, v = \alpha\,(x_1, y_1, z_1) + \beta\,(x_2, y_2, z_2)$

$= (\alpha\, x_1, \alpha\, y_1, \alpha\, z_1) + (\beta\, x_2, \beta\, y_2, \beta\, z_2,)$

$= (\alpha\, x_1 + \beta\, x_2, \alpha\, y_1, + \beta\, y_2, \alpha\, z_1 + \beta\, z_2)$

Here $= a(\alpha\, x_1 + \beta\, x_2) + b(\alpha\, y_1, + \beta\, y_2) + c\,(\alpha\, z_1 + \beta\, z_2)$

$= a\,\alpha\, x_1 + a\,\beta\, x_2, b\,\alpha\, y_1, + b\,\beta\, y_2 + c\,\alpha\, z_1 + c\,\beta\, z_2)$

$= \alpha(a\, x_1 + b\, y_1 + c\, z_1) + \beta\,(a\, x_2, + b\, y_2 + c\, z_2)$

$= \alpha\,(0) + \beta\,(0)$

$= 0$ [Using (1)]

$\therefore \quad \alpha\, u + \beta\, v = (a\,\alpha\, x_1 + \beta\, x_2, \alpha\, y_1, + \beta\, y_2 + \alpha\, z_1 + \beta\, z_2)$

$\in W$

Hence W is a subspace of V_3 (F).

Remark : The above example implies that any plane passing through (0, 0, 0) is a subspace of R^3.

Example 2 : Let $V = \{A \mid A = [a_{i\,j}]_{n \times n}, a_{i\,j} \in R\}$ be a vector space over reals. Show that W, the set consisting of all the symmetric matrices (i.e. matrices $[a_{i\,j}]_{n\times n}$ for which $a_{i\,j} = a_{j\,i}$) is a subspace of V.

Solution : It is given that $V = \{A \mid A = [a_{i\,j}]_{n \times n}, a_{i\,j} \in R\}$ is a vector pace over R.

We have to prove W $\{[a_{i\,j}]_{n \times n}, \mid a_{i\,j} = a_{j\,i}$ for $a_{ij} \in R\}$ is a subspace of V (R)

Clearly $O = [0]_{n\times n} \in W$ [Since $0_{i\,j} = 0 = 0_{j\,i}$ for $I \leq i, j \leq n$]

$\Rightarrow \quad W \neq \phi$

Also $W \subset V$

Let $P, Q \in W$ and $\alpha, \beta \in R$

$\therefore \quad P = [p_{i\,j}]_{n\times n}$ for which $p_{i\,j} = p_{j\,i}$

and $Q = [q_{i\,j}]_{n\times n}$ for which $q_{i\,j} = q_{j\,i}$

Then $\alpha\, P + \beta\, Q = \alpha\,[p_{i\,j}] + \beta(q_{i\,j}]$

$= [\alpha\, p_{i\,j} + \beta\, q_{i\,j}]_{\,n\times n}$

[By matrix addition and scalar multiplication]

$= [r_{ij}]_{n\times n}$ where $r_{ij} = \alpha\, p_{ij} + \beta\, q_{ij}$

Here $r_{ij} = \alpha\, p_{ij} + \beta\, q_{ij}$

$= \alpha\, p_{ji} + \beta\, q_{ji}$ [$\because$ $p_{ij} = p_{ji}$ and $q_{ij} = q_{ji}$]

$= r_{ji}$

$\therefore$ $[r_{ij}]_{n\times n}$ is a symmetric matrix

$\Rightarrow [r_{ij}] \in W$

$\Rightarrow \alpha P + \beta Q \in W$.

Hence W is a subspace of V (R).

Example 3: Show that U_1, U_2 are subspaces of R^4.

(i) $U_1 = \{(a, b, c, d)] \mid b + c + d = 0\}$

(ii) $U_2 = \{(a, b, c, d)] \mid a + b = 0, c = 2d\}$

Solution : (i) Since $(1, 0, 0, 0) \in U_1$ as $b + c + d = 0 + 0 + 0 = 0$; $0 \in R$.

$\Rightarrow$ $U_1 \neq \phi$

Clearly $U_1 \subset R^4$

Let $\alpha, \beta \in R$ and $u, v \in U_1$

Then $u = (a_1, b_1, c_1, d_1)$ and $v = (a_2, b_2, c_2, d_2)$

Where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in R$

Such that $b_1 + c_1 + d_1 = 0$ and $b_2 + c_2 + d_2 = 0$ ....(1)

Now $\alpha u + \beta v = \alpha (a_1, b_1, c_1, d_1) + \beta (a_2, b_2, c_2, d_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, \alpha c_1, \beta c_2, \alpha d_1 + \beta d_2)$

where $(\alpha b_1 + \beta b_2) + (\alpha c_1 + \beta c_2) + \alpha d_1 + \beta d_2)$

$= \alpha(b_1 + c_1 + d_1) + \beta (b_2 + c_2 + d_2)$

$= \alpha(0) + \beta(0) = 0$ (Using 1)

$\therefore$ $\alpha u + \beta v \in U_1$

Hence U_1 is a subspace of R^4.

(ii) Since $(0, 0, 2, 1) \in U_2$ as $a + b = 0 + 0 = 0$; $c = 2, 2(1) = 2d$.

$\Rightarrow$ $U_2 \neq \phi$

Clearly $U_2 \subset R^4$

Let $\alpha, p \in R$ and $u, v \in U_1$

Then $u = (a_1, b_1, c_1, d_1)$ and $v = (a_2, b_2, c_2, d_2)$

Where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in R$

Such that $a_1 + b_1 = 0, a_2 + b_2 = 0$ and $c_2 = 2 d_1, c_2 = 2 d_2$ ....(1)

Now $\alpha u + \beta v = \alpha (a_1, b_1, c_1, d_1) + \beta (a_2, b_2, c_2, d_2)$

$= (\alpha a_1 + \beta a_2, \alpha b_2 + \beta b_2, \alpha c_1, \beta c_2, \alpha d_1 + \beta d_2)$

where $(\alpha a_1 + \beta a_2) + (\alpha b_1 + \beta b_2) - \alpha (a_1 + b_1) + \beta(a_2 + b_2)$

$= \alpha(0) + \beta(0) = 0$

and $(\alpha c_1 + \beta c_2) = (\alpha 2 d_1) + \beta (2 d_2) = 2(\alpha d_1 + \beta d_2)$ (Using 1)

$\therefore$ $\alpha u + \beta v \in U_2$, Hence U_2 is a subspace of R^4.

Example 4 : Let V be a vector space in R^3.

Examine whether the following are subspaces or not

(i) $W = \{(a, b, c) \mid c \text{ is an integer}\}$

(ii) $W = \{(a, b, c) \mid a, \geq b \geq c\}$

(iii) $W = \{(a, b, c) \mid a = b - c \text{ and } 2a + 3b - c = 0\}$

Solution : (i) Let $(a, b, c) \in W$ where c is an integer

and $\alpha = \sqrt{2} \in R$

then $\alpha (a, b, c) = (\alpha a, \alpha b, \alpha c)$

$= (\sqrt{2} a, \sqrt{2} b, \sqrt{2} c)$

$\notin W$ [$\because$ c is an integer. But $\sqrt{2}$ c is not an integer]

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of V_3 (R).

(ii) Let $(a, b, c) \in W$, where $a, \geq b \geq c$

and $\alpha = - 2 \in R$

Then $\alpha (a, b, c) = (\alpha a, \alpha b, \alpha c)$

$= (-2 a, -2 b, -2 c)$

$\notin W$ [since $a \geq b \geq c \Rightarrow -2 a \leq -2 b \leq -2 c$]

$\therefore$ W is not closed under scalar multiplication.

Hence W is not a subspace of V_3 (R).

(iii) $\alpha \in r$ and $x, y \in w$.

then

$x = (a_1, b_1, c_1)$

$y = (a_2, b_2, c_2)$ such that

$$\left.\begin{aligned} a_1 = b_1 - c_1,\ 2a_1 + 3b_1 - c_1 = 0 \\ \text{and}\quad a_2 = b_2 - c_2,\ 2a_2 + 3b_2 - c_2 = 0 \end{aligned}\right\} \qquad - (1)$$

Now

$$x - y = (a_1, b_1, c_1) - (a_2, b_2, c_2)$$
$$= (a_1 - a_2, b_1 - b_2, c_1 - c_2)$$

such that

$$a_1 - a_2 = (b_1 - c_1) - (b_2 - c_2) \qquad \text{(Using (1))}$$
$$= (b_1 - b_2) - (c_1 - c_2)$$
$$= (2a_1 + 3b_1 - c_1) - (2a_2 + 3b_2 - c_2)$$
$$= 0 - 0 = 0 \qquad \text{(Using (1))}$$

$\therefore \quad x - y \in W$

and $\alpha x = \alpha (a_1, b_1, c_1) = (\alpha a_1, \alpha b_1, \alpha c_1)$

such that

$$\alpha \cdot a_1 = \alpha(b_1 - c_1) = \alpha b_1 - \alpha c_1$$

and $2(\alpha a_1) = 3. (\alpha b_1) - (\alpha c_1)$

$$= \alpha (2a_1 + 3b_1 - c_1)$$
$$= \alpha (0) = 0$$

$\therefore \alpha x \in W$

Hence W is a subspace of V_3 (R).

3.4 Self Check Exercises

Q.1 Set V be a vector space in R^3.

Examine whether

W = {(a, b, c,), c integer} is a subspace or not?

Q.2 Let R b a field of reals and V is a vector space of all infinite sequences < an >, an $\in$ R. under addition and scalar multiplication defined term wise. Show that

W = {<bn> | {bn^2 is convergent is a subspace of V.

3.5 Summary

We have learnt the following concepts in this unit.

(i) Vector subspaces

(ii) Some important theorems shows how a non-empty subset of a vector space become a vector space

(iii) Intersection of his subspaces of a vector space is a vector space.

3.6 Glossary

1. **Improper subspaces :** {0} and V are subspaces of V called improper subspaces or trivial subspaces.
2. **Proper subspaces :** Subspaces other than {0} and V are called proper subspaces or non-trivial subspaces of V(F).

3.7 Answers to Self Check Exercises

Ans.1 Take $\alpha = \sqrt{2} \in$ r, then proceed.

Ans.2 Take $<\alpha n>$, $<\beta n> \Rightarrow \{\alpha n^2, < \beta\ n^2$ are conversent then proceed.

3.8 Reference/Suggested Reading

1. Stephan H:, Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Edition, Prentice Hall of India Pvt. Ltd, New Delhi, 2004.
2. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.
3. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
4. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.

3.9 Terminal Questions

1. Discuss whether or not R^2 is a subspace of R^3.
2. Which of the following are set of vectors

   $x = (x_1, x_2, ...x_n) \in R^n$

   are subspaces of R^n ? $(n \geq 3)$

   (i) All x s. t. $x < 0$

   (ii) All x s. t. $x_2 = x_1^2$

   (iii) All x s.t. x_3 is rational.
3. Prove that set of straight lines passing through origin i.e. sets of the form

   $L = \{(x, y) \mid l\, x + my = 0, l, m \in R, l^2 + m^2 \neq 0\}$

   is a subgroup of R^2 over R.

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Unit - 4

Algebra of Subspaces

Structure

4.1 Introduction

Learning Objectives

4.2 Sum of Subspaces (Linear Sum of Two Subspaces)

4.3 Director Sum of Two Subspaces

4.4 Complementary Subspaces

4.5 Disjoint Subspaces

4.6 Identical Subspaces

4.7 Self Check Exercise

4.8 Summary

4.9 Glossary

4.10 Answers to self check exercises

4.11 References/Suggested Readings

4.12 Terminal Questions

4.1 Introduction

Dear students, having knowledge of vector spaces and vector subspaces from out previous units, we shall here discuss about algebra of subspaces. In this unit we shall study the concept of linear sum of two vector subspaces, direct sum of two subspaces and disjoint subspaces etc.

Objectives

The main objectives of this unit are

(i) to find the linear sum of two subspaces W_1 and W_2 of a vector space V(F).

(ii) to find the direct sum of two subspaces W_1 and W_2 of a vector spaces V(F).

(iii) to define complementary subspaces, disjoint subspaces and identical subspaces etc.

4.2 Sum of Subspaces

Linear sum of two subspace

Let W_1 and W_2 be two subspaces of a vector space V(F). Then linear sum of W_1 and W_2 is denoted by $W_1 + W_2$ and is defined as

$W_1 + W_2 = \{u + v \mid u \in W_1, v \in W_2\}$

Note : We have $u \in W_1 \Rightarrow u + O \in W_1 + W_2$

[O is addative identity of W_2]

$\Rightarrow u \in W_1 + W_2$

[$\because$ u + O = u, as O is also addative identity of W_1]

$\therefore W_1 \subseteq W_1 + W_2$

Similarly $W_2 \subseteq W_1 + W_2$

so that $W_1 \cup W_2 \subseteq W_1 + W_2$

For example

(i) Let $W_1 = \{(1, 3), (2, 0)\}$ and $W_2 = \{(1, 1), (-3, 1), (3, 4)\}$ be subspaces of V_2 (R).

$$\text{Then } W_1 + W_2 = \begin{Bmatrix} (1,3)+(1,1), (1,3)+(-3,1), (1,3)+(3,4) \\ (2,0)+(1,1), (2,0)+(-3,1), (2,0)+(3,4) \end{Bmatrix}$$

$= \{(2, 4), (-2, 4), (4, 7), (3, 1), (-1, 1), (5, 4)\}$

(ii) Let $W_1 = \left\{ \begin{bmatrix} x & y \\ 0 & w \end{bmatrix} \mid x, y, w \in R \right\}$

$W_2 = \left\{ \begin{bmatrix} x & 0 \\ z & 0 \end{bmatrix} \mid x, z \in R \right\}$,

Which are subspaces of v, a vector space of 2×2 matrices over R.

Then $W_1 + W_2 = \left\{ \begin{bmatrix} 2x & y \\ z & w \end{bmatrix} \mid x, y, z, w \in R \right\} = V$

and $W_1 \cap W_2 = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \mid z \in R \right\}$

and $W_1 \cup W_2 = \left\{ \begin{bmatrix} x & y \\ 0 & w \end{bmatrix}, \begin{bmatrix} x & 0 \\ z & 0 \end{bmatrix} \mid x, y, z, w \in R \right\}$

(iii) Let $W_1 = \{x, y, z) \mid x + y + z = 0$

$W_2 = \{x, y, z) \mid x = z\}$ be subspaces of V_3 (R)

(Prove yourself)

Now any vector $(x, y, z) \in V_3$ (R) can be written as

$$(x, y, z) = \left(\frac{x}{2}, -z, z - \frac{x}{2}\right) + \left(\frac{x}{2}, y + z, \frac{x}{2}\right)$$

where $\left(\frac{x}{2}, -z, z - \frac{x}{2}\right) \in W_1$ and $\left(\frac{x}{2}, y + z, \frac{x}{2}\right) \in W_2$

i.e. $V_3(R) = W_1 + W_2$

Note : The elements of $V_3(R)$ cannot be uniquely expressed as sum of elements of W_1 and W_2.

Theorem : If W_1 and W_2 are subspaces of vector space V(F). Prove that $W_1 + W_2$ is a subpace of V(F).

Proof : Let W_1 and W_2 be subspaces of a vector space V(F)

Take $w \in W_1 + W_2 \Rightarrow w = w_1 + w_2$ for $w_1 \in W_1$ and $w_2 = W_2$

$\Rightarrow w = w_1 + w_2$ for $w_1, w_2 \in V$ $\quad (\because W_1, W_2 \subset V)$

$\Rightarrow w \in V$

$(\because$ V is a vector space so closed under vector addition)

$\therefore \quad W1 + W2 \subset V$

Now let $u_1, + u_2 \in W1$ and $v_1, v_2 \in W_2$

Then $x = u_1 + v_1$ and $y = u_2 + v_2 \in W_1 + W_2$

For $\alpha, \beta \in F$ and $u_1, u_2 \in W_1 \Rightarrow \alpha u_1 + \beta u_2 \in W_1$

$(\because W_1$ is a subspace of V)

And $\alpha, \beta \in F$ and $v_1, v_2 \in W_2 \Rightarrow \alpha v_1 + \beta v_2 \in W_2$

$(\because W_2$ is a subspace of V)

$\therefore \quad \alpha u_1 + \beta u_2 \in W_1$ and $\alpha v_1 + \beta v_2 \in W_2$

$\Rightarrow \quad (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2) \in W_1 + W_2$ ...(1)

finally, for $\alpha, \beta \in F$; $x, y \in W_1 + W_2$

we have $\alpha x + \beta y = \alpha (u_1 + v_1) + \beta (u_2 + v_2)$

$= \alpha u_1 + \alpha v_1 + \beta u_2 + \beta v_2$

$= (\alpha u_1 + \beta u_2) + (\alpha v_1 + \beta v_2)$

$\in W_1 + W_2$ (Using (1))

Hence $W_1 + W_2$ is a subspace of V(F).

Note : The above theorem can be extended for n subspaces i.e. if W_1, W_2, ...W_n are subspaces of V(F). then $W_1 + W_2 + ... + W_n$ is also a subspace of V.

Theorem : Prove that $W_1 + W_2 = <W_1 \cup W_2>$

i.e. the linear sum of two subspaces W_1 and W_2 of a vector space V(F) is a subspace generated by union of W_1 and W_2.

Proof : We have $u \in W_1 \quad \Rightarrow u + O \in W1 + W_2$

($\because$ O is addative identity of W_2)

$\Rightarrow u = W_1 + W_2$ ($\because u + O = u$)

$\therefore \quad W1 \subseteq W_1 + W_2$

Similarly $W_2 \subseteq W_1 + W_2$

so that $W_1 \cup W_2 \subseteq W_1 + W_2$ ...(i)

Since $< W_1 \cup W_2 >$ is the smallest subspace of V containing $W_1 \cup W_2$ and $W_1 + W_2$ is a subspace containing $W_1 \cup W_2$ (by i)

so that $< W_1 \cup W_2 > \subseteq W_1 + W_2$ ...(ii)

Now let $x = u_1 + u_2 \in W_1 + W_2$ where $u_1 \in W_1$

So that $u_1, u_2 \in W_1 \cup W_2 \quad u_2 \in W_2$

there $x = u_1 + u_2 = 1 \, . u_1 + 1 \, . u_2$

i.e. x is a L.C. of elements $u_1, u_2 \in W_1 \cup W_2$

$\Rightarrow x \in < W_1 \cup W_2 >$

$\therefore W_1 + W_2 \subseteq < W_1 \cup W_2>$ ....(iii)

Thus from (ii) and (iii), we get

$W_1 + W_2 = < W_1 \cup W_2 >$

Hence the result.

Remarks : (i) W_1 and W_2 both are contained in $W_1 + W_2$ and is the smallest subspace of V containing W_1 and W_2.

(ii) $W_1 + W_1 = W_1$ and if $W_2 \subseteq W_1$, then $W_1 + W_2 = W_1$.

Theorem : If W_1, W_2, W_3 are subspaces of a vector space V such that $W_1 \supseteq W_2$.

Prove: $W_1 \cap (W_2 + W_3) = W_2 + (W_1 \cap W_3)$

Proof: Let $x \in W_1 \cap (W_2 + W_3)$

$\Rightarrow \quad x \in W_1 \quad$ and $\quad x \in W_2 + W_3$

$\Rightarrow \quad x \in W_1 \quad$ and $\quad x = x_3$ for $x_2 \in W_2, x_3 \in W_3$ .....(1)

$\Rightarrow \quad x \in W_1 \quad$ and $\quad x - x_2 = x_3$

Here $x_2 \in W_2 \Rightarrow x_2 \in W_1 \quad (\because W_1 \supseteq W_2)$

$\therefore \quad x.\ x_2 \in W_1 \Rightarrow x - x_2 \in W_1 \quad (\because W_1$ is a subspace)

$\Rightarrow \quad x_3 \in W_1$

so that $x_3 \in W_1$ and $x_3 \in W_3$

$\Rightarrow \quad x_3 \in W_1 \cap W_3$

$\therefore \quad (1) \Rightarrow x = x_2 + x_3 \in W_2 + (W_1 \cap W_3)$

so that $W_1 \cap (W_2 + W_3) \subset W_2 + (W_1 \cap W_3)$ .....(2)

Further, take $y \in W_2 + (W_1 \cap W_3)$

$\therefore \quad y = y_2 + z$ where $y_2 \in W_2 \subset W_1$ and $z \in w_1 \cap W_3$

i.e. $\quad y = y_2 + z$ where $y_2 \in W_1$ and $z \in W_3$

$\therefore \quad y \in W_1 \quad (\because y_2, z \in W_1$ and W_1 is a subspace so $y_2 + z \in y_2 \in W_1)$

Also $\quad y = y_2 + z \in W_2 + W_3 \quad (\because y_2 \in W_2$ and $z \in W_3)$

$\therefore \quad y \in W_1 \cap (W_2 + W_3)$

so that $W_2 + (W_1 \cap W_3) \subset W_1 \cap (W_2 + W_3)$ ....(3)

Combining (2) and (3) we get $W_1 \cap (W_2 + W_3) = W_2 + W_1 \cap W_3)$

4.3 Direct sum of Two Subspaces

A vector space V(F) is said to be direct sum of its two subspaces W_1 and W_2 ; denoted by $W_1 \oplus W_2$ if every vector $v \in V$ can be unequally expressed as $v = u_1 + u_2$ where $u_1 \in W_2$.

Note: $V = W_1 \oplus W_2 \Rightarrow V = W_1 + W_2$ but not conversely.

4.4 Complementary Subspaces

Any two subspaces W_1 and W_2 of V(F) are said to be complementary if every vector $v \in V$ can be written in one and only one way as

$v = u_1 + u_2$ where $u_1 \in W_1$ and $u_2 \in W_2$

i.e. $\quad V = W_1 \oplus W_2$

For example

(i) In $V_2(R)$, $W_1 = <(1, 0)>$ and $W_2 = <(0, 1)>$ are subspaces of $V_2(R)$. Any vector $(x, y) \in V_2(R)$ can be written in one and only one way as

$(x, y) + (X, 0) + (0, y)$ where $(x, 0) \in W_1$ and $(0, y) \in W_2$

$\therefore \quad V_2(R) = W_1 \oplus W_2$

and hence W_1 and W_2 are complementary subspace.

(ii) In $V_3(R)$, let W_1 be the yz-plane and W_2 be the x-axis.

i.e. $W_1 = \{(0, y, z) | y, z \in R\}$ and $W_2 = \{(x, 0, 0)\ x \in R\}$

Any vector $(x, y, z) \in V_3(R)$ can be written in one and only one way as

$(x, y, z) = (0, y, z) + (x, 0, 0$ where $(0, y, z) \in W_1$ and $(x, 0, 0) \in W_2$

$\therefore$ $V_3(R) = W_1 \oplus W_2$ and hence W_1, W_2 are complementary subspaces.

(iii) In $V_3(R)$; Let W_1 $\{(x, y, 0) | x, y \in R\}$

W_2 $\{(0, y, z)\ y, z \in R\}$

i.e. xy-plane and yz-plane respectively.

Then $V_3 \neq W_1 \oplus W_2$ but $V_3 = W_1 + W_2$ since any vector $(x, y, z) \in V_3(R)$ can be written indifferent ways as

$$(x, y, z) = \left(x, \frac{3y}{4}, 0\right) + \left(0, \frac{y}{4}, z\right)$$

where $\left(x, \frac{3y}{4}, 0\right) \in W_1$ and $\left(0, \frac{y}{4}, z\right) \in W_2$

and $(x, y, z) = \left(x, \frac{3y}{4}, 0\right) + \left(0, \frac{2y}{3}, z\right)$ where $\left(x, \frac{y}{4}, 0\right) \in W_1$ and $\left(x, \frac{2y}{3}, z\right)$

Here W_1 and W_2 are not complementary subspaces.

(iv) In V = $\left\{\begin{bmatrix} x & y \\ z & w \end{bmatrix} | x, y, z, w \in R\right\}$, vector space of 2×2 matrices.

Let $W_1 = \left\{\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} | y \in R\right\}$ and $W_2 = \left\{\begin{bmatrix} x & 0 \\ z & w \end{bmatrix} | x, z, \omega \in R\right\}$

be subspaces of V(R)

Then any elt $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in$ V can be written in one and only one way as

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x & 0 \\ z & w \end{bmatrix}$$

$\therefore$ $V = W_1 \oplus W_2$

and hence W_1 and W_2 are complementary subspace

4.5 Disjoint Subspaces

Any two subspaces W_1 and W_2 of a vector space V(F) are said to be disjoint if their intersection is zero space i.e. $W_1 \cap W_2 = \{0\}$.

For example: In examples (i), (ii) and (iv) given after Def. 11 ; W_1 and W_2 are disjoint subspaces.

4.6 Identical Subspaces

Two vector spaces U and V (of the same dimension) are called identical iff each is a subspace of the other.

Theorem. Let W_1 and W_2 be subspaces of vector space V(F). Prove that $V = W_1 \oplus W_2$ if and only if (i) $V=W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$

OR

prove that, the necessary and sufficient conditions for a vector space V(F) to be a direct sum of its subspaces W_1 and W_2 are

(i) $V = W_1 + W_2$ (ii) $W_1 \cap W_2 = \{0\}$

Proof: Firstly, let $V = W_1 \oplus W_2$

$V = W_1 + W_2$ ....(1)

which proves (i) necessary condition.

$\because$ by def. of direct sum, every vector $v \in V$ can be written uniquely as $v = u_1 + u_2$ where $u_1 \in W_1$ and $u_2 \in W_2$

Further let $v \neq 0$ and $v \in W_1 \cap W_2$

Also $v \in V$ $(\because W_1 \cap W_2 \subset V)$

and $v = 0 + v$ where $0 \in W_1$ and $v \in W_2$

$v = v + 0$ where $v \in W_1$ and $0 \in W_2$

which implies that $v \in V$ can be expressed in two different ways, which is a contradiction to (1)

$\therefore$ our supposition is wrong so only $0 \in W_1 \cap W_2$

i.e. $W_1 \cap W_2 = \{0\}$, which proves (ii) necessary condition.

Conversely:

Let $V = W_1 + W_2$ and $W_1 \cap W_2$ are subspaces of V(F)

Take $v \in V$ be any element

$\therefore$ $v = u_1 + u_2 = w_1 + w_2$

$\Rightarrow$ $u_1 - w_1 = w_2 - u_2$

Since W_1, W_2 are subspaces, so $u_1, w_1 \in W_1$ $\Rightarrow$ $u_1 - w_1 \in W_1$

and $u_2, w_2 \in W_2 \Rightarrow W_2 - u_2 \in W_2$

$\Rightarrow u_1 - w_2 = w_2 - u_2 \in w_1 \cap w_2$

But we have $W_1 \cap W_2 = \{0\}$

$\therefore u_1 = w_1 = w_2 - u_2 = 0 \Rightarrow w_1 = u_1$ and $w_2 = u_2$

so that any vector $v \in V$ can be written in one and only one way as

$v = u_1 + u_2$ where $u_1 \in W_1$ and $u_2 \in W_2$

Hence $V = W_1 \oplus W_2$

Theorem: If W_1 is a subspace of vector space V(F), then prove that $\exists$ a subspace W_2 of V(F) such that $V = W_1 \oplus W_2$

OR

Every subspace of a vector space has a direct summand. Prove it.

Proof: Let $B_1 = \{v_1, v_2, \ldots, v_n\}$ be a basis of $W_1 \subset V$.

As B_1 is L.I. subset of W_1 and so of V

so extend B_1 to a basis $B = \{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m\}$ of V

Take $B_2 = \{w_1, w_2, \ldots, w_m\}$ and W_2 be subspace of V generated by elements of B_2.

We claim $V = W_1 \oplus W_2$

(i) Let $v \in V$, then as B is basis of V

so $v = \sum_{i=1}^{n} a_i v_i + \sum_{j=1}^{m} b_j w_j$ for $a_i, b_j \in F$

$\Rightarrow n \in W_1 + W_2$

so that $V \subset W_1 + W_2$ ....(1)

Also we have $W_1 + W_2 \subset V$ ....(2)

Combining (1) and (2), we get $V = W_1 + W_2$

(ii) Let $v \in W_1 \cap W_2 \Rightarrow v \in W_1$ and $v \in W_2$

$\therefore v = \sum_{i=1}^{n} a_i v_i$ and $v = \sum_{j=1}^{m} b_j w_j$ for $a_i, b_j \in F$

$$\Rightarrow \sum_{i=1}^{n} a_i v_i = \sum_{j=1}^{m} b_j w_j$$

$\Rightarrow \quad \sum_{i=1}^{n} a_i v_i - \sum_{j=1}^{m} b_j w_j = 0$

$\Rightarrow \quad a_i = 0$ and $b_j = 0$ for $1 \leq i \leq n$. $1 \leq j \leq m$ as B is L.I. set being basis of V

$\therefore \quad v = 0$

$\Rightarrow \quad W_1 \cap W_2 = \{0\}$

so (i) and (ii) $\Rightarrow \quad V = W_1 \oplus W_2$ i.e., every subspace of a vector space has a direct summand.

Some Illustrative Examples

Example: If S and T are any subsets of V(F). prove that

L(S U T) = L(S) + L(T)

Solution: Let $v \in$ L(S U T) be any element

$\Rightarrow \quad \exists\, v_1, v_2, \ldots\ldots v_n \in$ S U T and $\alpha_1, \alpha_2, \ldots\ldots \alpha n \in F$ such that

$$v = \sum_{i=1}^{n} a_i v_i$$

$$\Rightarrow \quad v = \sum_{i=1}^{n} a_i v_i = \sum a_i v_i + \sum a_k v_k$$

where v_i's $\in$ S and v_k's $\in$ T

$$\left[\begin{array}{l} \because Each\, v, is\, either\, an\, elt\, of\; S\, or\, an\, elt\, of\; T\, or\, an\, element\, of\; both \\ S\; and\; T; so\, dividing\; elements\, v_i\; \text{int}\, o\, elements\, v_j\,'s\, belonging\; to \\ S\; and\; the\, elements\, v_k\,'s\, to\, T \end{array}\right]$$

$\Rightarrow \quad v \in$ L(S) + L(T)

$\therefore \quad$ L(S U T) $\subset$ L(S) + L(T) ....(i)

Now let $z \in$ L(S) + L(T)

$\Rightarrow \quad z = x + y$ where $x \in$ L(S) and $y \in$ L(T)

$\Rightarrow \quad z = \sum a_j v_i + \sum a_k v_k$ where v_j's $\in$ S and v_k's $\in$ T and $\alpha_j, \alpha_k \in F$.

$\Rightarrow \quad z = \sum a_j v_i$ ($\because$ {v} = {v_j} U {v_k})

$\Rightarrow \quad z \in$ L(S U T)

$\therefore \quad$ L(S) + L(T) $\subset$ L(S U T) ....(ii)

From (i) and (ii), we have

$L(S \cup T) = L(S) + L(T)$

Hence the result

Example 2: Let V (R) be a vector space of all functions from R to R

Show that W_1 and W_2 are subspaces of V (R)

where $W_1 = \{f | f \in V$ and $f(-x) = f(x)\}$

= The set of all even functions

and $W_2 = \{f | f \in V$ and $f(-x) = -f(x)\}$

- The set of all odd functions

Also show that

(iii) $V = W_1 + W_2$ (iv) $W_1 \cap W_2 = \{0\}$ (v) $V = W_1 \oplus W_2$

Solution: (i) Given $W_1 = \{f | f \in V$ and $f(-x) = f(x)\}$

Clearly $f(x) = x_4 + x_2 \in W_1$ $[\because f(-x) = f(x)]$

$\therefore$ W_1 is non-empty set

Now let $\alpha, \beta \in R$ and $f, g \in W_1$

$\Rightarrow$ $f(-x) = f(x)$ and $g(-x) = g(x)$ for all $x \in R$

$\therefore$ $(\alpha f + \beta g)(-x)$

$= (\alpha f)(-x) + (\beta g)(-x)$

$= \alpha.f(-x) + \beta.g(-x)$

$= \alpha.f(x) + \beta.g(x)$

$= (\alpha f)(x) + (\beta g)(x)$

$= (\alpha f + \beta g)(x)$

$\therefore$ $\alpha f + \beta g \in W_1$

$\Rightarrow$ W1 is a subspace of V.

(ii) Given $W_2 = \{f | f \in V$ and $f(-x) = f(x)\}$

Clearly $f(x) = x_3 + x \in W_2$ $[\because f(-x) = -f(x)]$

$\therefore$ W_2 is a non-empty set

Now let $\alpha, \beta \in R$ and $f, g \in W_2$

$\Rightarrow$ $f(-x) = -f(x)$ and $g(-x) = g(x)$

for all $x \in R$

$\therefore \quad (\alpha f + \beta g)(-x)$

$= (\alpha . f)(-x) + (\beta\, g)(-x)$

$= \alpha . f(-x) + \beta . g(-x)$

$= \alpha\,(-f(x)) + \beta\,(-g(x))$

$= -(\alpha . f(x) + \beta . g(x))$

$= -(\alpha f + \beta g)(x)$

$\therefore \quad \alpha f - \beta g \in W_2$

$\Rightarrow \quad W_2$ is a subspace of V.

(iii) To show that $V = W_1 - W_2$

Let $f \in V$

Then, we can write f as

$$f(x) = \frac{1}{2}(f(x) + f(-x) + \frac{1}{2}(f(x) - f(-x)) \;\forall\; x \in R$$

$$= F(x) + G(x), \qquad \text{where } F(x) = \frac{1}{2}(f(x) + f(x))$$

$$G(x) = \frac{1}{2}\,(f(x) - f(-x))$$

Check that $F(-x) = F(x)$

and $G(-x) = -G(x)$

$\therefore \quad F(x) \in W_1$ and $G(x) \in W_2$

Hence $f = F + G$ where $F \in W_1$ and $G \in W_2$

$\therefore \quad V = W_1 + W_2$

(iv) Let $f \in W_1 \cap W_2 \quad \Rightarrow \quad f \in W_1$ and $f \in W_2$

$\Rightarrow \quad f(-x) = f(x)$ and $f(-x) = -f(x)$

$\Rightarrow \quad f(x) = -f(x) \quad \Rightarrow \quad 2f(x) = 0$

$\Rightarrow \quad f(x) = 0$

so that $W_1 \cap W_2 = \{0\}$

(v) From (iii) and (iv), we get

$$V = W_1 + W_2 \text{ and } W_1 \cap W_2 = \{0\}$$

$\Rightarrow \quad V = W_1 \oplus W_2$

Hence the result

Example 3: If

$A = \{(0, 0, z) : z \in R\}$

$B = \{(x, y, x) : x, y \in R\}$

Then show that $R^3 = A \oplus B$

Solution: Let us take

$v = (x, y, z) \in R^3$

$\therefore$ $(x, y, z) = (0, 0, z-x) + (x, y, x)$

Here $(0, 0, z-x) \in A$ and $(x, y, x) \in B$

$\therefore$ $R^3 = A + B$ ....(1)

Claim: $A \cap B = \{0\}$

Let $v = (x, y, z) \in A \cap B$ be any vector

$\Rightarrow$ $(x, y, z) \in A$ and $(x, y, z) \in B$

$\Rightarrow$ $x = 0, y= 0$ and $z = x$

$\Rightarrow$ $x = 0, y = 0$ and $z = 0$

$\Rightarrow$ $v (x, y, z) = (0, 0, 0)$

$\Rightarrow$ $A \cap B = \{0\}$ ....(2)

Hence from (1) and (2), we have

$R^3 = A \oplus B$

Hence the result

Now you can try the following self check exercises.

4.7 Self Check Exercises

Q.1 Let V be a vector space of $n\times n$ matrices over a field F. Let W_1 and W_2 be subspaces of upper triangular and lower triangular matrices. Find

(i) $W_1 + W_2$ (ii) $W_1 \cap W_2$

Q.2 Let $V = F_3$, $x_1 = (1, 0, 0)$, $x_2 = (1, 1, 0)$ $x_3 = (1, 1, 1)$ and Fx_1, Fx_2, Fx_3 are subspaces generated by x_1, x_2 and x_3 resp. show that

$F = Fx_1 \oplus Fx_2 \oplus Fx_3$

4.8 Summary

We have learnt the following concepts in this unit.

(i) Algebra of subspaces where we have studied sum of subspaces, direct sum of subspaces. etc

(ii) Definition of complementary subspaces, disjoint subspaces and identical subspaces.

4.9 Glossary

1. **Disjoint Subspaces :** W_1, $W_2 \in V$ (F), then W_1, W_2 are said to be disjoint subspaces if $W_1 \cap W_2 = \{0\}$ i.e. their intersection is zero space.

2. **Identical Subspaces :** Two subspaces W_1 and W_2 of the same dimensions are called identical subspaces iff each is a subspace of other.

4.10 Answers to Self Check Exercises

Ans.1 Take $W_1 = \{B : B = [b_{ij}]_{n\times n}, b_{ij} = 0$ for $i < j$

and $W_2 = \{C : C = [C_{ij}]_{n\times n} = c_{ij} = 0$ for $i > j$

Now proceed.

Ans.2 First show $F^3 = Fx_1 + Fx_2 + Fx_3$ and then proceed to prove the result.

4.11 Reference/Suggested Reading

1. Stephan H:, Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Edition, Prentice Hall of India Pvt. Ltd, New Delhi, 2004.

2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

3. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.

4. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

4.12 Terminal Questions

1. If W1 = $\{(0, 0, z) : z \in R\}$ be a subspace of V_3 (R). Show that $W_2 = \{(x, y, 0) : x, y \in R\}$ is complements of W1.

2. Produce three examples of subspaces W_1, W_2 and W_3 of a vector space V s.t.

   $W_1 \oplus W_2 = W_1 \oplus W_3 = V, W_2 \neq W_3$.

3. Let V be a vector space of $n\times n$ matrices over the field R and W_1, W_2 are subspaces of symmetric and skew symmetric matrices of order n resp. Then show that

   $V = W_1 \oplus W_2$

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Unit - 5

Quotient Spaces

Structure

5.1 Introduction

5.2 Learning Objectives

5.3 Cosets

5.4 Quotient Space

5.5 Self Check Exercise

5.6 Summary

5.7 Glossary

5.8 Answers to self check exercises

5.9 References/Suggested Readings

5.10 Terminal Questions

5.1 Introduction

Dear students, in this unit we shall learn about another important space namely. Quotient space. To study the concept of quotient spaces we feel it is important to have the idea of linear independent vectors dimension and basis of a vector space in brief. The detained discussion of these ideas shall be taken in the next unit.

5.2 Learning Objectives

The main objectives of this unit are to:

(i) Learn about the concept of left cosets and right cosets.

(ii) To study the concept of Quotient space for which idea of linear independence, basis and dimension of a vector space will be taken in brief. However, detailed discussion of these concepts will be taken up in the next unit.

Linear Independence and linear Dependence of vectors

Let V be a vector space over field F, then vectors.

$v_1, v_2.........v_n$ $\in$ V are called linearly dependent. (LD) if $\exists$ scalars $\propto_1$, $\propto_2$ .... $\propto_n$ $\in$ F, not all zero, s.t.

$\propto_1$, $\nu_1 + \propto_2 \nu_2 + ... + \propto_n \nu_n = 0$, fir atleast one of And otherwise they are called linearly independent (LI) i.e. of $\propto_1 \nu_1 + \propto_2 \nu_2 + ... + \propto_n \nu_n = 0$ for all $\propto_i$'s zero.

Basis - Let V(F) be a vector space. Then a subset B of V is called basis of V iff.

(i) B is linearly independence

(ii) L(B) = V i.e. B spans V.

Finite Dimensional -

A vector space V(F) is called finite dimensional iff a finite subset S of V s.t.

L(S) = V i.e. Linear span of S is equal to V.

Dimension of a vector space -

The dimension of finitely generated vector space (or finite dimensional vector space) V(F) is defined as the number of element in a basis of V(F) and is denoted by dim V.

If any basis of V contains n elements, we say din V = n and in that case V is called n - dimensional vector space.

Extension Theorem (Statement only)

Any L.I. set in V(F) can be extended to a basis of V

Note: The extension theorem will be proved in the next unit.

5.3 Cosets

Let W be a subspace of a vector space V (F). For any element $\nu \in V$, the set

$\nu + W = \{\nu + \omega / \omega \in W\}$ is said to be Left Coset of W in V.

and the set $W + \nu = \{\omega + \nu / \omega \in W\}$ is said to be Right Coset of W in V. Since the vector space V (F) is abelian under '+'

$\therefore \quad \nu + \omega = \omega + \nu \ \forall \ \omega \in W$

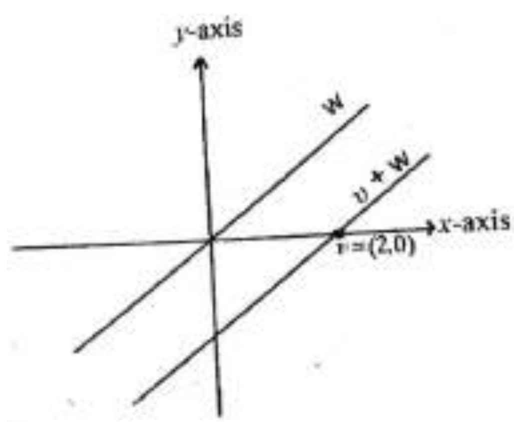
$\Rightarrow \quad \nu + W = W + \nu$

so that each set is known as a Coset of W in V generated by ν.

Example 1. (i) Let $W = \{(a, b) \mid a = b\}$ be a subspace of R^2

$\Rightarrow$ W is the line y = x through origin in R^2

Take the vector $\nu = (2,0)$ The set $\nu + W = (2, 0)$



The set ν + W = (2, 0) + W is viewed as translation of line W, which we shall obtain by addition of (2, 0) to every point of line W.

Thus ν + W is the line y = x - 2

$\therefore$ Coset of W in R^2 is the line parallel to W.

(ii) Let W = {(x, y, z)| 4x - 5y + z = 0} be a subspace

$\Rightarrow$ W is a plane through origin in R^3

The cosetsof W are planes parallel to W

i.e. ν + W = {(x, y, z)| 4x - 5y + z = K, K $\in$ R}

(iii) Let W = {(x, y, z)| z = 0, x, y $\in$ R} be a subspace of R^3

$\Rightarrow$ W is the x y - plane

The cosets of W are planes parallel to the xy-plane through the point (α, β, γ) at "height" γ

i.e. ν + W = {(x, y, z)|z = y, x, y $\in$ R}

Def. Let the set {ν + W/$\nu \in$ V} of all cosets of W in V(F) and it is denoted by V/W.

Def. Now we define vector addition and scalar multiplication in this set of cosets as below:

$(\nu + W) + (\nu' + W) = (\nu + \nu') + W$ for $\nu, \nu' \in V$

and $\alpha (\nu + W) = \alpha\nu + W$ for all $\alpha \in F$ $\nu \in V$

Def.4. Let $\eta \in$ V/W be a coset i.e. $\eta = \nu + W$, for $\nu \in V$

Then ν is known as representative of η.

Theorem: Let W be a subspace of vector space V (F) and $\nu, \nu' \in V$, prove

$\nu + W = \nu' + W$ iff $\nu - \nu' \in W$.

Proof: Firstly, let $\nu + W - \nu' + W$

$\Rightarrow$ If $\nu + w \in \nu + W$, then there must be an element $\nu' + w' \in \nu' + W$ such that

$\nu + w = \nu' + w'$; $\omega, \omega' \in W$

$\Rightarrow$ $\nu - \nu' = \omega' - \omega$

As W is a subspace, so $\omega' - \omega \in W$ for $\omega, \omega' \in W$

$\Rightarrow$ $\nu - \nu' \in W$

Hence the result

Conversely

Let $\nu - \nu' \in W$ for $\nu, \nu' \in V$

Now for $\omega \in W$, $\nu + \omega = O + \nu + \omega = (\nu' - \nu') + \nu + \omega$

$= \nu' + (\nu - \nu') + \omega$

$= \nu' + \omega'$ where $\omega= (\nu - \nu') + \omega \in W$

[as W is a subspace and $\nu - \nu', \omega \in W$]

$\Rightarrow \quad \nu + \omega \in \nu' + W$

$\Rightarrow \quad \nu + W \subseteq \nu' + W$

Similarly $\quad \nu' + W \subseteq \nu + W$

Hence $\nu + W = \nu' + W$

Theorem: Let W be a subspace of a vector V (F). Prove that the set

$V/W = \{\nu + W | \nu \in V\}$ of all cosets of W in V is a vector space w.r.t. the vector addition and scalar multiplication defined as

$(\nu_1 + W) + (\nu_2 + W) = (\nu_1 + \nu_2) + W$

and $\alpha (\nu_1 + W) = \alpha \nu_1 + W$ for $\alpha \in F$ and $\nu_1, \nu_2 \in V$

Proof: Given W be a subspace of a vector space V (F)

and $V/W = \{\nu + W / \nu \in V\}$

Here vector addition and scalar multiplication is defined as

$(\nu_1 + W) + (\nu_2 + W) = (\nu_1 + \nu_2) + W$

and $\alpha (\nu_1 + W) = \alpha\nu_1 + W$ for $\alpha \in F$ and $\nu_1, \nu_2 \in V$

Firstly we shall prove these compositions are well defined.

(i) Vector Addition is well defined : For this, we shall prove that if

$\nu_1 + W = \nu_2 + W$ and $u_1 + W = u_2 + W$ where $\nu_1, \nu_2, u_1, u_2 \in V$

Then $(\nu_1 + u_1) + W = (\nu_2 + u_2) + W$

By Theorem 1. $\nu_1 + W = \nu_2 + W \Rightarrow \nu_1 - \nu_2 \in W$

and $\quad u_1 + W = u_2 + W \Rightarrow u_1 - u_2 \in W$

But W is a vector subspace

So that $(\nu_1 - \nu_2) - (u_1 - u_2) \in W$

$\Rightarrow \quad (\nu_1 + u_2) + W = (\nu_2 + u_2) + W$)using Theorem 1)

Hence addition is well defined

(ii) Scalar Multiplication is well defined:

For this we shall prove if $\nu_1 + W = \nu_2 + W$

Then $\alpha \nu_1 + W = \alpha \nu_2 + W$ for $\alpha \in F, \nu_1, \nu_2 \in V$

By Theorem 1. $\nu_1 + W = \nu_2 + W \Rightarrow \nu_1 - \nu_2 \in W$

But W is a vector subspace

so that $\alpha(\nu_1 - \nu_2) \in W$ for $\alpha \in F$

$\Rightarrow \quad \alpha\, \nu_1 = \alpha\, \nu_2 + W$ (using Theorem 1.)

Hence scalar multiplication is well defined.

Now, we shall prove that V/W is a vector space

I. Properties under addition

A-1. Closure : Let $\nu_1 + W, \nu_2 + W \in V/W$ where $\nu_1, \nu_2 \in V$

As V is a vector space,

so $\nu_1 + \nu_2 \in V$ for $\nu_1, \nu_2 \in V$

$\Rightarrow \quad (\nu_1 + \nu_2) + W \in V/W$

$\Rightarrow \quad (\nu_1 + W) + (\nu_2 + W) \in V/W$

Thus V/W is closed under addition

A-2. Associative

Let $\nu_1, \nu_2, \nu_3 \in V$

and $\nu_1 + W, \nu_2 + W, \nu_3 + W \in V/W$

Now $[(\nu_1 + W) + (\nu_3 + W)] + (\nu_1 + \nu_2) + W] + (\nu_3 + W)$

(By def. of given addition composition)

$= [(\nu_1 + \nu_2) + \nu_3] + W$ [By def. of given addition composition as $\nu_1 + \nu_2 \in V$]

$= [(\nu_1 + (\nu_2 + \nu_3)] + W$

(Associative property holds for vector space V)

$= (\nu_1 + W) + [(\nu_2 + \nu_3) + W]$

$= (\nu_1 + W) + [(\nu_2 + W) + (\nu_3 + W)]$

Hence Addition is associative in V/W.

A-3. Existence of addative identity

For each $\nu_1 + W \in V/W$ for $\nu_1 \in V$

there is O + W = W E V/W for $O \in V$

such that $(O + W) + (\nu_1 + W) = (O - \nu_1) + W - \nu_1 = W$

($\because$ O is an additive identity of V)

and $(\nu_1 + W) + (O + W) = (\nu_1 + O) + W = \nu_1 + W$

($\because$ O is an additive identity of V)

$\therefore$ $(O + W) + (\nu_1 + W) = (\nu_1 - W) + (O + W) = \nu_1 + W$

$\Rightarrow$ $O + W = W$ is the additive identity in V/W

A-4. Existence of addative inverse

For each $\nu_1 + W \in V/W$ for $\nu_1 \in V$

there is $-\nu_1 + W \in V/W$ ($\because$ $-\nu_1 \in V$ as V is vector space)

such that $(-\nu_1 + W) + (\nu_1 + W) = ((-\nu_1) + \nu_1) + W = O + W = W$

($\because$ $-\nu_1$ is addative inverse of ν_1 in V)

and $(\nu_1 + W) + (-\nu + W) = (\nu_1 + (-\nu_1)) + W = O + W = W$

$\therefore$ $(-\nu_1 + W) + (\nu_1 + W = (\nu_1 + W) + (-\nu_1 + W) = W$

$\Rightarrow$ $-\nu_1 + W$ is the addative inverse of $\nu_1 + W$ in V/W.

A-5 Commutative:

Let $\nu_1 + W, \nu_2 + W \in V/W$ for ν_1, ν_2 E V

Now $(\nu_1 + W) + (\nu_2 + W) = (\nu_1 + \nu_2) + W$ (by def of addition)

$= (\nu_2 + \nu_1) + W$

($\because$ Commutative Property holds in V)

$= (\nu_2 + W) + (\nu_1 + W)$

Hence addition is commutative in V/W.

II. Properties under scalar Multiplication

M-1 Let $\alpha \in F, \nu \in V$ $\Rightarrow \alpha\, \nu \in V$

$\Rightarrow \alpha\, \nu + W \in V/W$

$\Rightarrow \alpha\, (\nu + W) \in V/W$

i.e. $\alpha \in F$ and $\nu + W \in V/W \Rightarrow \alpha\, (\nu + W) \in V/W$

$\therefore$ V/W is closed under scalar multiplication

M-2 Let $\alpha, \beta \in F$ and $\nu + W \in V/W$

Now $(\alpha + \beta)(\nu + W) = (\alpha + \beta)\, \nu + W$

$= (\alpha\nu + \beta\nu) + W$

$= (\alpha\nu + W) + (\beta\nu + W)$

$= \alpha\, (\nu + W) + \beta(\nu + W)$

$\therefore$ $(\alpha + \beta)(\nu + W) = \alpha\, (\nu + W) + \beta\, (\nu + W)$ for $\nu + W \in V/W$

Then $\nu + W = \sum_{k=1}^{m} a_k w_k + \sum_{t=1}^{n-m} b_t v_t + W$

$= w + \sum_{t=1}^{n-m} b_t v_t + W \left(\sum_{k=1}^{m} a_k w_k = w \Rightarrow w \in W \right)$

$= \sum_{t=1}^{n-m} b_t v_t + (\omega + W)$

$= \sum_{t=1}^{n-m} b_t v_t + W \qquad (\because \omega + W = W \text{ for } \omega \in W)$

$= b_1 \nu_1 + b_2 \nu_2 + \text{..........} + b_{n-m} \nu_{n-m} + W$

$= (b_1\nu_1 + W) + (b_2\nu_2 + W) + \text{.....} + (b_{n-m} \nu_{n-m} + W)$

(by def of addition in V/W)

$= b_1 (\nu_1 + W) + b_2 (\nu_2 + W)$

$+ \text{.......} + b_{n-m} + W)$ for $b, \in F$

$\Rightarrow$ Any element $\nu + W$ of V/W can be expressed as a L.C. of vectors in S

$\Rightarrow$ L (S) = V/W

Hence S is a basis of V/W

$\Rightarrow$ dim (V/W) = No. of elements in S = n-m

= dim V - dim W.

Hence the result

Example: Let W be subspace of vector space V_2 (R) = V generated by (1. 5) Find V/W and its basis.

Solution: The singleton set {(1, 5)} is L.I, which forms a basis of subspace W of V

$\therefore$ It can be extended to form a basis of V_2 (R)

Consider the set {(1, 5) (1, 0)}, which is L.I set

$\therefore$ It is a basis of V_2. $(\because (1, 0) \neq \lambda (1, 5)$ for any nonzero $\lambda)$

Now V/W $= \{\nu + W | \nu \in V\}$

$= \{(\alpha, \beta) + W | \alpha, \beta \in R\}$

$= \left\{ \left(\alpha - \frac{\beta}{5}, 0 \right) + \left(\frac{\beta}{5}, \beta \right) \middle| \alpha, \beta \in R \right\}$

$= \left\{ \left(\alpha - \frac{\beta}{5} \right) (1, 0) + \frac{\beta}{5} (1, 5) + W \middle| \alpha, \beta \in R \right\}$

$$= \left\{\left(\left(\alpha - \frac{\beta}{5}\right)(1,0) + W\right) + \left(\frac{\beta}{5}(1,5) + W\right) \middle| \alpha, \beta \in R\right\}$$

$$= \left\{a(1,0) + W \mid a = \alpha - \frac{\beta}{5} \in R\right\} \qquad [\because (1, 5) \in W]$$

$\Rightarrow$ V/W = {a (1, 0) + W| a $\in$ R}

Its basis is {(1, 0) + W}

Example 2. Let W be subspace of vector space

V = V_3 (R) generated by {(1, 0, 0) (1, 1, 0)}

Find V/W and its basis

Solution : The set {(1, 0, 0) (1, 1, 0) is L.I., which forms a basis of subspace W of V.

($\because$ (1, 1, 0) $\neq \lambda$ (1, 0, 0)) for any non zero λ)

$\therefore$ It can be extended to form a basis of ν

Consider the set {(1, 0, 0) (1, 1, 0) (0, 0, 1)}, which is L.I set (check yourself)

$\Rightarrow$ This set is a basis of V_3 (R)

Now V/W = {ν + W | $\nu \in$ V}

= {α (0, 0, 1) + W | $\alpha \in$ R} ($\because$ (1, 0, 0), (1, 1, 0) $\in$ W)

$\Rightarrow$ V/W = {(0, 0, α) + W | $\alpha \in$ R}

Example 3. Let W = {α, 0, 0, | $\alpha \in \beta$} be a subspace V = V_3 (R). Find basis of V/W and verify dim V/W = dim V - dim W.

Solution : The singleton set {(1, 0, 0)} is L.I, which forms a basis of subspace W of V = V_3

$\therefore$ By extension theorem, it can be extended to form a basis of V_3

Consider the set B = {(1, 0, 0) (0, 1, 0) (0, 0, 1)} which form a basis of V_3(R)

(already proved)

Now V/W = {α, β, γ) + W} where ν = (α, β, γ) $\in$ V

= {(α, 0, 0) + W) + (α, β, γ) + W)}

= {(α, 0, 0) + W) + (0, β, γ) + W)}

= {(0, β, γ) + W} ($\because$ (α, 0, 0) $\in$ W)

= {(0, β, 0) + W) + (0, 0, γ) + W)}

= {(β (0, 1, 0) + W) + (γ (0, 0, 1) + W)}

= {β {0, 1, 0) + W} + γ {0, 0, 1) + W}}

$\therefore$ {(0, 1, 0) + W, (0, 0, 1) + W} form a basis of V/W

$\Rightarrow$ din V/W = 2

Here dim V = dim V_3 = 3 and dim W = 1.

$\Rightarrow$ dim V/W = dim V - dim W is verified

Hence the result.

5.4 Quotient space

The vector space

V/W = $\{\nu + W : \nu \in V\}$ is called the quotient space of v with respect to W. In other words, the set of all cosets of W in v is called as the quotient space of V w.r.t. W.

Dimension of a Quotient space

Theorem : If W is a subspace of a finite dimensional vector space V(F), then show that

dim V/W = dim V - dim W.

Proof : Since W is a subspace of finite dimensional vector space V(F) therefore

W is also a finite dimensional vector space

Set dim V = n and dim W = m, m < n.

Set B = $\{w_1, w_2 \ldots\ldots., w_n\}$ be a basis of w.

$\therefore$ By extension theorem, it can be extended to form basis of v.

Set $B_1 = \{w_1, w_2 \ldots\ldots., w_n . \nu_1, \nu_2 \ldots\ldots\ldots \nu_{n-m}\}$,

having n elements be a basis of ν.

($\because$ dim ν = n)

Set as consider

$S = \{\nu_1 + w, \nu_2 + w, \ldots\ldots\ldots\ldots\ldots\ldots, \nu_{n-m} + w\}$, the set of all cosets of w.

Clearly $S \subseteq V/W$ and number of elements in

S = n - m

Now our claim is that S is a basis of V/W.

(i) We first show that S is sincerely independent (I)

As we known that W is the additive identity in V/W i.e. here zero vector is W.

$\therefore$ Consider

$\alpha_1 (\nu_1 + W)\ \alpha_2 (\nu_2 + W) + \ldots\ldots\ldots + \alpha_{n-m} (\nu_{n-m} + W)$

$= O = W$

$\Rightarrow$ $(\alpha_1 \nu_1 + W) + (\alpha_2 \nu_2 + W) + \ldots\ldots\ldots (\alpha_{n-m} \nu_{n-m} + W) = W$

$\Rightarrow$ $\alpha_1 \nu_1 + \alpha_2 \nu_2 + \ldots\ldots\ldots + \alpha_{n-m} \nu_{n-m} + W = W$

$\Rightarrow$ $\alpha_1 \nu_1 + \alpha_2 \nu_2 + \ldots\ldots\ldots + \alpha_{n-m} \nu_{n-m} \in W$

$\therefore$ It can be written as linear combination of elements of basis B of W

i.e. $\alpha_1 \nu_1 + \alpha_2 \nu_2 + \ldots\ldots.. + \alpha_{n-m} \nu_{n-m}$

$= \beta_1 w_1 + \beta_2 w_2 + \ldots.. + \beta_m w_m$ for

scalars β'j ($1 \leq j \leq m$)

$\Rightarrow \quad \beta_1 w_1 + \beta_2 w_2 + \ldots.. + \beta_m w_m + (-\alpha_1) \nu_1 + (-\alpha_2) \nu_2 + \ldots\ldots.. + (-\alpha_{n-m}) \nu_{n-m} = 0$

$\Rightarrow \quad \beta_1 = 0, \beta_2 = 0 \ldots\ldots \beta_m = 0, -\alpha_1 = 0, \ldots\ldots -\alpha_{n-m} = 0$

($\because$ β1 is αI set as it is basis_

$\Rightarrow \quad \alpha_1 = \ldots\ldots.. = \alpha_{n-1} = 0$

so that

$\alpha_1 (\nu_1 + W)\ \alpha_2 (\nu_2 + W) + \ldots\ldots.. + \alpha_{n-m} (\nu_{n-m} + W) = 0$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \ldots\ldots.. = \alpha_{n-m} = 0$ for $\nu_i + W \in S$

($1 \leq i \leq n-m$)

$\therefore$ S is L. I. set.

To show L(s) = V/W

Set $\nu + W \in V/W$ be any element, $\nu \in$ v.

As β1 is a basis of v, so that

$\nu = \sum_{k=1}^{m} a_k w_k + \sum_{t=1}^{n-m} b_t \upsilon_t$ for $a_k, b_t \in F$

Then

$$\nu + W = \sum_{k=1}^{m} a_k w_k + \sum_{t=1}^{n-m} b_t \upsilon_t + W$$

$$= w + \sum_{t=1}^{n-m} b_t \upsilon_t + W$$

$$= \sum_{t=1}^{n-m} b_t \upsilon_t + (w + W)$$

$$= \sum_{t=1}^{n-m} b_t \upsilon_t + W \qquad (\because\ w + W = w \text{ for } w \in W)$$

$= b_1 \nu_1 + b_2 \nu_2 + \ldots\ldots b_{n-m} \nu_{n-m} + w$

$= (b_1 \nu_1 + W) + (b_2 \nu_2 + w) \ldots\ldots + (b_{n-m} \nu_{n-m} + w)$

$= b_1 (\nu_1 + W) + b_2 (\nu_2 + w) \ldots\ldots + b_{n-m} (\nu_{n-m} + w)$ for $b_t \in F$

$\Rightarrow$ any element v + W of V/W can be expressed as a linear combination of vectors in S.

