B.A.: 2nd Year
Mathematics

Course Code: MATH310TH
Course Credits: 04 (SEC)

VECTOR CALCULUS

UNITS: 1 to 20

Dr. Aarti Manglesh

Centre for Distance and Online Education (CDOE)
Himachal Pradesh University,
Summer Hill, Shimla - 171005

Syllabus

Course Code	MATH310TH
Credits=4	L4, T-0, P-0
Name of the Course	Vector Calculus
Type of the Course	Skill Enhancement Course
Continuous Comprehensive Assessment: Based on Assignments	Max. Marks : 30
End Semester Examination	Max Marks: 70 Maximum Times: 3 hrs.

Instructions

Instructions for paper setter: The question paper will consist of two Sections A & B of 70 marks. Section A will be Compulsory and will contain 8 questions of 16 marks (each of 2 marks) of short answer type having two questions from each Unit of the syllabus. Section B of the question paper shall have four Units 1, II, III, and IV. Two questions will be set from each unit of the syllabus and the candidates are required to attempt one question from each of these units. Each question in Units I, II, III and IV shall be of 13.5 marks each.

Instructions for Candidates: Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

SEC 3.4: Vector Calculus

Unit -I

Scalar and vector product of three vectors. Product of four vectors. Reciprocal vectors. Vector differentiation, Scalar valued point functions, vector valued point functions. Derivative along a curve, directional derivatives.

Unit-II

Gradient of a scalar point function. Geometrical interpretation of gradient of a scalar point function (gradϕ). Divergence and curl of a vector point function. Character of divergence and curl of a vector point function. Gradient, Divergence and Curl of sums and products and their related vector identities. Laplacian operator.

Unit-III

Orthogonal curvilinear coordinates. Conditions for orthogonality. Fundamental triads of mutually orthogonal unit vectors. Gradient, Divergence, Curl and Laplacian operators in terms of orthogonal curvilinear coordinators.

Unit - IV

Vector integration: line integral, surface integral, Volume integral

Theorems of Gauss, Green and Stokes (without proof) and the problems based on these theorems.

Books Recommended

1. G. B. Thomas and R. L. Finney, Calculus, 9th Ed., Pearson Education, Delhi, 2005.

2. H. Anton, 1. Bivens and S. Davis, Calculus, John Wiley and Sons (Asia) P. Ltd.2002.

3. P.C. Matthew's, Vector Calculus, Springer Verlag London Limited, 1998.

Contents

Unit- 1

Basic of Vectors Calculus

Structure

1.1 Introduction

Dear student, in this unit we will study about some basic concepts related to vector calculus, which will be helpful throughout this course. We will study about vector quantity, vector representation Operations on vectors i.e. addition and subtraction of vector, we will also revise the concept direction cosine. Scalar and vector product of vectors will also be studied in this unit.

1.2 Learning Objectives

After studying this unit students will be able to

1. define vectors quantity.
2. represent graphically a vector.
3. addition and subtraction of vectors.
4. define direction cosines.
5. define and evaluate dot and cross product of vectors.

1.3 Vectors

Since a physical quantiting can be classified mainly into two classes on the basis of direction, known as scalar and vector, which are defined as.

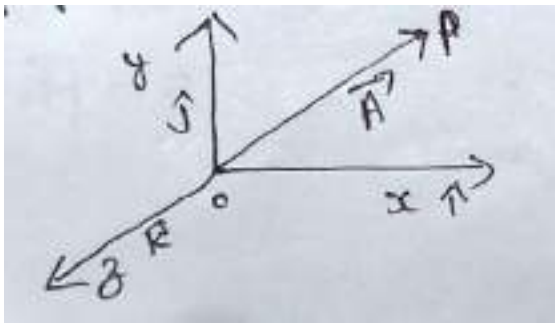
Scalar Quantity : A quantity which does not require any direction for its representation. Such quantity has only magnitude and unit. The basic rule of algebra will be apply for adding such quantity.

Mass, distance, time, temperature are examples of scalar quantity.

Vector Quantity : A physical quantity which require both magnitude as well as direction for its representation. These quantities can be added according to the vector law of addition.

Displaerat, velocity, acceleration, force are example of sector quantity.

Representation of Vector : Let a physical quantity is represented by an allow shaped straight line, with suitable length which represents it magnitude and the direction of arrow recusants direction, such physical quantity is known as vector quantity.



Magnitude of Vector : The magnitude of a vector quantity $\hat{A}$ is denoted by or mod $|\hat{A}|$.

So, a vector quantity is mathematically represented by $\vec{A}$.

Unit vector : A unit vector of vector quantity is that vector which has Unit magnitude. It is defined as $\hat{A} = \frac{\hat{A}}{|A|}$

Cartesian Representation of Vector : The Cartesian coordinate system, the unit vector along x1 y and 3 axis are represented by $\hat{I}, \hat{J}$ and $\hat{k}$ respectively. So, a vector quantity in its Cartesian representation or in component form is written as

$\hat{A} = A_x\hat{I} + A_y\hat{J} + A_z\hat{k}$, where $\hat{I}$, $\hat{J}$ and $\hat{k}$ are unit vectors and A_x, A_y, and A_z are component of vector $\hat{A}$ along x, y and 3 direction respectively.

Magnitude of Vector : If $\hat{A} = A_x + \hat{I} + A_y\hat{J} + A_z\hat{k}$ then magnitude of vector is defined as

$$|\hat{A}| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$$

Unit Vector : If $A = A_x + \hat{I} + A_y\hat{J} + A_z\hat{k}$ then unit vector is defined as,

$$\hat{A} = \frac{\hat{C}}{|A|} = \frac{Ax\hat{i} + Ay\hat{j} + Az\hat{k}}{\sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}}$$

Zero Vector : A vector which has magnitude equal to zero, is known as zero vector. Zero vector is also know as null vector. So if $\hat{A}$ is a zero vector then.

$$|\hat{A}| = 0$$

Equal Vectors : Two vectors are said to be equal vector if they have same magnitude as well as same direction.

Negative Vectors : A vector is called negative vector with reference to another vector if both vectors have same magnitude but have opposite direction.

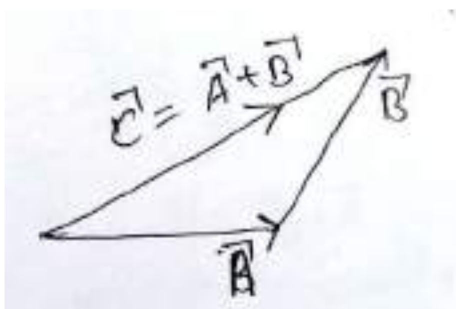
Like Vectors : If two vectors have same direction but have different directors then such vectors are known as like vectors.

Collinear Vectors : The vectors which are parallel to each others are known as collinear vectors.

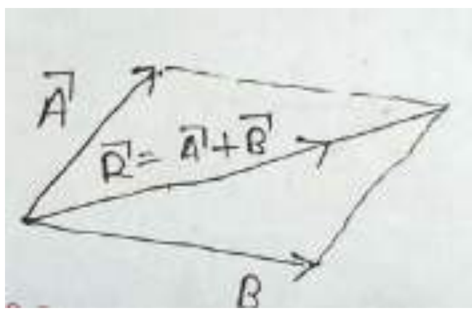
Coplaner Vectors. The vector which lies on the same plan is called coplaner vectors.

Vector Addition : The addition of two vectors are done by using following law:

Triangle Law : If two vectors represents the sides of triangle then the third side of triangle represents the resultant of these two vectors. Graphically, we can, see this as.



Parallelogram Law: If $\hat{A}$ and $\hat{B}$ represents two adjacent sides of a parallelogram then the sum of these two vectors i.e. resultant of these two vectors is represented by the diagonal of parallelogram. Graphically



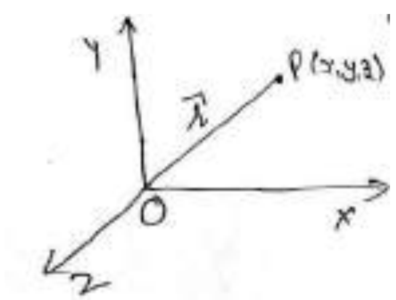
Difference of Vectors: The difference of vectors $\hat{A}$ and $\hat{B}$ is denoted by $\hat{A}-\hat{B}$, is in turn is $\hat{C}$. So $\hat{C}=\hat{A}-\hat{B}=\hat{A}+(-\hat{B})$

Scalar Multiplication: Multiplication of a vector $\hat{A}$ by a scalar m, gives a vector m$\hat{A}$, which has magnitude |m| limes the magnitude of $\hat{A}$. The direction of m$\hat{A}$ depends upon the value of m (positive or negatives. If m has positive then m$\hat{A}$ has same direction as of $\hat{A}$. If m is negative then m$\hat{A}$ has opposite direction as of $\hat{A}$.

Laws of Vector Algebra

1. $\left(\hat{A}+\hat{B}\right)+\hat{C}=\left(\hat{A}+\hat{B}\right)+\hat{C}$ Associative under addition
2. $\hat{A}+\hat{O}=\hat{O}+\hat{A}=A$ Exist no of zero element
3. $\hat{A}+\left(-\hat{A}\right)=\hat{O}=\left(-\hat{A}\right)+\left(-\hat{A}\right)$ Existeres of negative vector
4. $\hat{A}+\hat{B}=\hat{B}+\hat{A}$
5. $m\left(\hat{A}+\hat{B}\right)=m\hat{A}+m\hat{B}$
6. $(m+n)=m\hat{A}+n\hat{A}$
7. $m\left(n\hat{A}\right)=\left(mn\hat{A}\right)$
8. $1\left(\hat{A}\right)=\hat{A}$

Position Vector: Consider a Cartesian coordinate system, with point P(x,y,z), then this point is represented by a vector $\hat{r}$, this $\hat{r}$ is known as position vector of point P mathematically.



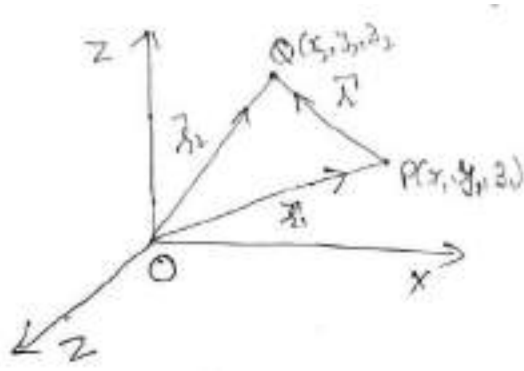
$\hat{r}=x\hat{i}+y\hat{j}+z\hat{k}$ where x, y, z are coordinates of point P w.r.t. origin, O. in x, y, and z directions.

If we have two vector $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ with position vector $\hat{r}_1$ and $\hat{r}_2$ respectively such that

$\hat{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\hat{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$PQ = OQ - \vec{P}$



Then the vector PQ is given by

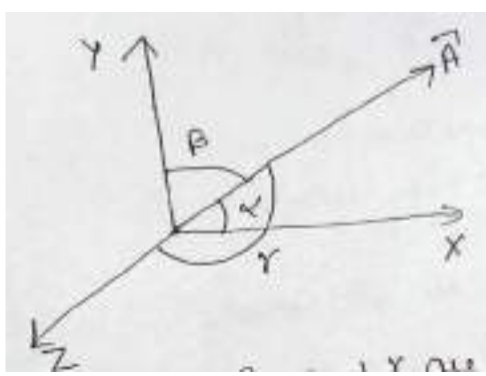
PQ = OQ - OP

$\hat{r} = \hat{r}_2 - \hat{r}_1$

= position vector of $\hat{Q}$ Position vector of $\vec{P}$

$\hat{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{J} + (z_2 - z_1)\hat{k}$

Direction Cosines: The direction cosines are the angle which $\hat{A}$ makes with three mutually perpendicular axes.



If $\propto$, β and γ are the angle which $\hat{A}$ makes with x, y and z axis respectively, then can $\propto$, can and cos γ are known as direction cosines and are given by

$$l = \cos \propto = \frac{A_x}{|\hat{A}|} = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$m = \cos\beta = \frac{A_y}{|\hat{A}|} = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

$$n = \cos\gamma = \frac{A_z}{|\hat{A}|} = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Where A_x, A_y, A_z are components of $\hat{A}$ along x, y and z axis respectively.

Also $l^2 + m^2 + n^2 = 1$ is an important result related to direction cosines.

Self Check Exercise - 1

Q.1 If $\hat{r}_1 = 3\hat{i} - 2\hat{j} + \hat{k}$

$\hat{r}_7 = 3\hat{i} + 4\hat{j} + a\hat{k}$

find $|\hat{r}_3| = \hat{r}_1 - \hat{r}_2 + \hat{r}_3$

Q.2 Find Unit Vector parallel to the resultant $\hat{R}$ of vectors = $\hat{r}_1 = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\hat{r}_2 = \hat{i} - 2\hat{j} + 3\hat{k}$.

$\frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Q.3 Calculate the direction cosines of a vector that makes angles $\propto = 30^0$, $\beta = 45^0$ and $\gamma = 60^0$ with coordinates axis.

$\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{1}}{2}\right)$

1.4 Scalar or Dot Product of Vectors

Consider the vectors $\hat{a}$ and $\hat{b}$ Let o be the angle between $\hat{a}$ and $\hat{b}$ then dot or Scalar product of two vector is denoted by $\hat{a}\,\hat{b}$ ($\hat{a}$ dot $\hat{b}$) is defined as the product of magnitude of $\hat{a}$ and $\hat{b}$ and the cosine of the angle θ between them. mathematically.

$\hat{a}.\hat{b}$ = $\hat{i}.\hat{i} = \hat{j}\,\hat{j} = \hat{k}.\hat{k} = 1$ $\cos\theta$, $o < \theta < \pi$

$\hat{a}.\hat{b}$ = 9.6 $\cos\theta$

Properties of Scalar Or Dot Product

1. Dot product is known as scalar product because the resultant is a scalar quantity.
2. $\hat{a}.\hat{b}$ = $\hat{b}.\hat{a}$ dot product is commutative

3. $\hat{a}.\left(\hat{b}+\hat{c}\right)=\hat{a}.\hat{b}+\hat{a}.\hat{c}$
$(\propto \hat{a}).\hat{b}=\propto\left(\hat{a}.\hat{b}\right)=\hat{a}.\left(\propto \hat{b}\right)$, $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

4. If $\hat{a}$ = $a_1 \pi/2$ + $a_2 \hat{j}$ + $a_3 \hat{k}$, $\hat{b}$ = $b_1 \hat{i}$ + $b_2 \hat{j}$ + $b_3 \hat{k}$

Then $\hat{a}.\hat{b}$ = $a_1b_1 + a_2b_2 + a_3b_3$

5. $\hat{a}.\hat{b}$ = 0 $\Rightarrow \hat{a}$ = 0 or $\hat{b}$ = 0 or θ = $\pi/2$

4. $\hat{a}.\hat{b}$ = 0, $\hat{a}$ and $\hat{b}$ are non zero vectors then $\hat{a}$ and $\hat{b}$ are perpendicular vectors.

5. $\hat{a}.\hat{b}$ are orthogonal $\hat{a}.\hat{b}$ = 0

6. $\hat{a}.\hat{a}$ = $\hat{a}^2$ = a^2

7. $\hat{a}.\hat{b}$ = (modulus of $\hat{a}$) (projection of $\hat{b}$ on $\hat{a}$)

8. Projection of $\hat{b}$ on $\hat{a}$ = $\dfrac{\hat{a}.\hat{b}}{|\hat{a}|}$

9. angle between two vectors $\hat{a}$ and $\hat{b}$ is given by cox θ = $\dfrac{\hat{a}.\hat{b}}{|\hat{a}||\hat{b}|}$

10. $\hat{a}.\left(\hat{b}+\hat{c}\right)=\hat{a}.\hat{b}+\hat{a}.\hat{c}$ distributive property

11. $\left(\hat{a}+\hat{b}\right)^2=\hat{a}^2+\hat{b}^2+2\hat{a}.\hat{b}$

12. $(\propto \hat{a}).\hat{b}=\propto\left(\hat{a}.\hat{b}\right)=\hat{a}.\left(\propto \hat{b}\right)$

13. If $\hat{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\hat{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

then cosθ = $\dfrac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2}}$

14. $\hat{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\hat{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

1. Then $\hat{a}$ and $\hat{b}$ are perpendicular when
$a_1b_1 + a_2b_2 + a_3b_3 = 0$

2. $\hat{a}$ and $\hat{b}$ are parallel when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Example 1: Given A = 4i + 2j - 3k, B = 5i - j - 2k,

C = 3i + j + 7k Then.

Solution : A-B = (4) (5) + (2) (-1) + (-3) (-2) = 20 - 2 + 6 = 24

A.C = 12 + 2 - 21 = -7

B.C = 15 - 1 - 14 = 0, A.A = $4^2 + 2^2 + (-3)^2 = 29$

Thus B and C vectors are perpendicular

Example 2: Find the angle between A = 2i + 2j - k

and B = 7i + 24k

Solution: We have A.B = |A| |B| Cos θ

|A| = $\sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$

and |B| = $\sqrt{(7)^2 + (0)^2 + (24)^2} = \sqrt{625} = 25$

$\therefore$ A.B = (2) (7) + (2) (0) + (-1) (24) = -10

Therefore,

$$\cos\theta = \frac{A.B}{|A||B|} = \frac{-10}{(3)(25)} = \frac{-2}{15} = -0.1333$$

And $\theta = 98^0$ (approximately)

Example 3: Determine the value of $\propto$ so that

A = 2i + $\propto$j + k and B = i + 3j - 8k are prependicular.

Solution: Give A and B are perpendicular

So that A.B = 0 Thus

A.B = (2) (1) + ($\propto$) (3) + (1) (-8) = 2 + 3$\propto$ - 8 = 0

$\propto = \frac{6}{3} = 2.$

Example 4: Show that the vectors A = -i + j, B = -i-j-2k, C = 2j + 2k form a right triangle.

Solution: First we show that the vectors form a triangle. From Fig. 2-3, we see that the vectors form a triangle if :

(a) one of the vectors, say (3), is the sum of (1) and (2) or

(b) the sum of the vectors (1) + (2) + (3) is zero.

According as (a) two vectors have a common terminal point, or (b) two vectors have a common terminal point, or (b) none of the vectors have a common terminal point. By trial, we find A = B + C So the vectors do form a triangle.

Since A.B = (-1)(-1) + (1) (-1) + (0), it follows that A and B are prependicular and the triangle is a right triangle.

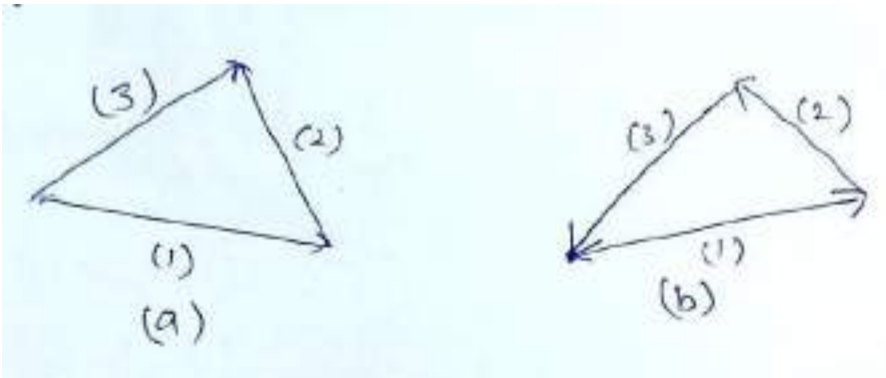


Fig. 2-3.

Example 5: Find the angles that the vector A = 4i - 8j + k makes with the co-ordinate axes.

Solution: Let $\propto$, β, γ be the angles that A makes with the positive x, y, z axes, respectively.

A.i = |A| (1) cos $\propto$ = $\sqrt{(4)^2+(-8)^2+(1)^2}$ cos $\propto$ = $\propto$ cos $\propto$

A.i = (4i - 8j + k) . i = 4

Then cos $\propto$ = 4/9 = 0.4444 and $\propto$ = 63.6^0 approximately.

Similarly, cos β = $\frac{-8}{9}$, β = 152. 70 and cos γ = $1/9$

γ = 836^0

The cosine of $\propto$, β, γ are called the direction cosine of the vector A.

Example 6: Find the projection of the vector

A = i - 2j + 3k on the vector B = i + 2j + 2k.

Solution: A unit vector in the direction of B is

b = $B/|B|$ = $\frac{(i+2j+2k)}{\sqrt{1+4+4}}=\frac{1}{3}i+\frac{2}{3}j+\frac{2}{3}k$

The projection of A on vector B is

A.b = (i - 2j + 3k). $\left(\frac{i}{3}+\frac{2j}{3}+\frac{2k}{3}\right)$= 1.

Self Check Exercise 2

Question 1: Find the value of a for which vectors

$\hat{A}$ and $\hat{B}$ are perpendicular where:

(a) $\hat{A}$ = ai - 2j + k and $\hat{B}$ = 2ai + aj - 4k

(b) $\hat{A}$ = 2i + j + ak and $\hat{B}$ = 2j + aj + k.

Question 2 : Find the projection of the vector A on the vector B where:

(a) A = 2i - 3j + 6k and B = i + 2j + 2k

(b) A = 2i + j - k and B = -6i + 2j - 3k.

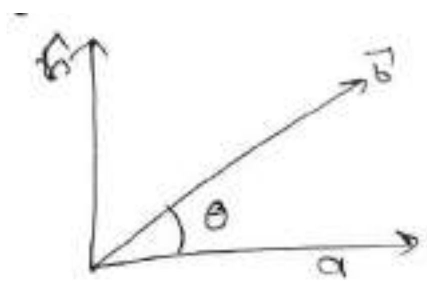
Question 3 : Show that A, B, and C are mutually orthogonal unit vector where:

(a) A = $\frac{2i-2j+k}{3}, B=\frac{i+2j+2k}{3}$ and C = $\frac{2i+j+2k}{3}$

(b) A = $\frac{12i-4j+3k}{13}, B=\frac{4i+3j+12k}{13}$ and C = $\frac{3i+12j+4k}{13}$

1.5 Vector or Cross Product of Vectors

Let $\hat{a}$ and $\hat{b}$ be two vectors such that Q be the angle between them. Let $\hat{n}$ be a unit vector which is perpendicular to $\hat{a}$, $\hat{b}$, $\hat{n}$ form a right handed system.



Then $\hat{a}$ cross product of $\hat{a}$ and $\hat{b}$ is defined as

$\hat{a}\times\hat{b} = |\hat{a}||\hat{b}|$ Sin Q $\hat{n}$

$\hat{a}\times\hat{b}$ = ab Sin Q $\hat{n}$

Properties of Vector or Cross Product of Vectors

1. $\vec{a}\times\vec{b}$ is a vector whose modulus is ab sin θ and its direction is prependicular to $\vec{a}$ as well as $\vec{b}$.

2. As $\vec{a}\times\vec{b}$ is a vector, therefore we call this product as vector product.

3. $\vec{a}\times\vec{b}$ = - $\vec{b}\times\vec{a}$ i.e. $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$.

4. $\vec{a}\times\vec{b}$ = 0 $\Rightarrow \vec{a}$ = 0 or $\vec{b}$ = 0 or $\vec{a}$ is parallel to $\vec{b}$ or $\vec{a}$ and $\vec{b}$ collinear.

5. $\hat{i}\times\hat{i}=\hat{0}$, $\hat{j}\times\hat{j}$ = 0, $\hat{k}\times\hat{k}$ = 0, $\hat{j}\times\hat{j}=\hat{k}$, $\hat{j}\times\hat{k}=\hat{i}$, $\hat{k}\times\hat{i}$ = $\hat{j}$.

6. Angle θ between $\hat{a}$ and $\hat{b}$ is given by

Sin θ = $\dfrac{|\hat{a}\times\hat{b}|}{|\hat{a}||\hat{b}|}$

7. Unit vectors perpendicular to the plane of $\hat{a}$ and $\hat{b}$ is

$\neq\left(\dfrac{\hat{a}\times\hat{b}}{|\hat{a}\times\hat{b}|}\right)$

8. Area of parallelogram with adjacent sides $\hat{a}$ and $\hat{b}$ is $|\hat{a}\times\hat{b}|$.

9. Area of triangle with adjacent sides $\hat{a}$ and $\hat{b}$ is $\frac{1}{2}|\hat{a}\times\hat{b}|$.

10. Area of parallelogram with diagonals $\vec{a}$ and $\vec{b}$ is $\frac{1}{2}\left(\vec{b}+\vec{c}\right)$.

11. If $\propto$ is a Scalar, then

$\propto\left(\vec{a}\times\vec{b}\right)$ = ($\propto\vec{a}$) x $\vec{b}$ = $\vec{a}$ x ($\propto\vec{b}$)

12. $\vec{a}$ x $\left(\vec{b}+\vec{c}\right)$ = $\vec{a}$ x $\vec{b}$ + $\vec{a}$ x $\vec{c}$

13. $\left(\vec{a}\times\vec{b}\right)^2=|\vec{a}|^2|\vec{b}|^2-\left(\vec{a}.\vec{b}\right)^2$

14. If $\vec{a}$ = $a_1\hat{i}$ + $a_2\hat{j}$ + $a_3\hat{k}$ and $b_1\hat{i}$ + $b_2\hat{j}$ + $b_3\hat{k}$

then $\vec{A}+\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 2 & -3\\ 3 & 5 & 2\end{vmatrix}$

$\left(3\hat{i}+5\hat{j}+2\hat{k}\right)$

Example 1: $\hat{A}$ = 4i + 2j - 3 $\hat{k}$ B = $\left(3\hat{i}+5\hat{j}+2\hat{k}\right)$

Find $\hat{A}$ x $\hat{B}$.

Solution:- $\vec{A}+\vec{B}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 2 & -3 \\ 3 & 5 & 2 \end{vmatrix}$

$\hat{i}(i+15)-\hat{j}(8+9)+\hat{k}(20-6)$

$\hat{A}$ x $\hat{B}$ = $19\hat{i}-17\hat{j}+14\hat{k}$

Example 2: Find the area of triangle whose vertices are P(1, 3, 2), Q(2,-1,1) R(-1,2,3)

Solution: $\overrightarrow{PQ}$ = $1\hat{i}-4\hat{j}-\hat{k}$

$\overrightarrow{PR}$ = $-\hat{j}+\hat{k}-2\hat{i}$

$\therefore$ area of triangle = $\frac{1}{2}$ |PQ x PR|

$\overrightarrow{PQ}$ x $\overrightarrow{PR}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -4 & -1 \\ -2 & -1 & 1 \end{vmatrix}$ = $\hat{i}$ (-4-1)- $\hat{j}$ (-1)+ $\hat{k}$ (-9)

$\therefore$ $\overrightarrow{PQ}$ x $\overrightarrow{PR}$ = -5 $\hat{i}$ + $\hat{j}$ -9 $\hat{k}$.

$\therefore$ | $\overrightarrow{PQ}$ x $\overrightarrow{PR}$ | = $\sqrt{(5)^2+(1)^2+(9)^2}=\sqrt{25+1+81}=\sqrt{107}$

$\therefore$ area of triangle = $\frac{\sqrt{107}}{2}$

Example 3: Determine a unit vector perpendicular to the plane of A = 2 to the plane of

A = $2\hat{i}-6\hat{j}-3\hat{k}$ and B = $4\hat{i}-3\hat{j}-\hat{k}$

Solution: A x B is a vector perpendicular to the plane A and B

$$A \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15\hat{i} - 10\hat{j} - 30\hat{k}$$

A unit vector parallel to A x B is $\frac{A\,x\,B}{|A\,x\,B|}$

$$= \frac{15\hat{i} - 10\hat{j} + 30\hat{k}}{(15)^2 + (-10)^2 + (30)^2}$$

$$= \frac{3}{7}\hat{i} - \frac{2}{7}\hat{j} - \frac{6}{7}\hat{k}$$

Question 1: Find the area of parellogram having diagonds: (a) $A = 3\hat{i}\ \hat{j} - 2\hat{k}$ **and** $B = 1\hat{i}\ 3\hat{j} - 4\hat{k}$

(b) $A = 2\hat{i}\ 4\hat{j}$ and $B = 4\hat{i} + 4\hat{k}$

Question 2: Find the area of a triangle with vertices at:

(a) (3, -2, 2), (1, -1, -3) and (4, -3, 1)

(b) (2, -3, -2), (-2, 3, 2) and (4, 3, -1)

1.6 Summary: Dear students, in this unit, we study that

1. A quantity which has only magnitude is a scalar.
2. A quantity having magnitude and direction is known as vector.
3. $\hat{a}\,.\hat{b}$ is scalar product of vectors.
4. $\hat{a}\,x\,\hat{b}$ is vector product of vectors.
5. Two vectors are perpendicular if $\hat{a}\,x\,\hat{b}$ = 0
6. Two vectors are parallel if $\hat{a}\,x\,\hat{b}$ = 0

1.7 Glossary

1. Scalar Product : $\vec{a}.\vec{b}$ = $|\vec{a}||\vec{b}|$ cosθ is a scalar quantity. It is a scalar quantity, hence known as scalar product.
2. Cross or vector Product : $\hat{a}\,x\,\hat{b}$ = $|\vec{a}||\vec{b}|$ = Sin $\theta\,\hat{n}$ is a vector quantity, so it is known as vector quantity.

1.8 Answers to Self Check Exercises

Self Check Exercise - 1

Q.1 $\sqrt{5},\quad 7\hat{i}+4\hat{j}+12\hat{k}$

Q.2 $\frac{1}{3}\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

Q.3 $\frac{\sqrt{3}}{2},\frac{\sqrt{2}}{2},\frac{\sqrt{1}}{2}$

Self Check Exercise - 2

Q.1 (a) a = 2, -1

(b) $\hat{a}$ = 2

Q.2 (a) $8/3$

(b) -1

Q.3 To show $\vec{A}.\vec{B}=0$, $\vec{B}\vec{C}=0$, $\vec{C}\vec{A}=0$

Self Check Exercise - 3

Q.1 (a) $5\sqrt{3}$

(b) 12

Q.2 (a) $\frac{\sqrt{165}}{2}$

(b) 21

1.9 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector analysis, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, Schand and Company Limited.
3. J.N. Sharma and A.R. Vasishtha, Vector Calculas, Krishna Prakashan Mandir.

1.10 Terminal Questions

1. Prove that (A.B x C) (a.b x c) = $\begin{vmatrix} A.a & A.b & A.c \\ B.a & B.b & B.c \\ C.a & C.b & C.c \end{vmatrix}$

2. Find a unit vector perpendicular to both vector A and B where :

   (a) A = 4i - $\hat{j}$ + 3k and B = $-2\hat{i}+\hat{j}-2\hat{k}$

****

Unit - 2

Scalar Triple Product

Structure

2.1 Introduction

Dear student, in this unit you will study about the scalar triple product. How you learn to multiply three vectors in such a way that the resulted is a scalar. You will also learn its geometrical interpretation as well as its properties such as to show three vectors are coplanar or not.In this unit you will also learn to find volume of a fetrahedran by using scalar triple product.

2.2 Learning Objectives

After studying this unit, students will be able to:

(1) Define scalar triple product.

(2) Find scalar triple product of given vector.

(3) Inter geometrically scalar triple product

(4) Prove the properties of scalar triple product

(5) Solve questions related to scalar triple product.

(6) Apply the properties of scalar triple product to show that given victors all coplar

(7) Able to find the volume of tetrahedron.

2.3 Definition and Component Form of Scalar Triple Product

Let $\hat{a}, \hat{b}, \hat{c}$ are three vectors. If we introduce dot and cross product between $\hat{a}, \hat{b}$ and $\hat{c}$, we have the following products:

$\left(\hat{a}.\hat{b}\right).\hat{c}$, $\left(\hat{a}.\hat{b}\right)x\hat{c}$, $\left(\hat{a}\,x\hat{b}\right).\hat{c}$ and $\left(\hat{a}\,x\hat{b}\,x\hat{c}\right)$

As in unit 1, we studied that if we take dot and class product are defined between of two vectors only and dot product of two vectors is a scalar. Taking there two points in our mind; two vectors is a scalar. Taking these two points in our mind ; two types of notation given above i.e. $\left(\hat{a}.\hat{b}\right).\hat{c}$ (as $\hat{a}\,.\hat{b}$ is a scalar and '.' is applicable only on two vectors) and $\left(\hat{a}xb\right).c$ (bythe same reason) are meaningless. In this unit we will study about one of the remaining two terms i.e. $\left(\hat{a}xb\right).c$ i.e. cross product of two vector and its dot product with the third one.

Definition of Scalar Product:

If $\hat{a}, \hat{b}, \hat{c}$ are three vectors, then the scalar triple product is defined as $\left(\hat{a}\,x\hat{b}\right).\vec{c}$ and is denoted by $\left[\hat{a}\,\hat{b}\,\hat{c}\right]$

Thus $\left[\hat{a}\,\hat{b}\,\hat{c}\right] = \left(\hat{a}\,x\hat{b}\right).\vec{c}$

As cross product $\left(\hat{a}\,x\hat{b}\right)$ will give a vector quantity and dot product of result of $\left(\hat{a}\,x\hat{b}\right)$ and $\hat{c}$ gives a scalar quantity so $\left[\hat{a}\,\hat{b}\,\hat{c}\right]$ is known as scalar triple product.

Component Form of Scalar Triple Product

Let $\hat{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\hat{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $c=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Then $\left[\hat{a}\,\hat{b}\,\hat{c}\right] = \left(\hat{a}\,x\hat{b}\right).\hat{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Taking $\left(\hat{a}\,x\hat{b}\right).\hat{c}$

Let $\hat{a}\,x\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ [Definition of cross product

$= \hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$

$\hat{i}(a_2b_3 - b_2a_3) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - b_1a_2)$

Now $(\hat{a}\times\hat{b}).\hat{c} = \left[\hat{i}(a_2b_3 - b_2a_3) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - b_1a_2)\right].\left[c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right]$

$= (a_2b_3 - b_2a_3)c_1 - (a_1b_3 - a_3b_1)c_2 + (a_1b_2 - b_1a_2).c_3$

[by definition of dot product]

$= a_2b_3c_1 - b_2a_3c_1 - a_1b_3c_2 + a_3b_1c_2 + a_1b_2c_3 - b_1a_2c_3$

$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1$

$= a_1\left(b_2c_3 - b_3c_2\right) + a_2\left(b_3c_1 - b_1c_3\right) + a_3\left(b_1c_2 - b_2c_1\right)$

$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ [Determinant Form]

Hence in component form $(\hat{a}\times\hat{b}).\hat{c}$ = $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ $(\hat{a}\hat{b}\hat{c})$ = $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Let us try to find scalar triple product of given then vectors by solving some examples.

Example 1 : If $\hat{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\hat{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\hat{c} = 3\hat{i} + \hat{j} + 2\hat{k}$ Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

Solution : Since in component form scalar triple product is given by

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$

= 2(4-1)-1(-2-3)+3(-1-6)

= 2(3) -1(-5) +3(-7)

= 6 + 5 - 21

= 11 - 21

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = -10

Example 2: If $\hat{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\hat{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\hat{c} = -3\hat{i} + \hat{j} + 2\hat{k}$ Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

Solution : Since $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ -3 & 1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix}$

= 2(4-1) -1(-2-3) +3(-1+6)

= 2(3) -1(1) +3(5)

= 6 - 1 + 15

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 21 - 1 = 20

Example 3 : Given $\vec{a} = 2\hat{i} - 3\hat{j}$, $\hat{b} = \hat{i} + \hat{j} - \hat{k}$ and $\hat{c} = 3\hat{i} - \hat{k}$ Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

Solution : Since we known that

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$

$= 2\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$

= 2(-1) +3(-1+3) +0

= -2 +3(2)

= -2 + 6

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 4

Self Check Exercise - 1

Q. 1 Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ if $\hat{a} = 10\hat{i} - 12\hat{j} - 4\hat{k}$, $\hat{b} = -6\hat{i} + 10\hat{j} - 6\hat{k}$ and $\hat{c} = -4\hat{i} + 12\hat{j} + 10\hat{k}$

Q. 2 Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ if $\hat{a} = -3\hat{i} + 7\hat{j} + 5\hat{k}$, $\hat{b} = -5\hat{i} + 7\hat{j} - 3\hat{k}$ and $\hat{c} = 7\hat{i} - 5\hat{j} - 3\hat{k}$

Q. 3 Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ if $\hat{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\hat{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\hat{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

Q. 4 Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ if $\hat{a} = \hat{i} + \hat{j}$, $\hat{b} = \hat{i} - \hat{j}$ and $\hat{c} = 5\hat{i} + 2\hat{j} + 3\hat{k}$

Q. 5 Find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ if $\hat{a} = -12\hat{i} + 3\hat{k}$, $\hat{b} = 3\hat{i} - \hat{k}$ and $\hat{c} = 2\hat{i} + \hat{j}15\hat{k}$

Q. 4 Geometrical Interpretation of $(\hat{a} \times \hat{b}).\hat{c}$

Scalar triple product of three vectors geometrically define the volume of the parallelopiped farmed by the three vectors originating from a common point.