$\Rightarrow$ L(s) = V/W

Hence S is a basis of V/W.

Thus dim (V/W) = no. of elements in S

= n - m

= dim v - dim W.

Hence the proof.

Some illustrative examples

Example 1 : Set W be a subspace of vector space V = V_3(R) generated by {(1, 0, 0), (1, 1, 0)}

Find V/W and its basis

Solution : Since (1, 1, 0) $\neq \lambda$ (1, 0, 0) for any non-zero λ, therefore the set

{(1, 0, 0), (1, 1, 0)} is L.I. which forms a basis of subspace W of V.

$\therefore$ It can be extended to form a basis of V

Consider the set {(1, 0, 0), (1, 1, 0), (0, 0, 1)} which is clearly L I

$\Rightarrow$ This set is a basis of V_3 (R)

Now V/W = {v +W : $v \in$ V}

= { α (0, 0, 1) + W : $\alpha \in$ R}

($\because$ (1, 0, 0), (1, 1, 0) $\in$ w)

$\Rightarrow$ V/W = {(0, 0, α) + W : $\alpha \in$ R}

Example 2. Set W be a subspace of vector space V_2 (R) = V generated by (1, 5). Find V/W and its basis.

Solution : The singleton set {(1, 5)} is L-I. Which form a basis of subspace W of V.

$\therefore$ it can be extended to form basis of V2(R)

Consider the set {(1, 5) (1, 0)} which is LI.

$\therefore$ It is a basis of v_2 ($\because$ (1, 0) $\neq \lambda$ (1, 5) for any $\lambda \neq 0$)

Now V/W = {v + W : $v \in$V}

= {(α, β) + W, α, $\beta \in$ R}

$$=\left\{\left(\alpha-\frac{\beta}{5},0\right)+\left(\frac{\beta}{5},\beta\right)\alpha,\beta\in R\right\}$$

$$= \left\{\left(\alpha - \frac{\beta}{5}\right)(1,0) + \frac{\beta}{5}(1,5) + W : \alpha, \beta \in R\right\}$$

$$= \left\{\left(\left(\alpha - \frac{\beta}{5}\right)(1,0) + W\right) + \left(\frac{\beta}{5}(1,5) + W\right) : \alpha, \beta \in R\right\}$$

$$= \left\{a(1,0) + W : \alpha = \alpha - \frac{\beta}{5} \in R\right\}$$

$\Rightarrow$ V/W = {a (1, 0) + W : a $\in$ R}

Its basis is {(1, 0) + W}

5.5 Self Check Exercises

Q.1 Let W = {(α, 0, 0) : $\alpha \in$ R} be a subspace V = V3 (R). Find basis of V/W and verify

dim V/W = dim V - dim W.

Q.2 Let W be a vector subspace of V(F) and (ν_1, ν_2, ...ν_n) in V is L.I. set and W ∩ L (ν_i) = {0}.

Show that set of cosets {ν_1 + W, ν_2 + W, ...ν_n + W} in V/W is L.I.

5.6 Summary

In this unit we have learnt the following concepts.

(i) Concept of cosets (Left coset and right cosets)

(ii) Definition of linearly independence, linearly dependence, dimension and basis in a vector space.

(iii) the quotient space and dimension of a quotient space.

5.7 Glossary

1. **Linear Combination of Vectors**

Set V be a vector space over F.

If ν_1, ν_2 ....$\nu_n \in$ V then any element v,

Written as $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + ...+\alpha_n \nu_n = \sum_{i=1}^{n} \alpha_i \upsilon_i$, $\alpha_i \in$ F $\nu_i \in$V, i $\leq$ i $\leq$ n is called linear combination of vector ν_1, ν_2 ..... ν_n over F.

2. **Identical Subspaces :** Two subspaces W_1 and W_2 of the same dimensions are called identical subspaces iff each is a subspace of other.

5.8 Answers to Self Check Exercises

Ans.1 Consider set B = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} a basis of V_3 (R)

Find V/W, then dim V/W = 2, dim v = 3, dim W = 1

Ans.2 Consider the set

$\alpha(\nu_1 + w) + \alpha_2 (\nu_2 + w) ..... + \alpha_n (\nu_{n+w}) = 0 = W$

and proceed.

5.9 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

5.10 Terminal Questions

1. Set W be a subspace of vector space V = V_2 (R) generated by (2, 0). Find V/W and its basis.
2. Show that any two cosets u_1 + U and u_2 + U of V are either identical or disjoint.
3. Let V be a vector space of all 2×2 matrices over R and

   W = $\left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}, \alpha, \beta, \gamma \in R \right\}$.

   Find a basis of V/W.

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Unit - 6

Linear Combinations of Vectors

Structure

6.1 Introduction

6.2 Learning Objectives

6.3 Linear Combination

6.4 Linear Span

6.5 Generator of a Vector Space

6.6 Smallest Subspace

6.7 Self Check Exercise

6.8 Summary

6.9 Glossary

6.10 Answers to self check exercises

6.11 References/Suggested Readings

6.12 Terminal Questions

6.1 Introduction

Dear students, in this unit we shall study the concept of linear combinations of vectors. A linear combination is an expression constructed from a set of terms by multiplying each term by a constant and adding the result. The concept of linear combination is central to linear algebra and related field of mathematics. In this unit we shall be dealing with linear combinations in the context of vector space over the field. An important application of linear combination is to wave function in quantum mechanics.

6.2 Learning Objectives

The main objectives of this unit are to:

(i) to study linear combination

(ii) to learn about linear span

(iii) how can we generate a vector space?

(iv) to know about smallest subspace

Each of above concepts will be followed by suitable example for better understanding.

6.3 Linear Combination

Let V be a vector space over the field F. As usual we call elements of V vectors and call elements of F scalars. If

ν_1, ν_2 ....$\nu_n \in$ V then any element ν, written as

$$\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + ...+\alpha_n \nu_n$$

$$= \sum_{i=1}^{n} \alpha_i \upsilon_i , \quad \alpha_i \in F ,\ i \leq i \leq n$$

is called linear combination of vector ν_1, ν_2 ..... ν_n over F.

We note the following

(i) Since v is a vector space, so by properly of addition and scalar multiplication in v, we have $\nu \in$ V.

(ii) For vectors ν_1, ν_2 ....ν_n , we shall get different linear combinations by taking different set of scalars α_1, α_2, ..., α_n.

For example

Let F be the field of real numbers R and let the vector space V be the Euclidean space R^3. Consider the vectors.

e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)

Then any vector in R^3 is a linear combination of e_1, e_2, e_3. To see this, take (a_1, a_2, a_3) in R^3 and write

$$(a_1, a_2, a_3) = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3)$$

$$= a_1 (1, 0, 0) + a_2 (0, 0, 1) + a_3 (0, 0, 1)$$

$$= a_1 e_1 + a_2 e_2 + a_3 e_3$$

6.4 Linear Span

If S is a non-empty subset of a vector space V(F), then the set of all linear combinations of any finite number of elements of S is called the linear span of S.

The linear span of S is denoted by L(S)

$$\therefore L(S) = \left\{ \sum_{i=1}^{n} a_i \upsilon_i : \upsilon_i \in s, \alpha_i \in F, 1 \leq i \leq n \right\}$$

Note : If S = ϕ. then L(S) = {0}.

6.5 Generator of a Vector Space

If S is a non-empty subset of a vector space V(F), then S is called a generator of the vector space V(F) if each element of V can be expressed as a liner combination of the elements of S.

Thus, if S is a generator of the vectors space V (F) and if $\nu \in$ V, then there exists,

ν_1, ν_2 ....$\nu_n \in$ S such that

$\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + ...+\alpha_n \nu_n$ for some αi's $\in$ F

where $1 \leq i \leq n$

e.g., Let the vector space V_2 (R) = $\{(\alpha, \beta) \mid \alpha, \beta \in R\}$

Take S = $\{(1, 0), (0,1)\} = \{e_1, e_2\}$

Let $v \in V_2$ be arbitrary element

Then $\nu = (\alpha, \beta)$ for some $\alpha, \beta \in R$

Clearly ν - (α, β)

$= \alpha (1, 0) + \beta (0, 1)$

$= \alpha e_1 + \beta e_2$

$\Rightarrow$ ν is a linear combination of e_1, e_2 (the elements of S)

$\therefore$ S generates V_2.

Theorem : Prove that the linear span L (S) of any subset S of a vector space V (F) is a subspace of V (F).

Proof. Let ν, w $\in$ L (S)

Then $\nu = \sum_{i=1}^{n} \alpha_i v_i$ where $\alpha_i \in F$, $\nu_i \in S$ for $1 \leq i \leq n$

and $w = \sum_{j=1}^{m} \beta_j w_j$ where $\beta_j \in F$, $w_j \in S$ for $1 \leq j \leq m$

To show L (S) is a subspace of V (F), we are to show that for a, b $\in$ F add ν, w $\in$ L (S) $\Rightarrow \alpha\ \nu + b\ w \in$ L (S)

Now $a\ \nu + b\ w = \alpha \left(\sum_{i=1}^{n} \alpha_i v_i\right) + \beta \left(\sum_{j=1}^{m} \beta_j w_j\right)$

$$= \sum_{i=1}^{n} a(\alpha_i v_i) + \sum_{j=1}^{m} b(\beta_j w_j)$$

$$= \sum_{i=1}^{n} (a\alpha_i)\nu_\iota + \sum_{j=1}^{m} (b\beta_j) w_j \text{ (By Associative Law)}$$

$= (a\alpha_1)\nu_1 + (a\alpha_2)\nu_2 + ... + (a\alpha_n)\nu_n + (b\beta_1)w_1 + (b\beta_2)w_2 + ... + (b\beta_m)w_m$

$\Rightarrow$ a ν w is expressed as a linear combination of finite number of vectors

$\nu_1, \nu_2, ..., \nu_n, w_1, w_2, ..., w_m$ of S

Hence $a\ \nu + b\ w \in$ L (S)

Thus, for a, b $\in$ F and ν, w $\in$ L (S) we have a ν + b w $\in$ L (S)

Hence L (S) is a subspace of V (F).

Remark: If $\nu_k \in$ S Then $\nu_k = 1. \nu_\kappa$ for k = 1, 2, ..., n

$\therefore$ $\nu_k \in$ L (S)

Thus L (S) is a subspace of V and S $\subset$ L (S)

6.6 Def. Smallest Subspace

Let V be a vector space over a field F and S $\subset$ V. Then a subspace W of V(F) is called smallest subspace of V containing S iff

(i) S $\subset$ W

(ii) If W_1 is a subspace of V(F) such that S $\subset W_1$

then W $\subset W_1$.

Notation:

The smallest subspace containing S is denoted by

<S> or {S}

Result:

For every subset S of a vector space V(F), there exists a unique smallest subspace of V(F) containing S.

Example. To show that L(S) = the linear span of S is the smallest subspace, where S is a subset of V(F).

Solution: Let W be a subspace of V such that S $\subset$ W

Take any element $\nu \in$ L (S)

$\Rightarrow$ $\nu = \sum_{i=1}^{n} \alpha_i v_i$ where $a_i \in$ F, $\nu_i \in$ S 1 < i < m.

Now S $\subset$ W $\Rightarrow$ ν_1, ν_2, ..., νm $\in$ W

$\Rightarrow$ $\alpha_1 \nu_1 + a_2 \nu_2$ + ... + am m $\in$ W

[$\because$ W is a subspace of V(F)]

$\Rightarrow$ $\sum_{i=1}^{n} \alpha_i v_i \in$ W

$\Rightarrow$ $\nu \in$ W

$\therefore$ L (S) $\subset$ W

$\Rightarrow$ L(S) is the smallest subspace

Hence L(S) = <S> or {S}

Theorem: If S and T are any subsets of a vector space V (F), Prove that

(i) $S \subset L(T) \Rightarrow L(S) \subset L(T)$

(ii) $S \subset T \Rightarrow L(S) \subset L(T)$

(iii) S is a subspace of V (F) $\leftrightarrow$ L (S) = S

(iv) L (L(S)) = L (S)

Proof. (i) Given is $S \subset L(T)$

Let $v \in L(S) \Rightarrow \exists\ v_1, v_2, \ldots\ldots, v_n, \in S ; \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

such that $v = \sum_{i=1}^{n} \alpha_i v_i$

$\Rightarrow$ $v_i \in L(T)$ for $1 \leq i \leq n$ [$\because S \subset L(T)$]

$\Rightarrow$ $\sum_{i=1}^{n} \alpha_i v_i \in L(T)$ for $1 \leq i \leq n$

[$\because$ L (T) is a subspace of V (F)]

$\Rightarrow$ $v \in L(T)$

$\therefore$ $L(S) \subset L(T)$

Hence the result

(ii) Given is $S \subset T$

Let $v \in L(S) \Rightarrow \exists\ v_1, v_2, \ldots\ldots, v_n \in S ; \alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

such that $v = \sum_{i=1}^{n} \alpha_i v_i$ for $1 < i < n$

$\Rightarrow$ $v = \sum \alpha_i v_i \in L(T)$ [$\because S \subset T$ so that $v_1, v_2, \ldots\ldots, v_n \in T$]

$\therefore$ $L(S) \subset L(T)$

Hence the result:

(iii) Given is S is a subspace of V (F)

We have to prove L (S) = S

Let $v \in L(S) \Rightarrow \exists\ v_1, v_2, \ldots\ldots, v_n \in S$

and $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \subset F$

such that $v = \sum_{i=1}^{n} \alpha_i v_i$

$\Rightarrow \quad \nu \in S$ [$\because$ S is a subspace of V (F) and it is closed under addition and scalar multiplication]

$\therefore \quad L(S) \subset S$ .... (A)

Also obviously $S \subset L(S)$ ..... (B)

[$\because$ If $\nu_k \in S$ Then $\nu_k = 1.\ \nu_k$]

From (A) and (B) we have L (S) = S

Hence the result

Conversely.

Given is that L (S) = S

Here, we have to prove that S is a subspace of V (F)

As L (S) is a subspace of V (F) [For Proof Theorem 6]

$\Rightarrow$ S is a subspace of V (F) [$\because$ S = L (S)]

Hence the result

(iv) As L (S) is a subspace of V (F) [By Theorem 6]

$\Rightarrow$ L (L(S)) = L (S) [By using the result of Part (iii)]

Hence the result

Example 1. Let $R^3 = \{(\alpha, \beta, \gamma) | \alpha, \beta, \gamma \in \text{Reals}\}$

and $\nu_1 = (1, 1, 1)$, $\nu_2 = (1, 2, 3)$, $\nu_3 = (2, -1, 1)$

Express vector $\nu = (1, -2, 5)$ as a linear combination of ν_1, ν_2, ν_3.

Solution: Let $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3$ for some scalars $\alpha_1, \alpha_2, \alpha_3$.

$$\Rightarrow \quad (1, -2, 5) = \alpha_1 (1, 1, 1) + \alpha_2 (1, 2, 3) + \alpha_3 (2, -1, 1)$$

$$= (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$$

$$= (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$$

By equality of vectors, we have

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 1 \quad \text{... (A)}$$

$$\alpha_1 + 2\alpha_2 - \alpha_3 = -2 \quad \text{... (B)}$$

$$\alpha_1 + 3\alpha_2 + \alpha_3 = 5 \quad \text{... (C)}$$

Now (A) - (B) gives $-\alpha_2 + 3\alpha_3 = 3$ ... (D)

(B) - (C) gives $-\alpha_2 - 2\alpha_3 = -7$

Further (D) - (E) gives $5\alpha_3 = 10 \quad \Rightarrow \quad \boxed{\alpha_3 = 2}$ ... (E)

Put in (D), we get $\boxed{\alpha_2 = 3}$

Put values of α_2, α_3 in (A), we get $\alpha_1 + 3 + 4 = 1 \quad \Rightarrow \quad \boxed{\alpha_1 = -6}$

$\therefore \quad \nu = -6\,\nu_1 + 3\nu_2 + 2\nu_3$

Hence ν is a linear combination of ν_1, ν_2 and ν_3

Example 2. Find the condition on a, b, c such that the matrix

E = $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ is a linear combination of the matrices

A = $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, B = $\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ and C = $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

Solution: The matrix E = $\begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ is a linear combination of give matrices A, B, C

if $\exists$ scalars $\alpha_1 + \alpha_2 + \alpha_3$ such that

$$E = \alpha_1 A + \alpha_2 B + \alpha_3 C.$$

if $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

if $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} a_1 & a_1 \\ 0 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2 & a_2 \\ -a_2 & 0 \end{bmatrix} + \begin{bmatrix} a_3 & a_3 \\ 0 & 0 \end{bmatrix}$

if $\begin{bmatrix} a & b \\ -b & c \end{bmatrix} = \begin{bmatrix} a_1 + a_2 + a_3 & a_1 + a_2 - a_3 \\ -a_2 & -a_1 \end{bmatrix}$

By equality of matrices, we must have

if $\alpha_1 + \alpha_2 + \alpha_3 = a$ ... (A)

$\alpha_1 + \alpha_2 - \alpha_3 = b$ ... (B)

$-\alpha_2 = -b$ ... (C)

$-\alpha_1 = c$ ... (D)

From (C), $\alpha_2 = b$

From (D), $\alpha_2 = -c$

Put these values in (B),

we get, $-c + b - \alpha_3 = b$

$\Rightarrow \quad \alpha_3 = -c$

Put these values of α_1, α_2 and α_3 in (A), we have -c + b - c = a

$\Rightarrow \quad a - b + 2c = 0$ which is required condition.

6.7 Self Check Exercises

Q.1 Examine whether (1, -3, 5) belong to linear space generated by S, where

S = {(1, 2), (1, 1, -1), (4, 5, -2)} or not?

Q.2 Find the condition on a, b, c so that the vector $\nu = (a, b, c) \in R^3$ belongs to the space generated by $\nu_1 = (2, 1, 0)$; $\nu 2 = (1, -1, 2)$; $\nu_3 = (0, 3, -4)$

6.8 Summary

We have learnt the following concepts in this unit:

(i) Linear combination of vectors

(ii) Linear span

(iii) Generator of a vector space

(iv) Smallest subspace

6.9 Glossary

1. **Sparring Set :** A subset S of a vector space V is called a spanning set for V if span (S) = V

2. **Vector Equation :** A vector equation is an equation involving a linear combination of vectors with possibly unlenourn coefficient.

6.10 Answers to Self Check Exercises

Ans.1 (1, -3, 5) does not belong to linear space of S.

Ans.2 2a - 4b - 3c = 0 is the required condition

6.11 Reference/Suggested Reading

1. Stephen H. tried berg, Arndd. J. Insel, Lawrence E. Spence, Linear Algebra, 4th Ed., Prantice Hall of India Pvt. Ltd., New Delhi, 2004.

2. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

3. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

4. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.

6.12 Terminal Questions

1. which of the polynomials are in

(i) $\langle x^3, x^2 + 2x, x^2 + 2, -x + 1\rangle$

(ii) $3x^2 + x + 5$

(iii) $x^3 + 3x^2 + 3x + 7$

2. Write the vector $\nu = (3, 2, 1)$ as a linear combination of vectors $\nu 1 = (2, -1, 0)$, $\nu_2 = (1, 2, 1)$ and $\nu_3 = (0, 2, -1)$.

3. Write the matrix

$E = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ as a linear combination of matrices

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & +2 \\ 0 & -1 \end{bmatrix}$

4. For what value of λ, will the vector $\nu = (1, \lambda, -4) \in \nu_3$ (R) is a linear combination of vectors $\nu_1 = (1, -3, 2)$ and $\nu_2 = (2, -1, 1)$

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Unit - 7

Linear Independence (LI)

And

Linear Dependence (L.D) Of Vectors

Structure

7.1 Introduction

7.2 Learning Objectives

7.3 Linear Dependence (L.D.)

7.4 Linear Independence (L.I)

7.5 Self Check Exercise

7.6 Summary

7.7 Glossary

7.8 Answers to self check exercises

7.9 References/Suggested Readings

7.10 Terminal Questions

7.1 Introduction

Dear students, in this unit, we shall study the concepts of linear independence and linear dependence of vectors in a vector space. The concept of linear combination, linear span discussed earlier will make you comfortable in better understanding of the concepts of L.I. and L.D resp.

7.2 Learning Objectives

The main objectives of this unit are

(i) to study the concept of linear dependence of vectors over a field F.

(ii) to study the concept of linear independence of vectors over the field F

(iii) later, we shall produce another treatment of linearly dependent and linearly independent vectors.

7.3 Linear Dependent (L.D.)

If V be a vector space over field F, then the vectors $v_1, v_2, ..., v_n \in V$ are called linearly dependence over F if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all of them zero (i.e., at least one of α_i's is non zero) such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

For Example

The vectors v_1 = (2, 3, 4), v_2 = (1, 0, 0), v_3 = (0, 1, 0) and v_4 = (0, 0, 1) are linearly dependent.

Solution: Here $1. v_1 + (-2) v_2 + (-3) v_3 + (-4) v_4$

$= 1 (2, 3, 4) + (-2) (1, 0, 0) + (-3) (0, 1, 0) + (-4) (0, 0, 1)$

$= (2, 3, 4) + (-2, 0, 0) + (0, -3, 0) + (0, 0, -4)$

$= (2, -2, 3 - 3, 4, -4) + (0, 0, 0) = 0$

So, we have $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4 = 0$

where $\alpha_1 = 1 \neq 0$, $\alpha_2 = -2 \neq 0$, $\alpha_3 = -3 \neq 0$, $\alpha_4 = -4 \neq 0$

Hence v_1, v_2, v_3, v_4 are Linearly Dependent vectors.

7.4 Linear Independent (L.I.)

If V is a vector space over a field F, then the vectors $v_1, v_2, ..., v_n \in V$ are called Linearly Independent over F if there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, all of them zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

For Example. The vectors $v_1 = (2, 0, 0)'$ $v_2 = (0, 3, 0)$ and $v_3 = (0, 0, 4)$ are linearly independent.

Solution: Consider $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$ for some scalars α_1, α_2 and α_n

$\Rightarrow \quad \alpha_1 (2, 0, 0) + \alpha_2 (0, 3, 0) + \alpha_3 (0, 0, 4) = 0$

$\Rightarrow \quad (2\alpha_1, 3\alpha_2, 4\alpha_3) = (0, 0, 0)$

$\therefore \quad 2\alpha_1 = 0, 3\alpha_2 = 0, 4\alpha_3 = 0$

$\Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0$, and $\alpha_3 = 0$

Hence vectors v_1, v_2, v_3 are L.I.

Note 1. If A = $\{v_1, v_2, ... v_n\}$. Then we call the set A as linearly dependent set or linearly independent set according as vectors $v_1, v_2, ..., v_n$ are linearly dependent or linearly independent.

2. An infinite subspace of A of a vector space is L.I. if every subset of A is L.I.

Theorem. Let V be a vector space over a field F. Then prove that

(i) Every non zero singleton subset of V is L.I. over F.

(ii) Every set containing only zero vector is L.D. over F.

(iii) Any subset S of V containing zero vector is L.D. over F.

(iv) If a subset S of V is L.I. then it cannot contain zero vector.

(v) Every super set of a L.D. set of vectors is L.D.

(vi) Any subset of a L.I. set of vectors is L.I.

Proof. (i) Let $\nu \neq 0$ and $\nu \in V$

Take $\alpha \in F$ such that $\alpha\, \nu = 0$

$\Rightarrow$ either $\alpha = 0$ or $\nu = 0$

$\Rightarrow$ $\alpha = 0$ [$\because$ $\alpha \neq O$, given]

$\therefore$ $\alpha\nu = 0$ $\Rightarrow$ $\alpha = 0$

Hence $\{\nu\}$ is L.I. set,

(ii) Let $S = \{0\} \subset V$

Take α be non zero element of F

Then $\alpha.0 = 0$ for $\alpha \neq 0 \in F$

$\Rightarrow$ S is L.D. set

(iii) Let $S = \{\nu_1, \nu_2, \ldots\ldots, \nu_n\}$ be a subset of vector space V (F) such that one of the vectors of S is zero vector.

Let $\nu_k = 0$ for some k, where $1 \leq k \leq n$.

Choose $\alpha_1 = \alpha_2 = \ldots. = \alpha_{k-1} = 0 = \alpha_{k+1} = \ldots.. = \alpha_n$ and $\alpha_k = 1$

$\therefore$ α_i's $\in$ F and not all are zero [$\because$ $\alpha_k = 1 \neq 0$]

Now $\alpha_1\nu_1 + \alpha_2\nu_2 + \ldots.. + \alpha_n\nu_n$

$= \alpha_1\nu_1 + \alpha_2\nu_2 + \ldots.. + \alpha_{k-1}\nu_{k-1} + \alpha_k\nu_k + \alpha_{k+1}\nu_{k+1} + \ldots.. + \alpha_n\nu_n$

$= 0\, \nu_1 + 0\, \nu_2 + \ldots.. + 0\, \nu_{k-1} + 1.\,(0) + 0\, \nu_{k+1} + \ldots.. + 0\, \alpha_n\nu_n$

$= 0 + 0 + \ldots.. + 0 + 0 + 0 + \ldots. + 0$

$= 0$

$\therefore$ we have scalars $\alpha_1, \nu_2, \ldots\ldots, \nu_n$, not all of them zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + \ldots.. + \alpha_n\nu_n = 0$

$\Rightarrow$ $\nu_1, \nu_2, \ldots\ldots, \nu_n$ are L.D.

$\Rightarrow$ S is L.D.

(iv) Suppose S is a L.I. subset of V (F)

If S contains zero vector, then S is L.D. which contradicts the given that S is L.I.

$\therefore$ S cannot have any zero vector.

$\Rightarrow$ there exists scalars $\alpha_1, \nu_2, \ldots\ldots, \nu_n$, not all of them zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + \ldots.. + \alpha_n\nu_n = 0$ ... (A)

Consider $T = \{\nu_1, \nu_2, \ldots\ldots, \nu_n, \nu_{n+1}, \nu_{n+2}, \ldots\ldots, \nu_{n+i}\}$ be a super set of S

Now Equation (A) implies

$\alpha_1 \nu_1 + \alpha_2 \nu_2 + ...+ \alpha_n \nu_n + 0\ \nu_{n+1} + 0\ \nu_{n+2} + ... + 0\ \nu_{n+1} = 0$ ... (B)

where $\alpha_1, \alpha_2, ..., \alpha_n, 0, 0, ... 0$, are not all zero

$\Rightarrow$ $\nu_1, \nu_2, ..., \nu_n, \nu_{n+1}, ..., \nu_{n+i}$ are L.D.

$\Rightarrow$ the set T is L.D.

(vi) Let S = $\{\nu_1, \nu_2, ..., \nu_m\}$ be L.I. set of vectors

$\therefore$ $\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_m \nu_m = 0$ ... (A)

$\Rightarrow$ $\alpha_1 = \alpha_2 = ... = \alpha_m = 0$

Consider T = $\{\nu_1, \nu_2, ..., \nu_p\}$ where $1 \leq p \leq m$

$\therefore$ clearly T is a subset of S

Consider $\alpha_1 \nu_1 + \alpha_p \nu_p = 0$

$\Rightarrow$ $\alpha_1 \nu_1 + \alpha_2 \nu_2 + ...+ \alpha_p \nu_p + 0\ \nu_{p+1} + ... + 0\ \nu_m = 0$

$\Rightarrow$ $\alpha_1 = \alpha_2 = ... = \alpha_p = 0$ [$\because$ S = $\{\nu_1, \nu_2, ..., \nu_m\}$ is L.I. set}

Hence T is L.I. set.

Theorem Let V be a vector space over F. Prove that

(i) The set $(\nu\}$ is L.D. iff $\nu = 0$

(ii) The set $(\nu_1, \nu_2\}$ is L.D. iff ν_1 and ν_2 are collinear i.e., iff one is a scalar multiple of the other.

(iii) The set $(\nu_1, \nu_2, \nu_3\}$ is L.D. iff ν_1, ν_2 and ν_3 are coplanar i.e., iff one is a linear combination of the other two.

Proof. (i) Let $\nu = 0$

Take $\alpha \neq 0 \in F$ Then $\alpha\ \nu = 0$ [$\because$ α. 0 = 0]

$\Rightarrow$ $\{\nu\}$ is L.D. set.

Conversely

Let $\{\nu\}$ be I.D. set

$\Rightarrow$ $\exists$ scalar $\alpha \neq 0$ such that

$\alpha\nu$ - 0

$\Rightarrow$ $\nu = 0$

Hence the result.

(ii) Let $(\nu_1, \nu_2\}$ be L.D. set

$\Rightarrow$ $\exists$ scalars α, β (not both zero, say $\alpha \neq 0$) such that

$\alpha\nu_1, \beta\nu_2 = 0$

$\Rightarrow$ $\alpha\nu_1 = - \beta\nu_2$

$\Rightarrow$ $\nu_1 = \frac{\beta}{\alpha}\ \nu_2$ $[\because\ \alpha \neq 0]$

$\Rightarrow$ ν_1 is a scalar multiple of ν_2

Hence ν_1, ν_2 are collinear vectors

Conversely

Let ν_1 and ν_2 be collinear

$\Rightarrow$ one of these, say ν_1 is a scalar multiple of ν_2

i.e., ν_1 k ν_2 for scalar k

$\Rightarrow$ $1.\ \nu_1 - k\nu_2 = 0$

$\Rightarrow$ $\alpha\nu_2 + \beta\nu_2 = 0$ where $\alpha.\ 1.\ \beta = - k$

Thus we have scalars $\alpha - 1 \neq 0, \beta$, not both zero such that

$\alpha\nu_1 + \beta\nu_2 = 0$

$\Rightarrow$ $\nu_1 + \nu_2$ are L.D. vectors

Hence $\{\nu_1, \nu_2\}$ is L.D.

(iii) Let $\{\nu_1, \nu_2, \nu_3\}$ be L.D. set

$\Rightarrow$ $\exists$ scalars $\alpha_1, \alpha_2, \alpha_3$, not all zero (i.e., at least one of them is non To, say $\alpha_1 \neq 0$)

such that $\alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 = 0$

$\Rightarrow$ $\alpha_1\nu_1 - \alpha_2\nu_2 - \alpha_3\nu_3$

Divide both sides by $\alpha 1$ ($\neq 0$)

$\Rightarrow$ $\nu_1 = \frac{\alpha_2}{\alpha_1}\ \nu_2 - \frac{\alpha_3}{\alpha_1}\ \nu_3$

$\Rightarrow$ $\nu_1 = \left(-\frac{\alpha_2}{\alpha_1}\right) \nu_2 + \left(-\frac{\alpha_3}{\alpha_1}\right)\nu_3$

$\therefore$ ν_1 is a linear combination of ν_2 and ν_3.

Hence ν_1, ν_2, ν_3 are coplanar vectors.

Conversely

Let ν_1, ν_2 and ν_3 be coplanar vectors

$\Rightarrow$ One of these say ν_1 is a linear combination of ν_2 and ν_3

$\therefore$ $\exists$ scalars α_2 and α_3 such that

$\nu_1 = \alpha_2\nu_2 + \alpha_3\nu_3$

$\Rightarrow$ $- I . \nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 = 0$

so there exists scalars $\alpha_1 = -1 \neq 0$, α_2, α_3 not all zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + \alpha_3\nu_3 = 0$

$\Rightarrow$ ν_1, ν_2, ν_3 are L.D. vectors

Hence $\{ \nu_1, \nu_2, \nu_3 \}$ is a L.D. set.

Theorem : If V(F) is a vector space then, prove that the set S of non zero vectors $\nu_1, \nu_2, ..., \nu_n \in V$ [i.e. $S = \{\nu_1, \nu_2, ..., \nu_n \} \subset V$] is L.D. iff some element of S is a linear combination of the others. Also show in this case that

$L\{\nu_1, \nu_2, ..., \nu_n\} . L \{\nu_1, \nu_2, ..., \nu_k , \nu_k + , ...\nu_n\}$

Proof : It is given that I. $\{\nu_1, \nu_2, ..., \nu_n\}$ is L.D. set

$\Rightarrow$ $\exists$ scalars $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ not all zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + ... \alpha_n\nu_n = 0$

Since α_j's are not all zero

Let $\alpha k \neq 0$ for $i \neq k$

Equation (A) can be written as

$\alpha_1\nu_1 + \alpha_2\nu_2 + ...+ \alpha_k \ I \ \nu_k ... 1 + \alpha_k\nu_k + \alpha_k + 1 \ \nu_k + 1 +.... + \alpha_n\nu_n = 0$

$\Rightarrow$ $\alpha_k\nu_k = (\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_k \ I \ \nu_k ... 1 + \alpha_k + 1 \ \nu_k + 1 +.... + \alpha_n\nu_n)$

Divide by $\alpha_k \neq 0$

$$\Rightarrow \quad \nu_k = \left(-\frac{\alpha_1}{\alpha_k}\right) \nu_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) \nu_2 + ...+ \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) \nu_k - 1$$

$$+ \left(-\frac{\alpha_{k+1}}{\alpha_k}\right)\nu_{k+1} +... + \left(-\frac{\alpha_n}{\alpha_k}\right)\nu_n$$

$\Rightarrow$ some $\nu_k \in s$ can be expressed as a linear combination of

$\nu_1, \nu_2, ..., \nu_{\lambda-1} , \nu_{k+1}, ..., \nu_n$

Conversely

Let some $\nu_k \in s$ can be expressed as a linear combination of

$\nu_1, \nu_2, ..., \nu_{\lambda-1} , \nu_{k+1}, ..., \nu_n$

i.e. $\nu_k = \beta_1\nu_1 + \beta_2\nu_2 + ...+ \beta_k \ I \ \nu_k ... 1 + \beta_k\nu_k + \beta_k + 1 \ \nu_k + 1 +.... + \beta_n\nu_n$

$\Rightarrow$ $\beta_1\nu_1 + \beta_2\nu_2 + ...+ \beta_{k-1} \ \nu_{k-1} + (-1) + \nu_k + \beta_k + 1 \ \nu_k + 1 +.... + \beta_n\nu_n = 0$

$\Rightarrow$ $\exists$ scalars $\beta_1, \beta_2, ..., \beta_{k-1}, \beta_k = -1 \neq 0, \beta_{k+1}, ..., \beta_n$, not all zero

Such that $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n = 0$

$\Rightarrow$ $v_1, v_2, ..., v_n$ are L.D. vectors

Hence $S = \{v_1, v_2, ..., v_n\}$ is L.D. set.

Note : The above theorem can also be stated as A non empty subset S of a vector space V (F) is linearly dependent iff there exist $v \in S$ such that $v \in L$ (S - '{v}).

Theorem : If V (F) is a vector space then prove that the set S of non zero vectors $v_1, v_2, ..., v_n \in V$ is linearly dependent if and only if some vector $vm \in S$, $2 < m < n$ can be expressed as a linear combination of its preceding vectors.

Proof : It is given that $S = \{v_1, v_2, ..., v_n\}$ is L.D.

Then there exists scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero such that

$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$ ...(A)

Let m be the largest suffix of α for which $\alpha_m \neq 0$

$\therefore$ Equation (A) $\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m + 0\, v_{m+1} + ... + \alpha_n v_n = 0$

$\Rightarrow$ $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_m v_m = 0$ ...(B)

If possible, let m = 1 Then Equation (B) gives

$\alpha_1 v_1 = 0$

$\Rightarrow$ $v_1 = 0$ $[\because \alpha_1 \neq 0]$

which contradicts given that all vectors of S are non zero.

$\therefore$ $m > 1$ or $2 \leq m \leq n$

Equation (B) can be written as

$\alpha_m v_m = -\alpha_1 v_1 - \alpha_2 v_2 - .... - \alpha_{m-1} v_{m-1}$

$$\Rightarrow \quad v_m = \left(-\frac{\alpha_1}{\alpha_m}\right) v_1 + \left(-\frac{\alpha_2}{v_m}\right) v_2 + ... + \left(-\frac{\alpha_{m-1}}{\alpha_m}\right) v_{m-1}$$

$[\because \alpha_m \neq 0]$

$\therefore$ v_m can be expressed as a linear combination of $v_1, v_2, ..., v_{m-1}$

Hence v_m can be expressed as a linear combination of its preceding vectors.

Conversely

Let $v_k \in S$ can be expresses as a linear combination of its preceding vectors i.e.,

$v_k = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_{k-1} v_{k-1}$ for scalars $\beta_1, \beta_2, ..., \beta_{k-1} \in F$

$\Rightarrow$ $\beta_1 v_1 + \beta_2 v_2 + ... + \beta_{k-1} v_{k-1} - v_k = 0$

$\Rightarrow$ $\beta_1\nu_1 + \beta_2\nu_2 + \ldots.+ \beta_{k-1}\ \nu_{k-1} + (-1) + \nu_k + 0\ \nu_{k+1} + 0\ \nu_k + 2 + \ldots. + 0\ \nu_n = 0$

$\therefore$ $\exists$ scalars $\beta_1, \beta_2, \ldots., \beta_{k-1}, \beta_k = -1 \neq 0$,

$\beta_{k+1} = 0, \ldots, \beta_n = 0$, not all zero such that

$\beta_1\nu_1 + \beta_2\nu_2 + \ldots.+ \beta_n\nu_n = 0$

$\Rightarrow$ $\nu_1, \nu_2, \ldots., \nu_n$ are L.D. vectors

Hence $S = \{\nu_1, \nu_2, \ldots., \nu_n\}$ is L.D. set.

Another treatment for Linearly Dependent and Linearly Independent Vectors.

Let $\nu_1 = (b_{11}, b_{12}, \ldots., b_{1n})$

$\nu_2 = (b_{21}, b_{22}, \ldots., b_{2n})$

....

....

$\nu_n = (b_{n\,1}, b_{n\,2}, \ldots, b_{n\,n})$

be n vectors of the vector space F^n (F).

These vectors are L.D. vectors iff there exists scalars

$\alpha_1, \alpha_2, \ldots., \alpha_n \in F$, not all zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + \ldots.+ \alpha_n\nu_n = 0$

$\Rightarrow$ $\alpha_1(b_{11}, b_{12}, \ldots., b_{1n}) + \alpha_2(b_{21}, b_{22}, \ldots., b_{2n}) + \ldots.+ \alpha_n(b_{n\,1}, b_{n\,2}, \ldots, b_{n\,n}) = 0$

$\Rightarrow$ $\alpha_1 b_{11}, \alpha_2 b_{21} + \ldots.+ \alpha_n b_{n\,1},\ \alpha_1 b_{12}, \alpha_2 b_{22} + \ldots.+ \alpha_n b_{n\,2}, \ldots.,$

$\alpha_1 b_{1\,n}, \alpha_2 b_{2\,n} + \ldots.+ \alpha_n b_{n\,n}, = (b_{n\,1}, b_{n\,2}, \ldots, b_{n\,n}) = (0, 0, \ldots, 0)$

$\therefore$ $\alpha_1 b_{11}, \alpha_2 b_{21} + \ldots.+ \alpha_n b_{n\,1} = 0$

$\alpha_1 b_{12}, \alpha_2 b_{22} + \ldots.+ \alpha_n b_{n\,2} = 0$

....

$\alpha_1 b_{1\,n}, \alpha_2 b_{2\,n} + \ldots.+ \alpha_n b_{n\,n} = 0$

These homogenous equations must have non-trivial (αi's are not all zero) solution.

By Theory of Equations, we know that above equations will have non-trivial solution iff the determinant of its coefficient matrix is zero

i.e., iff $\begin{vmatrix} b_{11} & b_{21} & \ldots & b_{n1} \\ b_{12} & b_{22} & \ldots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} & \ldots & b_{nn} \end{vmatrix} = 0$

Hence vectors $\{\nu_1, \nu_2, \ldots., \nu_n \in F^n$ are L.D. over F

$$\text{iff} \begin{vmatrix} b_{11} & b_{21} & \dots & b_{n1} \\ b_{12} & b_{22} & \dots & b_{n2} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{vmatrix} = 0$$

$$\text{or iff} \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} \end{vmatrix} = 0$$

[$\because$ value of det. remains unchanged if rows and columns are interchanged]

and vector are L.I. over F

iff determinant $\neq 0$.

Example 1 : If u, v, w are L.I. vectors in a vector space V (F), then show that the vectors u + v, u - v, u - 2v + w are L.I.

Solution : (a) Let $\alpha_1, \alpha_2, \alpha_3$ be scalars of F such that

$\alpha_1 (u + v) + \alpha_2 (v + w) + \alpha_3 (w + u) = O$ ... (I)

$\Rightarrow \quad (\alpha_1 + \alpha_3) u + (\alpha_1 + \alpha_2) v + (\alpha_2 + \alpha_3) w = O$

$\Rightarrow \quad \alpha_1 + \alpha_3 = 0$ ... (A) [Since u, v, w are L.I.]

$\alpha_1 + \alpha_2 = 0$ ... (B)

$\alpha_2 + \alpha_3 = 0$ ... (C)

Now (A) - (B) gives $\alpha_3 - \alpha_2 = 0$ ... (D)

Adding (C) and (D), we get, $2\alpha_3 = 0$

or $\alpha_3 = 0$

Putting in (A) and (C) we get $\alpha_1 = 0, \alpha_2 = 0$

$\therefore$ Equation (I) is true only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow \quad$ u + v, (v + w), w + u are L.I. vectors.

(b) Let $\alpha_1, \alpha_2, \alpha_3$ be scalars of F such that

$\alpha_1 (u - v) + \alpha_2 (u - v) + \alpha_3 (u - 2v + w) = O$ ... (I)

$\Rightarrow \quad (\alpha_1 + \alpha_2 + \alpha_3) u + (\alpha_1 - \alpha_2 - 2\alpha_3) v + \alpha_3 w = O$

$\Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = 0$ ... (A)

[Since it is given that u, v, w are L.I.]

$\alpha_1 - \alpha_2 - 2\alpha_3 = 0$ ... (B)

$\alpha_3 = 0$ ... (C)

Put value of α_3 from (C) in (A) and (B)

we get $\alpha_1 - \alpha_2 = 0$ ... (D)

$\alpha_1 - \alpha_2 = 0$ ... (E)

Adding (D) and (E), we get

$2\,\alpha_1 = 0 \quad \Rightarrow \quad \alpha_1 = 0$

Then (D) implies $\alpha_2 = 0$

$\therefore$ Equation (I) is true only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$

$\Rightarrow$ u + v, u - v, u - 2v + w are L.I. vectors.

Example 2 : If $V_1, V_2, ..., V_n$ are L.I. $n\times 1$ column vectors and A is a $n \times n$ singular matrix. Prove that $AV_1, AV_2, ..., AV_n$ are L.D.

Solution : Since A is $n \times n$ matrix and νi (i = i, 2, ..., n) is of type $n\times 1$

$\therefore$ AV_i (for i = 1, 2, ..., n) is defined and will be of type $n \times 1$

Consider $a_1 (AV_1) + a_2(AV_2) + ... + a_n (AV_n) = O$ ....(I)

for some scalars $a_1, a_2, ..., a_n$

$\Rightarrow$ $A(a_1 V_1 + a_2 V_2 + ... + a_n V_n) = O$

$\Rightarrow$ $AX = O$

where $X = a_1 V_1 + a_2 V_2 + ... + a_n V_n$

Since A is a singular matrix, so the equation (II) have a non zero solution

Let $X = X_1$ be non zero solution (II)

$\therefore$ $X_1 = a_1 V_1 + a_2 V_2 + ... + a_n V_n$ for some scalars $a_1, a_2, ..., a_n$

As $X1 \neq O$ so that $a_1 V_1 + a_2 V_2 + ... + a_n V_n \neq O$

$\Rightarrow$ all of ai's cannot be zero since otherwise X1 will be equal to zero

$\therefore$ $a_1 (AV_1) + a_2(AV_2) + ... + a_n (AV_n) = O$

$\Rightarrow$ the scalars $a_1, a_2, ..., a_n$ are not all zero

Hence $AV_1, AV_2, ..., AV_n$ are L.D. vectors.

Example 3 : Show that the vectors (a, b) and (c, d) in V2 (C) where C is set of complex numbers, are Linearly Dependent iff a d = b c.

Solution : It is given that (a, b) and (d, d) are L.D.

$\Rightarrow$ There is complex number $\lambda \in C$ such that

$(c, d) = \lambda\, (a, b) = (\lambda\, a, \lambda\, b)$

$\Rightarrow$ c = λ a ....(I)

and d = λ b ....(II)

If $\alpha \neq 0$, (I) gives λ = c a ^{I}

Putting this value of λ in (II), we get

d = (c a^{-1}) b = b (c a^{-1}) d = (b c) $a^{-1} \Rightarrow$ d = a^{-1} (b c)

$\Rightarrow$ a d = b c.

And If a = 0 Then (I) $\Rightarrow$ c = λ (0) = 0

$\therefore$ a d = 0 (d) = 0

and b c - b (0) = 0

$\Rightarrow$ a d = b c

Hence a d = b c.

Another Proof : Let α (a, b) + β(c, d) = (0, 0) for any scalars α, β

$\Rightarrow$ (α a + β c, α b + β d) = (0, 0)

$\therefore$ α a + β c = 0

and α b + β d = 0

since given vectors are L.D., so α, β are not both zero.

$\therefore$ the above homogeneous equation will have non-trivial solution if

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ if a d - b c = 0 i.e. a d = b c.

Hence the result.

Conversely

Given a d = b c

$\Rightarrow$ b = (a d)c^{-1} if c $\neq$ 0

$\Rightarrow$ b = (a c^{-1}) d ($\because$ a, b, c, d $\in$ field C)

$\Rightarrow$ b = λ d, λ = ac^{-1}

Also then a = λ c

$\therefore$ (a, b) = (λ c, λ d) ($\because$ ad = bc)

$\Rightarrow$ (a, b), (c, d) are L.D. vectors

Now if c = 0 $\Rightarrow$ ad = b(0) = 0

$\Rightarrow$ either a = 0 or d = 0

If d = 0, then one vector (c, d) = (0, 0) = 0

$\Rightarrow$ (a b) and (c, d) are L.D.

If a = 0 then (a, b) = (0, b)

and (c, d) = (0, d)

and one is multiple of other s.t. (0, d) = ab-1 (0, b) where b $\neq$ 0

$\Rightarrow$ (a, b), (c, d) are L.D.

Also if b = 0, then one vector is (a b) = (0, 0) zero vector

$\Rightarrow$ (a, b), (c, d) are L.D.

Hence the result.

Some More Illustrative Examples

Example 4 : Prove that the set of vector (ν_1, ν_2, ...ν_n) form a L.D. set iff atleast one of the vector is a zero vector.

Solution : Set ν_1 = 0, be a zero vector, and

$\alpha_1 \neq 0$, $\alpha_i = 0$ for i = 2, 3, ... n.

Now

$$\sum_{i=1}^{n} \alpha_i\nu_i = \alpha_1\nu_1 + \alpha_2\nu_2 + ...+ \alpha_n\nu_n$$

$$= \alpha_1(0) + 0\nu_2 + 0\nu_3 + ...+ 0\nu_n$$

$$= 0$$

Thus we have scalars $\alpha 1 \neq 0$, $\alpha 2 = \alpha 3 ....\alpha n = 0$ not all zero s.t. $\sum_{i=1}^{n} \alpha_i\nu_i = 0$

or $\alpha_1\nu_1 + \alpha_2\nu_2 + ...+ \alpha_n\nu_n$

Therefore the set of vectors {ν_1 , ν_2 + ...+ ν_n} is L.D.

Example 5 : Find λ so that the vectors

$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}$ are sincerely dependant.

Solution : Let α_1, α_2, α_3 are scalars in R s.t.

$$\alpha_1 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + \alpha_3 \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix} = 0$$

Where α_1, α_2, α_3 are not all zero and 0 is a 3×1 zero matrix.

$$\therefore \quad \begin{bmatrix} \alpha_1 \\ -\alpha_1 \\ 3\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 \\ 2\alpha_2 \\ -2\alpha_2 \end{bmatrix} + \begin{bmatrix} \lambda\alpha_3 \\ 0 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \quad \begin{bmatrix} \alpha_1 + \alpha_2 + \lambda\alpha_3 \\ .-\alpha_1 + \alpha_2 \\ -3\alpha_1 - 2\alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $\alpha_1 + \alpha_2 + \lambda\alpha_3 = 0$ ...(1)

$-\alpha_1 + \alpha_2 = 0$ ...(2)

$-3\alpha_1 - 2\alpha_2 + \alpha_3 = 0$ ...(3)

solving (1) - (3) we get

$$2\alpha_1 \left(\frac{3}{4} - \lambda\right) = 0$$

$\Rightarrow$ $\lambda = \dfrac{4}{3}$, since $\alpha_1 \neq 0$. Hence the result.

7.5 Self Check Exercise

Q. 1 Find the condition on scalar λ s.t. the vectors

$(\lambda, 1, 0)$, $(1, \lambda, 1)$ and $(0, 1, \lambda)$ in ν_3 (R) are linearly dependent.

Q. 2 Prove that the vectors

$\nu_1 = (1, 2, -3)$, $\nu_2 = (1, -3, 2)$ and $\nu_3 = (2, -1, 5)$ or ν_3 (R) is linearly independent.

7.6 Summary

We have learnt the following concepts in this unit

(i) Linear dependence and linear independence of vectors.

(ii) Theorems are proved to show L.D. and L.I. of vectors over the given field.

(iii) Enough examples are given to understand the above concepts of L.J. and L.D. of vectors

7.7 Glossary

Facts about linear independence -

(1) Two vectors are L.D. iff they are collinear i.e. one vector is a scalar multiple of other vector.

(2) Any set containing zero vector is L.D.

(3) If a subset $\{\nu_1\ \nu_2 - \ldots.\nu_k\}$ is L.D. then the set $\{\nu_1\ \nu_2 - \ldots.\nu_k\}$ is L.D. as well.

7.8 Answers to Self Check Exercises

Ans.1 $-\lambda\ (2 - \lambda^2) = 0 \Rightarrow \lambda = 0$ or $\lambda^2 - 2$ is the req. condition.

Ans.2 Try to show $\alpha_1 = \alpha_2 = 0$, which implies ν_1, ν_2, ν_3 are L.I.

7.9 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.
3. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.
4. Stephen H. tried berg, Arndd. J. Insel, Lawrence E. Spence, Linear Algebra, 4th Ed., Prantice Hall of India Pvt. Ltd., New Delhi, 2004.

7.10 Terminal Questions

1. Find the condition on scalar $\lambda \in$ C so that the vectors

   $(1 - \lambda, 1 + \lambda)$, $(1 + \lambda, 1 - \lambda)$ in ν_2(C) are L.D.

2. If ν_1, ν_2, are vectors of ν(F) and $\alpha, \beta \in$ F.

   Show that ν_1, ν_2, $\alpha\nu_1 + \beta\nu_2$ are L.D.

3. Prove that the vectors of ν3(R) are L.I.

   (i) x = (1, -1, 2, 0); y = (1, 1, 2, 0)

   z = (3, 0, 0, 1); w = (2, 1, -1, 0)

   (ii) x = (1, 5, 2), y = (0, 0, 1), z = (1, 1, 0)

4. Show that three column vectors of the snatrio

   $$\begin{bmatrix} 2 & 7 & 5 \\ 3 & -6 & 2 \\ 1 & 1T & T \end{bmatrix}$$

   are L.D. what about its three row vectors?

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Unit - 8

Basis and Dimensions of A Vector Space

Structure

8.1 Introduction

8.2 Learning Objectives

8.3 Basis

8.4 Finite Dimensional

8.5 Ordered Basis

8.6 Existence Theorem

8.7 Invariance of number of Elements of a Basis

8.8 Replacement Theorem

8.9 Self Check Exercise

8.10 Summary

8.11 Glossary

8.12 Answers to self check exercises

8.13 References/Suggested Readings

8.14 Terminal Questions

8.1 Introduction

Dear students, we have already discussed the concepts of linear combination, linear independence, linear dependence and linear span of vectors in our previous units. Now, it is time to understand the concepts of basis and dimensions in this unit.

A subset of a vector space is a basis of its elements are L.I. and span the vector space. Every vector space has atleast one basis, or many in general. Moreover all the basis of a vector space have the same cardinality, called dimension of the vector space. Basis are fundamental tool for the study of vector spaces, especially when the dimension is finite. In the infinite dimensional case, the existence of infinite basis, often called Hamel basis, depends upon the union of choice. Therefore, it follows that, in general, no base can be explicitly described. For instance, the real number form an infinite dimensional space over Q (the rationales), for which no specific basis is known.

8.2 Learning Objectives

The main objectives of this unit are

(i) to study basis of a vector space.

(ii) to learn bout finite dimensional vector space

(iii) to define ordered basis

(iv) to prove existence theorem

(v) to study invariance of numbers of elements of a basis.

(vi) to prove replacement theorem

(vii) to define coordinates of a vector relative to the basis etc.

8.3 Basis

Let V (F) be a vector space. A subset B of V is called a basis of V iff

(i) B is linearly Independent set

(ii) L (B) = V i.e., B generates (spans) V.

or in other words every element in V is a linear combination of the elements of B.

Note : (i) A set of vectors having zero vector is always L.D. set, so it cannot be basis of a vector space. Thus a zero vector cannot be an element of a basis of a vector space.

(ii) Since $L(\phi) = \{0\}$ and ϕ is L.I.

$\therefore$ ϕ is a basis of $\{0\}$

(iii) $\{0\}$ is not a basis of $\{0\}$

e.g., (a) The set B = {(1, 0, 0); (0, 1, 0); (0, 0, 1)}, a subset of V_3 (R) is a basis set of V_3 (R).

(i) Firstly, we shall prove B is L.I.

Let a, b, c $\in$ R such that

$a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = O$

$\Rightarrow$ $(a, b, c) = (0, 0, 0)$

$\Rightarrow$ $a = 0, b = 0$ and $c = 0$

$\therefore$ the set B is L.I.

(ii) To show that L (B) = V_3 (R)

We know that L (B) $\subset$ V_3 (R) ... (1)

Now for each $(\alpha, \beta, \gamma) \in V_3$ where $\alpha, \beta, \gamma \in$ R

we have $(\alpha, \beta, \gamma) = (\alpha, 0, 0) + (0, \beta, 0) + (0, 0, \gamma)$

$= \alpha (1, 0, 0) + \beta (0, 1, 0) + \gamma (0, 0, 1)$

$\Rightarrow$ each element of V_3 can be expressed as a linear combination of elements of B

$\therefore$ V3 (R) $\subset$ L (B) ...(2)

From (1) and (2), we get L (B) = V_3 (R)

Hence the set B is basis of V_3 (R)

(b) The set $B_1 = \{(1, 0, 0), (1, 1, 0), (1, 2, 3), (0, 1, 0)\}$, a subset of V_3 (R) is not a basis set of V_3 (R).

(i) To check whether the set B_1 is L.I. or not.

Let a, b, c, d $\in$ R such that

$$a (1, 0, 0) + b (1, 1, 0) + c (1, 2, 3) + d (0, 1, 0) = O \quad .....(A)$$

$\Rightarrow$ $(a + b + c, b + 2 c + d, 3 c) = (0, 0, 0)$

$\therefore$ $a + b + c = 0$ ...(1)

$b + 2 c + d = 0$ ...(2)

$3 c = 0$ ...(3)

From Eq. (3), se have c = 0

Putting in Eq. (1) and (2) we get a + b = 0 and b + d = 0

$\Rightarrow$ a = -b, d = -b

Let b = -k $\neq$ 0 real

$\therefore$ a = k, d = k, c = 0

Thus we have scalars a = k, b = -k, c = 0, d = k such that Eq. (A) holds.

$\therefore$ the vectors (1, 0, 0) ; (1, 1, 0) ; (1, 2, 3) ; (0, 1, 0) are not L.I.

Hence B_1 is not a basis set of V_3 (R).

Note. In general, any subset of V_m (F) where F is any field, having more that m elements will be L. Dependent and so cannot be basis of V_m.

8.4 Finite Dimensional.

A vector space V(F) is called finite dimensional or finitely generated iff there exists a finite subset S of V such that L(S) = V i.e., linear span of S is equal to V.

Note. If there exists no finite subset which generates V, then V is called an infinite dimensional vector space.

e.g., {a} Let $B_1 = \{(1, 0, 0) ; (1, 1, 0) ; (1, 2, 0) ; (0, 1, 0)\}$ be a subset of V_3 (R)

We have already shown that B_1 is not a basis set of V_3 (R)

Here, we shall show that L (B_1) = V_3 (R)

We know that L(B_1) $\subset$ V_3 (R) ...(1)

Let any vector $(\alpha_1, \alpha_2, \alpha_3) \in V_3$ (R)

Consider $(\alpha_1, \alpha_2, \alpha_3)$ = a (1, 0, 0) + b (1, 1, 0) + c (1, 2, 3) + d (0, 1, 0) for a, b, c, d $\in$ R

= (a + b + c, b + 2c + d, 3 c)

$\Rightarrow$ $a + b + c = \alpha_1$

$b + 2 c + d = \alpha_2$

$3c \quad = \alpha_3$

Solving these equations, we get c = $\frac{\alpha_3}{3}$

$a = \alpha_1 - \alpha_2 + \frac{1}{3}\ \alpha_3 + d$

$b = \alpha_2 = \frac{2}{3}\ \alpha_3 - d$

Thus a non zero solution exists

$\therefore \quad (\alpha_1, \alpha_2, \alpha_3) \in V_3 (R) \quad \Rightarrow \quad (\alpha_1, \alpha_2, \alpha_3) \in I. (B_1)$

so that $V_3 (R) \subset L (B_1)$

From (1) and (2) we get $V_3 (R) = L (B_1)$

$\Rightarrow$ V_3 (R) is a finite dimensional vector space

Hence the result

8.5 Ordered Basis

An ordered set S = $\{\nu_1, \nu_2, ..., \nu_m)$ which is basis of finite dimensional vector space V(F) is called an ordered basis of V.

Def. Co-ordinates of a Vector Relative to the basis :

If S = $\{\nu_1, \nu_2, ..., \nu_m)$ is an ordered basis of V(F) and $\nu \in V$ such that

$\nu = a_1 \nu_1 + a_2\nu_2 + ... + a_m \nu_m$ for a_i's $\in F$

Then co-ordinates of ν relative to ordered basis S are a_1, a_2, am and denoted as

$[\nu]_s = (a_1, a_2, a_{im})$.

Theorem. Let B = $\{\nu_1, \nu_2, ..., \nu_n)$ is a basis of finite dimensional vector space V (F) iff every $\nu \in$ V can be uniquely expressed as

$\nu = \alpha_1 \nu_1 + \alpha_2\nu_2 + ... + \alpha_n \nu_n, \alpha_i \in F, 1 < i < n.$

Proof. Given B = $\{\nu_1, \nu_2, ..., \nu_n)$ be a basis of V (F)

$\therefore \quad L (B) = V$

so that $\nu \in V \Rightarrow \nu \in L (B)$

$\Rightarrow \quad \nu = \alpha_1 \nu_1 + \alpha_2\nu_2 + ... + \alpha_n \nu_n$, for scalars $\alpha_1, \alpha_2, ..., \alpha_n$

To show uniqueness

If possible, let $\nu = \beta_1 \nu_1 + \beta_2 \nu_2 + ... + \beta_n \nu_n$ for scalars $\beta_1, \beta_2, ..., \beta_n \in F$

$\therefore \quad \alpha_1 \nu_1 + \alpha_2\nu_2 + ... + \alpha_n \nu_n = \beta_1 \nu_1 + \beta_2 \nu_2 + ... + \beta_n \nu_n$

$(\alpha_1 - \beta_1) \nu_1 + (\alpha_2 - \beta_2) \nu_2 + ... + (\alpha_n - \beta_n) \nu_n = O$

$\because$ $\{v_1, v_2, ..., v_n$ are L.I. {as $\{v_1, ..., v_n\}$ is a basis set)

so that $\alpha_1 - \beta_1 = 0$, $\alpha_2 - \beta_2 = 0$, ..., $\alpha_n - \beta_n = 0$

$\Rightarrow$ $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_n = \beta_n$

$\Rightarrow$ every vector $v \in V$ can be uniquely expressed as

$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$; $\alpha_i \in F$.

Conversely

Let every vector $v \in V$ be uniquely expressed as

$v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$; $\alpha_i \in F$, $1 \leq i \leq n$.

To Prove. $B = \{v_1, v_2, ..., v_n)$ be a basis of V (F)

(i) To show B is L.I.

Consider $\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = O$ for scalars $\alpha_1, \alpha_2, ... \alpha_v \in F$ ....(1)

We can also write vector O as

$O = 0\, v_1 + 0\, v_2 + ... + 0\, v_n$ ...(2)

$\because$ representation of vector $O \in V$ is unique

from (1) and (2), we have

$\alpha_1 = 0, \alpha_2 = 0, ..., \alpha_n = 0$

$\Rightarrow$ the vectors $v_1, v_2, ..., v_n$ are L.I.

$\therefore$ the set $B = \{v_1, v_2, ..., v_n\}$ is L.I.

(ii) To show that L (B) = V

We know that $L(B) \subset V$ ....(3)

Let $v \in V$

$\Rightarrow$ by given, there exists αi $(1 \leq i \leq n) \in F$

such that $v = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n$

$\Rightarrow$ $v \in L(B)$

$\therefore$ $V \subset L(B)$ ....(4)

From (3) and (4) V = L (B)

Hence B is a basis of V (F)

8.6 Theorem (Existence Theorem)

Prove that there exists a basis for each finitely generated vector space.

OR

If S = {ν_1, ν_2, ..., ν_m} spans a vector space V(F). prove $\exists$ a subset of S which is a basis of V.

Proof. Let V (F) be a finite dimensional vector space

$\Rightarrow$ there exists a finite subset S = {ν_1, ν_2, ..., ν_n} of V such that

L (S) = V ....(A)

Without any loss of generality, we may suppose there that all vectors in the subset S are non zero because contribution of zero vector in any linear combination of the vectors of S is zero.

Since S $\subset$ V so either S is L.I. or L.D.

If S is L.I., then S is a basis of V (F) [$\because$ by (A), it is given L (S) = V]

Hence the result

If S is L.D., then exists m, 2 $\leq$ m $\leq$ n such that $\nu m \in$ S is a linear

combination of ν_1, ν_2, ..., ν_m - 1

i.e., $\nu_m - \sum_{i=1}^{m-1} \alpha_i \nu_1$ for α_i's $\in$ F (B)

Consider the set

S_1 = {ν_1, ν_2, ..., ν_m-1, ν_m+1, ..., ν_n}

We shall show that L (S_1) = V

Clearly $S_1 \subset$ S L (S_1) $\subset$ L (S)

$\Rightarrow$ L (S_1) $\subset$ V ....(1) [$\because$ By (A), L (S) = V]

Now let $\nu \in$ V

$\Rightarrow$ ν is a linear combination of the elements of S [$\because$ L (S) = V]

$\therefore$ $\nu = \beta_1 \nu_1 + \beta_2 \nu_2 + ... + \beta_n \nu n$ for β_j's $\in$ F

$= \beta_1\nu_1 + \beta_2\nu_2 + ... + \beta_{m-1} \nu_{m-1} + \beta_m\nu_m + \beta_{m+1}\nu_{m+1} ..... + \beta_n\nu_n$

$= \beta_1\nu_1 + \beta_2\nu_2 + ... + \beta_{m-1} \nu_{m-1} + \beta_m\left(\sum_{i=1}^{m-1} \alpha_i \nu_i\right) + \beta_{m+1}\nu_{m+1} ..... + \beta_n\nu_n$

[Using (B)]

$= \beta_1\nu_1 + \beta_2\nu_2 + ... + \beta_{m-1} \nu_{m-1} + \beta_m(\alpha_1\nu_1 + \beta_2\nu_2 + ... + \alpha_{m-1} \nu_{m-1}) + \beta_{m+1}\nu_{m+1} ..... + \beta_n\nu_n$

$= (\beta_1 + \beta_m\alpha_1) \nu_1 + (\beta_2 + \beta_m\alpha_2) \nu_2 + ... + (\beta_{m-1} + \beta_m \alpha_{m-1}) \nu_{m-1} + \beta_{m+1} \nu_{m+1} + ... + \beta_n\nu_n$

$\Rightarrow$ ν is a linear combination of the vectors

ν_1, ν_2, ..., ν_{m-1}, ν_{m+1}, ..., ν_n of set S_1

$\therefore$ $\nu \in$ L (S_1) ....(2)

so that $V \subset L(S_1)$

From (1) and (2), we get $L(S_1) = V$ .... (C)

If S_1 is L.I., then S_1 is a basis of V (F)

Hence the result. [$\because$ by (C), $L(S_1) = V$]

If S_1 is L.D., then proceeding as above we can get a set S_2 of (n - 2) vectors such that $L(S_2) = V$ and if S_2 is L.I., then S_2 is a basis set. But if S_2 is L.D., we repeat the above procedure till we get a set S_m for some $m > 2$, $m \in$ Naturals which is L.I. and $L(S_m) = V$, so that S_m is basis set for V(F)

At the most by repeating the procedure, we are left with a subset having a single non zero vector which generates V. But we know that a singleton set having non-zero vector is L.I. Hence there exists a basis for each finite dimensional vector space.

Remark 1. The above Theorem is also true for an arbitrary vector space V (F) i.e., each vector space has a basis set. However its proof is beyond the scope of this syllabus.

2. The basis of a vector space need not be unique. In fact, a vector space generally have more than one bases (Later on, we shall give such examples). But the number of elements in all the bases of a finite dimensional vector space is same, this we shall prove in the next theorem.

8.7 Invariance of Number of Elements of a Basis

Theorem : Prove that any two bases of a finite dimensional vector space have same number of elements.

Proof: Let V(F) be a finite dimensional vector space, so it must have a basis set.

($\because$ Each finite dimensional vector space has a basis set)

Take $\beta_1 = \{x_1, x_2, ..., x_m\}$

and $\beta_2 = \{y_1, y_2, ..., y_n\}$ be two bases of V(F)

We have to prove that m = n

If possible, let m > n

Since B_1 is basis of V, so $x_1, x_2, ..., x_m \in V$ and B_2 is also a basis of V.

$\therefore$ Each x_i (1 < i < m) can be written as a L.C. of elements of B_2 .....($*$)

$$\Rightarrow \left.\begin{array}{l} x_1 = \beta_{11}y_1 + \beta_{12}y_2 + ...+ \beta_{1n}y_n \\ x_2 = \beta_{21}y_1 + \beta_{22}y_2 + ...+ \beta_{2n}y_n \\ .... \\ .... \\ x_m = \beta_{m1}y_1 + \beta_{m2}y_2 + ...+ \beta_{mn}y_n \end{array}\right] \quad ...(I)$$

where $\beta_{ij} \in F$

Consider a system of n equations in m unknowns i.e. $\beta_{11}z_1 + \beta_{21}z_2 + ... + \beta_{m2}\, z_m = 0$

$\beta_{12}z_2 + \beta_{22}z_2 + ... + \beta_{m2}\, z_m = 0$

....

....

$\beta_{1n}\, z_1 + \beta_{2n}\, z_2 + ... + \beta_{mn}\, z_m = 0$ where $\beta_{ij} \in F$

As m > n or n < m

i.e. No. of equations are less than number of unknowns.

$\therefore$ The above system of equations have a non zero solution say

$K_1, K_2, ..., K_m$. not all zero.

so that $\beta_{11}\, K_1 + \beta_{21}K_2 + ... + \beta_{m1}\, K_m = 0$

$\beta_{12}K_1 + \beta_{22}K_2 + ... + \beta_{m2}\, K_m = 0$

....

....

$\beta_{1n}\, K_1 + \beta_{2n}\, K_2 + ... + \beta_{mn}\, K_m = 0$

Multiply these equations by $y_1, y_2, ..., y_n$ respectively and then adding vertically, we get

$K_1(\beta_{11}y_1 + \beta_{12}y_2 + ... + \beta_{1n}\, y_n) + K_2\, (\beta_{21}y_1 + \beta_{22}y_2 + ... + \beta_{2n}\, y_n)$

$+ ..... + K_m\, (\beta_{m1}y_1 + \beta_{m2}y_2 + ... + \beta_{mn}\, y_n) = 0$

$\Rightarrow$ $K_1x_1 + K_2x_2 + ... + K_mx_m = 0$

i.e. $K_1x_1 + K_2x_2 + ... + K_mx_m = 0$

where $K_1, K_2, ..., K_m$ are not all zero.

$\Rightarrow$ $\{x_1, x_2, ..., x_m\}$ is L.D. set

which contradicts that B_1 is a basis of V(F)

$\therefore$ our supposition is wrong so that

$m \ngtr n$

Similarly $n \ngtr m$

Hence m = n

$\Rightarrow$ Any two bases of a finitely generated vector space have the same number of elements.

8.8 Theorem (Replacement Theorem)

If V(F) be a vector space which is generated by a finite set B_2 of vectors $y_1, y_2, ..., y_n$. Prove that any linearly independent set of vectors in V contains not more than n elements.

OR

If $B_2 = \{y_1, ..., y_n\}$ spans V(F), then prove that any (n + 1) vectors in V are L.D.

Proof: Given $B_2 = \{y_1, y_2, ..., y_n\}$ is a generating set of V i.e. $L(B_2) = V$

Let $B_1 = \{x_1, x_2, ..., x_m\}$ be L.I set in V(F)

We shall prove that $m \leq n$

If possible, let $m > n$

Since B_1 is L.I. set in V so $x_1, x_2, ... x_m \in V$ and given $L(B_2) = V$

Then proceed as in Theorem 17 from step (∗) till the end of step (∗∗)

we get $m \not> n \Rightarrow m \leq n$

Hence the result

Some Illustrative Examples

Example 1: Show that the vectors (1, 1, 0), (1, 0, 1) and (1, -1, -1) of R^3 form a basis of R^3 (R).

Solution: Since dim $R^3 = 3$, therefore it suffices to show that given three vectors are linearly independent once R.

Let $\propto (1, 1, 1) + \beta (1, 0, 1) + \gamma(1, -1, -1) = 0$

for $\propto, \beta, \gamma \in R$

$\Rightarrow$ $(\propto + \beta + \gamma, \propto - \gamma, \propto + \beta - \gamma) = (0, 0, 0)$

$\Rightarrow$ $\propto + \beta + \gamma = 0$

$\propto - \gamma = 0$

$\propto + \beta - \gamma = 0$

We can put these eqs in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \propto \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $Ax = 0$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

$\therefore$ $\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = 1 (0 + 1) - 1 (-1 + 1) + (1 - 0) = 1 - 0 + 1 = 2 \neq 0$

$\therefore$ given vectors are L.I. over R

Hence the given vectors form a basis of R^3 (R).

Example 2: show that the set of vectors

$S = \{1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis of R^3 (R)

Solution: For any a, b, c $\in R^3$, we have

(a, b, c) = a (1, 0, 0) + b (0, 1, 0) + c (0, 0, 1),

so any (a, b, c) $\in R^3$ is a linear combination of elements in S.

$\therefore$ R^3 = Span (S)

Also S is linearly independent, since

a (1, 0, 0) + b (0, 1, 0) + c (0, 0, 1) = (0, 0, 0)

$\Rightarrow$ a = b = c = 0

$\Rightarrow$ set of vectors S form a basis of R^3 (R)

Hence the result

8.9 Self Check Exercise

Q. 1 Show that

$B = \{(1, 1, 1), (1, -1, 0), (0, 1, 1)\}$ is a basis of R^3.

Q. 2 Examine whether the set of vectors in V_3 (R(form a basis or not

(i) (1, 0, -1), (1, 2, 1), (0, -3, 2)

(ii) (1, 1, 1), (1, 2, 3), (-1, 0, 1)

8.10 Summary

We have learnt the following concepts in this unit

(i) Basis and dimension of a vector space

(ii) Basis and dimension of a vector space

(iii) Existence theorem and replacement theorems

(iv) Invariance of number of elements of a basis

8.11 Glossary

1. Standard Basis of R^n -

A basis of n - triples R^n given by set

$S = \{e_1, e_2 \text{ .... } en\}$

where $e_1 = (1, 0, \ldots\ldots 0)$

$e_2 = (0, 1, \ldots\ldots 0)$

$e_n = (0, 0, \ldots\ldots 1)$

is called the standard basis of R^n.

2. Standard Basis of Vector space of all polynomial of degree m over reals the set

$B = \{1, x, x^2, ... x^m\}$ of (m + 1) polynomial is a basis set for the vector space P_m (R) of all polynomials of degree m over reals (R).

8.12 Answers to Self Check Exercises

Ans.1 Easy to prove

Ans.2 (i) Basis (ii) Not a basis

8.13 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007.
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.

8.14 Terminal Questions

1. Extend {(1, 1, 1, 1), (1, 2, 1, 2) to a basis of R^4 (R)
2. Examine whether the set of vectors in V3 (R) form a basis or not

   (1, 0, 0), (0, 1, 0) (1, 1, 0) (1, 1, 1)
3. Given examples of two different basis of V_2(R).

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Unit - 9

Basis and Dimensions of A Vector Space (Continued)

Structure

9.1 Introduction

9.2 Learning Objectives

9.3 Maximal Linear Independence Set

9.4 Dimension of A vector Space

9.5 Extension Theorem

9.6 Use of Matrices

9.7 Self Check Exercise

9.8 Summary

9.9 Glossary

9.10 Answers to self check exercises

9.11 References/Suggested Readings

9.12 Terminal Questions

9.1 Introduction

Dear students,in proceeding unit we have discussed the concepts of basis, finite dimensions and ordered basis. We extend our discussion on basis and dimension of a vector space here this unit, to study the concept of maroinal linear independent set, dimension of a vector space and use of matrices etc.

9.2 Learning Objectives

The main objectives of this unit are to study

(i) Maximal linear independent set

(ii) The concept of dimension of a vector space

(iii) Extension theorem

(iv) The use of matrices, echelon matrices etc.

9.3 Maximal Linearly Independent Set

Let T be a subset of a vector space V(F). Then T is called a maximal linearly independent subset of V if there is no superset of T, other than T itself which is L.I.

Theorem : Let V(F) is a finitely generated vector space, prove that any maximal linearly independent subset of V is a basis of V.

Proof: Let $T = \{u_1, u_2, ..., u_n\}$ be a maximal L.I. set in V.

i.e. T is L.I.

To show T is a basis of V, it is sufficient to show that L(T) = V

Given T is a L.I. set in V

i.e. $T \subset V$

$\therefore$ $L(T) \subset V$ ....(i) ($\because$ V is a vector space)

Let $\nu \in V$ be any element

As $T = \{u_1, ..., u_n\}$ is a maximal linearly independent set

so that $S = \{u_1, ..., u_n, \nu\}$ is L.D.

$\Rightarrow$ $\exists$ scalars $\alpha_1, \alpha_2 .... \alpha_n, \alpha$ not all zero] ....(ii)

s.t. $\alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n + \alpha\nu = 0$ ....(iii)

If $\alpha = 0$, Then $\alpha_1 u_1 + \alpha_2 \upsilon_2 + ... + \alpha_n u_n = 0$

$\Rightarrow$ $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ ($\because$ T is L.T.)

so that $\alpha_1 = \alpha_1 = ... = \alpha_n = \alpha = 0$

which contradicts (ii)

$\therefore$ $\alpha \neq 0$

so (iii) $\Rightarrow$ $\nu = \left(-\frac{\alpha_1}{\alpha}\right)u_1 + \left(-\frac{\alpha_2}{\alpha}\right)u_2 + ... + \left(-\frac{\alpha_n}{\alpha}\right)u_n$

$\Rightarrow$ ν is a L.C. of $u_1, u_2, ..., u_n$

$\Rightarrow$ ν is a L.C. of elts of T

$\Rightarrow$ $\nu \in L(T)$

$V \subset L(T)$ ....(iv)

From (i) and (iv), L(T) = V

Hence the result

9.4 Dimension of a vector space

The dimension of a finitely generated vector V(F) is defined as the number of elements in a basis of V (F) and is denoted by dim V.

i.e. if any basis of V contains n elements we say dim V - n and thus V is n-dimensional vector space.

e.g. If $V = R^n$, then dim V = n since $B = \{e_1, e_2, ..., e_n\}$ where

ei = (0, 0, ... 1, ...0) 0 occurs at ith place, is a basis of V having n elts.

Thus R^n is n-dimensional vector space

Note: (i) dim {0} = 0, as basis of zero space is empty set which contains no element.

(ii) If vector space V(F) is not a finitely generated vector space, then it is called to be infinite dimensional vector space and dim V = ∞.

(iii) The dimension f V is always less than or equal to number of elements in any generating set of V, as a basis can be chosen from the given generating set for vector space V (F).

(iv) If a field F is taken as a vector space over the same field F, then dim F = 1 and {1} is a bases of F where 1 is unity element of F.

9.5 Theorem (Extension Theorem)

Let V (F) be a finite dimensional vector space and if

$S_1 = (w_1, w_2, ..., w_p\}$ is any L.I. set of vectors in V, prove that, unless S_1 is a basis, we can find vectors $w_{p+1}, w_{p+2}, ..., w_{p+r}$ in V such that

$\{w_1, w_2, ... w_p, w_{p+1}, ..., w_{p+r})$ is a basis of V (F)

(OR)

Prove that any L.I., set in V (F) can be extended to a basis of V.

Proof. Given V (F) be a finite dimensional vector space

$\Rightarrow$ $\exists$ a set $B = \{v_1, v_2, ... v_n\} \subset V$, which is a basis of V (F).

Then B is L.I. and L (B) = V

Also, we are given that the set

$S = \{w_1, w_2, ..., w_p\}$ is L.I. set in V

Consider the set $S_2 = \{w_1, w_2, ..., w_p, v_1, v_2, ..., v_n\}$.

Let $wj \in S_1$ $\Rightarrow$ $wj \in V$ for $1 < j < p$ (Since $S_1 \subset V$)

$\Rightarrow$ w_i can be expressed as L.C. of the elements of B ($\because$ L (B) = V)

$\Rightarrow$ $w_j = \sum_{i=1}^{n} a_{ij} v_i$, for $1 < j < p$ and $a_{ij} \in F$ ... (1)

For each $v \in V$ $\Rightarrow$ $v = \sum_{i=1}^{n} a_{ij} v_i$ for b_i's $\in F$ (Since L (B) = V)

$\Rightarrow$ $v = \sum_{i=1}^{n} b_{ij} v_i$ where $O_j = O$

$\Rightarrow$ v is L.C. of $w_1, w_2, ..., w_p, v_1, v_2, ..., v_n$

$\Rightarrow$ $v \in L (S_2)$

so that $V \subset L (S_2)$ .... (2)

Also $S_2 \subset V$ and V is a vector space

$\Rightarrow$ $L (S_2) \subset V$ [by Theorem 7 (i) of Chapter 2] .... (3)

From (2) and (3), we get V = L (S_2)

The relation (1) can be written as

$$w_j = \sum_{t \neq j} O_j w_t + \sum_{i=1}^{n} a_{ij} v_i \text{ , where t = 1, 2, ..., p}$$

$\Rightarrow$ one element w_j of S_2 can be expressed as a linear combination of the remaining elements of S_2, so that S_2 is a L.D. set

[because of Theorem 13 of Chapter 2]

$\Rightarrow$ By Theorem 14 of Chapter 2, one of the vectors in S_2 is a linear combination of the preceding vectors and this vector cannot be any one of w_j's (since S1 is L.I.) so this vector must be one of the ν_i's and let it be $\nu_k \in B$

$$\therefore \quad \nu_k = \sum_{j=1}^{p} \beta_j w_j + \sum_{i=1}^{k-1} \alpha_i v_i \text{ for } \alpha_i, \beta_j \in F \qquad \text{.... (4)}$$

Now consider the set

$S_3 = \{w_1, w_2, ..., w_p, \nu_1, \nu_2, ..., \nu_{k-1}, \nu_{k+1}, ..., \nu_n\}$

Clearly $S_3 \subset S_2$ $\Rightarrow$ L (S_3) $\subset$ L (S_2)

$\Rightarrow$ L (S_3) $\subset$ V ($\because$ L (S_2) = V already done) ..... (5)

Now let any y $\in$ V

$$\Rightarrow \quad y = \sum_{i=1}^{n} c_i v_i \qquad \text{for scalars } c_i \in F$$

$$= \sum_{i \neq k} c_j v_i + c_k v_k = \sum_{i \neq k} c_i v_i + c_k \left(\sum_{j=1}^{p} \beta_j w_j + \sum_{i=1}^{k-1} \alpha_i v_i \right)$$

[Using (4)]

$$= \sum_{i=1}^{k-1} c_i v_i + \sum_{i=k+1}^{n} c_i v_i + \sum_{i=1}^{k-1} (c_k \alpha_i) v_i + \sum_{j=1}^{p} (c_k \beta_i) w_j$$

$$= \sum_{i=1}^{k-1} (c_i + c_k \alpha_i) v_i + \sum_{i=k+1}^{n} c_i v_i + \sum_{i=k+1}^{n} c_i v_i + \sum_{j=1}^{p} (c_k \beta_i) w_j$$

$\Rightarrow$ y is expressed as a linear combination of the vectors of S_3.

$\therefore$ y $\in$ L (S_3) .... (6)

so that V $\subset$ L (S_3)

From (5) and (6), we get V = L (S_3)

Since $S_3 \subset$ V, so either it is L.D. or L.I.

If S_3 is L.I., then S_3 is a basis set, which has been obtained by extending L.I. set S_1 of V.

But if S_3 is L.D. set, then we go on repeating the above procedure to get a new set S_k such that S_k is L.I. and L. (S_k) = V.

Thus S_k, which is an extension of S_1 is basis set of V. Hence any L.I. set in V can be extended as a basis set.

At the most by repeating this procedure, we shall get the set $\{w_1, w_2, ..., w_p\}$ which is given as L.I. set and so becomes a basis of V.

Theorem: Prove that each finite subset S of a finite dimensional vector space V (F) which generates V can be reduced to a basis of V i.e. there exists a basis B of V such that $B \subset S$.

Proof. As an empty set is L.I. and subset of any set S

$\therefore$ S contains a linearly independent subset.

Let p be the largest non-negative integer such that there is a subset B of S, having p elements and B is L.I.

We wish to show that L (B) = L (S) .... (i)

As $B \subset S \Rightarrow L(B) \subset L(S)$

Now let $y \in S$

We want to check $y \in L(B)$

If possible, let $y \notin L(B)$ .... (ii)

Consider a set $B_1 = B \cup \{y\}$

Let $\alpha y + b_1 y_1 + b_2 y_2 + ... + b_p y_p = O$, $y_i \in B$ and $b_i \in F$ for $1 < i < p$ ..... (iii)

If $\alpha \neq 0$ then $\alpha y = -(b_1 y_1 + b_2 y_2 + ... + b_p y_p)$

$$\Rightarrow \quad y = \left(\frac{-b_1}{\alpha}\right) y_1 + \left(\frac{-b_2}{\alpha}\right) y_2 + ... + \left(\frac{-b_p}{\alpha}\right) y_p$$

$\therefore$ $y \in L(B)$ which contradicts (ii)

so that $\alpha = 0$

(iii) $\Rightarrow$ $b_1 y_1 + b_2 y_2 + ... + b_p y_p = O$

$\Rightarrow$ $b_1 = b_2 = ... = b_p = 0$ ($\because$ B is L.I. set)

Hence $\alpha y + b_1 y_1 + ... + b_p y_p = O$

$\Rightarrow$ $\alpha = b_1 = b_2 = ... = b_p = 0$

so that $B_1 = \{y, y_1, y_2, ..., y_p\}$ having p+1 elements is L.I.

But this is impossible

($\because$ A L.I. subset of S cannot have more than p elements)

$\therefore$ our supposition (ii) is false

so that $y \in L(B)$

$\therefore$ $S \subset L(B)$

$\Rightarrow$ $L(S) \subset L(L(B))$

$\Rightarrow$ $L(S) \subset L(B)$ $(\because L(L(B)) = L(B))$

From (1) and (iv) we have $L(B) = L(S)$

But given is that $L(S) = V$

$\therefore$ $L(B) = v$

Also B is L.I. set

$\Rightarrow$ B is a basis of V such that $B \subset S$

$\Rightarrow$ the set S has been reduced to a basis of V.

Theorem. If a basis of vector space V (F) contains n elements, prove that

(a) A subset W of V having more than n elements is L.D.

(b) A L.I. subset W of V cannot have more than n elements

(c) A subset W of V which generates V must have atleast n elements

(d) A subset W of V having n elements is a basis iff W is L.I. iff $L(W) = V$

Proof. (a) Let $B = \{x_1, x_2, \ldots, x_{n+1}\}$ be a subset of W, having (n+1) elements.

We shall show that B is L.D.

If possible, let B be L.I. set

Then either this set B is a basis or can be extended to a basis of V (F) by Extension Theorem. But, in both cases, the basis will contain (n+1) or more vectors which contradicts given that a basis of V contains n elements.

$\therefore$ B is L.D. set

and $B \subset W$ or $W \supset B$

$\Rightarrow$ W is L.D. set ($\because$ super set of a L.D. set is L.D.)

(b) Let W be a L.I. subset of V, if it contains more than n elts then W is L.D. set

[Using part (i)]

Thus, a L.I. subset W of V cannot have more than n elements.

Hence the result

(c) Let K be a subset of V such that $V = L(W)$

If W is an infinite set, then result is obvious

($\because$ An infinite set always have more than n elements)

If W is finite set, then it can be reduced to a basis B of V such that

$B \subset W$

$\Rightarrow$ number of elements of B = n as a basis of V contains n elements

(By Theorem 7, Each finite subset of a finite dimensional vector space V which generates V can be reduced to a basis of V)

Now as $B \subset W$ or $W \supset B$

$\Rightarrow$ number of elements in W $\geq$ n

$\Rightarrow$ W contains at least n elements.

(d) Firstly let K be a basis of V (F)

$\Rightarrow$ K is L.I. and V = L (W)

$\therefore$ K is L.I. or L (W) = V

Conversely

Let W be L.I. set or L (W) = V.

If W is L.I. then by Extension Theorem, it can be extended to a basis of V i.e., $\exists$ a basis B of V such that $W \subset B$.

Since any two bases of a vector space have same number of elements.

Thus number of elements in B = n

Since $W \subset B$ and both W, B have n elements

$\therefore$ W = B

$\Rightarrow$ W is a basis of V (F)

If L (W) = V, then, since a finite subset which generates V can be reduced to a basis of V,

$\therefore$ there exists a basis B of V such that $B \subset W$

But number of elements in B = n

Since $B \subset W$ and both W, B have n elements

$\therefore$ W = B

$\Rightarrow$ W is a basis of V (F).

9.6 Use of Matrices

Elementary row and column operations

Let A be any $m \times n$ matrix over a field F, then we recall that the elementary row (column) operations are as below:

(a) Interchange of any two rows (columns) [say ith and jth] of a matrix A and is denoted as $R_i \leftrightarrow R_j$ ($C_i \leftrightarrow C_j$)

(b) Multiplication of any row (column) [say ith] of a matrix A by a non zero element (say λ) and is denoted as $R_i \leftrightarrow \lambda R_i$ ($C_i \leftrightarrow C_i$).

(c) The addition to the elements of a row (column) [say ith] by scalar (Non zero element λ) times the corresponding elements of another row (column) [say jth] and is denoted as $R_i \leftrightarrow R_i + R_j$ ($C_i \leftrightarrow C_i + \lambda C_j$).

Echelon Matrix: Let $A = [a_{ij}]_{m\times n}$ be a matrix over a field F. Then the matrix A is called an echelon matrix iff the number of zeros after the non zero elements of a row increases row by row. The elements of the last row or rows may be all zeros.

e.g., $\begin{bmatrix} -5 & 0 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 6 & 7 \\ 0 & \frac{3}{2} & -2 & 3 \\ 0 & 0 & 9 & 1 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are all echelon matrices (or the matrices in the echelon form).

Note. The first non zero entries in the rows of an echelon matrix are called distinguished entries (elements) of that matrix.

e.g., In the above matrices.

The distinguished elements in the first matrix are -5, 1, 6

and in the second matrix are 1, 7

and in the third matrix are 1, $\frac{3}{2}$, 9, 8.

Row Reduced Echelon Matrix

An echelon matrix A is called a row reduced echelon matrix iff.

(i) The distinguished elements are each equal to 1.

(ii) These elements (distinguished) are the only non zero elements in their respective columns.

e.g., $\begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ are all row reduced echelon matrices.

Theorem 9. Prove that the non-zero rows of an echelon matrix are L.I.

Proof. Let $R_1, R_2, \ldots, R_p$ be p, non-zero of an echelon matrix.

If possible, let $(R_p, R_{p-1}, \ldots, R_2, R_1)$ be L.D. set. Then one of rows, say R_k is a linear combination of the preceding rows

i.e., $R_k = \alpha_{k+1} R_{k+1} + \alpha_{k+2} R_{k+2} + \ldots + \alpha_{p-1} R_{p-1} + \alpha_p R_p$ ....(A)

Let us suppose that ith component, of R_k is the first non-zero entry of this row, then by definition of an echelon matrix, the ith components of $R_{k+1}, \ldots R_{p-1}, R_p$ are all zero

$\therefore$ ith component of R.H.s. of Eq. (A) is

$\alpha_{k+1} (0) + \alpha_{k+2} (0) + \ldots + \alpha_{p-1} (0) + \alpha_p (0) = 0$

which is a contradiction to the assumption that ith component of R_k [i.e., L.H.S. of Eq. (A)] is non-zero

Hence our supposition is wrong

$\therefore$ non-zero rows $R_1, R_2, \ldots R_p$ of an echelon matrix are L.I.

Some Illustrative Examples

Example 1. Show that the set

$B = \{e_1, e_2, \ldots, e_n\}$ where ei = (0, 0, ..., 1, 0, ..., 0), 1 occurs at ith place is a basis of $V_n (R)$

Solution: We know that

$V_n (R) = (\nu/\nu = (\alpha_1, \alpha_2, \ldots, \alpha_n)\ ;\ \alpha_i \in F$ where $1 \leq i \leq n\}$

(i) To Prove B is L.I. set

Let $a_1 e_1 + a_2 e_2 + \ldots + a_n e_n = O$, for a_i's $\in F$

$\Rightarrow$ $a_1 (1, 0, \ldots, 0) + a_2 (0, 1, 0, \ldots, 0) + \ldots + a_n (0, 0, \ldots, 1) = (0, 0, \ldots, 0)$

$\Rightarrow$ $(a_1, 0, \ldots, 0) + (0, a_2, 0, \ldots, 0) + \ldots + (0, 0, \ldots, a_n) = (0, 0, \ldots, 0)$

$\Rightarrow$ $(a_1, a_2, \ldots, a_n) = (0, 0, \ldots, 0)$

$\Rightarrow$ $a_1 = 0, a_2 = 0, \ldots, a_n = 0$

Thus B is L.I. set.

(ii) To Prove $L (B) = V_n (R)$

Let $\nu \in V_n (R)$

$\Rightarrow$ $\nu = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for α_i's $\in F$, where $1 \leq i \leq n$

$\Rightarrow$ $\nu = (\alpha_1, 0, \ldots, 0) + (0, \alpha_2, \ldots, 0) + \ldots + (0, 0, \ldots, \alpha_n)$

$= \alpha_1 (1, 0, \ldots, 0) + \alpha_2 (0, 1, \ldots, 0) + \ldots + a_n (0, 0, \ldots, 1)$

$= \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n$

$\Rightarrow$ ν is a linear combination of $e_1, e_2, ..., e_n$, the elements of B

$\Rightarrow$ $\nu \in L(B)$

$\therefore$ $V_n(R) \subset L(B)$ ... (1)

Obviously, we have $L(B) \subset V_n(R)$ .... (2)

From (1) and (2), $L(B) = V_n(R)$ [as $V_n(R)$ is a vector space]

Hence B is a basis of $V_n(R)$

Note. The above basis is known as standard basis of $V_n(R)$.

Example 2: Show that the vectors (1, 1, 1), (1, 0, 1) and (1, -1, -1) of R^3 form a basis of R^3 (R). Also find the co-ordinate vector of (-3, 5, 7) relative to this basis.

Solution: As dim R^3 3, thus to show the given three vectors form a basis of R^3, it is sufficient to check these vectors are L.I. over R.

Let $\alpha(1, 1, 1) + b(1, 0, 1) + c(1, -1, -1) = O$ for $a, b, c \in R$

$\Rightarrow$ $(a + b + c, a - c, a + b - c) = (0, 0, 0)$

$\therefore$ $a + b + c = 0$

$a - c = 0$

$a + b - c = 0$

These eqs. can be put in the form of matrix as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or $AX = O$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

Now $\det A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}$

$$= 1 \begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$= 1(0 + 1) - 1(-1 + 1) + (1 - 0)$

$= 1 - 0 + 1 = 2 \neq 0$

$\therefore$ given vectors are L.I. over R.

Hence the given vectors form a basis of R^3 (R).

(IInd part) Let x, y, z, $\in$ R such that

$$(-3, 5, 7) = x\,(1, 1, 1) + y\,(1, 0, 1) + z\,(1, -1, -1)$$
$$= (x + y + z,\ x - z,\ x + y - z)$$

$\therefore$ $x + y + z = -3$

$x - z = 5$

$x - y - z = 7$

Solving these, we get x = 0, y = 2, z = -5

$\therefore$ $(-3, 5, 7) = 0\,(1, 1, 1) + 2\,(1, 0, 1) + (-5)\,(1, -1, -1)$

$\Rightarrow$ the required co-ordinate vector is (0, 2, -5)

Example 3. Show that the set B = $\{1, x, x^2, ..., x^m\}$ of (m + 1) polynomials is a basis set for the vector space P_m (R) of all polynomials of degree m over R (reals).

Solution: Given the set B = $\{1, x, x^2, ..., x^m\}$

(i) Firstly, we shall show B is L.I. set

Let $\alpha_0, 1 + \alpha_1 . x + \alpha_2 x^2 + ... + a_m x^m = 0$ (zero polynomial)

for α_i's $\in$ R, $0 \leq i \leq m$

$\Rightarrow$ $\alpha_0 .1 + \alpha_1 .x + ... + \alpha_m x^m = 0.1 + 0.x + 0.x^2 + ... + 0.x^m$

By equality of polynomials,

we have $\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 0, ..., \alpha_m = 0$

$\Rightarrow$ $1, x, x^2, ..., x^m$ are L.I.

Hence B = $\{1, x, x^2, ..., x^m\}$ is L.I. set

(ii) To show L (B) = P_m (R)

Let f(x) be a polynomial of degree m over R

i.e. $f(x) \in P_m$ (R)

Then $f(x) = b_0 + b_1 x + b_2 x^2 + ... + b_m x^m$ for b_j's $\in$ R

$= b_0 .1 + b_1 .x + b_2 .x2 + ... + b_m .\ x^m$

$\Rightarrow$ f(x) is expressed as L. combination of the elements of B

$\Rightarrow$ $f(x) \in$ L (B)

$\therefore$ P_m (R) $\subset$ L (B) ... (1)

Obviously, L (B) $\subset$ P_m (R)

($\because$ P_m is a vector space)

From (1) and (2), we have

$$L(B) = P_m(R)$$

Hence B is a basis of $P_m(R)$

Example 4: Show that the set

B = {a + b, b + c, c + d} is a basis of $V_3(R)$, given that S = {a, b, c} is a basis of $V_3(R)$.

Solution: (i) We show that B is L_j.

Consider $\propto(a + b) + \beta(b + c) + \gamma(c + d) = 0, \propto, \beta, \gamma \in R$

$\Rightarrow \quad (\propto + \gamma)\, a + (\beta + \propto)\, b + (\beta + \gamma)\, c = O\,(0, 0, 0)$

$\Rightarrow \quad \propto + \gamma = 0, \beta + \propto = 0, \gamma + \beta = 0$

on solving, we have $\propto = 0 = \beta = \gamma$

$\therefore$ a + b, b + c, c + d are L.I. vectors

$\Rightarrow$ B is L.I.

(ii) To show $L(B) = V_3(R)$

Let $z = (p, q, r) \in V_3(R)$

The set S = {a, b, c} is a basis of $V_3(R)$

$\therefore$ z can be expressed as a linear combination of elemtns of S i.e.

$Z = (\propto + \gamma)\, a + (\propto + \beta)\, b + (\beta + \gamma)\, c$

$= \propto(a + b) + \beta(b + c) + \gamma(c + a)$

$\Rightarrow$ z can be expressed as linear combinations or elements of B

$\Rightarrow \quad z \in L(B)$

$\Rightarrow \quad V_3(R) \subset L(B) \qquad .... (1)$

$\Rightarrow \quad L(B) \subset V_3(R) \qquad .... (2)$

Hence from (1) and (2) we get

$$L(B) = V_3(R)$$

□ Thus B is a basis of $V_3(R)$.

9.7 Self Check Exercise

Q. 1 Show that the matrices

$\begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$ form a basis of V(R), where V is a vector space of 2×2 matrices over reals.

Also find the coordinate vector of the matrix

A = $\begin{bmatrix} -3 & 7 \\ 7 & -2 \end{bmatrix}$ relative to above basis.

B = {(1, 1, 1), (1, -1, 0), (0, 1, 1)} is a basis of R^3.

Q. 2 Show that the dimension of the vector space Q $\left(\sqrt{3}\right)$ over Q is 2.

9.8 Summary

We have learnt the following concepts in this unit

(i) Maximal Linearly Independent Set. we have proved a theorem that any maximal so I subset of V is a basis of V

(ii) Dimension of a vector space

(iii) Extension theorem which proves that any LI set in V(F) can be extended to a basis V.

(iv) Elementary row and column operations and define what is called an Echelon Matrix.

9.9 Glossary

1. Row Reduced Echelon Matrix -

   An Echelon matrix A is called a row reduced Echelon matrix iff

i. The distinguished elements are each equal to 1.

ii. These elements are the only non-zero elements in their respective columns

9.10 Answers to Self Check Exercises

Ans.1 dim V = 3, coordinate vector of $\begin{bmatrix} -3 & 7 \\ 7 & -2 \end{bmatrix}$ relative to above basis is $\left(5, -11, \frac{7}{2}\right)$

Ans.2 consider B = $\left\{1, \sqrt{3}\right\}$,

dim $\left(Q\left(\sqrt{3}\right)\right)$ = 2

9.11 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. David C. Lay, Linear Algebra and its Applications, 3rd Ed.,, Pearson Education, Asia, Indian Reprint, 2007.
3. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

9.12 Terminal Questions

1. Give examples of this different basis of V3 (R) or R3

2. Show that set B = $\{\propto + i\beta, \gamma + i\,\delta\}$ is a basis set of ((R) iff $\propto\delta \neq \beta\gamma$.

3. Find coordinate of the vector (2, 6, 4) relative to the basis

$\nu_1 = (1, 1, 2)$, $\nu_2 = (2, 2, 1)$, $\nu_3 = (1, 2, 2)$

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Unit - 10

Dimension of Subspace

Structure

10.1 Introduction

10.2 Learning Objectives

10.3 Dimension of A Subspace

10.4 Self Check Exercise 1

10.5 Existence of Complementary Subspace of A finite Dimensional Vector Space

10.6 Self Check Exercise 2

10.7 Summary

10.8 Glossary

10.9 Answers to self check exercises

10.10 References/Suggested Readings

10.11 Terminal Questions

10.1 Introduction

Dear students, continuing our discussion on vector space we shall discuss dimension of a subspace and existence of complementary subspace of a finite dimensional vector space in this unit.

10.2 Learning Objectives

The main objectives of this unit are

(i) to prove a theorem based on dimension of a subspace. In another theorem we shall prove that for U_1, U_2 finite dimensional spaces $U_1 + U_2$ is also a finite dimensional and dim $(U_1 + U_2)$ = dim U_1 + dim U_2 - dim $(U_1 \cap U_2)$

(ii) to study existence of complementary subspace of a finite dimensional vector space. Here also we shall prove a result that if dim V (F) = m+n and U_1 is m - dimensional vector space of V, then $\exists$ n-dimensional vector subspace U_2 of V s.t. $V = U_1 \oplus U_2$ etc.

10.3 Dimension of Subspace

Theorem: If W is a subspace of a finite dimensional vector space V (F), prove that dim. W $\leq$ dim. V. Moreover W = V iff dim. W = dim V.

Proof. Let dim. V = n.

To show W is finite dimensional

If possible, suppose it is not so i.e., W is not finite dimensional

Take B_1 be a infinite basis of W

$\therefore$ B_1 is L.I. set in W

$\Rightarrow$ B_1 is L.I. set V $(\because W \subset V)$

But the set B_1 is infinite

Thus B_1 is a L.I. subset of V having more than n elements.

$\Rightarrow$ B_1 is L.D. set which is a contradiction

$\therefore$ our supposition is wrong

Hence W is a finite dimensional

$\therefore$ Take dim. W = p

Now we have to show that $p \leq n$

Let $B = \{w_1, w_2, ..., w_p\}$ be a basis of W

$\Rightarrow$ B is L.I. set in W

$\Rightarrow$ B is L.I. set in V $(\because W \subset V)$

As every L.I. subset of a vector space can be extended to a basis, so there exists a basis S of V such that $B \subset S$

$\Rightarrow$ No. of elts in B $\leq$ No. of elts in S

$\Rightarrow$ $P \leq N$

I.e., dim. W $\leq$ dim. V

Hence the result

IInd Part. If V = W

Then if W is a subspace of V, V is a subspace of W.

$\Rightarrow$ dim W $\leq$ dim V and dim V $\leq$ dim W $\Rightarrow$ dim V = dim W.

and if dim. V = dim. W = n (say)

Let B a basis of W

$\Rightarrow$ L (B) = W and B has n elements

Also B is a subset of V $(\because B \subset W \subset V)$

and has n L.I. vectors

$\Rightarrow$ B shall be basis of V

$\Rightarrow$ L (B) = V

Hence V = W

Theorem 2. If U_1, U_2 are finite dimensional subspaces of a finite dimensional vector space V (F), Prove $U_1 + U_2$ is also a finite dimensional and dim. $(U_1 + U_2)$ = dim. U_1 + dim. U_2 - dim. $(U_1 \cap U_2)$

Proof. Since it is given that V is finite dimensional vector space and $U_1 + U_2 \cap U_2$ are vector subspace of V (F)

$\therefore$ $U_2 + U_2$ and $U_1 \cap U_2$ are finite dimensional

Let $B_1 = \{w_1, w_2, ..., w_r\}$ be a basis of $U_1 \cap U_2$

so dim, $(U_1 \cap U_2) = r$

$\Rightarrow$ B_1 is a L.I. subset of $U_1 \cap U_2$

$\Rightarrow$ B_1 is a L.I. subset of U_1 and U_2 both

[$\because$ $U_1 \cap U_2 \subseteq U_1$ and $U_1 \cap U_2 \subseteq U_2$]

$\therefore$ the set B_1 can be extended to form a basis of U_1 and U_2 both

Let $B_2 = \{w_1, w_2,...., w_r, u_1, u_2,..., u_{n-r}\}$

and $B_3 = \{w_1, w_2, ..., w_r, \nu_1, \nu_2,..., \nu_{m-r}\}$

be bases of U_1 and U_2 respectively so dim $B_2 = n$ and dim $B_3 = m$.

Consider the set

$S = B_2 \cup B_3 = \{w_1, w_2,.....w_r, u_1, u_2, ..., u_{n-r}, \nu_1, \nu_2, ..., \nu_{m-r}\}$

Now, we shall show that S is a basis of $U_1 + U_2$

(i) Firstly we shall show $L(S) = U_1 + U_2$

Since $B_2 \subset U_1 \subset U_1 + U_2$

and $B_3 \subset U_2 \subset U_1 + U_2$

$\Rightarrow$ $B_2 \cup B_3 \subset U_1 + U_2$

$\Rightarrow$ $S \subset U_1 + U_2$

$\Rightarrow$ $S \subset U_1 + U_2$ [$\because$ $B_2 \cup B_3 = S$]

$\Rightarrow$ $L(S) \subset U_1 + U_2$ ... (i) [by Theorem 7 (i)]

Now, to show $U_1 + U_2 \subset L(S)$

Let $y \in U_1 + U_2$

$\Rightarrow$ $y = y_1 + y_2$ where $y_1 \in U_1$ and $y_2 \in U_2$

$\because$ $y_1 \in U_1 \Rightarrow$ $y_1 \in L(B_2)$ [$\because$ B_2 is a basis of U_1]

$\Rightarrow$ $y_1 = \sum_{t=1}^{r} a_t w_t + \sum_{p=1}^{n-r} b_p u_p$ for scalars a_1 and b_p.

[$\because$ y_1 can be expressed as L.C. of elements of B_2]

and $y_2 \in U_2$ $\Rightarrow$ $y_2 \in L(B_3)$ [$\because$ B_3 is a basis of U_3]

$\Rightarrow$ $y_2 = \sum_{t=1}^{r} c_t w_t + \sum_{q=1}^{m-r} d_q v_q$ for scalars c_t and d_q

[$\because$ y_2 can be expressed as L.C. of elements of B_3]

so that $y = y_1 + y_2$

$$= \sum_{t=1}^{r}(a_t + c_t) w_t + \sum_{p=1}^{n-r} b_p u_p + \sum_{q=1}^{m-r} d_q v_q$$

$\Rightarrow$ y is expressed as L.C. of $w_1, w_2, ..., w_r, u_1, u_2, ..., u_{n-r}, \nu_1, ..., \nu_{m-r}$

$\Rightarrow$ y is expressed as L.C. of the elements of S

$\Rightarrow$ $y \in L(S)$

$\therefore$ $U_1 + U_2 \subset L(S)$ ....(2)

From (1) and (2) we have $L(S) = U_1 + U_2$

(ii) Secondly, we shall show that S is L.I. set

Let $\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_r w_r + \beta_1 u_1 + \beta_2 u_2 + ... + \beta_{n-r} u_{n-r}$

$+ \gamma_1 \nu_1 + \gamma_2 \nu_2 + ... + \gamma_{m-r} \nu_{m-r} = O$

for scalars α_i's, β_j's, γ_k's $\in F$ .... (3)

Put $x = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_r w_r + \beta_1 u_1 + \beta_2 u_2 + ... + \beta_{n-r} u_{n-r}$ .... (4)

$\Rightarrow$ $x \in U_1$ (Since $B_2 \subseteq U_1$)

$\therefore$ (3) becomes $x + \gamma_1 \nu_1 + \gamma_2 \nu_2 + ... + \gamma_{m-r} \nu_{m-r} = O$

$\Rightarrow$ $x = -\gamma_1 \nu_1 - \gamma_2 \nu_2 - \gamma_3 \nu_3 - .... - \gamma_{m-r} \nu_{m-r}$ .... (5)

$\Rightarrow$ $x \in U_2$

[Since $\nu_1, \nu_2, ..., \nu_{m-r} \in U_2$ and U_2 is a vector space]

$\therefore$ $\in U_1 \cap U_2$ [as $x \in U_1$ and $\nu \in U_2$ both]

$\Rightarrow$ x can be expressed as a linear combination of the elements of B_1

[$\because$ B_2 is a basis of $U_1 \cap U_2$]

i.e., $x = t_1 w_1 + t_2 w_2 + ... + t_r w_r$ for scalars t's $\in F$

$\Rightarrow$ $-\gamma_1 \nu_1 - \gamma_2 \nu_2 - \in_3 \nu_3 - .... - -\gamma_{m-r} \nu_{m-r} = t_1 w_1 + t_2 w_2 + ... + t_r w_r$

[Using (5)]

$\Rightarrow$ $t_1 w_1 + t_2 w_2 + ... + t_r w_r + \gamma_2 \nu_1 + y\gamma_2 \nu_2 + ... + \gamma_{m-r} \nu_{-r} = O$

$\Rightarrow$ $t_1 = t_2 = ... = t_r = 0 = \gamma_1 = \gamma_2 = ... = \gamma_{m-r}$

[As B_3 being basis of V_2 is L.I. set]

Putting these values in (3), we get

$\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_r w_r + \beta_1 u_1 + \beta_2 u_2 + ... + \beta_{n-r} u_{n-r} = O$

$\Rightarrow \quad \alpha_1 = \alpha_2 = ... = \alpha_r = 0 = \beta_1 = \beta_1 = ... \beta_{n-r}$

[As B_2 being basis of U_1 is L.I. set]

So Eq. (3) implies that

$\alpha_1 = \alpha_2 = ... = \alpha_r = 0$, $\beta_1 = \beta_2 = ... = \beta_n = r\ 0 = \gamma_1 = \gamma_2 = ... = \gamma_{m-r}$ which gives the vectors $w_1, w_2, ..., w_r, u_1, u_2, ..., u_{n-r}, \nu_1, \nu_2, ..., \nu_{m-r}$ are L.I.

i.e., The set S is L.I. set

Hence S is a basis of $U_1 - U_2$

$\therefore \quad \dim (U_1 + U_2)$ = Number of elements in set S

$= r + n = r = m = r$

$= n + m = r$

$= \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2)$

Hence the result

Cor. 1. If U_1 and U_2 are finite dimensional subspaces of a finite dimensional vector space V (F) such that $U_1 \cap U_2 = \{0\}$

Then prove that $\dim (U_1 \oplus U_2) = \dim U_1 + \dim U_2$

Proof: $\dim (U_1 \oplus U_2) - \dim (U_1 + U_2)$

$= \dim U_1 + \dim U_2 = \dim (U_1 \cap U_2)$

$= \dim U_1 + \dim U_2 - 0 \qquad (\because \text{ given } U_1 \cap U_2 = \{0\})$

$= \dim U_1 + \dim U_2$

Hence the result

Cor. 2. If a finite dimensional vector space V(F) is direct sum of its subspaces U_1 and U_2. Prove $\dim V = \dim U_1 + \dim U_2$.

Proof: Given $V = U_1 \oplus U_2$

$\Rightarrow \quad V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$

$\Rightarrow \quad V = U_1 + U_2$ and $\dim (U_1 \cap U_2) = 0$

$\therefore \quad \dim V = \dim (U_1 + U_2) = \dim U_1 + \dim U_2 - \dim (U_1 \cap U_2)$

$= \dim U_1 + \dim U_2$

Hence the result

Theorem 3. If U_1 and U_2 are finite dimensional subspaces of a finite dimensional vector space V(F) and $V = U_1 + U_2$, $\dim V = \dim U_1 + \dim U_2$

Prove that $V = U_1 \oplus U_2$

Proof: Let $\dim U_1 = m$ and $\dim U_2 = n$

and $B_1 = \{\nu_1, \nu_2, \ldots\ldots \nu_m\}$, $B_2 = \{w_1, w_2, \ldots., w_n\}$

be bases of U_1 and U_2 respectively.

Let $B = \{\nu_1, \nu_2, \ldots\ldots \nu_m, w_1, w_2, \ldots., w_n\}$

Now we shall show that B is a basis of $V = U_1 + U_2$

Let $x \in V \Rightarrow x = \nu + w$ where $\nu \in U_1$ and $w \in U_2$ ($\because$ $V = U_1 + U_2$)

Since B_1 is a bases of $U_1 \Rightarrow L(B_1) = U_1$

$\therefore \quad \nu \in U_1 \Rightarrow \exists\ \alpha_i \in F$ s.t. $\nu = \sum_{i=1}^{m} \alpha_i v_i$

Also B_2 is a basis of $U_2 \Rightarrow L(B_2) = U_2$

$\therefore \quad w \in U_2 \Rightarrow \exists \quad \beta_j \in F$ s.t. $w = \sum_{j=1}^{n} \beta_i w_j$

so that $x = \nu + w = \sum_{i=1}^{m} \alpha_i v_i + \sum_{j=1}^{n} \beta_i w_j$

$\in L(B)$

$\therefore \quad x \in V \Rightarrow x \in L(B) \Rightarrow V \subset L(B)$

Clearly $B \subset V \Rightarrow L(B) \subset V$ ($\because$ V is a vector space)

Hence L(B)

$\Rightarrow$ B is a basis of V

($\because$ B contains m + n elts

and dim V = dim U_1 + dim U_2 = m + n)

Further we shall show $U_1 \cap U_2 = \{0\}$

Let $u \in U_1 \cap U_2$ be any element

$\Rightarrow \quad u \in U_1$ and $u \in U_2$

$\Rightarrow \quad u = \sum_{i=1}^{m} \alpha_i v_i$ and $u = \sum_{j=1}^{n} b_j w_j$

($\because$ $L(B_1) = U_1$ and $L(B_2) = U_2$)

$\Rightarrow \quad \sum_{i=1}^{m} \alpha_i v_i = \sum_{j=1}^{n} b_j w_j$

$\Rightarrow \quad a_1 \nu_1 + a_2 \nu_2 + \ldots.. + a_m \nu_m - b_1 w_1 - b_2 w_2 \ldots. b_n w_n = 0$

But $B = \nu_1, \ldots. \nu_m, w_1, \ldots\ldots w_m\}$ is L.I. set (by i)

$\therefore \quad a_1 = a_2 = \ldots\ldots = a_m = b_1 = -b_2 = \ldots.. = -b_n = 0$

$\Rightarrow \quad a_i = 0 \ \forall \ 1 \leq i \leq m$ and $bj = 0 \ \forall \ 1 \leq j \leq n$

$\Rightarrow \quad u = 0$

Hence $U_1 \cap U_2 = \{0\}$

So we have $V = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$

$\Rightarrow \quad V = U_1 \oplus U_2$

The subspaces $U_1, U_2 .... U_n$ of a vector space V(F) are called L.I. iff

$u_1 + u_2 + ... + u_n = 0,$

$\Rightarrow \quad u_1 - u_1 - ..... - u_n - 0$ for u_t's $\in U_t$'s for all $1 < i < n$.

10.4 Self Check Exercise 1

Q.1 Let $V = M_n = \{A : A$ is $n \times n$ matrix over reals$\}$ be a vector space over R. Find basis and dimension of the subspace W of V, where.

$W = \{A \in V : A$ is diagonal matrix$\}$

Q.2 Find a basis and dimension of the subspace W-generated by the vectors

(1, -1, 1), (8, 4, 2) (2, 2, 0), (3, 9, -3) of R3

Also extend this basis to a basis of R_3

10.5 Existence of Complementary Subspace of a finite dimensional vector space.

Theorem 4. For every subspace U_1 of a finite dimensional vector space V(F), $\exists$ a subspace U_2 of V such that

$V = U_1 \oplus U_2$, i.e. $\exists$ a complementary subspace U_2

OR

If dim {V(F)} = m+n and U_1 is m-dimensional vector subspace of V, then prove that there exists a n-dimensional subspace U_2 of V such that

$V = U_1 \oplus U_2$

Proof: Take $B = \{u_1, u_2, ... u_m\}$ be basis of $U_1 \quad \Rightarrow \quad$ B is L.I. subset of V

$(\because U_1 \subset V)$

and by Extension Theorem, it can be extended to form a basis of V.

Take $B_1 = (u_1, u_2, ..., u_m, \nu_1, \nu_2, ... \nu_n\}$ be a basis of V

and take $B_2 = \{\nu_1, \nu_2, ... \nu_n\}$, which generates a vector space U_2 (say)

$\Rightarrow \quad L(B_2) = U_2, L(B_1) = V$

Now $B_2 \subset B_1 \quad \Rightarrow \quad L(B_2) \subset L(B_1)$

$\Rightarrow \quad U_2 \subset V$ and U_2 is a vector space

$\Rightarrow \quad U_2$ is vector subspace of V (F)

We know every subset of a L.I. set is L.I. so B_2 is L.I. and $B_2 \subset B_1$

$\Rightarrow$ B_2 is a basis of U_2 $(\because L(B_2) = U_2)$

$\therefore$ $\dim U_2 = n$

Hence there exists n-dimensional vector subspace U_2 of V

Also we have $\dim U_1 = m$ and $\dim V = m + n$

$\Rightarrow$ $\dim V = \dim U_1 + \dim U_2$ ... (i)

Now to show $V = U_1 + U_2$

Let $x \in V$ be any element

and $L(B_1) = V$

$\therefore$ $\exists$ scalars $\alpha\iota, \beta F$ $(1 \le i \le m, 1 \le j \le n)$

such that $x = \sum_{i=1}^{m} \alpha_i u_i + \sum_{j=1}^{n} \beta_j v_j$

$= y + z$ say where $y = \sum \alpha_i u_i \in L(B) = U_1$

and $z = \sum \beta_j v_j \in L(B_2) = U_2$

$\therefore$ $x \in V \Rightarrow$ there exists $y \in U_1$, $z \in U_2$

$\Rightarrow$ $V = U_1 + U_2$ ... (ii)

From (i) and (ii), $\dim (U_1 + U_2) = \dim U_1 + \dim U_2$

$\Rightarrow$ $U_1 \cap U_2 = \{0\}$

Hence (ii) and (iii) $\Rightarrow V = U_1 \oplus U_2$

Def. : The subspaces $U_1, U_2 \ldots\ldots U_n$ of a vector space V (F) are called L.I. iff

$u_1 + u_2 + \ldots\ldots + u_n = 0,$

$\Rightarrow$ $u_1 = u_2 = \ldots\ldots = u_n = 0$ for u_i's $\in U_i$'s for all $1 \le i \le n$

Example 1: Show that X=axis is a subspace of R^3 (R). Find its dimension and a basis.

Solution: Let B = {(1, 0, 0)}, which contains a unit vector of R^3 along X-axis

$\because$ $B \neq \phi$ and $B \subset R^3$

Now Linear span of B = L (B)

$= \{\lambda (1, 0, 0) | \lambda \in P\}$

$= \{(\lambda, 0, 0) | \lambda \in R\}$

= The set of all points on X-axis

But, we know that L(B) is a subspace of R^3 (R).

IInd Part. Since each set having single non-zero vector is L.I.

$\therefore$ the set B = {(1, 0, 0)} is L.I. and the only independent vector along X-axis is (1, 0, 0)

[All other L.I. vectors are scalar multiple of this vector]

$\Rightarrow$ B is a basis of X-axis and dim B = 1.

Hence {(1, 0, 0)} is a basis and dimension of this subspace is].

Example 2: Extend {(-1, 2, 5)} to two different bases of R^3 (R)

Solution: Let B = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} be a basis of R^3 (R)

Consider a set S = {(-1, 2, 5), (1, 0, 0), (0, 1, 0), (0, 0, 1)}

Given set {(-1, 2, 5)} is L.I.

(Every set having single non-zero element is L.I.)

and (1, 0, 0) $\notin$ Linear span of this set

($\because$ Linear span = α (-1, 2, 5) = ($-\alpha$, 2α, 5α)

$\therefore$ {(-1, 2, 5), (1, 0, 0) is L.I.

Let S_1 = {(-1, 2, 5), (1, 0, 0), (0, 1, 0)}

This set S_1 is L.I. (check it)

and its contains 3 elements

Hence S1 is a basis of R^3 (R) [$\because$ dim R^3 = 3]

Let S_2 = {(-1, 2, 5), (0, 1, 0), (0, 0, 1)}

This set S_2 is L.I. (check it)

and it contains 3 elements

Hence S_2 is also a basis of R^3 (R) [dim R^3 = 3]

Thus S_1 = {(-1, 2, 5), (1, 0, 0), (0, 1, 0)}

and S_2 = {(-1, 2, 5), (0, 1, 0), (0, 0, 1)}

are basis of R_2 (R) which are extension of given set {(-1, 2, 5)}.