Let us prove this

Consider a parallelopiped having edges OA, OB and OC with same extent. Let $\overrightarrow{OA} = \hat{a}$, $\overrightarrow{OB} = \hat{b}$ and $\overrightarrow{OC} = \hat{c}$. So, here edges of parallelopiped equal to three vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$. Let $\hat{n}$ be a unit vector perpendicular to the plane of $\hat{a}$ and $\hat{b}$ and θ be the angle between $\hat{c}$ and $\hat{n}$. From point c, draw a perpendicular CL to the plane of $\hat{a}$ and $\hat{b}$ such that CL to the plane of $\hat{a}$ and $\hat{b}$ such that CL = P is the height of parallelopiped. Now, $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = (\hat{a} \times \hat{b}).\hat{c}$

= (Area of the parallelogram OADB) $\hat{n}$. $\hat{c}$

= (Area of the parallelogram OADB) ($\hat{n}$. $\hat{c}$)

= Area of the parallelogram OADB $|\hat{n}||\hat{c}|\cos\theta$

[by definition of dot product]

= (Area of the parallelogram OADB) $|\hat{c}|\cos\theta$

[as $\hat{n}$ is unit vector and $|\hat{n}| = 1$]

= (Area of the parallelogram OADB). OC $\cos\theta$

[$\because |\hat{c}| = OC$]

= (Area of the parallelogram OADB). CL

[$\because \theta\cos\theta = CL$]

= (Area of the base of parallelogram)×(height Þ)

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = Volume of parallelopiped with edges of same extent $\hat{a}$, $\hat{b}$ and $\hat{c}$.

Therefore, scalar triple product of three vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ represents the volume of the parallelopiped having coterminous edges (edges of same extent) given by vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$.

Let us try to find the volume of parallelopipe by using scalar triple product.

Example 1 : Find the volume of the parallelopiped whose coterminous edges are given by $\hat{a} = 3\hat{i} + 4\hat{j}$, $\hat{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\hat{c} = 5\hat{k}$. Since we know that volume of a parallelopiped having edges $\hat{a}$, $\hat{b}$ and $\hat{c}$ is given by

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 3 & 4 & 0 \\ 2 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 3\begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} - 4\begin{vmatrix} 2 & 4 \\ 0 & 5 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 0 & 0 \end{vmatrix}$$

= 3(15) -4(1) +0

= 45-40

= 5 cubic units.

Example 2 : Find the volume of parallelopiped whose side are given by the vectors $\hat{a} = 2\hat{i} - 3\hat{j}$, $\hat{b} = \hat{i} + \hat{j} - \hat{k}$ and $\hat{c} = 3\hat{i} - \hat{k}$.

Solution : Since volume of parallelopiped $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 3 & 0 \end{vmatrix}$$

= 2(-1) +3(-1+3) +0

= -2 + 6

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 4 cubic units = Volume of parallelopiped.

Example 3 : Find the volume of parallelopiped with coterminal edges AB, AC and AD, where A = (3, 2, 1) B = (4, 2, 1), C = (0, 1, 4) and D = (0, 0, 7).

Solution : Since in his question the coterminal edges are not given by vector, but we are given the coordinates of points A, B, C and D. Here we first have to find the edges AB, AC and AD having starting point A. Let $\hat{a}$, $\hat{b}$ $\hat{c}$ and $\hat{d}$ be the position vectors of A (3, 2, 1), B(4, 2, 1), C(0, 1, 4) and D(0, 0, 7)

So, $\hat{a} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\hat{b} = 4\hat{i} + 2\hat{j} + \hat{k}$, $\hat{c} = \hat{j} + \hat{k}$ and $\hat{d} = 7\hat{k}$

Now $\overrightarrow{AB}$ = Position vector of B - Position vector of A

= $(4\hat{i} + 2\hat{j} + \hat{k})$ - $(3\hat{i} + 2\hat{j} + \hat{k})$

$\overrightarrow{AB}$ = $\hat{i} + 0\hat{j} + \hat{k}$

Similarly = $\hat{i}$

$\overrightarrow{AC}$ = Position vector of $\hat{c}$ - Position vector of A

= $(\hat{j} + 4\hat{k})$ - $(3\hat{i} + 2\hat{j} + \hat{k})$

$\overrightarrow{AC}$ = $-3\hat{i} - \hat{j} + 3\hat{k}$

and

$\overrightarrow{AD}$ = Position vector of $\hat{D}$ - Position vector of A

= $7\hat{k}$ - $(3\hat{i} + 2\hat{j} + \hat{k})$

$\overrightarrow{AD} \quad = \quad -3\hat{i} - \hat{j} + 6\hat{k}$

Therefore, volume of parallelopiped = $\left[\overline{AB}.\overline{AC}\,\overline{AD}\right]$

$$= \begin{vmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & -2 & 6 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & 3 \\ -2 & 6 \end{vmatrix} - 0 + 0$$

Volume of parallelopiped = 1(-6+6) = 0 cubic units.

Example 4 : The volume of the parallelopiped whose edges are $-12\hat{i} + \lambda\hat{k}$, $3\hat{j} - \hat{k}$ $3\hat{i} + \hat{j} - 15\hat{k}$ is 546 cubic units.

Solution : Given that volume of parallelopiped is 546 cubic units.

Let $\hat{a} = -12\hat{i} + \lambda\hat{k}$, $\hat{b} = 3\hat{j} - \hat{k}$ and $\hat{c} = 2\hat{i} + \hat{j} - 15\hat{k}$. Since we know that, if $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three edges of parallelopiped then of parallelopiped is given by $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$.

So, $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 546$

$$= \begin{vmatrix} -12 & 0 & -1 \\ 0 & 3 & -1 \\ 2 & 1 & -0 \end{vmatrix}$$

$$= -12\begin{vmatrix} 3 & -1 \\ - & -15 \end{vmatrix} - 0\begin{vmatrix} 0 & -1 \\ 2 & -15 \end{vmatrix} + \lambda\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} = 546$$

$\Rightarrow$ -12(-45+1) -0+ λ(-6) = 546

$\Rightarrow$ -12(-44) - 6 λ = 546

$\Rightarrow$ 528 - 6 λ = 546

$\Rightarrow$ - 6 λ = 546 - 528

$\Rightarrow$ - 6 λ = 18

$\Rightarrow \lambda = \dfrac{18}{6} = 3$

$\Rightarrow \lambda = 3$

Example 5 : Find the height of a parallelopiped whose base is given by parallelogram $\hat{a}$ and $\hat{b}$, where $\hat{a} = \hat{i} + \hat{j} + \hat{k}$, $\hat{b} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\hat{c} = \hat{i} + \hat{j} + 3\hat{k}$ are edges of parallelopiped.

Solution : Since we known that

Volume of parallelopiped = (area of bas of parallelogram). height of paralelopiped.

Height of parallelopiped = $\dfrac{\text{Volume of parallopiped}}{\text{area of base of parallelopiped}}$

Here $\hat{a}$, $\hat{b}$ and $\hat{c}$ are three edges of parallelopiped. Therefore, volume of parallelopiped $= \left[\hat{a}\ \hat{b}\ \hat{c}\right]$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$= 1\begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix}$$

= 1(12+1) -1 (6+1) +1 (2-4)

= 13 - 7 - 2

= 13 - 9

Volume of parallelopiped = 4 cubic units.

Now to find the area of base

Since area of base = $\left|\hat{a} \times \hat{b}\right|$

$$\text{So } \hat{a} \times \hat{b} = = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 4 & -1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}$$

$= \hat{i}$ (-1-4) - $\hat{j}$ (-1-2) + $\hat{k}$ (4-2)

$= -5\hat{i} + 3\hat{j} + 2\hat{k}$

$|\hat{a} \times \hat{b}| = |-5\hat{i} + 3\hat{j} + 2\hat{k}|$

$= \sqrt{(-5)^2 + (3)^2 + (2)^2}$

$= \sqrt{25 + 9 + 4}$

$|\hat{a} \times \hat{b}| = \sqrt{38}$ square unit

Now, Height of parallelopiped = $\dfrac{\text{Volume of parallopiped}}{\text{area of base}}$

Hence, Height of parallelopiped = $\dfrac{4}{\sqrt{38}}$ unit.

Self Check Exercise-2

Find the volume of parallelopiped whose sides are given by.

Q. 1 $\hat{i} + \hat{j} + \hat{k}$, $\hat{k}$, $3\hat{i} - \hat{j} + 2\hat{k}$

Q. 2 $2\hat{i} - 3\hat{j} + \hat{k}$, $\hat{i} - \hat{j} + 2\hat{k}$, $2\hat{i} + \hat{j} - \hat{k}$

Q. 3 $7\hat{i} - 5\hat{j} - 3\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$, $-3\hat{i} + 7\hat{j} + 5\hat{k}$

Q. 4 $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$, $-\hat{i} + 2\hat{j} - 3\hat{k}$

Q. 5 $2\hat{i} + 3\hat{j} + 4\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$, $3\hat{i} - \hat{j} + 2\hat{k}$

Q. 6 $11\hat{i}$, $2\hat{j}$, $13\hat{k}$

2.5 Properties of Scalar Triple product

Dear students, in this section we will discuss some important properties of scalar triple product which can be used as generalized results.

Property : If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are cyclically permuted, then value of scalar triple product remains same, i.e. $[\hat{a}\ \hat{b}\ \hat{c}] = [\hat{b}\ \hat{c}\ \hat{a}] = [\hat{c}\ \hat{a}\ \hat{b}]$

Proof : Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ represents the coterminous edges of a parallelopiped such that they form a right handed system. Then volume of parallelopipe is given by V = $(\hat{a} \times \hat{b}).\hat{c}$

Now $\hat{b},\ \hat{c},\ \hat{a}$ as well as $\hat{c},\ \hat{a},\ \hat{b}$ forms a right handed system of vectors and represents the coterminous edges of same parallelopiped.

Therefore, V = $\left(\hat{b}\times\hat{c}\right).\hat{a}$ and V = $\left(\hat{c}\times\hat{a}\right).\hat{b}$

Hence $\left(\hat{a}\times\hat{b}\right).\hat{c}$ = $\left(\hat{b}\times\hat{c}\right).\hat{a}$ = $\left(\hat{c}\times\hat{a}\right).\hat{b}$

Or = $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = $\left[\hat{b}\ \hat{c}\ \hat{a}\right]$ = $\left[\hat{c}\ \hat{a}\ \hat{b}\right]$

Hence the value of scalar triple product remains the same of the cyclic order of $\hat{a}$, $\hat{b}$ and $\hat{c}$ remains unchanged.

Property 2 : In scalar triple product, the position of dot and cross can be interchanged provided that the cyclic order of the vectors remains same.

Proof : Since from property -1, we know that

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = $\left[\hat{b}\ \hat{c}\ \hat{a}\right]$

$\left(\hat{a}\times\hat{b}\right).\hat{c}$ = $\left(\hat{b}\times\hat{c}\right).\hat{a}$

$\Rightarrow\left(\hat{a}\times\hat{b}\right).\hat{c}$ = $\hat{a}\ .\left(\hat{b}\times\hat{c}\right)$ [as dot product is commutative i.e. $\hat{a}\ .\ \hat{b}$ = $\hat{b}\ .\ \hat{a}$]

Hence dot and cross can interchange their position without changing the value of the product.

Property 3 : With the changes of cyclic order of vectors in scalar triple product, sign of scalar triple product also changes but the magnitude remains the same. Mathematically,

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = -\left[\hat{b}\ \hat{c}\ \hat{a}\right] = -\left[\hat{c}\ \hat{b}\ \hat{a}\right] = -\left[\hat{a}\ \hat{c}\ \hat{b}\right]$

Proof : Since we know that

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ $=\left(\hat{a}\times\hat{b}\right).\hat{c}$

$\Rightarrow$ $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = - $\left(\hat{b}\times\hat{a}\right).\ \hat{c}$ [cross product is not commutative i.e. $\hat{a}\times\hat{b} = -\hat{b}\times\hat{a}$]

= $\left\{\left(\hat{b}\times\hat{a}\right).\ \hat{c}\right\}$

$\Rightarrow$ $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ $=\left[\hat{b}\ \hat{a}\ \hat{c}\right]$ (1) [definition of scalar triple product]

Again $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = $\left[\hat{b}\ \hat{c}\ \hat{a}\right]$

$\Rightarrow \quad \left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left(\hat{b}\times\hat{c}\right).\hat{a}$

$= -\left(\hat{c}\times\hat{b}\right).\hat{a} \qquad \because \hat{b}\times\hat{c} = -\ \hat{c}\times\hat{b}$

$= -\left\{\left(\hat{c}\times\hat{b}\right).\hat{a}\right\}$

$= -\left[\hat{c}\ \hat{b}\ \hat{a}\right]$

Hence $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = -\left[\hat{c}\ \hat{b}\ \hat{a}\right]$ (2)

Again $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left[\hat{c}\ \hat{a}\ \hat{b}\right]$

$= \left(\hat{c}\times\hat{a}\right).\hat{b}$

$= -\left(\hat{a}\times\hat{c}\right).\hat{b} \qquad [\hat{c}\times\hat{a} = -\hat{a}\times\hat{c}]$

$= -\left\{\left(\hat{a}\times\hat{c}\right).\hat{b}\right\}$

$= -\left[\hat{a}\ \hat{c}\ \hat{b}\right]$ (3)

From (1), (2) and (3), we have

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = -\left[\hat{b}\ \hat{a}\ \hat{c}\right] = -\left[\hat{c}\ \hat{b}\ \hat{a}\right] = -\left[\hat{a}\ \hat{c}\ \hat{b}\right]$

Property 4 : Scalar triple product of three vectors is zero of any two of from are equal.

Proof : Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be any three vectors, then three cases arises :

Case I, when $\hat{a} = \hat{b}$

So, $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left(\hat{a}\times\hat{b}\right).\hat{c} \qquad$ [as $\hat{b}\times\hat{a} = \hat{a}\times\hat{a} = 0$]

$= \left(\hat{a}\times\hat{a}\right).\hat{c}$

$= 0.\hat{c}$

$= 0$

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = 0$

Case II when $\hat{b} = \hat{c}$

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left[\hat{b}\ \hat{c}\ \hat{a}\right] \qquad$ [using property 2]

$= \left(\hat{b} \times \hat{c}\right).\hat{a}$

$= \left(\hat{c} \times \hat{c}\right).\hat{a}$ $\qquad \hat{c} \times \hat{c} = 0$

$= \hat{a} \times \hat{a}$

$\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = 0$

Case III : When $\hat{c} = \hat{a}$

Then $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = \left[\hat{c}\ \hat{a}\ \ \hat{b}\right]$ [using property 2)

$= \left(\hat{c} \times \hat{a}\right).\hat{b}$

$= \left(\hat{c} \times \hat{c}\right).\hat{b}$ $\qquad$ [$\because \hat{c} = \hat{a}$]

$= \hat{0}\ .\ \hat{b}$

$\Rightarrow \left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = 0$

Hence $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$ = 0, if any two vectors are equal.

Property 5 : For any three vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ and for scalar λ, $\left[\lambda\hat{a}\ \hat{b}\ \ \hat{c}\right] = \lambda\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Proof : Taking L.H.S., $\left[\lambda\hat{a}\ \hat{b}\ \ \hat{c}\right] = \left(\lambda\,\hat{a} \times \hat{b}\right).\hat{c}$

$= \lambda\left(\hat{a} \times \hat{b}\right).\hat{c}$ $\qquad$ [$\because \lambda\,\hat{a} \times \hat{b} = \lambda\,(\,\hat{a} \times \hat{b}\,)$]

$= \lambda\left\{\left(\hat{a} \times \hat{b}\right).\hat{c}\right\}$

$\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = \lambda\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Hence $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = \lambda\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Property 6 : For any three vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ and for any three scalars l , m, and n.

$[l\ \hat{a}\ m\hat{b}\ n\hat{c}] = l\,mn\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Proof : Taking L.H.S.,

$[l\ \hat{a}\ m\hat{b}\ n\hat{c}] = \left(l\,\hat{a} \times m\hat{b}\right).n\hat{c}$

$= \ell m\left(\hat{a}\times\hat{b}\right).n\hat{c}$

$= \ell mn\left\{\left(\hat{a}\times\hat{b}\right).\hat{c}\right\}$

$= \ell mn\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Hence $[\ell\ \hat{a}\ m\hat{b}\ n\hat{c}] = \ell mn\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$

Property 7 : Scalar triple product of three vectors is zero of any two of them are parallel or collinear.

Proof : Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be any three vectors, let $\hat{a}$ is parallel (or collinear) to $\hat{b}$. Then

$\hat{a} = \lambda\hat{b}$ for some scalar λ

Now $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right]$ $= \left(\hat{a}\times\hat{b}\right).\hat{c}$

$= \left(\lambda\ \hat{b}\times\hat{b}\right).\hat{c}$

$= \lambda\left(\hat{b}\times\hat{b}\right).\hat{c}$

$= \lambda\ (0)\ .\ \hat{c}$ $\qquad [\because \hat{b}\times\hat{b} = 0]$

$= \hat{0}\ .\hat{c}$

$= 0$

Hence for two collinear or parallel vector, scalar triple product is zero.

Property 8 : The necessary and sufficient condition for three non zero, non-collinear vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ to be coplanar is $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = 0$

Or $\hat{a}\ \hat{b}\ \hat{c}$ are coplanar iff $\left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = 0$

Proof : Let $\hat{a}$, $\hat{b}\ \hat{c}$ are three non zero, non collinear, coplanar vectors.

Since $\hat{a}\times\hat{b}$ is perpendicular to the plane of $\hat{a}$ and $\hat{b}$.

Also $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar vector. So $\hat{a}\times\hat{b}$ is perpendicular to $\hat{c}$.

$\Rightarrow \quad \left(\hat{a}\times\hat{b}\right).\hat{c} = 0$ $\qquad$ [if two vectors are perpendicular then $\hat{a}\ .\hat{b} = 0$

$\Rightarrow \quad \left[\hat{a}\ \hat{b}\ \ \hat{c}\right] = 0$

So if $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar then $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0

Conversely : Let $\hat{a}$, $\hat{b}$, $\hat{c}$ be three non zero, non collinear vectors such that $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0, to prove $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar.

Given $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0

$\Rightarrow \left(\hat{a}\times\hat{b}\right).\hat{c}$ = 0

$\Rightarrow \hat{a}\times\hat{b}$ = $\hat{0}$ or $\hat{c}$ = $\hat{a}$ or $\hat{a}\times\hat{b}$ is perpendicular to $\hat{c}$

as $\hat{a}$, $\hat{b}$, $\hat{c}$ are non zero and non collinear vector so $\hat{a}\times\hat{b} \neq 0$ and $\hat{c} \neq \hat{0}$.

Therefore, $\hat{a}\times\hat{b}$ is perpendicular to $\hat{c}$.

Also $\left(\hat{a}\times\hat{b}\right)$ is a vector perpendicular to the plane $\hat{a}$ and $\hat{b}$.

$\therefore \left(\hat{a}\times\hat{b}\right)$ is perpendicular to $\hat{c} \Rightarrow \hat{c}$ lies in the plane of $\hat{a}$ and $\hat{b}$.

$\therefore \hat{a}$ $\hat{b}$ and $\hat{c}$ are coplanar.

Therefore, $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0 $\Rightarrow \hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar vectors.

Hence proved.

Property 9 : For given $\hat{a}$, $\hat{b}$, $\hat{c}$ three vectors,

$$\begin{vmatrix} \hat{a}.\hat{a} & \hat{a}.\hat{b} & \hat{a}.\hat{c} \\ \hat{b}.\hat{a} & \hat{b}.\hat{b} & \hat{b}.\hat{c} \\ \hat{c}.\hat{a} & \hat{c}.\hat{b} & \hat{c}.\hat{c} \end{vmatrix} = \left[\hat{a}\ \hat{b}\ \hat{c}\right]^2$$

Proof : Let $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\hat{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\hat{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

Now taking components of L.H.S.

$\hat{a} . \hat{a} = a_1^2 + a_2^2 + a_3^2$

$\hat{b} . \hat{b} = b_1^2 + b_2^2 + b_3^2$

$\hat{c}\,.\,\hat{c}$ = $c_1^2 + c_2^2 + c_3^2$

$\hat{a}\,.\,\hat{b}$ = $a_1b_1 + a_2b_2 + a_3b_3$ = $\hat{b}\,.\,\hat{a}$

$\hat{b}\,.\,\hat{c}$ = $b_1c_1 + b_2c_2 + b_3c_3$ = $\hat{c}\,.\,\hat{b}$

$\hat{c}\,.\,\hat{a}$ = $c_1a_1 + c_2a_2 + c_3a_3$ = $\hat{a}\,.\,\hat{c}$

How R.H.S.

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right]^2 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Let us try some examples based on these properties.

Example 1 : Find $\hat{a}\,.\left(\hat{c}\times\hat{a}\right)$ if $\hat{a} = 2\hat{i} - \hat{j} + \hat{k}$, & $\hat{c} = -\hat{i} + 3\hat{j} + \hat{k}$.

Solution : Using the property of scalar triple product its value will be zero, as it contain same vector twice. Let us check it by calculation.

$$\hat{c}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix}$$

$= \hat{i}$ (3+1) – $\hat{j}$ (-1-2) + $\hat{k}$ (1-6)

$\hat{c}\times\hat{a}$ = $4\hat{i} + 3\hat{j} - 5\hat{k}$

Now $\hat{a}\,.\left(\hat{c}\times\hat{a}\right)$ = $(2\hat{i} - \hat{j} + \hat{k})\,.\,(4\hat{i} + 3\hat{j} - 5\hat{k})$

= $(2\times4)\,\hat{i}\,.\,\hat{i} - (1\times3)\,\hat{j}\,.\,\hat{j} - (1\times5)\,\hat{k}\,.\,\hat{k}$

= 8 - 3 - 5 $\quad [\therefore \hat{i}^{\,2} = \hat{j}^{\,2} = \hat{k}^{\,2} = 1]$

= 8 - 8

= 0

$\therefore \hat{a} . (\hat{c} \times \hat{a})$ = 0

Example 2 : If $\hat{a}$ = $\hat{i}$ +2 $\hat{j}$ + $\hat{k}$, $\hat{b}$ = 3 $\hat{i}$ +2 $\hat{j}$ -7 $\hat{k}$, $\hat{c}$ = 5 $\hat{i}$ +6 $\hat{j}$ -5 $\hat{k}$

Show that $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar.

Solution : Since three vector are coplanar iff $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0

Therefore, $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & -7 \\ 5 & 6 & -5 \end{vmatrix}$

$= 1\begin{vmatrix} 2 & -7 \\ 6 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -7 \\ 5 & -5 \end{vmatrix} + 1\begin{vmatrix} 3 & 2 \\ 5 & 6 \end{vmatrix}$

= 1(-10+42) -2(-15+35)+1(18-10)

= 32-40+8

= 40-40

= 0

Since $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0, so $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar vectors.

Example 3 : Show that following vectors are coplanar, $\hat{a}$ = 10 $\hat{i}$ -12 $\hat{j}$ -4 $\hat{k}$, $\hat{b}$ = -16 $\hat{i}$ +22 $\hat{j}$ - 2 $\hat{k}$, $\hat{c}$ = 2 $\hat{i}$ -8 $\hat{j}$ +16 $\hat{k}$

Show that $\hat{a}$, $\hat{b}$, $\hat{c}$ are coplanar.

Solution : We just have to prove $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0

Now $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 10 & -12 & -4 \\ -16 & 22 & -2 \\ 2 & -8 & 16 \end{vmatrix}$

$= 10\begin{vmatrix} 22 & -2 \\ -8 & 16 \end{vmatrix} + 12\begin{vmatrix} -16 & -2 \\ 2 & 16 \end{vmatrix} - 4\begin{vmatrix} -16 & 22 \\ 2 & -8 \end{vmatrix}$

= 10(352-16) +12(-256+4)-4(128-44)

= 10(336)+12(-252)-4(84)

= 3360-3024-336

= 3360-3360

$= 0$

Therefore $\hat{a}$, $\hat{b}$ and $\hat{c}$ are coplanar as $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0

Example 4 : Let a, b, c are distinct non negative numbers.

If the vector $a\hat{i}$ $+a\ \hat{j}$ $+c\,\hat{k}$, $\hat{j}+\hat{k}$ and $c\,\hat{i}$ $+c\ \hat{j}$ $+b\,\hat{k}$ lies in a plane then show that c^2 = ab i.e. c is geometric mean of a and b.

Solution : Given $a\hat{i}$ $+a\ \hat{j}$ $+c\,\hat{k}$, $\hat{j}+\hat{k}$ and $c\,\hat{i}$ $+c\ \hat{j}$ $+b\,\hat{k}$ these three vectors lies in a plane, means these vector are coplanar. Therefore their scalar triple product should be equal to zero.

$$\Rightarrow \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix}$$

$$\Rightarrow a\begin{vmatrix} 0 & 1 \\ c & b \end{vmatrix} - a\begin{vmatrix} 1 & 1 \\ c & b \end{vmatrix} + c\begin{vmatrix} 1 & 0 \\ c & c \end{vmatrix} = 0$$

$\Rightarrow$ a(-c) -a(b-c) +c(c-0) = 0

$\Rightarrow$ -ac - ab + ac + c^2 = 0

$\Rightarrow$ c^2 - ab = 0

$\Rightarrow$ c^2 = 96

$\Rightarrow$ c = $\sqrt{96}$

Hence c^2 = 96, as a, b, c are distinct non negative number, so c is geometric mean of a and b as c^2 = 96.

Example 5 : Show that the four points whose position vectors are $6\hat{i}$ -7 $\hat{j}$, $16\hat{i}$ -19 $\hat{j}$ -4 $\hat{k}$, 3 $\hat{j}$ -6 $\hat{k}$, $2\hat{i}$ -5 $\hat{j}$ +10 $\hat{k}$ are coplanar.

Solution : From the given position vector of four points, first of all we have to find three vectors having same initial point. Let $6\hat{i}$ -7 $\hat{j}$, $16\hat{i}$ -19 $\hat{j}$ -4 $\hat{k}$, 3 $\hat{j}$ -6 $\hat{k}$ and $2\hat{i}$ -5 $\hat{j}$ +10 $\hat{k}$ represents four points A, B, C & D respectively. Then

$\overrightarrow{AB}$ = Position vector of B - Position vector of A

= ($16\hat{i}$ -19 $\hat{j}$ -4 $\hat{k}$) - ($6\hat{i}$ -7 $\hat{j}$)

$\Rightarrow \overrightarrow{AB}$ = $10\hat{i}$ -12 $\hat{j}$ -4 $\hat{k}$

Now, $\overrightarrow{AC}$ = Position vector of C - Position vector of A

$= (3\hat{j} - 6\hat{k}) - (6\hat{i} - 7\hat{j})$

$\overrightarrow{AC} = -6\hat{i} + 10\hat{j} - 6\hat{k}$

and $\overrightarrow{AD}$ = Position vector of D - Position vector of A

$= (2\hat{i} - 5\hat{j} + 10\hat{k}) - (6\hat{i} - 7\hat{j})$

$\Rightarrow \overrightarrow{AD} = -4\hat{i} + 2\hat{j} - 10\hat{k}$

Now to prove $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar it is sufficient to prove $\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$ = 0

So $\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = \begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 2 & 10 \end{vmatrix}$

$= 10\begin{vmatrix} 10 & -6 \\ 2 & 10 \end{vmatrix} + 12\begin{vmatrix} -6 & -6 \\ -4 & 10 \end{vmatrix} - 4\begin{vmatrix} -6 & 10 \\ -4 & 2 \end{vmatrix}$

= 10 (100+12) +12(-60-24)-4(-12+40)

= 10(112) +12(-84) -4(28)

= 1120 - 1008 - 112

= 1120 - 1120

$\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right] = 0$

Example 6 : Find the value of λ so that the vectors $\hat{a} = 2\hat{i} - 7\hat{j} + \lambda\hat{k}$, $\hat{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\hat{c} = 3\hat{i} - 5\hat{j} + 2\hat{k}$ are coplanar

Solution : Since we know that three vectors are coplanar if $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = 0$, so

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 2 & -7 & \lambda \\ 1 & 2 & -1 \\ 3 & -5 & 2 \end{vmatrix} = 0$$

$$\Rightarrow \quad 12\begin{vmatrix} 2 & -1 \\ -5 & 2 \end{vmatrix} + 7\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + \lambda\begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} = 0$$

$\Rightarrow$ 72(4-5) +7(2+3) +λ(-5-6) =0

$\Rightarrow$ 2(-2) +7(5) +λ(-11) = 0

$\Rightarrow$ -2 + 35 - 11λ = 0

$\Rightarrow$ 11λ = 33

$\Rightarrow$ λ = 3

Example 7 : The position vectors of the points A, B, C and D all $3\hat{i}-2\hat{j}-\hat{k}$, $2\hat{i}+3\hat{j}-4\hat{k}$, $-\hat{i}+\hat{j}+2\hat{k}$ and $4\hat{i}+5\hat{j}+\lambda\hat{k}$ respectively. Find the value of λ if these points lies on a plane.

Solution : From the position vectors of the point A, B, C and D we have to fine the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$. So

$\overrightarrow{AB}$ = Position vectors of B - Position vector of A

$= (2\hat{i}+3\hat{j}-4\hat{k}) - (3\hat{i}-2\hat{j}-\hat{k})$

$\Rightarrow \overrightarrow{AB} = -\hat{i}+5\hat{j}-3\hat{k}$

Now,

$\overrightarrow{AC}$ = Position vectors of C - Position vector of A

$= (-\hat{i}+\hat{j}+2\hat{k}) - (3\hat{i}-2\hat{j}-\hat{k})$

$\Rightarrow \overrightarrow{AC} = -4\hat{i}+3\hat{j}+3\hat{k}$

Also

$\overrightarrow{AD}$ = Position vectors of D - Position vector of A

$= (4\hat{i}+5\hat{j}+\lambda\hat{k}) - (3\hat{i}-2\hat{j}-\hat{k})$

$\Rightarrow \overrightarrow{AD} = \hat{i}+7\hat{j}+(\lambda+1)\hat{k}$

In order to prove that $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar [the vectors $[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}] = 0$

$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow -1\begin{vmatrix} 3 & 3 \\ 7 & \lambda+1 \end{vmatrix} - 5\begin{vmatrix} -4 & 3 \\ 1 & \lambda+1 \end{vmatrix} - 3\begin{vmatrix} -4 & 3 \\ 7 & 7 \end{vmatrix}$$

$\Rightarrow$ -1(3λ+3-21) -5(-4λ-4-3) -3(-28-3) = 0

$\Rightarrow$ -1(3λ-18)-5(-4λ-7)-3(-31) = 0

$\Rightarrow -3\lambda+18 + 20\lambda +35 + 93 = 0$

$\Rightarrow 17\lambda + 146 = 0$

$\Rightarrow\lambda = \frac{146}{17}$

Example 8 : If four points whose position vectors are $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ are coplanar, show that

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left[\hat{a}\ \hat{b}\ \hat{d}\right] + \left[\hat{a}\ \hat{d}\ \hat{c}\right] + \left[\hat{d}\ \hat{b}\ \hat{c}\right]$$

Solution : Let $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ be position vectors of A, B, C & D respectively. Then

$\overrightarrow{AB}$ = Position vectors of B - Position vector of A = $\hat{b} - \hat{a}$

$\overrightarrow{AC}$ = Position vectors of C - Position vector of A = $\hat{c} - \hat{a}$

$\overrightarrow{AB}$ = Position vectors of D - Position vector of A = $\hat{d} - \hat{a}$

Since four given points are coplanar so, $[\overrightarrow{AB}\ ,\ \overrightarrow{AC}\ ,\overrightarrow{AD}] = 0$

$\Rightarrow [\hat{b} - \hat{a}\ \ \hat{c} - \hat{a}\ \ \hat{d} - \hat{a}] = 0$

$\Rightarrow(\hat{b} - \hat{a}).\ [(\hat{c} - \hat{a}) \times (\hat{d} - \hat{a})] = 0$

$\Rightarrow (\hat{b} - \hat{a})\ .[(\hat{c} - \hat{a}) \times \hat{d}\ - (\hat{c} - \hat{a}) \times \hat{a}] = 0$

$\Rightarrow (\hat{b} - \hat{a})\ .[(\hat{c} \times \hat{d}) - (\hat{a} \times \hat{d}) - (\hat{c} \times \hat{a}) + \hat{a} \times \hat{a}] = 0$

$\Rightarrow (\hat{b} - \hat{a})\ .[(\hat{c} \times \hat{d}) - (\hat{a} \times \hat{d}) - (\hat{c} \times \hat{a})] = 0 \qquad \therefore \hat{a} \times \hat{a} = 0$

$= \hat{b}\ .(\hat{c} \times \hat{d}) - \hat{b}\ .\ (\hat{a} \times \hat{d}) - \hat{b}\ .\ (\hat{c} \times \hat{a}) - \hat{a}\ (\hat{c} \times \hat{d}) - \hat{a}\ (\hat{a} \times \hat{d}) + \hat{a}\ .(\hat{c} \times \hat{a})] = 0$

$= \left[\hat{b}\ \hat{c}\ \hat{d}\right] - \left[\hat{b}\ \hat{a}\ \hat{d}\right] - \left[\hat{b}\ \hat{c}\ \hat{a}\right] - \left[\hat{a}\ \hat{c}\ \hat{d}\right] = 0$

$[\hat{a}\ (\hat{a} \times \hat{d}) = \hat{a}\ .(\hat{c} \times \hat{a}) = 0$ using property 2]

$\Rightarrow\left[\hat{d}\ \hat{b}\ \hat{c}\right] + \left[\hat{a}\ \hat{b}\ \hat{d}\right] - \left[\hat{a}\ \hat{b}\ \hat{c}\right] + \left[\hat{a}\ \hat{d}\ \hat{c}\right] = 0$

$\Rightarrow\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \left[\hat{a}\ \hat{b}\ \hat{d}\right] + \left[\hat{a}\ \hat{d}\ \hat{c}\right] + \left[\hat{d}\ \hat{b}\ \hat{c}\right]$

Example 9 : Show that the vectors $\hat{a} - 2\hat{b} + 3\hat{c}$,$-2\hat{a} + 3\hat{b} - 4\hat{c}$ and $-\hat{b} + 2\hat{c}$ are coplanar.

Solution : Let $\hat{p} = \hat{a} - 2\hat{b} + 3\hat{c}$, $\hat{q} = -2\hat{a} + 3\hat{b} - 4\hat{c}$ and $\hat{r} = -\hat{b} + 2\hat{c}$ $\left[\hat{p}\ \hat{q}\ \hat{r}\right] = 0$ so $\hat{p}$, $\hat{q}$, $\hat{r}$ are coplanar.

$$\left[\hat{p}\ \hat{q}\ \hat{r}\right] = \hat{p} . \left[\hat{q}\times\hat{r}\right]$$

Now $\hat{q}\times\hat{r}$ $= (-2\hat{a} + 3\hat{b} - 4\hat{c})\times(-\hat{b} + 2\hat{c})$

$= 2(\hat{a}\times\hat{b}) - 4(\hat{a}\times\hat{c}) -3(\hat{b}\times\hat{b}) +6(\hat{b}\times\hat{c}) +4(\hat{c}\times\hat{b}) +8(\hat{c}\times\hat{c})$

$= 2(\hat{a}\times\hat{b}) - 4(\hat{a}\times\hat{c}) +6(\hat{b}\times\hat{c}) +4(\hat{c}\times\hat{b}) +8(\hat{c}\times\hat{c})$ [$\because \hat{b}\times\hat{b} = \hat{c}\times\hat{c} = 0$

$= 2(\hat{a}\times\hat{b}) - 4(\hat{a}\times\hat{c}) +6(\hat{b}\times\hat{c}) -4(\hat{b}\times\hat{c})$ [$\because (\hat{c}\times\hat{b}) = -(\hat{b}\times\hat{c})$]

$\Rightarrow \hat{q}\times\hat{r} = 2(\hat{a}\times\hat{b}) - 4(\hat{a}\times\hat{c}) +2(\hat{b}\times\hat{c})$

Now $\hat{p} . \left[\hat{q}\times\hat{r}\right] = (\hat{a} - 2\hat{b} + 3\hat{c}) . [2(\hat{a}\times\hat{c}) - 4(\hat{a}\times\hat{c}) + 2(\hat{b}\times\hat{c})]$

$= 2\hat{a} . (\hat{a}\times\hat{b}) - 4\hat{a} . (\hat{a}\times\hat{c}) +2\hat{a}(\hat{b}\times\hat{c}) -4\hat{b} . (\hat{a}\times\hat{b}) + 8\hat{b} . (\hat{a}\times\hat{c})$

$- 4\hat{b} . (\hat{b}\times\hat{c}) +6\hat{c} . (\hat{a}\times\hat{b}) - 12\hat{c}(\hat{a}\times\hat{c}) + 6\hat{c}\ (\hat{b}\times\hat{c})$ [$\because \hat{a}\times\hat{a} = \hat{b}\times\hat{b} = \hat{c}\times\hat{c} = 0$]

$= 2(0) - 4(0) +2[\hat{a}\ \hat{b}\ \hat{c}] -4(0) + 8[\hat{b}\ \hat{a}\ \hat{c}]$

$- 4(0) +6[\hat{c}\ \hat{a}\ \hat{b}] - 12(0) + 6(0)$

$= 2[\hat{a}\ \hat{b}\ \hat{c}] + 8[\hat{b}\ \hat{a}\ \hat{c}] +6[\hat{c}\ \hat{a}\ \hat{b}]$

$= 2[\hat{a}\ \hat{b}\ \hat{c}] - 8[\hat{a}\ \hat{b}\ \hat{c}] +6[\hat{a}\ \hat{b}\ \hat{c}]$ [To maintain cyclic order]

$= 0$

$\therefore$ $\hat{p} . \left[\hat{q}\times\hat{r}\right] = \left[\hat{p}\ \hat{q}\ \hat{r}\right] = 0$ are coplanar

Example 10 : Show that $[\hat{a}+\hat{b}, \hat{b}+\hat{c}, \hat{c}+\hat{a}] = 2[\hat{a}\ \hat{b}\ \hat{c}]$

Solution : Taking L.H.S.