ALITER

Let B = {(1, 0, 0), (0, 1, 0), (0, 0, 1)} = {e_1, e_2, e_3} be a basis of R^3 (R)

Given set {(-1, 2, 5) L.I. over R, being non-zero vector

$\therefore$ the vectors ν, e_1, e_2, e_3 span R^3 where ν = (-1, 2, 5)

Since dim $R^3 = 3$, so any basis of R^3 contains exactly three L.I. vectors

To find these vectors,

Let $\quad A = \begin{bmatrix} -1 & 2 & 5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\qquad$ Operate $R_2 \to R_2 + R_1$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \to 2\,R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{Operate } R_4 \to R_4 + \frac{1}{5} R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

which is echelon form of A, having three non zero rows, so they form a basis of R^3

$\therefore$ The set $B_1 = \{(-1, 2, 5), (0, 2, 5), (0, 0, -5)\}$ is a basis of R^3 (R).

Also the set $B_2 = \{(-1, 2, 5), (0, 1, 0),), (0, 0, 1)\}$ is a basis of R^3 (R)

[$\because$ these form rows of a echelon matrix]

Thus, we have two different basis, which are extension of {(-1, 2, 5)}

Note.- Here answer may differ, since there are infinite such L.I. sets having three vectors of R^3, one of these being the given vector.

10.6 Self Check Exercise-2

Q. 1 Let W_1 be the subspaces generated by (-1, 2, 1), (2, 0, 1) and (-8, 4, -1) in R^3 (R) and W_2 be the subspaces of all vectors (a, 0, b) for all reals a, b. Find a basis and dimension of

(i) W_1 (ii) W_2 (iii) $W_1 + W_2$

Also find dimension of W1 ∩ W_2

Q. 2 Let M and N be two subspaces of R^4,

M = {(a, b, c, d) : b + c + d = 0}

and N = {a, b, c, d) : a + b = 0, c = 2d}

Find a basis and dimension of

(i) M (ii) N (iii) M ∩ N

10.7 Summary

In this unit we have proved the important theorems on subspaces of a finite dimensional vector space.

(i) W is ↓ finite dimensional V (F) then subspace of dim W < dim V. Also W = V iff dim W = dim V

(ii) If U_1, U_2 are finite dimensional subspaces of a vector space (finite dimensional) and $V = U_1 + U_2$, dim V = dim U_1 + dim U_2 then $V = U_1 \oplus U_2$.

10.8 Glossary

1. The subspaces U1, U2 ...., Un of a vector space V (F) is called linear independence iff

   $U_1 + U_2 + ... + U_n = 0$

   i.e. $U_1 = U_2 = ... = U_n = 0$ for u_i's ∈ U_i's for all < i < n

2. If a finite dimensional vector space V(F) is direct sum of its subspaces U_1 and U_2. Then dim V = dim U_1 + dim U_2.

10.9 Answers to Self Check Exercises

Ans.1 B = {E_1, E_2 .... E_n} is a basis and

dim W = no. of elements in B = n prove it.

Self Check Exercise - 2

Ans.1 dim w_1 = 2, dim w_2 = 2, dim ($w_1 + w_2$) = 3, dim ($w_1 \cap w_2$)

Ans. 2 Basis of sol. space M ∩ N = {(3, -3, 2, 1)} and dim (m ∩ N) = 1.

10.10 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

10.11 Terminal Questions

1. Find a basis for the row space of matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

2. Construct two subspaces of R^4 (R) s.t. dim $w_1 = 2$, dim $w_2 = 2$, dim $(w_1 \cap w_2) = 1$

3. Find ordered basis of V_4 relative to which the vector (-1, 3, 2, 1) has coordinates 4, 1, -2 and 7

4. Show that three finite dimensional subspaces of a vector space are L.I iff sum of their dimensions is equal to dimensions of their sum.

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Unit - 11

Linear Transformations

Structure

11.1 Introduction

11.2 Learning Objectives

11.3 Linear Transformation

11.4 Self Check Exercise 1

11.5 Properties of Linear Transformations

11.6 Self Check Exercise 2

11.7 Summary

11.8 Glossary

11.9 Answers to self check exercises

11.10 References/Suggested Readings

11.11 Terminal Questions

11.1 Introduction

Dear students, in this unit, we shall discuss about linear transformation. A simple example if a linear transformation is y = 3x, where the input x is a real number and so is y. Thus for example an input is 3 units causes an output of units. A linear transformation is also called vector space homomorphism or linear mapping.

11.2 Learning Objectives

The main learning objectives of this unit are

(i) to define a linear transformation

(ii) to define a linear operator and linear functional

(iii) to study the properties of a linear transformation

(iv) to study identity operator and negative of a linear transformation etc.

11.3 Linear Transformation:

If V (F) and W (F) are two vector spaces, then a mapping T from V to W i.e., $T : V \rightarrow W$ is said to be a linear transformation (or vector space homomorphism or linear mapping) if and only if

(i) $T(\nu + w) = T(\nu) + T(w) \; \forall \; \nu, w \in V$

i.e., T takes the sum $\nu + w$ of V to the sum of the T-images (which is $T(\nu) + T(w)$) in W.

(ii) $T(a\nu) = a\,T(\nu)\ \forall\ \nu \in V$ and $\in F$.

i.e., T takes the scalar multiplication $a\,\nu$ of V to the scalar multiplication $a\,T(\nu)$ in W.

Note. (a) The property (i) is known as Addative Property of T and (ii) is known as Homogenous Property of T.

(b) A linear transformation is abbreviated as L.T.

Definition Linear Operator : If V (F) is a vector space. Then the linear transformation $T : V \to V$ is called linear operator (L.O.)

Definition Linear Functional : if V (F) is a vector space. Then the linear transformation $T : V \to F$ is called linear functional.

Theorem. If V (F) and W (F) are vector spaces. Then prove that

$T : V \to W$ is a linear transformation if and only if

$T(\alpha\nu + \beta w) = \alpha T(\nu) + \beta\,T(w)\ \forall\ \nu, w \in V$ and $\alpha, \beta, \in F$.

Proof. Let $T : V \to W$ be a linear transformation.

Let $\nu, w \in V$ and $\alpha, \beta, \in F$

$\therefore$ $T(\alpha\nu + \beta w) = T(\alpha\nu) + T(\beta w)$ [By Addative property of T]

$= \alpha\,T(\nu)\ \beta\,T(w)$

[By Homogenous property of T]

Thus $T(\alpha\nu + \beta w) = \alpha T(\nu) + \beta\,T(w)\ \forall\ \alpha, \beta \in F$ and $\nu, w \in V$

Conversely. It is given that

$T(\alpha\nu + \beta w) = \alpha\,T(\nu) + \beta\,T(w)\ \forall\ \alpha, \beta \in F$ and $\nu, w \in V$

Firstly Take $\alpha = \beta = 1$

we get $T(1.\ \nu + 1.w) = 1.T(\nu) + 1.T(w)$

$\Rightarrow$ $T(\nu + w) = T(\nu) + T(w)$

Secondly Take $\beta = 0$

Then given implies that

$T(\alpha\nu + 0.w) = \alpha.T(\nu) + 0.T(w)$

$\Rightarrow$ $T(\alpha\nu) = \alpha\,T(\nu)$

Thus $T : V \to W$ satisfies (i) $T(\nu + w) = T(\nu) + T(w)$

(ii) $T(\alpha\nu) = \alpha\,T(\nu)$ for $\alpha \in F, \nu, \in V$

$\therefore$ T is a linear Transformation.

Note. (i) We shall use the result of above Theorem to show a given mapping T is a L.t.

(i) If we want to check whether a given mapping is a L.T. or not then we shall check both properties separately for that mapping.

Some Illustrative Examples

Example 1: Show that the following mapping are linear transformations:

(i) $T : R^2 \to R^3$ defined by $T (x, y) = (x + y, x - y, y)$

(ii) $T : R^3 \to R$ defined by $T (x, y, z) = x + 3y - 4z$

Solution: (i) Let $u = (x_1, y_1)$ and $\nu = (x_2, y_2) \in V_2 (R) = R^2$

and α, β be any real numbers

$\therefore \quad \alpha u + \beta\nu = \alpha (x_1, y_1) + \beta (x_2, y_2)$

$= (\alpha x_1, \alpha y_1) + (\beta x_2, \beta y_2)$

$= (\alpha x_1 - \beta x_2, \alpha y_1 + \beta y_2)$

Now $T (\alpha u + \beta\nu) = T (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2)$

$= ((\alpha x_2 + \beta x_2) + (\alpha y_1 + \beta y_2), (\alpha x_1 + \beta x_2) - (\alpha y_1 + \beta y_2), \alpha y_1 + \beta y_2)$

[by def. of given T]

$= (\alpha (x_1 + y_1) + \beta (x_2 + y_2), (\alpha (x_1 - y_1) + \beta (x_2 - y_2), \alpha y_1 + \beta y_2$

$= (\alpha (x_1 + y_1), \alpha (x_1 - y_1), \alpha y_1) + (\beta (x_2 + y_2), \beta (x_2 - y_2), \beta y_2)$

$= \alpha(x_1 + y_1 . x_1 - y_1 . y_1) + \beta (x_2 + y_2 . x_2 - y_2 . y_2)$

$= \alpha T(x_1 , y_1) + \beta T (x_2 , y_2)$

$= \alpha T(u) + \beta T (\nu)$

Hence T is a Linear Transformation.

(ii) Let $u = (x_1, y_1, z_1)$ and $\nu = (x_2, y_2, z_2) \in R^3$

and α, β any real numbers

$\therefore \quad \alpha u + \beta \nu = \alpha (x_1, y_1, z_1) + \beta (x_2, y_2, z_2)$

$= (\alpha x_1, \beta x_2 , \alpha y_1+ \beta y_2, \alpha z_1 + \beta z_2)$

Now $T (\alpha u + \beta \nu) = T (\alpha x_1, \beta x_2 , \alpha y_1+ \beta y_2, \alpha z_1 + \beta z_2)$

$= \alpha x_1, \beta x_2 + 3 (\alpha y_1+ \beta y_2) - 4(\alpha z_1 + \beta z_2)$

[by def. of given T]

$= \alpha x_1, 3\alpha y_1 - 4 \alpha z_1 + \beta x_2 + 3 \beta y_2 - 4 \beta z_2$

$= \alpha (x_1 + 3 y_1 - 4 z_1) + \beta (x_2 + 3 y_2 - 4 z_2)$

$= \alpha T(x_1, y_1, z_1) + \beta (x_2, y_2, z_2)$

$= \alpha T (u) + \beta T (\nu)$

Hence T is Linear Transformation.

Example 2 : Show that the following mappings are not linear transformations :

(i) $T : R^2 \to R^3$ defined by $T(x, y) = (x + 1, 2y, x + y)$

(ii) $T : R^2 \to R$ defined by $T(x, y) = |2x - 3y|$

Solution : (i) Let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$

Ten $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1, + x_2, y_1 + y_2)$

$\in R^2$

$\therefore$ $T(u + v) = T(x_1, + x_2, y_1 + y_2)$

$= (x_1 + x_2 + I, 2(y_1 + y_2), (x_1 + x_2) + (y_1 + y_2))$

$= (x_1 + x_2 + I, 2y_1 + 2y_2, (x_1 + y_1) + (x_2 + y_2))$ ...(I)

and $T(u) + T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= (x_1 + 1, 2y_1 x_1 + y_1) + (x_2 + 1, 2y_{2\,,} x_2 + y_2)$

$= ((x_1 + 1) + (x_2 + 1), 2y_1 + 2y_2, (x_1 + y_1) + (x_2 + y_2))$

$= ((x_1 + x_2 + 2, 2y_1 + 2y_2, (x_1 + y_1) + (x_2 + y_2))$ ...(II)

From (I) and (II) $T(u + v) \neq T(v)$

Hence T is not a Linear Transformation.

Altier. We know if $T : V \to W$ is a L.T.

Then T takes the zero vector or V into zero vector of W.

Here $T(O) = T(0, 0) = (0 + 1, 2(0), 0 + 0)$

$= (1, 0, 0)$

$\neq O$

Hence T is not a Linear transformation.

(ii) Let $u = (x_1, y_1)$ and $v = (x_2, y_2) \in R^2$

$\therefore$ $u + v = (x_1, y_1) + (x_2, y_2) = (x_1, + x_2, y_1 + y_2) \in R^2$

Now $T(u + v) = T(x_1, + y_2, y_1 + y_2)$

$= |2(x_1 + x_2) - 3(y_1 + y_2)|$

$= |2x_1 + 2x_2 - 3y_1 - 3y_2|$

$= |(2x_1 - 3y_1) + (2x_2 - 3y_2)|$ ...(I)

and $T(u) + T(v) = T(x_1, y_1) + T(x_2, y_2)$

$= |2x_1 - 3y_1| + |2x_2 - 3y_2|$ ...(II)

From (I) and (II)

$$T(u + v) \neq T(u) + T(v)$$

[$\because$ $|a + b| = |a| + |b|$ does not hold for all reals a, b]

Hence T is not a Linear Transformation.

Example 3 : Let R be the field of reals and V be the space of all functions from R which are continuous.

Define T by $(T f)\, x = \int_0^x f(t)\, dt$.

Prove T is a linear operator from V into V.

Solution : Let $f(x)$ and $g(x)$ be two functions in V

and α, β be any real numbers

Then $\alpha f(x) + \beta g(x) \in V$ [$\because$ V is a vector space]

And $(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x)$

$$\therefore \quad (T(\alpha f + \beta g))(x) = \int_0^x (\alpha f + \beta g)(t)\, dt$$

$$= \int_0^x (\alpha f(t) + \beta g(t))\, dt$$

$$= \int_0^x (\alpha f(t) dt + \int_0^x \beta g(t))\, dt$$

$$= \alpha \int_0^x f(t) dt + \int_0^x g(t))\, dt$$

$$= \alpha (Tf)(x) + \beta (T g)(x).$$

Hence T is a linear transformation.

Example 4. Show that the mapping $T : V(R) \to P_2(x)$ where V is the vector space of square matrices defined by

$$T(A) = \alpha + (\beta + \gamma)\, \xi + \delta x^2 \text{ for } A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in V \text{ is a L.T}$$

Solution: Let $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ and $B = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \in V$

and a, b, $\in$ R

$\therefore \quad \alpha\, A + b\, B = \begin{bmatrix} a\alpha & a\beta \\ a\gamma & a\delta \end{bmatrix} + \begin{bmatrix} b\alpha_1 & b\beta_1 \\ b\gamma_1 & b\delta_1 \end{bmatrix}$

$$= \begin{bmatrix} a\alpha + b\alpha_1 & a\beta + b\beta_1 \\ a\gamma + b\gamma_1 & a\delta + b\delta_1 \end{bmatrix}$$

$\Rightarrow \quad T(\alpha A + bB) = a\alpha + b\alpha_1 + (a\beta + b\beta_1 + a\gamma + \beta\gamma_1)\ \xi + (\alpha\delta + \beta\delta_1)x^2$

$= a\alpha + b\alpha_1 + (a(\beta + \gamma) + b(\beta_1 + \gamma_1))\ x + (a\delta + b\delta_1)x^2$

$= (a\alpha + \alpha\ (\beta + \gamma)\ x + a\ \delta\ x^2) + (b\alpha_1 + b(\beta_1 + \gamma_1)\ x + b\ \delta_1\ x^2$

$= a(\alpha + (\beta + \gamma)\ x + \delta\ x^2) + b\ (\alpha_1 + (\beta_1 + \gamma_1)\ x\ \delta_1\ x^2)$

$= a\ T(A) + b\ T(B)$

Hence $T : V\ (R) \rightarrow P_2(x)$ is a L.T.

Example 5. If V and W are two vector spaces over the same field F.

Show $T : V \rightarrow W$ is a L.T.

iff $T(\lambda\ \nu_1 + \nu_2) = \lambda\ T(\nu_1) + T\ (\nu_2)\ \forall\ \nu 1m\ \nu_2 \in V$ and $\alpha \in F$

Solution: Let $T : V \rightarrow W$ be a L.T.

Take $\nu_1, \nu_2 \in V$ and $\lambda \in F$

$\therefore \quad T(\nu_1 + \nu_2) = T\ (\lambda\nu_1) + T(\nu_2)$ (By addative property of T)

$= \lambda\ T(\nu_1) + T(\nu_2)$ (By Homogonous property of T)

$\Rightarrow \quad T(\lambda\nu_1 + \nu_2) = \lambda\ T\ (\nu_1) + T\ (\nu_2)\ \forall\ \nu_1, \nu_2 \in V$ and $\alpha \in F$

Conversely it is given that

$T\ (\lambda\nu_1 + \nu_2) = \lambda\ T(\nu_1) + T(\nu_2)\ \forall\ \lambda \in F$ and $\nu_1, \nu_2 \in V$

Firstly Take $\lambda = 1$, we get

$T(1.\ \nu_1 + \nu_2) = 1.T(\nu_1) + T(\nu_2)$

$\Rightarrow \quad T(\nu_1 + \nu_2) = T(\nu_1) + T(\nu_2)$

Secondly Take $\nu_2 = 0 \in V$

Then $\quad T(\lambda\nu_1 + 0) = \lambda T(\nu_1) + T(0)$

$\Rightarrow \quad T(\lambda\nu 1) = \lambda T(\nu 1) + 0$

$[\because\ T(0) = T(0 + 0) = T(0) + T(0) \Rightarrow 0 + T(0) = T(0) + T(0) \Rightarrow T(0) = 0]$

$\Rightarrow \quad T(\lambda \because 1) = \lambda T\ (\because 1)$ .... (ii)

so (i) and (ii) $\Rightarrow$ T is a L.T.

$\Rightarrow \quad \alpha\ x + \beta\ y \in V$ $[\because\ V$ is vector space$]$

$\therefore \quad T(\alpha x + \beta y) = 0$ [by def. of T]

$= \alpha . 0 + \beta . 0$

$= \alpha\, T(x) + \beta\, T(y)$ [$\because$ of (1)]

Hence T is a linear transformation

Note: It is called Zero Transformation denoted by O

i.e. $O(x) = 0 \ \forall\ x \in V$

Theorem Identity Operator :

If V (F) is a vector space, then the mapping T defined as $T(x) = x \ \forall\ x \in V$ is linear operator on V

Proof. Let $x, y \in V$ and $\alpha, \beta \in F$, so that $T(x) = x$ and $T(y) = y$ ... (I)

$\Rightarrow \quad \alpha x + \beta y \in V$ [$\because$ V is a vector space]

$\therefore \quad T(\alpha x + \beta y) = \alpha x + \beta y$ [by def. of T]

$= \alpha T(x) + \beta T(y)$ [$\because$ of (1)]

Hence T is a linear operator

Note. It is called identity operator denoted by 1

i.e. $I(x) = x \ \forall\ x \in V$

Theorem 4. Negative of a linear Transformation. If V (F) and W (F) are vector spaces and $T : V \to W$ is a linear transformation, Then show that mapping $-T : V \to W$ defined as $(-T)(x) = -[T(x)] \ \forall\ x \in V$ is a linear transformation.

Proof. Given $T : V \to W$ is a linear transformation

$\therefore \quad T(x) \in W$ for $x \in V$

$\Rightarrow \quad -T(x) \in W$ [$\because$ W is a vector space]

Let $\alpha, \beta \in F$ and $x, y \in V$

$\Rightarrow \quad \alpha\, x + \beta\, y \in V$ [$\because$ V is a vector space]

$\therefore \quad (-T)(\alpha x + \beta y) = -[T(\alpha x + \beta y)]$ [by def. of T]

$= - \propto [T(x) + \beta\, T(y)$

($\because$ T is Linear)

$= - \propto T(x) - \beta\, T(y)$

$= \propto (-T(x)) + \beta\, (-T(y))$

$\Rightarrow \quad -T : V \to W$ is a Linear Transformation

Theorem. Zero Transformation (or operator)

If V (F) and W(F) are vector spaces then a mapping

$T : V \to W$ defined as

$T(x) = 0 \ \forall\ x \in V$

is a linear transformation protect

Proof. Set $x, y \in V, \propto, \beta \in F$ so that

$T(x) = 0$ and $T(y) = 0$ ... (1)

$\Rightarrow$ $\propto x + \beta y\ t\ V$

$\therefore$ $T(\propto x + \beta y) = 0 = \propto .(0) + \beta.(0)$

$= \propto T(x) + \beta T(y)$

$\therefore$ T is a Linear transformation

Note: This is called zero transformation denoted by O, i.e.,

$O(x) = 0 \ \forall\ x \in V$.

11.4 Self Check Exercise - 1

Q.1 Find out which of the following are L.T.

(i) $T : R \to R^2$ defined by $T(x) = (2x, 3x)$

(ii) $T : R^2 \to R$ defined by $T(x, y) = x - y$

Q.2 Show that the following maps are not L.T.

(i) $T : V_3(R) \to V_2(R)$, defined by

$T(x, y, z) = (|y|, 0)$

(ii) $T : R^2 \to R$ defined by $T(x, y) = xy$

11.5 Theorem (Properties of linear transformations)

If $T : V \to W$ is a linear transformation from V (F) to W (F). Then

(i) $T(0) = 0$, where 0 on left hand $\in V$ and 0 on right hand $\in W$

(ii) $T(-x) = -T(x) \ \forall\ x \in V$

(iii) $T(x - y) = T(x) - T(y) \ \forall\ x, y \in V$

(iv) $T(p\,x) = p\,T(x) \ \forall\ x \in V, p \in 1$

Proof. (i) Let $T(x) = w$ for $x\ V, w \in W$ .... (i)

Then $T(x) = T(x + 0)$

$= T(x) + T(0)$ [$\because$ T is L.T.]

$\Rightarrow$ $w = w + T(0)$ [$\because$ of (i)]

$\Rightarrow$ $w + 0 = w + T(0)$ [by cancellation Law]

$\Rightarrow$ $0 = T(0)$

$\Rightarrow \quad T(0) = 0$

(ii) $\quad T(-x) = T((-1)x)$

$= (-1)\, T(x) \qquad [\because T \text{ is a L.T.}]$

$= -T(x)$

(iii) $\quad T(x - y) = T(x + (-y))$

$= T(x) + T(-y) \qquad [\because T \text{ is a L.T.}]$

$= T(x) - T(y) \qquad$ [because of (ii)]

(iv) We shall prove this result by principle of induction on p

Case I. When p is a + ve integer

For $p = 1$, $T(1.\, x) = T(x) = 1.\, T(x)$ so that result is true for $p = 1$.

Suppose result is true for $p = m$, m is a + ve integer

i.e., $\quad T(m\,x) = m\,T(x)$

Now $T((m + 1) = T(m\,x + x)$

$= T(m\,x) + T(x)$

$= m\,T(x) + T(x) \qquad [\because T(m\,x) = m\,T(x)]$

$= (m + 1)\,T(x)$

$\therefore$ Result is true for $p = m + 1$

Hence by induction, $T(p\,x) = p\,T(x)$ for all + ve integers p.

Case II. When $p = 0$, then $T(0.\, x) = T(0) = 0 = 0\,.\,T(x)$

$\therefore$ Result is true for $p = 0$

Case III. When p is a negative integer, let $p = -q$ where q is a + ve integer

$\therefore \quad T(p\,x) - 1((-q)\,x) = 1(q(-x))$

$= q\,T(-x) = q(-T(x))$

$= (-q)\,T(x) = pT(x)$

$\therefore$ Result is true for negative integers p.

Hence the result is true for all integers p.

Theorem. Let V and W be vector spaces over the same field F. Let B1 = $\{\nu_1, \nu_2, \ldots \nu_n\}$ be a basis of V and let $w_1, w_2, \ldots, w_n$ be any vectors in W. Prove then there exists a unique linear mapping $T : V \rightarrow W$ such that $T(\nu_1) = w_i$, $1 \leq i \leq n$.

OR

Let $T : V \rightarrow W$ be a linear transformation of finitely generated vector spaces V and W. Prove T is completely determined if we know images of a basis of V under T.

Proof. We shall prove the theorem in three steps

(i) Define a mapping T : V $\to$ W such that T (ν_1) = wi, 1 < i < n

(ii) To show T is linear

(iii) To show T in Unique

Step (i) Existence of T

Let $\nu \in$ V. Since B = {ν_1, ν_2, ..., ν_n} is a basis of V, so there exists unique scalars α_1, α_2, ..., $\alpha_n \in$ F s.t.

$$\nu = \alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n$$

Let T : V $\to$ W be defined as

T (ν) = α_1 w$_1$ + α_2w$_2$ + ... + α_nw$_n$

$\because$ α_i's ($1 \leq i \leq n$) are unique, the mapping T from V to W is well defined

Now each $\nu_i \in$ V can be expressed as a linear combination of vectors of basis B

i.e., $\nu_i = 0\nu_1 + 0\nu_2 + ... + 1.\ \nu_i + ... + 0\nu_n$

$\therefore$ T (νi) = 0 w$_1$ + 0 w$_2$ + ... + 1.w$_i$ + ... + 0 w$_n$

[Because of (I)]

i.e., T (ν_i) = w$_i$, for i, 1, ..., n.

Step (ii) To show that T is linear

Let u, $\nu \in$ V and α, $\beta \in$ F where u and ν can be written as

u = $\sum_{i=1}^{n} b_i v_i$ and ν = $\sum_{i=1}^{n} c_i v_i$ for b$_i$-, c$_i$'s $\in$ F

[$\because$ B = {ν_1, ν_2, ..., ν_n} is a basis of V]

$\Rightarrow$ T (u) = $\sum_{i=1}^{n} b_i w_i$ and T (ν) = $\sum_{i=1}^{n} c_i w_i$

$\therefore$ α u + β $\nu \in$ V [$\because$ V is a vector space]

Now T (α u + β ν) = T $\left(\alpha \sum_{i=1}^{n} b_i v_i + \beta \sum_{i=1}^{n} c_i v_i \right)$

$$= T \left(\sum_{i=1}^{n} (\alpha b_i) v_i + \sum_{i=1}^{n} (\beta c_i) v_i \right)$$

$$= T \left(\sum_{i=1}^{n} (\alpha b_i + \beta c_i) v_i \right)$$

$$= \sum_{i=1}^{n} (\alpha b_i + \beta c_i) w_i \qquad \text{[by def. of T]}$$

$$= \sum_{i=1}^{n} (\alpha b_i)(w_i) + \sum_{i=1}^{n} (\beta c_i)(w_i)$$

$$= \alpha \sum_{i=1}^{n} b_i w_i + \beta \sum_{i=1}^{n} c_i w_i$$

$= \alpha$ T (u) + β. T (ν)

Hence T is a linear mapping.

Step (iii) To show that T is unique.

Let S : V $\rightarrow$ W be another linear mapping such that

S (νi) = w_i, i = 1, 2, ... n

If $\nu = \alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n \in V$ (F)

Then S (ν) = S ($\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n$)

$= \alpha_1$ S (ν_1) + α_2 S (ν_2) + ... + α_n S (ν_n) [$\because$ S is linear]

$= \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n$

= T (ν) [by def. of T]

$\therefore$ S (ν) = T (ν) $\forall\ \nu \in$ V so that S = T. Thus T is unique.

Theorem Let T : V (F) $\rightarrow$ W (F) be a linear transformation.

Prove that (i) if the vectors $\nu_1, \nu_2, ... \nu_n \in V$ are L.D. over F

Then T (ν_1), T (ν_2), ..., T (ν_n) $\in$ W are also L.D. over F.

(ii) If the vectors $\nu_1, \nu_2, ..., \nu_n \in V$ are such that their images T (ν_1), T (ν_2), ... T (ν_n) $\in$ W are L.I. over F. Then $\nu_1, \nu_2, ..., \nu_n \in V$ are L.I. over F

OR

Let T : V(F) $\rightarrow$ W(F) Prove that image of a L.D. set is L.D. and pre-image of L.I. set is L.I.

Proof: (i) Since $\nu_1, \nu_2, ..., \nu_n \in V$ are L.D. over F

$\therefore$ $\exists$ scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$, not all zero such that

$\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n = 0$

$\Rightarrow$ T ($\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n$) = T (0)

$\Rightarrow$ α_1 T (ν_1) + α_2 T (ν_2) + ...+ α_n T (ν_n) = 0 [$\because$ T is linear]

Thus T (ν_1), T (ν_2), ..., T (ν_n) $\in$ W are L.D.

Hence the result

(ii) Suppose, $\exists$ scalars $\alpha_1, \alpha_2, ..., \alpha_n \in F$ such that

$\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n = 0$ for ν_i's $\in V$ $(1 < i < n)$

Then $T(\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n) = T(0)$

$\Rightarrow$ $\alpha_1 T(\nu_1) + \alpha_2 T(\nu_2) + ... + \alpha_n T(\nu_n) = 0$ [$\because$ T is linear]

Given $T(\nu_1), T(\nu_2), ... T(\nu_n)$ are L.I. over F

$\therefore$ $\alpha_1 - \alpha_2 - ..... - \alpha_n - 0$

so that $\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_n \nu_n = 0 \Rightarrow \alpha_1 = \alpha_2 = ... = \alpha_n = 0$

$\therefore$ $\nu_1, \nu_2, ..., \nu_n$ are L.I.

Hence the result

Example 6: Find T (x, y) where $T : R_2 \to R_3$ is defined as

$T(2, -5) = (-1, 2, 3)$ and $T(3, 4) = (0, 1, 5)$

Solution: Firstly we shall show that given vectors (2, - 5) and (3, 4) of domain of T form a basis for R_2 (= domain of T).

(α) To show (2, - 5) and (3, 4) are L.I.

Consider $\alpha(2, -5) + \beta(3, 4) = 0$ for α, β any scalars

$\Rightarrow$ $(2\alpha, -5\alpha) + (3\beta, 4\beta) = (0, 0)$

$\Rightarrow$ $(2\alpha + 3\beta, -5\alpha + 4\beta) = (0, 0)$

$\therefore$ $2\alpha + 3\beta = 0$ and $-5\alpha + 4\beta = 0$

$\Rightarrow$ $\alpha = \beta = 0$

Thus (2, - 5) and (3, 4) are L.I.

(b) To show (2, - 5) and (3, 4) span R^2

Let $(x, y) \in R^2$

Let $(x, y) = \alpha(2, -5) + \beta(3, 4)$

$= (2\alpha + 3\beta, -5\alpha + 4\beta)$

$\therefore$ $2\alpha + 3\beta = x$ and $-5\alpha + 4\beta = y$

$\Rightarrow$ $\alpha = \dfrac{4x-3y}{7}$ and $\beta = \dfrac{5x+2y}{23}$

Thus $(x, y) = \dfrac{4x-3y}{23}(2, -5) + \dfrac{5x+2y}{23}(3, 4)$ .... (I)

Hence (2, 5) and (3, 4) span R^2

$\therefore$ T (x, y) = T $\left(\frac{4x-3y}{23}(2,-5)+\frac{5x+2y}{23}(3,4)\right)$ [by (I)]

= $\frac{4x-3y}{23}$ T (2, - 5) + $\frac{5x+2y}{23}$ T (3, 4) [$\because$ T is linear]

= $\frac{4x-3y}{23}$ (-1, 2, 3) + $\frac{5x+2y}{23}$ (0, 1, 5) [because of given]

= $\left(\frac{4x-3y}{23},\frac{8x-6y}{23},\frac{12x-9y}{23}\right)$ + $\left(0,\frac{5x+2y}{23},\frac{25x+10y}{23}\right)$

= $\left(\frac{-4x+3y}{23},\frac{13x-4y}{23},\frac{37x+y}{23}\right)$

so that T (x, y) = $\left(\frac{-4x+3y}{23},\frac{13x-4y}{23},\frac{37x+y}{23}\right)$ is the required linear transformation.

Example 7: Find a linear transformation T : $R^2 \to R^2$ such that T (1, 0) - (1, 1) and T (0, 1) = (-1, 2).

Prove that T maps the square with vertices (0, 0) (1, 0), (1, 1) and (0, 1) into a parallelogram.

Solution: Let (x, y) $\in R^2$. Firstly we shall find T (x, y) under given conditions T (1, 0) = (1, 1) and T (0, 1) = (- 1, 2)

We know {(1, 0), (0, 1)} is a basis set for R^2 (Prove here)

$\therefore$ any vector (x, y) $\in R^2$ can be expressed as a linear combination of elts of this basis.

And (x, y) = x (1, 0) + y (0, 1)

$\Rightarrow$ T (x, y) = T (x (1, 0) + y (0, 1))

= x T (1, 0) + y T (0, 1) [$\because$ T is a L.T.]

= x (1, 1) + y (- 1, 2) [because of given]

= (x - y, x + 2y)

so that T (x, y) = (x - y, x + 2y) is the required L.T. .... (I)

IInd Part. Let the vertices of square be P, Q, R, S resp. and their T-images be A, B, C, D resp.

$\therefore$ A = T (P) = T (0, 0) = (0, 0) [$\because$ Put x = 0, y = 0 in (I)]

B = T (Q) = T (1, 0) = (1, 1) [$\because$ Put x = 1, y = 0 in (I)]

C = T (R) = T (1, 1) = (0, 3) [$\because$ Put x = 1, y = 1 in (I)]

D = T (S) = T (0, 1) = (-1, 2) [$\because$ Put x = 0, y = 1 in (I)]

Mid point of [AC] = $\left(\frac{0+0}{2}, \frac{0+3}{2}\right)$ = $\left(0, \frac{3}{2}\right)$

Mid point of [BD] = $\left(\frac{1+(-1)}{2}, \frac{1+2}{2}\right)$ = $\left(0, \frac{3}{2}\right)$

$\therefore$ Mod point of [AC] = Mid point of [BD]

Thus ABCD is a parallelogram

Hence T maps square PQRS into a parallelogram ABCD

Example 8: A linear transformation T : $R^3 \to R^3$ is defined by

T (e_1) = $e_1 + e_2 + e_3$, T (e_2) = $e_2 + e_3$, T (e_3) = $e_2 - e_3$ where e_1, e_2, e are unit vectors of R^3.

(i) Find the transformation (image) of (2, - 1, 3) under T.

(ii) Describe explicitly the linear transformation T

Solution: Since e_1, e_2, e_3 are unit vectors of R^3

so that e_1 = (1, 0, 0), e_2 = (0, 1, 0) and e_3 = (0, 0, 1)

Now given T (e_1) = $e_1 + e_2 + e_3$

$\Rightarrow$ T (1, 0, 0) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) = (1, 1, 1)

T (e_2) = $e_2 + e_3$ = (0, 1, 0) + (0, 0, 1)

= (0, 1, 1)

T (e_3) = $e_3 - e_3$ $\Rightarrow$ T (0, 0, 1) = (0, 1, 0) (0, 0, 1) = (0, 1, -1)

Also we know e_1, e_2, e_3 form a basis of R^3 [Prove her]

$\therefore$ every vector of R^3 can be uniquely expressed as the I Combination of e_1, e_2, e_3.

(i) Here (2, -1, 3) = 2 (1, 0, 0) + (- 1) (0, 1, 0) + 3 (0, 0, 1)

= 2 e_1 + (- 1) e_2 + 3 e_3

$\therefore$ T (2, -1, 3) = T (2 e_1 - e_2 + 3 e_3)

= 2 T (e_1) - T (e_2) + 3 T (e_3) [$\because$ T is a L.T.]

= 2 T (1, 0, 0) - T (0, 1, 0) + 3 T (0, 0, 1)

= 2 (1, 1, 1) - (0, 1, 1) + 3 (0, 1, -1)

= (2, 4, -2)

so that image (transform) of (2, - 1, 3) under T is (2, 4, -2).

(ii) To find the linear transformation T explicitly

Let (x, y, z) $\in$ R^3

Then (x, y, z) = (x, 0, 0) + (0, y, 0) + (0, 0, z)

$= x (1, 0, 0) + y (0, 1, 0) + z (0, 0, 1)$

$\therefore$ $T (x, y, z) = T (x (1, 0, 0) + y (0, 1, 0) + z (0, 0, 1))$

$= x T (1, 0, 0) + y T (0, 1, 0) + z T (0, 0, 1)$

[$\because$ T is a L.T]

$= x (1, 1, 1) + y (0, 1, 1) + z (0, 1, - 1)$

$= (x, x + y + z, x + y - z)$

Hence $T (x, y, z) = (x, x + y + z, x + y - z)$ is the required linear transformation

11.6 Self Check Exercise-2

Q. 1 A linear transformation

$T : R^3 \to R^2$ defined by

$T (e_1) = e_1 + e_2 + e_3$

$T (e_2) = e_2 + e_3$

$T (e_3) = e_2 + e_3$, where e_1, e_2, e_3

are unit vectors of R^3

(i) Find the transformation (image) of (2, -1, 3) under T

(ii) Describe explicitly the linear transformation T.

Q. 2 Find a L.T. which transforms (3, -1, -2), (1, 1, 0), (-2, 0, 2) in R^3 to twice the elementary vectors $2e_1, 2e_2, 2e_3$, in R^3, here e_1, e_2, e_3 are elementary vectors.

11.7 Summary

We have learnt the following concepts in this unit

(i) Linear Transformation, Linear operator, and linear functional.

(ii) Zero Transformation (operator)

(iii) Identity operator

(iv) Negative of a linear Transformation

11.8 Glossary

1. Linear operator - If V(F) is a vector space. Then the L.T. $T : V \to V$ is called linear operator L.O.

2. Linear Functional - If V(F) is a vector space. Then L.T. $T : V \to F$ is called linear Functional.

11.9 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 (i) L.T. (ii) L.T.

Ans. 2. Show it

Self Check Exercise - 2

Ans.1 (i) Image of (2, -1, 3) under T is (2, 4, -2)

(ii) T (x, y, z) = (x, x + y + z, x + y - z, x + y - z) is L.T.

Ans. 2 T (x, y, z) = (x - y + z, x + y + z, x - y + 2z) is reqd. L.T.

11.10 Reference/Suggested Reading

1. Sliephen H. Friedberg, Arhold J. Insel. Lawrence E. Spence, Linear Algebra, 4th Ed., Prantice Hall of India. Pvt. Ltd., New Delhi. 2004.
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

11.11 Terminal Questions

1. Find T (x, y), T : $R^2 \to R^3$ defined by T (1, 2) = (3 -1, -5)

   and T (0, 1) = (2, 1, -1)

2. Find L.T., T : P_3 (x) $\to$ P_3 (x) s.t.

   T (1 + x) = 1 + x

   T (2 + x) = x + $3x^2$

   T (x^2) = 0

3. Find a L.T., T : $R^2 \to R^2$ s.t.

   T (1, 2) = (3, 4)

   T (0, 1) = (0, 0)

4. Let V (R) be a vector space of integrable functions on R. Prove that T : V $\to$ R defined as

   T (f) = $\int_c^d f(x)dx$, $f \in$V, c, d $\in$ R is a linear functional.

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Unit - 12

Rank and Nullity of Linear Transformations

Structure

12.1 Introduction

12.2 Learning Objectives

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12.1 Introduction

Dear students, our aim in this unit is to study the concepts of rank and nullity of a linear transformation. The rank of a matrix is the number of L.I. row or column vector of a matrix and the dimension of a null space or kernal of the given matrix is called nullity of the matrix. We shall first define the concept of range and null space of a linear transformation in a vector space. Later on we shall prove a theorem, namely Rank-Nullity theorem or Sylyester's Law of nullity.

12.2 Learning Objectives

The main learning objectives of this unit are

(i) Range of a L.T.

(ii) Null space or Kernal of L.T.

(iii) Rank of a L.T.

(iv) Nullity of L.T.

(v) Invariant row spaces

(vi) to prove rank-nullity theorem

12.3 Definition (Range)

If V (F) and W (F) are vector spaces and $T : V \to W$ is a Linear Transformation. Then the image set of V under T is R (T) or T (V) i.e., Range $T = \{T(\nu) \mid \nu \in V\}$

Theorem : Let V (F) and W (F) be vector spaces and $T : V \to W$ is a linear trans formation prove that the range of T is a subspace of W (F) i.e., T (V) is a subspace of W (F) i.e., T (V) is a subspace of W (F).

Proof. We know Range T = T (V) = $(w : w = T(\nu) \ \forall \ \nu \in V\}$

Since $T : V \to W$ is a transformation (mapping)

and $\forall \ \nu \in V \Rightarrow T(\nu) \in W$

$\Rightarrow w \in W$

$\therefore$ Range $T \in W$

Let $w_1, w_2 \in$ Range T, Then

$w_1 \in$ Range $T \Rightarrow \exists \ \nu_1 \in V$ s.t. $T(\nu_1) = w_1$

and $w_2 \in$ Range $T \Rightarrow \exists \ \nu_2 \in V$ s.t. $T(\nu_2) = w_2$

Let $\alpha, \beta \in F \quad \Rightarrow \alpha \ \nu_1 + \beta\nu_2 \in V$

[$\therefore$ V is a vector space]

$\therefore \quad T(\alpha\nu_1 + \beta\nu_2) - \alpha \ T(\nu_1) + \beta \ T(\nu_2)$ [$\because$ T is a L.T.]

$= \alpha \ w_1 + \beta \ w_2$

$\Rightarrow \quad \alpha \ w_1 + \beta \ w_2 \in$ Range T

so that for $w_1, w_2 \in$ Range T and $\alpha, \beta \in F$ we have $\alpha \ w_1 + \beta \ w_2 \in$ Range T

Hence Range T is a subspace of W (F).

Note. Range T is also called RANGE SPACE

($\because$ R (T) is a vector space)

12.4 Definition (Null space or Kernel)

If V (F) and W (F) are two vector spaces and $T : V \to W$ is a linear transformation then the set of all those vectors in V whose image under T is zero, is called Kernel or Null space of T, which is denoted by N (T), i.e.,

Null space of T = N (T) = $\{\nu \in V \ ; \ T(\nu) = 0 \in W\}$

Theorem : Let V (F) and W (F) be vector spaces and $T : V \to W$ is a linear transformation. Prove that the null space of T is a subspace of V.

Proof. We know that

Null space of T = N (T) = $\{\nu : \nu \in V$ and $T(\nu) = 0\}$

Obviously N (T) $\subset$ V

Let $\nu_1, \nu_2 \in N(T)$ and $\alpha, \beta \in F$

$\therefore \quad T(\nu_1) = 0$ and $T(\nu_2) = 0 \quad$ [by def. of N (T)] .....(I)

And $\nu_1, \nu_2 \in N(T) \quad \Rightarrow \quad \nu_1, \nu_2 \in V$

$\Rightarrow \quad \alpha\,\nu_1 + \beta\nu_2 \in V$ [$\because$ V is a vector space]

$\therefore \quad T(\alpha\,\nu_1 + \beta\nu_2) = \alpha\,T(\nu_1) + \beta\,T(\nu_2) \quad$ [$\because$ T is L.T.]

$= \alpha.\,0 + \beta.\,0$ [Using I]

$= 0$

$\Rightarrow \quad \alpha\nu_1 + \beta\nu \in N(T)$

Hence N (T) is a subspace of V.

12.5 Self Check Exercise - 1

Q.1 Let V be a vector space of 2×2 matrice over R and

$$P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

Let $T : V \to V$ be a L. T defined by

$T(A) = PA \ \forall\ A \in V$

Find basis and dimension of

(i) Null space of T

(ii) Range space of T

Q.2 Find a L.T $T : R^3 \to R^3$ whose range space is generated by (1, 2, 3) and (4, 5, 6)

12.6 Rank

If V (F) and W (F) be vector spaces and $T : V \to W$ be a L.T, then the dimension of range space of T is called the rank of T and is denoted by P (T)

$\therefore \quad P(T) = \text{Rank of } T = \dim(\text{Range } T)$

12.7 Nullity

If V (F) and W (F) be vector spaces and $T : V \to W$ is a L.T. the dimension of null space of T is called Nullity of T and is denoted by V (T)

$\therefore \quad V(T) = \text{Nullity} = \dim(\text{Null space of } T)$

Theorem: (Range-Nullity Theorem or Sylvester's Law of Nullity)

If V(F) and W(F) are vector spaces and $T : V \to W$ is a linear transformation.

Let V is of dimension n (i.e. V is finite dimensional) prove that

Rank T + Nullity T = dim V.

Proof. Since Null space of T (= N (T)) is a subspace of V (F) and V (F) is finite dimensional

$\therefore$ N (T) is also finite dimensional

Let dim. N (T) = p where $p \leq n$ = dim. V

$\Rightarrow$ Nullity T = p

$\therefore$ N (T) has a basis having p elements, say

$B_1 = \{\nu_1, \nu_2, ... \nu_p)$ be basis of N (T)

$\therefore$ $T(\nu_1) = 0, T(\nu_2) = 0, ... T(\nu_p) = 0$ [by def. of N (T)] .....(I)

Also, we can extend the basis set B_1 to the basis set B_2 of V (F), having n elements.

Let $B_2 = \{\nu_1, \nu_2, ..., \nu_p, \nu_{p+1}, \nu_{p+2}, ... \nu_n\}$, be the basis of V(F).

Consider the set

$B_3 = \{T(\nu_{p+1}), T(\nu_{p+2}) ....T(\nu_n)\}$

We shall now show it to be the basis set of R (T)

In order to prove it, we have to prove that

(i) The set B_3 is L.I.

(ii) The set B_3 spans the range of T (=R (T))

To show B_3 is L.I.

Consider $a_{p+1} T(\nu_{p+1}) + a_{p+2} T(\nu_{p+2}) + ... + a_n T(\nu_n) = 0$ for a_i's, scalar

$\Rightarrow$ $T(a_{p+1} \nu_{p+1}) + T(a_{p+2} \nu_{p+2}) + ... + T(an\ \nu n) = 0$

[$\because$ T is linear]

$\Rightarrow$ $T(a_{p+1} \nu_{p+1} + a_{p+2} \nu_{p+2} + ... + a_n \nu_n) = 0$

[$\because$ T is linear]

$\Rightarrow$ $a_{p+1} \nu_{p+1} + a_{p+2} \nu_{p+2} + ... + a_n \nu_n \in N(T)$

[by def. of Null space of T]

$\Rightarrow$ $\nu \in N(T)$ where $\nu = \alpha_{p+1} \nu_{p+1} + \alpha_{p+2} \nu_{p+2} + ... + \alpha_n \nu_n$

Since B_1 is a basis of N (T)

$\therefore$ $\nu \in N(T)$ can be written as linear combination of the elements of B_1.

So let $\nu = b_1 \nu_1 + b_2 \nu_2 + ... + b_p\nu_p$ for bi scalars

$\Rightarrow$ $a_{p+1} \nu_{p+1} + a_{p+2} \nu_{p+2} + ... + a_n \nu_n = b_1 \nu_1 + b_2\nu_2 + ... + b_p \nu_p$

$\Rightarrow$ $b_1\nu_1 + b_2\nu_2 + ... + b_p\nu_p - a_{p-1} \nu_{p+1} - a_{p+2}\nu_{p+2} - ... - a_n\nu_n = 0$

[$\because$ $B_2 = \{\nu_1, \nu_2, ..., \nu_p, \nu_{p+1}, ..., \nu_n\}$ is a basis set so B_2 is L.I. set]

$\therefore$ $a_{p+1} T(\nu_{p+1}) + a_{p+2} T(\nu_{p+2}) + ... + a_n T(\nu_n) = 0$

$\Rightarrow$ $a_{p+1} = 0, a_{p+2} = 0, ..., a_n = 0$

$\therefore$ The set B_3 is L. Independent.

To show B_3 spans R (T)

Let $y \in R(T)$ be any element

$\therefore$ there exists $x \in V$ such that $y = T(x)$

Since $x \in V$ and B_2 is a basis set of V

$\therefore$ x can be written as linear combination of elements of B_3

Let x $= \alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_p \nu_p + \alpha_{p+1} \nu_{p+1} + ... + \alpha_n \nu_n$ for α_i's scalar.

$\Rightarrow$ $T(x) = T(\alpha_1 \nu_1 + \alpha_2 \nu_2 + ... + \alpha_p \nu_p + \alpha_{p+1} \nu_{p+1} + ... + \alpha_n \nu_n)$

$\Rightarrow$ $y = \alpha_1 T(\nu_1) + \alpha_2 T(\nu_2) + ... + \alpha_p T(\nu_p) + \alpha_{p+1} T(\nu_{p+1})$

$+ .... + \alpha_n T(\nu_n)$ [$\because$ T is L.T.]

$= \alpha_1 . 0 + \alpha_2 . 0 + ... + \alpha_p . 0 + \alpha_{p+1} T(\nu_{p+1})$

$+ ..... + \alpha_n T(\nu_n)$ [Using I]

$= \alpha_{p+1} T(\nu_{p+1}) + \alpha_{p+2} T(\nu_{p+2}) + ... + \alpha_n T(\nu_n)$.

so that y is a linear combination of the elements of B_3.

$\therefore$ the set B_3 spans the range of T,

Hence B_3 is a basis of R (T), having n - p elements

$\therefore$ dim (R (T)) = n - p

$\Rightarrow$ Rank T = n - Nullity T

$\Rightarrow$ Rank T + Nullity T = n

$\Rightarrow$ Rank (T) + Nullity (T) = dim V.

Hence the theorem is proved.

Note. Let $B = \{\nu_1, \nu_2, ... \nu_m\}$ be basis set of V (F),

Then the set $T(B) = \{T(\nu_1), T(\nu_2), ... T(\nu_m)\}$ spans R (T) and the number of linearly independent vectors in T (B) is the rank of T.

Theorem: Let $T : V \rightarrow W$ be a L.T.

Prove $p(T) \leq$ Min (dim V, dim W)

Proof: Let dim V = n and dim W = m

$\because$ R(T) {= range space of T) is a subspace of W

so dim (R(T)) $\leq$ dim W

$\Rightarrow$ $p(T) \leq m$ ....(i)

Now let any set of n + 1 vectors i.e. $w_1, w_2, ... w_{n+1}$ in R(T) so $\exists\ \nu_1, \nu_2, ... \nu_{n+1} \in V$

s.t. $T(\nu_1) = w_i$ for $1 \leq i \leq n + 1$

Here $\{\nu_1, \nu_2, ... \nu_{n+1}\}$ is L.D. set [$\because$ dim V = n]

$\Rightarrow$ $\{T(\nu_1), T(\nu_2), ..., T(\nu_n), T(\nu_{n+1})$, then set of images is also L.D.

$\Rightarrow$ $\{w_1, w_2, ..., w_n, w_{n+1}\}$ is a L.D. set

so that R(T) can not have (n+1) L.I. vectors

$\therefore$ dim $R(T) \leq n$ $\Rightarrow$ $p(T) \leq n$ ... (ii)

Combining (i) and (ii),

$p(T) \leq$ Min (m, n) = Min. (n, m)

$\Rightarrow$ $p(T) \leq$ Min. (dim V, dim W).

Definition (Invariant Subspace)

Let $T : V \to V$ be a L.T. Let U be subspace of V,

Then U is said to be invariant under T iff for all $u \in U$

$\Rightarrow$ $T(u) \in U$

Theorem: Prove that range space and null space of a L, transformation

$T : V \to V$ are invariant subspaces under T.

Proof. (i) We have Range of T = R (T) = $(w|w = T(\nu)\ \forall\ \nu \in V]$

$\therefore$ $w \in R(T)$ $\Rightarrow$ $w \in V$ [$\because$ R (T) is a subspace of V]

$\Rightarrow$ $T(w) \in R(T)$ [by def. of T (T)]

so that $w \in R(T)$ $\Rightarrow$ $T(w) \in R(T)$

$\Rightarrow$ R (T) is invariant under T.

(ii) Now Null space of T = N (T) = $\{\nu\ |\nu \in V, T(\nu) = 0\}$

Let'$\nu \in N(T)$ $\Rightarrow$ $T(\nu) = 0$ [by def. of N (T)]

$\therefore$ $T(T(\nu)) = T(0) = 0$ [$\because$ T (0) = 0 as T is linear]

$\Rightarrow$ $T(\nu) \in N(T)$ [by def. of N (T)]

so that $\nu \in N(T)$ $\Rightarrow$ $T(\nu) \in N(T)$

$\Rightarrow$ N (T) is invariant under T.

Some Illustrative Examples

Example 1. Let the linear transformation $T : R^3 \to R^3$ be defined as

$T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$

Verify Rank-Nullity Theorem for T.

Solution: We know that usual basis for R^3 is

$B = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

(i) Firstly we shall find basis for range T

As B is a basis for R^3

so $B_1 = \{T(e_1), T(e_2), T(e_3)\}$ generates range T.

Here $T(e_1) = T(1, 0, 0) = (2, 4, 2)$

$T(e_2) = T(0, 1, 0) = (0, -1, 3)$

$T(e_3) = T(0, 0, 1) = (0, 0, -1)$ (by def. of T)

$\therefore$ $B_1 = \{(2, 4, 2), (0, -1, 3), (0, 0, -1)\}$ generates range T

To find basis for range T, we have to find the L.I. vectors from $T(e_1)$, $T(e_2)$, $T(e_3)$. For this consider the matrix A, whose rows are generators of T and reduce it to echelon matrix.

i.e. $A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ Apply $R_1 \to \frac{1}{2} R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$ Apply $R_1 \to R_1 + + 2 R_2$

$\sim \begin{bmatrix} 1 & 0 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore$ (1, 0, 7), (0, -1, 3), (0, 0, -1) form L.I. set of vectors which generals range T which is basis for range T

$\therefore$ dim (R (T)) = 3

To find basis for Null space of T

Let $(x, y, z) \in R^3$

such that $T(x, y, z) = (0, 0, 0)$

$\Rightarrow$ $(2x, 4x - y, 2x + 3y - z) = (0, 0, 0)$

$\Rightarrow$ $2x = 0, 4x - y = 0, 2x + 3y - z = 0$

$\Rightarrow$ $x = 0, y = 0, z = 0$

$\therefore$ null space of $T = \{(0, 0, 0)\}$

$\Rightarrow$ nullity T = dim (null space of T) = 0

$\therefore$ Rank T + Nullity T = 3 + 0 = 3 = dim R^3

Hence Rank-Nullity theorem is verified.

Example 2 : Let V be vector space 2 $\times$ 2 matrices over R and

$$P = \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}$$

Let T : V $\rightarrow$ V be a linear transformation defined by

T (A) = PA $\forall \in$ V

Find a basis and dimension of (i) Null space of T

(ii) Range space of T.

Solution : To find basis of Null space of T

We shall find the matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

T (A) = O

i.e. PA = O

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ a - c = 0, b - d = 0

and - 2 a + 2 c = 0, - 2 b + 2 d = 0

$\Rightarrow$ a - c = 0 and b - d = 0

$\Rightarrow$ a = c and b = d

Here c and d are independent variables

$$\therefore \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ c & d \end{bmatrix} = \begin{bmatrix} c & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \\ 0 & d \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} c + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} d$$

Thus basis of null space of T = $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ and dim.

(null space of T) = 2.

To find basis of range space of T

We know $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$= \{E_1, E_2, E_3, E_4\}$ is a basis set of V

$\therefore$ $B_1 = \{T(E_1), T(E_2), T(E_3), T(E_4)\}$ generates range T

By def. of T,

$$T(E_1) = PE_1 = \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$T(E_2) = PE_2 = \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

$$T(E_3) = PE_3 = \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T(E_4) = PE_4 = \begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$$

To find basis for range T, we have to find the L.I. vectors from $T(E_1)$, $T(E_2)$, $T(E_3)$, $T(E_4)$. For this consider the matrix A, whose rows are generators of T and reduce it to echelon form

$$\text{i.e. } A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

Operator $R_3 \to R_3 + R_1$, $R_4 \to R_4 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ which is echelon form}$$

Thus (1, 0, -2, 0) (0, 1, 0, -2) form a set of L.I. vectors which generates Range T.

$\therefore$ $B2 = \left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$ is a basis of range T

$\therefore$ dim. (R (T)) = 2.

Hence the result.

Example 3 : Find range, rank null space and nullity for zero transformation and the identity transformation on a finite dimensional vector space V.

Solution : (a) Let $O : V \rightarrow V$ be zero transformation, defined as $O(\nu) = 0$ for all $\nu \in V$

Null space of zero transformation

$= \{\nu \in V \mid O(\nu) = 0\}$

$= \{\nu \in V \mid 0 = 0\}$ [by def. of O]

$= \{\nu \in V\} = V$

so that dim. (Null space) = dim. V

$\Rightarrow$ Nullity of zero transformation = dim. V

And Range of zero Transformation

$= \{w \in V \mid w = O(\nu) \text{ for all } \nu \in V\}$

$= \{w \in V \mid w = 0\}$ [by def. of O]

$= \{0\}$

so that dim. (Range space) = 0

$\Rightarrow$ Rank of zero transformation = 0

(b) Let $I : V \rightarrow V$ be identity transformation, defined as

$I(\nu) = \nu \ \forall \ \nu \in V$

Null space of identity transformation

$= \{\nu \in V \mid I(\nu) = 0\}$

$= \{\nu \in V \mid \nu = 0\}$ [by def. of I]

$= \{0\}$

so that dim. (Null space) = 0

$\Rightarrow$ Nullity of identity transformation = 0

And Range of identity transformation

$= \{w \in V \mid w = I(\nu) \text{ for all } \nu \in V\}$

$= \{w \in V \mid w = \nu \text{ for } \nu \in V\}$ [by def. of I]

$= \{\nu \mid \nu \in V\} = V$

so that dim. (Range space) = dim. V

$\Rightarrow$ Rank of identity transformation = dim V.

12.8 Self Check Exercise - 2

Q. 1 Find a linear map $T : R^4 \rightarrow R^3$ whose null space is generated by (Kernal is spasmed) (0, 1, 2, 3) and (-1, 2, 3, 0).

Q. 2 Give an example of a L.T $T : V \rightarrow V$ s.t. its range space and null space are identical

12.9 Summary

In this unit we have learnt the following concepts :

(i) What is range of a L.T.?

(ii) How we define Null space (or kernal) of L.T.

(iii) Rank of a linear transformation

(iv) Nullity of a L.T.

(v) Invariant subspace

12.10 Glossary

1. Range Space : Range T is called range space.

2. Invariant Subspace : If $T : V \rightarrow V$ is a L.T. Let U be a subspace of V. Then U is called an invariant subspace under T iff $\forall\, u \in U \Rightarrow T(u) . \in U$.

12.11 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 basis of null space of T = $\left\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$

and dim (Null space of T) = 2

dim (R (T)) = 2

Ans. 2. $T(x, y, z) = (x + 4y, 2x + 5y, 3x + 6y)$ is L.T.

Self Check Exercise - 2

Ans.1 $T(x, y, z, t) = (-x - 2y + z, -6x - 3y + t, 0)$ is req. L.T.

Ans. 2 Set $T : R^2 \rightarrow R^2$ defined by $T(x, y) = (y, 0)$

$\forall\, (x, y) \in R^2$ then $N(T) = \{(x, 0) : x \in R\}$

$R(T) = \{x, 0) : x \in R\} \therefore N(T) = R(T)$

12.12 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.

12.13 Terminal Questions

1. Find a linear map $T : R^4 \rightarrow R^3$ whose null space is generated by (1, 2, 3, 4) and (0, 1, 1, 1).

2. Let $L : R^4 \rightarrow R^3$ is defined by

   $L(x, y, z, w) = (x + y, y - z, z - w)$

   verify Rank - Nullity theorem for L.

3. Find a basis and dimension of (i) range (ii) null space of the Linear map $T : V \rightarrow W$ defined by

   (i) $T : R^2 \rightarrow R^2$ s.t. $T(x, y) = (x + y, x - y, y)$

   (ii) $T : R^3 \rightarrow R^3$ s.t. $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z)$

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Unit - 13

Linear Transformations and Matrices

Structure

13.1 Introduction

13.2 Learning Objectives

13.3 Matrix Representation of a Linear Transformation

13.4 Self Check Exercise-1

13.5 Theorems on Linear Operators

13.6 Self Check Exercise-2

13.7 Summary

13.8 Glossary

13.9 Answers to self check exercises

13.10 References/Suggested Readings

13.11 Terminal Questions

13.1 Introduction

Dear students, in this unit we shall study the concept of linear transformation and matrices. In linear algebra, linear transformation can be represented by matrices. Matrices allow arbitrary linear transformation to be displayed in a consistent format, suitable for computation. This also allows transformation to be composed easily (by multiplying their matrices). A 4×4 transformation matrices are widely used in computer graphics.

13.2 Learning Objectives

The main learning objectives of this unit are

(i) to study matrix representation of a linear transformation relative to ordered basis.

(ii) two important theorems are proved for linear operators.

13.3 Matrix Representation of a Linear Transformation

Matrix representation of a linear transformation relative to ordered basis.

Let $T : V \to W$ be a linear transformation, where V and W are vector spaces over a field F

and dim V = n and dim W = m.

Let $B_1 = \{v_1, v_2, ..., v_n\}$

and $B_2 = \{w_1, w_2, ..., w_n\}$ be ordered bases of V and W respectively

$\because$ T : V $\to$ W is a L.T. (i.e., linear mapping) so that for every $\nu \in$ V, we have T(ν) $\in$ W.

Since B_2 is a basis of W, so each T (ν) $\in$ W can be uniquely written as a linear combination of the elements of B_2

In particular, each T (ν_j) $\in$ W where $1 \le j \le n$, can be expressed as follows :

Let $T(\nu_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + \ldots + \alpha_{m1} w_m$

$T(\nu_2) = \alpha_{12} w_1 + \alpha_{22} w_2 + \ldots + \alpha_{m2} w_m$

....

....

$T(\nu_n) = \alpha_{1n} w_1 + \alpha_{2n} w_2 + \ldots + \alpha_{mn} w_m$ where $\alpha_{ij} \in F$

for $1 \le i \le m$, $1 \le j \le n$.

i.e., $T(\nu j) = \sum_{i=1}^{m} \alpha_{ij} w_i$, $1 \le j \le n$.

Then the coefficient matrix of the above equations is

$$\begin{bmatrix} \alpha_{11} & \alpha_{21} & \ldots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \ldots & \alpha_{m2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{1n} & \alpha_{2n} & \ldots & \alpha_{mn} \end{bmatrix}$$

The transpose of the above coefficient matrix is defined as the matrix of linear transformation T, relative to the bases B_1 and B_2.

Notation

The matrix of linear transformation T w.r.t. the basis B_1 and B_2 is denoted by

[T : B_1, B_2] or simply by [T]

$$\therefore \quad [T] \begin{bmatrix} \alpha_{11} & \alpha_{21} & \ldots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{m1} & \alpha_{m2} & \ldots & \alpha_{mn} \end{bmatrix}_{m \times n}$$

i.e., $[T] = [\alpha_{ij}]_{m \times n}$.

Particular Case

If W = V i.e., T : V $\to$ V is a linear operator then from above discussion, we have the matrix of w.r.t., basis B = ($B_1 = B_2$) as $[T] = [\alpha_{ij}]_{m\times n}$ i.e., matrix of order $n \times n$.

Theorem : Show that to every matrix [αij] of m n scalars ($1 \le i \le m$, $1 \le j \le n$) where αij $\in$ F (any field). There corresponds a linear transformation T from V into W, where V and W are vector spaces of dimensions n and m respectively over a field F.

Proof : Given T : V $\to$ W be a L.T.

Let $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$

and $B_2 = \{w_1, w_2, ..., w_n\}$ be ordered bases of V and W respectively

Then $[T; B_1, B_2] = [\alpha_{i\,j}]_{m\times n}$ where αij are scalars in F

where $T(\nu_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$, where $1 \le j \le n$ ...(1)

Firstly, we shall prove that the above relation completely determines T i.e., T (ν) can be uniquely written as a linear combination of the vectors of B_2 for all $\nu \in$ V.

Since B_1 is a basis of V. so each $\nu \in$ V can be uniquely written as a linear combination of the elements (vectors) of B_1

i.e., $\nu = \sum_{j=1}^{n} b_j v_j$ where b_j's $\in$ F

$$\text{Then } T(\nu) = T\left[\sum_{j=1}^{n} b_j v_j\right]$$

$$= \sum_{j=1}^{n} b_j T(v_j) \qquad (\because \text{ T is linear})$$

$$= \sum_{j=1}^{n} b_j \left\{\sum_{i=1}^{m} \alpha_{ij} b_i\right\} \qquad \text{(using (1))}$$

$$= \sum_{i=1}^{m} \left\{\sum_{j=1}^{n} \alpha_{ij} b_j\right\} \text{wi}$$

$$= \sum_{i=1}^{m} p_i w_i,\ p_i = \sum_{j=1}^{n} \alpha_{ij} b_j \in F$$

Since by are unique so that p_i's are also unique.

Hence T (ν) can be uniquely written as a linear combination of elements of B_2.

$\Rightarrow$ T (ν) is uniquely defined for all $\nu \in$ V. Secondly, for each linear combination $\sum_i l_{ij} w_i \in$ W, there is a L.T. from V into W s.t.

$T(\nu_j) = \sum_{i=1}^{m} \propto_{ij} w_i$ for j = 1, 2, ...n.

$\Rightarrow$ for every matrix. $[\propto_{ij}]_{m\times n}$, $\exists$ a L.T. T : V $\rightarrow$ W defined by relation (1).

Remark: Here $[\nu_1, B_1]$ is the coordinate matrix of ν w. r.t. basis B_1 and $[T(\nu); B_2]$ is the coordinate matrix of $T(\nu]$ w. r.t. basis B_2.

13.4 Self Check Exercise - 1

Q.1 Let T be a linear operator on R^2 defined by

T (x, y) = (4x - 2y, 2x + y) Find the matrix of T relative to basis B = {(1, 1) ; (-1, 0)}

Also verify that

$[T : B] [\nu : B] = [T(\nu); B]$ for any vector $\nu \in R^2$

Q.2 Let T be a linear operator on R^3 defined by

T (x, y, z) = (2y + z, x - 4y, 3x)

Find a matrix of T relative to basis

B = {(1, 1, 1), (1, 1, 0), (1, 0, 0)}.

Also verify that

$[T : B] [\nu : B] = [T(\nu) : B]$

$\forall \nu \in R^3$

13.5 Theorems on Linear Operator

Theorem: Let T : V $\rightarrow$ V be a linear operator, V is a finite dimensional vector space over F (a field). Suppose $B = (\nu_1, \nu_2, ..., \nu_n)$ is a basis of V (F). Prove that for any vector $\nu \in V$

$[T ; B] [\nu ; B] = [T(\nu) ; B]$ for any vector $\nu \in V$.

Proof. It is given that $B = \{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V (F)

and T : V $\rightarrow$ V is a linear operator

Now each element $\nu \in V$ can be uniquely expressed as a linear combination of the elements of B [$\because$ B is a basis of V $\Rightarrow$ L (B) = V]

Let $\nu = \beta_1 \nu_1 + \beta_2 \nu_2 + ... + \beta_n \nu_n$ for β_j's $\in$ F

$$= \sum_{j=1}^{n} \beta_j \nu_j \qquad .... (1)$$

$\therefore$ $[\nu ; B] = [\beta_1, \beta_2, ..., \beta_n]^t$

$T(\nu n) = \alpha 1n\nu 1 + \alpha 2n\ \nu 2 + ... + \alpha nn\ \nu n$

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$\Rightarrow$ $[\nu ; B] = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$ .... (2)

$\because$ B = $\{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V

$\Rightarrow$ $\nu_1, \nu_2, ..., \nu_n \in V$

$\Rightarrow$ each T (ν_1), T (ν_2), ..., T $(\nu_n) \in V$ can be uniquely written as linear combination of the elements of basis B.

Let $T(\nu_1) = \alpha_{11}\nu_1 + \alpha_{21}\nu_1 + ... + \alpha_n 1 \nu_n$

$T(\nu_2) = \alpha_{12}\nu_1 + \alpha_{22}\nu_2 + ... + \alpha_n 2 \nu_n$

....

....

$T(\nu_n) = \alpha_1 n\nu_1 + \beta_{2n}\nu_2 + ... + \alpha_{n\ n}\nu_n$

i.e., $T(\nu_j) = \sum_{i=1}^{n} \alpha_{ij} v_i$; j = 1, 2, ..., n .... (3)

$\Rightarrow$ $[T ; B] = [\alpha_{ij}]_{n\times n} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$ .... (4)

Now, we have T (ν) = T $\left(\sum_{j=1}^{n} \beta_j v_j \right)$ [Using (1)]

$= \sum_{j=1}^{n} \beta_j T\left(v_j\right)$ [$\because$ T is a L.T.]

$$= \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{n} \alpha_{ij} v_i \right) \qquad \text{[Using (3)]}$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \alpha_{ij} \beta_j \right) v_i$$

$$= \sum_{i=1}^{n} \left(\alpha_{ij} \beta_1 + \alpha_{i2} \beta_2 + ... + \alpha_{in} \beta_n \right) v_i$$

$\Rightarrow$ $T(v) = (\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + ... + \alpha_{1n}\beta_n)\, v_1 + (\alpha_{21}\beta_1 + \alpha_{22}\beta_1 + ...$

$\alpha_2 v \beta_n) v_2 + ... + (\alpha_{n1}\,\beta_1 + \alpha_{n2}\beta_2 + ... + \alpha_{n\,n}\,\beta_n)\, v_n$

$\therefore$ $[T(v)\,;\,B] = [\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + ... + \alpha_{1n}\beta_n, \alpha_{21}\beta_1 + \alpha_{22}\beta_2 + ...$

$+ \alpha_{2n}\beta_n, ..., \alpha_{n1}\beta_1 + \alpha_{n2}\beta_2 + ... + \alpha_{n\,n}\beta_n]^t$

$$= \begin{bmatrix} \alpha_{11}\beta_1 + & \alpha_{12}\beta_2 + & +...+ & \alpha_{1n}\beta_n \\ \alpha_{21}\beta_1 + & \alpha_{22}\beta_2 & +...+ & \alpha_{2n}\beta_n \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha_{n1}\beta_1 + & \alpha_{n2}\beta_2 + & +...+ & \alpha_{nn}\beta_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha_{n1} & \alpha_{n2} & ... & \alpha_{nn} \end{bmatrix}$$

$= [T\,;\,B]\,[v\,;\,B]$ [Using (2) and (4)]

Hence $[T(v)] = [T\,;\,B]\,[v\,;\,B]$

i.e., $[T(v)] = [T]\,[v]$

Theorem. Let T_1 and T_2 be two linear operators on vector space V (F) whose dimension is n. Let $B = \{v_1, v_2, ..., v_n\}$ be an ordered basis of V (F). Prove that

(i) $[T_1 + T_2\,;\,B] = [T_1\,;\,B] + [T_2\,;\,B]$

(ii) $[\lambda T_1\,;\,B] = \lambda\,[T_1\,;\,\beta]$ for all $\lambda \in F$

(iii) $[T_1\,T_2] = [T_1]\,[T_2]$

Proof. It is given that $B = \{v_1, v_2, ..., v_n\}$ is a basis of V (F)

Let $[T_1 ; B] = [\alpha_{ij}]_{n\times n}$ where $T_1(\nu_j) = \sum_{i=1}^{n} \alpha_{ij} v_i$, $1 \leq j \leq n$

and $[T_2 ; B] = [\beta_{ij}]_{n\times n}$ where $T_2(\nu_j) = \sum_{i=1}^{n} \beta_{ij} v_i$, $1 \leq j \leq n$

(i) Now $(T_1 + T_2)(\nu_j) = T_1(\nu_j) + T_2(\nu_j)$

$$= \sum_{i=1}^{n} \alpha_{ij} v_i + \sum_{i=1}^{n} \beta_{ij} v_i$$

$$= \sum_{i=1}^{n} \left(\alpha_{ij} + \beta_{ij}\right) v_i, \ 1 \leq j \leq n$$

$\Rightarrow$ $[T_1 + T_2 ; B] = [\alpha_{ij} + \beta_{ij}] = [\alpha_{ij}] + [\beta_{ij}]$

$= [T_1 ; B] + [T_2 ; B]$

Hence the result

(ii) And $(\lambda T_1)(\nu j) = \lambda T_1(\nu j)$, $1 \leq j \leq n$ and $\lambda \in F$

$$= \lambda \sum_{i=1}^{n} \alpha_{ij} v_i$$

$$= \sum_{i=1}^{n} \left(\lambda \alpha_{ij}\right) v_i$$

$\Rightarrow$ $[\lambda T_1 ; B] = [\lambda \alpha ij] = \lambda [\alpha ij]$

$= \lambda (T_1 ; \beta]$

Hence the result

(iii) And $(T_1 T_2)(\nu j) = T_1(T_2(\nu j))$

$$= T_1 \left(\sum_{k=1}^{n} \beta_{kj} v_k \right)$$

$$\left[Since\, T_2(v_1) = \sum_{i=1}^{n} \beta_{ij} v_i \text{ and replace the suffix } i \text{ by } k \right]$$

$$= \sum_{k=1}^{n} \beta_{kj} T\left(v_k\right) \qquad [\because \text{ T is a L.O.}]$$

$$= \sum_{k=1}^{n} \beta_{kj} \left(\sum_{i=1}^{n} \alpha_{ik} v_i \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} \beta_{kj} \right) v_i$$

$\therefore$ $[T_1\, T_2 ; B] = \left[\sum_{k=1}^{n} \alpha_{ik} \beta_{kj} \right]$ where $1 \leq i \leq n$, $1 \leq j \leq n$

$= [\alpha_{ik}]\, [\beta_{kj}]$ where $1 \leq i, j, k \leq n$

$= [\alpha_{ij}]\, \beta_{ij}]$

$= [T_1 ; B]\, [T_2 ; B]$

Hence the result

Theorem. Let V (F) be a vector space and B = $\{v_1, v_2, ..., v_n\}$ be a basis of V (F). Prove that matrices of linear operators I (identity) and O (zero) and $[\delta_{ij}]$ and $[0_{ij}]$ respectively

where $\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$ and $0_{ij} = 0$ for $1 \leq i, j \leq n$.

Proof. It is given that B = $\{v_1, v_2, ..., v_n\}$ is a basis of V (F)

(a) To find matrix of identity operator 1 : V $\rightarrow$ V relative to basis B

Here I $(v_j) = v_j$ for all j = 1, 2, ..., n. [by def. of identity operator]

$\therefore$ I $(v_j) = 0.\, v_1 + 0.\, v_2 + ... + 1.\, v_j + ... + 0\, v_n$

$= \sum_{i=1}^{n} \delta_{ij} v_i$ where $\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$

$$\Rightarrow \quad [I ; B] = [\delta_{ij}]_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \end{bmatrix}_{n\times n}$$

which is called unit matrix (or order n)

(b) To find matrix of zero operator O : V $\rightarrow$ V relative to basis B

Here O (vj) = 0 for all j = 1, 2, ..., n

$= 0.\, v_1 + 0.\, v_2 + ... + 0 v_n$

$= \sum_{i=1}^{n} 0_{ij} v_i$ where $O_{ij} = 0$, for $1 \leq j \leq n$

$\Rightarrow$ $[O\ ;\ B] = [0_{ij}]_{n\times n} = \begin{bmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}_{n\times n}$

which is called zero matrix (of order n)

Theorem. Let V (F) be an n-dimensional vector space over the field F and let L (V, V) be the set of all linear operators on V and M be the vector space of all $n \times n$ matrices over F.

Prove that L (V, V) $\cong$ i.e., L (V, V) is isomorphic to M.

Proof. Since V (F) is an n-dimensional vector space

$\therefore$ $B = \{\nu_1, \nu_2, ..., \nu_n\}$ is an ordered basis of V

Suppose T be any linear operator on V

i.e., $T \in L(V, V)$

Let $T(\nu_j) = \sum_{i=1}^{n} \alpha_{ij} v_i$ where $1 \leq j \leq n$.

$\therefore$ $[T] = [T\ ;\ B] = [\alpha_{ij}]_{n\times n}$

$\Rightarrow$ $[T] \in M$

Thus $T \in L(V, V) \Rightarrow [T] \in M$

Now we define a function f as

$f : L(V, V) \to M$

such that $f(T) = [T] = [\alpha_{ij}] \in M$

Suppose $T_1, T_2 \in L(V, V)$

and $[T_1] = [\alpha_{ij}]_{n\times n}$, $[T_2] = [b_{ij}]_{n\times n}$

where $T_1(\nu_j) = \sum_{i=1}^{n} \alpha_{ij} v_i$ and $T_2(\nu_j) = \sum_{i=1}^{n} b_{ij} v_i$ for j = 1, 2, ..., n

(i) Firstly to show that f is linear :

Let $T_1, T_2 \in L(V, V)$ and $\alpha, \beta \in F$

$\Rightarrow$ $\alpha T_1 + \beta T_2 \in L(V, V)$ [Since L (V, V) is a vector space]

$\therefore$ $f(\alpha T_1 + \beta T_2) = [\alpha T_1 + \beta T_2]$

$= [\alpha T_1] + [\beta T_2]$

$= \alpha [T_1] + \beta [T_2]$

$= \alpha f(T_1) + \beta f(T_2)$

$\Rightarrow$ f is a L. Transformation

(ii) To show that f is 1 - 1 (one - one)

Let $T_1, T_2 \in L(V, V)$

such that $f(T_1) = f(T_2)$

$\Rightarrow$ $[T_1] = [T_2]$

$\Rightarrow$ $[\alpha_{ij}] = [b_{ij}]$ for all i, j = 1, 2, ..., n

$\Rightarrow$ $\sum_{i=1}^{n} \alpha_{ij} v_i = \sum_{i=1}^{n} b_{ij} v_i$ for all j = 1, 2, ..., n

$\Rightarrow$ $T_1(\nu_j) = T_2(\nu_j)$ for each $\nu_j \in B$

$\Rightarrow$ $T_1 = T_2$

$\therefore$ f is one-one

(iii) To show that f is onto

We know that for each matrix $[\alpha_{ij}]_{n\times n} \in M$, there exists a linear transformation $T \in L(V, V)$ such that $T(\nu_j) = \sum_{i=1}^{n} \alpha_{ij} v_i$; $1 \le j \le n$

$\Rightarrow$ $[T] = [\alpha_{ij}]$

$\Rightarrow$ $f(T) = [T] = [\alpha_{ij}]$

$\therefore$ f is onto

From (i), (ii) and (iii), we get L (V, V) is isomorphic to M

i.e., $L(V, V) \cong M$

Some Illustrated Examples

Example 1: Let T be a linear operator on R^2 defined by

$T(x, y) = (4x - 2y, 2x + y)$

(i) Find the matrix of T relative to the basis B = {(1, 1) [(-1, 1)}

(ii) Also verify that $[T ; B] [\nu ; B] = [T(\nu) ; B]$ for any vector $\nu \in R^2$.

Solution: Firstly, we shall express any element

$\nu_1 = (\alpha, \beta) \in R^2$ as a linear combination of the element of basis B.

Let $(\alpha, \beta) = a(1, 1) + b(-1, 0)$ for reals a and b

$\Rightarrow$ $(\alpha, \beta) = (a - b, a)$

$\therefore$ $\alpha = a - b, \alpha = a$

$\Rightarrow$ a = β and b = β - α

$\therefore$ $(\alpha, \beta) = \beta\,(1, 1) + (\beta - \alpha)\,(-1, 0)$ ... (1)

Given T : $R^2 \rightarrow R^2$ defined as

T (x, y) = (4x - 2y, 2x + y)

and B = {(1, 1), (-1, 0)} is a basis of R^2

Now T (1, 1) = (4 - 2, 2 + 1) = (2, 3) = 3 (1, 1) + (3 - 2) (-1, 0)

[Using (1)]

= 3 (1, 1) + 1 (-1, 0)

and T (-1, 0) = (-1 -0, 2 (-1) + 0)

= (-4, - 2) = (- 2) (1, 1) + (-2 + 4) (-1, 0) [using (1)]

= -2 (1, 1) + 2 (-1 , 0)

$\therefore$ $[T ; B] = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

which is the matrix of T relative to the basis B.

To verify [T ; B] [ν ' B] = [T (ν) ' B]

Let $\nu = (x, y) \in R^2$

Then ν = (x, y) = y (1, 1) + (y - x) (-1, 0) [Using (1)]

$\therefore$ $[\nu ; B] = [y, y - x]^t = \begin{bmatrix} y \\ y-x \end{bmatrix}$

Now T (ν) = T (x, y)

= (4x - 2y, 2x + y) (by def. of T]

= (2x + y) (1, 1) + (2x + y - 4x + 2y) (-1, 0)

[Using (1)]

= (2x + y) (1, 1) + (-2x + 3y) (-1, 0)

$\therefore$ $[T(\nu) ; B] = [2x + y, -2x + 3y]^t = \begin{bmatrix} 2x+y \\ -2x+3y \end{bmatrix}$

L.H.S. = [T ; B] [ν ; B]

$= \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ y-x \end{bmatrix}$

$$= \begin{bmatrix} 3y - 2(y-x) \\ y + 2(y-x) \end{bmatrix}$$

$$= \begin{bmatrix} 2x + y \\ -2x + 3y \end{bmatrix} = [T(\nu) ; B] = \text{R.H.S.}$$

Hence the result is verified.

Example 2: Let T be a linear operator on R^3 defined by

T (x, y, z) = (2y + z, x - 4y, 3x)

(i) Find the matrix of T relative to the basis

B = {(1, 1, 1), (1, 1, 0), (1, 0, 0)}

(ii) Verify that $[T ; B] [\nu ; B] = [T(\nu) ; B] \ \forall \ \nu \in R^3$

Solution: (i) Firstly, we shall express any element

$\nu_1 = (\alpha, \beta, \gamma) \in R^3$ as a linear combination of the elements of basis B

Let $(\alpha, \beta, \gamma) = \alpha$ (1, 1, 1) + b (1, 1, 0) + c (1, 0, 0) for some reals a, b, c

= (a + b + c, a + b, a)

$\Rightarrow$ a + b + c = α, a + b = β, $\alpha = \gamma$

Solving these, we get, $\alpha = \gamma$, b = $\beta - \gamma$, c = $\alpha - \beta$

$\therefore$ $(\alpha, \beta, \gamma) = \gamma$ (1, 1, 1) + $(\beta - \gamma)$ (1, 1, 0) + $(\alpha - \beta)$ (1, 0, 0) .... (1)

Given T : $R^3 \to R^3$ is a linear operator defined as

T (x, y, z) = (2y + z, x -4 y, 3x)

and B = {(1, 1, 1), (1, 1, 0), (1, 0, 0)} is a basis of R^3

Now T (1, 1, 1) = (2 + 1, 1 -4, 3) = (3, -3, 3)

= 3 (1, 1, 1) + (-3 -3) (1, 1, 0) + (3 + 3) (1, 0, 0) [Using (1)

= 3 (1, 1, 1) + (-6) (1, 1, 0) + (6) (1, 0, 0)

T (1, 1, 0) = (2 + 0, 1 - 4, 3) = (2, -3, 3)

= 3 (1, 1, 1) + (-3 -3) (1, 1, 0) + (2 + 3) (1, 0, 0) [Using (1)]

= 3 (1, 1, 1) + (-6) (1, 1, 0) + 5 (1, 0, 0)

T (1, 0, 0) = (0 + 0, 1 - 0, 3) - (0, 1, 3)

= 3 (1, 1, 1) + (1 - 3) (1, 1, 0) + (0 - 1) (1, 0, 0) [Using (1)]

= 3 (1, 1, 1) + (-2) (1, 1, 0) + (-1) (1, 0, 0)

$\therefore$ [T ; B] = $\begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}$ = $\begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

(ii) To verify that [T ; B] [ν ; B] = [T (ν) ; B] $\forall$ $\nu \in R^3$

Let ν = (x, y, z) $\in R^3$

Then ν = (x, y, z) = z (1, 1 1) + (y - z) (1, 1, 0) + (x - y) (1, 0, 0) [Using (1)]

$\therefore$ [ν ; B] = [z, y - z, x - y] = $\begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$

Now T (ν) = T (x, y, z)

= (2y + z, x - 4y, 3x)

= (3x (1, 1, 1) + (x - 4ν - 3x) (1, 1, 0) + (2y + z - x + 4y) (1, 0, 0) [Using (1)]

= 3x (1, 1, 1) + (-2x - 4y) (1, 1, 0) + (-x + 6y + z) (1, 0, 0)

$\therefore$ [T (ν) ; B] = $[3x - 2x - 4y - x + 6y + z]^t$ = $\begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$

L.H.S. = [T ; B] [ν ; B]

$$= \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} 3z+3(y-z)+3(x-y) \\ -6z-6(y-z)-2(x-y) \\ 6z+5(y-z)-1(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} 3z+3y-3z+3x-3y \\ -6z-6y+6z-2x+2y \\ 6z+5y-5z-x+y \end{bmatrix} = \begin{bmatrix} 3x \\ -2x-4y \\ -x+6y+z \end{bmatrix}$$

13.6 Self Check Exercise - 2

Q. 1 Let V(R) be vector space of all 2×2 matrices and T be a linear operator on V(R) s.t. T (ν) = Mν, $\nu \in$ V (R) and M = $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Find the matrix T relative to basis

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of V (F)

Q. 2 Let $V = R^3$. Find the matrix of standard basis $(e_1\ e_2\ e_3)$ relative to

(f_1, f_2, f_3), $f_1 = (1, \cos x, \sin x)$

$f_2 = (1, 0, 0)$, $f_3 = (1, -\sin x, \cos x)$

13.7 Summary

In this unit we have learnt the following concepts :

(i) Matrix representation of a linear transformation motive representation of a linear transformation relative to ordered basis.

13.8 Glossary

1. Coordinate Matrix : $[\nu, B,]$ is the coordinate matrix of ν. w.r.t. basis B_1 and $[T(\nu)\ ;\ B_2]$ is the coordinate matrix of $T(\nu)$ w.r.t. basis B_2.

2. Notation : The matrix of L.T. T w.r. the basis B_1 and B_2 is denoted by

$[T\ ;\ B_1, B_2]$ or simply by $[T]$

$$\therefore \quad [T] = \begin{bmatrix} \propto_{11} & \propto_{12} & \cdots & \propto_{1n} \\ \propto_{21} & \propto_{22} & \cdots & \propto_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \propto_{m1} & \propto_{m2} & \cdots & \propto_{mn} \end{bmatrix}_{m\times n}$$

i.e. $[T] = [\propto_{ij}]_{m\times n}$

13.9 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 $[T\ ;\ B] = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}^t = \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

Ans. 2. $[T\ ;\ B] = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

Self Check Exercise - 2

Ans.1 $[T ; B] = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$

Ans. 2 Reqd matrix $A = \begin{bmatrix} 0 & \cos x & \sin x \\ 1 & \sin x - \cos x & -\sin x - \cos x \\ 0 & -\sin x & \cos x \end{bmatrix}$

13.10 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.
4. Stephen H. Friedbery, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Ed., Prantice Hall of India Pvt. Ltd., New Delhi, 2004.

13.11 Terminal Questions

1. Find the matrix representation of $T : R^2 \rightarrow R^2$ defined as T (x, y) = (3x - 4y, x + 5y) w r.t. basis B = {(1, 3), (3, 4)}
2. For the matrix $\begin{bmatrix} \frac{1}{2} & 1 \\ \frac{2}{3} & 4 \end{bmatrix}$, find the corresponding linear operator T on R^2 relative to basis B = {(1, 0), (1, 1)}
3. Find the matrix representation of

   $T : R^2 \rightarrow R^2$ defined as

   T (x, y) = (2x - 3y, x + y) w.r.t. basis

   B = {(1, 2), (2 3)}
4. Let V = R3 Find the matrix of Standard basis {e1 e_2 e_3} relative to (f_1 f_2 f_3) where f_1 = (2, 1, 0), f_2 = (0, 2, 1), f_3 (0, 1, 2)

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Unit - 14

Linear Transformations and Matrices (Continued)

Structure

14.1 Introduction

14.2 Learning Objectives

14.3 Matrix Of An Inverse Operator

14.4 Self Check Exercise-1

14.5 Change of Coordinate Matrix

14.6 Self Check Exercise-2

14.7 Summary

14.8 Glossary

14.9 Answers to self check exercises

14.10 References/Suggested Readings

14.11 Terminal Questions

14.1 Introduction

Dear students, Continuing our discussion on Linear transformation and matrices, we shall, in this unit, discuss the concepts of matrix on an inverse operator and change of coordinate matrix (Transion matrix). Some important theorems to prove the inevitability of a linear transformations are also discussed in the present unit.

14.2 Learning Objectives

The main Learning objectives of this unit are

(i) to study the concept of matrix of an inverse operator. Here we shall find condition for a linear transformation to be invertible

(ii) to study the concept of change of coordinate matrix where the concept of transion matrix is discussed. We shall prove some theorems here to show that the transion matrix from basis B_1 to B_2 is invertible.

14.3 Matrix of an Inverse Operator

Theorem : Let V(F) be n-dimensional vector space and B = $\{v_1, v_2, ...v_n\}$, a basis of V(F).

If T be a linear operator on V such that

$[T : B] = [\alpha_{ij}]_{n\times n}$ for $\alpha_{ij} \in F$. Prove that T is invertible iff [T]B is invertible

and $[T^{-1} ; B] = [T : B]^{-1}$

Proof : Let T be invertible

$\Rightarrow$ $\exists$ an inverse operator T^{-1} on V

s.t. $T^{-1} T = 1 = T T^{-1}$

$\Rightarrow$ $[T^{-1}]_B = [1]_B = [T T^{-1}]_B$

$\Rightarrow$ $[T^{-1}]_B [T]_B = 1 = [T]_B [T^{-1}]_B$

$\Rightarrow$ [T]B is invertible

and $[T^{-1}]_B = [T]^{-1}_B$

i.e. $[T^{-1} : B] = [T ; B]^{-1}$

Conversely

Let $[T]_B$ be invertible

and $[T^{-1} ; B] = [T ; B]^{-1}$

Since $[T ; B]^{-1}$ is a matrix, so there exists a linear operator S on V such that

$[S ; B] = [T ; B]-1 = [T]_B^{-1}$

$\Rightarrow$ $[S]_B [T]_B = [T]_B = [T]_B [S]_B$

$\Rightarrow$ $[ST]_B = [T]_B = [TS]_B$

$\Rightarrow$ $ST = I = TS$

$\Rightarrow$ T is invertible operator

Hence the result.

Theorem : Let $T : V \to W$ be a linear transformation, where V and W are vector spaces over F (a field) of dimensions n and m respectively. Prove that for any vector $v \in V$

$[T] [v] = [T (v)]$

Proof : since dim V = n and dim W = m (given)

$\therefore$ suppose $B_1 = \{v_1, v_2, ..., v_n\}$

and $B_2 = \{w_1, w_2, ..., w_m\}$ be ordered basis of V and W respectively.

Now each element $v \in V$ can be uniquely expressed as a linear combination of the elements of B_1

[$\because$ B_1 is a basis of V $\Rightarrow L (B_1) = V$]

Let $v = \beta_1 v_1 + \beta_2 v_2 + ... + \beta_n v_n$ for Bi's $\in$ F

$$\sum_{j=1}^{n} \beta_j \, v_j \qquad ...(1)$$

$$\therefore \qquad [\nu \,;\, B_1] = [\beta_1, \beta_2, ..., \beta_n]^t = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix} \qquad ...(2)$$

Since $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V

$\Rightarrow \quad \nu_1, \nu_2, ..., \nu_n \in V$

$\Rightarrow$ each. $T(\nu_1), T(\nu_2), ..., T(\nu_n) \in W$ can be uniquely written as linear combination of the element of basis B_2.

Let $T(\nu_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + ... + \alpha_{m1} w_m$

$T(\nu_2) = \alpha_{12} w_1 + \alpha_{22} w_2 + ... + \alpha_{m1} w_m$

....

....

$T(\nu_n) = \alpha_{1n} w_1 + \alpha_{2n} w_2 + ... + \alpha_{mn} w_m$ for scalars α_{ij}'s $\in F$

i.e., $T(\nu_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$; j = 1, 2, ..., n ...(3)

$$\Rightarrow \qquad [T \,;\, B_1, B_2] = [\alpha_{ij}]_{m \times n} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha_{m1} & \alpha_{m2} & ... & \alpha_{mn} \end{bmatrix} \qquad .... (4)$$

Now, we have $T(\nu) = T\left(\sum_{j=1}^{n} \beta_j \nu_j\right)$ [Using (1)]

$$= \sum_{j=1}^{n} \beta_j T(\nu_j) \qquad [\because \text{ T is a L.T.}]$$

$$= \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{m} \alpha_{ij} w_i\right) \qquad [\text{Using (3)}]$$

$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} \alpha_{ij} \beta_j\right) w_i$$

$$= \sum_{i=1}^{m} \left(\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + ... + \alpha_{in}\beta_n\right) w_i$$

$\therefore$ $T(\nu) = (\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + ... + \alpha_{1n}\beta_n)\, w_1 + (\alpha_{21}\beta_1 + \alpha_{22}\beta_1 + ... + \alpha_2\nu\beta_n)\, w_2 +$

$... + (\alpha_{m1}\beta_1 + \alpha_{m2}\beta_2 + ... + \alpha_{m\,n}\beta_n)\, w_n$

$\Rightarrow$ $[T(\nu) ; B_2] = [\alpha_{11}\beta_1 + \alpha_{12}\beta_2 + ... + \alpha_{1n}\beta_n, \alpha_{21}\beta_1 + \alpha_{22}\beta_2 + ...+ \alpha_{2n}\beta_n,$

$\alpha_{m\,1}\beta_1 +.... + \alpha_{m\,n}\beta_n]$

$$= \begin{bmatrix} \alpha_{11}\beta_1 + & \alpha_{12}\beta_2 + & +...+ & \alpha_{1n}\beta_n \\ \alpha_{21}\beta_1 + & \alpha_{22}\beta_2 & +...+ & \alpha_{2n}\beta_n \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha_{m1}\beta_1 + & \alpha_{m2}\beta_2 + & +...+ & \alpha_{mn}\beta_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & ... & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & ... & \alpha_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ \alpha_{m1} & \alpha_{m2} & ... & \alpha_{mn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_n \end{bmatrix}$$

$= [T ; B_1, B_2]\,[\nu ; B_1]$ [Using (2) and (4)]

Hence $[T(\nu) : B_2] = [T ; B_1, B_2]\,[\nu ; B_1]$

i.e., $[T(\nu)] = [T]\,[\nu]$

Theorem : Let V(F) and W(F) be finite dimensional vector space of dimensions n and m respectively and T_1 and T_2 be any linear transformations from V into W.

Show that (i) $[T_1 + T_2] = [T_1] + [T_2]$

(ii) $[\lambda T_1] = \lambda [T_1], \lambda \in F$.

Proof. Since dim V = n and dim W = m

$\therefore$ suppose $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$ and $B_2 = \{w_1, w_2, ..., w_n\}$ be ordered basis of V and W respectively

Since $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V

$\Rightarrow$ $\nu_1, \nu_2, ..., \nu_n \in V$

$\Rightarrow$ each $T_1(\nu_1), T_1(\nu_2), ..., T_1(\nu_n) \in W$ can be uniquely written as linear combination of the elements of basis B_2.

Let $T_1(\nu_1) = \alpha_{11} w_1 + \alpha_{21} w_2 + ... + \alpha_m 1\, w_m$

$T_1(\nu_2) = \alpha_{12} w_1 + \alpha_{22} w_2 + ... + \alpha_m 2\, w_m$

....

....

$T_1(\nu_n) = \alpha 1_n\, w_1 + \alpha 2_n\, w_2 + ... + \alpha_m\, w_n$ for scalars α_{ij}'s $\in F$

i.e., $T_1(\nu_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$; j = 1, 2, ..., n ....(1)

$$\Rightarrow \quad [T_1 ; B_1. B_2] = [\alpha_{ij}]_{m\times m} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix}_{m\times n} \quad .... (2)$$

Similarly each $T_2(\nu_1)$, $T_2(\nu_2)$, ..., $T_2(\nu_n) \in W$ can be uniquely written as linear combination of the elements of basis B_2.

Let $T_2(\nu_1) = \beta_{11}w_1 + \beta_{21}w_2 + ... + \beta_m 1 w_m$

$T_2(\nu_2) = \beta_{12}w_1 + \beta_{22}w_2 + ... + \beta_m\, 2\, w_m$

....

....

$T_1(\nu_n) = \beta 1_n\, w_1 + \beta 2_n\, w_2 + ... + \beta_m\, w_n$ for scalars β_{ij}'s $\in F$

i.e., $T_1(\nu_j) = \sum_{i=1}^{m} \beta_{ij} w_i$; j = 1, 2, ..., n ....(3)

$$\Rightarrow \quad [T_1 ; B_1, B_2] = [\beta_{ij}]_{m\times n} = \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} \end{bmatrix}_{m\times n} \quad .... (4)$$

(i) We have $(T_1 + T_2)\{\nu_j) = T_1(\nu_j) + T_2(\nu_j)$ for $1 \leq j \leq n$

$$= \sum_{i=1}^{m} \alpha_{ij} w_i + \sum_{i=1}^{m} \beta_{ij} w_i$$

$$= \sum_{i=1}^{m} \left(\alpha_{ij} + \beta_{ij}\right) w_i$$

$\therefore$ $[T_1 + T_2 ; B_1 ; B_2] = [\alpha_{ij} + \beta_{ij}]_{m\times n}$ for $1 \leq i \leq m$

$= [\alpha_{ij}]_{m\times n} + [\beta_{ij}]_{m\times n}$

$= [T_1 ; B_1, B_2] + [T_2 ; B_1, B_2]$

Hence the result

(ii) We have $(\lambda T_1)(\nu_j)\ \lambda T_1(\nu_j)$ for $\lambda \in F$ and $1 \leq j \leq n$

$$= \lambda \left(\sum_{i=1}^{m} \alpha_{i\,j}\, w_i \right)$$

$$= \sum_{i=1}^{m} \left(\lambda\, \alpha_{i\,j} \right) w_i$$

$\therefore$ $[\lambda T_1 ; B_1, B_2] = [\lambda\, \alpha_{ij}]_{m\times n}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$

$= \lambda\, [\alpha_{ij}]_{m\times n}$

$= \lambda[T_1 ; B_1 , B_2]$

Hence the result

Theorem. Let U (F) and W (F) be three finite dimensional vector spaces of dimensions m, n, p respectively. Let $T_1 : U \to V$ and $T_2 : V \to W$ be linear transformations.

Prove that $[T_2 T_1] = [T_2] [T_1]$

Proof. Since dim U = m, dim V = n, dim W = p (given)

$\therefore$ suppose $B_1 = \{u_1, u_2, ..., u_m\}$

$B_2 = \{\nu_1, \nu_2, ..., \nu_n\}$

and $B_3 = \{w_1, w_2, ..., w_p\}$

be ordered basis of V, V, W respectively

Since $B_1 = \{u_1, u_2, ..., u_m\}$ is a basis of U

$\Rightarrow$ $u_1, u_2, ..., u_m \in U$

$\Rightarrow$ each $T_1(u_1), T_1(u_2), ..., T_1(u_m) \in V$ [$\because$ $T_1 : U \to V$ is a mapping]

can be uniquely written as a linear combination of elements of basis B_2

$\therefore$ let $T_1(uj) = \sum_{j=1}^{n} \alpha_j\ \nu_j$; $1 \leq j \leq m$ .... (1)

$\Rightarrow$ $[T_1 ; B_1, B_2] = [\alpha_{ij}]_{n\times m}$

And $B_2 = \{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V

$\Rightarrow$ $\nu_1, \nu_1, ..., \nu_n \in V$

$\Rightarrow$ each $T_2(\nu_1), T_2(\nu_2), ..., T_2(\nu_n) \in W$

[$\because$ $T_2 : V \to W$ is a mapping] can be uniquely written as a linear combination of elements of basis B_3

$\therefore$ let $T_2(v_r) = \sum_{t=1}^{p} \beta_{tr} w_t$; $1 \leq r \leq n$ .... (2)

$\Rightarrow$ $[T_2 ; B_2, B_3] = [\beta_{tr}]_{p \times n}$

Since $T_1 : U \to V$ and $T_2 : V \to W$ are linear mappings

$\Rightarrow$ $T_2 T_1$ is defined

[$\because$ range of T_1 = Domain of T_2]

and $T_2 T_1 : U \to W$ is a linear mapping

Now $(T_2 T_1)(u_j) = T_2(T_1(u_j))$

$$= T2\left(\sum_{i=1}^{n} \alpha_{ij} v_i\right) \qquad \text{[Using (1)]}$$

$$= \sum_{i=1}^{n} \alpha_{ij} T_2(v_i) \qquad [\because T^2 \text{ is linear}]$$

$$= \sum_{i=1}^{n} \alpha_{ij} \left(\sum_{t=1}^{p} \beta_{ti} w_i\right)$$

[Using (2) on replacing r by i]

$$= \sum_{t=1}^{p} \left(\sum_{i=1}^{n} \beta_{ti}\, \alpha_{ij}\right) w_i$$

$\therefore$ $[T_2 T_1 ; B_1 B_2] = \left(\sum_{i=1}^{n} \beta_{ti}\, \alpha_{ij}\right)$ where $1 \leq t \leq p$

$1 \leq j \leq m$

$= [\beta_{ti}]_{p \times n} [\alpha_{ij}]_{n \times m}$

$= [T_2 ; B_2, B_3] [T_1 ; B_1, B_2]$

i.e., $[T_2 T_1] = [T_2] [T_1]$

Hence the result

Theorem. If P be a matrix representation of an operator T on a vector space V (F). Prove that f (P) is the matrix representation of f (T) for any polynomial f(x) over F.

Proof. Let V (F) be a vector space of dimension n, and L (V, V) be the set of all linear operators on V and M be the set of all $n \times n$ matrices over F.

Let P be matrix of operator T relative to basis B i.e., [T ; B] = P

We define a map $G : L(V, V) \to M$ as

$G(T) = [T] = P$

To show $G(f(T)) = [f(T)] = f(P)$ .... (1)

Firstly we shall show that G is a Linear Transformation.

Let $T_1, T_2 \in L(V, V)$

and $[T_1] = P_1$ and $[T_2] = P_2$

Suppose $\alpha, \beta \in F$

Then $\alpha T_1 + \beta T_2 \in L(V, V)$

Now $G(\alpha T_1 + \beta T_2) = [\alpha T_1 + \beta T_2]$ (by def. of G)

$= [\alpha T_1] + [\beta T_2]$

$= \alpha [T_1] + \beta [T_2]$

$= \alpha P_1 + \beta P_2$

$= \alpha G(T_1) + \beta G(T_2)$

$\Rightarrow$ G is a L.T.

$\therefore$ $G(T_1 T_2) = [T_1 T_2] = [T_1][T_2] = G(T_1) G(T_2)$

Let $f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_2 x^2 + \ldots + \alpha_n x^n$

We shall prove the result (1) by induction on n.

Let $n = 0$ Then $f(x) = \alpha_0$ for some $\alpha_0 \in F$

$\Rightarrow$ $f(T) = \alpha_0 I'$ where I' is identity operator on V

and $f(P) = \alpha_0 I$ where I is identity matrix of order n.

Now $G(f(T)) = G(\alpha_0 I')$

$= [\alpha_0 I']$ [by def. of G]

$= \alpha_0 [I']$

$= \alpha_0 I$

($\because$ matrix of identity operator I' is identity matrix I)

$= f(P)$

i.e., $G(f(T)) = [f(T)] = f(P)$

$\Rightarrow$ the result holds for $n = 0$

Suppose the result (1) is true for all polynomials over F of degree less than n.

Now we shall show that the result (1) is true for polynomial f (x) of degree n.

Here $G (f(T)) = G (\alpha_0 I' + \alpha_1 T + ... + \alpha_{n-1} T^{n-1} \alpha_n T^n)$

$(\because G (T_1 + T_2) = G (T_1) + G (T_2))$

$= G (\alpha_0 I' + \alpha_1 T + ... + \alpha_{n-1} T^{n-1}) + \alpha_n G (T_n)$

$= G (\alpha_0 I' + \alpha_1 T + ... + \alpha_{n+1} T^{n-1}) + \alpha_n G (T) G (T^{n-1})$

$(\because G (T_1 T_2) = G (T_1) G (T_2))$

$= [\alpha_0 I' + \alpha_1 T + ... + \alpha_{n-1} T^{n-1}] + \alpha_n [T] [T^{n-1}]$

(by def. of G)

$= (\alpha_0 I + \alpha_1 P + ... + \alpha_{n-1} P^{n-1}) + \alpha_n (P) (P^{n-1})$

(because of (2))

$= \alpha_0 I + \alpha_1 P + ... + \alpha_{n-1} P^{n-1} + \alpha_n P^n$

$= f(P)$

$\Rightarrow$ $G (f(T)) = [f(T)] = f(P)$

$\Rightarrow$ $f(P)$ is the matrix representation of f (T)

i.e., result is true for polynomial $f(x)$ of degree n.

Hence the result

Some Illustrative Examples

Example 1: Let $T : R_3 \to R_2$ be the linear transformation defined by

$T (x, y, z) = (2x + y - z, 3x - 2y + 4z)$

Find the matrix of T relative to ordered basis

$B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

and $B_2 = \{(1, 3), (1, 4)\}$ of R3 and R2 respectively

Solution: Given $T = R^3 \to R^2$ defined by

$T(x, y, z) = (2x + y - z, 3x - 2y + 4z)$

and $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$B_2 = \{(1, 3), (1, 4)\}$ are ordered basis of R^3 and R^2 respectively.

To find $[T ; B_1, B_2]$

Firstly we shall express any vector $\nu = (\alpha, \beta)$ as a linear combination of the elements of basis B_2

Let $(\alpha, \beta) = \alpha (1, 3) + b (1, 4)$ for some scalars a and b

$= (a + b, 3a + 4b)$

$\Rightarrow$ $a + b = \alpha$ and $3a + 4b = \beta$

$\Rightarrow$ $\alpha = 4\alpha - \beta$ and $b = \beta - 3\alpha$

$\therefore$ $(\alpha, \beta) = (4\alpha - \beta)(1, 3) + (-3\alpha + \beta)(1, 4)$ ... (1)

Now $T(1, 1, 1) = (2 + 1 - 1, 3 - 2 + 4)$ [by def of T]

$= (2, 5)$

$= (8 - 5)(1, 3) + (-6 + 5)(1, 4)$ [Using (1)]

$= 3(1, 3) + (-1)(1, 4)$

$T(1, 1, 0) = (2 + 1 - 0, 3 - 2 + 0)$

$= (3, 1)$

$= (12 - 1)(1, 3) + (-9 + 1)(1, 4)$ [Using (I)]

$= 11(1, 3) + (-8)(1, 4)$

$T(1, 0, 0) = (2 + 0 - 0, 3 - 0 + 0)$

$= (2, 3)$

$= (8 - 3)(1, 3) + (-6 + 3)(1, 4)$ [Using (I)]

$= 5(1, 3) + (-3)(1, 4)$

$$\therefore \quad [T ; B_1, B_2] = \begin{bmatrix} 3 & -1 \\ 11 & -8 \\ 5 & -3 \end{bmatrix}^t = \begin{bmatrix} 3 & 11 & 5 \\ -1 & -8 & -3 \end{bmatrix}$$

Example 2: Let $T : R^3 \to R2$ be the linear transformation defined by

$T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$

(i) Find the matrix of T in the following bases of R^3 and R^2 :

$B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$B_2 = \{(1, 3), (2, 5)\}$

(ii) Verify that the action of T is preserved by its matrix representation

i.e., $[T ; B_1, B_2]\,[\nu ; B_1] = [T(\nu) ; B_2]$ for all $\nu \in R^3$

Solution: Given $T : R^3 \to R^2$ defined by

$T(x, y, z) = (3x + 2y - 4z, x - 5y + 3z)$

and $B_1 = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$

$B_2 = \{(1, 3), (2, 5)\}$ be ordered basis for R^3 and R^2 respectively

(α) To find $[T ; B_1, B_2]$

Firstly we shall express any vector $\nu_1 = (\alpha, \beta) \in R^2$ as a linear combination of the elements of basis B_2.

Let $(\alpha, \beta) = \alpha (1, 3) + b (2, 5)$ for some scalars a, b

$= (a + 2b, 3a + 5b)$

$\Rightarrow$ $a + 2b = \alpha$ and $3a + 5b = \beta$

$\Rightarrow$ $a = -5 \alpha + 2\beta$ and $b = 3\alpha - \beta$

$\therefore$ $(\alpha, \beta) = (-5 \alpha + 2 \beta) (1, 3) + (3 \alpha - \beta) (2, 5)$ .... (1)

Now $T (1, 1, 1) = (3 + 2 - 4, 1 - 5 + 3)$ [by def. of T]

$= (1, - 1)$

$= (-5 -2) (1, 3) + (3 + 1) (2, 5)$ [Using (1)]

$= (-7) (1, 3) + 4 (2, 5)$

$T (1, 1, 0) = (3 + 2 - 0, 1 - 5 + 0)$

$= (5, - 4)$

$= (- 25 - 8) (1, 3) + (15 + 4) (2, 5)$ [Using (1)]

$= (- 33) (1, 3) + 19 (2, 5)$

$T (1, 0, 0) = (3 + 0 - 0, 1 - 0 1 - 0 + 0) = (3, 1)$

$= (- 15 + 2) (1, 3) + (9 - 1) (2, 5)$ [Using (1)]

$= (- 13) (1, 3) + 8 (2, 5)$

$$\therefore \quad [T ; B_1, B_2] = \begin{bmatrix} -7 & 4 \\ -33 & 19 \\ -13 & 8 \end{bmatrix}^t = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$$

(ii) To verify [T ; B1, B2] [ν ; B1] = [T (ν) ; B2] for all $\nu \in$ R3

To find $[\nu ; B_1]$

Let $\nu = \{x, y, z) \in R^3$

Suppose $\nu = p (1, 1, 1) + q (1, 1, 0) + r (1, 0, 0)$ for some scalars p, q, r

$\Rightarrow$ $(x, y, z) = (p + q + r, p + q, p)$

$\therefore$ $p + q + r = x, p + q = y, p = z$

$\Rightarrow$ $p = z, q = y - z, r = x - y$

$\therefore \qquad \nu = z\,(1, 1, 1) + (y - z)\,(1, 1, 0) + (x - y)\,(1, 0, 0) \qquad .... (2)$

$\Rightarrow \qquad [\nu\ ;\ B_1] = [z\ \ y - z\ \ x - y]^t = \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$

To find $[T(\nu)\ ;\ B_2]$

Here $\quad T(\nu) = T(x, y, z)$ for all $\nu = (x, y, z) \in R^3$

$= (3x + 2y - yz,\ x - 5y + 3z)$

$\Rightarrow \qquad T(\nu) = (-5\,(3x + 2y - 4z) + 2\,(x - 5y + 3z)\,(1, 3)$

$+ (3\,(3x + 2y - 4z) - (x - 5y + 3z))\,(2, 5) \qquad$ [Using (1)]

$= (-13x - 20y + 26z)\,(1, 3) + (8x + 11y - 15z)\,(2, 5)$

$\therefore \qquad [T[\nu)\ ;\ B_2] = -13x - 20y + 26z\ \ 8x + 11y - 15z]$

$$= \begin{bmatrix} -13x - 20y + 26z \\ 8x + 11y - 15z \end{bmatrix}$$

L.H.S. $= [T\ ;\ B_1, B_2]\,[\nu\ ;\ B_1]$

$$= \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix}$$

$$= \begin{bmatrix} -7z - 33(y-z) - 13(x-y) \\ 4z + 19(y-z) + 8(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} -13x - 20y + 26z \\ 8x + 11y - 15z \end{bmatrix} = [T(\nu)\ ;\ B_2] = \text{R.H.S.}$$

Hence the result

14.4 Self Check Exercise - 1

Q. 1. Let $T : V \rightarrow W$ be a linear transformation defined as

$$T[F(x)] = \int_1^n f(t)\ dt,$$

$V = \{F(x)\ :\ F(x)$ is a polynomial over R and def $F(x) < 3$ or $F(x) = 0\}$

Let $\quad B_1 = \{1, 1 + x, 1 - x + x^2\}$

$B = \{1, x, x^2, x^3\}$

Find the matrix representation of T relative of basis B1 and B2.

Q. 2 Find a linear map

$T : R^2 \to R^3$ determined by the matrix

$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 3 \end{bmatrix}$ w.r.t., the ordered basis

$B_1 = \{ (1, 2), (0, 3) \}$ and

$B_2 = \{ (1, 1, 0), (0, 1, 1), (1, 1, 1) \}$

For R^2 and R^3 respectively

14.5 Change of coordinate Matrix

Def. Transion Matrix

Let $B_1 = \{\nu_1, \nu_2, ...\nu_n\}$

$B_2 = \{w_1, w_2, ...w_n\}$ be bases of vector space V(F)

$\therefore$ $w_1, w_2, ...w_n \in V$ and B_1 is a basis.

$\Rightarrow$ Each w_j can be expressed as a linear combination of the elements of basis B_1.

Let $w_1 = \alpha_{11}\, \nu_1 + \alpha_{21}\, \nu_2 + ... + \alpha_{n\,1}\, \nu_n$

$w_2 = \alpha_{12}\, \nu_1 + \alpha_{22}\, \nu_2 + ... + \alpha_{n\,2}\, \nu_n$

....

....

$w_n = \alpha_{1n}\, \nu_1 + \alpha_{2n}\, \nu_2 + ... + \alpha_{n\,n}\, \nu_n$

i.e., $w_j = \sum_{p=1}^{n} \alpha_{p\,j}\, v_p$; $1 \leq j \leq n$

The coefficient matrix of above equations is

$$\Rightarrow \quad [T ; B_1, B_2] = [\alpha_{i\,j}]_{m \times n} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \ldots & \alpha_{n1} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{1n} & \alpha_{2n} & \ldots & \alpha_{nn} \end{bmatrix}$$

Then $P = \{\alpha_{i\,j}\}$, which is the transpose of above coefficient matrix is called transition matrix from the basis B_1 to the basis B_2.

Theorem : Let $B_1 = \{\nu_1, \nu_2, \ldots.\nu_n\}$ and $B_2 = \{w_1, w_2, \ldots.w_n\}$ be two ordered basis of V(F).

Prove that the transition matrix from the basis B_1 to B_2 is invertible.

Proof : Given $B_1 = \{\nu_1, \nu_2, \ldots.\nu_n\}$

and $B_2 = \{w_1, w_2, \ldots.w_n\}$ be basis of V(F)

Let $w_j = \sum_{p=1}^{n} \alpha_{p\,j}\, v_{\,p}$, $1 < j < n$ ....(1)

and $\nu_1 = \sum_{j=1}^{n} \beta_{j\,i}\, w_j$, $1 < i < n$ ....(2)

Then $P = [\alpha_{ij}]$ and $Q = [\beta_{ij}]$ are transition matrices from basis B_1 to B_2 and B_2 to B_1 respectively.

We want to show that $PQ = QP = 1_n$

where 1_n is unit matrix of order n.

Now $\nu_1 = \sum_{j=1}^{n} \beta_{j\,i}\, w_j = \sum_{j=1}^{n} \beta_{j\,i} \left(\sum_{p=1}^{n} \alpha_{p\,j}\, v_{\,p} \right)$

$= \sum_{p=1}^{n} \left(\sum_{j=1}^{n} \alpha_{p\,j}\, \beta_{\,j\,i} \right) \nu_p$

$= \sum_{p=1}^{n} \gamma_{pi}\, v_{\,p}$ where $\gamma_{\,pi} = \sum_{j=1}^{n} \alpha_{p\,j}\, \beta_{\,ji}$

$\Rightarrow$ $\nu_1 = \sum_{p=1}^{n} \gamma_{pi}\, v_{\,p}$

Since B1 is a basis, so that $\nu_1, \nu_2, \ldots.\nu_n$ are L.I.

$\Rightarrow$ $\gamma_{i\,i} = 1$ and $\gamma_{p\,i} = 0$ for $p \neq i$

$\therefore$ $PQ = [\gamma_{p\,i}] = I_n$

Hence the result.

Note. The inverse of matrix P, i.e., P^{-1} is the transition matrix from basis B_2 to B_1.

Theorem : Let $B_1 = \{\nu_1, \nu_2, \ldots.\nu_n\}$ and $B_2 = \{w_1, w_2, \ldots.w_n\}$ be two ordered basis of V(F). If P is transition matrix from bases B_1 to B_2. Then prove $P\,[\nu; B_2] = [\nu; B_1]$ for all $\nu \in V$.

Proof : Let $P = [\alpha_{ij}]$ be transition matrix from basis B_1 to B_2

so that $w_j = \sum_{p=1}^{n} \alpha_{p\,j}\, v_{\,p}$, $1 \leq j \leq n$ ...(1)

Take $\nu = \sum_{j=1}^{n} \alpha_j w_j$ for scalar $a_j \in F$ ....(2)

$\therefore \quad [\nu; B_2] = [a_1\ a_2\ \dots\ a_n]^t = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

so that L.H.S. = $P[\nu; B_2] = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

$$= \begin{bmatrix} a_1\alpha_{11} + a_2\alpha_{12} + \dots + a_n\alpha_{1n} \\ a_1\alpha_{21} + a_2\alpha_{22} + \dots + a_n\alpha_{2n} \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_1\alpha_{n1} + a_2\alpha_{n2} + \dots + a_n\alpha_{nn} \end{bmatrix}$$

Further $\nu = \sum_{j=1}^{n} \alpha_j w_j = \sum_{j=1}^{n} \alpha_j \left(\sum_{p=1}^{n} \alpha_{pj} v_p \right)$ [Using (1) and (2)]

$$= \sum_{p=1}^{n} \left(\sum_{j=1}^{n} \alpha_j \alpha_{pj} \right) \nu_p$$

$$= \left(\sum_{j=1}^{n} \alpha_j \alpha_{1j} \right) \nu_1 + \left(\sum_{j=1}^{n} \alpha_j \alpha_{2j} \right) \nu_2 + \dots + \left(\sum_{j=1}^{n} \alpha_j \alpha_{nj} \right) \nu_n$$

$\therefore \quad$ R.H.S. = $[\nu; B_1]$

$$= \left[\sum_{j=1}^{n} \alpha_j \alpha_{1j}, \sum_{j=1}^{n} \alpha_j \alpha_{2j}, \dots, \sum_{j=1}^{n} \alpha_j \alpha_{nj} \right]^t$$

$$= \begin{bmatrix} \sum_{j=1}^{n} \alpha_j \, \alpha_{1j} \\ \sum_{j=1}^{n} \alpha_j \, \alpha_{2j} \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ \sum_{j=1}^{n} \alpha_j \, \alpha_{nj} \end{bmatrix} = \begin{bmatrix} a_1\alpha_{11} + a_2\alpha_{12} + \ldots + a_n\alpha_{1n} \\ a_1\alpha_{21} + a_2\alpha_{22} + \ldots + a_n\alpha_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1\alpha_{n1} + a_2\alpha_{n2} + \ldots + a_n\alpha_{nn} \end{bmatrix}$$

$\therefore$ L.H.S. = R.H.S.

Hence Theorem is proved.

Theorem : Let V be a finite dimensional vector space over F and T : V → V be a linear operator. Let [T; B_1] = P and [T; B_2] = Q where B_1, B_2 are two ordered bases of V (F) prove that Q = A^{-1} PA where A is transition matrix from basis B_1 to B_2.