$[\hat{a}+\hat{b}\ \hat{b}+\hat{c}\ \hat{c}+\hat{a}] = (\hat{a}+\hat{b}) . [(\hat{b}+\hat{c})\times(\hat{c}+\hat{a})]$

$= (\hat{a}+\hat{b}) . [(\hat{b}\times\hat{c}) + (\hat{b}\times\hat{a}) +(\hat{c}\times\hat{c}) +(\hat{c}\times\hat{a})]$

$= (\hat{a}+\hat{b}) . [(\hat{b}\times\hat{c}) + (\hat{b}\times\hat{a}) +(\hat{c}\times\hat{a})]$ $\because \hat{c}\times\hat{c} = 0$

$= \hat{a} . (\hat{b}\times\hat{c}) + \hat{a} . (\hat{b}\times\hat{a}) + \hat{a} . (\hat{c}\times\hat{a})\hat{b} . (\hat{b}\times\hat{c})$

$+\hat{b}\,.\,(\hat{b}\times\hat{a})+\hat{b}\,.\,(\hat{c}\times\hat{a})$

$= \hat{a}\,.\,(\hat{b}\times\hat{c}) + 0 + 0 + 0 + 0 + \hat{b}\,.\,(\hat{c}\times\hat{a})$

[$\because$ value of scalar triple product with two equal vectors is zero]

$= \hat{a}\,.\,(\hat{b}\times\hat{c}) + \hat{b}\,.\,(\hat{c}\times\hat{a})$

$= [\hat{a}\ \hat{b}\ \hat{c}] + [\hat{b}\ \hat{c}\ \hat{a}]$

$= [\hat{a}\ \hat{b}\ \hat{c}] + [\hat{a}\ \hat{b}\ \hat{c}]$

$= 2[\hat{a}\ \hat{b}\ \hat{c}]$ = R.H.S.

Hence $[\hat{a}+\hat{b}\ \ \hat{b}+\hat{c}\ \ \hat{c}+\hat{a}] = 0$

Example 11 : Find a unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$, $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$.

Solution : Let $\hat{a} = \hat{i}+\hat{j}+2\hat{k}$, $\hat{b} = \hat{i}+2\hat{j}+\hat{k}$ and let $\hat{r} = x\hat{i}+y\hat{j}+z\hat{k}$ be the required vector.

Given $\hat{r}$ is coplanar with $\hat{a}$ and $\hat{b}$

$\Rightarrow$ $\hat{r} = \lambda\hat{a}+\lambda\hat{b}$ [Definition of coplanar vector]

$\Rightarrow$ $x\hat{i}+y\hat{j}+z\hat{k} = (\lambda\hat{i}+\lambda\hat{j}+2\lambda\hat{k}) + r\hat{i}+2r\hat{j}+r\hat{k}$

$\Rightarrow$ $x\hat{i}+y\hat{j}+z\hat{k} = (\lambda+r)\hat{i} + (\lambda+2r)\hat{j} + (2+r)\hat{k}$

$\Rightarrow$ $x = \lambda + r$ (1)

$\Rightarrow$ $y = \lambda + 2r$ (2)

$\Rightarrow$ $z = 2\lambda + r$ (3)

Also given the given vector is perpendicular to $\hat{i}+\hat{j}\ \hat{k}$

Therefore, $(x\hat{i}+y\hat{j}+z\hat{k})\,.\,(\hat{i}+\hat{j}+\hat{k}) = 0$

[$\because \hat{a}\,.\,\hat{b} = 0 \Rightarrow a \perp b$]

$\Rightarrow x + y + z = 0$

Using (1, (2) and (3)

$\lambda + r + \lambda + 2r + 2\lambda + r = 0$

$\Rightarrow$ $4\lambda + 4r = 0$

$\Rightarrow$ $4(\lambda + r) = 0$

$=$ $\lambda + r = 0$ (4)

Using (4), in (1), (2) and (3) we get

$x = 0, \; y = (\lambda + r) + r, \; z = \lambda + (\lambda + r)$

$\Rightarrow$ $x = 0, \; y = r, \; z = \lambda = -r$

$\therefore$ $\hat{r} = 0\hat{i} + r\hat{j} + \lambda\hat{k}$

$\hat{r} = r\hat{j} - r\hat{k} \quad = r(\hat{j} - \hat{k})$

Since r is a unit vector

$\hat{r} = |\hat{r}| = 1 = \sqrt{(r^2) + r^2} = 1$

$\Rightarrow 2r^2 = 1$

$\Rightarrow r^2 = \dfrac{1}{2}$

$\Rightarrow r = \pm\dfrac{1}{\sqrt{2}}$

$\therefore$ $r = \pm\dfrac{1}{\sqrt{2}}(\hat{j} - \hat{k})$ is the required unit vector.

Self Check Exercise - 3

Q. 1 Show that the vector (2, 1, 3), (0, 5, 5), (-1, 2, 1) are coplanar.

Q. 2 Show that the four points having position vectors $6\hat{i} - 7\hat{j} >$ $16\hat{i} - 19\hat{j} - 4\hat{k}$, $2\hat{i} + 5\hat{j} + 10\hat{k}$ are not coplanar.

Q. 3 Do the points (4, -2, 1), (5, 1, 6), (2, 2, -5) and (3, 5, 0) be in a plane.

Q. 4 Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be non zero and non coplanar vectors. Show that $2\hat{a} - \hat{b} + 3\hat{c}$, $\hat{a} + \hat{b} - 2\hat{c}$ and $\hat{a} + \hat{b} - 3\hat{c}$ are non coplanar vectors.

Q. 5 For what value of λ for vectors $\hat{i} + 2\hat{j} + 3\hat{k}$, $\lambda\hat{i} - \hat{j} - \hat{k}$ and $3\hat{i} - 4\hat{j} + 3\hat{k}$ are coplanar.

Q. 6 Find λ such that vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$, $\hat{i} - 4\hat{j} + \lambda\hat{k}$ are coplanar.

Q. 7 Show that $[\hat{a} - \hat{b} \;.\; \hat{b} - \hat{c} \;\; \hat{c} - \hat{a}] = 0$

Q. 8 Let $\hat{a} = \hat{i} - \hat{j}$, $\hat{b} = \hat{j} - \hat{k}$, $\hat{c} = \hat{k} - \hat{i}$, d is a unit vector such that $\hat{a} . \hat{d} = 0 = \left[\hat{b} \;\; \hat{c} \;\; \hat{d}\right]$

then show that $|\hat{d}| = \pm \dfrac{\hat{j}+\hat{j}-2\hat{k}}{\sqrt{6}}$.

2.6 Volume of Tetrahedron

Scalar triple product is also used to find the volume to Tetrahedron. We will now prove as given below

(1) The volume of a tetrahedron whose coterminous edges are $\hat{a}$, $\hat{b}$ and $\hat{c}$ is given by $\frac{1}{6}[\hat{a}\ \hat{b}\ \hat{c}]$

Proof : Let OABC be a tetrahedron whose coterminus edges OA, OB, OC represents vectors $\hat{a}\ \hat{b}\ \hat{c}$ respectively.

$\therefore$ area of Δ OBC = $\frac{1}{2}\left(\hat{b}\, x\, \hat{c}\right)$

$\therefore$ Volume of tetrahedron Δ ABC = $\frac{1}{3}$ (area of base) height

= $\frac{1}{3}$ (area of Δ OBC) height

= $\frac{1}{3}$ x $\frac{1}{2}\left(\hat{b}\, x\, \hat{c}\right)$ â

= $\frac{1}{6}\{(\hat{b}\, x\, \hat{c}).\hat{a}\}$

= $\frac{1}{6}(\hat{a}\,\hat{b}\,\hat{c})$

Therefore, volume of tetrahedron = $\frac{1}{6}(\hat{a}\,\hat{b}\,\hat{c})$

Let us try some examples to have more understanding.

Example 1. Find the volume of tetrahedron whose coterminous edges are $2\hat{i}+2\hat{j}+6\hat{k}$, $-1\hat{i}+3\hat{j}+2\hat{k}$ and $-1\hat{i}+5\hat{j}+5\hat{k}$

Solution: Let $\hat{a} = 2\hat{i}+2\hat{j}+6\hat{k}$, $\hat{b} = -1\hat{i}+3\hat{j}+2\hat{k}$ and $\hat{c}$ $-1\hat{i}+5\hat{j}+5\hat{k}$ are three coterminous edges of tetrahedron.

Then volume of tetrahedron = $\frac{1}{6}(\hat{a}\,\hat{b}\,\hat{c})$

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \frac{1}{6}\begin{vmatrix} 2 & 2 & 6 \\ 1 & 3 & 2 \\ -1 & 5 & 5 \end{vmatrix}$$

$$= \frac{1}{6}\left[2\begin{vmatrix} 3 & 2 \\ j & j \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -1 & j \end{vmatrix} + 6\begin{vmatrix} 1 & 3 \\ -1 & j \end{vmatrix}\right]$$

$$= \frac{1}{6}\left[2(15-10) - 2(5+2) + 6(5+3)\right]$$

$$= \frac{1}{6}\left[2(5) - 2(7) + 6(8)\right]$$

$$= \frac{1}{6} [10 - 14 + 48]$$

$$= \frac{1}{6} [58 - 14]$$

$$= \frac{1}{6} [44]$$

$$= \frac{44}{6} \text{ cubic units.}$$

$\therefore$ Volume of the tetrahedron = $\frac{44}{6}$ cubic units.

Example 2: Find the volume of the tetrahedron formed by the point (1,1,1), (2,1,3), (3,2,2) and (3,3,4)

Solution: Let O be the origin and A, B, C, D be votias of tetrahedron given by (1,1,1), (2,1,3), (3,2,2) and (3,3,4) respectively.

Then $\overrightarrow{OA} = \hat{i} + \hat{j} + \hat{k}$, $\overrightarrow{OB} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\overrightarrow{OC} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ $\overrightarrow{OD} = 3\hat{i} + 3\hat{j} + 4\hat{k}$

Now $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 2\hat{i} + \hat{j} + 3\hat{k}$, $-(\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = 3\hat{i} + 2\hat{j} + 2\hat{k}$, $-(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + \hat{j} + 2\hat{k}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 3\hat{i} + 3\hat{j} + 4\hat{k}$, $-(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 3\hat{k}$

Therefore volume of tetrahedron ABCD = $\frac{1}{6}\left[\overline{AB}\ \overline{AC}\ \overline{AD}\right]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$= \frac{1}{6}\left[1\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}\right]$$

$$= \frac{1}{6}\left[1(3-2)-0+2(4-2)\right]$$

$$= \frac{1}{6}\,[1 + 4]$$

$$= \frac{5}{6} \text{ cubic units.}$$

Therefore volume of tetrahedron is $\frac{5}{6}$ cubic units.

Self Check Exercise - 4

Q.1 Find the volume of tetrahedron whose vertices are the point A (2,-1,-3), B (4, 1, 3), C (3, 2, -1) and D (1, 4, 2).

Q.2 Find the volume of tetrahedron whose edges are given by the vectors $\hat{i} + 2\hat{k}$, $2\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$.

2.17 Summary

Dear students in this unit you learn about

(1) Scalar triple product of there vectors.

(2) Component from of scalar triple products.

(3) Geometrical interpretation of scalar triple product.

(4) Scalar triple product properties.

(5) Volume of tetrahedron and volume of parallelepiped using scalar triple products.

2.8 Glossary

(1) Tetrahedron: A Solid having four plane triangular faces or triangular pyramidal.

(2) Vertices: A point where two lines meet to form an angle.

2.9 Answers to Self Check Exercises

Self Check Exercise - 1

Q 1 840

Q. 2 264

Q. 3 -7

Q. 4 -6

Q. 5 510

Self Check Exercise - 2

Q.1 4 Cubic units

Q.2 14 (in magnitude) cubic units

Q.3 90 Cubic units

Q.4 8 Cubic units

Q.5 37 (in magnitude) Cubic units

Q.6 286 Cubic unite

Self Check Exercise - 3

Q.1 $\begin{vmatrix} 2 & 1 & 3 \\ 0 & 5 & 5 \\ -1 & 2 & 1 \end{vmatrix} = 0$

Q. 2 $\begin{vmatrix} 10 & -12 & -4 \\ -6 & 10 & -6 \\ -4 & 12 & 10 \end{vmatrix} = 840 \neq 0$,

where $\overrightarrow{AB} = 10\hat{i} - 12\hat{j} - 4\hat{k}$

$\overrightarrow{AC} = -6\hat{i} - 10\hat{j} - 6\hat{k}$

$\overrightarrow{AD} = -4\hat{i} + 12\hat{j} + 10\hat{k}$

Q.3 $\begin{vmatrix} 1 & 3 & 5 \\ -2 & 4 & -6 \\ -1 & 7 & 1 \end{vmatrix} = 0$

Where $\overrightarrow{AB} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\overrightarrow{AC} = -2\hat{i} + 4\hat{j} - 6\hat{k}$

$\overrightarrow{AD} = \hat{i} + 7\hat{j} - \hat{k}$

Q.4 Just to show $\hat{p}.(\hat{a} \, x \, \hat{r}) = 0$, by taking $\hat{p} = 2\hat{a} - \hat{b} + 3\hat{c}$, $\hat{q} = \hat{a} + \hat{b} + 2\hat{c}$

and $\hat{r} = \hat{a} + \hat{b} + 3\hat{c}$

Q.5 $\lambda = {}^{-2}\!/_{9}$

Q.6 $\lambda = {}^{27}\!/_{5}$

Q.7 Same as Example 19

Q.8 Take d = $\lambda\left(\hat{i} + \hat{j} + l\,\hat{k}\right)$

then we $\left[\hat{b}\,\hat{c}\,\hat{a}\right]$ = 0 gives $\lambda = \dfrac{1}{\sqrt{6}}$

Self Check Exercise - 4

Q.1 $\dfrac{22}{3}$ Cubic unite

Q. 2 ${}^{5}\!/_{6}$ Cubic unite

2.10 References/Suggested Readings

(1) R. Murray, S. Lipschutz, D. Spellman, Vector analysis, Schaum's outlines.

(2) S. Narayan and P.K. Mittal, Vector Calculas, Schand and Company limited.

(3) J.N. Sharma and A.R. Vasishtha, Vector Calculas, Krishna Prakashan Mandir.

2.11 Terminal Questions

1. Interpret geometrically $\left[\hat{i}\ \hat{j}\ \hat{k}\right]$ after evaluating it.

2. Let $\hat{a}\ \hat{b}\ \hat{c}$ be three nor zero vectors such that $\hat{c}$ is a unit vector perpendicular to both $\hat{a}$ and $\hat{b}$. If the and to between $\hat{a}$ and $\hat{b}$ is ${}^{\pi}\!/_{6}$ show that $[\hat{a}\ \hat{b}\ \hat{c}]^2 = \dfrac{1}{4}|\hat{a}|^2|\hat{b}|^2$.

3. Find the value of λ for which $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} + \hat{j} - \hat{k}$ and $\lambda\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar.

4. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$ = 0 and the vectors $\hat{A}$ = (1, a, a^2), $\hat{B}$ (1, b, b^2) and C = (1, c, c^2) are non coplanar, prove that abc = -1

5. What you concludes about four non zero vector $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ for which $\left|\hat{a}\, x\, \hat{b}, \hat{c}\right|$ + $\left|\left(\hat{b}\, x\, \hat{c}\right).\hat{d}\right|$ = 0

6. Show that $\left[\hat{a}, \hat{b}, \hat{c} + \hat{d}\right]$ = $\left[\hat{a}, \hat{b}, \hat{c}\right]$ + $\left[\hat{a}, \hat{b}, \hat{d}\right]$

7. Prove that $\left[\left(\hat{a}.\hat{b}\, x\, \hat{c}\right)\right]^2$= a^2 b^2 c^2, when $\hat{a}, \hat{b}, \hat{c}$ are perpendicular to each other.

8. Simply $\{(\hat{b} - \hat{c})x(\hat{c} - \hat{a})(\hat{a} - \hat{b})\}$

9. Prove that the normal to the prove containing three points A, B and C whose position vectors are $\hat{a}$ $\hat{b}$ and $\hat{c}$ lies in the direction of $\hat{b}\, x\, \hat{c} + \hat{c}\, x\, \hat{a} + \hat{a}\, x\, \hat{b}$.

10. $\hat{a}, \hat{b}, \hat{c}$ are three non collinear unit vector such that the angle between any two is $\propto$. If $\hat{a}\, x\, \hat{b} + \hat{b}\, x\, \hat{c} = l\hat{a} + m\hat{b} + n\hat{c}$ then find l, m, n in terms of $\propto$.

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Unit - 3

Vector Triple Product

Structure

3.1 Introduction

Dear student, in this unit we will study about vector triple product. There is cross product between three vector. We will learn to define vector triple product as well as expand it. Vector triple product has some of its properties which are used in other fields of mathematic.

3.2 Learning Objectives:

After studying this unit, students will be able to:

1. Define vector triple product
2. Calculate vector triple product of given vectors
3. Under what vector triple product shows physically.
4. Apply the properties of vector triple products.

3.3 Vector Triple Product

Let $\hat{a}$ $\hat{b}$ and $\hat{c}$ are any three vectors then $\left(\hat{a} \times \hat{b}\right)\times\hat{c}$ or $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ are known as vector triple product of $\hat{a}$, $\hat{b}$ and $\hat{c}$. Since cross product of two vector is a vector, So in a vector triple product the resultend is a vector quantity. Hence it is known as vector triple product.

Also, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)\neq\left(\hat{a}\times\hat{b}\right)\times\hat{c}$, in general. The vector triple product $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ is a vector which lies in the plane of $\hat{b}$ and $\hat{c}$ and is perpendicular to $\hat{a}$. Simmilarly $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ is a vector which lies in the plane of $\hat{a}$ and $\hat{b}$ and is perpendicular to λ.

Expansion Formula for Vector Triple Product

(1) $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}\times\hat{b}\right)\times\hat{c}$

Proof: Let
$$\hat{a}=a_1\hat{i}+a_2\,\hat{j}+a_3\,\hat{k}$$
$$\hat{b}=b_1\,\hat{i}+b_2\,\hat{j}+b_3\,\hat{k}$$
$$\hat{c}=c_1\,\hat{i}+c_2\,\hat{j}+c_3\,\hat{k}$$

L.H.S. Now, $\hat{b}\,x\,\hat{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$=\hat{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \hat{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \hat{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_3 \end{vmatrix}$$

$= \hat{i}\,(b_2\,c_3 - c_2\,b_3) - \hat{j}\,(b_1\,c_3 - c_1\,b_3) + \hat{k}\,(b_1c_2 - c_1\,b_2)$

Now, $\hat{a}\,x\left(\hat{b}\,x\,\hat{c}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_2c_3-c_2b_3 & -b_1c_3+c_1b_3 & b_1c_2-c_1b_2 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} a_2 & a_3 \\ -b_1c_3+c_1b_3 & b_1c_2-c_1b_2 \end{vmatrix} = -\hat{j}\begin{vmatrix} a_1 & a_3 \\ b_2c_3-c_2b_3 & b_1c_2-c_1b_2 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_2c_3-c_2b_3 & -b_1c_3+c_1b_3 \end{vmatrix}$$

$=\hat{i}\,(a_2b_1c_2 - a_2c_1b_2 + b_1c_3a_3 - a_3c_1b_3) - \hat{j}\,(a_1b_1c_2 - a_1c_2b_2 - a_3b_2c_3 + a_3c_2b_3) + \hat{k}\,(-a_1b_1c_3 + a_1c_1b_3 - a_2b_2c_3 + c_2b_3a_2)$ ....(1)

Now taking R.H.S

$\hat{a}\,.\,\hat{c}\,\left(\hat{a}_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)$

$\Rightarrow$ $\hat{a}\,.\,\hat{c} = a_1c_1 + a_2c_2 + a_3c_3$

Now, $(\hat{a}.\hat{c})\hat{b} = (a_1c_1 + a_2c_2 + a_3c_3)\,.\,(b_1\hat{i} + b_2\,\hat{j} + b_3\hat{k})$

$= a_1c_1b_1\hat{i} + a_1c_1b_2\,\hat{j} + a_1c_1b_3\hat{k} + a_2c_2b_1\hat{i} + a_2c_2b_2\,\hat{j} + a_2c_2b_3\hat{k}$

$+ a_3c_3b_1\hat{i} + a_3c_3b_2\hat{j} + a_3c_3b_3\hat{k}$.

also $\left(\hat{a}.\hat{b}\right)=\left(\hat{a}_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right).\left(b_1\hat{i}+b_2\hat{j}+b_3\hat{k}\right)$

$\Rightarrow \quad \hat{a}.\hat{b} = a_1b_1 + a_2b_2 + a_3b_3$

Now $\left(\hat{a}.\hat{b}\right).\hat{c}=\left(a_1b_1+a_2b_2+a_3b_3\right).\left(\hat{c}_1\hat{i}+c_2\hat{j}+c_3\hat{k}\right)$

$= a_1b_1c_1\hat{i} + a_1b_1c_2\hat{j} + a_1b_1c_3\hat{k} + a_2b_2c_1\hat{i} + a_2b_2c_2\hat{j}$

$+a_2b_2c_3\hat{k}+a_3b_3c_1\hat{i}+a_3b_3c_3\hat{j}+a_3b_3c_3\hat{k}$

Now $\left(\hat{a}.\hat{c}\right).\hat{b} - \left(\hat{a}.\hat{b}\right).\hat{c} = a_1b_1c_1\hat{i} + a_1c_1b_2\hat{j} + a_1c_1b_3\hat{k} + a_2c_2b_1\hat{i}$

$+ a_2c_2b_1\hat{i} + a_2c_2b_2\hat{j} + a_2c_2b_3\hat{k} + a_3c_3b_1\hat{i}$

$+ a_3c_3b_2\hat{j} + a_3c_3b_3\hat{k} - \{a_1b_1c_1\hat{i} + a_1b_1c_2\}$

$+ a_1b_1c_3\hat{k} + a_2b_2c_1\hat{i} + a_2b_2c_2\hat{j} + a_3c_3b_1\hat{k}$

$+ a_3b_1c_3\hat{i} + a_3b_3c_2\hat{j} + a_3b_3c_3\hat{k}$

$= \hat{i}\,[a_1b_1c_1 + a_2c_2b_1 + a_3c_3b_1 - a_1b_1c_1 - a_2b_2c_1 - a_3b_3c_1]$

$+ \hat{j}\,[a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2 - a_1b_1c_2 - a_2b_2c_2 - a_3b_3c_3]$

$+ \hat{k}\,[a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3 - a_1b_1c_3 - a_2b_2c_3 - a_3b_3c_3]$

$= \hat{i}\,[a_2c_2b_1 - a_2b_2c_1 + a_3b_1c_3 - a_3b_3c_1] + \hat{j}\,[a_1b_2c_1 + a_3c_3b_2 - a_1b_1c_2 - a_3b_3c_2]$

$+ \hat{k}\,[a_1c_1b_3 + a_2c_2b_3 - a_1b_1c_3 - a_2b_2c_3]$ ....(2)

From (1) and (2)

$$\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c}$$

(2) $\left(\hat{a}\times\hat{b}\right)\times\hat{c} = \left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{b}.\hat{c}\right)\hat{a}$

Proof : Let $\hat{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\hat{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$

$\hat{c} = c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Taking L.H.S. $\left(\hat{a}\times\hat{b}\right)\hat{c}$

Now $\left(\hat{a}\times\hat{b}\right)$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$=\hat{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

= $\hat{i}$ $(a_2b_3 - a_3b_2)$ - $\hat{j}$ $(a_1b_3 - a_3b_1)$ + $\hat{k}$ $(a_1b_2 - a_2b_1)$

$\Rightarrow\left(\hat{a}\times\hat{b}\right)$ = $\hat{i}$ $(a_2b_3 - a_3b_2)$ - $\hat{j}$ $(a_3b_1 - a_1b_3)$ + $\hat{k}$ $(a_1b_2 - a_2b_1)$

Now $\left(\hat{a}\times\hat{b}\right)\hat{c}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} a_3b_1 - a_1b_3 & a_1b_2 - a_2b_1 \\ c_2 & c_3 \end{vmatrix} - \hat{j}\begin{vmatrix} a_2b_3 - a_3b_2 & a_1b_2 - a_2b_1 \\ c_1 & c_3 \end{vmatrix}$$

$$= +\hat{k}\begin{vmatrix} a_2b_3 - a_3b_2 & a_3b_1 - a_1b_3 \\ c_1 & c_2 \end{vmatrix}$$

= $\hat{i}$ $[a_3c_3b_1 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2]$ - $\hat{j}$ $[a_2b_3c_3 - a_3b_2c_3 - a_1b_2c_1 + a_2b_1c_1]$

+ $\hat{k}$ $[a_2b_3c_2 - a_3b_2c_2 - a_3b_1c_1 + a_1b_3c_1]$

= $\hat{i}$ $[a_3c_3b_1 - a_1b_3c_3 - a_1b_2c_2 + a_2b_1c_2]$ + $\hat{j}$ $[a_3b_2c_3 + a_1b_2c_1 - a_2b_3c_3 - a_2b_1c_1]$

+ $\hat{k}$ $[a_2b_3c_2 - a_3b_2c_3 - a_3b_1c_1 + a_1b_3c_1]$

Now taking R.H.S.

$$\left(\hat{a}.\hat{c}\right) = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right).\left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$$

$\Rightarrow$ $\hat{a}.\hat{c}$ $= \left(a_1c_1 + a_2c_2 + a_3c_3\right)$

Now, $\left(\hat{a}.\hat{c}\right)\hat{b}$ $= \left(a_1c_1 + a_2c_2 + a_3c_3\right)\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right)$

$$= a_1c_1b_1\hat{i} + a_1c_1b_2\hat{j} + a_1c_1b_3\hat{k} + a_2c_2b_1\hat{i} + a_2c_2b_2\hat{j} + a_2c_2b_3\hat{k}$$

$$a_3c_3b_1\hat{i} + a_3c_3b_2\hat{j} + a_3c_3b_3\hat{k}$$

$$= \hat{i}\left(a_1c_1b_1 + a_2c_2b_1 + a_3c_3b_1\right) + \hat{j}\left(a_1c_1b_2 + a_2c_2b_2 + a_3c_3b_2\right)$$

$$\hat{k}\left[a_1c_1b_3 + a_2c_2b_3 + a_3c_3b_3\right]$$

Again $\left(\hat{b}.\hat{c}\right) = \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) . \left(c_1\hat{i} + c_2\hat{j} + c_3\hat{k}\right)$

$\Rightarrow \quad \hat{b}.\hat{c} = \left(b_1c_1 + b_2c_2 + b_3c_3\right)$

Now, $\left(\hat{b}.\hat{c}\right)\hat{a} = \left(b_1c_1 + b_2c_2 + b_3c_3\right)\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)$

$$= b_1c_1a_1\hat{i} + b_1c_1a_2\hat{j} + b_1c_1a_3\hat{k} + b_2c_2a_1\hat{i} + b_2c_2a_2\hat{j} + b_2c_2a_3\hat{k}$$

$$b_3c_3a_1\hat{i} + b_3c_3a_2\hat{j} + b_3c_3a_3\hat{k}$$

$$= \hat{i}\left(b_1c_1a_1 + b_2c_2a_1 + b_3c_3a_1\right) + \hat{j}\left(b_1c_1a_2 + b_2c_2a_2 + b_3c_3a_2\right)$$

$$\hat{k}\left[b_1c_1a_3 + b_2c_2a_3 + b_3c_3a_3\right]$$

Therefore, $\left(\hat{a}.\hat{c}\right)\hat{b}$ - $\left(\hat{b}.\hat{c}\right)\hat{a} = \hat{i}\left[a_2c_2b_1 + a_3c_3b_1 - b_2c_2a_1 - b_3c_3a_1\right]$

$$= \hat{j}\left[a_1c_1b_2 + a_3c_3b_2 - b_1c_1a_2 - b_3c_3a_2\right]$$

$$= \hat{k}\left[a_1c_1b_3 + a_2c_2b_3 - b_1c_1a_3 - b_2c_2a_3\right]$$

Hence $\left(\hat{a}\times\hat{b}\right)\hat{c} = \left(\hat{a}.\hat{c}\right)\hat{b}$ - $\left(\hat{b}.\hat{c}\right)\hat{a}$

Hence the result.

Let us trey to evaluate vector triple product using the exertion of it.

Example 1: Find $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ where $\hat{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$, $\hat{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\hat{c} = \hat{i} + \hat{j} - \hat{k}$

Solution : Here $\hat{a} = 2\hat{i} + 4\hat{j} - 5\hat{k}$,

$$\hat{b} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\hat{c} = \hat{i} + \hat{j} - \hat{k}$$

Now $\hat{b}\times\hat{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

= $\hat{i}$ (-2-3) - $\hat{j}$ (-1-3) + $\hat{k}$ (1-2)

$\Rightarrow \quad \hat{b}\times\hat{c}$ = -5 $\hat{i}$ +4 $\hat{j}$ - $\hat{k}$

Now, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 4 & -5 \\ -5 & 4 & -1 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} 4 & -5 \\ 4 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & -5 \\ -5 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 4 \\ -5 & 4 \end{vmatrix}$$

= $\hat{i}$ (-4+20) - $\hat{j}$ (-2-25) + $\hat{k}$ (8+20)

= 16 $\hat{i}$ +27 $\hat{j}$ +28 $\hat{k}$

$\therefore \ \hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = 16 $\hat{i}$ +27 $\hat{j}$ +28 $\hat{k}$

Example 2 : Calculate $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ where $\hat{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\hat{b} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

Solution : Here $\hat{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\hat{b} = 2\hat{i} + \hat{j} - 3\hat{k}$

$\hat{c} = -3\hat{i} + \hat{j} + 2\hat{k}$

Now $\hat{a}\times\hat{b}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} -2 & 3 \\ 1 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 2 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$$

= $\hat{i}$ (6-3) - $\hat{j}$ (-3-6) + $\hat{k}$ (1+4)

$\Rightarrow \quad \hat{a}\times\hat{b}$ = 3 $\hat{i}$ +9 $\hat{j}$ +5 $\hat{k}$

Now, $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 9 & 5 \\ -3 & 1 & 2 \end{vmatrix}$

= $\hat{i}\begin{vmatrix} 9 & 5 \\ 1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 5 \\ -3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & 9 \\ -3 & 1 \end{vmatrix}$

= (18-5) $\hat{i}$ - $\hat{j}$ (6+15) + $\hat{k}$ (3+27)

$\therefore$ $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = 13 $\hat{i}$ - 21 $\hat{j}$ + 30 $\hat{k}$.

Example 3 : Verify $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = $\left(\hat{a}.\hat{c}\right)\hat{b}$ - $\left(\hat{a}.\hat{b}\right)\hat{c}$ where $\hat{a}=\hat{i}+2\hat{j}+3\hat{k}$, $\hat{b}=2\hat{i}-\hat{j}+\hat{k}$ and $\hat{c}=3\hat{i}+2\hat{j}-5\hat{k}$

Solution : Here $\hat{a}=\hat{i}+2\hat{j}+3\hat{k}$

$\hat{b}=2\hat{i}-\hat{j}+\hat{k}$

$\hat{c}=3\hat{i}+2\hat{j}-5\hat{k}$

Taking L.H.S., $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

Now $\hat{b}\times\hat{c}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 3 & 2 & -5 \end{vmatrix}$

= $\hat{i}\begin{vmatrix} -1 & 1 \\ 2 & -5 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 3 & -5 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix}$

= $\hat{i}$ (5-2) - $\hat{j}$ (-10-3) + $\hat{k}$ (4+3)

$\Rightarrow$ $\hat{b}\times\hat{c}$ = 3 $\hat{i}$ + 13 $\hat{j}$ + 7 $\hat{k}$.

Now, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 13 & 7 \end{vmatrix}$

= $\hat{i}\begin{vmatrix} 2 & 3 \\ 13 & 7 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 3 & 13 \end{vmatrix}$

$= \hat{i}$ (14-39) - $\hat{j}$ (7-9) + $\hat{k}$ (13-6)

$\Rightarrow \hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = -25$\hat{i}$ + 2 $\hat{j}$ + 7$\hat{k}$ . ...(1)

Taking right hand side, $(\hat{a}.\hat{c})\hat{b}$ - $\left(\hat{a}.\hat{b}\right)\hat{c}$

Now, $(\hat{a}.\hat{c})$ = (1$\hat{i}$ + 2 $\hat{j}$ + 3$\hat{k}$) . (3$\hat{i}$ + 2 $\hat{j}$ -5 $\hat{k}$)

= 3 + 4 - 15

= 7 - 15

= -8

$\therefore (\hat{a}.\hat{c})\hat{b}$ = -8(2$\hat{i}$ - $\hat{j}$ + $\hat{k}$)

$\Rightarrow (\hat{a}.\hat{c})\hat{b}$ = -16$\hat{i}$ + 5 $\hat{j}$ -8$\hat{k}$

Now $\left(\hat{a}.\hat{b}\right)\hat{c}$ = ($\hat{i}$ + 2 $\hat{j}$ + 13$\hat{k}$) . (2$\hat{i}$ - $\hat{j}$ + $\hat{k}$)

= 2 - 2 + 3

= 3

$\therefore \left(\hat{a}.\hat{b}\right)\hat{c}$ = 3(3$\hat{i}$ +2 $\hat{j}$ - 5$\hat{k}$)

$\Rightarrow \left(\hat{a}.\hat{b}\right)\hat{c}$ = (9$\hat{i}$ +6 $\hat{j}$ - 15$\hat{k}$)

Now. $(\hat{a}.\hat{c})\hat{b}$-$\left(\hat{a}.\hat{b}\right)\hat{c}$ = (-16$\hat{i}$ +8 $\hat{j}$ - 8$\hat{k}$)-(9$\hat{i}$ +6 $\hat{j}$ - 15$\hat{k}$)

= (-16-9)$\hat{i}$ + $\hat{j}$ (8-6) + $\hat{k}$ (-8+15)

$(\hat{a}.\hat{c})\hat{b}$ - $\left(\hat{a}.\hat{b}\right)\hat{c}$ = -25$\hat{i}$ +2 $\hat{j}$ + 7$\hat{k}$ ...(2)

From (1) and (2) we have

$$\hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$$

Hence verified.

Example 4 : Prove that $\hat{a}\times\left(\hat{b}\times\hat{c}\right)+\hat{b}\times(\hat{c}\times\hat{a})$ + $\hat{c}\times\left(\hat{a}\times\hat{b}\right)$ = 0

Solution : Taking L.H.S.

$$\hat{a}\times\left(\hat{b}\times\hat{c}\right)+\hat{b}\times(\hat{c}\times\hat{a}) + \hat{c}\times\left(\hat{a}\times\hat{b}\right)$$

Using expansion formulate of $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$, we get

$= (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} + \left(\hat{b}.\hat{a}\right)\hat{c} - \left(\hat{b}.\hat{c}\right)\hat{a} + \left(\hat{c}.\hat{b}\right)\hat{a} - (\hat{c}.\hat{a})\hat{b}$

Using the property of dot product $\hat{a}.\hat{b} = \hat{b}.\hat{a}$, we get

$= (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} + \left(\hat{a}.\hat{b}\right)\hat{c} - \left(\hat{b}.\hat{c}\right)\hat{a} + \left(\hat{b}.\hat{c}\right)\hat{a} - (\hat{a}.\hat{c})\hat{b}$

= 0 = R.H.S.

Hence $\hat{a}\times\left(\hat{b}\times\hat{c}\right) + \hat{b}\times(\hat{c}\times\hat{a}) + \hat{c}\times\left(\hat{a}\times\hat{b}\right) = 0$

Example 5 : Show that for any vector $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\hat{i} \times (\hat{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\hat{a} \times \hat{k}) = \overrightarrow{2a}$$

Solution : Taking the L.H.S.

$$\hat{i} \times (\hat{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\hat{a} \times \hat{k})$$

Using expansion formula of $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$, we get

$= (\hat{i}.\hat{i})\hat{a} - (\hat{i}.\hat{a})\hat{j} + (\hat{j}.\hat{j})\hat{a} - (\hat{j}.\hat{a})\hat{j} + (\hat{k}.\hat{k})\hat{a} - (\hat{k}.\hat{a})\hat{k}$

= as $\hat{i}.\hat{i} = 1$, $\hat{j}.\hat{j} = 1$, $\hat{k}.\hat{k} = 1$ and $\hat{a}.\hat{b} = \hat{b}.\hat{a}$ we have

$= \hat{a} - (\hat{a}.\hat{i})\hat{i} + \hat{a} - (\hat{a}.\hat{j})\hat{j} + \hat{a} - (\hat{a}.\hat{k})\hat{k}$

$= 3\hat{a} - \{(\hat{a}.\hat{i})\hat{i} + (\hat{a}.\hat{j})\hat{j} + (\hat{a}.\hat{k})\hat{k}\}$ ...(1)

as $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\left.\begin{array}{l} \text{so } \hat{a}.\hat{i} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\hat{i} = a_1 \\ \hat{a}.\hat{j} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\hat{j} = a_2 \\ \hat{a}.\hat{k} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).\hat{k} = a_3 \end{array}\right\} \quad ...(2)$$

Using (2) in (1), we get

$\hat{i} \times (\hat{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\hat{a} \times \hat{k}) = 3\hat{a} - \{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\}$

$= 3\hat{a} - \hat{a} = \overrightarrow{2a}$

Hence $\hat{i} \times (\hat{a} \times \hat{i}) + \hat{j} \times (\hat{a} \times \hat{j}) + \hat{k} \times (\hat{a} \times \hat{k}) = \overrightarrow{2a}$

Example 6 : Prove that $\hat{a}\times\left\{\hat{a}\times\left(\hat{a}\times\hat{b}\right)\right\} = (\hat{a}.\hat{a})\left(\hat{b}\times\hat{a}\right)$.

Solution : L.H.S. $\hat{a}\times\left\{\hat{a}\times\left(\hat{a}\times\hat{b}\right)\right\} = \hat{a}\times\left\{\left(\hat{a}.\hat{b}\right)\hat{a}-(\hat{a}.\hat{a})\hat{b}\right\}$

$$= \left(\hat{a}\times\hat{a}\right)\hat{a}.\hat{b} - \left(\hat{a}.\hat{a}\right)\hat{a}\times\hat{b}$$

$$= 0 - (\hat{a}.\hat{a})\left(\hat{a}\times\hat{b}\right)$$

$\Rightarrow \quad \hat{a}\times\left\{\hat{a}\times\left(\hat{a}\times\hat{b}\right)\right\} = (\hat{a}.\hat{a})\left(\hat{b}\times\hat{a}\right) \qquad \because \left(\hat{a}\times\hat{b}\right) = -\left(\hat{b}\times\hat{a}\right)$

Try to do these questions.

Self Check Exercise - 1

Q. 1 If $\hat{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\hat{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$, $\hat{c} = 4\hat{i} - 3\hat{j} + 5\hat{k}$ State which of the following are meaningful and evaluate any one of these that are meaningful; $\left(\hat{a}.\hat{b}\right)\times\hat{c}$, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)\hat{a}.\left(\hat{b}\times\hat{c}\right)$

Q. 2 If $\hat{a} = 3\hat{i} - \hat{j} + \hat{k}$, $\hat{b} = \hat{i} + 3\hat{j} - \hat{k}$, $\hat{c} = -\hat{i} + \hat{j} + 3\hat{k}$ State which of the following are meaningful and evaluate any one of these that are meaningful; $\left(\hat{a}.\hat{b}\right)\times\hat{c}$, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)\left(\hat{a}\times\hat{b}\right).\hat{c}$

Q. 3 Verify $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$

$\hat{a} = \hat{i} + \hat{j} + \hat{k}$, $\hat{b} = \hat{j} - \hat{j} + \hat{k}$, $\hat{c} = 2\hat{i} + \hat{k}$

Q. 4 Verify $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ by the expansion where

$\hat{a} = 15\hat{i} - 3\hat{j} + 7\hat{k}$, $\hat{b} = 49\hat{i} - 31\hat{j} - 45\hat{k}$, $\hat{c} = \hat{i} - \hat{k}$

3.4 Properties of Vector Triple Product

Property : The vector $\hat{r} = \hat{a}\times\left(\hat{b}\times\hat{c}\right)$, is a vector which is perpendicular to $\vec{a}$ and lies inthe plane of $\hat{b}$ and $\hat{c}$. By using this property of vector triple product, we can prove that a given vector is parallel to product of two vectors let us try to understand it by doing this example.

Example 1 : If $\hat{q} = \hat{c}\times\hat{a}$ and $\hat{r} = \hat{a}\times\hat{b}$, show that $\hat{a}$ is parallel to $\hat{q}\times\hat{r}$.

Solution : Since we know that if $\hat{a}\times\hat{b}$ = 0 then two vectors are parallel to each other. To show $\hat{a}$ is parallel to $\hat{q}\times\hat{r}$, it is sufficient to prove $\hat{a}\times\left(\hat{q}\times\hat{r}\right)$ = 0

Given $\hat{q} = \hat{c}\times\hat{a}$ and $\hat{r} = \hat{a}\times\hat{b}$

So $\hat{a}\times\left(\hat{q}\times\hat{r}\right) = \left(\hat{a}.\,\hat{r}\right)\hat{q} - \left(\hat{a}.\,\hat{q}\right)\hat{r}$ [using the expansion formula]

$= \hat{a}.\left(\hat{a}\times\hat{b}\right)\left(\hat{c}\times\hat{a}\right) - \left\{\hat{a}.\left(\hat{c}\times\hat{a}\right)\right\}\left(\hat{a}\times\hat{b}\right)$

$= 0\left(\hat{c}\times\hat{a}\right) - 0\left(\hat{a}\times\hat{b}\right)$

$\therefore$ value of scalar triple product with two equal vectors is zero

= 0

$\Rightarrow$ $\hat{a}\times\left(\hat{q}\times\hat{r}\right)$ = 0

Hence $\hat{a}$ is parallel to $\left(\hat{q}\times\hat{r}\right)$.

Example 2 : If $\hat{r} = \hat{a}\times\hat{b}$, $\hat{p} = \hat{b}\times\hat{c}$, prove that is show that $\hat{b}$ is parallel to $\hat{r}\times\hat{p}$.

Solution : To prove $\hat{b}$ is parallel to $\hat{r}\times\hat{p}$ if is sufficient to prove $\hat{b}\times\left(\hat{r}\times\hat{p}\right)$ = 0

So taking $\hat{b}\times\left(\hat{r}\times\hat{p}\right) = \left(\hat{b}.\hat{p}\right)\hat{r} + \left(\hat{b}.\hat{r}\right)\hat{p}$ (using expansion formula)

$\left\{\hat{b}.\left(\hat{b}\times\hat{c}\right)\right\}\hat{r} + \left\{\hat{b}.\left(\hat{a}\times\hat{b}\right)\right\}\hat{p}$

[using $\hat{r} = \hat{a}\times\hat{b}$, $\hat{p} = \hat{b}\times\hat{c}$]

= 0($\hat{r}$) + 0($\hat{p}$)

[$\because$ Value of scalar triple product with two equal vectors is zero]

= 0

$\Rightarrow$ $\hat{b}\times\left(\hat{r}\times\hat{p}\right)$ = 0

Hence $\hat{a}$ is parallel to $\left(\hat{r}\times\hat{p}\right)$.

Self Check Exercise - 2

Q. 1 Show that $\hat{c}$ is parallel to $\left(\hat{r}\times\hat{p}\right)$, where $\hat{p} = \hat{b}\times\hat{c}$ and $\hat{q} = \hat{c}\times\hat{a}$.

Property -2 : Vector triple product of three vectors is not associative. Mathematically

$\hat{a} \times (\hat{b} \times \hat{c}) \neq (\hat{a} \times \hat{b}) \times \hat{c}$, in general

Let us try to verify this property by following example

Example 3 : If $\hat{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\hat{b} = 2\hat{i} - \hat{j} + \hat{k}$, $\hat{c} = \hat{i} + \hat{j} - 2\hat{k}$

Compute $(\hat{a} \times \hat{b}) \times \hat{c}$ and $\hat{a} \times (\hat{b} \times \hat{c})$ check associatively in vector triple product.

Solution : Given $\hat{a} = \hat{i} - 2\hat{j} + \hat{k}$,

$\hat{b} = 2\hat{i} - \hat{j} + \hat{k}$,

$\hat{c} = \hat{i} + \hat{j} - 2\hat{k}$

To Find

Now $(\hat{a} \times \hat{b}) \times \hat{c}$

Taking $\hat{a} \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & -1 & 1 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}$$

$= \hat{i}\,(-2+1) - \hat{j}\,(1 - 2) + \hat{k}\,(-1 + 4)$

$\Rightarrow \quad \hat{a} \times \hat{b} = \hat{i} + \hat{j} + 3\hat{k}$

Now, $(\hat{a} \times \hat{b}) \times \hat{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 3 \\ 1 & 1 & -2 \end{vmatrix}$

$$= \hat{i} \begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - \hat{j} \begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$$

$= \hat{i}\,(-2 - 1) - \hat{j}\,(2 - 3) + \hat{k}\,(-1 - 1)$

$= -5\hat{i} + \hat{j} - 2\hat{k}$

$\Rightarrow \quad (\hat{a} \times \hat{b}) \times \hat{c} = -5\hat{i} + \hat{j} - 2\hat{k}$

Now to evaluate $\hat{a} \times (\hat{b} \times \hat{c})$

Taking $\hat{b}\times\hat{c}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

$=\hat{i}\begin{vmatrix} 1 & 3 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}$

= $\hat{i}$ (-2 -3) - $\hat{j}$ (2 - 3) + $\hat{k}$ (-1 -1)

= -5 $\hat{i}$ + $\hat{j}$ - 2 $\hat{k}$

$\Rightarrow$ $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = -5 $\hat{i}$ + $\hat{j}$ - 2 $\hat{k}$

Now to evaluate $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

Taking $b\times c = \begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 1 & 1 & -2 \end{vmatrix}$

$=\hat{i}\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$

= $\hat{i}$ (2 -1) - $\hat{j}$ (-4 - 1) + $\hat{k}$ (2 + 1)

$\Rightarrow$ $\hat{b}\times\hat{c}$ = $\hat{i}$ + 5 $\hat{j}$ + 3 $\hat{k}$

Now $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 5 & 3 \end{vmatrix}$

$=\hat{i}\begin{vmatrix} -2 & 1 \\ 5 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 1 & 5 \end{vmatrix}$

= $\hat{i}$ (-6-5) - $\hat{j}$ (3-1) + $\hat{k}$ (5 + 2)

$\Rightarrow$ $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ =-11 $\hat{i}$ -2 $\hat{j}$ + 7 $\hat{k}$

Since $\left(\hat{a}\times\hat{b}\right)\times\hat{c} =$ -5 $\hat{i}$ + $\hat{j}$ - 2 $\hat{k} \neq$ -11 $\hat{i}$ -2 $\hat{j}$ + 2 $\hat{k}$ = $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

Hence cross product is not associative.

Example 4 : Show that $\hat{a}\times\left(\hat{b}\times\hat{c}\right)\neq\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ for

$\hat{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\hat{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\hat{c} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Solution : Given $\hat{a} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\hat{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$,

$\hat{c} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

Taking L.H.S. $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

$$\hat{b}\times\hat{c} = \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$=\hat{i}\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9)$

$\Rightarrow \quad \hat{b}\times\hat{c} = -\hat{i} + 2\hat{j} - \hat{k}$

Now, $$\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix}$$

$$=\hat{i}\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix}$$

$= \hat{i}(-2 - 6) - \hat{j}(-1 + 3) + \hat{k}(2 + 2)$

$\Rightarrow \quad \hat{a}\times\left(\hat{b}\times\hat{c}\right) = -8\hat{i} - 2\hat{j} + 4\hat{k}$

Now, R.H.S. $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$

$$\left(\hat{a}\times\hat{b}\right) = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$$

= $\hat{i}$ (8 - 9) - $\hat{j}$ (4 - 6) + $\hat{k}$ (3 - 4)

= $\hat{i}$ + 2 $\hat{j}$ - $\hat{k}$

Now, $\left(\hat{a}\times\hat{b}\right)\times\hat{c} = \begin{vmatrix} i & j & k \\ -1 & 2 & -1 \\ 3 & 4 & 5 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} - \hat{j}\begin{vmatrix} -1 & -1 \\ 3 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix}$$

= $\hat{i}$ (10 + 4) - $\hat{j}$ (-4 + 3) + $\hat{k}$ (-4 - 6)

$\Rightarrow$ $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = 14 $\hat{i}$ + $\hat{j}$ -10 $\hat{k}$

Since $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$=- 8 $\hat{i}$ -2 $\hat{j}$ +4 $\hat{k}$ $\neq\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = 14 $\hat{i}$ + $\hat{j}$ + 10 $\hat{k}$

Hence $\hat{a}\times\left(\hat{b}\times\hat{c}\right)\neq\left(\hat{a}\times\hat{b}\right)\times\hat{c}$

The vector or cross product is not associative put under some condition the associative but under some condition the associativity hold. Let us try to understand that conditions by following examples.

Example 5 : The associativity hold in vector product i.e. $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$= $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$, and only if $\hat{a}\,and\,\hat{c}$ are collinear.

Solution : Firstly taking $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

Let $\hat{a}\,and\,\hat{c}$ are collinear

$\Rightarrow$ $\hat{c}$ = λ â , where λ is a scalar

Now, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$= $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$

= $\left(\hat{a}.\lambda\hat{a}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\lambda\hat{a}$

= $\lambda\left(\hat{a}.\hat{a}\right)\hat{b}-\lambda\left(\hat{a}.\hat{b}\right)\hat{a}$

$$\Rightarrow \quad \hat{a}\times\left(\hat{b}\times\hat{c}\right) = \lambda\left[(\hat{a}.\hat{a})\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{a}\right]$$

Now, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)= (\hat{a}.\hat{c})\hat{b}-\left(\hat{b}.\hat{c}\right)\hat{a}$

$$= (\hat{a}.\lambda\hat{a})\hat{b}-\left(\hat{b}.\lambda\hat{a}\right)\hat{a}$$

$$= \lambda(\hat{a}.\hat{a})\hat{b}-\lambda\left(\hat{b}.\hat{a}\right)\hat{a}$$

$$= \lambda\left[\lambda(\hat{a}.\hat{a})\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{a}\right]$$

$$\Rightarrow \quad \left(\hat{a}\times\hat{b}\right)\times\hat{c} = \lambda\left[(\hat{a}.\hat{a})\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{a}\right]$$

Hence $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \left(\hat{a}\times\hat{b}\right)\times\hat{c}\,\#\,\hat{a}\, and\, \hat{c}$ are collinear or parallel.

Example 6 : $\hat{a}\times\left(\hat{b}\times\hat{c}\right)=\left(\hat{a}\times\hat{b}\right)\times\hat{c}\ \#\ \hat{c}=\left(\dfrac{\hat{c}.\hat{b}}{\hat{a}.\hat{b}}\right)\hat{a}$

i.e. $\hat{a}$ and $\hat{c}$ are collinear.

Solution : Let us assume that $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \left(\hat{a}\times\hat{b}\right)\times\hat{c}$

$$\Rightarrow \quad \hat{a}\times\left(\hat{b}\times\hat{c}\right)= -\hat{c}\times\left(\hat{a}\times\hat{b}\right) \qquad \because \hat{a}\times\hat{b} = -\left(\hat{b}\times\hat{a}\right)$$

$$\Rightarrow \quad (\hat{a}.\hat{c})\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c} = -\left[\left(\hat{c}.\hat{b}\right)\hat{a}-(\hat{c}.\hat{a})\hat{b}\right]$$

$$\Rightarrow \quad (\hat{c}.\hat{a})\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c} = (\hat{c}.\hat{a})\hat{b}-\left(\hat{c}.\hat{b}\right)\hat{a} \qquad [\because \hat{c}.\hat{a} = \hat{a}.\hat{c}]$$

$$\Rightarrow \quad -\left(\hat{a}.\hat{b}\right)\hat{c} \qquad = -\left(\hat{c}.\hat{b}\right)\hat{a}$$

$$\Rightarrow \quad \hat{c}=\left(\frac{\hat{c}.\hat{b}}{\hat{a}.\hat{b}}\right)\hat{a}$$

$\Rightarrow \quad \hat{c} = \lambda\,\hat{a}$ [$\because$ dot product of two vectors is a scalar quantity]

$\Rightarrow \quad \hat{c}$ is collinear to $\hat{a}$ or parallel to $\hat{a}$.

Example 7 : Prove that $\left(\hat{a}\times\hat{b}\right)\times\hat{c}= \hat{a}\times\left(\hat{b}\times\hat{c}\right)$ if and only if $(\hat{c}.\hat{a})\times\hat{b} = \vec{0}$

Solution, Let $\left(\hat{a}\times\hat{b}\right)\times\hat{c}= \hat{a}\times\left(\hat{b}\times\hat{c}\right)$

iff $-\hat{c}\times\left(\hat{a}\times\hat{b}\right)=\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ $\quad\left[\because \hat{a}\times\hat{b}=-\left(\hat{b}\times\hat{a}\right)\right]$

$\Rightarrow$ iff $-\left[\left(\hat{c}.\hat{b}\right).\hat{a}-\left(\hat{c}.\hat{a}\right)\hat{b}\right]=\left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c}$

$\Rightarrow$ iff $\left(\hat{c}.\hat{a}\right)\hat{b}-\left(\hat{c}.\hat{b}\right)\hat{a}=\left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c}$

iff $\left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{c}.\hat{b}\right)\hat{a}=\left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c}$

iff $-\left(\hat{c}.\hat{b}\right)\hat{a}=-\left(\hat{a}.\hat{b}\right)\hat{c}$

iff $-\left(\hat{a}.\hat{b}\right)\hat{c}+\left(\hat{c}.\hat{a}\right)\hat{a}=\hat{0}$

iff $-\left(\hat{b}.\hat{a}\right)\hat{c}+\left(\hat{b}.\hat{c}\right)\hat{a}=0$ $\quad\because\left[\hat{a}.b=\hat{b}.\hat{a}\right]$

iff $-\hat{b}\times\left(\hat{c}\times\hat{a}\right)=\hat{0}$

iff $\left(\hat{c}\times\hat{a}\right)\times\hat{b}=\hat{0}$

Hence the result.

$\hat{a}\left(\hat{b}\times\hat{c}\right)$

Vector triple product gives the vector which is coplanar to $\left(\hat{b}\times\hat{c}\right)$ and perpendicular to $\hat{a}$. Using this concept we can find the unit vector which is coplanar to fixed and perpendicular to $\hat{a}$. Let us try these examples.

Example 8 : Find a unit vector coplanar with $\hat{i}+\hat{j}+2\hat{k}$, $\hat{i}+2\hat{j}+\hat{k}$ and perpendicular to $\hat{i}+\hat{j}+\hat{k}$

Solution :

Since $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ is a vector which coplanar to $\left(\hat{b}\times\hat{c}\right)$ and perpendicular to $\hat{a}$. Hence, we choose

$\hat{a}=\hat{i}+\hat{j}+\hat{k}$

$\hat{b}=\hat{i}+\hat{j}+2\hat{k}$

$\hat{c}=\hat{i}+2\hat{j}+\hat{k}$

Now, $\hat{b}\times\hat{c}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix}$

$$=\hat{i}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$$

= $\hat{i}$ (1 - 4)- $\hat{j}$ (1 - 2) + $\hat{k}$ (2 - 1)

$\Rightarrow$ $\hat{b}\times\hat{c} = -3\,\hat{i} + \hat{j} + \hat{k}$

Now, $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$= $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ -3 & 1 & 1 \end{vmatrix}$

$$=\hat{i}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 1 \\ -3 & 1 \end{vmatrix}$$

= $\hat{i}$ (1 - 1)- $\hat{j}$ (1 +3) + $\hat{k}$ (1+3)

$\Rightarrow$ $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = -4\,\hat{j} + 4\,\hat{k}$

Since Unit vector = $\dfrac{Vector}{|Vector|}$ i.e. $\hat{i} = \dfrac{\hat{i}}{|\hat{i}|}$

So $\left|\hat{a}\times\left(\hat{b}\times\hat{c}\right)\right|$ = $\sqrt{-4\hat{i}+4\hat{k}}$

= $\sqrt{(-4)^2+(4)^2}$

= $\sqrt{16+16}$

= $\sqrt{32}$

= $4\sqrt{2}$

$\therefore$ Required unit vector = $\dfrac{-4\hat{j}+4\hat{k}}{4\sqrt{2}}$

= $\dfrac{-4}{4\sqrt{2}}\hat{j} + \dfrac{4}{4\sqrt{2}}\hat{k}$

$\therefore$ Required unit vector = $\frac{1}{\sqrt{2}}\hat{j}+\frac{4}{\sqrt{2}}\hat{k}$

Example 9 : Find a unit vector which is perpendicular to $\hat{a}$ and coplanar with vectors $\hat{a}$ = $2\hat{i}+\hat{j}+\hat{k}$, $\hat{b}$ = $\hat{i}+2\hat{j}+\hat{k}$.

Solution : Using the definition of vector triple product $\hat{a}\times\left(\hat{a}\times\hat{b}\right)$ is the vector which is perpendicular to $\hat{a}$ and coplanar to vector $\hat{a}\, and\, b$.

So, $$\hat{a}\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$$=\hat{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}-\hat{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}+\hat{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

= $\hat{i}$ (-1 -2) - $\hat{j}$ (-2 -1) + $\hat{k}$ (4 - 1)

$$\hat{a}\times\hat{b} = -3\hat{i}+3\hat{j}+3\hat{k}$$

Now, $$\hat{a}\times\left(\hat{a}\times\hat{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

$$=\hat{i}\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}-\hat{j}\begin{vmatrix} 2 & 1 \\ -3 & 3 \end{vmatrix}+\hat{k}\begin{vmatrix} 2 & 1 \\ -3 & 3 \end{vmatrix}$$

= $\hat{i}$ (3 - 3) - $\hat{j}$ (6 + 3) + $\hat{k}$ (6 + 3)

$$\Rightarrow \quad \hat{a}\times\left(\hat{a}\times\hat{b}\right) = -9\hat{j}+9\hat{k}$$

Since Unit vector = $\frac{Vector}{|Vector|}$

$\therefore$ The required unit vector is = $\frac{-9\hat{j}+9\hat{k}}{\left|-9\hat{i}+9\hat{k}\right|}$

$$=\frac{-9\hat{j}+9\hat{k}}{\sqrt{81+81}}$$

$$= \frac{-9\hat{j}+9\hat{k}}{\sqrt{162}}$$

$$= \frac{-9\hat{j}+9\hat{k}}{9\sqrt{2}}$$

$$= \frac{-\hat{j}}{\sqrt{2}} + \frac{\hat{k}}{\sqrt{2}}$$

So $= \dfrac{-\hat{j}}{\sqrt{2}} + \dfrac{\hat{k}}{\sqrt{2}}$ is the required unit vector.

Self Check Exercise - 2

Q. 2 Find the unit vector which is perpendicular to $\hat{a}$ and coplanar to $\hat{b}$ and $\hat{c}$ given $\hat{a} = \hat{i} - 2\hat{j} - 3\hat{k}$ and $\hat{b} = 2\hat{i} + \hat{j} - \hat{k}$ and $\hat{c} = \hat{i} + 3\hat{j} - 2\hat{k}$.

Q.3 Find the unit vector which is perpendicular to $\hat{a}$ and coplanar to $\hat{b}$ and $\hat{c}$ for $\hat{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ $\hat{b} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ and $\hat{c} = 4\hat{i} - 3\hat{j} + 5\hat{k}$.

Q.4 Find $\left|\left(\hat{a}\times\hat{b}\right)\times\hat{c}\right|$, when $\hat{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\hat{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\hat{c} = \hat{i} + \hat{j} + 2\hat{k}$.

Let us try some more question related to vector product:

Example 10: Show that $\left[\hat{b}\times\hat{c}, \hat{c}\times\hat{a}, \hat{a}\times\hat{b}\right] = [a.bc]^2$

Also prove that if a, b, c are non coplanar, So as $\hat{b}\times\hat{c}$, $\hat{c}\times\hat{a}$ and $\hat{a}\times\hat{b}$

Solution: L.H.S. = $\left[\hat{b}\times\hat{c} \quad \hat{c}\times\hat{a} \quad \hat{a}\times\hat{b}\right]$ is a scalar triple product of three vector. So by definition of scalar triple product we have,

$$\left[\hat{b}\times\hat{c} \quad \hat{c}\times\hat{a} \quad \hat{a}\times\hat{b}\right] = \left\{\left(\hat{b}\times\hat{c}\right)\times\left(\hat{c}\times\hat{a}\right)\right\}.\left(\hat{a}\times\hat{b}\right)$$

taking $\left(\hat{b}\times\hat{c}\right)\times\left(\hat{c}\times\hat{a}\right)$, Let A = $\hat{b}\times\hat{c}$, so

$$\Rightarrow \quad \vec{A} \times \left(\hat{c}\times\hat{a}\right)$$

$$= \left(\hat{A}.\hat{a}\right)\hat{c} - \left(\hat{A}.\hat{c}\right)\hat{a} \qquad \text{[using expansion of } \hat{A}\times\left(\hat{B}\times\hat{C}\right)$$

$$= \left\{\left(\hat{b}\times\hat{c}\right).\hat{a}\right\}\left\{\hat{c} - \left(\hat{b}\times\hat{c}\right).\hat{c}\right\}\hat{a}$$

$$= \left\{\hat{c}.\left(\hat{b}\times\hat{c}\right)\right\}\hat{c} - \left\{\hat{c}.\left(\hat{b}\times\hat{c}\right)\right\}\vec{a} \quad [\because a.b = \hat{b}\times\hat{a}]$$

$$\left\{\left(\hat{a}\ \hat{b}\ \hat{c}\right)\hat{c} - \left[\hat{c}\ \hat{b}\ \hat{c}\right]\hat{a}\right\}$$

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{c} - 0\ (\hat{a})$$

$\because$ Scalar triple product is zero if two vector are same.

$$\therefore \qquad \left(\hat{b}\times\hat{c}\right)\times\left(\hat{c}\times\hat{a}\right) = \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{c}$$

Now taking $\left(\hat{b}\times\hat{c}\right)\times\left(\hat{c}\times\hat{a}\right).\left(\hat{a}\times\hat{b}\right)$

$$= \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{c}\ .\ \left(\hat{a}\times\hat{b}\right)$$

$$= \left[\hat{a}\ \hat{b}\ \hat{c}\right]\left\{\hat{c}.\left(\hat{a}\times\hat{b}\right)\right\}$$

$$= \left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

$$= \left[\hat{a}\ \hat{b}\ \hat{c}\right]^2$$

Hence $\left[\hat{b}\times\hat{c} \quad \hat{c}\times\hat{a} \quad \hat{a}\times\hat{b}\right] = = \left[\hat{a}\ \hat{b}\ \hat{c}\right]^2$

Again given $\hat{a}$, $\hat{b}$, $\hat{c}$ are non coplanar. So using property of scalar triple product $\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$ as $\left[\hat{a}.\hat{b}\ \hat{c}\right] \neq 0$

So $\left[\left(\hat{b}\times\hat{c}\right)\ \left(\hat{c}\times\hat{a}\right)\ \left(\hat{a}\times\hat{b}\right)\right] \neq 0$

Therefore, $\left(\hat{b}\times\hat{c}\right)$, $\left(\hat{c}\times\hat{a}\right)$ and $\left(\hat{a}\times\hat{b}\right)$ are non coplanar

Example 11 : If given $\hat{a}$, $\hat{b}$, $\hat{c}$ be three unit vectors such that $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \frac{1}{2}\hat{b}$. Then find the angles which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$ such that $\hat{b}$ and $\hat{c}$ being non-paraller.

Solution : Since given, $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \frac{1}{2}\hat{b}$

$$\Rightarrow \left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} = \frac{1}{2}\hat{b} \qquad \text{[using expansion formula of } \hat{a}\times\left(\hat{b}\times\hat{c}\right)\text{]}$$

$$\Rightarrow \left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} = \frac{1}{2}\hat{b} = \hat{O}$$

$$\Rightarrow \left[(\hat{a}.\hat{c}) - \tfrac{1}{2} \right] \hat{b} - \left(\hat{a}.\hat{b} \right) \hat{c} = \hat{O}$$

Since $\hat{b}$ and $\hat{c}$ being are non parallel. So coefficients of $\hat{b}$ and $\hat{c}$ should vanish separately.

$\Rightarrow \hat{a}.\hat{c} - \tfrac{1}{2} = 0$ and $\hat{a}.\hat{b} = 0$

or $\hat{a}.\hat{c}\ \tfrac{1}{2}$ and $\hat{a}.\hat{b} = 0$

Let θ and be the ϕ be the angle which $\hat{a}$ makes with $\hat{b}$ and $\hat{c}$ respectively.

Then $\hat{a}.\hat{b}$ $= |\hat{a}||\hat{b}|$ Cos $\theta = 0$ using $\hat{a}.\hat{b} = |\hat{a}||\hat{b}|$ Cos θ

$= 1 . 1$ Cos $\theta = 0$ [as a and b all unit vector)

= Cos $\theta = 0$

= Cos θ = Cos 90^0

$\Rightarrow\theta$ = Cos 90^0

So $\hat{a}$ makes on angle 900 with $\hat{b}$.

Again $\hat{a}.\hat{c} = \tfrac{1}{2}$

$\Rightarrow |\hat{a}||\hat{c}|$ Cos $\phi = \tfrac{1}{2}$

$\Rightarrow |1||1|$ Cos $\phi = \tfrac{1}{2}$

$\Rightarrow$ Cos ϕ = Cos 60^0

$\Rightarrow\phi = 60^0$

So $\hat{a}$ makes an angle 60^0 with $\hat{c}$.

Self Check Exercise-2

Q. 1 If $\hat{a}$, $\hat{b}$, $\hat{c}$ are there unit vectors such that $\hat{b}\times(\hat{c}\times\hat{a}) = \tfrac{1}{2}\hat{c}$. Find angles which $\hat{b}$ makes with $\hat{c}$ and $\hat{a}$, given $\hat{c}$ and $\hat{a}$ are non parallel.

Q. 2 If $\hat{a}$, $\hat{b}$, $\hat{c}$ are non coplanar unit vectors such that $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \dfrac{\hat{b}\times\hat{c}}{\sqrt{2}}$, then find the angle between $\hat{a}$ and $\hat{b}$, given $\hat{b}$ and $\hat{c}$ are non parallel.

3.5 Summary

In this unit, we studied

1. To define vector triple product of three vector.

2. $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = $\left(\hat{a}\,.\,\hat{c}\right)\hat{b}$ - $\left(\hat{a}.\,\hat{b}\right)\hat{c}$

3. $\left(\hat{a}\times\hat{b}\right)\times\hat{c}$ = $\left(\hat{a}\,.\,\hat{c}\right)\hat{b}$- $\left(\hat{b}\,.\,\hat{c}\right)\hat{a}$

4. Vector triple product gives us a vector quantity.

5. Vector product is not associative.

6. $\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ i.e. vector triple product is a vector which is perpendicular to $\hat{a}$ and lies in the plane of $\hat{b}$ and $\hat{c}$.

3.6 Glossary

- **Triple product :** In geometry and algebra, the triple product is a product at three dimessional vector.

3.7 Answer to Self Check Exercises

Self Check Exercise - 1

Q. 1 $\left(\hat{a}\,.\,\hat{b}\right)\times\hat{c}$ is not meaningful as $\hat{a}.\hat{b}$ is a scalar and cross product is defined only between two vectors. While other two are meaningful.

$\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = 175 $\hat{i}$ + 26 $\hat{j}$ - 68 $\hat{k}$, $\hat{a}.\left(\hat{b}\times\hat{c}\right)$ = -63

Q. 2 $\left(\hat{a}\,.\,\hat{b}\right)\times\hat{c}$ is not meaningful.

$\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = -2 $\hat{i}$ - 2 $\hat{j}$ + 4 $\hat{k}$ and $\left(\hat{a}\times\hat{b}\right)\hat{c}$ = 36

Q. 3 Result is verified

Q. 4 Result is verified by using the expansion

$\hat{a}\times\left(\hat{b}\times\hat{c}\right)$ = $\left(\hat{a}\,.\,\hat{c}\right)\hat{b}$ - $\left(\hat{a}.\,\hat{b}\right)\hat{c}$

Self Check Exercise - 2

Q. 1 Prove that $\hat{c}\times(\hat{p}\times\hat{q})$ = 0

Q. 2 $= \dfrac{-\hat{i}-8\hat{j}+5\hat{k}}{\sqrt{90}} = \dfrac{-\hat{i}-8\hat{j}+5\hat{k}}{3\sqrt{10}}$

Q. 3 $\dfrac{175\hat{i}+26\hat{j}-68\hat{k}}{\sqrt{(175)^2+(26)^2+(-68)^2}}$

Q. 4 $\sqrt{350} = 5\sqrt{14}$

Q. 5 Angle between $\hat{b}$ and $\hat{c}$ is 90^0 and angle between $\hat{b}$ and $\hat{a}$ is 60^0.

Q. 6 $\frac{3\pi}{4}$

3.8 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector analysis, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, Schand and Company Limited.
3. J.N. Sharma and A.R. Vasishtha, Vector Calculas, Krishna Prakashan Mandir.

3.9 Terminal Questions

Q. 1. Prove that A. (B×C) = B.(C×A) = C.(A×B).

Q. 2 Show that, A.(A×C) = 0

Q. 3 Find the value of a, so that the vectors are coplanar, where and $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{B} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{C} = 3\hat{i} + a\hat{j} + 5\hat{k}$.

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Unit - 4

Quadruple Products

Structure

4.1 Introduction

Dear student, in this unit we will study about the product of four vector. The product of four non zero vectors is known as quadruple product of these four vector. This product is again of two types that is scalar product of four vectors and vector product of four vector. In this unit we will study about such products along with their properties.

4.2 Learning Objectives

After studying this unit, students will be able to

1. define scalar product of four vector.
2. understand and apply properties of scalar product of four vectors.
3. define vector product of four vectors.
4. understand and apply properties of vector product of four vectors.
5. evaluate scalar and vectors product of four vectors.

4.3 Scalar Product of Four Vectors

If $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are four non-zero vectors then the scalar product of $\hat{a} \times \hat{b}$ and $\hat{c} \times \hat{d}$ i.e. $\left(\hat{a} \times \hat{b}\right) . \left(\hat{c} \times \hat{d}\right)$ is called scalar product of four vectors.

It is called the scalar product of Four vectors because the result of $\left(\hat{a} \times \hat{b}\right) . \left(\hat{c} \times \hat{d}\right)$ is a scalar quantity.

Properties of Scalar Product of Four Vectors

Property : $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{a}.\hat{c} & \hat{b}.\hat{c} \\ \hat{a}.\hat{d} & \hat{b}.\hat{d} \end{vmatrix} = (\hat{a}.\hat{c})\left(\hat{b}.\hat{d}\right) - \left(\hat{b}.\hat{c}\right)\left(\hat{a}.\hat{d}\right)$

Proof : Taking L.H.S.

$\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \left(\hat{a}\times\hat{b}\right).\hat{A}$, Where $\hat{A} = \hat{c}\times\hat{d}$

$= \hat{a}.\left(\hat{b}\times\hat{A}\right)$ [$\left(\hat{a}\times\hat{b}\right).\hat{A} = \hat{a}.\left(\hat{b}\times\hat{A}\right)$ as dot and cross are inter

$= \hat{a}.\left\{\hat{b}\times\left(\hat{c}\times\hat{d}\right)\right\}$ changeable in scalar triple product.]

$= \hat{a}.\left\{\left(\hat{b}.\hat{d}\right)\hat{c}-\left(\hat{b}.\hat{c}\right)\hat{d}\right\}$

$= (\hat{a}.\hat{c})\left(\hat{b}.\hat{d}\right) - \left(\hat{a}.\hat{d}\right)\left(\hat{b}.\hat{c}\right)$

Hence $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} (\hat{a}.\hat{c}) & \left(\hat{b}.\hat{c}\right) \\ \left(\hat{a}.\hat{d}\right) & \left(\hat{b}.\hat{d}\right) \end{vmatrix}$

Property 2 : $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = 0$ when $\hat{a}$ and $\hat{b}$ lie in a plane normal to plane containing $\hat{c}$ and $\hat{d}$.

Proof : Let $\hat{a}$ and $\hat{b}$ are vectors lie in the plane α and $\hat{c}$ and $\hat{d}$ lie in plane $\hat{B}$.

Given that α plane is normal to plane B.

Since $\hat{a}\times\hat{b}$ is a vector which is perpendicular to plane containing $\hat{a}$ and $\hat{b}$ i.e. plane α and $\hat{c}\times\hat{d}$ is a vector which is perpendicular to plane containing $\hat{c}$ and $\hat{d}$ that is plane B.

Also plane α and B are perpendicular.

Therefore, $\left(\hat{a}\times\hat{b}\right)$ is perpendicular to $\left(\hat{c}\times\hat{d}\right)$

$\Rightarrow \quad \left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = 0$ [$\because$ $\hat{a}.\hat{b} = 0 \Leftrightarrow a \perp b$.

Hence proved

Let us try to learn more about scalar product of four vectors by these examples.

Example 1 : If $\hat{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\hat{b} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\hat{c} = \hat{i} - \hat{j} + \hat{k}$ and $\hat{d} = 3\hat{i} + \hat{j} + 2\hat{k}$. Find $\left(\hat{a}\times\hat{b}\right)$. $\left(\hat{c}\times\hat{d}\right)$. Also verify the result by $\left(\hat{a}\times\hat{b}\right)$. $\left(\hat{c}\times\hat{d}\right)$ = $\begin{vmatrix} \hat{a}.\hat{c} & \hat{b}.\hat{c} \\ \hat{a}.\hat{d} & \hat{b}.\hat{d} \end{vmatrix}$

Solution : Given $\hat{a} = \hat{i} + 2\hat{j} - \hat{k}$,

$\hat{b} = 2\hat{i} + \hat{j} + 3\hat{k}$,

$\hat{c} = \hat{i} - \hat{j} + \hat{k}$

$\hat{d} = 3\hat{i} + \hat{j} + 2\hat{k}$.