Proof : Let $B_1 = \{\nu_1, \nu_2, \ldots.\nu_n\}$ and $B_2 = \{w_1, w_2, \ldots.w_n\}$ be two ordered basis of V(F).

Take $P = [p_{ij}]_{n\times n}$ so $T(\nu_j) = \sum_{i=1}^{n} p_{ij} \, v_i$, $1 \leq j \leq n$

and $Q = [q_{kj}]_{n\times n}$ so $T(w_j) = \sum_{i=1}^{n} q_{ij} \, w_i$, $1 \leq j \leq n$

Take $w_j = \sum_{\ell=1}^{n} a_{\ell j} \, v_\ell$ so $A = [a_{ij}]_{n\times n}$ is matrix from basis B_1 to B_2.

and $\nu_i = \sum_{K=1}^{n} b_{ki} w_j$ so $A^{-1} = [b_{ij}]_{n\times n}$ is matrix from basis B_2 to B_1.

$$\therefore \quad T(w_j) = T\left(\sum_{\ell=1}^{n} a_{\ell j} \, v_\ell\right) = \sum_{2=1}^{n} a_{\ell j} \, T(v_\ell)$$

$$= \sum_{\ell=1}^{n} a_{\ell j} \left(\sum_{2=1}^{n} p_{i\ell} \, v_i\right)$$

$$= \sum_{\ell=1}^{n} \sum_{i=1}^{n} a_{\ell j} \, p_{i\ell} \left(\sum_{k=1}^{n} b_{ki} w_j\right)$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki} \, p_{i\ell} \, a_{\ell j}\right) \mathrm{w}_j \qquad \ldots.(1)$$

Already we have T $(w_j) = \sum_{j=1}^{n} q_{ij} w_i$ ...(2)

From (1) and (2), we have $q_{ij} = \sum_{i=1}^{n}\sum_{K=1}^{n} b_{ki} p_{i\ell} a_{\ell j}$

$\Rightarrow$ $Q = A^{-1} PA$

Hence proved.

Note : The above theorem can also be stated as :

Let V be a finite dimensional vector space over F and T : V $\rightarrow$ V be a linear operator.

Let $[T; B_1] = P$ and $[T; B_2] = Q$ where B_1, B_2 are two ordered bases of V(F). Prove P and Q are similar matrices.

($\because$ Two matrices P, Q (of same order) are similar if $\exists$ invertible matrix A sit $Q = A^{-1} PA$).

Some Illustration Examples

Example 3 : Consider the following bases of R^2

$B_1 = \{(1, 0), (0, 1)\}$, $B_2 = \{(1, 2), (2, 3)\}$

(a) Find the transition matrices P and Q from basis B_1 to B_2 and B_2 to B_1 respectively. Verify $Q = P^{-1}$.

(b) Show $[\nu : B_1] = P [\nu; B_2]$ for any vector $\nu \in R^2$.

(c) Show $[T : B_2] = P^{-1} [T; B_1] P$

where (i) T (x, y) = (2x - 3 y, x + y)

(ii) T (x, y) = (5 x + y, 3 x - 2 y)

(d) Verify that (i) $[T ; B_2] [\nu ; B_2] = [T (\nu) ; B_2]$

(ii) $[T ; B_1] [\nu ; B_1] = [T (\nu) ; B_1]$

Solution : Let $B_1 = \{(1, 0), (0, 1)\} = \{\nu_1, \nu_2\}$

$B_2 = \{(1, 2), (2, 3)\} = \{w_1, w_2\}$

(a) To find transition matrix P from basis B_1 to B_2

Now $w_1 = (1, 2) = 1 (1, 0) + 2 (0, 1) = 1 . \nu_1 + 2 . \nu_2$

$w_2 = (2, 3) = 2 (1, 0) + 3 (0, 1) = 2 . \nu_1 + 3 . \nu_2$

$\therefore$ $P = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^t = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

which is the transition matrix from basis B_1 to B_2.

To find transition Matrix Q from basis B_2 to B_1

Let $(a, b) \in R^2$

Let $(a, b) = \alpha (1, 2) + \beta (2, 3)$

$= (\alpha + 2 \beta), 2 \alpha + 3 \beta)$

$\therefore \quad \alpha + 2 \beta = \alpha$ and $2 \alpha + 3 \beta = b$

$\alpha = - 3 a + 2 b$ and $\beta = 2 a - b$

$\therefore \quad (a, b) = (- 3 a + 2 b) (1, 2) + (2 a - b) (2, 3)$ ....(1)

Now $\quad \nu_1 = (1, 0) = (-3 + 0) (1, 2) + (2 -0) (2, 3)$ [Using (1)]

$= - 3 (1, 2) + 2 (2, 3)$

$\nu_2 = (1, 0) = (0 + 2) (1, 2) + (0 -1) (2, 3)$

$= 2 (1, 2) + (-1) (2, 3)$

$\therefore \quad Q = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}^t \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$ which is the transition matrix from B_2 to B_1.

To verify $Q = P^{-1}$

Here $PQ = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} -3+4 & 2-2 \\ -6+6 & -4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

and $QP = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} -3+4 & -6+6 \\ 2-2 & -4-3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$\therefore \quad PQ = QP = 1$

Hence $Q = P^{-1}$

(b) Let $\nu = (a, b) \in R^2$

Then $\nu = (a, b) = a (1, 0) + b (0, 1) = a \nu_1 + b \nu_2$

$\therefore \quad [\nu ; B_1] = [a\ b]^t = \begin{bmatrix} a \\ b \end{bmatrix}$.

And $\nu = (a, b) = (-3 a + 2 b) (1, 2) + (2 a - b) (2, 3)$

[because of (1)]

$= (- 3 a + 2 b) w_1 + (2 a - b) w_2$

$\therefore \quad [\nu ; B_2] = [-3 a + 2 b\ 2 a - b]^t = \begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$

R.H.S. $= P [\nu : B_2]$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$$

$$= \begin{bmatrix} -3a+2b+4a-2b \\ -6a+4b+6a-3b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = [\nu \,;\, B_1] = \text{L.H.S.}$$

Hence P $[\nu \,;\, B_2] = [\nu \,;\, B_1]$

(c) (i) Given T(x, y) = (2 x - 3 y, x + y)

To find $[T \,;\, B_1]$

Here T (1, 0) = (2, 1) = (1, 0) + 1 (0, 1) = 2 $\nu_1 + \nu_2$

T (0, 1) = (-3, 1) = -3 (1, 0) + 1 (0, 1) = - 3 $\nu_1 + \nu_2$

$$\therefore \quad [T \,;\, B_1] = \begin{bmatrix} 2 & 1 \\ -3 & 1 \end{bmatrix}^t = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$$

To find $[T \,;\, B_2]$

Here T (1, 2) = (-4, 3) [by def of T]

= (12 + 6) (1, 2) + (-8 -3) (2, 3) [Using (1)]

= 18 (1, 2) + (-11) (2, 3) = 18 w_1 + (-11) w_2.

T (2, 3)= (-5, 5)

= (15 + 10) (1,2) + (-10 -5) (2, 3) = 25 w_1 + (- 15) w_2

$$\therefore \quad [T \,;\, B_2] = \begin{bmatrix} 18 & -11 \\ 25 & -25 \end{bmatrix}^t = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix}$$

Now R.H.S. = $P^{-1} [T \,;\, B_1] P$

$$= \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 & 9+2 \\ 4-1 & -6-1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 \\ 3 & -7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+22 & -8+33 \\ 3-14 & 6-21 \end{bmatrix} = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix}$$

= $[T \,;\, B_2]$ = L.H.S.

Hence the result.

(ii) Do yourself.

(d) (i) Here T (ν) = T (a, b)

= (2 a - 3 b, a + b) (by def to T)

= (-6 a + 9 b + 2 a + 2 b) (1, 2) + (4 a - 6 b - a - b) (2, 3)

[Using (1)]

= (-4 a + 11 b) (1, 2) + (3 a - 7 b) (2, 3)

$\therefore$ $[T(\nu) ; B_2] = \begin{bmatrix} -4a+11b \\ 3a-7b \end{bmatrix}$

Now $[T ; B_2] [\nu ; B_2] = \begin{bmatrix} 18 & 25 \\ -11 & -15 \end{bmatrix} \begin{bmatrix} -3a+2b \\ 2a-b \end{bmatrix}$

$= \begin{bmatrix} -54a+36b+50a-25b \\ 33a-22b-30a+15b \end{bmatrix}$

$= \begin{bmatrix} -4a+11b \\ 3a-7b \end{bmatrix}$

$= [T(\nu) ; B_2]$

Hence the result.

(ii) T (ν) = T (a, b) = (2 a - 3 b, a + b) [by def of T]

= (2 a - 3 b) (1, 0) + (a + b) (0, 1)

$\therefore$ $[T(\nu) ; B_1] = \begin{bmatrix} 2a-3b \\ a+b \end{bmatrix}$

Now $[T ; B_1] [\nu ; B_1] = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$= \begin{bmatrix} 2a-3b \\ a+b \end{bmatrix}$

$= [T(\nu) ; B_1]$

Hence the result.

14.6 Self Check Exercise - 2

Q. 1 Let $T : R^3 \rightarrow R^3$ be a linear operator defined by

T (x, y, z) = (x + y + z, x + y, z)

If $[T ; B_1] = P$ and $[T ; B_2] = Q$

Where

$B_1 = \{ (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$

and

$B_2 = \{ (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ are basis of R^3.

Then show that P, Q are similar by binding an invertible matrix A such that $Q = A^{-1} PA$.

Q. 2 Let $S = \{(1, 2), (0, 1)$ and $T = \{(1, 1), (2, 3)\}$ be basis for R^2. What is the transion matrix from basis T to basis S.

14.7 Summary

We have learnt the following concepts in this unit :

(i) matrix of an inverse operator.

(ii) change of coordinate matrix where we have studied transion matrix

14.8 Glossary

1. If B_1 and B_2 are two ordered basis of V(F) then transion matrix from B_1 to B_2 is invertible.

14.9 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 $[T ; B_1, B_2] = \begin{bmatrix} -1 & -3/2 & -5/6 \\ 1 & 1 & -1 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/3 \end{bmatrix}$

Ans. 2. $(x, y) = \left(\dfrac{-4x+5y}{3}, \dfrac{x+4y}{3}, \dfrac{2x+2y}{3} \right)$

Self Check Exercise - 2

Ans.1 Compute T^{-1} = , find A^{-1} PA,

$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

then P, Q will be similar as $\exists$ an invertible matrix A s.t. $Q = A^{-1} PA$.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Ans. 2 $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

14.10 Reference/Suggested Reading

1. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.
2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
3. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

14.11 Terminal Questions

1. If $B_1 = \{1, i\}$

   $B_2 = \{1+i, 2i+1\}$ are basis of vector space C(R). Find the transition matrix A from B_1 to B_2 and matrix B from B_2 to B_1.

2. Let $T : R^3 \rightarrow R^3$ be a linear operator defined as

   T (x, y, z) = (3x + z, -2x + y, -x + 2y + 4z)

   (i) Find matrix T relative to basis

   $B_1 = \{(1, 0, 0), (0, 1, 0) (0, 0, 1)\}$ for R^3

   & $B_2 = \{(1, 0, 1), (-1, 2, 1) (2, 1, 1)\}$ for R^3

   (i) Also find transion matrix P from B_1 to B_2 and verify that

   $P^{-1} [T ; B_1] P = [T ; B_2]$

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Unit - 15

Algebra of Linear Transformations

Structure

15.1 Introduction

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15.1 Introduction

Dear students, we have already learnt the concept of linear transformation, rank and nullity of L.T., matrix of linear transformations etc. in our previous units. Now, in this unit we shall learn the concept of algebra of linear transformation where we shall discuss about algebra of linear transformation (properties due to addition and scalar multiplication) and product to two L.Ts.

15.2 Learning Objectives

The main Learning objectives of this unit are

(i) to study algebra of linear transformation where properties of addition, scalar multiplication etc are studied.

(ii) to discuss product of two linear transformations

(iii) some theorems related to above two concepts are also proved.

15.3 Algebra of Linear Transformation

Let V and W be vector spaces over the same field F. Let L (V, W) be the set of all linear transformations from V to W. We share, now see that $L(\nu, w)$ is also a vector space over the same field F.

Theorem : Prove that the set L (V, W) or Hom. (V, W) of all linear transformations from V (F) into W (F) is a vector space over the field F with addition scalar multiplication defined by

$(T_1 + T_2)(x) = T_1(x) + T_2(x) \ \forall \ x \in V$ and $T_1, T_2 \in L(V, W)$

$(\alpha T_1)(x) = \alpha T_1(x) \ \forall \ x \in V$ and $\alpha \in F, T_1 \in L(V, W)$

Proof : To show that L (V, W) is a vector space, we have to verify all the properties of a vector space.

1. Properties due to addition:

(a) Closure property. Let $T_1 : V \to W$ and $T_2 : V \to W$ be two linear transformations. Then to show that $T_1 + T_2$ defined as

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for all $x \in V$

is also a linear transformation

Since $T_1, T_2 \in L(V, W)$

i.e., T_1 and T_2 are transformation from V into W

$\therefore$ $T_1(x), T_2(x) \in W$

$\Rightarrow$ $T_1(x) + T_2(x) \in W$ [$\because$ W is a vector space]

so $T_1 + T_2 : V \to W$ is a mapping

Let $\alpha, \beta, \in F$ and $y, z \in V$

$\Rightarrow$ $\alpha y + \beta z \in V$ [$\because$ V is a vector space]

$\therefore$ $(T_1 + T_2)(\alpha y + \beta z) = T_1(\alpha y + \beta z) + T_2(\alpha y + \beta z)$

$= [\alpha T_1(y) + \beta T_1(z) + [\alpha T_2(y) + \beta T_2(z)]$

[$\because$ T_1 and T_2 are linear transformations]

$= \alpha (T_1(y) + T_2(y)) + \beta (T_1(z) + T_2(z))$

$= \alpha[(T_1 + T_2)(y)] + \beta [(T_1 + T_2)(z)]$

$= \alpha [(T_1 + T_2)(y)] + \beta [T_1 + T_2)(z)]$

$= \alpha (T_1 + T_2)(y) + \beta (T_1 + T_2)(z)$

$\therefore$ $T_1 + T_2 : V \to W$ is a linear transformation

Hence closure property is verified

i.e., $T_1, T_2 \in L(V, W) \Rightarrow T_1 + T_1 \in L(V, W)$

(b) Commutative Property.

For each $T_1, T_2 \in L(V, W)$ and $x \in V$, we have

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$

$= T_2(x) + T_1(x)$ [$\because$ Addition is commutative in W]

$= (T_2 + T_1)\ (x)$

Hence $T_1\ T_2 = T_2 + T_1$, thus commutative property is verified

(c) Associative Property.

For each $T_1\ T_{22}, T_3 \in (V, W)$ and $x \in V$

$(T_1 + T_2) + T_3]\ (x) = (T_1 + T_2)\ (x) + T_3\ (x)\ \forall\ x \in V$

$= [T_1\ (x) + T_2\ (x)] + T_2\ (x)] + T_3\ (x)$

$= T_1\ (x) + [T_2\ (x) + T_3\ (x)]$ [$\because$ Addition is associative in W]

$= T_1\ (x) + (T_2 + T_3)\ (x)$

$= [T_1 + (T_2 + T_3)]\ (x)$

$\therefore$ $(T_1 + T_2) + T_3 = T_1 + (T_2 + T_3)$

Hence associative property holds in L (V, W)

(d) Existence of addative identity. Let us define a zero mapping

$O : V \rightarrow W$ as

$O\ (x) = 0\ \forall\ x \in V$

Let $\alpha, \beta \in F$ and $x, y \in V$

$\Rightarrow$ $\alpha\ x + \beta\ y \in V$ [$\because$ V is a vector space]

$\therefore$ $O\ (\alpha\ x\ \beta\ y) = 0$ [by def. of zero mapping]

$= \alpha.\ 0 + \beta.\ 0 = \alpha\ O\ (x) + \beta\ O\ (y)$

so that $O : V \rightarrow W$ is a Linear Transformation

i.e., $O \in L\ (V, W)$

Now for all $T_1, O \in L\ (V, W)$

$(T_1 + O)\ (x) = T_1\ (x) + O\ (x)\ \forall\ x \in V$

$= T_1\ (x) + 0 = T_1\ (x)$

$\therefore$ $(T_1 + O)\ (x) = T_1\ (x)\ \forall\ x \in V$

And $(O + T_1)\ (x) = O\ (x) + T_1\ (x)\ \forall\ x \in V$

$= 0 + T_1\ (x) = T_1\ (x)$

$\therefore$ $(O + T_1)\ (x) = T_1\ (x)\ \forall\ x \in V$

Hence $T_1 + O = T_1 = O + T_1$

$\therefore$ O is additive identity for L (V, W)

(e) Existence of Additive Inverse.

For each $T_1 \in L\ (V, W)$, we define a mapping,- $T_1 : V \rightarrow W$ as $(-T_1)\ (x)\ \forall\ x \in V$

To show, $-T_1 : V \to W$ is linear mapping.

Let α, '$\beta \in F$ and $x, y \in V$

$\Rightarrow$ $\alpha x + \beta y \in V$ [$\because$ V is a vector space]

$\therefore$ $(-T_1)(\alpha x + \beta y) = -T_1(\alpha x + \beta y)$ [by def. of '$-T_1$']

$= -[\alpha T_1(x) + \beta T_1(y)]$ [$\because$ T1 is linear]

$= -\alpha T_1(x) - \beta T_1(y)$

$= \alpha(-T_1)(x) + \beta(-T_1)(y)$

$\therefore$ $-T_1 : V \to W$ is a linear transformation

so that $-T_1 \in L(V, W)$.

Now $[T_1 + (-T_1)](x) = T_1(x) + (-T_1)(x) \ \forall x \in V$

$= T_1(x) - T_1(x) = 0 \in W$

$= O(x)$

$\therefore$ $[T_1 + (-T_1)](x) = O(x) \ \forall x \in V$

And $[(-T_1) + T_1](x) = (-T_1)(x) + T_1(x) \ \forall \in V$

$= -T_1(x) + T_1(x) = 0 \in W$

$= O(x)$

$\therefore$ $[(-T_1) + T_1](x) = O(x) \ \forall x \in V$

Thus $T_1 + (-T_1) = O = (-T_1) + T_1$

Hence $-T_1$ is the additive inverse of T_1

II. Properties due to scalar multiplication

(f) Closure Property. We have,

$(\alpha T_1)(x) = \alpha T_1(x) \ \forall x \in V, \alpha \in F$

Since $T_1 \in L(V, W)$

$\Rightarrow$ T_1 is a mapping from V into W

$\Rightarrow$ $T_1(x) \in W$ for all $x \in V$

$\Rightarrow$ $\alpha T_1 : V \to W$ is a transformation

Let $s, t \in F$ and $x, y \in V$

$\Rightarrow$ $s x + t y \in V$ [$\because$ V is a vector space]

$\therefore$ $(\alpha T_1)(s x + t y) = \alpha . T_1(s x + t y)$

$= \alpha . [s\ T_1 (x) + t\ T_1 (y)]$ $\quad$ [$\because$ T_1 is linear]

$= \alpha\ [s\ T_1 (x)] + \alpha\ [t\ T_1 (y)]$

$= (\alpha\ s)\ T_1 (x) + (\alpha\ t)\ T_1 (y)$

$= (s\ \alpha)\ T_1 (x) + (t\ \alpha)\ T_1 (y)$

$= s\ (\alpha\ T_1 (x)) + t\ (\alpha\ T_1 (y))$

$= s\ [(\alpha\ T_1) (x)] + t\ [(\alpha\ T_1) (y)]$

$= s\ (\alpha\ T_1) (x) + t\ (\alpha\ T_1) (y)$

$= s\ (\alpha\ T_1) (x) + t\ (\alpha\ T_1) (y)$

$\therefore$ $\alpha\ T_1$ is a linear transformation from V into W

Hence $\alpha\ T_1 \in L\ (V, W)$

(g) For each $T_1 \in L\ (V, W)$, $\alpha, \beta \in F$, we have

$[(\alpha + \beta)\ T_1] (x) = (\alpha + \beta)\ T_1 (x)\ \forall\ x \in V$

$= \alpha\ T_1 (x) + \beta\ T_1 (x)$

[By Property of scalar multiplication]

$= (\alpha\ T_1) (x) + (\beta\ T_1) (x)$

$= [\alpha\ T_1 + \beta\ T_1] (x)$

$\therefore$ $(\alpha + \beta)\ T_1 = \alpha\ T_1 + \beta\ T_1$

(h) For each $T_1, T_2 \in L\ (V, W)$, $\alpha \in F$, we have

$[\alpha\ (T_1 + T_2)] (x) = \alpha\ (T_1 + T_2) (x)\ \forall\ x \in V$

$= \alpha\ [T_1 (x) + T_2 (x)]$

$= \alpha\ T_1 (x) + \alpha\ T_2 (x)$

[By distributive law in W]

$= (\alpha\ T_1) (x) + (\alpha\ T_2) (x)$

$= [\alpha\ T_1 + \alpha\ T_2] (x)$

$\therefore$ $\alpha\ (T_1 + T_2) \quad = \alpha\ T_1 + \alpha\ T_2$

(i) For each $T_1 \in L\ (V, W)$ and $\alpha, \beta \in F$, we have

$[(\alpha\ \beta)\ T_1] (x) = (\alpha\ \beta)\ T_1 (x)\ \forall\ x \in V$

$= \alpha\ [\beta\ (T_1 (x)]$

$= \alpha\ [(\beta\ T_1) (x)]$

$$= [\alpha (\beta T_1)] (x)$$

$\therefore \quad (\alpha \beta) T_1 = \alpha (\beta T_1)$

(ii) For each $T_1 \in L (V, W)$ there exists $1 \in F$ such that

$$(1 . T_1) (x) = 1. T_1 (x) \ \forall\ x \in V$$

$$= T_1 (x)$$

$\therefore \quad 1. \ T_1 = T_1$

Thus all the properties of vector space are satisfied by elements of L (V, W)

$\therefore$ L (V, W) is a vector space over F.

Cor. The set L (V, V) of all linear operators on V i.e., linear transformations V into V forms a vector space with respect to addition and scalar multiplication compositions defined above.

Proof. Replace W by V in the above given proof.

Note. L (V, W) is defined only when V and W are vector spaces over the same field.

Theorem : Prove that, if V (F) and W (F) are finite dimensional, then the vector space of all linear transformations from V to W is also a finite dimensional and its dimension is equal to [dim. (V)] [dim. W)].

Proof. We know

$$L (V, W) = \{T \mid T : V \rightarrow W \text{ is a L.T., } T (x) \in W \text{ for all } x \in V\}$$

Let $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$ and $B_2 = \{w_1, w_2, ..., w_m\}$ be basis sets for V (F) and W (F) respectively,

Then dim. V = n and dim. W = m.

Now, define a mapping T_{ij} as

$T_{ij} : V \rightarrow W$ such that for $1 \leq i \leq n$, $1 \leq j \leq m$

$$T_{ij} (\nu p) = \begin{cases} w_j, i = p \\ 0, i \neq p \end{cases}$$

We shall prove that T_{ij} is linear transformation

Let $x_1, x_2 \in V$ and $\alpha\ \beta, \in F$

$\therefore \quad x_1 = \sum_{p=1}^{n} \alpha_p v_p$ and $x_2 = \sum_{p=1}^{n} \beta_p v_p$

[$\because$ B_1 is a basis of V (F)]

$$\Rightarrow \quad \alpha x_1 + \beta x_2 = \alpha \left(\sum_{p=1}^{n} \alpha_p v_p \right) + \beta \left(\sum_{p=1}^{n} \beta_p v_p \right)$$

$$= \sum_{p=1}^{n}(\alpha\alpha_p)v_p + \sum_{p=1}^{n}(\beta\beta_p)v_p$$

$$= \sum_{p=1}^{n}(\alpha\alpha_p + \beta\beta_p)v_p \qquad \text{... (i)}$$

Also $T_{ij}(x_1) = T_{ij}\left(\sum_{p=1}^{n}\alpha_p v_p\right) = \alpha_i w_j$ [by def. of T_{ij}]

and $T_{ij}(x_2) = T_{ij}\left(\sum_{p=1}^{n}\beta_p v_p\right) = \beta_i w_j$ [by def. of T_{ij}]

$\therefore$ $T_{ij}(\alpha x_1 + \beta x_2) = T_{ij}\left[\sum_{p=1}^{n}(\alpha\alpha_p + \beta\beta_p)v_p\right]$ [Using (i)]

$= (\alpha\alpha_i + \beta\beta_i) w_j$ [by def. of T_{ij}]

$= \alpha\alpha_i w_j + \beta\beta_i w_j$

$= \alpha(\alpha_i w_j) + \beta(\beta_i w_j)$

$$= \alpha\left(T_{ij}\left(\sum_{p=1}^{n}\alpha_p v_p\right)\right) + \beta\left(T_{ij}\left(\sum_{p=1}^{n}\beta_p v_p\right)\right)$$

$= \alpha T_{ij}(x_1) + \beta T_{ij}(x_2)$

$\therefore$ T_{ij} is a linear transformation from V into W

Hence $T_{ij} \in L(V, W)$

Now if we fix j and vary i, we get $T_{1j}, T_{2j}, ..., T_{nj}$ as n linear transformations and if now we vary j from 1 to m, we shall have in all m n transformations and now we shall prove that these m n linear transformations form a basis of L (V, W)

To Show Linear Independence.

Let αij be set of m n scalars, where $1 \leq i \leq n$ and $1 \leq j \leq m$

such that $\sum_{i=1}^{n} \sum_{j=1}^{m}\alpha_{ij}T_{ij} = O$

We have to show that $\alpha_{ij} = 0$ for all i and j,

Now $\nu_p \in V$ for each p = 1, 2, ... n

and $O(\nu_p) = 0 \in W$

$\therefore \quad \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} T_{ij}\right) (\nu_p) = O\ (\nu_p)$

$\Rightarrow \quad \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{ij} T_{ij}\ (\nu_p) = 0 \in W$

$\Rightarrow \quad \sum_{j=1}^{m} \alpha_{ij} T_{ij} \left(v_p\right) + ... + \alpha_{pj}\ T_{pj}\ (\nu_p) + ... + \alpha_{nj}\ T_{nj}\ (\nu_p)) = 0$

$\Rightarrow \quad \sum_{j=1}^{m}\ (\alpha_{ij} .0 + \alpha 2_j.\ 0 + ... + \alpha_{pj}\ w_j + ... + \alpha_{nj}\ 0) = 0$

[by def. of T_{ij}]

$\Rightarrow \quad \sum_{j=1}^{m}\ \alpha_{pj}\ w_j = 0$

$\Rightarrow \quad \alpha p_1\ w_1 + \alpha p_2\ w_2 + ... + \alpha_{pm}\ w_m = 0$

$\Rightarrow \quad \alpha_{p1} = \alpha_{p2} = ... = \alpha_{pm} = 0$

[$\because$ $B_2 = \{w_1, w_2, ...w_m\}$ is a basis for W, so L.I]

where $1 \leq p \leq n$

$\therefore$ aij = 0 for $1 \leq i \leq n$ and $1 \leq j \leq m$

$\therefore$ the set $\{T_{ij}\}$ is a linearly Independent.

To show $\{T_{ij}\}$ spans L (V, W).

We have to show that any linear transformation T of (V, W) is expressible as a linear combination of T_{ij}.

Let $T \in L\ (V, W)$ be any linear transformation so that $T\ (\nu_p) \in W$

and $B_2 = \{w_1, w_2\ ...., w_m\}$ is a basis of W

Thus $T\ (\nu_p)$ can be expressed as linear combination of elements of B_2 and let

$$T\ (\nu_p) = \sum_{j=1}^{m} \beta_{pj} w_j \qquad ...\ (ii)$$

Consider $S = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} T_{ij}$

Since S is a linear combination of elements of $\{T_{ij}\} \subset L\ (V, W)$ and L (V, W) is a vector space, so that S is also a linear transformation in L (V, W).

Now we shall prove S = T

We have $S(\nu_p) = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} T_{ij} \right)(\nu_p)$

$= \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{ij} T_{ij} (\nu_p)$

$= \sum_{i=1}^{m} (\beta_{1j} T_{1j} (\nu_p) + \beta_{2j} T_{2j} \{\nu_p) + ... + \beta_{pj} T_{pj} (\nu_p) + ...$

$+ \beta_{nj} T_{nj} (\nu_p))$

$= \sum_{i=1}^{m} (\beta_{ij} . 0 + \beta_{2j} .0 + ... + \beta_{pj} w_j + ... + \beta_{nj} -)$

$= \sum_{i=1}^{m} (\beta_{pj} w_j = T(\nu_p)$ [Using ii]

$\therefore$ $S(\nu_p) = T(\nu_p)$ for all p = 1, 2, ... n

$\Rightarrow$ S = T.

Thus each element of L (V, W) is a linear combination of T_{ij}.

Hence the set $\{T_{ij}\}_{\substack{1 \le i \le n \\ 1 \le i \le m}}$ forms a basis for L (V, W) and it contains a finite number of elements m n.

$\therefore$ L (V, W) is a finite dimensional vector space having m n elements

Thus dim. L (V, W) - m n = dim. W dim. V

Hence the result

Cor. if T : V → V is a linear operator, Then L (V, V), the vector space of all linear operators on V is finite dimensional and

dim [L (v, V)] = (dim V)2

Proof : In above proof, let W - V and m = n.

15.4 Self Check Exercise-1

Q. 1 Let $T_1 : R^3 \to R^2$ and $T_2 : R^3 \to R^2$ be defined by

$T_1 (x, y, z) = (3x + y, z)$

and

$T_2 (x, y, z) = (-y + z, x - y)$

Compute $T_1 + T_2$, $4 T_1$, $3T_1 - T_2$.

Q. 2 Let T_1 and T_2 be linear operator defined on R^2 defined by

$T_1(x, y) = (y, x)$

$T_2(x, y) = (x, 0)$

Compute $T_1 + T_2$, $2T_1 - T_2$

15.5 Product of Two Linear Transformations

Definition

Let U, V, W be three vectors spaces other the same field F and $T : V \to W$, $S : U \to V$ be two Linear transformations. Then the composite mapping.

$TS : U \to W$ or $ToS : U \to W$ is defined as

$T \circ S(u) = (TS)(u) = T(S(u))$ for all $u \in U$.

Note : TS is defined only when range of S $\subseteq$ Domain of T or not otherwise.

In general $TS \neq ST$

For example : Let V = vector space of all polynomials ones reals. Define linear operator D and T as

$$D(f(t)) = \frac{d\ f(t)}{dt} \text{ and } T(f(t)) = \int_0^t f(t)\,dt$$

Show that DT = I and TD ≠ I, I is identity operator.

Solution : We have

$V = \{f(t) : f(t) = a_0 + a_1t + \ldots\ldots\}$ u's are reals. Then

$$(DT)(f(t)) = D[T(f(t))] = D\left[\int_0^t f(t)dt\right]$$

$$= D\left[\int_0^t (a_0 + a_1t + a_2t^2 + \ldots.)\right]dt$$

$$= D\left[\left\{a_0t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3} + \ldots\right\}_0^t\right]$$

$$= D\left[\left(a_0t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3} + \ldots\right) - (0 + 0 + \ldots)\right]$$

$= a_0 + a_1t + a_2t^2 + \ldots\ldots = f(t) = I(f(t))$

$\therefore \quad (DT)(f(t)) = I(f(t))$

$\Rightarrow \quad DT = I\ \forall\ f(t) \in V$

Now $\quad (TD)(f(t)) \quad = T[D(f(t))]$

$= T[D (a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...)]$

$= T[\alpha_1 + 2 \alpha_2 t + 3 \alpha_3 t^2 + ...]$

$$= \int_0^t (\alpha_1 + 2\alpha_2 t + 3\alpha_3 t^2 + ...)dt$$

$$= \left[\alpha_1.1 + 2\alpha_2 \frac{t^2}{2} + 3\alpha_3 \frac{t^3}{3} + ...\right]_0^t$$

$= \alpha_1 t + \alpha^2 t2 + \alpha_3 t^3 + ...$

$\neq f(t)$

$\Rightarrow$ (TD) f(T)) $\neq$ 1 (f(t)) [$\because$ f (t) = 1 (f(t))]

$\therefore$ TD $\neq$ I.

Hence DT $\neq$ TD

Theorem : Let U, V, W be vector spaces over the same field F and $T_1 : V \to W$ and $T_2 : U \to V$ be linear transformations. Prove that $T_1 T_2 : U \to W$ is a linear transformation.

Proof: Since $T_1 : V \to W$ and $T_2 : U \to V$ are linear transformation, so the composite mapping $T_1 T_2 : U \to W$ is defined by

[$T_1 T_2$ is defined as Range T_2 = Domain T_1]

$(T_1 T_2)(u) = T_1 [T_2 (u)] \ \forall \ u \in U$

Let $u_1, u_2 \in U$ and $\alpha, \beta \in F$

$\Rightarrow$ $\alpha u_1 + \beta u_2 \in U$ [$\because$ U is a vector space]

Then $(T_1 T_2)(\alpha u_1 + \beta u_2) = T_1 [T_2 (\alpha u_1 + \beta u_2)]$

$= T_1 [\alpha T_2 (u_1) + \beta T_2 (u_2)]$ [$\because$ T_2 is linear]

$= T_1 (\alpha T_2 (u_1)) + T_1 (\beta T_2 (u_2))$ [$\because$ T_1 is linear]

$= \alpha T_1 (T_2 (u_1)) + \beta T_1 (T_2 (u_2))$ [T_1 is linear]

$= \alpha T_1 T_2) (u_1) + \beta (T_1 T_2) (u_2)$

Hence $T_1 T_2 : U \to W$ is a linear transformation

Theorem : Let U, V, W be three vector spaces over the same field F. Let $T_1 : U \to V$ and $T_2 : U \to V$ be linear transformations. Also let $S_1 : V \to W$ and $S_2 : V \to W$ be linear transformations. Then prove

(i) $S_1 (T_1 + T_2) = S_1 T_1 + S_1 T_2$

(ii) $(S_1 + S_2) T_1 = S_1 T_1 + S_2 T_1$

(iii) $\alpha (S_1 T_1) = (\alpha S_1) T_1 = S_1 (\alpha T_1)$ for $\alpha \in F$

Proof : (i) For all $u \in U$, we have

$[S_1 (T_1 + T_2)] (u) = S_1 [(T_1 + T_2) (u)]$

[by def. of composite mapping]

$= S_1 [T_1 (u) + T_2 u)]$

$= S_1 (T_1 (u)) + S_1 (T_2 (u))$ [$\because$ S_1 is a L.T.]

$= (S_1 T_1) (u) + (S_1 T_2) (u)$

$= [S_1 T_1 + S_1 T_2] (u)$

Hence $S_1 (T_1 + T_2) = S_1 T_1 + S_1 T_2$

(ii) For all $u \in U$, we have

$[(S_1 + S_2) T_1] (u) = (S_1 + S_2) (T_1 (u))$

$= S_1 (T_1 (u)) + S_2 (T_1 (u))$

$= (S_1 T_1) (u) + (S_2 T_1) (u)$

$= S_1 T_1 + S_2 T_1) (u)$

Hence $(S_1 + S_2) T_1 = S_1 T_1 + S_2 T_1$

(iii) For all $u \in U$, we have

$[\alpha(S_1 T_1) (u) = \alpha (S_1 T_1 (u)$

$= \alpha S_1 [T_1 (u)]$

$=[(\alpha S_1) T_1] (u)$ .... (1)

Also $[S_1(\alpha T_1)] (u) = S_1 [(\alpha T_1) u]$

$= S_1 [\alpha T_1 (u)]$

$= \alpha S_1 [T_1 (u)]$

$= [(\alpha S_1) T_1] (u)$ .... (2)

From (1) and (2)

$\alpha (S_1 T_1) = (\alpha S_1) T_1 = S_1 (\alpha T_1)$

Hence the result

Theorem (Properties of multiplication of linear operators)

Let R, s, T be three linear operators on a vector space V (F) and O and I are the zero and identity operators on V. Then prove that

(i) RO = OR = O (ii) RI = IR = R

(iii) R (S + T) = RS + RT (iv) (R + S) T = RT + ST

(v) R (ST) = (RS) T (vi) k (RS) = (kR) S = R (kS)

where k is any scalar

Proof : Let $\nu \in V$. Then $O (\nu) = 0$ and $I (\nu) = \nu$

(i) $(RO)(v) = R[O(v)] = R(O) = 0 = O(v)$

$\Rightarrow$ $RO = O$

and $(RO)(v) = O[R(v)] = O[v_1] = 0$ where $v_1 = R(v) \in V$

$= O(v)$

$\Rightarrow$ $OR = O$

Hence $RO = OR = O$

(ii) $(RI)(v) = R(I(v)) = R(v) = R(v)$

$\Rightarrow$ $RI = R$

and $(IR)(v) = I(R(v)) = I(v_1)$ where $v_1 = R(v) \in V$

$= v_1 = R(v)$

$\Rightarrow$ $IR = R$

Hence $IR = RI = R$

(iii) $[R(S + T)](v) = R[S + T)(v)]$

$= R[S(v) + T(v)]$

$= R(S(v)) + R(T(v))$ [$\because$ R is a L.T.]

$= (RS)(v) + (RT)(v)$

$= (RS + RT)(v)$

Hence $R(S + T) = RS + RT$

(iv) $[(R + S)T](v) = (R + S)(T(v))$

$= R(T(v)) + S(T(v))$

$= (RT)(v) + (ST)(v)$

$= (RT + ST)(v)$

Hence $(R + S)T = RT + ST$

(v) $[R(ST)](v) = R[(ST)(v)] = R[S(T(v))]$

$= [RS][T(v)]$

$= [(RS)T](v)$

Hence $R(ST) = (RS)T$

(vi) $[k(RS)](v) = k(RS)(v) = kR(S(v))$

$= (kR)(S(v)) = [(kR)S](v)$

And $[R(kS)](v) = R((kS)(v))$

$= R(k.S(v))$

$= k [R (S (\nu))]$

$= [k (RS)] (\nu)$

Hence $k (RS) = (k R) S = R (k S)$

15.6 Definition (Linear Algebra Or Algebra)

Let V be a vector space over the field F. Then V (F) is said to be an algebra over F if it is equipped with another composition, known as multiplication of vectors, satisfying the following

(I) For all $x, y, z \in V$, $(x y) z = x (y z)$ [Multiplication is Associative]

(II) For all $x, y, z \in V$, $x (y + z) = xy + xz$

$(x + y) = xz + yz$

[Multiplication is distributive]

(III) For all $x, y \in V$, $\alpha \in F$, $\alpha (x y) = (\alpha x) y = x (\alpha y)$

[Associative under scalar multiplication]

Note. (i) If the elements of V are commutative for multiplication i.e., $xy = yx$ for all $x, y \in V$, then it is called Commutative algebra.

(ii) If there exists an element $1 \in V (F)$ such that

$1.x = x.1 = x$ for $x \in V$,

then V (F) is called linear algebra with unity.

Theorem : Let V (F) be a vector space. Then L (V, V), the set of linear operators on V is an algebra with unity. Prove.

Proof: We know that L (V, V) is a vector space

[Prove, Theorem 12 Cor.]

Also by results of Theorem 16, we have

(i) For $R, S, T \in L (V, V)$, $(RS) T = R (ST)$ [Associativity]

(ii) For $R, S, T \in L (V, V)$, $R (S + T) = RS + RT$

$(R + S) T = RT + ST$ [Distributive]

(iii) For $R, S, \in L (V, V)$, $\alpha \in F$, $\alpha (RS) = (\alpha R) S = R (\alpha S)$

[Associative under scalar multiplication]

(iv) For $R \in L (V, V)$, there is $I \in L (V, V)$ such that $RI = R = IR$

Hence L (V, V) is a algebra with unity

Definition

Let $T : V \to V$ be a linear operator, V is a vector space over the field F.

We define $T^2 : V \to V$ as $T^2 (\nu) = T (T (\nu))$ i.e., $T^2 = T.T$

Similarly $T^3 = T^2. T, ... T^n = T^{n-1}. T.$

Also we define $T^0 = I$ if $T \neq O$ and $T^1 = T$.

Theorem : If $T : V (F) \to V (F)$ is a linear operator, then prove $T^n : V (F) \to V (F)$ is a linear operator, for every positive integer n.

Proof: We shall prove it by induction on n.

Step I. For n = 1, $T^n = T$, which is a linear operator on V (Given)

$\therefore$ result holds for n = 1.

Step II. Suppose result is true for n = p, a natural number.

i.e., T^p is a linear operator on V.

Step III. Verify that T^{p+1} is a linear operator on V

Since $T^{p+1} = T^p T$

and by Theorem 14, $T^p T$ (Product of linear operators)

is a linear operator on V

$\therefore$ T^{p+1} is a linear operator on V

$\therefore$ result is true for n = p + 1

Hence by mathematical induction, $T^n : V \to V$ is a linear operator, for all naturals n.

Note 1. $T_m T_n = T_{m+n}$ and $(T_m)_n = T_{mn}$ for + ve integers m, n

2. Polynomial linear transformation in T over F, is written

as $P(T) = \alpha_0 I + \alpha_1 T + \alpha_2 T^2 + ... + \alpha_n T^n$ for α_i's $\in F$

when $p (t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ... + \alpha_n t^n$, a polynomial in real variable t.

Theorem : Prove that if $T_1, T_2 \in L (V, W)$ then

(a) $\rho (\alpha T_1) = \rho(T_1)$ for $\alpha \in F, \alpha \neq 0$

(b) $|\rho (T_1) - \rho (T_2)| \leq \rho (T_1 + T_2) \leq \rho (T_1) + \rho(T_2)$

Proof:

(a) We know $T_1 (V)$ i.e. Range space of T_1 is a subspace of W

so that $\alpha T_1 (V) \subseteq T_1 (V)$ for $0 \neq \alpha \in F$ .... (i)

Similarly $\alpha^{-1} T_1 (V) \subseteq T_1 (V)$

$\Rightarrow$ $\alpha (\alpha^{-1} T_1 (V) \subseteq \alpha T_1 (V)$

$\Rightarrow$ $(\alpha\alpha^{-1} T_1 (V) \subseteq \alpha T_1 (V)$

$\Rightarrow$ $T_1 (V) \subseteq \alpha T_1 (V)$ ...(ii)

Combining (i) and (ii), we get $T_1 (V) = \alpha T_1 (V)$

$\Rightarrow \quad T_1 (V) = (\alpha T_1) (V)$

$\Rightarrow$ Rang space of T_1 = Range space of αT_1

$\Rightarrow$ dim (range space of T_1) = dim (Range space of αT_1)

$\Rightarrow \quad \rho(T_1) = \rho (\alpha T_1)$

Which proves

(b) Let $\nu \in V$ be any vector

We have $(T_1 + T_2) (\nu) = T_1 (\nu) + T_2 (\nu)$

$\Rightarrow \quad (T_1 + T_2) (V) \subseteq T_1 (V) + T_2 (V)$

$\Rightarrow \quad \dim ((T_1 + T_2) V) \leq \dim (T_1 (V) + T_2 (V))$

$\Rightarrow \quad \dim ((T_1 + T_2) V) \leq \dim T_1 (V) + \dim T_2 (V)$

$[\dim (W_1 + W_2) = \dim W_1 + \dim W_2 - \dim (W_1 \cap W_2) < \dim W_1 + \dim W_2]$

$\Rightarrow$ dim (range space of $(T_1 + T_2)$) < dim (range space of T_1)
+ dim (range space of T_2)

$\Rightarrow \quad \rho(T_1 + T_2) \leq \rho (T_1) \rho (T_2)$

Further T1 = $(T_1 + T_2) - T_2$

$\Rightarrow \quad \rho (T_1) = \rho ((T_1 + T_2) + (-T_2))$

$\leq \rho(T_1 + T_2) + \rho (-T_2)$ (Using (iii))

$\Rightarrow \quad \rho(T_1) \leq \rho (T_1 + T_2) + \rho (T_2)$ $(\because \rho (\alpha T_2) = \rho (T_2)$ Take $\alpha = -1)$

$\Rightarrow \quad \rho(T_1) - \rho(T_2) \leq \rho (T_1 + T_2)$ ...(iv)

Similarly starting with T2, we get

$\rho(T_2) - \rho (T_1) \leq \rho (T_1 + T_2)$ ...(v)

From (iv) and (v), we have $| \rho(T_2) - \rho (T_2) | \leq \rho (T_1 + T_2)$ ...(vi)

Combining (vi) and (iii), we have $| \rho(T_1) - \rho (T_2) | \leq \rho (T_1) + \rho(T_2)$ which proves (b)

Theorem : Let $T_1 : U \rightarrow W$ and $T_2 : V \rightarrow U$ be linear transformations where U, V, W are vector spaces over the same field F

Prove $\rho (T_1T_2) \leq \text{Min} \{\rho (T_1), \rho (T_2)\}$

Proof : As $T_1 : U \rightarrow W$ and $T_2 : V \rightarrow U$ are L.T.'s so that $T_1 T_2 : U \rightarrow W$ is a L.T.

Now $\rho (T_1T_2)$ = dim (range space of T_1T_2)

$= \dim (T_1T_2) (V))$

$= \dim (T_1 (T_2 (V)))$

$\leq$ Min {dim T_2 (V), dim W}

$\leq$ dim (T_2 (V)) = dim (R (T_2)) = ρ (T_2)

$\Rightarrow$ $\rho (T_1T_2) \leq \rho (T_2)$ ...(i)

Further (T_1T_2) (V) = (T_1 (T_2 (V)) $\subset$ T_1 (U) ($\because T_2 (V) \subset U$)

$\Rightarrow$ dim ((T_1T_2) (V)) $\leq$ dim (T_1 (U))

$\Rightarrow$ dim (range space of T_1T_2) < dim (Range T_1)

$\Rightarrow$ $\rho (T_1T_2) \leq \rho (T_1)$ ...(ii)

Combining (i) and (ii), $\rho (T_1T_2) \leq$ Min ($\rho (T_1), \rho (T_2)$)

Hence the theorem.

Cor : If T_1 or T_2 is an isomorphism (i.e. linear, one-one and onto)

Prove $\rho (T_1T_2) = \rho (T_1)$ or $\rho (T_2)$

Proof : Let T_2 be an isomorphism so that T_2^{-1} exists

$\therefore$ $T_1 = (T_1T_2) T_2^{-1} \Rightarrow \rho (T_1) \leq$ Min {$\rho (T_1T_2)$, $\rho (T_2^{-1})$}

$\Rightarrow$ $\rho (T_1) \leq \rho (T_1T_2)$ ...(i)

But $\rho (T_1T_2) \leq$ Min {$\rho (T_1), \rho (T_2)$}

$\leq \rho (T_1)$

$\Rightarrow$ $\rho (T_1T_2) \leq \rho (T_1)$ ....(ii)

From (ii) and (ii) we have $\rho (T_1T_2) = \rho (T_1)$

Similarly if T1 is an isomorphism

Then $\rho (T_1T_2) = \rho (T_2)$.

Lemma : Let W be a subspace of a vector space V(F) and T : V $\rightarrow$ V is a linear operator prove dim (T (W)) $\geq$ dim W - dim (N (T)) where N (T) is null space of T.

Proof : Given W is a subspace of a vector space V so there exists subspace U of V such that V = W $\oplus$ U

$\Rightarrow$ dim V = dim W + dim U ....(i)

Also we know

dim (W + U) = dim W + dim U - dim (W $\cap$ U)

$\leq$ dim W + dim U

$\Rightarrow$ dim (W + U) $\leq$ dim W + dim U ...(ii)

Since V = W $\oplus$ U

$\Rightarrow \quad V = W + U \Rightarrow T(V) = T(W + U)$

$= T(W) + T(U)$

($\because$ T is a L.O.)

$\Rightarrow \quad \dim (T (V)) = \dim (T (W) + T (U))$

$\leq \dim (T (W)) + \dim (T (U))$

$\Rightarrow \quad \dim (R (T)) \leq \dim (T (W)) + \dim (T (U))$ ....(iii)

[$\because$ T (V) = Range of T and ρ (T) = dim (Range of T]

$\because \quad$ T (U) is a subspace of U $\Rightarrow \dim (T (U)) \leq \dim U$

so (iii) $\Rightarrow \dim (R (T)) \leq \dim (T (W)) + \dim U$

$\Rightarrow \quad \dim (R (T)) - \dim U \leq \dim (T (W))$

$\Rightarrow \quad \dim W = \dim (N (T)) < \dim (T (W))$

$$\left[\begin{array}{ll} \because & \text{Rank } T = \text{Nullity } T = \dim V = \dim W + \dim U \\ \Rightarrow & \text{Rank } T + \dim U = \dim W - \text{Nullity } T = \dim W - \dim (N (T)) \end{array}\right]$$

$\Rightarrow \quad \dim (T (W) > \dim W - \dim (N (T))$

Hence the lemma.

Theorem : Let $T_1 : V \to V$ and $T2 : V \to V$ be linear operators and dim V = n.

Prove (a) $\rho (T_1T_2) \geq \rho (T_1) + \rho (T_2) - n$

(b) $\quad \nu (T_1T_2) \leq \nu (T_1) + \nu (T_2)$ where ν (T) = Nullity T.

Proof : (a) Using above lemma,

$\dim (T_1 (W)) > \dim W - \dim (N (T1))$ ..(i)

Let $W = T_2 (V)$

$\therefore \; \dim T_1 (W) = \dim T_1 (T_2 (V)) = \dim (T_1T_2) (V)$

$= \dim (R (T_1T_2))$

$= \rho (T_1T_2)$

and $\dim W = \dim T_2 (V) = \dim (R (T_2) = \rho (T_2)$

so (i) becomes $\rho (T_1T_2) \geq \rho (T_2) = \nu (T_1)$

$\Rightarrow \quad \rho (T_1T_2) \geq \rho (T_2) - (n - \rho (T_1))$ $\qquad (\rho (T_1) + \nu (T_1) = \dim V = n)$

$\Rightarrow \quad \rho (T_1T_2) \geq \rho (T_1) + \rho (T_2) - n$

(b) Using ρ (T) + ν (T) = n

We have $\rho (T_1T_2) = n - \nu (T_1T_2)$, $\rho (T_1) = m - \nu (T_1)$

and $\rho (T_2) = n - \nu (T_2)$

$\therefore$ Result (a) becomes

$n - \nu (T_1T_2) \geq n - \nu (T_1) + n - \nu (T_2) - n$

$\Rightarrow$ $- \nu (T_1T_2) \geq - \nu (T_1) - \nu (T_2)$

$\Rightarrow$ $\nu (T_1T_2) \leq \nu (T_1) + \nu (T_2)$

Hence the result.

Note : Result (b) is a lso known as Sylvester's Law of nullity.

Some Illustrative Examples

Example 1 : Let T : $R^2 \rightarrow R^2$ and S : $R^2 \rightarrow R^3$ be Liner Transformations defined by T (x, y, z) = (x - 3 y - 2 z, y - 4 z) and S (x, y) = (2 x, 4 x - y, 2 x + 3 y)

Find ST, TS is product commutative.

Solution : Since of S = R^3 = Domain of T

$\therefore$ TS is defined.

and (TS) (x, y) = T [S (x, y)] for (x, y) $\in R^2$

= T (2 x, 4 x - y, 2 x + 3 y) [by def. of S]

= (2 x - 3 (4 x - y) - 2 (2 x + 3 y), 4 x - y - 4 (2 x + 3 y))

[by def. of T]

= (2 x - 12 x + 3 y - 4 x - 6 y, 4 x - y - 8 x - 12 y)

= (- 14 x - 3 y, - 4 x - 13 y)

Also Range of T = R^2 = Domain of S

$\therefore$ ST is defined

and (ST) (x, y, z) = S [T (x, y, z)] for (x, y, z) $\in$ R3

= S (x - 3 y - 2 z, y - 4 z) [by def. of T]

= (2 (x - 3 y - 2 z), 4 (x - 3 y - 2 z) - (y - 4 z), 2 (x - 3 y - 2 z) + (y - 4 z))

[by def. of S]

= (2 x - 6 y - 4 z, 4 x - 13 y - 4 z, 2 x - 3 y - 16 z).

Example 2 : Show that the following linear mappings T, S, U are linearly mappings were T,S, U $\in$ L (R^3, R^2) defined as T(x, y, z) = (x + y + z, x + y)

$S(x, y, z) = (2y + z, x + y)$ and $U(x, y, z) = (2x, y)$.

Solution : Let $\alpha T + \beta S + \gamma U = 0$ for scalars $\alpha, \beta, \gamma \in R$

Where O is a zero transformation from R^3 into R^2.

For $e_1 = (1, 0, 0) \in R^3$, we have

$(\alpha T + \beta S + \gamma U)(1, 0, 0) = O(1, 0, 0)$

$\Rightarrow \quad \alpha T(1, 0, 0) + \beta S(1, 0, 0) + \gamma U(1, 0, 0) = (0, 0)$

$\Rightarrow \quad \alpha (1 + 0 + 0, 1 + 0) + \beta (0 + 0, 1 + 0) + (2, 0 = (0, 0)$

$\Rightarrow \quad (\alpha + 2\gamma, \alpha \beta) = (0, 0)$

$\therefore \quad \alpha + 2\gamma, \alpha + \beta = 0 \qquad ....(1)$

For $e_2 = (0, 1, 0) \in R^3$, we have

$(\alpha T + \beta S + \gamma U)(0, 1, 0) = O(0, 1, 0)$

$\Rightarrow \quad \alpha T(0, 1, 0) + \beta S(0, 1, 0) + \gamma U(0, 1, 0) = (0, 0)$

$\Rightarrow \quad \alpha (0 + 1 + 0, 0 + 1) + \beta (2 + 0, 0 + 1) + \gamma (2(0), 1) = (0, 0)$

$\Rightarrow \quad (\alpha + 2\beta, \alpha + \beta + \gamma) = (0, 0)$

$\therefore \quad \alpha + 2\beta = 0, \alpha + \beta + \gamma = 0 \qquad ....(2)$

Solving (1) and (2), we get $\alpha = \beta = \gamma = 0$

Hence, T, S, U are linearly independent.

15.7 Self Check Exercise - 2

Q. 1 Let $T_1 : R^3 \to R^3$, $T_2 : R^3 \to R^2$ defined by

$T_1(x, y, z) = (3x + y, z)$

$T_2(x, y, z) = (-y + z, x + y)$

Compute T_1T_2, T_2T_1 if possible.

Q. 2 Let T be Linear Operator on R^3 defined by

$T(x\ y) = (x + 2y, 3x + 4y)$

Find $\rho(T)$, $\rho(T) = t^2 - 5t - 2$

15.8 Summary

In this unit we have learnt the following concepts:

(i) Algebra of Linear transformations

- (a) properties due to addition
- (b) properties due to scalar multiplication

(ii) Product of two linear transformation

(a) Properties of multiplication of linear operators

(b) Linear algebra or algebra etc.

15.9 Glossary

1. Linear Algebra with unity - If $\exists$ an element $1 \in V(F)$ s.t.

$1.\ x = x - 1.\ x$ for $x \in V$

then V (F) is called linear algebra with unity.

2. Let $T : V \rightarrow V$ be a linear operator then

$Tn : V(F) \rightarrow V(F)$ is a Linear operator for every positive n.

3. $T^2 : V \rightarrow V$ as $T^2(x) = T(T(u)$ i.e. $T^2 = T.T$

$T^3 = T^2\ T \ldots\ldots T^n = T^{n-1}.\ T,\ T^0 = 1,\ T \neq 0$

$T^1 = T$.

15.10 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 $(T_1 + T_2)(x, y, z) = (3x + z, x - y + z)$

$(4T_1)(x, y, z) = (12x + 4y, 4z)$

$(3T_1 - T_2)(x, y, z) = (9x + 4y - z, -x + y + 3z)$

Ans. 2. $(x + y, x)$

Self Check Exercise - 2

Ans.1 $T_1 T_2$ is not defined since Range of T_2 $(= R^2)$ is not a subset of Domain of T_1 $(= R^3)$

Ans. 2 $\rho(T) = O$, a zero operator on R^2. $\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$

15.11 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.
4. Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence, Linear Algebra, 4th Ed., Prentice-Hall of India Pvt. Ltd., New Delhi, 2004.

15.12 Terminal Questions

1. Let V be a vector space of differentiable functions and

   B = {sin x, cos x} as basis of V.

   Let D be differential operator on V

   Show that D is a zero of F (t) = $t^2 + 1$

2. Let $T_1 : R^3 \rightarrow R^2$, $T_2 : R^3 \rightarrow R^2$ be two LTs defined as

   T_1 (x, y, z) = (3x, y + z)

   T2 (x, y, z) = (2x - 3 z, y)

   Compute $T_1 + T_2$, $5 T_1$, $4T_1 - 5 T_2$, $T_1 T_2$, $T_2 T_1$.

3. Give an example of a L.T. T s.t.

   $T \neq O$, $T^2 \neq O$, ... $T^{n-1} \neq O$ but $T^n = O$.

4. If T : V → V is L. Operator s.t.

   $T^2 (I - T) = T (I - T)^2$

   Prove that T is a projection.

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Unit - 16

Dual Spaces, Dual Basis and Double Dual

Structure

16.1 Introduction

16.2 Learning Objectives

16.3 Dual Space

16.4 Dual Basis

16.5 Self Check Exercise-1

16.6 Double Dual

16.7 Self Check Exercise-2

16.8 Summary

16.9 Glossary

16.10 Answers to Self Check Exercises

16.11 References/Suggested Readings

16.12 Terminal Questions

16.1 Introduction

Dear students, in this unit we are going to discuss the concept of dual spaces. In mathematics any vectors space V has a corresponding dual vector space consisting of a linear transformation on V, together with vector space structure of pointwise addition and scalar multiplication of constants. Dual space finds application in tensor analysis with finite dimensional vector space.

16.2 Learning Objectives

The main Learning objectives of this unit are

(i) to define Linear Functional.

(ii) dual space or conjugate space of V

(iii) dual basis

(iv) to prove some important theorems related to dual space and dual basis etc.

Linear Functional

Let V (F) be a vector space over a field F, then a function,

$f : V \rightarrow F$ is called a linear functional on V iff

$f(\alpha \nu_1 + \beta \nu_2) = \alpha f(\nu_1) + \beta f(\nu_2) \ \forall \ \alpha, \beta \in F$

$\nu_1, \nu_2 \in V$

Theorem : Prove that set V* of all linear functionals 'f' form V to F (where V is a vector space over field F) is a vector space over the field F with addition and scalar multiplication defined as

$(f_1 + f_2)(x) = f_1(x) + f_2(x) \ \forall \ x \in V$ and $f_1, f_2 \in V^*$

$(\alpha f_1)(x) = \alpha f_1(x) \ \forall \ x \in V, \alpha \in F, f_1 \in V^*$

Proof : Similar to Theorem 13 of chapter 4 (page 367), replace the notation L(V, W) by V*

16.3 Dual Space

The vector space V* of all linear functional from V to F is called dual space or conjugate space of V.

Note : The other notations for dual space are V' or $\hat{V}$

Theorem : Let V be an n dimensional vector space over F and B = $\{\nu_1, \nu_2, ... \nu_n\}$ is an ordered basis. Prove that for any ordered set S = $\{a_1, a_2, ... a_n\}$ of scalars, there is a unique linear functional f on V such that $f(\nu 1) = a1$ for $1 \leq i \leq n$.