Now, $\left(\hat{a}\times\hat{b}\right)$. $\left(\hat{c}\times\hat{d}\right)$

$$\left(\hat{a}\times\hat{b}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

= $\hat{i}$ (6+1) - $\hat{j}$ (3+2) + $\hat{k}$ (1-4)

$\Rightarrow$ $\left(\hat{a}\times\hat{b}\right)$ = 7$\hat{i}$ -5 $\hat{j}$ -3$\hat{k}$

$$\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

= $\hat{i}$ (-2-) - $\hat{j}$ (2-3) + $\hat{k}$ (1+3)

$\Rightarrow$ $\left(\hat{c}\times\hat{d}\right)$ = -3$\hat{i}$ + $\hat{j}$ +4$\hat{k}$

Now $\left(\hat{a}\times\hat{b}\right)$. $\left(\hat{c}\times\hat{d}\right)$ = (7$\hat{i}$ -5 $\hat{j}$ -3$\hat{k}$) . (-3$\hat{i}$ + $\hat{j}$ +4$\hat{k}$)

= 7×(-3) + (-5) ×1 + (-3) × 4

= -21 - 5 - 12

$\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right)$ = -38

Now to verify the result $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} (\hat{a}.\hat{c}) & (\hat{b}.\hat{c}) \\ (\hat{a}.\hat{d}) & (\hat{b}.\hat{d}) \end{vmatrix}$

Now, $\hat{a}.\hat{c}$ $= (\hat{i} + 2\hat{j} - \hat{k}).(\hat{i} - \hat{j} + \hat{k})$

$= 1 + 2 \times (-1) + (-1)\times(1)$

$= 1 - 2 - 1$

$\Rightarrow \hat{a}.\hat{c} = -2$

$\hat{a}.\hat{d} = (\hat{i} + 2\hat{j} - \hat{k}).(3\hat{i} + \hat{j} + 2\hat{k})$

$= (1)(3) + (2)(1) + (-1)(2)$

$= 3 + 2 - 2$

$\Rightarrow \hat{a}.\hat{d} = 3$

Now, $\hat{b}.\hat{c}$ $= (2\hat{i} + \hat{j} + 3\hat{k}).(\hat{i} - \hat{j} + \hat{k})$

$= (2)(1) + (1)(1) + (3)(2)$

$= 6 + 1 + 6$

$\Rightarrow \hat{b}.\hat{c} = 13$

Now, = $\begin{vmatrix} \hat{a}.\hat{c} & \hat{b}.\hat{c} \\ \hat{a}.\hat{d} & \hat{b}.\hat{d} \end{vmatrix} = \begin{vmatrix} -2 & 4 \\ 3 & 13 \end{vmatrix}$

$= -26 - 12$

$= -38 = \left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right)$

Hence $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{a}.\hat{c} & \hat{b}.\hat{c} \\ \hat{a}.\hat{d} & \hat{b}.\hat{d} \end{vmatrix}$

Example 2 : Find the scalar product of given four vectors $\hat{a} = \hat{i}+\hat{j}+\hat{k}$, $\hat{b} = 2\hat{i}+3\hat{j}+2\hat{k}$, $\hat{c} = 2\hat{i}+\hat{j}+3\hat{k}$ and $\hat{d} = 4\hat{i}+\hat{j}-\hat{k}$.

Solution : Given $\hat{a} = \hat{i}+\hat{j}+\hat{k}$

$\hat{b} = 2\hat{i}+3\hat{j}+2\hat{k}$

$$\hat{c} = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$\hat{d} = 4\hat{i} + \hat{j} - \hat{k}$$

Since $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{a}.\hat{c} & \hat{b}.\hat{c} \\ \hat{a}.\hat{d} & \hat{b}.\hat{d} \end{vmatrix}$

Now $\hat{a}.\hat{c}$ = ($\hat{i} + \hat{j} + \hat{k}$) . (- $2\hat{i} + \hat{j} + 3\hat{k}$) = - 2 + 1 + 3 = 2

$\hat{b}.\hat{c}$ = ($2\hat{i} + 3\hat{j} + 2\hat{k}$) . (- $2\hat{i} + \hat{j} + 3\hat{k}$) = - 4 + 3 + 6 = 5

$\hat{a}.\hat{d}$ = ($\hat{i} + \hat{j} + \hat{k}$) . ($4\hat{i} + \hat{j} - \hat{k}$) = 4 + 1 - 1 = 4

$\hat{b}.\hat{d}$ = ($2\hat{i} + 3\hat{j} + 2\hat{k}$) . ($4\hat{i} + \hat{j} - \hat{k}$) = 8 + 3 - 2 = 9

$\therefore \quad \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} 2 & 5 \\ 4 & 9 \end{vmatrix}$

= 18 - 20

$\Rightarrow \quad \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ = - 2

Example 2: Since we know that $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right)$ and prove that $\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) + \left(\hat{b}\times\hat{c}\right).\left(\hat{a}\times\hat{d}\right)$ + $\left(\hat{c}\times\hat{a}\right).\left(\hat{b}\times\hat{d}\right)$ = 0

Solution : Since we know that

$$\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \left(\hat{a}.\hat{c}\right)\left(\hat{b}.\hat{d}\right) - \left(\hat{b}.\hat{c}\right)\left(\hat{a}.\hat{d}\right) \qquad (1)$$

[Example 1]

Similarly we can prove that

$$\left(\hat{b}\times\hat{c}\right).\left(\hat{a}\times\hat{d}\right) = \left(\hat{b}.\hat{a}\right)\left(\hat{c}.\hat{d}\right) - \left(\hat{c}.\hat{a}\right)\left(\hat{b}.\hat{d}\right) \qquad (2)$$

and

$$\left(\hat{c}\times\hat{a}\right).\left(\hat{b}\times\hat{d}\right) = \left(\hat{c}.\hat{b}\right)\left(\hat{a}.\hat{d}\right) - \left(\hat{a}.\hat{b}\right)\left(\hat{c}.\hat{d}\right) \qquad (3)$$

Using these expression in

$$\left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = \left(\hat{b}\times\hat{c}\right).\left(\hat{a}\times\hat{d}\right) + \left(\hat{c}\times\hat{a}\right).\left(\hat{b}\times\hat{d}\right)$$

$$= \left(\hat{a}.\hat{c}\right)\left(\hat{b}.\hat{d}\right) - \left(\hat{b}.\hat{c}\right)\left(\hat{a}.\hat{d}\right) + \left(\hat{b}.\hat{a}\right)\left(\hat{c}.\hat{d}\right) - \left(\hat{c}.\hat{a}\right)\left(\hat{b}.\hat{d}\right)$$

$+ \left(\hat{c}.\hat{b}\right)\left(\hat{a}.\hat{d}\right) - \left(\hat{a}.\hat{b}\right)\left(\hat{c}.\hat{d}\right)$

$= \left(\hat{a}.\hat{c}\right)\left(\hat{b}.\hat{d}\right) - \left(\hat{b}.\hat{c}\right)\left(\hat{a}.\hat{d}\right) + \left(\hat{a}.\hat{b}\right)\left(\hat{c}.\hat{d}\right) - \left(\hat{a}.\hat{c}\right)\left(\hat{b}.\hat{d}\right)$

$+ \left(\hat{b}.\hat{c}\right)\left(\hat{a}.\hat{d}\right) - \left(\hat{a}.\hat{b}\right)\left(\hat{c}.\hat{d}\right)$ [$\because \hat{a}.\hat{b} = \hat{b}.\hat{a}$]

Self Check Exercise - 1

Q. 1 Show that $\left(\hat{b}\times\hat{c}\right).\left(\hat{a}\times\hat{d}\right) + \left(\hat{c}\times\hat{a}\right).\left(\hat{b}\times\hat{d}\right) + \left(\hat{a}\times\hat{b}\right).\left(\hat{c}\times\hat{d}\right) = 0$

4.4 Vector Product of Four Vectors

If $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ be you vectors, then the product $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ is called vector product of four vectors. It is called vector product of four vectors because the result of $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ is a vectors quantity.

Properties of vector product of four vectors

1. The vector $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ is a vector which is perpendicular to $\left(\hat{a}\times\hat{b}\right)$ and coplanar with $\hat{c}$ and $\hat{d}$. It is also perpendicular to $\hat{c}\times\hat{d}$ and coplanar with $\hat{a}$, $\hat{b}$.
2. The vector $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ is parallel to the line of intersection of plane parallel to $\hat{a}$ and $\hat{b}$ with other plane parallel to $\hat{c}$ and $\hat{d}$.

To understand more about vector product of four vectors let us do some examples.

Example 1 : Prove that $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = [\hat{a}\ \hat{b}\ \hat{c}]\,\hat{c} - [\hat{a}\ \hat{b}\ \hat{c}]\ \hat{d}$

Solution : Taking L.H.S.

$= \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$, Let $\hat{a}\times\hat{b}$ = A

$= \hat{A}\times\left(\hat{c}\times\hat{d}\right)$

$= \left(\hat{A}.\hat{d}\right)\hat{c} - \left(\hat{A}.\hat{c}\right)\hat{d}$ [$\because \hat{Q}\times\left(\hat{b}\times\hat{c}\right) = \left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$]

$= \{ \left(\hat{a}\times\hat{b}\right).\hat{d} \}\hat{c} - \{ \left(\hat{a}\times\hat{b}\right).\ \hat{c} \}\hat{d}$

$= [\hat{a}\ \hat{b}\ \hat{c}]\,\hat{c} - [\hat{a}\ \hat{b}\ \hat{c}]\ \hat{d}$

= R.H.S.

Hence $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ = $[\hat{a}\ \hat{b}\ \hat{c}]\hat{c}$ - $[\hat{a}\ \hat{b}\ \hat{c}]\ \hat{d}$

Example 2 : Prove that $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ = $[\hat{a}\ \hat{c}\ \hat{d}]\hat{b}$ - $[\hat{b}\ \hat{c}\ \hat{d}]\ \hat{d}$

Solution : Taking L.H.S. = $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$, Let $\left(\hat{c}\times\hat{d}\right)$= B

Then $\left(\hat{a}\times\hat{b}\right)\times\ \hat{B}$

= $-\hat{B}\times\left(\hat{a}\times\hat{b}\right)$ $\qquad$ $[\because \hat{a}\times\hat{b} = -\left(\hat{b}\times\hat{a}\right)]$

= $\left\{\left(\hat{B}.\hat{b}\right)\hat{d}-\left(\hat{B}.\hat{a}\right)\hat{d}\right\}$

= $-\left\{\left(\hat{c}\times\hat{d}\right).\hat{b}\right\}\hat{a}\ +\left\{\left(\hat{c}\times\hat{d}\right).\hat{a}\right\}\hat{b}$

= $-[\hat{c}\ \hat{d}\ \hat{b}]\ \hat{a}\ +[\hat{c}\ \hat{d}\ \hat{a}]\ \hat{b}$

= $[\hat{c}\ \hat{d}\ \hat{a}]\ \hat{b}-[\hat{c}\ \hat{d}\ \hat{b}]\ \hat{a}$

Hence $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ = $[\hat{c}\ \hat{d}\ \hat{a}]\ \hat{b}-[\hat{c}\ \hat{d}\ \hat{b}]\ \hat{a}$

$\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$= $[\hat{a}\ \hat{c}\ \hat{d}]\ \hat{b}-[\hat{b}\ \hat{c}\ \hat{d}]\ \hat{a}$

Example 3 : If $\hat{a}$, $\hat{b}$, $\hat{c}$ be three non-coplanar vectors then any vector can be expressed in terms of If $\hat{a}$, $\hat{b}$ and $\hat{c}$ as

$$\hat{r}=\frac{[\hat{r}\ \hat{b}\ \hat{c}]\hat{a}+[\hat{r}\ \hat{c}\ \hat{a}]\hat{b}+[\hat{r}\ \hat{a}\ \hat{b}]\hat{c}}{[\hat{a}\ \hat{b}\ \hat{c}]}$$

Solution : $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ = $[\hat{a}\ \hat{b}\ \hat{d}]\hat{c}$ - $[\hat{a}\ \hat{b}\ \hat{c}]\hat{d}$ (1)

and $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$= $[\hat{a}\ \hat{c}\ \hat{d}]\ \hat{b}-[\hat{b}\ \hat{c}\ \hat{d}]\ \hat{a}$ (2)

$\Rightarrow$ $[\hat{a}\ \hat{b}\ \hat{d}]\hat{c}$ - $[\hat{a}\ \hat{b}\ \hat{c}]\hat{d}$ = $[\hat{a}\ \hat{c}\ \hat{d}]\ \hat{b}-[\hat{b}\ \hat{c}\ \hat{d}]\ \hat{a}$

Replacing $\hat{d}$ by $\hat{r}$ we get

$\Rightarrow$ $[\hat{a}\ \hat{b}\ \hat{r}]\hat{c}$ - $[\hat{a}\ \hat{b}\ \hat{c}]\hat{r}$ = $[\hat{a}\ \hat{c}\ \hat{r}]\ \hat{b}-[\hat{b}\ \hat{c}\ \hat{r}]\ \hat{a}$

$\Rightarrow$ $-[\hat{a}\ \hat{b}\ \hat{c}]\hat{r}$ = $[\hat{a}\ \hat{c}\ \hat{r}]\ \hat{b}$ - $[\hat{b}\ \hat{c}\ \hat{r}]\ \hat{a}-[\hat{a}\ \hat{b}\ \hat{r}]\hat{c}$

$\Rightarrow$ $[\hat{a}\ \hat{b}\ \hat{c}]\hat{r}$ = $-[\hat{b}\ \hat{c}\ \hat{r}]\ \hat{a}$ - $[\hat{a}\ \hat{c}\ \hat{r}]\ \hat{b}+[\hat{a}\ \hat{b}\ \hat{r}]\hat{c}$

$\Rightarrow \quad [\hat{a}\ \hat{b}\ \hat{c}]\hat{r} = [\hat{r}\ \hat{b}\ \hat{c}]\ \hat{a} + [\hat{r}\ \hat{c}\ \hat{a}]\ \hat{b} + [\hat{r}\ \hat{a}\ \hat{b}]\hat{c}$ $\quad [\because [\hat{a}\ \hat{c}\ \hat{r}] = -[\hat{r}\ \hat{c}\ \hat{a}]]$

$$\hat{r} = \frac{[\hat{r}\ \hat{b}\ \hat{c}]\hat{a} + [\hat{r}\ \hat{c}\ \hat{a}]\hat{b} + [\hat{r}\ \hat{a}\ \hat{b}]\hat{c}}{[\hat{a}\ \hat{b}\ \hat{c}]}$$

Hence the result.

Example 4 : If $\hat{a} = \hat{i} + 2\hat{j} - \hat{k}$, $\hat{b} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\hat{c} = \hat{i} - \hat{j} + \hat{k}$ and $\hat{d} = 3\hat{i} + \hat{j} + 2\hat{k}$. Find $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$.

Solution : Since $\hat{a} = \hat{i} + 2\hat{j} - \hat{k}$

$\hat{b} = 2\hat{i} + \hat{j} + 3\hat{k}$

$\hat{c} = \hat{i} - \hat{j} + \hat{k}$

$\hat{d} = 3\hat{i} + \hat{j} + 2\hat{k}$

Then $\hat{a}\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$

$= \hat{i}\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

$= \hat{i}\ (6+1) - \hat{j}\ (3+2) + \hat{k}\ (1-4)$

$\Rightarrow \quad \hat{a}\times\hat{b} = 7\hat{i} - 5\hat{j} - 3\hat{k}$

Now $\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

$= \hat{i}\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$

$= \hat{i}\ (-2-1) - \hat{j}\ (2-3) + \hat{k}\ (1+3)$

$= -3\hat{i} + \hat{j} + 4\hat{k}$

$\Rightarrow \left(\hat{c}\times\hat{d}\right) = -3\hat{i} + \hat{j} + 4\hat{k}$

Now, $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -5 & -3 \\ -3 & 1 & 4 \end{vmatrix}$

$= \hat{i}\begin{vmatrix} -5 & -3 \\ 1 & 4 \end{vmatrix} - \hat{j}\begin{vmatrix} 7 & -3 \\ -3 & 4 \end{vmatrix} + \hat{k}\begin{vmatrix} 7 & -5 \\ -3 & 1 \end{vmatrix}$

$= \hat{i}$ (-20+3) - $\hat{j}$ (28-9) + $\hat{k}$ (7-15)

$= \hat{i}$ (-17) - $\hat{j}$ (19) $\hat{k}$ (-8)

$\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = -17\hat{i} - 19\hat{j} - 8\hat{k}$

Example 5 : If the vectors $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are coplanar then $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \hat{0}$

Solution : Using the property of $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)$ that $\left(\hat{a}\times\hat{b}\right)$ is a vector perpendicular to the plane of $\hat{a}$ or $\hat{b}$ also $\left(\hat{c}\times\hat{d}\right)$ is a vector perpendicular to the plane of $\hat{c}$ and $\hat{d}$.

But $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$ are coplanar.

$\therefore \hat{a}\times\hat{b}$ and $\hat{c}\times\hat{d}$ are both perpendicular to same plane.

Therefore, $\hat{a}\times\hat{b}$ and $\hat{c}\times\hat{d}$ are both parallel vectors and vector product of parallel vector is zero vector.

Hence $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \hat{0}$

Example 6 : Prove that $[\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)] \cdot \hat{d} = (\hat{a}\cdot\hat{d})\,[\hat{a}\ \hat{b}\ \hat{c}]$

Solution : Taking L.H.S. =

$[\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)]\cdot\hat{d}$

Let $\hat{a}\times\hat{b} = \hat{A}$

$= \left[\hat{A}\times\left(\hat{a}\times\hat{c}\right)\right]\cdot\hat{d}$

$= \left[\left(\hat{A}.\hat{c}\right)\hat{a} - \left(\hat{A}.\hat{a}\right)\hat{c}\right]\cdot\hat{d}$ $\quad [\because \hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}]$

$= \left[\left\{\left(\hat{a}\times\hat{b}\right).\hat{c}\right\}\hat{d} - \left\{\left(\hat{a}\times\hat{b}\right).\hat{d}\right\}\hat{c}\right]\cdot\hat{d}$

$= \left\{\left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{a} - 0\right\}.\ \hat{d}$ [$\because \left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 0, as two vectors are same]

$\left[\hat{a}\ \hat{b}\ \hat{c}\right]\left(\hat{a}.\hat{d}\right) = \left(\hat{a}.\hat{d}\right)\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

= R.H.S.

Hence proved.

Example 7 : Show that

$$\left[\left(\hat{a}\times\hat{b}\right)\ \left(\hat{c}\times\hat{d}\right)\ \left(\hat{e}\times\hat{f}\right)\right] = \left[\hat{a}\ \hat{b}\ \hat{d}\right]\left[\hat{c}\ \hat{e}\ \ \hat{f}\right] - \left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{d}\ \hat{e}\ \ \hat{f}\right]$$

Solution : L.H.S. $\left[\hat{a}\times\hat{b}\ \ \hat{c}\times\hat{d}\ \ \hat{e}\times\hat{f}\right]$

$= \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right).\left(\hat{e}\times\hat{f}\right)$

$= \left[\left\{\hat{a}\times\hat{b}.\hat{d}\right\}\hat{c} - \left\{\left(\hat{a}\times\hat{b}\right).\hat{c}\right\}\hat{d}\right].\ \left(\hat{e}\times\hat{f}\right)$

[$\because \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \left[\hat{a}\ \hat{b}\ \hat{d}\right]\hat{c} - \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{d}$]

$= \left\{\left[\hat{a}\ \hat{b}\ \hat{d}\right]\hat{c} - \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{d}\right\}.\ \left(\hat{e}\times\hat{f}\right)$

$= \left[\hat{a}\ \hat{b}\ \hat{d}\right]\hat{c}.\left(\hat{e}\times\hat{f}\right) - \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{d}.\left(\hat{e}\times\hat{f}\right)$

$= \left[\hat{a}\ \hat{b}\ \hat{d}\right]\left[\hat{c}\ \hat{e}\ \ \hat{f}\right] - \left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{d}\ \hat{e}\ \ \hat{f}\right]$

= R.H.S.

Hence Proved.

Example 8 : Given $\hat{x} = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$, $\hat{y} = \dfrac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$ and $\hat{z} = \dfrac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$ where $\hat{a}$, $\hat{b}$, $\hat{c}$ are non coplanar vectors. Show that $\hat{x}$, $\hat{y}$, $\hat{z}$ also form a non coplanar system. Find the value of $\hat{x}\left(\hat{a}+\hat{b}\right) + \hat{y}\left(\hat{b}+\hat{c}\right) + \hat{z} = \left(\hat{c}+\hat{a}\right)$.

Solution : Since $\hat{a}$, $\hat{b}$, $\hat{c}$ are non coplanar vectors So $\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$

Now to prove $[\hat{x}\ \hat{y}\ \hat{z}] \neq 0$ [Form a non coplanar system]

So $[\hat{x}\ \hat{y}\ \hat{z}] = \hat{x}.[\hat{y}\times\hat{z}]$

$$= \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} \cdot \left\{\frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} \times \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\right\}$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left(\hat{c}\times\hat{a}\right)\times\left(\hat{a}\times\hat{b}\right)\right\}\right]$$

Let $\hat{c}\times\hat{a} = \hat{A}$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\hat{A}\times\left(\hat{a}\times\hat{b}\right)\right\}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left(\hat{A}.\hat{b}\right)\hat{a} - \left(\hat{A}.\hat{a}\right)\hat{b}\right\}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left(\hat{c}\times\hat{a}.\hat{b}\right)\hat{a} - (\hat{c}\times\hat{a}).\hat{a}\right\}\hat{b}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left[\hat{c}\ \hat{a}\ \hat{b}\right]\hat{a} - [\hat{c}\ \hat{a}\ \hat{a}]\right\}\hat{b}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left[\hat{c}\ \hat{a}\ \hat{b}\right]\hat{a} - 0\right\}\right]$$

[$\because$ $\left[\hat{c}\ \hat{a}\ \hat{a}\right]$ = as two vectors are same]

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left(\hat{b}\times\hat{c}\right).\left\{\left[\hat{c}\ \hat{a}\ \hat{b}\right]\hat{a}\right\}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\left[\hat{a}\ \hat{b}\ \hat{c}\right] - \left\{\hat{a}\ \left(\hat{b}\times\hat{c}\right)\right\}\right]$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3}\left[\hat{a}\ \hat{b}\ \hat{c}\right]^2$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Since $\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$

Hence $[\hat{x}\ \hat{y}\ \hat{z}] \neq 0$

So $\hat{x}$, $\hat{y}$, and $\hat{z}$ from a non coplanar system.

Now, $\hat{x}.\left(\hat{a}+\hat{b}\right) = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}.\left(\hat{a}+\hat{b}\right)$

$= \dfrac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\left\{\left(\hat{b}\times\hat{c}\right).\left(\hat{a}+\hat{b}\right)\right\}$

$= \dfrac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\left\{\left(\hat{b}\times\hat{c}\right).\hat{a}+\left(\hat{b}\times\hat{c}\right).\hat{b}\right\}$

$= \dfrac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\left[\hat{a}\ \hat{b}\ \hat{c}+0\right]$

$= \dfrac{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} = 1$

$\therefore \quad \hat{x}.\left(\hat{a}+\hat{b}\right) = 1 \qquad (1)$

Similarly $\hat{y}.\left(\hat{b}+\hat{c}\right) = 1 \qquad (2)$

and $\hat{z}.\left(\hat{c}+\hat{a}\right) = 1 \qquad (3)$

Adding (1), (2) and (3), we get

$\hat{x}.\left(\hat{a}+\hat{b}\right)+\hat{y}.\left(\hat{b}+\hat{c}\right)+\hat{z}.\left(\hat{c}+\hat{a}\right) = 3$

Self Check Exercise-2

Q. 1 Express $\hat{b}\times\hat{c}$, $\hat{c}\times\hat{a}$, $\hat{a}\times\hat{b}$ in terms of $\hat{a}$, $\hat{b}$, $\hat{c}$.

Q. 2 If the vectors $\hat{b}$, $\hat{c}$, $\hat{a}$ are not coplanar, then prove that

$\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right)+\left(\hat{a}\times\hat{c}\right)\times\left(\hat{d}\times\hat{b}\right)+\left(\hat{a}\times\hat{d}\right)\times\left(\hat{b}\times\hat{c}\right)$ is parallel to $\hat{a}$.

4.5 Summary

In this unit, we studied about

1. Scalar product of four vectors which is a scalar quantity.
2. Properties of scalar product of four vectors.
3. Evaluate the value of scalar product of given four vectors.
4. Vector product of four vector which is a vectors quantity.
5. Properties of vector product of four vectors.
6. Evaluate the value of vector product of given four vectors.

4.6 Glossary

- **Quadruple Product :** It is a product of four vectors in three dimensional Euclidean Space.
- **Euclidean Space :** A space of finite dimension in which points are represented by co-ordinates and the distance between two point is given by distance formula.

4.7 Answers to Self Check Exercises

Self Check Exercise-1

Q. 1 Same as example 3.

Self Check Exercise-2

Q. 1 Consider $\hat{b}\times\hat{c} = \lambda\ \hat{a}$, m$\hat{b}$, n$\hat{c}$ and multiplying scalarly it by $\hat{b}\times\hat{c}$, we get

$$\lambda = \frac{\left(\hat{b}\times\hat{c}\right).\left(\hat{b}\times\hat{c}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Similarly for the next two terms.

Q. 2 Given $\left[\hat{b}\ \ \hat{c}\ \ \hat{d}\right] \neq 0$

Using the property $\left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = \left[\hat{c}\ \hat{d}\ \hat{a}\right]\hat{b} - \left[\hat{c}\ \hat{d}\ \hat{b}\right]\hat{a}$

then adding the terms we get the result.

$= -2\left[\hat{b}\ \ \hat{c}\ \ \hat{d}\right]\hat{a}$.

4.8 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector analysis, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, Schand and Company Limited.
3. J.N. Sharma and A.R. Vasishtha, Vector Calculas, Krishna Prakashan Mandir.

4.9 Terminal Questions

Q. 1. If $\hat{A}$, $\hat{B}$, $\hat{C}$ are vectors such that $|\hat{B}| = |\hat{C}|$ then prove that

$$[(\hat{A}+\hat{B})\times(\hat{A}+\hat{C})] \times (\hat{B}\times\hat{C}) \, . \, (\hat{B}+\hat{C}) = 0$$

Q. 2 Show that,

$$\left(\hat{b}\times\hat{c}\right)\times\left(\hat{a}\times\hat{d}\right) + \left(\hat{c}\times\hat{a}\right)\times\left(\hat{b}\times\hat{d}\right) + \left(\hat{a}\times\hat{b}\right)\times\left(\hat{c}\times\hat{d}\right) = -2\left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{d}$$

Q. 3 Show that

$$[\hat{a}\times\hat{p}\ \ \hat{b}\times\hat{q}\ \ \hat{c}\times\hat{r}] + [\hat{a}\times\hat{q}\ \ \hat{b}\times\hat{r}\ \ \hat{c}\times\hat{p}] + [\hat{a}\times\hat{r}\ \ \hat{b}\times\hat{p}\ \ \hat{c}\times\hat{q}] = 0$$

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Unit - 5

Reciprocal system of Vectors

Structure

5.1 Introduction

Dear student, in this unit we will study about the reciprocal system of vectors. Using the definition of reciprocal system of vector we will evaluate the reciprocal vector of given and will try to prove some result of reciprocal system.

5.2 Learning Objectives

After studying this unit, students will be able to

1. define a reciprocal system of given vectors.
2. find a reciprocal system of given vector.
3. solve the equalities based on reciprocal system of vectors.

5.3 Reciprocal System of Vectors

Let $\hat{a}$, $\hat{b}$, $\hat{c}$ are three non zero vectors which are non collinear and non coplanar, then three vectors $\hat{a}^1$, $\hat{b}^1$, $\hat{c}^1$ are known as reciprocal system of vectors of $\hat{a}$, $\hat{b}$, $\hat{c}$ and are given of $\hat{a}^1 = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$ $\hat{b}^1 = \dfrac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$ $\hat{c}^1 = \dfrac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

To have more understanding of reciprocal system of vectors let us do some exercise.

Example1: Find the reciprocal triad $\hat{a}^{1}, \hat{b}^{1}, \hat{c}^{1}$ for $\hat{a} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\hat{b} = 2\hat{i} + \hat{j} - \hat{k}, \hat{c} = \hat{i} + 3\hat{j} - \hat{k}$. Also verify that $\left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{a}^{1}\ \hat{b}^{1}\ \hat{c}^{1}\right] = 1$

Solution: Given $\hat{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$\hat{b} = 2\hat{i} + \hat{j} - \hat{k},$

$\hat{c} = \hat{i} + 3\hat{j} - \hat{k}$

Since $\hat{a}^{1} = \dfrac{\hat{b} \times \hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$\hat{b}^{1} = \dfrac{\hat{c} \times \hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

and $\hat{c}^{1} = \dfrac{\hat{a} \times \hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

Hence we have to find $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$, $\hat{b} \times \hat{c}$, $\hat{c} \times \hat{a}$, $\hat{a} \times \hat{b}$.

So, $\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 2 & -1 & 3 \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix}$

$$= 2\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix}(-1)\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

= 2 (-1 + 3) +1 (-2 + 1) +3 (6 - 1)

= 4 - 1 + 15

18

$\Rightarrow \left[\hat{a}\ \hat{b}\ \hat{c}\right] = 18$

Now $\hat{b} \times \hat{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 3 & -1 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \hat{j}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= i(-1+3) - \hat{j}(-2+1) + \hat{k}(6-1)$$

$$\Rightarrow \quad \hat{b}\times\hat{c} = 2\hat{i} + \hat{j} - 5\hat{k}$$

Now $\quad \hat{c}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$$\hat{i}\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix}$$

$$= i(9-1) - \hat{j}(3+2) + \hat{k}(-1-6)$$

$$\Rightarrow \quad \hat{c}\times\hat{a} = 8\hat{i} - 5\hat{j} - 7\hat{k}$$

Now $\quad \hat{a}\times\hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 2 & 1 & -1 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} -1 & 3 \\ 1 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} + \hat{k}\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= i(1-3) - \hat{j}(-2-6) + \hat{k}(2+2)$$

$$\Rightarrow \hat{a}\times\hat{b} = -2\hat{i} + 8\hat{j} + 4\hat{k}$$

Now Therefore $\quad \hat{a}^1 = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$= \frac{2\hat{i} + \hat{j} + \hat{j}\hat{k}}{18}$$

$$= \frac{1}{18}\left[2\hat{i} + \hat{j} + 5\hat{j}\right]$$

$$\hat{b}^1 = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$= \frac{8\hat{i} - 5\hat{j} - 7\hat{k}}{18}$$

$$=\frac{1}{18}\left[8\hat{i}-5\hat{j}-7\hat{k}\right]$$

$$\hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$=\frac{2\hat{i}+8\hat{j}+4\hat{k}}{18}$$

$$\hat{c}^1=\frac{1}{18}\left[-2\hat{i}+8\hat{j}+4\hat{k}\right]$$

Now to prove that $\left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]$ = 1

Taking $\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]=\frac{1}{(18)^3}\begin{vmatrix}2 & 1 & 5\\ 8 & -5 & -7\\ -2 & 8 & 4\end{vmatrix}$

$$=\frac{1}{(18)^3}\left\{2\begin{vmatrix}-5 & -7\\ 8 & 4\end{vmatrix}-1\begin{vmatrix}8 & -7\\ -2 & 4\end{vmatrix}+5\begin{vmatrix}8 & -5\\ -2 & 8\end{vmatrix}\right\}$$

$$=\frac{1}{(18)^3}\left\{2(-20+56)-1(32-14)+5(64-10)\right\}$$

$$=\frac{1}{(18)^3}\left\{-12-18+270\right\}$$

$\Rightarrow$ $\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]=\frac{1}{(18)^3}$ (324) = $\frac{18\times18}{18\times18\times18}=\frac{1}{18}$

Hence $\left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]$ = 18 × $\frac{1}{18}$ = 1

Hence $\left[\hat{a}\ \hat{b}\ \hat{c}\right]\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]$ = 1

Example 2: Given $\hat{a}=2\hat{i}+3\hat{j}-3\hat{k},\hat{b}=\hat{i}-\hat{j}-2\hat{k},\hat{c}=-\hat{i}+2\hat{j}+2\hat{k}$

Does reciprocal system exists? If so find it.

Solution: Given $\hat{a}=2\hat{i}+3\hat{j}-3\hat{k}$

$$\hat{b}=\hat{i}-\hat{j}-2\hat{k},$$

$$\hat{c} = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\left[\hat{a}\ \hat{b}\ \hat{c}\right] = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} -1 & -2 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

= 2 (-2 + 4) -3 (2 -2) -1 (2 - 1)

= 2 (2) -3 (0) - 1 (1)

= 4 - 1 - 1

$\Rightarrow$ $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ = 3 $\neq$

Since $\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$, so reciprocal system exists.

Since, we know that $\hat{a}^1 = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{b}\times\hat{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -2 \\ -1 & 2 & 2 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -1 & -2 \\ 2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= \hat{i}\left(-2+4\right) - \hat{j}\left(2-2\right) + \hat{k}\left(2-1\right)$$

$$= 2\hat{i} + 0\hat{j} + \hat{k} = 2\hat{i} + \hat{k}$$

Therefore $\hat{a}^1 = \dfrac{2\hat{i}+\hat{k}}{3} = \dfrac{1}{3}\left(2\hat{i}+\hat{k}\right)$

Now, $\hat{b}^1 = \dfrac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{c}\times\hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$=\hat{i}\begin{vmatrix}2 & 2\\3 & -1\end{vmatrix}-\hat{j}\begin{vmatrix}-1 & 2\\2 & -1\end{vmatrix}+\hat{k}\begin{vmatrix}-1 & 2\\2 & 3\end{vmatrix}$$

$$= \hat{i}\left(-2-6\right)-\hat{j}\left(1-4\right)+\hat{k}\left(-3-4\right)$$

$\Rightarrow \qquad \hat{c}\times\hat{a} = -8\hat{i}+3\hat{j}-7\hat{k}$

Hence $\hat{b}^1 = \dfrac{-8\hat{i}+3\hat{j}-7\hat{k}}{3} = \dfrac{1}{3}\left(-8\hat{i}+3\hat{j}-7\hat{k}\right)$

Now, $\quad \hat{c}^1 = \dfrac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{a}\times\hat{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\2 & 3 & -1\\1 & -1 & -2\end{vmatrix}$$

$$=\hat{i}\begin{vmatrix}3 & -1\\-1 & -2\end{vmatrix}-\hat{j}\begin{vmatrix}2 & -1\\1 & -2\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\1 & -1\end{vmatrix}$$

$$= \hat{i}\left(-6-1\right)-\hat{j}\left(-4+1\right)+\hat{k}\left(-2-3\right)$$

$\hat{a}\times\hat{b} \quad = -7\hat{i}+3\hat{j}-5\hat{k}$

$\therefore \qquad \hat{c}^1 = \dfrac{7\hat{i}+3\hat{j}-5\hat{k}}{3} = \dfrac{1}{3}\left(7\hat{i}+3\hat{j}-5\hat{k}\right)$

Self Check Exercise - 1

Q.1 Find reciprocal system of vectors for $\left(-\hat{i}+2\hat{j}+2\hat{k}\right)$, $\left(2\hat{i}+3\hat{j}+\hat{k}\right)$ and $\left(\hat{i}-\hat{j}-2\hat{k}\right)$

Q.2 Find reciprocal system of vector for

$\hat{a}$ = (1, 0, 0) $\quad \hat{b}$ = (1, 1, 0) $\quad \hat{c}$ = (1, 1, 1)

5.3 Properties of Reciprocal System of Vectors

Property 1: If $\hat{a}\ \hat{b}\ \hat{c}$ are There non coplanar vector and $\hat{a}^1, \hat{b}^1, \hat{c}^1$ are reciprocal system of $\hat{a}\ \hat{b}\ and\ \hat{c}$ respectively then $\hat{a}.\hat{a}=\hat{b}.\hat{b}^1=\hat{c}.\hat{c}^1$= 1

Proof : Given $\hat{a}\ \hat{b}\ and\ \hat{c}$ are non coplanar so

$\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$

Also we know that reciprocal system of $\hat{a}\ \hat{b}\, and\, \hat{c}$ is given by

$$\hat{a}^1 = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

then $\hat{a}.\hat{a}^1 = \hat{a}.\frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$= \frac{\hat{a}.\left(\hat{b}\times\hat{c}\right)}{\left[\hat{a}\hat{b}\hat{c}\right]}$$

$$= \frac{\left[\hat{a}\hat{b}\hat{c}\right]}{\left[\hat{a}\hat{b}\hat{c}\right]}$$

$\Rightarrow \quad \hat{a}.\hat{a}^1 = 1$

Similarly $\hat{b} = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{b}.\hat{b}^1 = \hat{b}.\frac{\left(\hat{c}\times\hat{a}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$= \frac{\hat{b}.\left(\hat{c}\times\hat{a}\right)}{\left[\hat{a}\hat{b}\hat{c}\right]}$$

$$= \frac{\left[\hat{a}\hat{b}\hat{c}\right]}{\left[\hat{a}\hat{b}\hat{c}\right]}$$

$\Rightarrow \quad \hat{b}.\hat{b}^1 = 1$

Now $\hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{c}.\hat{c}^1 = \hat{c}.\frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$= \frac{\hat{c}.\left(\hat{a}\times\hat{b}\right)}{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}$$

$$= \frac{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}$$

$\hat{c}.\hat{c}^1 \quad = 1$

Property 2: If $\hat{a}^1, \hat{b}^1, \hat{c}^1$ are the reciprocal system of non-coplanar vectors $\hat{a}$, $\hat{b}$ and $\hat{c}$ respectively then $\hat{a}.\,\hat{b}^1 = \hat{a}.\hat{c}^1 = \hat{b}.\hat{a}^1 = \hat{b}.\hat{c}^1 = \hat{c}.\hat{a}^1 = \hat{c}.\,\hat{b}^1 = 0$

Solution: Since for non coplanar vectors $\hat{a}\ \hat{b}\, and\, \hat{c}$

$$\hat{a}^1 = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]},\ \hat{b}^1 = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]},\ \hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Now $\quad \hat{a}.\,\hat{b}^1 = \hat{a}\,.\,\dfrac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$= \frac{\hat{a}.\left(\hat{c}\times\hat{a}\right)}{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}$$

$$= \frac{\left(\hat{a}\ \hat{c}\ \hat{a}\right)}{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}$$

$\Rightarrow \quad \hat{a}.\,\hat{b}^1 = 0 \qquad [\because$ Scalar triple product with two equal vectors is zero]

Similarly $\hat{a}.\hat{c}^1\ \hat{a}\,.\,\dfrac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\frac{\left(\hat{a}\,\hat{c}\,\hat{b}\right)}{\left[\hat{a}\,\hat{b}\,\hat{c}\right]}$$

$\hat{a}.\hat{c}^1 = 0$

Similarly $\quad \hat{b}.\hat{a}^1 = 0$

$\hat{b}.\hat{c}^1 = 0,\ \hat{a}.\hat{c}^1 = 0,\ \hat{c}.\,\hat{b}^1 = 0$

Property 3: $\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \dfrac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

Proof: Since $\hat{a}^1 = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

$$\hat{b}^1 = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

then $\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \dfrac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} \cdot \left\{\dfrac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\times\dfrac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}\right\}$

$$\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \cdot \left\{(\hat{c}\times\hat{a})\times\left(\hat{a}\times\hat{b}\right)\right\}$$

Let $\hat{c}\times\hat{a} = A$

$$\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \frac{\hat{c}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \cdot \left\{A\times\left(\hat{a}\times\hat{b}\right)\right\}$$

using $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = (\hat{a}.\hat{c})\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$, we get

$$\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \left\{\left(\hat{A}.\hat{b}\right)\hat{a} - \left(\hat{A}.\ \hat{a}\right)\hat{b}\right\}$$

$$= \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \left\{(\hat{c}\times\hat{a})\hat{b}\right\}\hat{a} - \left\{(\hat{c}\times\hat{a}).\hat{a}\right\}\hat{b}$$

$$= \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \left[\left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{a} - 0\right]$$

[$\because$ $\hat{c}\times\hat{a}$. $\hat{a}$ = 0 value of scalar triple product is zero for two equal vectors

$$= \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^3} \cdot \left[\hat{a}\ \hat{b}\ \hat{c}\right]\hat{a}$$

$$= \frac{\left(\hat{b}\times\hat{c}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^2}$$

$$= \frac{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]^2}$$

$$= \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\Rightarrow \quad \left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right] = \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

From above result, we can also have

$$\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right]\left[\hat{a}\ \hat{b}\ \hat{c}\right] = 1$$

Property 4 If $\hat{a}\ \hat{b}\ \hat{c}$ are non coplanar, so the reciprocal vector $\hat{a}^1, \hat{b}^1, \hat{c}^1$ are also non-coplanar

Proof: Since from property 3

$$\left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right] = \frac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

As $\hat{a}^1, \hat{b}^1$ and $\hat{c}^1$ are non coplanar so $\left[\hat{a}\ \hat{b}\ \hat{c}\right] \neq 0$

$$\Rightarrow \quad \left[\hat{a}^1\ \hat{b}^1\ \ \hat{c}^1\right] \neq 0$$

Hence $\hat{a}\ \hat{b}$ and $\hat{c}$ are also non-coplanar.

Property 5: The orthonormal vector triad $\hat{i}\ \ \hat{j}$ and $\hat{k}$ are non-coplanar. Then $\hat{i}^1$, $\hat{j}^1$, $\hat{k}^1$ be reciprocal vector of $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively such that

$$\hat{i}^1 = \frac{\hat{j}\times\hat{k}}{\left[\hat{i}\ \hat{j}\,\hat{k}\right]}$$

Since $\hat{j}\times\hat{k} = \hat{i}$ and $\left[\hat{i}\ \hat{j}\,\hat{k}\right] = \hat{i}\ .\ \left[\hat{j}\times\hat{k}\right] = \hat{i}\ .\ \hat{i} = 1$

so $\quad \hat{j}^1 = \hat{i}$

Now $\hat{j}^1 = \dfrac{\hat{k}\times\hat{i}}{\left[\hat{i}\ \hat{j}\ \hat{k}\right]} = \dfrac{\hat{j}}{1} = \hat{j}$

and $\hat{k}^1 = \dfrac{\left[\hat{i}\times\hat{j}\right]}{\left[\hat{i}\ \hat{j}\ \hat{k}\right]}$

$= \dfrac{\hat{k}}{1}$ = k

Hence $\hat{i}^1 = \hat{i}$, $\hat{j}^1 = \hat{j}$, $\hat{k}^1 = \hat{k}$, so orthonormal vector triad form self reciprocal system.

Property 6: Reciprocal vector triad are both right handed or both left handed.

Proof: Since vector tried $\hat{a}\,\hat{b}$ and $\hat{c}$ is right handed if $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ is positive and is left handed if $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$ is negative.

Since from property 3,

$\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right] = \dfrac{1}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

So $\left[\hat{a}^1\ \hat{b}^1\ \hat{c}^1\right]$ has the same sign as of $\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

So, reciprocal vector triad are both handed or both left handed.

Property 7: If $\hat{a}^1, \hat{b}^1$ and $\hat{c}^1$ respectively then $\hat{a}.\hat{a}+\hat{b}.\hat{b}+\hat{c}.\hat{c}$ = 3

Proof: Using Property 1, $\hat{a}.\hat{a}^1$ = 1

$\hat{b}.\hat{b}^1$ = 1

$\hat{c}.\hat{c}^1$ = 1

Hence $\hat{a}.\hat{a}^1 + \hat{b}.\hat{b}^1 = \hat{c}.\hat{c}^1$ = 1+1+1 = 3.

Property 8: If $\hat{a}^1, \hat{b}^1$ and $\hat{c}^1$ are reciprocal vectors of $\hat{a}\ \hat{b}$ and $\hat{c}$

then $\hat{a}\times\hat{a}^1 + \hat{b}\times\hat{b}^1 + \hat{c}\times\hat{c}^1$ = 0

Proof: For given $\hat{a}\ \hat{b}$ and $\hat{c}$ the reciprocal vector are

$$\hat{a}^1 = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\hat{b}^1 = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

then $\hat{a}\times\hat{a}^1 + \hat{b}\times\hat{b}^1 + \hat{c}\times\hat{c}^1$

$$= = \hat{a}\times\frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} + \hat{b}\times\frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} + \hat{c}\times\frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Using $\hat{a}\times\left(\hat{b}\times\hat{c}\right) = \left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c}$

$$= \frac{\left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} + \left(\hat{b}.\hat{a}\right)\hat{c} - \left(\hat{b}.\hat{c}\right)\hat{a} + \left(\hat{c}.\hat{b}\right)\hat{a} - \left(\hat{c}.\hat{a}\right)\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$= \frac{\left(\hat{a}.\hat{c}\right)\hat{b} - \left(\hat{a}.\hat{b}\right)\hat{c} + \left(\hat{a}.\hat{b}\right)\hat{c} - \left(\hat{b}.\hat{c}\right)\hat{a} + \left(\hat{b}.\hat{c}\right)\hat{a} - \left(\hat{a}.\hat{c}\right)\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

[∵ $\hat{a}.\hat{b} = \hat{b}.\hat{a}$

$\hat{a}.\hat{c} = \hat{c}.\hat{a}$

$\hat{b}.\hat{c} = \hat{c}.\hat{a}$

$$= \frac{0}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Hence $\hat{a}\times\hat{a}^1 + \hat{b}\times\hat{b}^1 + \hat{c}\times\hat{c}^1 = 0$

Property 9: If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are system of non-coplanar vectors and $\hat{a}^1, \hat{b}^1$ and $\hat{c}^1$ are the reciprocal system of vectors then any vector $\hat{r}$ can be expressed as

$$\left(\hat{r}.\hat{a}^1\right)\hat{a} + \left(\hat{r}.\hat{b}^1\right)\hat{b} + \left(\hat{r}.\hat{c}^1\right)\hat{c}$$

Proof : For $\hat{a}$, $\hat{b}$, $\hat{c}$ non-coplanar Vector $\hat{a}^1 = \frac{\hat{b}\times\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$, $\hat{b}^1 = \frac{\hat{c}\times\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$, $\hat{c}^1 = \frac{\hat{a}\times\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

Let $\hat{r} = x\hat{a} + y\hat{b} + z\hat{c}$ (1)

Multiplying both sides of (1) by $\hat{b}\times\hat{c}$, we get using dot product

$$\hat{r}\,.\,\left(\hat{b}\times\hat{c}\right) = x\hat{a}.\left(\hat{b}\times\hat{c}\right)+y\hat{b}.\left(\hat{b}\times\hat{c}\right)+z\hat{c}.\left(\hat{b}\times\hat{c}\right)$$

$$= x\hat{a}.\left(\hat{b}\times\hat{c}\right) + 0 + 0$$

$$= x\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

[∵ value of scalar triple product is zero for two same vectors.]

So $\left[\hat{r}\ \hat{b}\ \hat{c}\right] = x\left[\hat{a}\ \hat{b}\ \hat{c}\right]$

$$\Rightarrow \quad x = \frac{\left[\hat{r}\ \hat{b}\ \hat{c}\right]}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} = \frac{\hat{r}.\left(\hat{b}\times\hat{c}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]} = \hat{x}.\hat{a}^{1}$$

Similarly on multiplying both side of (1) by $\hat{c}\times\hat{a}$ using dot product

$$\hat{r}.\left(\hat{c}\times\hat{a}\right) = x\hat{a}.\left(\hat{c}\times\hat{a}\right)+y\hat{b}.\left(\hat{c}\times\hat{a}\right)+z\hat{c}.\left(\hat{c}\times\hat{a}\right)$$

$$= 0 + y\left[\hat{b}\ \hat{c}\ \hat{a}\right] + 0$$

$$= y\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

$$\Rightarrow \quad \hat{r}.\left(\hat{c}\times\hat{a}\right) = y\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

$$\Rightarrow \quad y = \frac{\hat{r}.\left(\hat{c}\times\hat{a}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\Rightarrow \quad y = \hat{r}\,.\,\hat{b}^{1}$$

Similarly on multiplying both side of (1) by $\hat{a}\times\hat{b}$ we get

$$\hat{r}.\left(\hat{a}\times\hat{b}\right) = x\hat{a}.\left(\hat{a}\times\hat{b}\right)+y\hat{b}.\left(\hat{a}\times\hat{b}\right)+z\hat{c}.\left(\hat{a}\times\hat{b}\right)$$

$$= 0 + 0 + z\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

$$\hat{r}.\left(\hat{a}\times\hat{b}\right) = z\left[\hat{a}\ \hat{b}\ \hat{c}\right]$$

$$\Rightarrow \quad z = \frac{\hat{r}.\left(\hat{a}\times\hat{b}\right)}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\Rightarrow \quad z = \hat{r}\,.\,\hat{c}^{1}$$

On putting the value of x, y and z in (1) we get

$$\hat{r} = \left(\hat{r}.\hat{a}^1\right)\hat{a} + \left(\hat{r}.\hat{b}^1\right)\hat{b} + \left(\hat{r}.\hat{c}^1\right)\hat{c}.$$

Self Check Exercise - 2

Q. 1 If $\hat{r}$ is any vector, show that

$$\hat{r} = \left(\hat{r}.\hat{a}\right)\hat{a}^1 + \left(\hat{r}.\hat{b}\right)\hat{b}^1 + \left(\hat{r}.\hat{c}\right)\hat{c}^1$$

Q. 2 If $\hat{a}$, $\hat{b}$, $\hat{c}$ denote the reciprocal triad of vectors show that

$$\left(\hat{a}^1 \times \hat{b}^1\right) + \left(\hat{b}^1 \times \hat{c}^1\right) + \left(\hat{c}^1 \times \hat{a}^1\right) = \frac{\hat{a} + \hat{b} + \hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

5.5 Summary

In this unit we studied about

1. Reciprocal system of vector.
2. Find the reciprocal system of vector for given vector.
3. Properties of reciprocal system of vectors.

5.6 Glossary

- **Orthonormal Vectors :** The vector which are orthogonal as well as normalized i.e. vectors are prependicular to each other, each having magnitude 1.
- **Reciprocal :** It is defined as inverse of a value or a number.

5.7 Answer to Self Check Exercises

Self Check Exercise-1

Q. 1 $\hat{a}^1 = \dfrac{-5\hat{i} + 5\hat{j} - 5\hat{k}}{5}$

$\hat{b}^1 = \dfrac{2\hat{i} + 10\hat{k}}{5}$

$\hat{c}^1 = \dfrac{-4\hat{i} + 5\hat{j} - 7\hat{k}}{5}$

Q. 2 $\hat{a}^1 = (1, -1, 0)$

$\hat{b}^1 = (0, 1, -1)$

$\hat{c}^1 = (0, 0, 1)$

Self Check Exercise - 2

Q. 1 Interchanging $\hat{a}$ by $\hat{a}^1$, $\hat{b}$ by $\hat{b}^1$ and $\hat{c}$ by $\hat{c}^1$ in property a.

Q. 2 Find

$$\hat{a}^1 \times \hat{b}^1 = \frac{\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\hat{b}^1 \times \hat{c}^1 = \frac{\hat{a}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

$$\hat{c}^1 \times \hat{a}^1 = \frac{\hat{b}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$$

Then an adding we get = $\dfrac{\hat{a}+\hat{b}+\hat{c}}{\left[\hat{a}\ \hat{b}\ \hat{c}\right]}$

5.8 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector analysis, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, Schand and Company Limited.
3. J.N. Sharma and A.R. Vasishtha, Vector Calculas, Krishna Prakashan Mandir.

5.9 Terminal Questions

1. Find the set of vector reciprocal to $\hat{a}$, $\hat{b}$ and $\hat{a} \times \hat{b}$
2. Find the set of vector reciprocal to

   (1) $2\hat{i} + 3\hat{j}$ - k, $\hat{i} - \hat{j} - 2\hat{k}$, $-\hat{i} + 2\hat{j} + 2\hat{k}$

   (2) $\hat{i} + 2\hat{j} - 3\hat{k}$, $5\hat{i} - \hat{j} - \hat{k}$, $\hat{i} + \hat{j} - \hat{k}$

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Unit - 6

Differentiation of Vectors

Structure

6.1 Introduction

Dear student, in this unit we will study about differentiation of vector which is same as differentiation of real valued function $f(x)$ of single variable. Here we extend the definition of derivative of real valued function of single variable to vector valued functions of single variable. We will also study about constant vector. On the basis of vector differentiation we will discuss the concept of velocity and acceleration as these are first and second order derivative of displacement vector respectively.

6.2 Learning Objectives

After studying this unit students will be able to

1. define and evaluate derivative of a given vector.
2. give physical interpretation of derivative of a vector.
3. define constant vector.
4. prove same results of constant vector.
5. define unit tangent vector.

6. define and evaluate velocity and acceleration for a given displacement vector.

6.3 Derivative of a Vector Function

Since we known that derivative of real valued function f(x) of single variable is given as

$$f^1(\text{x}) = \frac{dt}{dr} = \underset{h \to 0}{Lim} \frac{f(x+h) - f(x)}{h}$$

To define derivative for vector function, we unit have to extend this definition to vector valued function of scalar variable.

So,

Vector valued function : It to each value of scalar variable t in same interval [a, b], there corresponds, by any law, a value of a variable vector $\hat{r}$, then we say that $\hat{r}$ is a vector function of scalar variable 't' defined in the interval [a, b] , and then we write

$$\hat{r} = \hat{f}\,(\text{t}) = f_1(\text{t})\,\hat{i} + f_2(\text{t})\,\hat{j} + f_3(\text{t})\,\hat{k}$$

Where $\hat{f}$ denotes the law of correspondence.

For example :

Let a particle is moving and at time t, it is at point P, whose position vector relative to origin 0 is given by $\hat{r}$, then $\hat{r}$ is a function of the scalar variable t. The velocity and acceleration of moving practice are also vector function of the scalar time t.

Before defining the derivative of a vector function, let us first definite limit and continuity of a vector function.

Limit of a Vector Function :

A vector function $\hat{f}$ (t) is said to tend to a limit $\hat{\ell}$, when t tends to, if for any given positive number $\in$, however small, there exists a positive number δ, depending on $\in$, such that | $\hat{f}$ (t) $\hat{\ell}$ | <$\in$ for 0 |<| t- to |<δ.

We write it as $\underset{h \to to}{Lim}\ \hat{f}$ (t) = $\hat{\ell}$

Here, also, we extend the definition of limit of real valued function of single variable to vector valued function. The results of limit of real valued function are applicable on limit of vector values function. These results are

1. $\underset{t \to t0}{Lim}\left[\hat{f}(t) + \hat{g}(t)\right] = \underset{t \to t0}{Lim}\,\hat{f}(t) + \underset{t \to t0}{Lim}\,\hat{g}(t)$

2. $\underset{t \to t0}{Lim}\left[\hat{f}(t) - \hat{g}(t)\right] = \underset{t \to t0}{Lim}\,\hat{f}(t) - \underset{t \to t0}{Lim}\,\hat{g}(t)$

3. $\underset{t \to t0}{Lim}\left[\hat{f}(t).\hat{g}(t)\right] = \left[\underset{t \to t0}{Lim}\,\hat{f}(t)\right].\left[\underset{t \to t0}{Lim}\,\hat{g}(t)\right]$

4. $\lim\limits_{t \to t0} \left[\hat{f}(t) \times \hat{g}(t) \right] = \left[\lim\limits_{t \to t0} \hat{f}(t) \right] \times \left[\lim\limits_{t \to t0} \hat{g}(t) \right]$

5. $\lim\limits_{t \to t0} \left[\phi(t) \times \hat{f}(t) \right] = \left[\lim\limits_{t \to t0} \phi(t) \right] \left[\lim\limits_{t \to t0} \hat{f}(t) \right]$

6. If $\hat{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ [if $\lim\limits_{t \to t0} \hat{f}(t) = \hat{\lambda}$]

then $\hat{\lambda} = \lambda_1 \hat{i} + \lambda_2 \hat{j} + \lambda_3 \hat{k}$

where $\lim\limits_{t \to t0} f_1(t) =$, $\lim\limits_{t \to t0} f_2(t) = \lambda_2$, $\lim\limits_{t \to t0} f_3(t) = \lambda_3$

Continuity of A Vector Function :

A vector function $\hat{f}(t)$ is said to be continuous at t = t_0, if for any given positive number $\in$, however small, there exists a positive number δ, depending upon $\in$ such that

$\left| \hat{f}(t) - \hat{f}(t_0) \right| < \in$ for $|t = t_0| < \delta$

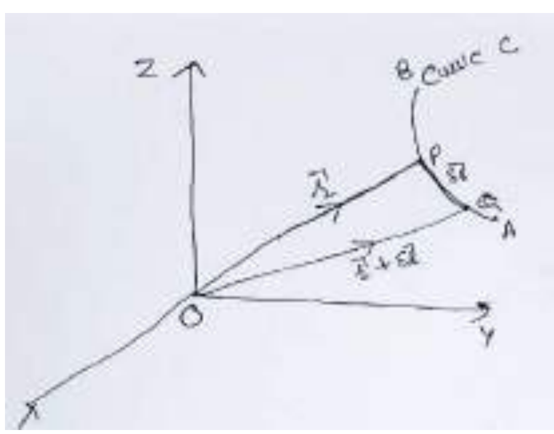
This definition is again an extension of continuity of real valued function of single variable to vector valued function.

Note : 1. A vector $\hat{f}(t)$ is said to be continuous of it is continuous for every value of t for which it is defined.

2. If $\hat{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ then $\hat{f}(t)$ is continuous of and only if $f_1(t)$, $f_2(t)$ and $f_3(t)$ are continuous functions of t.

Derivative of a Vector Function

Let O be the origin. Let the position vector of a point P is given $\hat{r} = \hat{f}(t)$. As time t varies continuously, point P trace out a curve C. Thus a vector function $\hat{r} = \hat{f}(t)$ represents a curve in space.



Let $\hat{r}$ = $\hat{f}(t)$ be the position vector of pint P. Let $\hat{r}$ +δ$\hat{r}$ is the position vector of neightauing point Q on the curve AB.

Then OP = $\hat{r}$ = $\hat{f}(t)$

OQ = $\hat{r}$ +δ$\hat{r}$ = $\hat{f}$ (t+δt)

∴ $\overrightarrow{PQ}$ = $\overrightarrow{OQ}$ - $\overrightarrow{OP}$

= $\hat{r}$ +δ$\hat{r}$ - = $\hat{r}$

= δ$\hat{r}$

∴ $\overrightarrow{PQ}$ = δ$\hat{r}$

∴ δ$\hat{r}$ = $\hat{f}$ (t+δt) - $\hat{f}$ (t)

Now,

$$\Rightarrow \quad \frac{\delta\hat{r}}{\delta t} = \frac{\hat{f}(t+\delta t)-\hat{f}(t)}{\delta t}$$

Now $\lim\limits_{\delta t\to 0}\frac{\hat{f}(t+\delta t)-\hat{f}(t)}{\delta t}$, if it exists, is called the derivative of vector function $\hat{r}$ with respect to t and is denoted by $\frac{d\hat{r}}{dt}$

$$\therefore \frac{d\hat{r}}{dt} = \lim_{\delta t\to 0}\frac{(\hat{r}+\delta\hat{r})-\hat{r}}{\delta t} = \lim_{\delta t\to 0}\frac{\hat{f}(t+\delta t)-\hat{f}(t)}{\delta t}$$

Physically, $\frac{\delta\hat{r}}{\delta t}$ is a vector parallel to the chord $\overrightarrow{PQ}$.

as $\delta t\rightarrow$, Q →P, the chord $\overrightarrow{PQ}$ become the tangent to the curve at point P.

Hence $\lim\limits_{\delta t\to 0}\frac{\delta\hat{r}}{\delta t}=\frac{d\hat{r}}{dt}$ is a vector parallel to the tangent at point P to the curve $\hat{r}$ = $\hat{f}(t)$.

Unit Tangent Vector

Replacing t by s, in above result we have $\frac{d\hat{r}}{ds}$ is a vector along the tangent at P to the curve and is in the direction of increasing S.

Then $\frac{d\hat{r}}{ds}$ is known as unit tangent vector and is denoted by $\hat{t}$.

So $\hat{t} = \dfrac{d\hat{r}}{ds}$

Notes: (1) If $\dfrac{d\hat{r}}{dt}$ exists, then $\hat{r}$ is said to be differentiable w.r.t. t.

(2) As $\hat{r}$ is a vector quantity so $\dfrac{d\hat{r}}{dt}$ is also a vector quantity.

(3) Every differentiable vector function is continuous but converse may or may not be free.

(4) $\dfrac{d\hat{r}}{dt}$, $\dfrac{d^2\hat{r}}{dt^2}$, $\dfrac{d^3\hat{r}}{dt^3}$ - are known as first, second, third derivative of vector function $\hat{r}$ and are given as $\dfrac{d^2\hat{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d^2\hat{r}}{dt}\right)$ and soon.

Now, Let us prove some theorems based on vector differentiation.

Theorem 1 : If $\hat{f}$ (t) = f_1(+)$\hat{i}$ + f_2(t) $\hat{j}$ + f_3(t) $\hat{k}$ are also drivable function and

$$\frac{d\hat{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Proof: Since $\dfrac{d\hat{f}}{dt} \underset{\delta t \to 0}{Lim} \dfrac{f(t+\delta t) - f(t)}{\delta t}$

$$\Rightarrow \quad \underset{\delta t \to 0}{Lim} \frac{\hat{f}(f+\delta t) - f(f)}{\delta t} = \underset{\delta t \to 0}{Lim} \frac{\hat{f}(t+\delta t) - f(t)}{\delta t}\hat{k}\,\hat{i} + \underset{\delta t \to 0}{Lim} \frac{f_2(t+\delta t) - f(t)}{\delta t}\hat{j}$$

$$+ \underset{\delta t \to 0}{Lim} \frac{f_3(t+\delta t) - f(t)}{\delta t}\hat{k}$$

then $\underset{\delta t \to 0}{Lim} \dfrac{\hat{f}(t+\delta t) - f(t)}{\delta t}$ exists only and only if the three limits on R.H.S. exists.

$$\Rightarrow \quad \frac{d\hat{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Limits on R.H.S. exists.

$$\Rightarrow \quad \frac{d\hat{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

Derivability in Relation to Algebraic Operation

Theorem 2: If $\hat{f}$ (t) and $\hat{g}$ (t) are two derivable functions of t then

(1) $\frac{d}{dt}\left(\hat{f}(t) \pm \hat{g}(t)\right) = \frac{d}{dt}\hat{f}(t) \pm \frac{d}{dt}\hat{g}(t)$

(2) $\frac{d}{dt}\left(\hat{f}(t).\hat{g}(t)\right) = \hat{f}(t)\frac{d}{dt}\hat{g}(t) + \frac{d}{dt}\hat{f}(t).\hat{g}(t)$

(3) $\frac{d}{dt}\left(\hat{f}(t)\times\hat{g}(t)\right) = \hat{f}(t)\times\frac{d}{dt}\hat{g}(t) + \frac{d}{dt}\hat{f}(t)\times\hat{g}(t)$

(4) $\frac{d}{dt}\left(\phi\hat{f}(t)\right) = \phi\frac{d}{dt}\hat{f}(t) + \frac{d\phi}{dt}\hat{f}(t)$

Where ϕ (t) is a derivable scalar function.

(1) $$\frac{d}{dt}\left[\hat{f}(t)+\hat{g}(t)\right] = \lim_{\delta t\to 0}\frac{\left[\hat{f}\left(t+\delta t\right)+\hat{g}(t)\delta t\right]-\left[\hat{f}\left(t\right)+\hat{g}(t)\right]}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\left(\hat{f}\left(t+\delta t\right)-\hat{f}(t)+\hat{g}\left(t+\delta t\right)-\hat{g}(t)\right)}{\delta t}$$

$$= \lim_{\delta t\to 0}\frac{\hat{f}\left(t+\delta t\right)-\hat{f}(t)}{\delta t} + \lim_{\delta t\to 0}\frac{\hat{g}\left(t+\delta t\right)-\hat{g}(t)}{\delta t}$$

$$\Rightarrow \quad \frac{d}{dt}\left[\hat{f}(t)+\hat{g}(t)\right] = \frac{d\hat{f}}{dt} + \frac{d\hat{g}}{dt}$$

Similarly, $\frac{d}{dt}\left[\hat{f}(t)+\hat{g}(t)\right] = \frac{d\hat{f}}{dt} + \frac{d\hat{g}}{dt}$

(2) $$\frac{d}{dt}\left[\hat{f}(t).\hat{g}(t)\right] = \lim_{\delta t\to 0}\frac{\left(\hat{f}\left(t+\delta t\right)\hat{g}(t)\left(t+\delta t\right)-\hat{f}(t)\hat{g}\right)}{\delta t}$$

adding and subtracting the term $\hat{f}\left(t+\delta t\right)\hat{g}(t)$ in R.H.S. of above, we get

$$\frac{d}{dt}\hat{f}\left(t+\delta t\right)\hat{g}(t) = \lim_{\delta t\to 0}$$
$$\frac{\left[\hat{f}\left(t+\delta t\right).\hat{g}\left(t+\delta t\right)-\hat{f}\left(t+\delta t\right)\hat{g}\left(t\right)+\hat{f}\left(t+\delta t\right)\hat{g}(t)-\hat{f}\left(t\right)\hat{g}(t)\right]}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\hat{f}(t+\delta t)\left(\hat{g}(t+\delta t) - \hat{g}(t)\right) + \hat{g}(t)\left(\hat{f}(t+\delta t) - \hat{f}(t)\right)}{\delta t}$$

$$= \lim_{\delta t \to 0} \hat{f}(t+\delta t) \cdot \frac{\left(\hat{g}(t+\delta t) - \hat{g}(t)\right)}{\delta t} + \lim_{\delta t \to 0} \hat{g}(t) \cdot \frac{\hat{f}(t+\delta t) - \hat{f}(t)}{\delta t}$$

applying the limits we get,

$$\frac{d}{dt}\left[\hat{f}(t).\hat{g}(t)\right] = \hat{f} \cdot \frac{d\hat{g}}{dt} + \hat{g}\ \frac{df}{dt}.$$

$$\left[\because \lim_{\delta t \to 0} \frac{\hat{g}(t+\delta t) - \hat{g}(t)}{\delta t} = \frac{d\hat{g}}{dt} \text{ and } \lim_{\delta t \to 0} \frac{\hat{f}(t+\delta t) - \hat{f}(t)}{\delta t} = \frac{df}{dt}\right]$$

(3) $\frac{d}{dt}(\hat{f} \times \hat{g}) = \hat{f} \times \frac{d\hat{g}}{dt} + \frac{d\hat{f}}{dt} \times \hat{g}$

Since $\frac{d}{dt}\left[\hat{f}(t) \times \hat{g}(t)\right] = \lim_{\delta t \to 0} \left[\frac{\left(\hat{f}(t+\delta t) \times \hat{g}(t+\delta t) - \hat{f}(t) \times \hat{g}(t)\right)}{\delta t}\right]$

Adding and subtracting the term $\hat{f}(t+\delta t)\hat{g}(t)$ in R.H.S. of above, we get

$$\frac{d}{dt}\left[\hat{f}(t) \times \hat{g}(t)\right] =$$

$$\lim_{\delta t \to 0} \frac{\hat{f}(t+\delta t) \times \hat{g}(t+\delta t) - \hat{f}(t+\delta t) \times \hat{g}(t) + \hat{f}(t+\delta t) \times \hat{g}(t) - \hat{f}(t) \times \hat{g}(t)}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\hat{f}(t+\delta t) \times \left[\hat{g}(t+\delta t) - \hat{g}(t)\right] + \left[\hat{f}(t+\delta t) - \hat{f}(t)\right] \times \hat{g}(t)}{\delta t}$$

$$\frac{d}{dt}\left[\hat{f}(t) \times \hat{g}(t)\right] = \lim_{\delta t \to 0} \hat{f}(t+\delta t) \times \frac{\left[\hat{g}(t+\delta t) - \hat{g}(t)\right]}{\delta t} \lim_{\delta t \to 0} \frac{\left[\hat{f}(t+\delta t) - \hat{f}(t)\right]}{\delta t} \times \hat{g}(t)$$

Applying the limits, we get,

$$\frac{d}{dt}\left[\hat{f}(t) \times \hat{g}(t)\right] = \hat{f}\text{ (t)} \times \frac{d\hat{g}}{dt} + \frac{d\hat{f}}{dt} \times \hat{g}(t)$$

(4) $\frac{d}{dt}\left[\phi(t)+\hat{f}(t)\right]= \underset{\delta t\to 0}{Lim}\ \frac{\phi\left(t+\delta t\right)\hat{f}\left(t+\delta t\right)-\phi(t)\hat{f}(t)}{\delta t}$

Adding and subtracting the term $\phi\left(t+\delta t\right)\hat{f}(t)$ in R.H.S of above, we get

$$\frac{d}{dt}\left[\phi(t)+\hat{f}(t)\right]= \underset{\delta t\to 0}{Lim}\ \frac{\underline{\phi\left(t+\delta t\right)\hat{f}(t)+\phi\left(t+\delta t\right)-\hat{f}(t)}}{\underline{\hat{f}\left(t+\delta t\right)-\hat{f}(t)}\ \ \underline{\phi\left(t+\delta t\right)-\phi(t)}}\ \delta t$$

$$= \underset{\delta t\to 0}{Lim}\ \phi\left(t+\delta t\right)\frac{\hat{f}\left(t+\delta t\right)-\hat{f}(t)}{\delta t} + \hat{f}(t)\frac{\phi\left(t+\delta t\right)-\phi(t)}{\delta t}$$

appying the limit, we get,

$$\frac{d}{dt}\left[\phi(t)\hat{f}(t)\right]= \phi(t)\frac{d\hat{f}}{dt}+ \hat{f}(t)\frac{d\phi}{dt}.$$

Theorem 3: Prove that $\hat{f}\ .\ \frac{d\hat{f}}{dt} = \left|\hat{f}\right|\frac{d\left|\hat{f}\right|}{dt}$

Proof:- Since $\hat{f}$ (t) . $\hat{f}$ (t) = $\left|\hat{f}\right|^2$

Differentiating both side and

applying $\frac{d}{dt}\left[\hat{f}(t).\hat{g}(t)\right]= \hat{f}$ (t), $\frac{d}{dt}\hat{g}(t)+\hat{g}(t).\frac{d}{dt}\hat{f}$ (t).

$$\hat{f}\text{ (t). }\frac{d\hat{f}}{dt}\text{ (t) + }\hat{f}\text{ (t) = 2 }\left|\hat{f}(t)\right|\frac{d\left|\hat{f}(t)\right|}{dt}$$

$$\Rightarrow \quad 2\hat{f}\text{ (t). }\frac{d\hat{f}}{dt}\text{ (t) = 2}\left|\hat{f}(t)\right|\frac{d\left|\hat{f}(t)\right|}{dt}$$

$$\Rightarrow \quad \hat{f}\text{ (t).}\frac{d\hat{f}(t)}{dt} = \left|\hat{f}(t)\right|\frac{d\left|\hat{f}(t)\right|}{dt}$$

or $\hat{f}\ .\frac{d\hat{f}}{dt} = \left|\hat{f}\right|\frac{d\left|\hat{f}\right|}{dt}$

Derivative of Triple Products

Theorem 4: If $\hat{f}$, $\hat{g}$ and $\hat{h}$ are three derivable functions of t then

(1) $\frac{d}{dt}\left[\hat{f}\,\hat{g}\,\hat{h}\right] = \left[\frac{d\hat{f}}{dt}\hat{g}\,\hat{h}\right] + \left[\hat{f}\,\frac{d\hat{g}}{dt}\hat{h}\right] + \left[\hat{f}\,\hat{g}\,\frac{d\vec{h}}{dt}\right]$

(2) $\frac{d}{dt}\left[\hat{f}\,\hat{g}\,\hat{h}\right] = \frac{d\hat{f}}{dt}\times\left(\hat{g}\times\hat{h}\right) + \hat{f}\times\left(\frac{d\hat{g}}{dt}\times\hat{h}\right) + \hat{f}\times\left[\left(\hat{g}\times\frac{d\hat{h}}{dt}\right)\right]$

Proof: (1) $\frac{d}{dt}\left[\hat{f}\,\hat{g}\,\hat{h}\right] = \frac{d}{dt}\left(\hat{f}\times\hat{g}.\hat{h}\right)$ [defining of scalar triple product]

$= \frac{d}{dt}\left(\hat{f}\times\hat{g}.\hat{h}\right) + \left(\hat{f}\times\hat{g}\right).\ \frac{d\hat{h}}{dt}\ \left[\because \frac{d}{dt}\hat{f} = \hat{g} = \hat{f}\,\frac{d\hat{g}}{dt} + \hat{g}.\frac{d\hat{f}}{dt}\right]$

$= \left[\hat{f}\times\frac{d\hat{g}}{dt} + \frac{d\hat{f}}{dt}\times\hat{g}\right].\ \hat{h} + \left(\hat{f}\times\hat{g}\right).\frac{d\hat{h}}{dt}$

$\left[\because \frac{d}{dt}\hat{f}\times\left(\hat{g}\times\hat{f}\right) = \hat{f}\times\frac{d\hat{g}}{dt} + \frac{d\hat{f}}{dt}\times\hat{g}\right]$

$= \hat{f}\times\frac{d\hat{g}}{dt}$. h + $\frac{d\hat{f}}{dt}\times\hat{g}.\hat{h} + \left(\hat{f}\times\hat{g}\right).\frac{d\hat{h}}{dt}$

$= \frac{d\hat{f}}{dt}\times\hat{g}.\hat{h} + \hat{f}\times\frac{d\hat{g}}{dt}.\vec{h} + \left(\hat{f}\times\hat{g}\right).\ \frac{d\hat{h}}{dt}$

$= \frac{d}{dt}\left[\hat{f}\,\hat{g}\,\hat{h}\right] = \left[\frac{d\hat{f}}{dt}g\,h\right] + \left[\hat{f}\,\frac{d\hat{g}}{dt}\hat{h}\right] + \left[\hat{f}\,\hat{g}\,\frac{d\hat{h}}{dt}\right]$

(2) $\frac{d}{dt}\left[\hat{f}\,\hat{g}\,\hat{h}\right] = \hat{f}\times\frac{d}{dt}\left(\hat{g}\times\hat{h}\right) + \frac{d}{dt}\ \hat{f}\times\left(\hat{g}\times\hat{h}\right)$

$\left[\because \frac{d}{dt}\left(\hat{f}\times\hat{g}\right) = \hat{f}\times\frac{d\hat{g}}{dt} + \hat{g}.\frac{d\hat{f}}{dt}\times\hat{f}\right]$

$= \hat{f}\times\left[\hat{g}\times\frac{d\hat{h}}{dt} + \frac{d}{dt}\hat{g}\times\hat{h}\right] + \frac{d\hat{f}}{dt}\times\left(\hat{g}\times\hat{h}\right)$

$= \hat{f}\times\left(\hat{g}\times\frac{d\hat{h}}{dt}\right) + \hat{f}\times\left(\frac{d\hat{g}}{dt}\times\hat{h}\right) + \frac{d\hat{f}}{dt}\times\left(\hat{g}\times\hat{h}\right)$

after rearranging the terms on R.H.S, we get

$$\Rightarrow \qquad \frac{d}{dt}\left[\hat{f} \times \left(\hat{g} \times \hat{h}\right)\right] = \frac{d\hat{f}}{dt} \times \left(\hat{g} \times \hat{h}\right) + \hat{f} \times \left(\frac{d\hat{g}}{dt} \times \hat{h}\right) + \hat{f} \times \left(\hat{g} \times \frac{d\hat{h}}{dt}\right)$$

Derivative of Vector Function in Component Form

Let $\hat{r}$ be the vector function of the scalar variable t such that $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where x, y and z are component of $\hat{r}$ in X, Y and Z directions and are scalar function of t and $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in three mutually perpendicular directions.