Proof : Firstly, we define $f : V \to F$ as follows

If $\nu \in V$, then ν can be uniquely expressed as a linear combination of elements of B i.e. $\exists$ unique scalars $k_1, k_2, ...k_n$ such that

$\nu = k_1\nu_1 + k_2 \nu_2 + ...k_n\nu_n$; k_i's $\in F$

We define f as

$f(\nu) = k_1a_1 + k_2 a_2 + ...k_n \ a_n \in F$

[$\because$ $k_i, a_i \in F \Rightarrow \sum k_i a_i \in F$ as F is a field]

(i) f is well defined

Since $k_1, k_2, ...k_n$ are unique for given $\nu \in V$

$\Rightarrow \quad k_1a_1 + k_2 a_2 + ...k_n \ a_n \in F$ for a_i's $\in F$

$f(\nu)$ is unique element of F for given $\nu \in V$

f is well defined

(ii) To show $f(\nu_i) = a_i$

Now $\nu_1 \in V$ can be uniquely expressed as L.C. of elements of B

i.e. $\nu_i = 0\nu_1 + 0\nu_2 + ... + 0\nu_{i-1} + 1.\nu_1 + 0\nu_{i+1} + ... + 0\nu_n$

$\Rightarrow \quad f(\nu_i) = 0 \ a_1 + 0 \ a_2 + ... + 0 \ a_{i-1} + 1. \ a_1 + 0 \ a_{i+1} + ... + 0 \ a_n$

$= a_i$ for $1 \leq i \leq n$

$\Rightarrow \quad f(\nu_i) = a_i$

(iii) To show f is linear functional

Let x, y $\in$ V; $\alpha, \beta \in$ F

$\Rightarrow$ x = $\sum_{i=1}^{n} l_i v_i$ and y = $\sum_{i=1}^{n} m_i v_i$ for l_i, $m_i \in$ F

$\therefore$ f(x) = $\sum_{i=1}^{n} l_i a_i$ and f (y) = $\sum_{i=1}^{n} m_i a_i$ (by def. of f)

Now f (αx + β y) = $f\left(\alpha\sum_{i=1}^{n} l_i v_i + \beta\sum_{i=1}^{n} m_i v_i\right)$

$= f\left(\sum_{i=1}^{n}(\alpha l_i + \beta\, m_i)v_i\right)$

$= \sum_{i=1}^{n}(\alpha l_i + \beta\, m_i) f(v_i) = \sum_{i=1}^{n}(\alpha l_i + \beta\, m_i)(a_i)$

$= \alpha\sum_{i=1}^{n} l_i a_i + \beta\sum_{i=1}^{n} m_i a_i = \alpha\, f(x) + \beta\, f(y)$

$\Rightarrow$ f is linear functional

(iv) To show f is unique

Let h : V $\to$ F be another linear functional such that h (v_i) = ai for $1 \leq i \leq n$.

$\therefore$ For any $v \in$ V, $v = \sum_{i=1}^{n} k_i v_i$

$\Rightarrow$ h (v) = h $\left(\sum_{i=1}^{n} k_i v_i\right) = \sum_{i=1}^{n} k_i h(v_i) = \sum_{i=1}^{n} k_i a_i = f(v)$

$\Rightarrow$ h (v) = f (v) $\forall$ $v \in$ V

so that h = f

$\Rightarrow$ f is unique

Hence the theorem is proved.

Theorem : Let B = $\{v_1, v_2, \ldots\ldots v_n\}$ be a basis of vector space V (F) (of dimension n), prove there is a basis B* = $\{f_1, f_2, \ldots\ldots f_n\}$ for V* such that

$$f_i(v_k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Proof : As B = $\{\nu_1, \nu_2, ...\nu_n\}$ is a basis of V(F) by previous theorem, $\exists$ a unique functional f_1 on V s.t.

$f_1(\nu_1) = 1$, $f_1(\nu_k) = 0$, $2 \leq k \leq n$

where S = { 1, 0, 0, ... 0} is an ordered set of n scalars.

So for each i, 1 < i < n : $\exists$ a unique linear functional $f1$ on V s.t.

fi (νk 0 = Sik = $\begin{cases} 1, & i = k \\ 0, & i = k \quad S_{ik} \in F \end{cases}$

We prove that

B* = $\{f_1, f_2, ...f_n\}$ is a basis of V*.

(i) We shall show that B* is L.I. set

Let $\propto_1 f_1 + \propto_2 f_2 + ... + \propto_n f_n = O$ for $\propto_i \in F$

$\Rightarrow$ $(\propto_1 f_1 + \propto_2 f_2 + ... + \propto_n f_n)\ \nu_k) = O\ (u_k) = 0$

($\because$ f_i's are linear functional)

$\Rightarrow$ $\propto_1 \delta_i k + \propto_2 \delta_2 k + ... + \propto_k \propto_{kk} + ... + \propto_n \delta_{nk} = 0$

$\Rightarrow$ $\propto_1 (0) + \propto_2 (0) + ... + \propto_k (1) + ... + \propto_n (0) = 0$

$\Rightarrow$ $\propto_k = 0 \quad \forall \quad k = 1, 2, ... n$

$\Rightarrow$ $\propto_1 = \propto_2 = ... = \propto_n = 0$

$\Rightarrow$ B* = $\{f_1, f_2 .....; f_n\}$ is a L.I. set

(ii) We shall now show that B* span V*.

Let $f \in$V* be any element

and $f(\nu_k) = \alpha_k$ for $1 \leq k \leq n$: $a_k \in F$, $\nu_k \in B$.

Let $\nu \in \nu \Rightarrow \nu = \sum_{k=1}^{n} b_k v_k$ [$\because$ B is a basis]

$\Rightarrow$ $f(\nu) = f\left[\sum_{k=1}^{n} b_k v_k\right] = \sum_{k=1}^{n} b_k f(v_k) = \sum_{k=1}^{n} b_k a_k$

$\Rightarrow$ $f(\nu) = \sum_{k=1}^{n} b_k a_k$ .... (1)

Now $f_i(\nu) = f_i\left[\sum_{k=1}^{n} b_k v_k\right]$

$$= \sum_{k=1}^{n} b_k f_i(v_k) = \sum_{k=1}^{n} b_k a_k$$

$= b_1 \delta_{i1} + b_2 \delta_{i2} + ... + b_n \delta_{in}$

$= b_1 (0) + b_2 (0) + ... + b_i (1) + ... b_n (0)$

$= b_i$

$\Rightarrow$ $b_k = f_k(\nu)$ (Change suffix i to k)

$\therefore$ From (1) we have

$$f(\nu) = \sum_{k=1}^{n} a_k f_k(v) = \left(\sum_{k=1}^{n} (a_k f_k) v\right)$$

$\Rightarrow$ $f = \sum_{k=1}^{n} a_k f_k$ for scalars. a_k

$\Rightarrow$ f is a L.C. of $f_1, f_2 .... f_n$

$\Rightarrow$ B* span V*

Hence B* is a basis of V*.

Note. (i) As B* contains n element $\therefore$ dim V* = n

Hence dim V* = dim V.

(ii) Since dim V* = dim V, $\therefore$ V* ≅ V* i.e. V is isomorphic to V*

(iii) The above defined δ_{ik} is known as Kronecker Delta.

16.4 Dual Basis

The basis B* = $\{f_1, f_2 .... f_n\}$ of V* (F) is called Dual basis (or transpose basis) of the basis B = $\{\nu_1, \nu_2 ... \nu_n\}$ of V(F).,

where $f_i(\nu_k) = \delta_{ik} = \begin{cases} i, i \neq k \\ 0 \; i = k \end{cases}$

Theorem : Let V (F) be n dimensional vector space and B = $\{\nu_1, \nu_2 ... \nu_n\}$; B* $\{f_1, f_2 .... f_n\}$ be basis of and V* resp. Then prove that

(i) any vector $\nu \in V$ can be expressed as

$$\nu = \sum_{i=1}^{n} v_i f_i(v)$$

(ii) any linear functional $f \in V^*$ can be expressed as

$$f = \sum_{i=1}^{n} \left(f(v) \right) f_i$$

Proof : Since

B = {v_1, v_2 ... v_n} is a basis of V and

B* = {f_1, f_2 .... f_n} is a dual basis of B so that

f_i (v_i) = $\begin{cases} i, i \neq k \\ 0, \ i = k \end{cases}$

= δ_{ik}

(i) As B is a basis of V, so any vector $v \in$ V can be expressed as

$v = \sum_{j=1}^{n} a_j v_j$ for scalars $\alpha_j \in$ F .... (1)

$\Rightarrow$ $f_i(v) = f_i\left[\sum_{j=1}^{n} a_j v_j\right] = \sum_{j=1}^{n} a_j f_i\left(v_j\right)$

($\because$ f_i is a linear functional)

$= \sum_{j=1}^{n} \alpha_j \delta_{ij} = \alpha_i$

$\Rightarrow$ $f_i(v) = \alpha_j$

$\Rightarrow$ $f_j(v) = \alpha_j$

(Changing suffix i by j)

Put value of α_i in (I), we get $v = \sum_{j=1}^{n} v_i f_i(v) v_j$

$\Rightarrow$ $v = \sum_{i=1}^{n} f_i(v) v_i = \sum_{i=1}^{n} v_i \ f_i(v)$

Hence the result

(ii) As B* is a basis of V*, so any linear functional $f \in$ V*, can be expressed as

$f = \sum_{i=1}^{n} a_i f_i$ for $f_i \in$ V*

$\Rightarrow$ $f(v_j) = \left(\sum f_i a_i\right)(v_j) = \sum a_i \ f_i(v_j) = \sum a_i \ \delta_{i\,j}$

$\Rightarrow$ $f(v_j) = \sum_{i=1}^{n} \alpha_i \delta_{ij} = \alpha_j$ for $1 \leq j \leq n$

$\therefore$ $f(v_i) = \alpha_i$ (Change suffix j by i)

Put value of αi in (II), we get $f = \sum_{L=1}^{n} \left(f\left(v_i\right)\right) f_i$

Hence the result

Cor. 1: Let V be a finite dimensional vector space of dim n over the field F and $\nu \in V$ ($\nu \neq 0$), then show that $\exists\, f \in V^*$ such that $f(\nu) \neq 0$

Proof : Given ν is a non zero vector of V

$\Rightarrow$ $\{\nu\}$ is L. I set

$\therefore$ It can be extended to form a basis of V

Let B = $\{\nu_1, \nu_2 \ldots \nu_n\}$ be a basis of V ($\because$ dim V = n)

call $\nu_i = \nu_1$

Take $\{f_1, f_2 \ldots f_n\}$ be corresponding dual basis.

$\therefore$ $f_i(\nu_k) = \delta_{ik} = \begin{cases} i \text{ if } i = k \\ 0, \text{ if } i \neq k \end{cases}$

$\Rightarrow$ $f_1(\nu_1) = \delta_{11} = 1$ .... (i)

Take $f = f_1 \Rightarrow f \in V^*$

and $f(\nu) = f_1(\nu 1)$ ($\because$ $\nu = \nu_1$ and $f = f_1$)

= 1

$\neq 0$

Hence $\exists$ a Linear functional $f \in V^*$ s.t. $f(\nu) \neq 0$.

Cor. 2. Let V be a finite dimensional vector space of dim n over F, then show that

$f(\nu) = 0 \quad \forall \quad f \in V^* \Rightarrow \nu = 0$ for $\nu \in V$.

Proof : If possible let $\nu \neq 0$, $\nu \in V$

then by cor. 1, $\exists$ a Linear functional $f \in V^*$ s.t.

$f(\nu) \neq 0$. which contradicts the fact that

$f(\nu) = 0 \;\forall\; \nu \in V$

$\therefore$ Our supposition is wrong

Hence $\nu = 0$

Cor. 3 Let V be a finite dimensional vector space of dim n over F. Show that if $\nu_1, \nu_2 \in V$ ($\nu_1 \neq \nu_2$) then $\exists$ a linear functional $f \in V^*$ s.t.

$f(\nu_1) \neq f(\nu_2)$

Proof : $\nu_1, \nu_2 \in V, \nu_1 \neq \nu_2$

$\Rightarrow$ $\nu_1 - \nu_2 \neq 0$ and $\nu_1 - \nu_2 \in V$

$\therefore$ by cor. 1, $\exists$ a Linear functional $f \in V^*$ s.t.

$f(\nu_1 - \nu_2) \neq 0$

$\Rightarrow$ $f(\nu_1) - f(\nu_2) \neq 0$ ($\because$ f is Linear)

$\Rightarrow$ $f(\nu_1) \neq f(\nu_2)$

Hence the results

16.5 Self Check Exercise - 1

Q.1 Find the dual basis for

$B = \{\nu_1, \nu_2\}$ of R_2 over R, where

$\nu_1 = (1, 2)$ and $\nu_2 = (1, 5)$

Q.2. Let V (R) be vector space of all polynomias of degree $\leq$ 1 and V* its dual space.

Let ϕ_1 $V \rightarrow R$ and $\phi_2 : V \rightarrow R$ s.t.

$\phi_1(f(t)) = \int_0^1 f(t)\,dt$, $\phi_2(f(t)) = \int_0^2 f(t)\,dt$

Find basis $\{\nu_1\ \nu_2\}$ of V which is dual is $\{\phi_1, \phi_2\}$.

16.6 Second Dual Space or Bidual or Double Space

We know that every vector space V over F has a dual space V*, which is a vector space having all linear functional on V.

$\therefore$ V* being a vector space, also must have a dual space (V*)* having all linear functional on V*. This dual space of V* is known as second dual or Bidual or Double space of V and is denoted by V*.

Note: If V is finite dimensional

Then dim V = dim V* = dim V**

and so $V \cong V^{**}$

Theorem : Let V (F) be a vector space of dim n, If $\nu \in V$ is any vector then show that the function L_1, on V* defined as $L_\nu(f) = f(f)\ \forall\ f \in V^*$ is a linear functional i.e. $L_\nu \in V^{**}$.

Also show that the function $\nu \rightarrow L_\nu$ is an isomorphism of V onto V**.

Proof : Firstly to show that L_ν is a linear functional

We define elements of V** as follows.

If $\nu \in V$ and $f \in V^*$, then $f(\nu)$ is a unique element of F.

$\therefore$ The rule defined by

$L_\nu (f) = f(\nu) \ \forall \ f \in V^*$ is a function from V* into F.

To show it is linear

Let $\alpha, \beta \in F$ and $f, g \in V^*$

$\Rightarrow \quad \alpha f + \beta g \in V^*$ ($\because$ V* is a vector space)

Now $L_\nu (\alpha f + \beta g) = (\alpha f + \beta g) (\nu)$

$= (\alpha f) (\nu) + (\beta g) (\nu)$

$= \alpha f (\nu) + \beta \ g (\nu)$ ($\because$ f, g are linear functional)

$\Rightarrow \quad L_\nu$ is a linear functional on V*

Secondly To show the function

$H : V \to V^{**}$ defined as $H (\nu) = L_\nu \quad \forall \quad \nu \in V$ is an isomorphism

(i) Firstly to show H is 1 - 1

Let $H (\nu_2) = H (\nu_2) \ \forall \ \nu_1, \nu_2 \in V$

$\Rightarrow \quad L\nu_1 = L\nu_2$

$\Rightarrow \quad L\nu_1 (f) = L\nu_2 (f)$ for $f \in V^*$

$\Rightarrow \quad f (\nu_1) = f (\nu_2)$

$\Rightarrow \quad f (\nu_1) - f (f_2) = 0$

$\Rightarrow \quad f (\nu_1 - \nu_2) = 0$ ($\because$ f is linear)

$\Rightarrow \quad \nu_1 = \nu_2$ (because of Cor. 2)

(ii) To show H is onto

We know dim V = dim V**

and H is one-one $\Rightarrow$ H is onto also

$\therefore$ H is onto

(iii) To show H is linear transformation

Let α, β and $u, \nu \in V$

$\Rightarrow \quad \alpha u + \beta \nu \Leftarrow V$ ($\because$ V is a vector space)

$\therefore \quad H (\alpha u + \beta \nu) = L_{\alpha u+\beta \nu}$ .... (I)

Now $L_{\alpha u+\beta \nu} (f) = f (\alpha u + \beta \nu)$ for $f \in V$

$= \alpha f (u) + \beta f(\nu)$ ($\because$ f is linear functional)

$= \alpha \ L_u (f) + \beta L_\nu (f) = (\alpha \ L_u + \beta \ L_\nu) (f)$

$\Rightarrow \quad L_{\alpha u+\beta \nu} = \alpha \ F(u) + \beta(\nu)$

so that (I) implies

$H(\alpha u + \beta \nu) = \alpha H(u) + \beta H(\nu)$

$\Rightarrow$ H is a linear transformation

Hence H is an isomorphism of V onto V** i.e. the function $\nu \to L_\nu$ is an isomorphism of V onto V**.

Note: $f(\nu)$ is also denoted as $[\nu, f]$ and V* as V'.

Def. Natural mapping:

The mapping $H : V \to V^{**}$ defined as $H(\nu) = L_\nu \ \forall \ \nu \in V$ where $L_\nu(f) = f(\nu) \ \forall \ f \in V^*$ is known as a natural mapping.

Def 6. Reflexivity:

We have shown in above theorem that natural mapping from V to V** is an isomorphism. This property of vector spaces is known as Reflexivity between V and V**. The vector spaces V and V** are called Reflexive.

Remark : If V(F) is not finite dimensional then natural mapping between V and V** cannot be onto

Theorem. Prove that every finite dimensional vector space is reflexive.

Proof. Reproduce the proof of above theorem and then give definition of reflexivity.

Theorem : Let V(F) be a vector space of dim n and V* be its dual space.

Prove for a linear function L on V*, there exists a unique vector $\nu \in V$ such that $L(f) = f(\nu) \ \forall \ f \in V^*$

Proof : Reproduce the proof of above Theorem i.e. the function $\nu \to L_\nu$ is an isomorphism (i.e. 1 - 1 and onto) between V and V**

$\therefore$ for $L \in V^{**}$, $\exists$ unique vector $\nu \in V$

such that $L = L_\nu \quad \Rightarrow \quad L(f) = L_\nu(f) = f(\nu) \ \forall \ f \in V^*$

Hence the result

Some Illustrative Examples

Example 1. Find the dual basis for $B = \{\nu_1, \nu_2\}$ of R^2 over R, where $\nu 1 = (1, 2)$ and $\nu_2 = (1, 5)$

Solution: Given $B = \{\nu_1, \nu_2\}$ is a basis of $V = R^2$ over R

$\because$ dim V* = dim V = 2

$\Rightarrow$ Dual basis of $V = R^2$ over R contains two functional

Let $B_1 = \{\phi_1, \phi_2\}$ be the dual basis of B

Let $\phi_1(x, y) = \alpha x + \beta y$ and $\phi_2(x, y) = \gamma x + \delta y$ for all $(x, y) \in R^2$ and $\alpha, \beta, \gamma\ \delta \in R$

By def $\phi_i(\nu_k) = \delta_{ik} = \begin{cases} 1 \ if \ i = k \\ 0, \ if \ i \neq k \end{cases}$ for $\nu_k \in B$

$\therefore$ ϕ_1 (ν1) = δ_{11} = 1, ϕ_1 (ν_2) = δ_{12} = 0

ϕ_2 (ν_1) = δ_{21} = 0, ϕ_2 (ν_2) = δ_{22} = 0

$\Rightarrow$ ϕ_1 (1, 2) = 1, ϕ_1 (1, 5) = 0, ϕ_2 (1, 2) = 0, ϕ_2 (1, 5) = 1

$\Rightarrow$ α + 2β - 1, α + 5β - ν. γ + 2δ = 0, γ + 5δ = 1

Solving, we get $\alpha = \frac{5}{3}$, $\beta = -\frac{1}{3}$, $\gamma = -\frac{2}{3}$, $\delta = \frac{1}{3}$

$\therefore$ ϕ1 (x, y) = $\frac{5}{3}$ x - $\frac{1}{3}$ y and ϕ_2 (x, y) = - $\frac{2}{3}$ x + $\frac{1}{3}$ y

$\Rightarrow$ $\left\{\frac{5}{3}x - \frac{1}{3}y, \frac{-2}{3}x + \frac{1}{3}y\right\}$ is dual basis of B.

Example 2. Let V(R) be vector space of all polynomials of degree $\leq$ 1 and V* its dual space

Let ϕ_1 : V $\to$ R and ϕ_2 : V $\to$ R such that

ϕ_1 (f(t)) = $\int_0^1 f(t)\,dt$ and ϕ_2 (f(t)) = $\int_0^2 f(t)\,dt$

Find basis {ν_1, ν_2} of V which is dual to {ϕ_1, ϕ_2}

Solution: Here V (R) - Vector space of all polynomials of degree $\leq$ 1

= {α + βt | α, β $\in$ R}

Also ϕ_1 (f(t)) = $\int_0^1 f(t)\,dt$ and ϕ_2 (f(t)) = $\int_0^2 f(t)\,dt$ are given where f(t) $\in$ V.

Let $\nu_1 = \alpha + \beta$t and $\nu_2 = \gamma + \delta$t

By def. ϕ_i (ν_k) = δ_{ik} = $\begin{cases} 1 \; if \; i = k \\ 0, \; if \; i \neq k \end{cases}$ for $\nu_k \in$ V

$\therefore$ ϕ_1 (ν_1) = δ_{11} = 1, ϕ_1 (ν_2) = δ_{12} = 0

ϕ_2 (ν_2) = δ_{21} = 0, ϕ_2 (ν_2) = δ_{22} = 1

But ϕ_1 (ν_1) = $\int_0^1 (\alpha + \beta t)\,dt = \alpha t + \beta \left.\frac{t^2}{2}\right]_0^1 = \alpha + \frac{\beta}{2}$

ϕ_1 (ν_2) = $\int_0^1 (\gamma + \delta t)\,dt = \gamma\, t + \left.\frac{\delta t^2}{2}\right]_0^1 = \gamma + \frac{\delta}{2}$

ϕ_2 (ν_1) = $\int_0^2 (\alpha + \beta t)\,dt = \alpha\, t + \left.\frac{\beta t^2}{2}\right]_0^2 = 2\,\alpha + 2\beta$

$$\phi_2(\nu_2) = \int_0^2 (\gamma + \delta t)\, dt = \gamma t + \frac{\delta t^2}{2}\Big]_0^2 = 2\gamma + 2\delta$$

so that $\alpha + \frac{\beta}{2} = 1, \gamma + \frac{\delta}{2} = 0$

$2\alpha + 2\beta = 0, 2\gamma + 2\delta = 1$

Solving, we get $\alpha = 2$, $\beta = -2$

$\gamma = -\frac{1}{2}, \delta = 1$

16.7 Self Check Exercise - 2

Q. 1 Show that dual of an n-dimemional vector space is n-dimemional

Q. 2 Let V be a vector space of are polynomials of degree $\leq$ 2 over reals. If f_1, f_2, f_3 are linear functional on V given by $f1(\phi(t)) = \int_0^1 \phi(t)\, dt$, $f2(\phi(t)) = \phi'(t)$ for $t = 1$ $f_3(\phi(t)) = \phi(0)$, $\phi(t) = \propto + \beta t + \gamma t_2 \in V$

and $\phi'(t) = \frac{d}{dt}\phi(t) = \beta + 2\gamma t$

Find basis $\{\phi_1(t)]\ \phi_2(t), \phi_3(t)\}$ which is dual to $\{f_1, f_2, f_3\}$

16.8 Summary

We have learnt the following concepts in this unit,

(i) dual space and linear functional

(ii) dual basis

(iii) double dual or Bidval

(iv) Natural mappings

(v) Reflexivity

16.9 Glossary

1. Natural Imbedding-

The mapping H : $V \to V^{**}$ defined as $H(\nu) = L_\nu\ \forall\ \nu \in V$, where $L_\nu(f) = f(\nu)\ \forall\ f \in V^*$ is called natural imbedding

2. Reflexivity - The property of vector space of being natural imbedding from V to V* an isomorphism, is called reflexivity. The vector space V and V** are called reflexive.

16.10 Answers to Self Check Exercises

Self Check Exercise - 1

Ans. $\left\{\frac{5}{3}x-\frac{1}{3}y, \frac{-2}{3}x+\frac{1}{3}y\right\}$ is dual basis of B

Ans. $\left\{2-2t, \frac{-1}{2}+t\right\}$ is a basis of V dual to $\{\phi_1, \phi_2\}$

Self Check Exercise - 2

Ans. Prove it.

Ans. $\{\phi_1, \phi_2, \phi_3\} = \left\{3t-\frac{3}{2}t^2, -\frac{1}{2}t+\frac{3}{4}t^2, 1-3t+\frac{3}{2}t^2\right\}$

16.11 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.

16.12 Terminal Questions

1. Define a non zero functional on a vector space V_3 (R) s.t. if ν = (1, 1, 1) ν = (1, 1, -1) when f (ν) = f (ν) = 0
2. Find a dual basis for B = $\{e_1, e_2\}$ on R_2 over R, where e_1 = (1, 0) and e_2 = (0, 1)
3. Let B = $\{\nu_1, \nu_2, \nu_3\}$ be a basis of R_3 where

   ν_1 = (1, 0, -1), ν_2 = (1, 1, 1), ν_3 = (2, 2, 0).

   Find a dual basis of B.
4. Let B = (ν_1, ν_2, ν_3) is a basis of vector space V_3 (R), where

   ν_1 = (1, -2, 3), $\nu_2 = (1, -1, 1)$ and

   $\nu_3 = (2, -4, 7)$. Find its dual basis

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Unit - 17

Eigen Values and Eigen Vectors

Structure

17.1 Introduction

17.2 Learning Objectives

17.3 Eigen Value And Vectors of Linear Operator

17.4 Self Check Exercise - 1

17.5 Eigen Value And Vectors of A Matrix

17.6 Characteristic Polynomial of A Matrix

17.7 Self Check Exercise-2

17.8 Summary

17.9 Glossary

17.10 Answers to Self Check Exercises

17.11 References/Suggested Readings

17.12 Terminal Questions

17.1 Introduction

Dear students, in this unit we shall discuss the concept of eigen value and eigen vector of a linear operator the same idea is extended to a matrix. Saler on, over main focus will be to discuss characterizes polynomial of a square matrix.

17.2 Learning Objectives

The main Learning objectives of this unit are

(i) to define eigen vectors and eigen values of a linear transformation

(ii) to study eigen values and vectors of a matrix

(iii) to discus chara clerisies polynomial of a matrix

17.3 Eigen Value and Eigen Vectors of Linear Operator-

Definition 1: Let T be a linear operator on vector space V(F). Then a scalar $\lambda \in F$ is called Eigen value of T iff $\exists$ a non-zero vector $\nu \in V$ s.t.

$$T(\nu) = \lambda\nu.$$

Here, Vector ν satisfying above equation is called Eigen Vector of T corresponding to characteristic root λ.

Note: **1.** Characteristic roots are also known as characteristic or proper or spectral or Latent roots (Values) and similarly eigen vectors are also called characteristic or proper or spectrol or latent vectors.

2. If ν is eigen vector of T corresponding to eigen root λ, then for each non zero $k \in F$, $k\nu$ is also eigen vector of T corresponding to eigen values λ.

Since $T(k\nu) = KT(\nu) = k(\lambda\nu) = \lambda(k\nu)$

For example

(i) Let D be a differential operator on vector space V of all differentiable functions.

Since $D(e^{6t}) = 6\, e^{6t}$ for $\nu = e^{6t} \in V \Rightarrow \lambda = 6$ is an eigen value of D and e^{6t} is the eigen vector corresponding to eigen value $\lambda = 6$.

(ii) Let I be the identity operator on vector space V. Since $I(\nu)$ for each non zero $\nu \in V$

$\Rightarrow$ $I(\nu) = 1.\nu$

$\Rightarrow$ $\lambda = 1$ is an eigen value of I and each non zero vector in V is an eigen vector corresponding to eigen value $\lambda = 1$.

Self Check Exercise - 1

Q.1 Let S and T be operator on V(F) and ν be any eigen vector of S and T both show that ν is an eigen vector of operator $\propto S + \beta T$, $\propto, \beta \in F$

Q.2 For each of the operator $T : R^2 \to R^2$ find all eigen values and eigen vectors and basis of eigen space.

(i) $T(x, y) = (3x + 3y, x + 5y)$

(ii) $T(x, y) = (y, -x)$

17.5 Eigen Value And Vectors of A Matrix

Definition 2. Let A be a square matrix of order n over F. Then a scalar $\lambda \in F$ is called eigen value of A if and only if there exists or non-zero vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} \text{ such that } AX = \lambda X \qquad \text{....(i)}$$

Here, the vector X satisfying (i) is called eigen vector of A corresponding to the eigen vector of a corresponding to the eigen value roots λ.

Note

(i) Eigen values of matrix A are also defined as roots of characteristic equation det $(\lambda I_n - A) = 0$

where I_n is identity matrix of order n.

In fact, equation (i) can be written as

$AX = \lambda I_n X$ or $O = (\lambda I_n - A) X$ or $(\lambda I_n - A) = O$

which are homogeneous equations and will have non zero solution ($X \neq O$) when det $(\lambda I_n - A) = O$

(ii) We denote det $(\lambda I_n - A)$ as $\Delta_A(\lambda)$ and call it characteristic polynomial of A

(iii) On putting $\lambda = O$ in $\Delta_A(\lambda) = \det(\lambda I_n - A)$

we get ΔA (O) = det (-A) = (-1)n det A, which is constant term of characteristic polynomial of A.

17.6 Characteristic Polynomial of A Matrix

If X is characteristic vector of A corresponding to characteristic root λ, then for each non-zero scalar $k \in F$, kX is also characteristic vector of X corresponding of characteristic root λ since $A(kX) = k(A x) = k(\lambda X) = \lambda(k X)$ where $kX \neq O$.

Definition (Spectrum) The set having elements as characteristic roots of a linear operator T (matrix A) is called Spectrum of T (A) (matrix A).

Theorem : Let $T : V \to V$ be a linear operator on a finite dimensional vector space V (F). Prove that if $\nu \in V$ is an eigen vector of T, then ν cannot be associated with more than one eigen value of T.

Proof : If possible, let ν correspond to two distinct eigen values λ_1, λ_2 of T,

$T(\nu) = \lambda_1 \nu$ and $T(\nu) - \lambda_2 \nu$

$\Rightarrow \quad \lambda_1 \nu = \lambda_2 \nu$

$\Rightarrow \quad (\lambda_1 - \lambda_2) \nu = O$

$\Rightarrow \quad \nu = O$ $\quad$ [$\because$ λ_1 and λ_2 are distinct i.e. $\lambda_1 \neq \lambda_2$]

which is a contradiction,

as ν being an eigen vector must be a non zero vector

$\therefore$ our supposition is wrong.

Hence the eigen vector ν cannot be associated with more than one eigen value of T.

Another form of Above theorem (For Matrices)

Theorem : If X is a characteristic vector of a matrix A, then prove that X cannot correspond to more than one characteristic roots of A

Proof : If possible, let X be characteristic vector of a matrix A corresponding to two characteristic roots λ_1, λ_2 where $\lambda_1 \neq \lambda_2$.

$\therefore \quad A X = \lambda_1 X$ and $AX = \lambda_2 X$

$\Rightarrow \quad \lambda_1 X = \lambda_2 X \Rightarrow (\lambda_1 - \lambda_2) X = 0 \Rightarrow \lambda_1 - \lambda_2 = 0$ $\quad$ ($\because X \neq 0$)

$\Rightarrow \lambda_1 = \lambda_2$

which is a contradiction

$\therefore$ Our supposition is wrong

Hence X cannot correspond to more than one characteristic roots of A.

Hence the theorem.

Theorem : Let λ be an eigen value of a linear operator T on a vector space V (F). Let V_λ denote the set of all eigen vectors of T corresponding to eigen value λ. Prove that V_λ is a subspace of V (F).

Proof : Here $V_\lambda = \{\nu \in V \mid \nu$ is an eigen vector of T corresponding to eigen value $\lambda\}$

$= \{\nu \in V \mid T(\nu) = \lambda \nu\}$.

Given is that λ be an eigen value of T

$\therefore$ $\exists$ a non zero vector ν' such that $T(\nu') = \lambda\nu'$.

so that $\nu' \in V_\lambda \Rightarrow V_\lambda \neq \phi$

i.e., $V\lambda$ is non-empty set.

Let $\nu_1, \nu_2 \in V_\lambda$ and $\alpha, \beta \in F$.

Since $\nu_1, \nu_2 \in V_\lambda \Rightarrow T\nu_1 = \lambda\nu_1$ and $T\nu_2 = \lambda \nu_2$

$$
\begin{aligned}
\text{Now } T(\alpha \nu_1 + \beta \nu_2) &= T(\alpha \nu_1) + (\beta \nu_2) && [\because \text{ T is a L.T}] \\
&= \alpha T(\nu_1) + \beta T(\nu_2) && [\because \text{ T is a L.T}] \\
&= \alpha (\lambda \nu_1) + \beta (\lambda \nu_2) \\
&= \lambda (\alpha \nu_1 + \beta \nu_2)
\end{aligned}
$$

$\therefore$ $T(\alpha \nu_1 + \beta \nu_2) = \lambda (\alpha \nu_1 + \beta \nu_2)$

$\Rightarrow$ $\alpha \nu_1 + \beta \nu_2$ is an eigen vector corresponding to eigen value λ

$\Rightarrow$ $\alpha \nu_1 + \beta \nu_2 \in V_\lambda$

Hence V_λ is a subspace of V.

Note. This subspace V_λ is called the eigen space or the characteristic spaces of the eigen value λ.

Theorem : Prove that the non zero eigen vectors corresponding to distinct eigen values of a linear operator are linearly independent.

Proof : Let $\nu_1, \nu_2, ...\nu_m$ be m non-zero eigen vectors of a linear operator $T : V \rightarrow V$ corresponding to distinct eigen values $\lambda_1, \lambda_2, ... \lambda_m$ respectively.

$\therefore$ $T(\nu_1) = \lambda_1 \nu_1, T(\nu_2) = \lambda_2 \nu_2, ... T(\nu_m) = \lambda_m \lambda_m$. ...(1)

We want to show that $\nu_1, \nu_2, ...\nu_m$ are L.I. vectors.

We shall prove this result by induction on m.

Step I. Let m = I,

Then ν_1 is L.I. since ν_1 is a non zero vector.

$\therefore$ the result is true for m = I.

Step II. Assume the result is true for the number of vectors less than m.

Step III. Now, we shall show the result is true for m vectors.

Let $a_1 \nu_1 + a_2 \nu_2 + \ldots + a_m \nu_m = O$ ...(2)

$\Rightarrow$ $T(a_1 \nu_1 + a_2 \nu_2 + \ldots + a_m \nu_m) = T(O)$

$\Rightarrow$ $a_1 T(\nu_1) + a_2 T(\nu_2) + \ldots + a_m T(\nu_m) = O$ [Since T is a L.T.]

$\Rightarrow$ $a_1 (\lambda_1 \nu_1) + a_2 (\lambda_2 \nu_2) + \ldots + a_m (\lambda_m \nu_m) = O$ [Using (1)]

$\Rightarrow$ $a_1 \lambda_1 \nu_1 + a_2 \lambda_2 \nu_2 + \ldots + a_m \lambda_m \nu_m = O$ ...(3)

Multiplying (2) on both sides by λm, we get

$a_1 \lambda_m \nu_1 + a_2 \lambda_m \nu_2 + \ldots + a_m \lambda_m \nu_m = O$ ...(4)

$\therefore$ Eq. (3) - Eq. (4) gives

$a_1 (\lambda_1 - \lambda_m) \nu_1 + a_2 (\lambda_2 - \lambda_m) \nu_2 + \ldots + a_m -1 (\lambda_m - 1 - \lambda_m) \nu_m - 1 = O$

$\Rightarrow$ $a_1 (\lambda_1 - \lambda_m) = 0,\ a_2 (\lambda_2 - \lambda_m) = 0, \ldots, a_m -1 (\lambda_m - 1 - \lambda_m) = 0$

[$\because$ $\nu_1, \nu_2, \ldots\ldots \nu_m - 1$ are L.I. because of Step II]

$\Rightarrow$ $a_1 = 0,\ a_2 = 0, \ldots, a_m - 1 = 0$

[$\because$ $\lambda_j - \lambda_m \neq 0$ for $1 \leq i \leq m - 1$ as λ_i are distinct]

Putting these in (2), we get

$a_m \nu_m = O$

$\Rightarrow$ $a_m = 0$ [$\because \nu_m \neq O$]

Thus we have $a_1 = a_2 = \ldots = a_m = 0$

$\therefore$ the vectors $\nu_1, \nu_2, \ldots\ldots \nu_m$ are L.I.

Hence the result.

Another form of above Theorem (For Matrices)

Theorem : Prove that the characteristic vectors corresponding to distinct characteristic roots of a matrix are Linearly Independent.

Proof : Let $X_1, X_2, \ldots\ldots X_p$ be a characteristic vector of a matrix A corresponding to distinct characteristic roots $\lambda_1, \lambda_2, \ldots\ldots \lambda_p$

$\therefore$ $A X_i = \lambda_i X_i$ for $1 \leq i \leq p$ ...(i)

To show $X_1, X_2, \ldots\ldots X_p$ are L.I.

If possible, let $X_1, X_2, ... X_p$ be L.D.

$\therefore$ $\exists$ k ($1 \leq k < p$) such that $X_1, X_2, ... X_k$ are L.I. (*)

but $X_1, X_2, ... X_k X_{k+1}$ are L.D.

so $\exists$ scalars $\alpha_1, \alpha_2, ... \alpha_k, \alpha k+1$, not all zero ...(ii)

such that $\alpha_1 X_1 + \alpha_2 X_2 +....... + \alpha_k X_k + \alpha_{k+1} X_{k+1} = O$ ...(iii)

Per-multiply both sides by A

$\Rightarrow$ $A(\alpha_1 X_1 + \alpha_2 X_2 +....... + \alpha_k X_k + \alpha_{k+1} X_{k+1}) = AO = O$

$\Rightarrow$ $\alpha_1 (A X_1) + \alpha_2 (A X_2) +....... + \alpha_k (A_k X_k) + \alpha_{k+1} (A X_{k+1}) = O$

$\Rightarrow$ $\alpha_1 (\lambda_1 X_1) + \alpha_2 (\lambda_2 X_2) +....... + \alpha_k (\lambda_k X_k) + \alpha_{k+1} (\lambda_{k+1} X_{k+1}) = O$

(using i) ...(iv)

Now, multiply (iii) by λ_{k+1}, we get

$\alpha_1 \lambda_{k+1} X_1 + \alpha_2 \lambda_{k+1} X_2 +....... + \alpha_k \lambda_{k+1} X_k + \alpha_{k+1} (\lambda_{k+1} X_{k+1}) = O$ ...(v)

so (iv) - (v) $\Rightarrow \alpha_1 (\lambda_1 - \lambda_{k+1}) X_1 + \alpha_2 (\lambda_2 - \lambda_{k+1}) X_2$

$+....... + \alpha_k (\lambda_k - \lambda_{k+1}) X_k = O$

But $X_1, X_2, ... X_k$ are L.I. (by *)

$\therefore$ $\alpha_1 (\lambda_1 - \lambda_{k+1}) = \alpha_2 (\lambda_2 - \lambda_{k+1}) = ...= \alpha_k (\lambda_1 - \lambda_{k+1}) = 0$

$\Rightarrow$ $\alpha_1 = \alpha_2 = ...= \alpha_k = 0$

($\because \lambda_1, \lambda_2, ... \lambda_{k+1}$ are all distinct $\Rightarrow \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq ..... \neq \lambda_{k+1}$)

But in (iii), we get $\alpha_{k+1} X_{k+1} = 0$

$\Rightarrow$ $\alpha_{k+1} = 0$ ($\because X_{k+1} \neq O$)

$\therefore$ $\alpha_1 = \alpha_2 = ...= \alpha_k = \alpha_{k+1} = 0$ which contradicts (ii)

So our supposition is wrong

Hence vectors $X_1, X_2, ... X_p$ are L.I.

Theorem 4 : Prove that zero is an eigen value of T iff T is singular operator on V (F).

Proof : Zero is an eigen value of T

iff $\exists$ a non zero vector $\nu \in V$

such that $T(\nu) = 0. \nu = O$

iff T is singular operator.

Hence the result.

Note. The above result implies that 0 (zero) cannot be an eigen value of a non-singular operator.

Another form of above theorem (For Matrices)

Theorem : Prove that 0 is an eigen value of an $n \times n$ matrix A over a field F if and only if A is singular.

Proof : Let f (t) - det (t I - A)

Now 0 is an eigen values of A

iff 0 is a root of $f(t) = 0$ [by cor.]

iff 0 is a root of $\det(tI - A) = 0$

iff $\det(0I - A) = 0$

iff $\det(-A) = 0$

iff $(-I)^n \det A = 0$

iff $\det A = 0$

iff A is singular matrix.

Hence the result.

Theorem : Let λ be an eigen value of an invertible operator T on a vector space V (F). Prove that λ^{-1} is an eigen value of T^{-1}.

Proof : Given T be invertible operator

$\Rightarrow$ T is non-singular

$\Rightarrow$ $\exists$ an eigen value $\lambda \neq 0$ [because of above Note]

$\Rightarrow$ λ^{-1} exists.

Since λ is an eigen value of T, therefore there exists a non zero vector $\nu \in V$ such that

$T(\nu) = \lambda \nu$ [by def. of eigen vector]

Operating T^{-1} on both sides

$\Rightarrow$ $T^{-1}(T(\nu)) = T^{-1}(\lambda \nu)$

$\Rightarrow$ $\nu = \lambda T^{-1}(\nu)$

[since T is linear operator $\Rightarrow$ T^{-1} is also linear operator]

$\Rightarrow$ $\dfrac{1}{\lambda}\nu = T^{-1}(\nu)$

or $T^{-1}(\nu) = \dfrac{1}{\lambda}\nu = \lambda^{-1}\nu$

$\Rightarrow$ λ^{-1} is an eigen value of T^{-1}.

Hence the result.

Theorem : Let A be a non-singular matrix over a field F and $\lambda \in F$ be an eigen value of A. Prove λ^{-1} is an eigen value of A^{-1}.

Proof : Since A is non singular and λ is an eigen value of A.

$\therefore \quad \lambda \neq 0 \qquad (\because$ 0 is an eigen value of A iff A is singular)

$\Rightarrow \quad \lambda$-1 exists.

Let non zero vector X be an eigen vector of A corresponding to eigen value λ

$\therefore \quad AX = \lambda X$

$\Rightarrow \quad A^{-1} (AX) = A^{-1} (\lambda X)$

$\Rightarrow \quad (A^{-1} A) X = \lambda (A^{-1} \lambda)$

$\Rightarrow \quad IX = \lambda (A^{-1} X)$

$\Rightarrow \quad X = \lambda (A^{-1} X)$

$\Rightarrow \quad \lambda^{-1} X = \lambda^{-1} (\lambda A^{-1} X)$

$= 1 (A X)$

$= A^{-1} X$

i.e., $\quad A^{-1} X = \lambda^{-1} X$

λ^{-1} is an eigen value of A^{-1} corresponding to the same eigen vector X.

Hence the result.

Theorem : Let $T : V \rightarrow V$ be a linear operator on a finite dimensional vector space V (F). Prove that the following statements the equivalent.

(i) $\lambda \in F$ is an eigen value of T

(ii) The operator λ 1 - T is singular and hence non-invertible

(iii) det. $(\lambda I T) = 0$

Proof : To show (i) $\Rightarrow$ (ii)

Since $\lambda \in F$ is an eigen value of T

$\therefore \quad \exists$ a non zero vector $\nu \in V$ such that

$T (\nu) = \lambda \nu$

$\Rightarrow \quad T (\nu) = (I (\nu)) \qquad [\because I (\nu = \nu$ for identity operator I]

$\Rightarrow \quad T (\nu) = (\lambda I) (\nu)$

$\Rightarrow \quad (\lambda I - T) (\nu) = O$ where ν is a non zero vector

$\Rightarrow \quad \lambda I - T$ is singular operator and hence non-invertible.

To show (ii) $\Rightarrow$ (iii)

Let λ I - T be a singular operator on V (F)

$\Rightarrow$ [λ I - T ; B] is a singular matrix relative to ordered basis B for V

$\Rightarrow$ det ([λ I - T ; B]) = 0

$\Rightarrow$ det (λ I - T) = 0.

To show (iii) $\Rightarrow$ (i)

Let det (λ I - T) = 0

$\Rightarrow$ det ($[\lambda I - T ; B] = 0$, where B is an ordered basis for V

$\Rightarrow$ [λI - T] is a singular matrix

$\Rightarrow$ $\lambda I - T$ is a singular operator on V

$\Rightarrow$ (λI - T) (ν) = O where $\nu \neq O \in V$

$\Rightarrow$ (λI) (ν) - T (ν) = O

$\Rightarrow$ λ.I (ν) = T (ν)

$\Rightarrow$ $\lambda\ \nu, = T\ (\nu)$

$\Rightarrow$ $T\ (\nu) = \lambda\ \nu$ for non zero vector $\nu \in V$

$\Rightarrow$ λ is an eigen value of T

Thus we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) Hence the result.

Cor. The eigen values of a linear operator T on a finite dimensional vector space V (F) are given by

det (λ I - A) = 0 where I is identity matirx and A = [T ; B],

where B is ordered basis for V

Proof. Here λ is an eigen value of T

iff det (λI - T) = 0

iff det [λI - T ; B] = 0

iff det ([λI ; B] - [T ; B]) = 0

iff det $(\lambda I - A) = 0$

Another Form of above theorem (for matrices)

Theorem : Let A be a matrix of order n over the field F. Prove $\lambda \in F$ i.e. characteristic not of A iff λI - A is singular i.e. | λI - A| = 0

Proof. Firstly Let λ be a characteristic root of a square matrix A

$\Rightarrow$ $\exists$ a non zero column vector X of A

such that AX = λX

$\Rightarrow \quad AX = \lambda IX \qquad (\because\ IX = X)$

$\Rightarrow \quad (\because I - A)\ X = O$

$\Rightarrow \quad \lambda I - A$ is singular $\qquad (\because\ X \neq 0)$

$\Rightarrow \quad |\lambda I - A| = 0$

Conversely:

Given $\lambda I - A$ is singular

i.e. $\quad |\lambda I - A| = 0$

$\Rightarrow \quad \exists$ a non zero solution X, such that

$$(\lambda I - A)\ X = O$$

or $\quad \lambda IX = AX$

or $\quad \lambda X = AX$

or $\quad AX = \lambda X$

$\Rightarrow \quad \lambda$ is characteristic root of A

Hence the result

Theorem : Let $T : V \to V$ be a linear operator on a finite dimensional vector space V (F). Prove that the number of eigen values of T cannot exceed the dimension of vector space V (F).

Proof: Given V is a finite dimensional vector space over F

Let us assume dim V = n.

Now λ is an eigen value of T

iff det $(\lambda I - T) = 0$

i.e., the eigen values of T are the roots of equation

$$\det (xI - T) = 0 \qquad \text{.... (1)}$$

Since dim V = n, so any matrix representation of T is of order $n \times n$

$\therefore \quad$ the matrix representation of $xI - T$ is also of order $n \times n$

$\Rightarrow \quad$ The det $(xI - T)$ is a polynomial of degree n in x.

But the eigen values of T are roots of this polynomial. $\qquad$ [Because of (1)]

$\therefore \quad$ number of eigen values cannot exceed the degree n of the polynomial det $(xI - T)$

Hence the number of eigen values of T cannot exceed the dim V.

How to find eigen values and eigen vectors of a Linear operator

(i) Firstly, Let on ordered basis B for vector space V (F)

(ii) Find the matrix T w.r.t. basis B and call it A and solve it for $\lambda \in F$.

(iii) Let $v \in V$ be an eigen vector of T corresponding to eigen value λ, then

$(\lambda I - T)(v) = 0$

$\Rightarrow \quad \left[(\lambda I - T)(v)\right]_B = O$

$\Rightarrow \quad \left[(\lambda I - T)\right]_B [v]_B = O$

$\Rightarrow \quad (\lambda U - A) X = O, \qquad X = [v]_B$

Now solve this matrix equation to find L.I. solution set and this L.I. set of solution will be a basis for eigen space of λ.

Some Illustrative Examples

Example 1. For each of the following matrices find characteristic polynomial

(i) $\begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & 1 \end{bmatrix}$

Solution:

(i) Let A = $\begin{bmatrix} 3 & -7 \\ 4 & 5 \end{bmatrix}$

$\therefore$ Characteristic polynomial if A is

$|t\,I - A| = \begin{vmatrix} t-3 & 7 \\ -4 & t-5 \end{vmatrix}$

$= (t - 3)(t - 5) + 28$

$= t^2 = 8t + 43$

$\therefore \quad \Delta(t) = t2 - Bt + 4.$

(ii) Let A = $\begin{bmatrix} 2 & 3 & -2 \\ 0 & 5 & 4 \\ 1 & 0 & 1 \end{bmatrix}$

The Characteristic Polynomial of A is,

$\Delta(t) = |tI - A| = \begin{bmatrix} t-2 & -3 & 2 \\ 0 & t-5 & -4 \\ -1 & 0 & t-1 \end{bmatrix}$

$= (t - 2)(t - 5)(t - 1) - (12 - 2t + 10)$

$= t^3 - 8t^2 + 17t - 10 - 22 + 2t$

$\Delta(t) = t^3 - 8\,t^2 + 19\,t - 32$

Example 2. Find all the eigen values and eigen vectors of following matrices

(a) $\begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Solution: (a) Let, A - $\begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$

The eigen values of A are the values of t such that, |t I - A| = 0

$\Rightarrow \quad \begin{vmatrix} t-1 & -1 \\ 0 & t-i \end{vmatrix} = 0$

$\Rightarrow \quad (t - 1)\,(t - i) = 0 \qquad \Rightarrow \qquad t = 1, i$

$\therefore$ t = 1, i are the two eigen values of A.

To find eigen vector associated to eigen value t = 1 :

Putting, t = 1 in (t I - A) = O

where, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ is eigen vector of A associated to eigen value t = 1

$\therefore \quad \begin{bmatrix} 0 & -1 \\ 0 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = O$

$\Rightarrow \quad \begin{bmatrix} -y \\ -i\,y \end{bmatrix} = O$

$\Rightarrow \quad \left.\begin{array}{r} -y = 0 \\ -i\,y = 0 \end{array}\right] \qquad \Rightarrow \qquad y = 0$

Thus, y = 0, x can have any value

$\therefore \quad X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is required eigen vector associated to eigen value t = 1

To find eigen vector associated to eigen value t = i :

Putting t = i in (t I - A) X = O

where, X = $\begin{bmatrix} x \\ y \end{bmatrix}$ is the eigen vector of A associated to eigen value t = i

$\therefore \qquad \begin{vmatrix} i-1 & -1 \\ 0 & 0 \end{vmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ (i - 1) x - y = 0

$\Rightarrow$ y = x (i - 1)

let, x = k, k $\neq$ 0

Thus, y = (i - 1) k.

$\Rightarrow$ X = $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ (i-1)k \end{bmatrix}$ is required eigen vector associated to eigen value t = i

(b) Given A = $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

The eigen values of A are the values of t such that |t I - A| = 0

$\Rightarrow \qquad \begin{bmatrix} t-2 & 0 & -1 \\ 0 & t-2 & 0 \\ -1 & 0 & t-2 \end{bmatrix} = 0$

Expand by C_2

$\Rightarrow$ (t - 2) $\begin{vmatrix} i-2 & -1 \\ -1 & t-2 \end{vmatrix} = 0$

$\Rightarrow$ (t - 2) ((t - 2)2 - 1) = 0

$\Rightarrow$ (t - 2) (t2 - 4t + 3) = 0

$\Rightarrow$ (t - 2) (t - 3) (t - 1) = 0

$\Rightarrow$ t = 1, 2, 3 which are eigen values of A.

To find eigen vector associated to eigen value = 1

Putting t = 1 in (t I - A) X = O

where X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector associated to eigen value t = 1

$$\therefore \quad \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ - x - z = 0 $\Rightarrow$ z = - x

and - y = 0 and y = 0

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x \Rightarrow$$ (1, 0, -1) is eigen vector corresponding to $\lambda = 1$

To find eigen vector associated to eigen value = 2

Putting t = 2 in (t I - A) X = O

where X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector associated to eigen value t = 2

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ - z = 0 and - x = 0

$\Rightarrow$ x = z = 0 and y can take any value

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y$$

$\Rightarrow$ (0 1 0) is eigen vector associated to eigen value t = 2

To find eigen vector of A associated to eigen value = 3

Putting t = 3 in (t I - A) X = O

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ x - z = 0, y = 0 and - x + z = 0

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x$$

$\Rightarrow$ (1, 0, 1) is eigen vector associated to eigen value t = 3

Hence basis of eigen space

17.7 Self Check Exercise - 2

Q. 1 find all the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$$

Q. 2 Find all the eigen values and eigen vectors of the matrix

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

17.8 Summary

We have learnt the following concepts in this unit :

(i) eigen values and eigen vectors of a linear operator

(ii) eigen values and eigen vectors of a matrix

(iii) characteristic polynomial of a square matrix

(iv) method to find eigen values and eigen vectors of a linear operator etc.

17.9 Glossary

1. Eigen Space - The subspace Vλ is called eigen space or the characteristic spaces of eigen value λ.

2. Spectrum - The set having elements as characteristic root of a linear operator T (matrix A) is called spectrum of T(A) (matrix A).

17.10 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 easy to show

Ans. 2. (i) 2, 6 ; {(3, -1)}, {(1, ν)}

(ii) No eigen value, no eigen vector

Self Check Exercise - 2

Ans.1 t - 1, i are two eigen values of A

and X = $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is req. eigen vector

Ans. 2 t = 1, 2, 3 are eigen values of A and (1, 0, 1) is the eigen vector corresponding to t = 3

and basis of eigen space

= {(1, 0, -1), (0, 1, 0), (1, 0, 1)}

17.11 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.

17.12 Terminal Questions

1. Find all eigen values and basis for each eigen space of linear operator

   $T : R^3 \rightarrow R^3$ defined by

   T (x, y, z) = (x + y + z, 2y + z, 2y + 3z)

2. Find characteristic equation and characteristic roots of zero and identity matrix of order n.

3. If λ is a characteristic root of a non-singular matrix A, prove that $\frac{|A|}{\lambda}$ is a characteristic voot of adj A.

4. Find eigen values and eigen vectors of the matrix

   A = $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

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Unit - 18

Diagonalizable of Operators and Matrices Eigen Values and Eigen Vectors (Continued)

Structure

18.1 Introduction

18.2 Learning Objectives

18.3 Diagonalizable Operators

18.4 Diagonalizable Matrix

18.5 Self Check Exercise

18.6 Summary

18.7 Glossary

18.8 Answers to self check exercises

18.9 References/Suggested Readings

18.10 Terminal Questions

18.1 Introduction

Dear students, continuing our discussion on the topic eigen values and eigen vectors, we shall, in this unit discuss the concept of diagonalizable of operators and matrices. An important theorem a linear operator be diagonalizable is also proved.

18.2 Learning Objectives

The main Learning Objectives of this unit are

(i) what are diagonalizable operators

(ii) diagonalizable matrix

(iii) A necessary and sufficient condition for a linear operator to be diagnosable diagonalizable is also proved.

18.3 Definition Diagonalizable Operator

Let $T : V \rightarrow V$ be a linear operator on a finite dimensional vector space V (F). Then T is called diagonalizable operator iff there exists a basis of V such that matrix representation of T relative to this basis is a diagonal matrix.

18.4 Definition Diagonalizable Matrix

Let A be a matrix of order n. Then A is called diagonalizable iff these exists a non singular matrix P such that P^{-1} AP = B where B is a diagonal matrix i.e. iff A is similar to a diagonal matrix

$$\begin{bmatrix} \because \text{ A square matrix } B \text{ is called similar to } A \text{ iff} \\ \exists \text{ a non singular matrix } P \text{ such that } B = P^{-1}AP \end{bmatrix}$$

Theorem: Prove that a linear operator T : V → V is diagonalizable iff there exists a basis of V all of whose elements are eigen vectors of V. And in this case, the diagonal elements of matrix of T are the corresponding eigen values.

Proof. Given T : V → V is a linear operator on V

Suppose dim V = n.

Firstly suppose T is diagonalizable operator

$\therefore$ $\exists$ a basis B = $\{v_1, v_2, \ldots, v_n\}$ of V such that matrix of T relative to basis B is a diagonal matrix. [by def.]

$$\text{Let } [T ; B] = \begin{bmatrix} d_1 & 0 & 0 & \ldots & 0 \\ 0 & d_2 & 0 & \ldots & 0 \\ 0 & 0 & d_3 & \ldots & 0 \\ \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & d_n \end{bmatrix}$$

$\therefore$ $T(v_1) = d_1 v_1 + 0\, v_2 + \ldots + 0\, v_n$

$T(v_2) = 0\, v_1 + d_2\, v_2 + \ldots + 0\, v_n$

....

....

$T(v_n) = 0\, v_1 + 0\, v_2 + \ldots + d_n\, v_n$

i.e., $T(v_1) = d_1\, v_1, T(v_2) = d_2\, v_2, \ldots, T(v_n) = d_n\, v_n$

$\Rightarrow$ the vectors $v_1, v_2, \ldots, v_n$ [which are non zero as no element of basis can be a zero] are eigen vectors of T corresponding to eigen values $d_1, e_2, \ldots, d_n$ respectively.

Hence the diagonal elements of [T ; B] are the corresponding eigen values.

Conversely.

Let $\exists$ a basis B = $\{v_1, v_2, \ldots, v_n\}$ of T such that each of the elements of basis B is an eigen vector of T

Suppose d_1, d_2, ..., d_n be eigen values of T belonging to eigen vectors ν_1, ν_2, ..., ν_n respectively

$T(\nu_1) = d_1 \nu_1 = d_1\nu_1 + 0\,\nu_2 + ... + 0\,\nu_n$

$\therefore$ $T(\nu_2) = d_2 \nu_2 = 0\,\nu_1 + d_2 \nu_2 + ... + 0\,\nu_n$

....

....

$T(\nu_n) = d_n \nu_n = 0\,\nu_1 + 0\,\nu_2 + ... + d_n \nu_n$

$$\Rightarrow \quad [T\,;\,B] = \begin{bmatrix} d_1 & 0 & 0 & ....0 \\ 0 & d_2 & 0 & .....0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & d_n \end{bmatrix} = \begin{bmatrix} d_1 & 0 & 0 & ....0 \\ 0 & d_2 & 0 & .....0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & d_n \end{bmatrix}$$

$\Rightarrow$ The matrix of T relative to basis B is diagonal and diagonal elements of matrix are the eigen values of T.

Hence T is diagonalizable operator.

Note: The above eigen values need not necessarily be distinct.

Another form of the Above Theorem

A square matrix A of order n is similar to a diagonal matrix B iff A has n linearly independent eigen vectors. And in this case, the diagonal elements of B are the corresponding eigen values.