Then $\frac{d\hat{r}}{dt} = \frac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$= \frac{d}{dt}(x\hat{i}) + \frac{d}{dt}(y\hat{j}) + \frac{d}{dt}(z\hat{k})$

$= \frac{dx}{xt}\hat{i} + x\frac{d\hat{i}}{dt} + \frac{dy}{dt}\hat{j} + y\frac{d\hat{j}}{dt} + \frac{dz}{dt}\hat{k} + z\frac{d\hat{k}}{dt}$

$\frac{d\hat{r}}{dt} = \frac{dx}{xt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$

$$\left[\because \frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = 0\, as\, i\, z\, and\, k\, are Unit Vectors\right]$$

Let us do some examples for implementation of vector differentiation.

Example 1: If $\hat{r} = (t + 1)\hat{i} + (t^2 + t + 1)\hat{j} + (t^3 + t^2 + t + 1)\hat{k}$

find $\frac{d\hat{r}}{dt}$ and $\frac{d^2\hat{r}}{dt^2}$

Solution: Given $\hat{r} = (t + 1)\hat{i} + (t^2 + t + 1)\hat{j} + (t^3 + t^2 + t + 1)\hat{k}$

Then $\frac{d\hat{r}}{dt} = \frac{d}{dt}(t + 1)\hat{i} + \frac{d}{dt}(t^2 + t + 1)\hat{j} + \frac{d}{dt}(t^3 + t^2 + t + 1)\hat{k}$

$\Rightarrow \quad \frac{d\hat{r}}{dt} = 1\hat{i} + (2t + 1)\hat{j} + (t^3 + t^2 + t + 1)\hat{k}$

Now $\quad \frac{d^2\hat{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{r}}{dt}\right)$

$= \frac{d}{dt}\left[1.\hat{i} + \left(2t+1\right)\hat{j} + \left(3t^2 + 2t + 1\right)\hat{k}\right]$

$= \frac{d}{dt} 1\hat{i} + \frac{d}{dt}(2t+1)\ \hat{j}\ \frac{d}{dt}(3t^2+2+1)\ \hat{k}$

$= 0 + 2\ \hat{j} + (6t+2)\ \hat{k}$

$\Rightarrow \quad \frac{d^2\hat{r}}{dt^2} = 2\ \hat{j} + (6t+2)\ \hat{k}$

Example 2: If $\hat{r}$ = (a cos t)$\hat{i}$ + (a sin t) $\hat{j}$ + t $\hat{k}$ then find

$\frac{d\hat{r}}{dt}$, $\frac{d^2\hat{r}}{dt^2}$ and $\left|\frac{d^2\hat{r}}{dt^2}\right|$.

Solution : Given $\hat{r}$ = (a cos t)$\hat{i}$ + (a sin t) $\hat{j}$ + t $\hat{k}$

Then $\frac{d\hat{r}}{dt} = \frac{d}{dt}(a \cos t)\hat{i} + \frac{d}{dt}(a \sin t)\ \hat{j} + \frac{d}{dt} t\ \hat{k}$

$= -a \sin t\hat{i} + a \cos t\ \hat{j} + \hat{k}$ $\quad \left[\because \frac{d}{dt} \cos t = -\sin t\right.$

$\left.\frac{d}{dt} \sin t = \cos t\right]$

$\Rightarrow \quad \frac{d\hat{r}}{dt} = -a \sin t\hat{i} + a \cos t\ \hat{j} + \hat{k}$

Now, $\frac{d^2\hat{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{r}}{dt}\right)$

$= \frac{d}{dt}(-a \sin t\hat{i} + a \cos t\ \hat{j} + \hat{k})$

$= -a \cos t\hat{i} - a \sin t\ \hat{j} + 0\ \hat{k}$

$\Rightarrow \quad \frac{d^2\hat{r}}{dt^2} = -a \cos t\hat{i} - a \sin t\ \hat{j}$

Now, $\left|\frac{d^2\hat{r}}{dt^2}\right| = \sqrt{(-a\cos t)^2 + (a \sin t)^2}$

$= \sqrt{a^2 \cos^2 t + a^2 \sin^2 t}$

$= \sqrt{a^2(\cos^2 t + \sin^2 t)}$

$= \sqrt{a^2}$

$\Rightarrow \quad \left|\frac{d^2\hat{r}}{dt^2}\right| = a \qquad \because \cos^2 t + \sin^2 t = 1$

Example 3: If $\hat{f} = t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k}$ and $\hat{g} = (2t - 3)\hat{i} + \hat{j} - \hat{k}$

Find $\frac{d}{dt}(\hat{f} \cdot \hat{g})$ when t = 1

Solution : Given $\hat{f} = t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k}$

and $\hat{g} = (2t - 3)\hat{i} + \hat{j} - t\,\hat{k}$

When $\frac{d}{dt}(\hat{f} \cdot \hat{g}) = \frac{d}{dt}\hat{f} \cdot \hat{g} + \hat{f} \cdot \frac{d}{dt}\hat{g}$ [by theorem]

$= \frac{d}{dt}[t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k}] \cdot ((2t - 3)\hat{i} + \hat{j} - t\,\hat{k})$

$+ t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k} \cdot \frac{d}{dt}[(2t - 3)\hat{i} + \hat{j} - t\,\hat{k}]$

$= (2t\,\hat{i} - \hat{j} + 2\,\hat{k}) \cdot ((2t - 3)\hat{i} + \hat{j} - t\,\hat{k})$

$+ t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k} \cdot (2\hat{i} + 0\,\hat{j} - \hat{k})$

$= 2t\,(2t - 3) - 1 - 2t + 2t^2 + 0 - (2t+1)$

$= 4t^2 - 6t - 1 - 2t + 2t^2 - 2t - 1$

$\frac{d}{dt}(\hat{f} \cdot \hat{g}) = 6t^2 - 10t - 2$

$\frac{d}{dt}(\hat{f} \cdot \hat{g})$ when t = 1 is

$= 6(1)^2 - 10(1) - 2$

$= 6 - 10 - 2$

$= -6.$

We can solve this question by another method i.e. first applying the dot product and then differentiate it i.e.

$\frac{d}{dt}(\hat{f} \cdot \hat{g}) = \frac{d}{dt}[(t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k}) \cdot ((2t - 3)\hat{i} + \hat{j} - t\,\hat{k})]$

$= \frac{d}{dt}$ [t^2 (2t-3) - t×1 + (2t+1)(-t)]

$= \frac{d}{dt}$ [$2t^3 - 3t^2$- t - $2t^2$- t]

$= \frac{d}{dt}$ [$2t^3 - 5t^2$- 2t]

$\frac{d}{dt}(\hat{f}\,.\,\hat{g})$ = 6t2 - 10t -2

and $\frac{d}{dt}(\hat{f}\,.\,\hat{g})$ when t = 1 = - 6.

* It is much easier to apply 2nd approach.

Example 4: If $\hat{f} = t^2\hat{i} - t\,\hat{j} + (2t+1)\,\hat{k}$ and $\hat{g} = 2t\,\hat{i} + \hat{j} - t\,\hat{k}$

then find $\frac{d}{dt}(\hat{f} \times \hat{g})$

Solution : To find $\frac{d}{dt}(\hat{f} \times \hat{g})$ we first find the value of $\hat{f} \times \hat{g}$ and then differentiate it.

$$\text{So, } \hat{f} \times \hat{g} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & 2t+1 \\ 2t & 1 & -t \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -t & 2t+1 \\ 1 & -t \end{vmatrix} - \hat{j}\begin{vmatrix} t^2 & 2t+1 \\ 2t & -t \end{vmatrix} + \hat{k}\begin{vmatrix} t^2 & -t \\ 2t & 1 \end{vmatrix}$$

$\Rightarrow \hat{f} \times \hat{g} = \hat{i}\,(t^2 - 2t - 1) - \hat{j}\,(-t^3 - 4t^2 - 2t) + \hat{k}\,(t^2 + 2t^2)$

Now $\frac{d}{dt}(\hat{f} \times \hat{g}) = (2t - 1)\,\hat{i} - (-3t^2 - 8t^2 - 2t)\,\hat{j} + (2t + 4t)\,\hat{k}$

$\Rightarrow \frac{d}{dt}(\hat{f} \times \hat{g}) = (2t - 2)\,\hat{i} + (3t^2 - 8t - 2t)\,\hat{j} + 6t\,\hat{k}$

Example 5: If $\hat{r} = t^3\hat{i} + \left(2t^3 \frac{1}{5t^2}\right)\hat{j}$ then show that

$\hat{r} \times \frac{d\hat{r}}{dt} = \hat{k}$

Solution : Given $\hat{r} = t^3\hat{i} + \left(2t^3\ \dfrac{1}{5t^2}\right)\hat{j}$

then $\dfrac{d\hat{r}}{dt} = t^3\hat{i} + \left(6t^2\ \dfrac{2}{5t^3}\right)\hat{j}$ [$\because \dfrac{d}{dt} t^{-2} = -2t^{-3}$]

Now $\hat{r} \times \dfrac{d\hat{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & \left(2t^3\ \dfrac{1}{5t^2}\right) & 0 \\ 3t^2 & \left(6t^2\ \dfrac{2}{5t^3}\right) & 0 \end{vmatrix}$

$= \hat{i}\,(0) - \hat{j}\,(0) + \hat{k}\left[t^3\left(6t^2\ \dfrac{2}{5t^3}\right) - 3t^2\left(2t^3\ \dfrac{1}{5t^2}\right)\right]$

$= \hat{k}\left[6t^5 + \dfrac{2}{5} - 6t^5 + \dfrac{3}{5}\right]$

$\Rightarrow \quad \hat{r} \times \dfrac{d\hat{r}}{dt} = \hat{k}$

Example 6: If $\hat{r}$ = a cos t$\hat{i}$ + a sin t $\hat{j}$ + (a + tan x) $\hat{k}$

find $\left|\dfrac{d\hat{r}}{dt} \times \dfrac{d^2\hat{r}}{dt^2}\right|$ and $\left[\dfrac{d\hat{r}}{dt}, \dfrac{d^2\hat{r}}{dt^2}, \dfrac{d^3\hat{r}}{dt^3}\right]$

Solution: If $\hat{r}$ = a cos t$\hat{i}$ + a sin t $\hat{j}$ + (a + tan x) $\hat{k}$

then $\dfrac{d\hat{r}}{dt}$ =- a cos t$\hat{i}$ + a sin t $\hat{j}$ + a tan x $\hat{k}$ [Here tan x is taken as constant]

Now $\dfrac{d^2\hat{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\hat{r}}{dt}\right) = \dfrac{d\hat{r}}{dt}$ = - a cos t$\hat{i}$ - a sin t $\hat{j}$

Now $\dfrac{d^3\hat{r}}{dt^3} = \dfrac{d}{dt}\left(\dfrac{d^2\hat{r}}{dt^2}\right) \dfrac{d}{dt}$(- a cos t$\hat{i}$ + a sin t $\hat{j}$)

$\Rightarrow \quad \dfrac{d^3\hat{r}}{dt^3}$ = a sin t$\hat{i}$ - a cos t $\hat{j}$

Now $\frac{d\hat{r}}{dt}\times\frac{d^2\hat{r}}{dt^2}$ = [- a cos t$\hat{i}$ + a sin t $\hat{j}$ + a tan x $\hat{k}$]× [- a cos t$\hat{i}$ - a sin t $\hat{j}$]

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & a\tan x \\ -a\sin t & -a\sin t & 0 \end{vmatrix}$$

$\Rightarrow$ $\frac{d\hat{r}}{dt}\times\frac{d^2\hat{r}}{dt^2}$ = $\hat{i}$ [a^2 tan x sin t] - $\hat{j}$ [a^2 tan x cos t] + $\hat{k}$ [a^2 sin t + a^2 cos t]

Now, $\left|\frac{d\hat{r}}{dt}\times\frac{d^2\hat{r}}{dt^2}\right| = \sqrt{\left(a^2\tan x\sin t\right)^2 - \hat{j}\left(a^2\tan x\cos t\right)^2 + (a^2)^2}$

$= \sqrt{a^4\tan^2 x\sin^2 t + a^4\tan^2 x\cos^2 t + a^4}$

$= \sqrt{a^4\tan^2 x(\sin^2 x + \cos^2 x) + a^4}$

$= \sqrt{a^4(1+\tan^2 x)}$

$= \sqrt{a^4\sec^2 x}$ [∵ 1+ $\tan^2$ x = $\sec^2$ x]

$= a^4\sec x$

$\Rightarrow$ $\left|\frac{d\hat{r}}{dt}\times\frac{d^2\hat{r}}{dt^2}\right| = a^4\sec x$

Now $\left[\frac{d\hat{r}}{dt},\frac{d^2\hat{r}}{dt^2},\frac{d^3\hat{r}}{dt^3}\right] = \left|\frac{d\hat{r}}{dt}\times\frac{d^2\hat{r}}{dt^2}\right|\cdot\frac{d^3\hat{r}}{dt^3}$ [defining of scalar triple product.]

= *[(a^2 sin t tan x)$\hat{i}$ - (a^2 cos t tan x) $\hat{j}$ + a^2 $\hat{k}$].[a sin t$\hat{i}$ - a cos t $\hat{j}$]*

= *a^3 $\sin^2$ t tan x +a^3 $\cos^2$ t tan x+ a^2*(0)

= *a^3 tan x ($\sin^2$+ $\cos^2$ t)*

= a^3 tan x

$\Rightarrow$ $\left[\frac{d\hat{r}}{dt}\frac{d^2\hat{r}}{dt^2}\frac{d^3\hat{r}}{dt^3}\right]$ = a^3 tan x

Example 7 : Find the unit tangent vector to any point on the curve,

$\hat{r}$ =a cos t$\hat{i}$ + a sin t $\hat{j}$ +b t $\hat{k}$

Solution : Since we know that $\frac{d\hat{r}}{dt}$ is the tangent vector at any point and $\left|\frac{d\hat{r}}{dt}\right|$ is the unit tangent vector.

Given $\hat{r}$ =a cos t$\hat{i}$ + a sin t $\hat{j}$ +b t $\hat{k}$

$\frac{d\hat{r}}{dt}$ = -a cos t$\hat{i}$ + a sin t $\hat{j}$ +b $\hat{k}$ is the tangent vector at any point.

Since we know that unit vector = $\frac{\text{Vector}}{\text{Magnitude of vector}}$

So unit tangent vector t = $\frac{-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}}{\sqrt{(-a\sin t)^2 + (a\cos t)^2 + (b)^2}}$

$$\hat{t} = \frac{-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}}{\sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}}$$

Unit tangent vector $\hat{t} = \frac{-a\sin t\hat{i} + a\cos t\hat{j} + b\hat{k}}{\sqrt{a^2 + b^2}}$

Example 8 : Find the unit tangent vector for the curve

$\hat{r}$ = 3 cos t$\hat{i}$ + 3 sin t $\hat{j}$ + 4 t $\hat{k}$

Solution : Given $\hat{r}$ = 3 cos t$\hat{i}$ + 3 sin t $\hat{j}$ + 4 t $\hat{k}$

then $\frac{d\hat{r}}{dt}$ gives the tangent vector so $\frac{d\hat{r}}{dt}$ = -3 cos t$\hat{i}$ + 3 sin t $\hat{j}$ + 4 t $\hat{k}$

Now, unit tangent vector, = $\hat{t} = \frac{-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}}{\sqrt{(-3\sin t)^2 + (3\cos t)^2 + (4)^2}}$

$$= \frac{-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}}{\sqrt{a\sin^2 t + a\cos^2 t + 16}}$$

$$= \frac{-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}}{\sqrt{9+16}}$$

Unit tangent vector = $\hat{t} = \frac{1}{3}\left(-3\sin t\hat{i} + 3\cos t\hat{j} + 4\hat{k}\right)$

Example 9 : Find the angle between the tangents to the curve

$\hat{r} = t^2\hat{i} + 2t\,\hat{j} - t^3\hat{k}$ at pints t = ± 1

Solution : Given $\hat{r} = t^2\hat{i} + 2t\,\hat{j} - t^3\hat{k}$

then $\frac{d\hat{r}}{dt}$ gives us the tangent. Taking t = 1 and t = -1 we get two tangents $\hat{t}_1$ and $\hat{t}_2$

So $\frac{d\hat{r}}{dt} = \frac{d}{dt}(t^2\hat{i} + 2t\,\hat{j} - t^3\hat{k})$

$\frac{d\hat{r}}{dt} = 2t^2\hat{i} + 2\,\hat{j} - 3t^2\hat{k}$

Now $\hat{t}_1 = \left(\frac{d\hat{r}}{dt}\right)$ t = 1 = $2(1)\hat{i} + 2\,\hat{j} - 3(t)^2\hat{k}$

$\hat{t}_1 = 2\hat{i} + 2\,\hat{j} - 3\,\hat{k}$

and $\hat{t}_2 = \left(\frac{d\hat{r}}{dt}\right)$ t = 1 = $2(-1)\hat{i} + 2\,\hat{j} - 3(-1)^2\hat{k}$

$\hat{t}_2 = -2\hat{i} + 2\,\hat{j} - 3\,\hat{k}$

Let Θ be angel between $\hat{t}_1$ and $\hat{t}_2$ then

$$\text{Cos } \hat{\Theta} = \frac{\hat{t}_1.\hat{t}_2}{|\hat{t}_1||\hat{t}_2|}$$

$$\text{Cos } \Theta = \frac{\left(2\hat{i}+2\hat{j}-3\hat{k}\right).\left(-2\hat{i}+2\hat{j}-3\hat{k}\right)}{\sqrt{(2)^2+(2)^2+(-3)^2}\sqrt{(-2)^2+(2)^2+(-3)^2}}$$

$$= \frac{-4+4+9}{\sqrt{17}\sqrt{17}}$$

$$\text{Cos } \Theta = \frac{9}{17}$$

⇒ $\Theta = \text{Cos}^{-1}\left(9/17\right)$

Self Check Exercise-1

Q. 1 If $\hat{r} = \sin t\hat{i} + \cos t\,\hat{j} + t\hat{k}$ then find

$\frac{d\hat{r}}{dt}$, $\frac{d^2\hat{r}}{dt^2}$, $\left|\frac{d\hat{r}}{dt}\right|$, $\left|\frac{d^2\hat{r}}{dt^2}\right|$

Q. 2 Verify the formula $\frac{d}{dt}\left(\hat{f}.\hat{g}\right) = \hat{f}.\frac{d\hat{g}}{dt} + \frac{d\hat{f}}{dt}.\hat{g}$ for

$\hat{f} = 3t^2\hat{i} + 2t\,\hat{j} + t^3\hat{k}$ and $\hat{g} = 5t^2\,\hat{j} + t\,\hat{k}$.

Q. 3 If $\hat{f} = 5t^2\hat{i} + t\,\hat{j} - t^3\hat{k}$ and $\hat{g} = \text{Sin } t\,\hat{i} - \text{Cos } t\,\hat{j}$

find $\frac{d}{dt}\left(\hat{f}.\hat{g}\right)$

Q. 4 Find unit tangent vector for $\hat{r} = 3t^2\hat{i} + (t^2 - 2t)\,\hat{j} + t^3\hat{k}$ at t = 1.

6.4 Constant Vector

A vector is said to be a constant vector of its magnitude and direction do not change i.e. both magnitude and direction are fixed. Let us understand, constant vector with following examples.

Example 1 : Check $\hat{f} = 2t\hat{i} + 3t\,\hat{j} + 4\,t^3\hat{k}$ is constant vector or not.

Solution : Given $\hat{f} = 2t\hat{i} + 3t\,\hat{j} + 4\,t^3\hat{k}$ then

$$\left|\hat{f}\right| = \sqrt{(2t)^2 + 3(t)^2 + (4t)^2}$$

$$= \sqrt{4t^2 + 9t^2 + 16t^2}$$

$$= \sqrt{29t^2}$$

$$\left|\hat{f}\right| = \sqrt{29t^2}$$

Since $\left|\hat{f}\right| = \sqrt{29t^2}$

So for different value of t value of $\left|\hat{f}\right|$ is different so $\hat{f}$ has not fixed magnitude.

Also the direction of $\hat{f}$ is fixed, as its direction is same for all value of t.

So $\hat{f}$ is not a constant vector, having fixed direction but verifying magnitude.

Example 2 : Check $\hat{f} = \text{Cos } t\hat{i} + \text{Sin } t\,\hat{j} + \hat{k}$ is constant vector or not.

Solution : Given $\hat{f} = \text{Cos } t\hat{i} + \text{Sin } t\,\hat{j} + \hat{k}$

then $\left|\hat{f}\right| = \sqrt{\cos^2 t + \sin^2 t + 1}$

$= \sqrt{1+1}$

$\left|\hat{f}\right| = \sqrt{2}$

Magnitude of $\hat{f}$ is fixed. But the direction of $\hat{f}$ varies as the value of t varies, so $\hat{f}$ is not a constant vector.

Example 2 : Check $\hat{f} = 3\hat{i} + 3\hat{j} + 4\hat{k}$ is constant vector.

Solution : Given $\hat{f} = 3\hat{i} + 3\hat{j} + 4\hat{k}$

then $\left|\hat{f}\right| = \sqrt{(3)^2 + (3)^2 + (4)^2}$

$= \sqrt{9+9+16}$

$\left|\hat{f}\right| = \sqrt{34}$

Magnitude of $\hat{f}$ is fixed. Also $\hat{f}$ represents to position vector (3, 3, 4), which is a fixed point in space. So direction of $\hat{f}$ is also fixed. So $\hat{f}$ is a constant vector.

Theorem 1 : Show that derivative of constant vector is zero vector.

Proof : Let $\hat{r}$ be 0 constant vector i.e.

$\hat{r} = \hat{c}$ (1)

$\therefore$ $\hat{r} + \delta\hat{r} = \hat{c}$ (2)

Subtracting (1) from (2)

$\hat{r} + \delta\hat{r} - \hat{r} = 0$

$\Rightarrow$ $\delta\hat{r} = 0$

or $\frac{\delta\hat{r}}{\delta t} = 0$

Taking $\delta t \to 0$, we get

$\underset{\delta t \to 0}{Lim} \frac{\delta\hat{r}}{\delta t} = 0$

$\Rightarrow$ $\frac{d\hat{r}}{dt} = 0$ [definition of derivative.]

Hence derivative of constant vector is zero.

Theorem 2 : The necessary and sufficient condition for a vector to be constant is $\frac{d\hat{f}}{dt} = 0$

Proof : Condition is necessary

Let $\hat{f}$ (t) is a constant vector function.

Such that $\hat{f}$ (t + δt) = $\hat{f}$ (t)

$\Rightarrow \hat{f}$ (t + δt) - $\hat{f}$ (t) = 0

dividing both side by δt

$$\Rightarrow \quad \frac{\hat{f}(t+\delta t) - \hat{f}(t)}{\delta t} = 0$$

Taking Lim δt → 0 both side

$$\Rightarrow \quad \lim_{\delta t \to 0} \frac{\hat{f}(t+\delta t) - \hat{f}(t)}{\delta t} = 0$$

$$\Rightarrow \quad \frac{d\hat{f}}{dt} = 0$$ [by defining of derivative of vector function]

Conversely : Let $\frac{d\hat{f}}{dt} = 0$, to prove $\hat{f}$ (t) is constant vector

Let $\hat{f}$ (t) = $\hat{f}_1 \hat{i} + \hat{f}_2 \hat{j} + \hat{f}_3 \hat{k}$

Now $\frac{d\hat{f}}{dt} = \frac{d\hat{f}_1}{dt}\hat{i} + \frac{d\hat{f}_2}{dt}\hat{j} + \frac{d\hat{f}_3}{dt}\hat{k}$

$$\Rightarrow \quad 0 = \frac{d\hat{f}_1}{dt}\hat{i} + \frac{d\hat{f}_2}{dt}\hat{j} + \frac{d\hat{f}_3}{dt}\hat{k}$$

Since two vector are equal is their components and direction all same

So $\quad \frac{d\hat{f}_1}{dt} = 0$, $\frac{d\hat{f}_2}{dt} = 0$, $\frac{d\hat{f}_3}{dt} = 0$

$\Rightarrow \quad d\hat{f}_1, d\hat{f}_2, d\hat{f}_3$ i.e. component of $\hat{f}$ in x, y and z directions are independent of variable t.

So $\hat{f}$ (t) is constant vector.

Theorem 3 : The necessary and sufficient condition for a vector $\hat{f}$ (t) to have constant magnitude is $\hat{f} \cdot \frac{d\hat{f}}{dt} = 0$.

Proof : Condition is necessary

Let $\hat{f}$ (t) constant magnitude, to prove

$\hat{f} \cdot \frac{d\hat{f}}{dt} = 0.$

Since $\hat{f}$ (t) has constant magnitude

$\Rightarrow \left|\hat{f}\right| =$ constant

Now, $\hat{f} \cdot \hat{f} = \left|\hat{f}\right|^2 =$ constant

$$\frac{d}{dt}(\hat{f} \cdot \hat{f}) = \frac{d\hat{f}}{dt} \cdot \hat{f} + \hat{f} \cdot \frac{d\hat{f}}{dt} = \frac{d}{dt}(\text{constant})$$

$\left[\because \frac{d}{dt}(\hat{f} \cdot \hat{g}) = \frac{d\hat{f}}{dt} \cdot \hat{g} + \hat{f} \cdot \frac{d\hat{f}}{dt}\right]$

$= \hat{f} \cdot \frac{d\hat{f}}{dt} + \hat{f} \cdot \frac{d\hat{f}}{dt} = 0$ [$\because$ derivative of constant = zero

and $\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{a}$]

$\Rightarrow 2\hat{f} \cdot \frac{d\hat{f}}{dt} = 0$

$\Rightarrow \hat{f} \cdot \frac{d\hat{f}}{dt} = 0$

Conversely : If $\hat{f} \cdot \frac{d\hat{f}}{dt} = 0$, then $\hat{f}$ has constant magnitude.

Let $\hat{f} \cdot \frac{d\hat{f}}{dt} = 0$

$\Rightarrow 2\hat{f} \cdot \frac{d\hat{f}}{dt} = 0$

$\Rightarrow \hat{f} \cdot \frac{d\hat{f}}{dt} + \hat{f} \cdot \frac{d\hat{f}}{dt} = 0$

$\Rightarrow \frac{d\hat{f}}{dt} \cdot \hat{f} + \frac{d\hat{f}}{dt} \cdot \hat{f} = 0$

$$\Rightarrow \frac{d}{dt}(\hat{f} \cdot \hat{f}) = 0$$

$$\Rightarrow (\hat{f} \cdot \hat{f}) = \text{Constant}$$

$$\Rightarrow \left|\hat{f}\right|^2 = \text{Constant}$$

$$\Rightarrow \left|\hat{f}\right| = \text{Constant}$$

Theorem 4 : The necessary and sufficient condition for a vector $\hat{f}$ (t) to have constant direction is $\hat{f} \times \frac{d\hat{f}}{dt} = 0$

Proof : Let $\hat{f}$ (t) = ϕ$\hat{F}$ where $\hat{F}$ is a vector function with modulus unity for every value of t then $\frac{d\hat{f}}{dt} = \frac{d\hat{F}}{dt} + \frac{d\hat{f}}{dt}\hat{F}$ and ϕ = ϕ(t) is magnitude of $\hat{f}$

Now $\hat{f} \times \frac{d\hat{f}}{dt} = (\phi\, \hat{f}) \times \left(\phi \frac{d\hat{F}}{dt} + \frac{d\phi}{dt}\hat{F}\right)$

$$= \phi^2 \hat{F} \times \frac{d\hat{F}}{dt} + \frac{d\hat{F}}{dt}\hat{F} \times \hat{F}$$

$$= \phi^2 \hat{F} \times \frac{d\hat{F}}{dt} + 0$$

$$\Rightarrow \quad \phi \times \frac{d\hat{F}}{dt} + = \phi^2 \hat{F} \times \frac{d\hat{F}}{dt} \qquad (1)$$

Condition is necessary : Let $\hat{f}$ (t) has a constant direction.

$\therefore \hat{F}$ is a constant vector

$$\Rightarrow \frac{d\hat{F}}{dt} = 0$$

So that from (1) $\hat{f} \times \frac{d\hat{f}}{dt} = \phi \hat{F} \times 0 = 0$

$$\Rightarrow \quad \hat{f} \times \frac{d\hat{f}}{dt} = 0$$

Condition is sufficient : Let $\hat{f} \times \frac{d\hat{f}}{dt} = 0$

then from (1) $= \phi^2 \hat{F} \times \frac{d\hat{F}}{dt} = 0$

$\Rightarrow \quad = \hat{F} \times \frac{d\hat{F}}{dt} = 0 \qquad (2)$

Since $\hat{F}$ has a constant length then

$= \hat{F} . \frac{d\hat{F}}{dt} = 0 \qquad (3)$ [by theorem 3]

From equation (2) and (3)

$$\hat{F} \times \frac{d\hat{F}}{dt} = \hat{F} . \frac{d\hat{F}}{dt} = 0$$

$\Rightarrow \quad \frac{d\hat{F}}{dt} = 0$

$\Rightarrow \hat{F}$ is a constant.

$\Rightarrow$ direction of $\hat{f}$ remain constant.

Example 4 : Show that derivative of a vector of constant magnitudes is perpendicular to the vector itself.

Solution : Let $\hat{r}$ is a given vector.

Also given $\hat{r}$ has a constant magnitude

$\Rightarrow |\hat{r}| = r =$ constant

So $|\hat{r}| = r^2$

or $\hat{r} . \hat{r} = r^2$

So differtiating both side

$\frac{d}{dt}(\hat{r} . \hat{r}) = 0$

$= \hat{r} . \frac{d\hat{r}}{dt} + \frac{d\hat{r}}{dt} . \hat{r} = 0$

$\Rightarrow 2\hat{r} . \frac{d\hat{r}}{dt} = 0$

$\Rightarrow \hat{r} \ . \ \frac{d\hat{r}}{dt} = 0$

Since dot product of two vector is zero, so the two vector are perpendicular to each other so $\hat{r} \perp \frac{d\hat{r}}{dt}$.

Example 5 : If $\hat{r}$ = $t^m \hat{a}$ + $t^n \hat{b}$ where $\hat{a}$ and $\hat{b}$ are constant vectors, then show that if $\hat{r}$ is parallel to $\frac{d\hat{r}}{dt}$ then m + n = 1 or m = n.

Solution : Given $\hat{r}$ = $t^m \hat{a}$ + $t^n \hat{b}$ (1)

So, $\frac{d\hat{r}}{dt}$ = $mt^{m-1} \hat{a}$ + $nt^{n-1} \hat{b}$ (2)

also $\frac{d^2\hat{r}}{dt^2}$ = $m(m-1)t^{m-1} \hat{a}$ + $n(n-1)t^{n-2} \hat{b}$ (3)

Given $\hat{r}$ and $\frac{d^2\hat{r}}{dt^2}$ are parallel, we know that if two vectors are parallel then we can write them as $\hat{r} = k \frac{d^2\hat{r}}{dt^2}$

$\Rightarrow$ $\hat{r}$ = k [$m(m-1)t^{m-2} \hat{a}$ + $n(n-1)t^{n-2} \hat{b}$]

$\Rightarrow$ $t^m \hat{a}$ + $t^n \hat{b}$ = k[$m(m-1)t^{m-2} \hat{a}$ + $n(n-1)t^{n-2} \hat{b}$]

Comparing the coefficients of $\hat{a}$ and $\hat{b}$, using the concept of equally of two vectors, we get

$t^m = k[m(m-1)t^{m-2}$

= $t^2 = mk(m-1) = k(m^2 - m) = km^2 - km$

$\Rightarrow$ $t^2 = k(m^2 - m)$ (4)

and $t^n = k(n(n-1))t^{n-2}$

$\Rightarrow$ $t^2 = k(n^2 - n)$ (5)

From equal (4) and (5)

$k(m^2 - m) = k(n^2 - n)$

$\Rightarrow$ $m^2 - m = n^2 - n$

$\Rightarrow$ $m^2 - n^2 - m + n = 0$

$\Rightarrow$ $(m^2 - n^2) - (m - n) = 0$

$\Rightarrow$ (m - n) (m + n) - (m - n) = 0

= (m - n) [m + n - 1] = 0

$\Rightarrow$ m - n = 0 m + n - 1 = 0

$\Rightarrow$ m - n or m + n = 1

Hence the result

Self Check Exercise - 2

Q. 1 If vector $\hat{a} = 4\hat{i} + 2\hat{j} - 3\hat{k}$ is constant

Q. 2 Show that $\hat{f} = 2t\hat{i} + 3t\hat{j} + 4t\hat{k}$ has constant direction.

Q. 3 Show that the vector $\hat{f}$ = Csos t$\hat{i}$ + Sin t $\hat{j} + \hat{k}$ has constant magnitude.

6.5 Velocity and Acceleration

Let $\hat{r}$ is the position vector of a proving particle at point p with respect to origen O. at time t. Then $\hat{r} = \hat{r}$(t). Let in small time δt the displacement of particle is given by $\delta\hat{r}$. Then $\frac{\delta\hat{r}}{\delta t}$ gives the average velocity of the particle during the time interval δt.

If we take $\hat{v}$ as velocity of particle at point P, then

$$\hat{v} = \lim_{\delta t \to 0} \frac{\delta \hat{r}}{\delta t} = \frac{d\hat{r}}{dt}$$

So, velocity $\hat{v} = \frac{d\hat{r}}{dt}$

and acceleration $\hat{a} = \frac{d\hat{v}}{dt} = \frac{d^2\hat{r}}{dt^2}$

So $\hat{v}$ and $\hat{a}$ are velocity and acceleration of moving particle having position vector $\hat{r}$, which are given by first and second order derivative of $\hat{r}$.

Example 1 : A particle is moving whose position vector point P w.r.t. origen O is giving by

$\hat{r} = 3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}$.

Find velocity and acceleration of moving particle at t = 1. Also find the magnitude of velocity and accellaration.

Solution : Given $\hat{r} = 3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}$.

In order to find velocity and acceleration, we have to find the first and second order derivatives of $\hat{r}$.

So $\hat{v} = \frac{d\hat{r}}{dt} = \frac{d}{dt}[3t^2\hat{i} + (t^2 - 2t)\hat{j} + t^3\hat{k}]$

$\hat{v} = 6t\hat{i} + (2t - 2)\hat{j} + 3t^2\hat{k}$

Velocity at t = 1 = $\underset{t=1}{\hat{v}} = 6(1)\hat{i} + [2(1) - 2]\hat{j} + 3(1)^2\hat{k}$

Magnitude of velocity $|\hat{v}| = \frac{6\hat{i}+3\hat{k}}{\sqrt{(6)^2+(3)^2}} = \sqrt{36+9} = \sqrt{45} = \sqrt[3]{5}$.

Also $\hat{a} = \frac{d^2\hat{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{r}}{dt}\right)$

$= \frac{d}{dt}(6t\hat{i} + (2t - 2)\hat{j} + 3t^2\hat{k})$

$\hat{a} \quad = 6\hat{i} + 2\hat{j} + 6t\hat{k}$

Acceleration at t = 1 = $\underset{t=1}{\hat{a}} = 6\hat{i} + 2\hat{j} + 6t\hat{k}$

Magnitude of acceleration = $|\hat{a}| = \sqrt{36+4+36} = \sqrt{76} = \sqrt[2]{19}$

Example 2 : A particle is moving along the curve $x = t^3+1$, $y = t^2$, $z = 2t + 5$, where t is time. Find the component of velocity and acceleration at t = 1 in the direction of $\hat{i} + \hat{j} + z\hat{k}$.