Proof. Firstly let A be a diagonalizable matrix over the field F

$\Rightarrow$ $\exists$ a nonsingular matrix P such that $P^{-1}AP = B$ where B is a diagonal matrix over the field F

$\Rightarrow$ $P(P^{-1}AB) = PB$

$\Rightarrow$ $(PP^{-1})AP = PB$

$\Rightarrow$ $I(AP) = PB \Rightarrow AP = PB$ ... (i)

Let $B = \text{diag}(d_1, d_2, ... d_n)$ i.e. $B = \begin{bmatrix} d_1 & 0 & 0 & ....0 \\ 0 & d_2 & 0 & .....0 \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & d_n \end{bmatrix}$

$\Rightarrow$ The eigen values of B are its diagonal entries d_1, d_2 .... d_n

($\because$ A and B are similar and similar matrices have same eigen values)

Let $C_1, C_2 \ldots. C_n$ be n columns of P

and det P $\neq$ 0 ($\because$ P is non singular)

$\Rightarrow$ columns $C_1, C_2 \ldots\ldots.. C_n$ are L.I [$\because$ of a result from matrix theory] ... (ii)

$\therefore$ (i) $\Rightarrow$ A $[C_1, C_2 \ldots. C_n] = [C_1\ C_2 \ldots.. C_n] \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_n \end{bmatrix}$

Equating columns on both sides

We get $A\,C_1 = d_1\,C_1$, $AC_2 = d_2\,C_2$, ... $AC_n = d_n\,C_n$

$\Rightarrow$ C^t_is ($1 \leq i \leq n$) are eigen vectors of A for corresponding eigen values d^t_is ... (iii)

$\Rightarrow$ Matrix A has n L.I eigen vectors ($\because$ of ii & iii)

Conversely Now suppose A has n L.I eigen vectors say $X_1, x_2, \ldots.. X_n$ corresponding to eigen values $\lambda_1, \lambda_2, \ldots. \lambda_n$ respectively

$\therefore$ $A\,X_i = \lambda_i\,X_i$ ($1 \leq i \leq n$) ... (iv)

Consider a matrix $P = [X_1\ X_2 \ldots.. X_n]$

$\Rightarrow$ P is non singular ($\because$ X^t_is are L.I)

so (iv) $\Rightarrow$ AP = PB where B = diag ($\lambda_1, \lambda_2, \ldots. \lambda_n$)

$\Rightarrow$ P^{-1} AP = B

$\therefore$ A is similar to diagonal matrix B

$\Rightarrow$ A is a diagonalizable matrix

Note: To find a diagonal matrix B such that A and B are similar, we take B = P^{-1} AP where P is the matrix whose columns are linearly independent eigen vectors.

Some Illustrative Examples

Example 1. Diagonalize the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

Solution: Let us take

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$\Rightarrow$ A - λ = $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ - $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

= $\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$

$\Rightarrow$ Characteristic equation of A is |A - λ I| = 0

or $\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$ = 0

$\Rightarrow$ (1 - λ) (2 - λ) - 6 = 0

or Y^2 - 3λ - 4 = 0

$\therefore$ λ = -1, 4

$\Rightarrow$ λ = -1, 4 are eigen values of A

To find eigen vectors

Let X = $\begin{bmatrix} x \\ y \end{bmatrix} \neq 0$ be eigen vector corresponding to eigen value λ = -1, given by

AX = λX or (A - λI) X = O

or (A - (-1) I) X = O

or $\begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 2+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (operate $R_2 \rightarrow R_2 - \frac{3}{2} R_1$)

Now coefficient matrix of these equations is of rank 1.

$\therefore$ This equation has only one L-I eigen vector corresponding to value -1.

The equn can be written as

2x + 2y = 0 or x = -y

Take y = -1, $\therefore$ x = 1

So X = $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to value -1.

Similarly the eigen vector X = $\begin{bmatrix} x \\ y \end{bmatrix} \neq$ O corresponding to value λ = 4 is

AX = 4X or (A - 4I) X = O.

$$\Rightarrow \quad \begin{bmatrix} 1-1 & 2-0 \\ 3-0 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ - 3x + 2y = 0 or $x = \frac{2}{3}y$

Take y = 3, Then x = 2, so

$X = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigen vector of λ corresponding to $\lambda = 4$.

$$\therefore \quad P = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

To find P^{-1}, Here det P = 3 + 2 = 5

adj $P = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$, $\therefore$ P-1 = $\frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

Now $P^{-1} AP = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 8 \\ 1 & 12 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} -5 & 0 \\ 0 & 20 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ which is diagonalizable at A.

Example 2. If P is 2×2 matrix over a field F, then show that characteristic eq=n of P is λ^2 - t r(P) λ + det (P) = 0, where t_r (P) = trace of P, det (P) = Determinant of P.

Solution: Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}_{2\times 2}$ over F

$\therefore$ Characteristic eq = n of P is

$$\begin{bmatrix} \lambda - p_{11} & -p_{12} \\ -p_{21} & \lambda - p_{22} \end{bmatrix} = O$$

$\Rightarrow$ $(\lambda - p_{11})(\lambda - p_{22}) - p_{12} p_{21} = 0$

$\Rightarrow$ $\lambda_2 - t_r(P)\lambda + \det(P) = 0$

[$\because$ t_r (P) = sum of diagonal entries = $p_{11} + p_{22}$

and = det (P) = $p_{11} p_{22} - p_{12} p_{21}$]

Hence proved

18.5 Self Check Exercise - 2

Q. 1 For the matrix A

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$, find an invertible matrix P s.t. P^{-1} AP is diagonal matrix.

Q. 2 Show that similar matrices have the same determinant

18.6 Summary

We have learnt the following concepts in this unit :

(i) diagonalizable operators

(ii) diagonalizable matrix

18.7 Glossary

Diagonalizable operators-

Let $T : V \to V$ be a L.O on a finite binominal vector space V(F). Then T is called diagonalizable operator iff $\exists$ a basis of V such that matrix representation of T relative to this basis is a diagonal matrix.

18.8 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 X1 = (1, 1, 0), X2 = (1, 0 1) are L I eigen vectors of A for eigen value -2

and (1, 1, 2) is the eigen vector associated to t = 4.

Ans. 2. Prove it.

18.9 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.

3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.

18.10 Terminal Questions

1. Prove that the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable over the field C.

2. Let $T : R^3 \to R^3$ be a linear operator defined as T (x, y, z) = (3x + y + 4z, 2y + 6z, 5z) Is T diagonalizable?

3. Find an invertible matrix P s.t. $P^{-1}AP$ is diagonal matrix, where

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Also write the diagonal matrix

4. Find the matrix of eigen vectors of matrix A, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Is this diagonalizable? Give reason.

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Unit - 19

Isomorphism And Invertibility

Structure

19.1 Introduction

19.2 Learning Objectives

19.3 Injective, Surjective And Bijective Transformation

19.4 Singular And Non-Singular Transformations

19.5 Self Check Exercise-1

19.6 Isomorphic Vector Space

19.7 Self Check Exercise - 2

19.8 Summary

19.9 Glossary

19.10 Answers to self check exercises

19.11 References/Suggested Readings

19.12 Terminal Questions

19.1 Introduction

Dear students, in this unit we shall extend our discussion on linear transformations to the concept of isomorphism and inevitability. An isomorphism is an invert table linear map. Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

19.2 Learning Objectives

In this unit we shall learn the following concepts

(i) one-one (injective) transformation

(ii) onto (Surjective) ttansformation

(iii) one-one onto or bijective transformation

(iv) singular and non-singular transformation

(v) isomorphic vector space etc.

19.3 Injective, Surjective and Bijective Transformation -

Definition - one-one (Injective) Transformation -

Let $T : V \rightarrow W$ be a L.T. then T is called one-one or injective iff for all x, y $\in$ V, T (x) = T(y) $\Rightarrow$ x = y.

Definition. Onto (Surjective) Transformation -

Let T : V $\rightarrow$ W be a L.T. then T is called onto or surjective iff for each w$\in$W, $\exists$ $\nu \in$ V s.t. w = T (ν) or

W = Range of T

Definition. One-one onto or Bijective Transformation -

Let T : V $\rightarrow$ W be a linear transformation then T is called one-one onto or bijective iff it is both one-one (injective) and onto (surjective)

Note: Bijective transformation is also called as isomorphism.

19.4 Definition Non Singular Transformation.

A L.T., T : (V) $\rightarrow$ W (F) is said to be non-singular iff the null space of T is the zero space {0}, i.e., the null space consists of only the zero element.

Thus if T (ν) = 0 $\Rightarrow$ ν = 0 for all $\nu \in$ V

or if $\nu \neq 0$ $\Rightarrow$ T (ν) $\neq$ 0 for all $\nu \in$ V

Then T is said to be non-singular.

Definition Singular Transformation.

A L.T., T : V (F) $\rightarrow$ W (F) is said to be singular iff the null space of T contains at least one non-zero vector.

Thus if $\nu \neq 0$ $\Rightarrow$ T (ν) = 0 for some $\nu \in$ V

Thus T is said to be singular.

Theorem 1. Prove that a linear transformation T : V $\rightarrow$ W is non-singular iff T is one-one.

Proof. Assume that T : V $\rightarrow$ W is non singular.

To Prove T is one-one.

Let T (ν_1) = T (ν_2) for $\nu_1, \nu_2 \in$ V

$\Rightarrow$ T (ν_1) - T (ν_2) = 0

$\Rightarrow$ T ($\nu_1 - \nu_2$) = 0 [$\because$ T is L.T.]

$\Rightarrow$ $\nu_1 - \nu_2 = 0$ [$\because$ T is non-singular]

$\Rightarrow$ $\nu_1 = \nu_2$

$\therefore$ T is one-one.

Conversely.

Let T be one-one

To prove T : V $\rightarrow$ W is non-singular

Let $\nu \in$ V such that T (ν) = 0

$\Rightarrow \quad T(\nu) = T(0) \qquad [\because$ T is L.T. so $0 = T(0)]$

$\Rightarrow \quad \nu = 0$

Thus $\quad T(\nu) = 0 \quad \Rightarrow \quad \nu = 0$ for $\nu \in V$

$\therefore \quad T : V \to W$ is non-singular.

Theorem 2. If V and W are finite dimensional vector spaces such that dim V = dim W. Then Prove that a linear transformation $T : V \to W$ is one-one iff T is onto.

Proof. T is one-one iff T is non-singular

iff N (T) = {0} where N (T) is the null space of T

iff Nullity T = 0

iff Rank T + 0 = dim V = dim W $\qquad [\because$ dim V = Rank T + Nullity T]

iff Rank T = dim W

iff T is onto

Hence the result

Theorem 3. Let V (F) and W (F) be two vector spaces and $T : V \to W$ is a L. Transformation. Assume that V (F) is finite dimensional. Prove that V and range space of T have the same dimensions iff T is non-singular.

Proof. Given dim V = dim (Range T) = Rank T

$\Rightarrow$ Rank T = Rank T + Nullity T

$\Rightarrow$ Nullity T = 0

$\Rightarrow$ Null space of T = {0}

$\Rightarrow$ T is non-singular

Conversely.

Given $T : V \to W$ is non-singular.

$\therefore$ Null space of T is the zero space

$\Rightarrow$ Nullity T = 0

We know dim V = Rank T + Nullity T

$\Rightarrow$ dim V = Rank T + 0

$\Rightarrow$ dim V = dim (Range T)

Hence the result

Theorem 4. Prove that a linear transformation $T : V \to W$ is non singular iff set of images of a linearly independent set is linearly independent.

Proof. Suppose $T : V \to W$ is non-singular L.T.

Let $\{\nu_1, \nu_2, ..., \nu_n\}$ be a L.I. subset of V

We have to show that $\{T(\nu_1) + \alpha_2 T(\nu_2) + ... + \alpha_n T(\nu_n) = 0$ for α_i's scalar

$\Rightarrow \quad T(\alpha_1\nu_1) + T(\alpha_2\nu_2) + ... + T(\alpha_n\nu_n) = 0$ [$\because$ T is a L.T.]

$\Rightarrow \quad T(\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n) = 0$ [$\because$ T is a L.T.]

$\Rightarrow \quad \alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n = 0$ [$\because$ T is non-singular]

$\Rightarrow \quad \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ [$\because \nu_1, \nu_2, ..., \nu_n\}$ is a L.I. set]

So that $\alpha_1 T(\nu_1) + \alpha_2 T(\nu_2) + ... + \alpha_n T(\nu_n) = 0$

$\Rightarrow \quad \alpha_1 = \alpha_2 = ... = \alpha_n = 0$ for α_i's scalar

$\therefore \quad \{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is L.I. set

Conversely : Suppose that the set of images of a L.I. set is L.I.

To Prove : T " $V \rightarrow W$ is non-singular

Suppose $\nu \in V$ such that $T(\nu) = 0$

To show T is non-singular, we shall prove $\nu = 0$

If possible suppose that $\nu \neq 0$

$\Rightarrow \quad \{\nu\}$ is a L.I. set

[$\because$ In vector space single non zero vector is L.I.]

$\Rightarrow \quad \{T(\nu)$ is also a L.I. set [By given]

$\Rightarrow \quad \{0\}$ is a L.I. set [$\because T(\nu) = 0$]

which is a contradiction [$\because$ Singleton zero set is L.D.]

$\therefore$ our supposition is wrong

so that $\nu = 0$

Thus $T(\nu) = 0 \Rightarrow \quad \nu = 0$ for $\nu \in V$

$\Rightarrow$ T is non-singular

Hence the result

Theorem 5. Let $T : V \rightarrow W$ be a linear Transformation. Prove that the following statements are equivalent

(i) T is non-singular

(ii) T is one-one

(iii) B is a Linearly Independent subset of V implies that T (B) is a Linearly Independent subset of W.

Proof. We shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) [See Theorem 19]

(ii) $\Rightarrow$ (iii) Let $B = \{\nu_1, \nu_2, ..., \nu_n)\}$ be L.I. subset of T

To show that T (B) = {T (ν_1), T (ν_2), ..., T (ν_n)} is L.I. subset of W

Consider α_1 T (ν_1) + α_2 T (ν_2) + ... + α_n T (ν_n) = 0 for α_i's scalar

$\Rightarrow$ T ($\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n$) = T (0) [$\because$ T is a L.T.]

$\Rightarrow$ $\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n = 0$ [$\because$ T is one-one]

$\Rightarrow$ $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ [$\because$ B is L.I. set]

Hence T (B) is L.I. subset of W.

(iii) $\Rightarrow$ (i) [See Converse of Theorem 4]

Some Illustrative Examples

Example 1. (i) Show that L.T., T : $R^3 \to R^2$ defined by

T (x, y, z) = (x - y, y - z) is onto but not 1 - 1

(ii) Show that L.T., T : $R^2 \to R^3$ defined by T (x, y) = (x, x - y, x + y) is one-one but not onto.

Solution: (i) Given L.T., T : $R^3 \to R^2$ is defined by

T (x, y, z) = (x - y, y - z)

To show T is onto Let (a, b) $\in R^2$ be any element

$\therefore$ T is onto if $\exists$ (x, y, z) $\in R^3$ such that

T (x, y, z) = (a, b)

or (x - y, y - z) = (a, b)

or x - y = a, y - z = b

which is true if x = a, y = 0, z = - b

so there exists (x, y, z) = (a, 0, - b) $\in R^3$

such that T (x, y, z) = T (a, 0, - b) = (a - 0, 0 - (- b)) = (a, b)

$\Rightarrow$ T is onto

To show T is not 1 - 1

Let (4, 5, 6) and (8, 9, 10) $\in R^3$

such that T (4, 5, 6) = (4 - 5, 5 - 6) = (- 1, - 1)

and T (8, 9, 10) = (8 - 9, 9 - 10) = (- 1, - 1)

$\Rightarrow$ T (4, 5, 6) = T (8, 9, 10)

where as (4, 5, 6) $\neq$ (8, 9, 10)

$\Rightarrow$ T is not 1 - 1.

(ii) To Show T is 1 - 1

Let (x, y) and (z, t) $\in R^2$ such that T (x, y) = T (z, t)

$\Rightarrow$ (x, x - y, x + y) = (z, z - t, z + t)

$\Rightarrow$ x = z, x - y = z - t, x + y = z + t

$\Rightarrow$ x = z and y = t

so that T(x, y) = T (z, t) $\Rightarrow$ (x, y) = (z, t)

$\therefore$ T is 1 - 1

To show T is not onto

Consider (10, 11, 12) $\in R^3$ but there is no (x, y) $\in R^2$ such that T (x, y) = (10, 11, 12)

$\because$ T (x, y) = (10, 11, 12)

$\Rightarrow$ (x, x - y, x + y) = (10, 11, 12)

$\Rightarrow$ x = 10, x - y = 11, x + y = 12

$\Rightarrow$ x = 10, y = -1 and y = 2

which is impossible

Hence T is not onto

Example 2. Show that linear transformation T : $R^3 \rightarrow R^2$ defined by

$T(e_1) = e_1 - e_2$, $T(e_2) = 2e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$

where $\{e_1, e_2, e_3\}$ is a standard basis of R^3, is neither one-one nor onto.

Solution: We know $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$

$\therefore$ $T(e_1) = e_1 - e_2$ $\Rightarrow$ T(1, 0, 0) = (1, 0, 0) - (0, 1, 0) = (1, -1, 0)

$T(e_2) = 2e_2 + e_3$ $\Rightarrow$ T(0, 1, 0) = 2 (0, 1, 0) + (0, 0, 1) = (0, 2, 1)

and $T(e_3) = e_1 + e_2 + e_3$

$\Rightarrow$ T(0, 0, 1) = (1, 0, 0) + (0, 1, 0) + (0, 0, 1) = (1, 1, 1)

Now we shall find T

Let (x, y, z) $\in R^3$ be any vector

Then (x, y, z) = x (1, 0, 0) + y (0, 1, 0) + z (0, 0, 1)

$\Rightarrow$ 1 (x, y, z) - x T (1, 0, 0) + y T (0, 1, 0) + z T (0, 0, 1)

= (1, -1, 0) + y (0, 2, 1) + z (1, 1, 1)

= (x + z, - x + 2 y + z, y + z)

To show T is not 1 - 1

Let T (x, y, z) = (0, 0, 0)

$\Rightarrow$ (x + z, - x + 2y + z, y + z) = (0, 0, 0)

$\therefore$ x + z = 0, - x + 2 y + z = 0, y + z = 0

$\Rightarrow \quad x = y = -z$

Take $z = -1$, $\Rightarrow \quad x = y = 1$

$\therefore \quad (x, y, z) = (1, 1, -1) \neq 0$

so that $T(x, y, z) = (0, 0, 0) \quad \Rightarrow \quad (x, y, z) \neq (0, 0, 0)$

$\Rightarrow$ T is singular

$\Rightarrow$ T is not 1 - 1

To show T is not onto

Consider $(0, 2, 0) \in R^3$ But there is no $(x, y, z) \in R^3$ such that

$T(x, y, z) = (0, 2, 0)$

$\because \quad T(x, y, z) = (0, 2, 0)$

$\Rightarrow \quad (x + z, -x + 2y + z, y + z) = (0, 2, 0)$

$\Rightarrow \quad x + z = 0, -x + 2y + z = 2, y + z = 0$

$\Rightarrow \quad x + z - x + 2y + z = 0 + 2, y + z = 0$

$\Rightarrow \quad y + z = 1, y + z = 0$

which is impossible

Hence T is not onto

Note: Another method to show T is not onto

Find a basis for Range T = (R (T))

$\because \quad B = \{e_1, e_2, e_3\}$ is a basis of R^3

$\Rightarrow \quad \{T(e_1), T(e_2), T(e_3)\}$ generates R(T)

$\Rightarrow \quad \{(1, -1, 0), (0, 2, 1), (1, 1, 1)\}$ generates R (T)

To find L.I. vectors from this set

Consider matrix A, whose rows are generators of T and reduce it to echelon matrix

i.e. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Operate $R_2 \to R_2 - R_1$

$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$ Operate $R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ which is echelon form}$$

$\therefore$ (1, -1, 0) (0, 2, 1) form L.I. set of vectors which generates R(T)

$\Rightarrow$ Range space of T = {(1, -1, 0), (0, 2, 1)} $\subset R^3$

$\therefore$ dim (R(T)) = 2

$\Rightarrow$ T is not onto.

19.5 Self Check Exercise - 1

Q.1 Show that the L.T.

$T : R^2 \rightarrow R^2$ defined by

$T(x, y) = (x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta)$ is an isomorphism

Q. 2 Show that there is no nonsingular L.T. from R^4 to R^3.

19.6 Isomorphic Vector Space

Definition - Two vectors V(F) and W(F) over the same field F are called isomorphic iff $\exists$ a linear transformation $T : V \rightarrow W$ such that it is one-one and onto.

Notation. $V \cong W$, is read as V is isomorphic to W.

Theorem 6. Prove that, two finite dimensional vector spaces V(F) and W(F) over the same field F are isomorphic iff they have the same dimensions i.e.

$V \cong W$ iff dim V = dim W.

Proof. Let $V \cong W$

$\Rightarrow$ $\exists$ a L.T. $T : V \rightarrow W$ s.t. T is one-one and onto

To show. dim V = dim W.

Since V is finite dimensional, so it has a finite basis set

Let $B_1 = \{v_1, v_2 \, ... \, v_n\}$ be a basis of V so that dim V = n

Now

$B_2 = \{T(v_1), T(v_2) \, ..., T(v_n)\}$

which is a subset of W, having n elements To show B_2 is a basis set of W.

To show B_2 is L.I.

Take $\propto_1 T(v_1) + \propto_2 T(v_2) + ... + T(v_n) = 0$

$\Rightarrow$ $T(\propto_1 v_1) + T(\propto_2 v_2) + ...+ T(\propto_n v_n) = 0$ ($\because$ T is L.T. $\propto T(0) = 0$)

$\Rightarrow$ $\propto_1 v_1 + \propto_2 v_2 + ... + \propto_n v_n = 0$ ($\because$ B is basis of V $\Rightarrow v_1, v_2 \, ... v_n$ are L.I.)

$\therefore$ $T(\nu_1), T(\nu_2)$ ..... $T(\nu_n)$ are L.I

$\therefore$ B_2 is L.I. set

To show B_2 spans W.

Let $y \in W \Rightarrow \exists\, x \in V$ s.t.

$y = T(x)$

$\because$ $B = \{\nu_1, \nu_2 ...., \nu_n\}$ is a basis of V, so x can be written as

$x = \beta, \nu_1 + \beta_2\nu_2 + ... + \beta_n\nu_n$, β_i's scalars.

$\Rightarrow$ $y = T(x)$

$\Rightarrow$ $y = T(\beta_1\nu_1 + \beta_2\nu_2 + ... + \beta_n\nu_n)$

$= \beta_1 T(\nu_1) + ... + \beta_n T(\nu_n)$ ($\because$ T is L.T.)

$\therefore$ y is a Linear combination of elements of B_2

Thus B_2 and W

Hence B_2 is a basis of W $\therefore$ dim W = n = dim V.

Conversely. Let dim V = dim W = n

To prove $V \cong W$

Since dim V = dim W = n, there exists basis of V and W, each having n elements.

Let $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$

and $B_2 = \{w_1, w_2, ..., w_n\}$ be basis of V and W respectively.

Let $\nu \in V \Rightarrow \exists$ scalars $\alpha_1, \beta_2, ..., \alpha_n \in F$

such that $\nu = \alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n$ [$\because$ B_1 generates V]

We defined $T(\nu) = \alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n$.

(i) T is well defined. Since $\alpha_1, \alpha_2, ..., \alpha_n$ are unique scalars

so $\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_n w_n$ is unique element of W

$\Rightarrow$ $T(\nu)$ is a unique element of W for given $\nu \in V$

$\Rightarrow$ T is well defined

(ii) T is L.T. Let $x, y, \in V$, $\alpha, \beta \in F$

$\therefore$ $x = \sum_{i=1}^{n} a_i v_i$ and $y = \sum_{i=1}^{n} b_i v_i$ for $a_i, b_i \in F$

$\Rightarrow$ $\alpha x + \beta y = \sum_{i=1}^{n} \left(a\, a_i + \beta b_i\right) v_i$

$\therefore$ T $(\alpha x + \beta y)$ = T $\left(\sum_{i=1}^{n} (a\, a_i + \beta b_i) v_i \right)$

$= \sum_{i=1}^{n} (a\, a_i + \beta b_i) w_i$ [by def. of T]

$= \sum_{i=1}^{n} a\, a_i w_i + \sum_{i=1}^{n} \beta\, a_i w_i$

$= \alpha \left(\sum_{i=1}^{n} a_i w_i \right) + \beta \left(\sum_{i=1}^{n} b_i w_i \right)$

$= \alpha T \left(\sum_{i=1}^{n} a_i v_i \right) + \beta . T \left(\sum_{i=1}^{n} b_i v_i \right)$ [by def. of T]

$= \alpha T(x) + \beta T(y)$

$\therefore$ T is a Linear Transformation

(iii) T is one-one. Let T (x) = T (y) for $x, y \in V$

$\Rightarrow$ $T \left(\sum_{i=1}^{n} a_i v_i \right) = T \left(\sum_{i=1}^{n} b_i v_i \right)$

$\Rightarrow$ $\sum_{i=1}^{n} a_i w_i = \sum_{i=1}^{n} b_i w_i$

$\Rightarrow$ $\sum_{i=1}^{n} (a_i - b_i) w_i = 0$

$\Rightarrow$ $a_i - b_i = 0$ for i = 1, 2, ...n [$\because$ $\{w_1, w_2, ..., w_n\}$ is L.I. set]

$\Rightarrow$ $a_i = b_i$

$\Rightarrow$ $\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$

$\Rightarrow$ x = y

$\therefore$ T is one-one

(iv) T is onto. Let $w \in W$ le arbitrary element

$\Rightarrow$ w is a linear combination of the elements of B_2

[$\because$ B_2 is a basis of W]

$\Rightarrow$ $w = c_1w_1 + c_2w_2 + ... + c_nw_n$ for c_1's $\in F$

Let $\nu = c_1v_1 + c_2\nu_2 + ... + c_n\nu_n \in V$ [$\because$ V is vector space] and $T(\nu) = T(c_1\nu_1 + c_2\nu_2 + ... + c_n\nu_n)$

$= c_1w_1 + c_2w_2 + ... + c_nw_n$ [by def. of T]

$= w$

Thus T is onto

$\therefore$ we have a L.T., $T : V \to W$, which is one-one and onto

$\Rightarrow$ $V \cong W$.

Theorem: Every n-dimensional vector space over the field F is isomorphic to the space F^n.

Proof: Let V be an n-dimensional vector space over the field F.

Take $B = \{\nu_1, \nu_2, ..., \nu_n\}$ be an ordered basis for V.

Let F^n be the vector-space of all n-tuples of the field F.

$\therefore$ $F^n = \{(\alpha_1, \alpha_2, ..., \alpha_\nu) \mid \alpha_i\text{'s} \in F\}$

We have to prove $V \cong F^n$

For this, we define $T :: V \to F^n$ as follows:

If $\nu \in V$, then ν can be uniquely expressed as a linear combination of elements of B

$\Rightarrow$ $\nu - \alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n$ [$\because$ B is a basis for V]

for some unique $\alpha_1, \alpha_2, ..., \alpha_n \in F$

We define T as

$T(\nu) = (\alpha_1, \alpha_2, ..., \alpha_n) \in F^n$.

(i) T is well defined. Since $\alpha_1, \alpha_2, ..., \alpha_n$ are unique for given $\nu \in V$

$\Rightarrow$ $(\alpha_1, \alpha_2, ..., \alpha_n)$ is an element of Fn for given $\nu \in V$

$\Rightarrow$ $T(\nu)$ is a unique element of Fn for given $\nu \in V$

$\Rightarrow$ T is well defined.

(ii) T is linear. Let $x, y \in V$; $\alpha, \beta \in F$

$\Rightarrow$ $x = \sum_{i=1}^{n} a_i v_i$ for some unique a_i's $\in F$

and $y = \sum_{i=1}^{n} b_i v_i$ for some unique b_i's $\in F$

$\therefore$ $T(x) = (a_1, a_2, ..., a_n)$

and $T(y) = (b_1, b_2, ..., b_n)$

And $\alpha x + \beta y = \alpha \sum_{i=1}^{n} a_i v_i + \beta \sum_{i=1}^{n} b_i v_i$

$$= \sum_{i=1}^{n} (\alpha a_i + \beta b_i) v_i$$

$\Rightarrow$ $T(\alpha x + \beta y) = (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \ldots\ldots, \alpha a_n + \beta b_n)$

[By def. of T]

$= (\alpha a_1, \alpha a_2, \ldots\ldots, \alpha a_n) + (\beta b_1, \beta b_2, \ldots\ldots, \beta b_n)$

$= \alpha (a_1, a_2, \ldots\ldots, a_n) + \beta (b_1, b_2, \ldots\ldots, b_n)$

$= \alpha T(x) + \beta T(y)$

$\Rightarrow$ T is a L.T.

(iii) T is one-one. Let $x, y \in V$ such that

$T(x) = T(y)$

$\Rightarrow$ $T\left(\sum_{i=1}^{n} a_i v_i\right) = T\left(\sum_{i=1}^{n} b_i v_i\right)$

$\Rightarrow$ $(a_1, a_2, \ldots\ldots, a_n) = (b_1, b_2, \ldots\ldots, b_n)$ [by def. of T]

$\Rightarrow$ $a_1, = b_1, a_2 = b_2, \ldots\ldots, a_n = b_n$

$\Rightarrow$ $\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$

$\Rightarrow$ $x = y$

$\Rightarrow$ T is one-one.

(iv) T is onto. Let $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n) \in F^n$

$\Rightarrow$ $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

$\Rightarrow$ $\nu = \alpha_1 \nu_1 + \alpha_2 \nu_2 + \ldots\ldots + \alpha_n \nu_n \in V$

$\Rightarrow$ $T(\nu) = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$

$\therefore$ for $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n) \in F^n$ there exists $\nu \in V$

such that $T(\nu) = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$

$\Rightarrow$ T is onto

$\therefore$ $V \cong F^n$

Hence the result.

Theorem : State and prove first ismorphism theorem of linear transformation.

OR

Let V and W we vector spaces over the same field F and T : V → W be a onto linear transformation with Kernel T = K, then $\frac{V}{K} \cong$ W

Proof : Let us define a mapping

$\psi : \frac{V}{K} \to$ W by ψ (K + x) = T(x) for all K + x $\in$ V/K

As K + x $\in$ V/K so that x $\in$ V $\Rightarrow$ T (x) $\in$ W

$\Rightarrow$ ψ (K + x) $\in$ W

(i) To show ψ is well defined

Take K + x, K + y $\in$ V/K such that

K + x = K + y

$\Rightarrow$ x - y $\in$ K

$\Rightarrow$ T (x - y) = 0 ($\because$ K = Kernel T)

$\Rightarrow$ T (x) - T (y) = 0

$\Rightarrow$ T (x) = T (y)

$\Rightarrow$ ψ (K + x) = ψ (K + y)

$\therefore$ ψ is well defined.

(ii) To show ψ is homomorphism

Take K + x, K + y $\in$ V/K so that x, y $\in$ V

Now ψ (K +x) + (K + y)) $\equiv$ ψ (K + (x + y))

= T (x + y)

= T (x) + T (y) ($\because$ T is L.T.)

= ψ (K + x) + ψ (K + y)

And ψ (λ (K + x)) = ψ (K + λ x) for $\lambda \in$ F and K + x $\in$ V/K

= T (λ x)

= λ T(x) ($\because$ T is L.T.)

= λ ψ (K + x)

$\therefore$ ψ is homomorphism.

(iii) To show ψ is 1 -1 : Let K + x, K + y $\in$ V/K

such that ψ (K + x) = ψ (K + y)

$\Rightarrow$ T (x) = T (y)

$\Rightarrow \quad T(x) = T(y) = 0$

$\Rightarrow \quad T(x - y) = 0 \qquad (\because$ T is L.T.)

$\Rightarrow \quad x - y \in K = \text{Kernel } T$

$\Rightarrow \quad K + x = K + y$

$\therefore \quad \psi$ is 1 -1.

(iv) To show ψ is onto

Let $y \in W$ be any element

As $T : V \to W$ is onto

so $\exists\, x \in V$ s.t. $y = T(x)$.

$\because \quad x \in V \Rightarrow K + x \in V/K$ and

$\psi(K + x) = T(x) = y$,

so for all $y \in W$, $\exists\, K + x \in V/K$

s.t. $\psi(K + x) = y$

$\Rightarrow \quad \psi$ is onto

Thus $\psi : V/K \to W$ is homomorphism, 1 - 1 and onto

$\Rightarrow \quad V/K \cong W$

Hence proved.

19.7 Self Check Exercise - 2

Q. 1 Let ν be a vector space of all polynomials in x over F. Show that the map

$T : V \to V$ defined by

$T(f(x)) = -f(x)\ \forall\ f(x) \in V$ is an isomorphism of V onto V.

Q. 2 Prove that the subset of matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ in M_2 (F) $\forall$ a ψ F (field) is a vector subspace over F, which is isomorphic to the field F.

19.8 Summary

We have learnt the following concepts in this unit :

(i) one-one or injective transformation

(ii) onto or surjective transformation

(iii) one-one, onto or bijective transformation

(iv) singular and non-singular transformation

(v) Isomorphic vector space etc.

19.9 Glossary

1. **Isomorphism :** A bijective transformation is called as isomorphism.

2. Every n-dimensional vector space over F is isomorphic to space F^n. (F being field).

19.10 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 First show T is one-one and then show T is onto

Ans. 2. Show it.

Self Check Exercise - 2

Ans.1 Show that T is linear, T is one-one, T is onto.

Ans. 2. Take A = $\begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}$, B = $\begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$ and V = $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, find α A + β B, Then proceed.

19.11 Reference/Suggested Reading

1. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
2. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.
3. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007

19.12 Terminal Questions

1. Show that L.T., T : $R^2 \to R$ defined by T (y, x) = x is onto but not one-one.

2. Show that T : $R^3 \to R^3$ defined by

   $T(x, y, z) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta)$

   is non-singular, θ is any angle.

3. Show that L.T.

   T : $R^2 \to R^2$ defined by

   $T(x, y) = (x - y, x + y) \ \forall \ (x, y) \in R^2$ is bijective

4. Prove that subjset of R^3 consisting of triplets (a, b, c) with c = 0 is a subspace of R^3 which is isomorphic to R^2.

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Unit - 20

Isomorphism And Invertibility

(Continued)

Structure

20.1 Introduction

20.2 Learning Objectives

20.3 Invertibility

20.4 Self Check Exercise

20.5 Summary

20.6 Glossary

20.7 Answers to self check exercises

20.8 References/Suggested Readings

20.9 Terminal Questions

20.1 Introduction

Dear students, we shall continue our discussion on isomorphism and invertibility in this unit too. The main emphasis in this unit will be to study the concept of invertibility. To understand the concept of invertibility, we shall discuss the concept of invertible operator.

20.2 Learning Objectives

The main learning objectives in this unit are

(i) to study the concept of invertible operator

(ii) to study uniqueness of inverse

(iii) to find conditions for invertibility of an operator

(iv) various theorems are proved to find the conditions of invertibility of an operator.

20.3 Invertibility

Definition (Invertible Operator)

A linear operator T : V (F) → V (F) is said to be invertible operator iff there exists an operator S : V (F) → V (F) such that TS = I = ST, where I is an identity operator. Here S is called ; the inverse of T and is denoted by T^{-1}.

Theorem 1 : (Uniqueness of Inverse)

Prove that inverse of an invertible operator is unique.

Proof : Let T : V → V be an invertible operator

If possible, let S_1 and S_2 be two inverse of T

Therefore, by def. $S_1 T = I = TS_1$ ...(1)

$S_2 T = I = TS_2$ ...(2)

where I is identity operator on V.

Now $S_2 = I. S_2$ ($\because$ I is identity operator)

$= (S_1 T) S_2$ (Using (1))

$= S_1 (TS_2)$ (By Associative Property)

$= S_1 . T$ (Using (2))

$= S_1$

$\therefore$ the inverse of an invertible operator must be unique.

Theorem 2 : Let V be a vector space over F and $T : V \to V$ be a linear operator. Prove that T is invertible if an only if T is one-one and onto.

Proof: Suppose $T : V \to V$ is one-one and onto

To Prove T is invertible.

We define $S : V \to V$ as follows

$S (y) = x$ iff $T (x) = y$

To show S is well-defined.

Since $T : V \to V$ is one-one and onto, so there exists a unique $x \in V$ such that $y = T (x)$

so that there exists a unique $x \in V$ such that

$S (y) = x$ [by def. of S]

$\therefore$ S is well-defined

To show S is a Linear operator.

Let $y_1, y_2 \in V$

And $S (y_1) = x_1$ so that $y_1 = T (x_1)$

$S (y_2) = x_2$ so that $y_2 = T (x_2)$

Let $\alpha, \beta \in F \Rightarrow \alpha x_1 + \beta x_2 \in V$

Then $T (\alpha x_1 + \beta x_2) = \alpha T (x_1) + \beta T (x_2)$ [$\because$ T is L.O.]

$= \alpha y_1 + \beta y_2$

$\Rightarrow S (\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$ [by def. of S]

$= \alpha S (y_1) \beta S (y_2)$

Hence $S(\alpha y_1 + \beta y_2) = \alpha S(y_1) + \beta S(y_2)$ for all $y_1, y_2 \in V$ and $\alpha, \beta \in F$

$\Rightarrow$ S is a Linear operator on V.

To show ST = 1 = TS.

Let $x \in V$, Let $y = T(x)$. Then $x = S(y)$

Now $(ST)(x) = S(T(x)) = S(y) = x = I(x)$

$\therefore$ $ST = I$

and $(TS)(y) = T(S(y)) = T(x) = y = I(y)$

$\therefore$ $TS = I$

Hence $ST = I = TS$

$\Rightarrow$ T is invertible and $S = T^{1}$.

Conversely: Let $T : V \to V$ be an invertible linear operator so there exists a linear operator $S : V \to V$ such that $TS = I = ST$.

To show T is one-one. Let $x_1, x_2 \in V$ such that

$$T(x_1) = T(x_2)$$

$\Rightarrow$ $S(T(x_1)) = S(T(x_2))$ [$\because$ S is a mapping]

$\Rightarrow$ $(ST)(x_1) = (ST)(x_2)$

$\Rightarrow$ $I(x_1) = I(x_2)$ [$\because$ ST = I]

$\Rightarrow$ $x_1 = x_2$

$\therefore$ $T : V \to V$ is one-one

To show T is onto. Let $y \in V$

$\Rightarrow$ $S(y)$ is a unique element of V [$\because$ $S : V \to V$ is a mapping]

Let $S(y) = x$, so that $x \in V$

$\therefore$ $T(S(y)) = T(x)$

$\Rightarrow$ $(TS)(y) = T(x)$

$\Rightarrow$ $I(y) = T(x)$ [$\because$ TS = I]

$\Rightarrow$ $y = T(x)$

Hence for given $y \in V$, there exists $x \in V$ such that $y = T(x)$

$\therefore$ T is onto

Hence $T : V \to V$ is one-one and onto

Theorem 3: Let $T : V(F) \to V(F)$ be invertible linear operator. Then show that the inverse mapping T^{-1} defined as $y = T(x)$ iff $x = \Gamma^{1}(y)$ is a linear transformation.

Proof: Given $T : V \to V$ is an invertible linear operator

Define $T^{-1} : V \to V$ be $T^{-1}(y) = x$ iff $y = T(x)$ $\forall$ $y \in V$

T^{-1} is well defined. Let $y \in V$

$\Rightarrow$ there exists a unique $x \in V$ such that $y = T(x)$

$\Rightarrow$ there exists a unique $x \in V$ such that $T^{-1}(y) = x$

[by def. of T^{-1}]

$\Rightarrow$ T^{-1} is will defined.

T^{-1} is linear transformation.

Let $y_1, y_2 \in V$, then $\exists$ unique $x_1, x_2 \in V$ such that

$T(x_1) = y_1$ and $T(x_2) = y_2$

$\Rightarrow$ $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$ [by def. of T^{-1}]

Let $\alpha, \beta, \in V$, $x_1\ x_2 \in V$ $\Rightarrow$ $\alpha x_1 + \beta x_2 \in V$

$\therefore$ $T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$ [$\because$ T is linear]

$= \alpha y_1 + \beta y_2$

$\Rightarrow$ $T^{-1}(\alpha y_1 + \beta y_2) = \alpha x_1 + \beta x_2$

$= \alpha T^{-1}(y_1) + \beta T^{-1}(y_2)$ [by def. of T^{-1}]

$\therefore$ T^{-1} is a linear transformation on V.

Theorem 4: If T, S, U are linear operators on V such that

$TS = UT = I$.

Prove T is invertible and $T^{-1} = S = U$.

Proof: Given T, S, U are linear operators on V such that

$TS = UT = I$

To show T is one-one. Let $x, y \in V$ such that

$T(x) = T(y)$

$\Rightarrow$ $U(T(x)) = U(T(y))$ [$\because$ U is a mapping]

$\Rightarrow$ $(UT)(x) = (UT)(y)$

$\Rightarrow$ $I(x) = I(y)$ [$\because$ UT = I]

$\Rightarrow$ $x = y$

$\therefore$ T is one-one

To show T is onto. Since $S : V \rightarrow V$ is a mapping so for each $y \in V$, there exists $x \in V$ such that $x = S(y)$

$\Rightarrow \quad T(x) = T(S(y))$

$\Rightarrow \quad T(x) = (TS)(y)$

$\Rightarrow \quad T(x) = I(y) \qquad [\because TS = I]$

$\Rightarrow \quad T(x) = I(y)$

$\Rightarrow \quad T(x) = y$

$\therefore$ for each, $y \in V$, $\exists\, x \in V$ such that $T(x) = y$

$\Rightarrow$ T is onto.

Therefore T is one-one and onto

$\Rightarrow$ T is invertible.

Now we show that $T^{-1} = S = U$

Given $TS = I$

$\Rightarrow \quad T^{-1}(TS) = T^{-1} I$

$\Rightarrow \quad (T^{-1} T)(S) = T^{-1}$

$\Rightarrow \quad IS = T^{-1}$

$\Rightarrow \quad S = T^{-1}$

Also given $UT = I$

$\Rightarrow \quad (UT) T^{-1} = I.\, T^{-1}$

$\Rightarrow \quad U(T T^{-1}) = T^{-1}$

$\Rightarrow \quad U.I = T^{-1}$

$\Rightarrow \quad U = T^{-1}$

Hence $T^{-1} = S = U$

Hence the result

Theorem 5: Let V (F) be a vector space and T, S be linear transformations on V. Then show that

(i) If T and S are invertible, then TS is also invertible

and $(TS)^{-1} = S^{-1} T^{-1}$.

(ii) If T is invertible and $0 \neq \alpha \in F$, Then αT is invertible and $(\alpha\, T)^{-1} = \dfrac{1}{\alpha}\, T^{-1}$

(iii) If T is invertible, then T^{-1} is also invertible and $(T^{-1})^{-1} = T$.

Proof: (i) Since T and S are invertible operators (Given)

$\therefore$ T^{-1} and S^{-1} exist such that

$TT^{-1} = I = T^{-1} T$

and $SS^{-1} = I = S^{-1} S$ ... (1)

To show $TS : V \to V$ is one-one ....(2)

Consider $(TS)(x) = (TS)(y)$ for $x, y \in V$

$\Rightarrow$ $T(S(x)) = T(S(y))$

$\Rightarrow$ $S(x) = S(y)$ [$\because$ T is one-one as T is invertible]

$\Rightarrow$ $x = y$ [$\because$ S is one-one as S is invertible]

$\therefore$ TS is one-one

To show $TS : V \to V$ is onto

Since T is onto $\therefore$ $\forall\, y \in V, \exists\, x \in V$ such that $y = T(x)$

Also S is onto $\therefore$ $\forall\, x \in V, \exists\, z \in V$ such that $x = S(z)$

$\therefore$ for all $y \in V, \exists\, z \in V$ such that

$y = T(x) = T(S(z)) = (TS)(z)$

$\therefore$ TS is onto

Thus $TS : V \to V$ is one-one and onto

$\Rightarrow$ TS is invertible operator on V.

Now, we shall show that $(TS)^{-1} = S^{-1} T^{-1}$

Here $(TS)(S^{-1} T^{-1}) = T(SS^{-1}) T^{-1}$

$= T(I) T^{-1} = TT^{-1} = I$

Also $(S^{-1} T^{-1})(TS)$ $= S^{-1}(T^{-1} T) S$

$= S^{-1}(I) S = S^{-1} S = I$

$\therefore$ $(TS)(S^{-1} T^{-1}) = (S^{-1} T^{-1})(TS) = I$

$\Rightarrow$ $(TS)^{-1} = S^{-1} T^{-1}$ [$\because$ TS is invertible]

(ii) To show $\alpha\, T \to V$ is one-one, α is non-zero scalar.

$\Rightarrow$ $\alpha\, .\, T(x) = \alpha\, .\, T(y)$ [by def. of Scalar Multiplication]

$\Rightarrow$ T (x) = T (y) [$\because$ $\alpha \neq 0$]

$\Rightarrow$ x = y

$\therefore$ α T is one-one

To show α T : V $\rightarrow$ V is onto

Let y $\in$ V be arbitrary element

Since $\alpha \in$ F and $\alpha \neq 0$

$\therefore$ $\frac{1}{\alpha} \in F \Rightarrow \frac{1}{\alpha} y \in V$ [$\because$ V is a vector space over F]

Since T is onto

$\therefore$ for $\frac{1}{\alpha} y \in V$, $\exists\, x \in V$ such that T (x) = $\frac{1}{\alpha}$ y

Now (α T) (x) = α, T (x) = $\alpha . \left(\frac{1}{\alpha} y\right) = \left(\alpha . \frac{1}{\alpha}\right)$ y = 1. y = y

so that for y $\in$ V, $\exists$ x $\in$ V such that

y = (α T) (x)

$\therefore$ α T is onto.

Thus α T : V $\rightarrow$ V is one-one and onto

$\Rightarrow$ α T is an invertible operator on V.

Now, we shall show that $(\alpha T)^{-1} = \frac{1}{\alpha} T^{-1}$

Here $(\alpha T)\left(\frac{1}{\alpha} T^{-1}\right) = \left(\alpha . \frac{1}{\alpha}\right) (TT^{-1})$

$= 1.\ TT^{-1} = I$ [by (1)]

Also $\left(\frac{1}{\alpha} T^{-1}\right) (\alpha T) = \left(\frac{1}{\alpha} \alpha\right) (T^{-1} T)$

$= 1.\ (T^{-1} T) = T^{-1} T = 1$ [by (2)]

$\therefore$ $(\alpha T)\left(\frac{1}{\alpha} T^{-1}\right) = \left(\frac{1}{\alpha} T^{-1}\right) (\alpha T) = 1$

$\Rightarrow$ (α T)-1 = $\frac{1}{\alpha} T^{-1}$ [$\because$ α T is invertible]

(iii) To show that T^{-1} is one-one.

Let $y_1 = T(x_1)$ for $x_1, y_1 \in V$

$\Rightarrow$ $x_1 = T^{-1}(y_1)$ [$\because$ T is invertible] .... (3)

and $y_2 = T(x_2)$ for $x_2, y_2 \in V$

$\Rightarrow$ $x_2 = T^{-1}(y_2)$ [$\because$ T is invertible] ..... (4)

For all $y_1, y_2 \in V$ s.t.

$T^{-1}(y1) = T^{-1}(y_2)$

$\Rightarrow$ $x_1 = x_2$ [because of (3) and (4)]

$\Rightarrow$ $T(x_1) = T(x_2)$ [$\because$ T is a mapping]

$\Rightarrow$ $y_1 = y_2$ [because of (3) and (4)]

To show T^{-1} is onto.

Since $T : V \to V$ is a mapping,

so for $x \in V$, $\exists\, y \in V$ such that $y = T(x)$

$\Rightarrow$ for all $x \in V$, there exists $y \in V$

such that $x = T^{-1}(y)$ [$\because$ T is invertible so $y = T(x) \Rightarrow x = T^{-1}(y)$]

Thus T^{-1} is onto

Thus $T^{-1}\ V \to V$ is one-one and onto

$\Rightarrow$ T^{-1} is invertible operator on V.

Also, we have $TT^{-1} = I = T^{-1}T$

Hence $(T^{-1})^{-1} = T$ [by (1)]

Theorem 6: Let $T : V \to W$ be a Linear Transformation where dim V = dim W. Prove that the following statements are equivalent:-

(i) T is invertible

(ii) T is non-singular

(iii) T is onto i.e., Range of T is W

(iv) If $\{\nu_1, \nu_2, \ldots, \nu_n\}$ be a basis of V, Then $\{T(\nu_1)\ T(\nu_2), \ldots, T(\nu_n)\}$ is basis of W.

Proof: We shall prove

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(α) To prove (i) $\Rightarrow$ (ii)

Given T is invertible

$\Rightarrow$ T is one-one

Let $\nu \in V$ such that $T(\nu) = 0$

$\Rightarrow$ $T(\nu) = T(0)$ [$\because$ $T(0) = 0$]

$\Rightarrow$ $\nu = 0$ [$\because$ T is 1 . 1]

$\therefore$ $T(\nu) = 0 \Rightarrow \nu = 0$

$\Rightarrow$ T is non-singular

(b) To prove (ii) $\Rightarrow$ (iii)

Given T is non-singular.

To prove $T : V \to W$ is onto

Let $\{\nu_1, \nu_2, ..., \nu_n\}$ be a basis of V

$\Rightarrow$ $\{\nu_1, \nu_2, ..., \nu_n\}$ is L.I. set

$\Rightarrow$ $\{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is L.I. subset of W.

[$\because$ images of elements of L.I. set under nonsingular L.T. are L.I.]

But dim W = dim V = n

Thus $\{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is a basis set of W

Let $y \in W$

$\Rightarrow$ y is a linear combination of $T(\nu_1), T(\nu_2), ..., T(\nu_n)$

$\therefore$ $y = \alpha_1 T(\nu_1) + \alpha_2 T(\nu_2) + ...+ \alpha_n T(\nu_n)$ for α_i's scalar

$= T(\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n)$ [$\because$ T is a L.T.]

$= T(x)$ where $x = \alpha_1\nu_1 + ... + \alpha_n\nu_n \in V$

$\therefore$ for $y \in W$, there is $x \in V$

such that $y = T(x)$

T is onto i.e., Range of T = W

(c) To prove (iii) $\Rightarrow$ (iv)

Given T is onto i.e., Range of T = W

Let $\{\nu_1, \nu_2, ..., \nu_n\}$ be a basis for V

$\Rightarrow$ $\{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ spans Range of T = W.

Also dim W = dim V = n

$\therefore$ any subset of n elements of W, which generates W, is also a basis of W.

thus $\{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is a basis of W.

(d) To prove (iv) $\Rightarrow$ (i)

Given, if $B_1 = \{\nu_1, \nu_2, ..., \nu_n\}$ is a basis of V.

Then $B_2 = \{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is a basis of W

To show T is one-one

Let $x, y \in V$ such that $T(x) = T(y)$

$\therefore$ $x, y \in V$ $\Rightarrow$ x, y are linear combinations of the elements of B_1.

Let $x = \sum_{i=1}^{n} \alpha_i v_i$ and $y = \sum_{i=1}^{n} \beta_i v_i$ for α_i's $\in$ F and β_i's $\in$ F

$\therefore$ $T(x) = T(y)$

$\Rightarrow$ $T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = T\left(\sum_{i=1}^{n} \beta_i v_i\right)$

$\Rightarrow$ $\sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} \beta_i T(v_i)$

$\Rightarrow$ $\sum_{i=1}^{n} (\alpha_i - \beta_i) T(v_i) = 0$

$\Rightarrow$ $\alpha_i - \beta_i = 0$ $[\because \{T(\nu_1), T(\nu_2), ..., T(\nu_n)\}$ is L.I.$]$

$\Rightarrow$ $\alpha_i = \beta_i$ for all $1 \leq i \leq n$

$\therefore$ $\sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} \beta_i v_i$

$\Rightarrow$ $x = y$

Thus T is one-one

To show T is onto

Let $y \in W$ be arbitrary element

$\Rightarrow$ y is linear combination of the elements of B_2

$\Rightarrow$ $y - \alpha_1 T(\nu_1) + \alpha_2 T(\nu_2) + ...+ \alpha_n T(\nu_n)$ for α_i's scalar

$\Rightarrow$ $y = T(\alpha_1\nu_1 + \alpha_2\nu_2 + ... + \alpha_n\nu_n)$ $[\because$ T is a L.T.$]$

$\Rightarrow$ $y = T(x)$ where $x = \alpha_1\nu_1 + ... + \alpha_n\nu_n \in V$

$\therefore$ for $y \in W$, there is $x \in W$ such that

$y = T(x)$

Thus T is onto

Hence T is one-one and onto

$\Rightarrow$ T is invertible.

Some Illustrative Examples

Example 1: Let $T : V_3(R) \to V_3(R)$ be defined as

$T(x, y, z) = (3x, x - y, 2x + y + z)$

Prove T is invertible and find T^{-1}.

Solution: We know that T is invertible iff T is one-one and onto

(i) To Prove T is one-one

Let $\nu_1 = (x_1, y_1, z_1)$ and $\nu_2 = (x_2, y_2, z_2) \in V_3(R)$

such that $T(\nu_1) = T(\nu_2)$

$\Rightarrow \quad T(x_1, y_1, z_1) = T(x_2, y_2, z_2)$

$\Rightarrow \quad (3x_1, x_1 - y_1, 2x_1 + y_1 + z_1) = (3x_2, x_2 - y_2, 2x_2 + y_2 + z_2)$

$\Rightarrow \quad 3x_1 = 3x_2 \quad \Rightarrow \quad x_1 = x_2$

$x_1 - y_1 = x_2 - y_2 \Rightarrow \quad y_1 = y_2 \qquad [\because x_1 = x_2]$

and $\quad 2x_1 + y_1 + z_1 = 2x_2 + y_2 + z_2 \quad \Rightarrow \quad z_1 = z_2$

$[\because x_1 = x_2 \text{ and } y_1 = y_2]$

$\therefore \quad (x_1, y_1, z_1) = (x_2, y_2, z_2)$

$\Rightarrow \quad \nu_1 = \nu_2$

$\therefore \quad T(\nu_1) = T(\nu_2) \Rightarrow \quad \nu_1 = \nu_2$

$\therefore$ T is one-one.

(ii) To Prove T is onto. Let $(a, b, c) \in V_3(R)$ and we shall show that there exists a vector $(x, y, z) \in V_3(R)$ such that

$T(x, y, z) = (a, b, c)$

$\Rightarrow \quad (3x, x - y, 2x + y + z) - (a, b, c)$

$\Rightarrow \quad 3x = \alpha, x - y = b, 2x + y + z = c$

$\Rightarrow \quad x = \frac{\alpha}{3}, y = \frac{\alpha}{3} - b, z = c - a + b$

Since $a, b, c \in R \quad \Rightarrow \quad x, y, z \in R$

$\therefore$ (x, y, z) = $\left(\frac{a}{3}, \frac{a}{3} - b, c - a + b\right) \in V_3$ (R)

Thus T is onto

Hence T is one-one and onto

$\Rightarrow$ T is invertible

$\therefore$ T (x, y, z) = (a, b, c)

$\Rightarrow$ T-1 (a, b, c) = (x, y, z)

$$= \left(\frac{a}{3}, \frac{a}{3} - b, c - a + b\right)$$

$\Rightarrow$ T^{-1} (a, b, c) = $\left(\frac{a}{3}, \frac{a}{3} - b, -a + b + c\right)$ is the required inverse of T.

Example 2: Let T be a linear operator on R^3 defined by

T (x, y, z) = (2x, 4x - y, 2x + 3y - z)

Show that T is invertible and find T^{-1}.

Solution: We know that T is invertible iff $\exists$ a linear operator S on R^3 such that ST = TS = I.

Let T (x, y, z) = (a, b, c)

$\Rightarrow$ (2x, 4x - y, 2x + 3y - z) = (a, b, c)

$\therefore$ 2x = a, 4x - y = b, 2x + 3y - z = c

$\Rightarrow$ x = $\frac{a}{2}$ $\left| \begin{array}{l} y = 4\left(\frac{a}{2}\right) - b \\ = 2a - b, \end{array} \right|$

z = 2x + 3y - c

= 2 $\left(\frac{a}{2}\right)$ + 3 (2 a - b) - c

= 7a - 3b - c

$\Rightarrow$ x = $\frac{a}{2}$, y = 2 a - b, z = 7 a - 3 b - c

Define: S : $R^3 \rightarrow R^3$ as

$S(a, b, c) = \left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$

(i) Check S is linear operator [Try yourself]

(ii) (ST) (x, y, z) = S (T (x, y, z))

= S (2x, 4x - y, 2x + 3y - z) [by def. of T]

$= \left(\frac{2x}{2}, 2(2x)-(4x-y), 7(2x)-3(4x-y)-(2x+3y-z)\right)$ [by def. of S]

= (x, y, z) = I (x, y, z)

And (TS) (a, b, c) = T (S (a, b, c))

$= T\left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$ [by def. of S]

$= \left(2\left(\frac{a}{2}\right), 4\left(\frac{a}{2}\right)-(2a-b), 2\left(\frac{a}{2}\right)+3(2a-b)-(7a-3b-c)\right)$

= (a, b, c) = I (a, b, c) [by def. of T]

$\therefore$ ST = TS = I

$\Rightarrow$ T is invertible and $T^{-1} = S$

i.e., $T^{-1}(a, b, c) = \left(\frac{a}{2}, 2a-b, 7a-3b-c\right)$

Example 3: Let T be a linear operator on R^3 defined by

T (x, y, z) = (x - 2y - z, y - z, x)

Show that T is invertible and find T^{-1}

Solution: We know that T is invertible iff T is non singular

To show T is non-singular

Let T (x, y, z) = (0, 0, 0) for (x, y, z) $\in R^3$

$\Rightarrow$ (x - 2y - z, y - z, x) = (0, 0, 0)

$\Rightarrow$ x - 2y - z = 0, y - z = 0 and x = 0

$\Rightarrow$ - 2y - z = 0, y - z = 0

$\Rightarrow$ x = 0, y = 0, z = 0

$\Rightarrow$ (x, y, z) = (0, 0, 0)

$\therefore$ [(x, y, z) - (0, 0, 0) $\Rightarrow$ (x, y, z) (0, 0, 0)

$\Rightarrow$ T is non-singular

$\Rightarrow$ T is invertible operator on R^3

To find T^{-1}

Let T (x, y, z) = (a, b, c)

$\Rightarrow$ (x, 2y - z, y - z, x) (a, b, c)

$\Rightarrow$ x - 2y - z = a, y - z = b, x = c

Solving x = c, y = $\frac{-a+b+c}{3}$, z = $\frac{-a-2b+c}{3}$

Thus T^{-1} is given by

T^{-1} (a, b, c) = (x, y, z)

$\Rightarrow$ T^{-1} (a, b, c) = $\left(c, \frac{-a+b+c}{3}, \frac{-a-2b+c}{3}\right)$

Note. For finding inverse of linear operator, three methods have been given in above solved examples.

20.4 Self Check Exercise

Q. 1 If T is a Linear operator on ν s.t. $T^2 - T + I = 0$. Prove that T is invertible.

Q. 2 If T, S be Linear operators on vector space V(F), show that T and S are invertible iff TS and ST are invertible.

20.5 Summary

We have learnt the following concepts in this unit :

(i) what is invertibility and what we call as invertible operators

(ii) various theorems are proved to find the condition of invertibility of an operator etc.

20.6 Glossary

T-1 - If T : V (F) $\rightarrow$ V (F) and

S : V (F) $\rightarrow$ V (F) s.t.

TS = I = ST, I is an identity operator, then S is called inverse of T, denoted by T^{-1}.

20.7 Answers to Self Check Exercises

Ans.1 $T^2 = T - I$, prove if T (x) = T(y) then x = y.

Ans. 2. Consider (TS) (x) = (TS) y, x y $\in$ V show x = y, they T is onto then result follows.

20.8 Reference/Suggested Reading

1. Gilbert Strang, Linear Algebra and its Applications, Thomson, 2007
2. S. Lang, Introduction to Linear Algebra, 2nd Ed., Springer, 2005.
3. David C. Lay, Linear Algebra and its Applications, 3rd Edition, Pearson Education. Asia, Indian Reprint, 2007.

20.9 Terminal Questions

1. Show that each of the following are invertible operators. Find T-1.
   (i) T (x, y, z) = (x - 3y - 2z, y - 4z, x)
   (ii) T (x, y, z) = (x + z, x - z, y)
2. Show that $T : R^2 \rightarrow R^2$ defined by T (x, y) = (ax, by), $a \neq 0$, $b \neq 0$ reals is invertible and $T^{-1}(x, y) = \left(\frac{x}{a}, \frac{y}{b}\right)$
3. If T, S are linear operators on V (F) show that T and S are invertible iff TS and ST are invertible.

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