Solution : Let $\hat{r}$ be position vector of any point P(x, y, z) on the given curve then

$\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\Rightarrow \quad \hat{r} = (t^3+1)\hat{i} + t^2\hat{j} + (2t + 5)\hat{k}$

Velocity = $\hat{v} = \frac{d\hat{r}}{dt} = \frac{d}{dt}[(t^3+1)\hat{i} + t^2\hat{j} + (2t + 5)\hat{k}]$

$\hat{v} = 3t^2\hat{i} + 2t\hat{j} + 2\hat{k}$

Acceleration = $\hat{a} = \frac{d^2\hat{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\hat{r}}{dt}\right)$

$= \frac{d}{dt}[3t^2\hat{i} + 2t\hat{j} + 2\hat{k}]$

$\hat{a} \quad = 6t\hat{i} + 2\hat{j}$

Now velocity and acceleration at time t = 1 is

$$\hat{v}_{t=1} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

and $\hat{a}_{t=1} = 6\hat{i} + 2\hat{j}$

Let $\hat{n}$ be to unit vector in the direction of $\hat{i} + \hat{j} + 3\hat{k}$

then $\hat{n} = \dfrac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{\hat{i}+\hat{j}+3\hat{k}}}$

$$= \frac{\hat{i}+\hat{j}+3\hat{k}}{\sqrt{1+1+9}}$$

$$\hat{n} = \frac{1}{\sqrt{11}}\left(\hat{i}+\hat{j}+3\hat{k}\right)$$

So component of velocity in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ time t = 1, is

$$\hat{v}\,.\,\hat{n} = (3\hat{i}+2\hat{j}+2\hat{k})\,.\,\frac{1}{\sqrt{11}}(\hat{i}+\hat{j}+3\hat{k}) = \frac{1}{\sqrt{11}}(3+2+6) = \frac{1}{\sqrt{11}} = \sqrt{11}$$

and component of acceleration in the direction of $\hat{i} + \hat{j} + 3\hat{k}$ at time t = 1, is

$$\hat{a}\,.\,\hat{n} = (6\hat{i}+2\hat{j})\,.\,\frac{1}{\sqrt{11}}(\hat{i}+\hat{j}+3\hat{k}) = \frac{1}{\sqrt{11}}(6+2) = \frac{8}{\sqrt{11}}$$

Example 3 : A particle is moving and its position vector is given as $\hat{r}$ = Cos w t$\hat{i}$ + Sin w t $\hat{j}$. Show that the velocity $\hat{v}$ is perpendicular to $\hat{r}$ and $\hat{r} \times \hat{v}$ is a constant vector.

Solution : Given $\hat{r}$ = Cos w t$\hat{i}$ + sin w t $\hat{j}$

$$\hat{v} = \frac{d\hat{r}}{dt} = \frac{d}{dt}[\text{Cos w t}\hat{i} + \text{Sin w t}\hat{j}]$$

$$= (-\text{ w Sin t }\hat{i} + \text{w Cos w t }\hat{j})$$

Since we know that two vector are perpendicular if there dot product is zero. So to prove velocity is perpendicular to $\hat{r}$ we have to show that

$$\hat{r}\,.\,\hat{v} = \hat{r}\,.\,\frac{d\hat{r}}{dt} = 0$$

So $\hat{r} \cdot \frac{d\hat{r}}{dt}$ = [Cos w t$\hat{i}$ + Sin w t $\hat{j}$] . (- w Sin t $\hat{i}$ + w Cos w t $\hat{j}$)

= -w Cos w t Sin w t + w Sin w t Cos w t

= 0

Hence $\hat{r} \cdot \frac{d\hat{r}}{dt} = \hat{r} \cdot \hat{v} = 0$

So velocity is perpendicular to $\hat{r}$.

To show $\hat{r} \times \hat{v}$ is constant

$$\hat{r} \times \hat{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos wt & \sin wt & 0 \\ -w\sin wt & w\cos wt & 0 \end{vmatrix}$$

= $\hat{i}$ (0) + $\hat{j}$ (o)+ $\hat{k}$ (w $\cos^2$ wt + w $\sin^2$ wt)

= (w $\cos^2$ wt + w $\sin^2$ wt) $\hat{k}$

$\hat{r} \times \hat{v}$ = w $\hat{k}$ = constant

Self Check Exercise - 3

Q. 1 Find velocity and acceleration at t = 0 and t = $\pi/2$, when

$\hat{r}$ = 4 cos t $\hat{i}$ + 4 sin t $\hat{j}$ + 6t $\hat{k}$.

Also find magnitude of velocity and acceleration.

Q. 2 Determine velocity and acceleration and their magnitude t = 0 if

$\hat{r}$ = $e^{-t}\hat{i}$ + 2cos 3t $\hat{j}$ + 2 sin 3t $\hat{k}$

Q. 3 If $\hat{r}$ = Cos w t$\hat{i}$ + sin w t $\hat{j}$ where w is constant show that velocity is perpendicular to

$\hat{r}$ and $\hat{r} \times \frac{d\hat{r}}{dt}$ is a constant vector.

6.6 Summary:

Dear students, in this unit, we study that

1. If $\hat{r}$ (t) is the position vector of point P w.r.t. asigen O then $\frac{d\hat{r}}{dt}$ is a vector parallel to the tangent at point P and is the derivative of vector $\hat{r}$.

2. $\frac{d\hat{r}}{dt}$ is a vector quantity.

3. $\frac{d\hat{r}}{dt}$, $\frac{d^2\hat{r}}{dt^2}$, $\frac{d^3\hat{r}}{dt^3}$ are first, second, third derivative of $\hat{r}$.

4. Unit tangent vector is given by

$$\hat{t} = \frac{d\hat{r}}{dt} = \frac{d\hat{r}/dt}{\left|\frac{d\hat{r}}{dt}\right|}$$

5. If $\hat{r}$ = $x\hat{i} + y\hat{j} + 3\hat{k}$ then

$$\frac{d\hat{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

6. A constant vector has fixed magnitude and direction.

7. A vector has constant magnitude of $\hat{f} \cdot \frac{d\hat{f}}{dt}$ = 0

8. A vector has constant direction of $\hat{f} \times \frac{d\hat{f}}{dt}$ = 0

9. If $\hat{r}$ (t) = $x\hat{i} + y\hat{j} + z\hat{k}$ where x, y, z are function of scalar variable time t, then $\frac{d\hat{r}(t)}{dt}$ gives the velocity and $\frac{d^2\hat{r}}{dt^2}$ gives the acceleration of making particle.

6.7 Glossary

1. Differentiation : The process of finding derivative is called differentiation.
2. Velocity : The rate of change of distance of a body with respect to time is known as velocity. It is a vector quantity.
3. Acceleration : It is change in velocity over change in time.
4. Constant : a Symbol or number having a fixed value.

6.8 Answers to Self Check Exercises

Self Check Exercise - 1

Q.1 $\frac{d\hat{r}}{dt}$ = con t$\hat{i}$ - sin t $\hat{j}$ + $\hat{k}$, $\frac{d^2\hat{r}}{dt^2}$ = sin t$\hat{i}$ - con t $\hat{j}$

$\left|\frac{d\hat{r}}{dt}\right| = \sqrt{2}$, $\left|\frac{d^2\hat{r}}{dt^2}\right| = 1$

Q.2 Apply the formula and verify it using $\hat{f}.\hat{g}$

Q. 3 (t^3 sin t - $3t^2$ con t)$\hat{i}$ - (t^3 con t - $3t^2$ sin t) $\hat{j}$ +($5t^2$ sin t - 11con t - sin t) $\hat{k}$

Q. 4 $\hat{t} = \frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$

Self Check Exercise - 2

Q.1 Since $|\hat{a}| = \sqrt{16+4+9} = \sqrt{2a}$ and direction is also constant.

Q.2 $\hat{f} \times \frac{d\hat{t}}{dt}$ = 0 so $\hat{f}$ has constant direction.

Q.3 $\hat{f} \times \frac{d\hat{t}}{dt}$ = 0 so, $\hat{f}$ has constant magnitude.

Self Check Exercise - 3

Q.1 $\hat{v}$ at 0 $\quad \hat{v} = 4\hat{j}+6\hat{k}$

$\hat{a} = -4\hat{j}$

at t = $\pi/2$ $\quad \hat{v}$ = -$4\hat{j}+6\hat{k}$

$\hat{a} = -4\hat{j}$

Q.2 $\hat{v} = -\hat{j}+6\hat{k}$, $|\hat{v}| = \sqrt{37}$

$\hat{a} = \hat{i}+18\hat{j}$ $|\hat{a}| = \sqrt[5]{13}$

Q. 3 Do same as example 3

6.9 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector Calculus, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, S Chand and Company Limited.

3. J.N. Sharma and A.R. Vasishtha, Vector Calculus, Krishna Prakashan Mandir.

6.10 Terminal Questions

Q. 1. Prove that $\frac{d}{dt}\left(\hat{u}\times\frac{d\hat{u}}{dt}\right) = \hat{u}\times\frac{d^2\hat{u}}{dt^2}$

Q. 2. If $\hat{r} = \hat{a}$ sin wt + $\hat{b}$ con wt where $\hat{a}$ and $\hat{b}$ and wt are constant then show that

$\frac{d^2\hat{r}}{dt^2}$ = -w^2 $\hat{r}$ and

$\hat{r}\times\frac{d\hat{r}}{dt}$ = - $\hat{w}$ $\hat{a}\times\hat{b}$

Q. 3 If $\hat{r}$ is unit vector in the direction of $\hat{r}$ then show that

$\hat{r}\times\frac{d\hat{r}}{dt} = \frac{1}{r^2}\,\hat{r}\times\frac{d\hat{r}}{dt}$

Q. 4 If $\hat{r}$ is unit vector, then prove that

$\left|\hat{r}\times\frac{d\hat{r}}{dt}\right| = \left|\frac{d\hat{r}}{dt}\right|$

Q. 5 Find velocity and acceleration of the moving particle for which

$\hat{r}$ = 3t$^3\hat{i}$ + (t^2 - 2t) $\hat{j}$ +t$^3\hat{k}$ at t = 1 in the direction of $\hat{i}+\hat{j}-\hat{k}$ vector.

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Unit - 7

Partial Derivatives of Vector Function

Structure

7.1 Introduction

Dear student, in this unit we will study about differentiation of vector function. In the last unit we studied the differentiation of a vector function of single variable. Just like the differentiation of function of two or more variable in calculus of real function, we extend this topic to differentiation of vector function of two or more variable, which is represented by partial derivatives. So, students in this unit we will study about partial derivative of a vector function, along with its component form. We will learn to apply chain rule in vector differentiation and also learn to evaluate higher order partial derivative.

7.2 Learning Objectives

After studying this unit, students will be able to

1. define partial derivative of a vector function.
2. devaluate partial derivative of a given vector function.
3. apply chain rule in vector function.
4. calculate the higher order partial derivatives of given vector function.

7.3 Partial Derivative of Vector :

Just like the function of several variable, if $\hat{r} = \hat{f}$ (x, y, z) i.e. $\hat{r}$ is a function of three independent variables x, y and z, then partial derivative of $\hat{r}$ w.r.t. x is defined as

$$\frac{\partial \hat{r}}{\partial x} = \lim_{\delta x \to 0} \frac{\hat{f}(x+\delta x, y, z) - \hat{f}(x, y, z)}{\delta x}$$, when this Limit exists.

Similarly $\frac{\partial \hat{r}}{\partial y}$ = $\underset{\delta y \to 0}{Lim} \frac{\hat{f}(x+x, \delta y, z) - \hat{f}(x, y, z)}{\delta y}$

and $\frac{\partial \hat{r}}{\partial z}$ = $\underset{\delta z \to 0}{Lim} \frac{\hat{f}(x+x, y, \delta z) - \hat{f}(x, y, z)}{\delta z}$

Note :

In order to find the partial derivative of a vector $\hat{r}$ = $\hat{r}$ (x, y, z) w.r.t. x variable we that y and z as constant. Similarly on differentiating partially w.r.t. y, x and z are beeted as constant and when we differentiate w.r.t. z, x and y are taken as constant.

Partial Derivative in Component Form

If $\hat{r}$ = f_1(x, y, z)$\hat{i}$ +f_2(x, y, z) $\hat{j}$ + f_3(x, y, z) $\hat{k}$

then $\frac{\partial \hat{r}}{\partial x}$ = $\frac{\partial f_1}{\partial x}\hat{i}$ + $\frac{\partial f_2}{\partial x}\hat{j}$ + $\frac{\partial f_3}{\partial x}\hat{k}$

$\frac{\partial \hat{r}}{\partial y}$ = $\frac{\partial f_1}{\partial y}\hat{i}$ + $\frac{\partial f_2}{\partial y}\hat{j}$ + $\frac{\partial f_3}{\partial y}\hat{k}$

and $\frac{\partial \hat{r}}{\partial z}$ = $\frac{\partial f_1}{\partial z}\hat{i}$ + $\frac{\partial f_2}{\partial z}\hat{j}$ + $\frac{\partial f_3}{\partial z}\hat{k}$

Here the operator $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$ is known as partial derivative operator, and is pronounced as curly operator or partial derivative w.r.t. x, y, z respectively.

Chain Rule

As we studied in function of real value, if $\hat{f}$ is a function of x and y and z and y are again function of some variable t, then in order find the derivative we use chain rule. Similarly, in vector calculus chain rule is applied.

If $\hat{r}$ = $\hat{r}$ (x, y) i.e. $\hat{r}$ is a function of x and y.

and x = x(t), x is a function of t

y = y(t) y is a function of t

then to find the derivative of $\hat{r}$ w.r.t. t i.e.

$$\frac{d\hat{r}}{dt} = \frac{\partial \hat{r}}{\partial x}\frac{dx}{dt} + \frac{\partial \hat{r}}{\partial y}\frac{dy}{dt}$$

* Rules for partial derivative of vector are same as those of ordinary differentiation of scalar function.

Total Derivative

Total derivative d$\hat{r}$ of $\hat{r}$ is given as

$$d\hat{r} = \frac{\partial \hat{r}}{\partial x} dx + \frac{\partial \hat{r}}{\partial y} dy + \frac{\partial \hat{r}}{\partial z} dz.$$

Higher Order Partial Derivative

If $\hat{r} = \hat{r}$ (x, y, z)

then $\frac{\partial \hat{r}}{\partial x}$, $\frac{\partial \hat{r}}{\partial y}$, $\frac{\partial \hat{r}}{\partial z}$ are first order partial derivatives w.r.t. x, y and z. Then

$$\frac{\partial^2 \hat{r}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \hat{r}}{\partial x}\right)$$

$$\frac{\partial^2 \hat{r}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \hat{r}}{\partial y}\right)$$

$\frac{\partial^2 \hat{r}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \hat{r}}{\partial z}\right)$, are second order partial derivatives w.r.t. x, y and z.

Also $\frac{\partial}{\partial x}\left(\frac{\partial \hat{r}}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \hat{r}}{\partial x}\right) = \frac{\partial^2 \hat{r}}{\partial x \partial y} = \frac{\partial^2 \hat{r}}{\partial y \partial x}$ is do second order partial derivatives w.r.t. x and y.

Here we first differential $\hat{r}$ partially w.r.t. x and then differentiate the resulting value partially w.r.t. y and vise versa, here $\hat{r}$ is a continuous function.

To have more understanding of this concept let us do following examples :

Example 1 : If $\hat{f} = (2x^2y - x^4)\,\hat{i} + (e^{xy} - y \text{ Sin } x)\,\hat{j} + (x^2 \text{ Cos } y)\,\hat{k}$

Find $\frac{\partial \hat{f}}{\partial x}$, $\frac{\partial \hat{f}}{\partial y}$, $\frac{\partial^2 \hat{f}}{\partial x^2}$ $\frac{\partial^2 \hat{f}}{\partial y^2}$. And verify that $\frac{\partial^2 \hat{f}}{\partial x \partial y} = \frac{\partial^2 \hat{f}}{\partial y \partial x}$.

Solution : Given $\hat{f} = (2x^2y - x^4)\,\hat{i} + (e^{xy} - y \text{ Sin } x)\,\hat{j} + (x^2 \text{ Cos } y)\,\hat{k}$

$$\hat{f} = f\text{ (x, y)}$$

Then $\frac{\partial \hat{f}}{\partial x}$, will be calculated by differentiating of w.r.t. x treating y as a constant.

$$\frac{\partial \hat{f}}{\partial x} = \frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_2}{\partial x}\hat{j} + \frac{\partial f_3}{\partial x}\hat{k}$$

$$\Rightarrow \quad \frac{\partial \hat{f}}{\partial x} = \frac{\partial}{\partial x}(2x^2y - x^4)\ \hat{i} + \frac{\partial}{\partial x}\ (e^{xy} - y \text{ Sin } x)\ \hat{j} + \frac{\partial}{\partial x}\ (x^2 \text{ Cos } y)\ \hat{k}$$

$$\frac{\partial \hat{f}}{\partial x} = (4xy - 4x^3)\ \hat{i} + (ye^{xy} - y \text{ Cos } x)\ \hat{j} + (2x \text{ Cos } y)\ \hat{k}$$

Now $\quad \frac{\partial \hat{f}}{\partial y} = \frac{\partial f_1}{\partial y}\ \hat{i} + \frac{\partial f_2}{\partial y}\ \hat{j} + \frac{\partial f_3}{\partial y}\ \hat{k}$

$$\Rightarrow \quad \frac{\partial \hat{f}}{\partial y} = \frac{\partial}{\partial y}(2x^2y - x^4)\ \hat{i} + \frac{\partial}{\partial y}\ (e^{xy} - y \text{ Sin } x)\ \hat{j} + \frac{\partial}{\partial y}(x^2 \text{ Cos } y)\ \hat{k}$$

$$\Rightarrow \quad \frac{\partial \hat{f}}{\partial y} = 2x^2\hat{i} + (xe^{xy} - \text{ Sin } x)\ \hat{j} + (x^2 \sin y)\ \hat{k}$$

Now, $\frac{\partial^2 \hat{f}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \hat{f}}{\partial x}\right)$

$$= \frac{\partial}{\partial x}(4xy - 4x^3)\ \hat{i} + \frac{\partial}{\partial x}(ye^{xy} - y \text{ Cos } x)\ \hat{j} + \frac{\partial}{\partial x}(2x \text{ Cos } y)\ \hat{k}$$

$$\Rightarrow \quad \frac{\partial^2 \hat{f}}{\partial x^2} = (4y - 12x^2)\ \hat{i} + \frac{\partial}{\partial x}(y^2e^{xy} - y \sin x)\ \hat{j} + (2 \text{ Cos } y)\ \hat{k}$$

and $\frac{\partial^2 \hat{f}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \hat{f}}{\partial y}\right)$

$$= \frac{\partial}{\partial y}(2x^2)\hat{i} + \frac{\partial}{\partial y}\ (xe^{xy} - \text{ Sin } x)\ \hat{j} - \frac{\partial}{\partial y}\ (x^2 \sin y)\ \hat{k}$$

$$\frac{\partial^2 \hat{f}}{\partial y^2} = 0 + x^2e^{xy}\ \hat{j} - x^2 \cos y\ \hat{k}$$

$$\Rightarrow \quad \frac{\partial^2 \hat{f}}{\partial y^2} = (x^2e^{xy})\ \hat{j} - x^2 \cos y\ \hat{k}$$

Now $\frac{\partial^2 \hat{f}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \hat{f}}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\ [2x^2\hat{i} + (xe^{xy} - \ \text{Sin } x)\ \hat{j} - x^2 \sin y\, \hat{k}\,]$$

$$\Rightarrow \qquad \frac{\partial^2 \hat{f}}{\partial x \partial y} = 4x\ \hat{i} + (xye^{xy} - \text{Cos } x)\ \hat{j} - 2x \sin y\, \hat{k} \qquad (1)$$

Again $\dfrac{\partial^2 \hat{f}}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \hat{f}}{\partial x}\right)$

$$= \frac{\partial}{\partial y}\,[(4xy - 4x^3)\ \hat{i} + (ye^{xy} - y\ \text{Cos } x)\ \hat{j} + (2x\ \text{Cos } y)\, \hat{k}\,]$$

$$= 4x\hat{i} + (xye^{xy} - \text{Cos } x)\ \hat{j} - 2x\,(-\sin y)\, \hat{k}$$

$$\frac{\partial^2 \hat{f}}{\partial y \partial x} = 4x\hat{i} + (xye^{xy} - \text{Cos } x)\ \hat{j} - 2x \sin y\, \hat{k} \qquad (2)$$

From (1) and (2), we have

$$\frac{\partial^2 \hat{f}}{\partial x \partial y} = \frac{\partial^2 \hat{f}}{\partial y \partial x}$$

Example 2 : If $\hat{f} = x^2\hat{i} - y\ \hat{j} + xz\,\hat{k}$ and $\hat{g} = y\hat{i} + x\ \hat{j} - xyz\,\hat{k}$

Then find $\dfrac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$

Solution : Given $\hat{f} = x^2\hat{i} - y\ \hat{j} + xz\,\hat{k}$

$\hat{g} = y\hat{i} + x\ \hat{j} - xyz\,\hat{k}$

then $(\hat{f} \times \hat{g}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x^2 & -y & xz \\ y & x & -xyz \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} -y & xz \\ x & -xyz \end{vmatrix} - \hat{j}\begin{vmatrix} x^2 & xz \\ y & -xyz \end{vmatrix} + \hat{k}\begin{vmatrix} x^2 & -y \\ y & x \end{vmatrix}$$

$$\Rightarrow \qquad (\hat{f} \times \hat{g}) = \hat{i}\,(xy^2z - x^2z) - \hat{j}\,(-x^3yz - \ xyz) + \hat{k}\,(x^3 + y^2)$$

$$\Rightarrow \qquad (\hat{f} \times \hat{g}) = \hat{i}\,(xy^2z - x^2z) + \hat{j}\,(x^3yz - \ xyz) + \hat{k}\,(x^3 + y^2)$$

Now $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g}) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\hat{f} \times \hat{g})\right)$

So $\frac{\partial}{\partial y}(\hat{f} \times \hat{g}) = \frac{\partial}{\partial y}[(xy^2z - x^2z)\hat{i} + (x^3yz - xyz)\hat{j} + (x^3 + y^2)\hat{k}]$

$= \frac{\partial}{\partial y}(xy^2z - x^2z)\hat{i} + \frac{\partial}{\partial y}(x^3yz + xyz)\hat{j} + \frac{\partial}{\partial y}(x^3 + y^2)\hat{k}$

$= (2xyz)\hat{i} + (x^3yz + xz)\hat{j} + (2y)\hat{k}$

$\Rightarrow \quad \frac{\partial}{\partial y}(\hat{f} \times \hat{g}) = 2xyz\hat{i} + (x^3yz + xz)\hat{j} + (2y)\hat{k}$

Now $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g}) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\hat{f} \times \hat{g})\right)$

$= \frac{\partial}{\partial x}[(2xyz)\hat{i} + (x^3yz + xz)\hat{j} + (2y)\hat{k}]$

$= \frac{\partial}{\partial x}(2yzx)\hat{i} + \frac{\partial}{\partial x}(x^3z + xz)\hat{j} + \frac{\partial}{\partial x}(2y)\hat{k}$

$= 2yz\hat{i} + (3x^2z + z)\hat{j} + 0\hat{k}$

$\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g}) = 2yz\hat{i} + (3x^2z + z)\hat{j}$

Example 3 : If $\hat{f} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$ and $\hat{g} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

Then find $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$ at (1, 1, 1)

Solution : Given $\hat{f} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$

and $\hat{g} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$

Then $(\hat{f} \times \hat{g}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2z & y & -x^2 \\ x^2yz & -2xz^3 & xz^2 \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} y & -x^2 \\ -2xz^3 & xz^2 \end{vmatrix} - \hat{j}\begin{vmatrix} 2z & -x^2 \\ x^2yz & xz^2 \end{vmatrix} + \hat{k}\begin{vmatrix} 2z & y \\ x^2yz & -2xz^3 \end{vmatrix}$$

$$= \hat{i}\ (xyz^2 - 2x^3z^3) - \hat{j}\ (2xz^3 + x^4yz) + \hat{k}\ (-4xz^4 - x^2y^2z)$$

Now, $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g}) \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\hat{f} \times \hat{g})\right)$

So, $\frac{\partial}{\partial y}(\hat{f} \times \hat{g}) = \frac{\partial}{\partial y}[(xyz^2 - 2x^3z^3)\hat{i} - (2xz^3 + x^4yz)\,\hat{j} + (-4xz^4 - x^2y^2z)\,\hat{k}\,]$

$$\frac{\partial}{\partial y}(\hat{f} \times \hat{g}) = xz^3\hat{i} - x^4z\,\hat{j} - 2x^2yz\,\hat{k}$$

Now $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}(\hat{f} \times \hat{g})\right) = \frac{\partial}{\partial x}\ [xz^3\hat{i} - x^4z\,\hat{j} - 2x^2yz\,\hat{k}\,]$

$$= z^2\hat{i} - 4x^3z\,\hat{j} - 4xyz\,\hat{k}$$

$$\Rightarrow \frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g}) = z^2\hat{i} - 4x^3z\,\hat{j} - 4xyz\,\hat{k}$$

Now $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$ at $x_1 = 1$, $y = 1$, $z = 1$ is

$\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$ at $(1, 1, 1) = \hat{i} - 4\,\hat{j} - 4\,\hat{k}$.

Self Check Exercise

Q. 1 If $\hat{f} = x^2yz\hat{i} - 2xz^3\,\hat{j} + xz^2\,\hat{k}$

$\hat{g} = 2z\hat{i} + y\,\hat{j} - x^2\,\hat{k}$

Then find $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$ at (0, 0, 2)

Q. 2 If $\hat{f} = x^2\hat{i} + y^2\,\hat{j} + x^2\,\hat{k}$

$\hat{g} = yz\hat{i} + zx\,\hat{j} + xy\,\hat{k}$

Then find $\frac{\partial^2}{\partial x \partial y}(\hat{A} \times \hat{B})$

6.6 Summary:

Dear students, in this unit, we studied

1. The derivative of a vector function which is function of more than two variable.

2. If $\hat{f} = \hat{f}$ (x, y, z) then $\frac{\partial \hat{f}}{\partial x}$, $\frac{\partial \hat{f}}{\partial y}$ and $\frac{\partial \hat{f}}{\partial z}$ represents the partial derivative of $\hat{f}$ with respect to x, y and z. These are also known as first order partial derivative.

3. If $\hat{r} = f_1(x, y, z)\hat{i} + f_2(x, y, z)\hat{j} + f_3(x, y, z)\hat{k}$

then $\frac{\partial \hat{r}}{\partial x} = \frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_2}{\partial x}\hat{j} + \frac{\partial f_3}{\partial x}\hat{k}$

$\frac{\partial \hat{r}}{\partial y} = \frac{\partial f_1}{\partial y}\hat{i} + \frac{\partial f_2}{\partial y}\hat{j} + \frac{\partial f_3}{\partial y}\hat{k}$

and $\frac{\partial \hat{r}}{\partial z} = \frac{\partial f_1}{\partial z}\hat{i} + \frac{\partial f_2}{\partial z}\hat{j} + \frac{\partial f_3}{\partial z}\hat{k}$

4. Total derivative of vector function $\hat{r}$ is given by

$d\hat{r} = \frac{\partial \hat{r}}{\partial x}dx + \frac{\partial \hat{r}}{\partial y}dy + \frac{\partial \hat{r}}{\partial z}dz.$

7.5 Glossary

1. Function of more variable : $f = f(x, y, z)$ is a function of three variable in x, y and z.

2. Partial derivative : Derivative with respect to one variable where others variables are constant.

7.6 Answers to Self Check Exercises

Self Check Exercise - 1

Q.1 0

Q. 2 $3y^2\hat{i} - 3x^2\hat{j}$

7.7 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector Calculus, Schaum's outlines:

2. S. Narayan, and P.K. Mittal, Vector Calculus, S Chand and Company Limited.

3. J.N. Sharma and A.R. Vasishtha, Vector Calculus, Krishna Prakashan Mandir.

7.8 Terminal Questions

Q. 1. If $\hat{r} = \frac{1}{2}a(u + v)\hat{i} + \frac{1}{2}(u - v)\hat{j} + \frac{1}{2}uv\hat{k}$

Then find $\left[\frac{\partial \hat{r}}{\partial u}, \frac{\partial \hat{r}}{\partial v}, \frac{\partial^2 \hat{r}}{\partial u^2}\right]$

Q. 2 If $\hat{f} = x^2yz\hat{i} - 2xz^3\hat{j} + xz^2\hat{k}$ and $\hat{g} = 2z\hat{i} + y\hat{j} - x^2\hat{k}$

Then find the $\frac{\partial^2}{\partial x \partial y}(\hat{f} \times \hat{g})$ at (1, 0, -2)

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Unit - 8

Gradient of Scalar Point Function

Structure

8.1 Introduction

Dear student, in this unit we will study about the partial differentiation of a scalar point function, or scalar field, which is known as gradient of scalar field. We will also learn some of the application of gradient like directional derivative and angle between two surfaces.

8.2 Learning Objectives:

After studying this unit, students will be able to

(1) explain the concept of scalar field.

(2) determine the gradient of scalar field.

(3) determine the directional derivative of a scalar field.

(4) find the again between two surfaces for a given field.

8.3 Gradient of a Scalar Field

Since we know that many physical quantities have different values at different points in space. Field a function which describes a physical quality at different points in space is called a field.

Also, we know that a physical quantity either have a nature of scalar or vector, so there may be a scalar or vector field. Here in this unit we will study about scalar field.

Scalar Field:

A scalar field is a function that assigns a unique scalar to every point in a given region.

For example, density of air (in earth) is a scalar quantity that charges with altitude above sea level. The temperature distribution in a headed body, gravitational potential of an object are example of scalar field.

Since we know that every point in space can be denoted by Cartesian coordinator (x_1 y_1 z). So we can write the scalar function or scalar field as $f = f(x_1\ y_1\ z)$. This means that for every point (x, y, z) in spaces, there exists a unique scalar field given by $f(x_1\ y_1\ z)$.

Contour Lines or Curves:

Contour lines or contour curves are the lines which connect those points which are at the same height above a fixed level. So contour curves are a pictorial representation of a scalar function.

So, contour curve is curve in two dimensions on which the value of scalar field $f(x_1\ y)$ is constant.

i.e. $f(x_1\ y) = c$

In order to define scalar field in three dimensions we need to define one more quantity i.e.

Contour Surfaces:

Contour surface are the surfaces on which the value of a three dimensional scalar field is constant. So, if a scalar field is defined by the function $f(x_1\ y_1\ z)$, so contour surface would be the collection of all those points (x_1 y_1 z) for which the value of f is constant say c. So contour surface is defined as $f(x_1\ y_1\ z) = c$.

Gradient of A scalar Field:

Since you know that slope of a function is related to the rate of change. When a quantity or function is dependent more than one variable then we use the concept of partial derivative for rate of change.

Let and Q are two neighboring points in a region in which scalar field $f(x_1\ y)$ is defined. The coordinate of point P are (x_1 y) and that of Q are ($x + \Delta x_1\ y+\Delta y$) respectively. $\hat{r}$ and $\hat{r}+\Delta\hat{r}$ represents the position vectors of the points P and Q. Then the change in $f(x, y)$ as one goes Q small distance from point P to Q is given by

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x_1\ y) \qquad ...(1)$$

adding and subtracting $f(x_1\ y + \Delta y)$ in (1) we get

$$\Delta f = \left[f\left(x+\Delta x, y+\Delta y\right) - f\left(x_1 y+\Delta y\right)\right] + \left[f\left(x_1\ y+\Delta y\right) - f\left(x_1\ y\right)\right] \qquad ...(2)$$

Multiplying and dividing R.H.S. by Δx and Δy respectively, we get

$$\Delta f = \frac{\left[f\left(x+\Delta x, y+\Delta y\right)-f\left(x_1 y+\Delta y\right)\right]}{\Delta x}\,\Delta x + \frac{\left[f\left(x_1 y+\Delta y\right)-f\left(x_1\, y\right)\right]\Delta y}{\Delta y} \quad ...(3)$$

taking limits of $\Delta x \rightarrow$ and $\Delta y \rightarrow 0$,

$$\underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\Delta f = \underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\frac{\left[f\left(x+\Delta x, y+\Delta y\right)-f\left(x_1 y+\Delta y\right)\right]}{\Delta x}\,\Delta x$$

$$+\underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\frac{\left[f\left(x_1 y+\Delta y\right)-f\left(x_1\, y\right)\right]\Delta y}{\Delta y}$$

Using the definition of partial derivative, we get

$$\underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\Delta f = \underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\frac{\partial f\left(x_1 y+\Delta y\right)}{\partial x}\,dx + \underset{\substack{\Delta x\rightarrow 0\\ \Delta y\rightarrow 0}}{Lim}\,\frac{\partial f\left(x_1 y+\Delta y\right)}{\partial y}\,dy$$

$$= \frac{\partial f\left(x_1 y\right)}{\partial x}\,dx + \frac{\partial f\left(x_1 y\right)}{\partial x}\,dy$$

$$\Rightarrow \qquad df = \frac{\partial f\left(x_1 y\right)}{\partial x}\,dx + \frac{\partial f\left(x_1 y\right)}{\partial x}\,dy \qquad ...(4)$$

Here df is known as total differential of f. So total differential of a two dimensional scalar field $f(x_1y)$ is

$$df = \frac{\partial f\left(x_1 y\right)}{\partial x}\,dx + \frac{\partial f\left(x_1 y\right)}{\partial x}\,dy$$

on generalizing the above result to three dimensional scalar field $f(x_1y)$, the total differential is given by

$$df = \frac{\partial f\left(x_1 y_{1z}\right)}{\partial x}\,dx + \frac{\partial f\left(x_1 y_{1z}\right)}{\partial x}\,dy + \frac{\partial f\left(x_1 y_{1z}\right)}{\partial x}\,dz \qquad ...(5)$$

Rewriting equation (5) as

$$df = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k}\right) . (dx\,\hat{i} + dy\,\hat{j} + dx\,\hat{k}) \qquad ...(6)$$

Where $dx\,\hat{i} + dy\,\hat{j} + dx\,\hat{k} = \overrightarrow{dr}$, ...(7)

is the change in position vector.

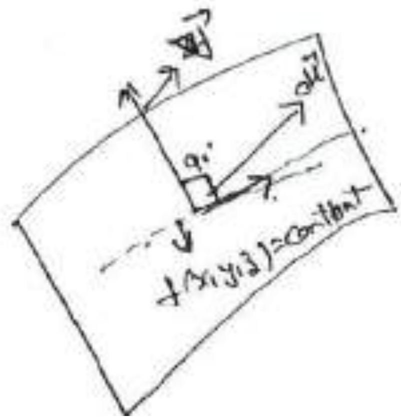
and the vector, $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k}$ = grad f = $\hat{\nabla} f$ ...(8)

is define as gradient of scalar field $f(x, y, z)$.

So $df = (\hat{\nabla} f) . \vec{dr}$ ...(9)

Where symbol $\hat{\nabla}$ is pronounced as 'del' and $(\hat{\nabla} f)$ as gradient of f.

From equation (8), we can say that gradient of a scalar field is a vector.



Physical Interpretation of Gradient of Scalar Field

Since $df = (\hat{\nabla} f) . \vec{dr}$

Let of the angle between $\hat{\nabla} f$ and $\vec{dr}$ is given by Θ.

Then $df = |\hat{\nabla} f| \, |\vec{dr}| \cos \Theta$

$df = |\hat{\nabla} f| \, dr \cos \Theta$ ...(10)

Following cases are true.

1. When $\theta = 90^0$ i.e. $\hat{\nabla} f$ is perpendicular to $\vec{dr}$

$df = |\hat{\nabla} f| \, dr \cos 90^0$

$= 0$

$\Rightarrow$ $df = 0$

$\Rightarrow$ f is constant

$\Rightarrow$ Value of scalar field is constant along the direction perpendicular to its gradient.

Note 1 :

$\Rightarrow$ The vector $\hat{\nabla} f$ is perpendicular (normal) to the surface $f(x, y, z)$ = constant.

2. When Θ = O, let us keep dr constant and find the change df in various directions by changing Θ. Then the maximum change will occurs when Θ = O.

i.e. df = $|\hat{\nabla} f|$ dr cos Θ

Dividing both sides by dr, we get

$$\frac{df}{dr} = |\hat{\nabla} f| \cos \Theta$$

$$\left(\frac{df}{dr}\right)_{\max} = |\hat{\nabla} f| \cos O = |\hat{\nabla} f| \quad ...(11)$$

Then maximum rate of increase of the scalar field f is along the direction of the gradient of the field $\hat{\nabla} f$ and its magnitude is given by equation (11).

Note 2 :

The magnitude of $\hat{\nabla} f$ gives us the maximum rate of change of the scalar field in space.

3. When Θ = 180^0 : When $\overrightarrow{dr}$ is in the direction opposite to $\hat{\nabla} f$ then Θ = 180^0, in that case

$$\frac{df}{dr} = |\hat{\nabla} f| \cos 1800$$

$$\frac{df}{dr} = -|\hat{\nabla} f| \quad ...(12)$$

Then this is the direction in which the rate of decrease of the field is maximum.

Note 3 :

Maximum rate of decrease of scalar field f is given by equation (12)

Theorem 1 : The necessary and sufficient condition for a scalar field f to be constant is $\hat{\nabla} f = 0$

Proof : Condition is necessary :-

Let f is a constant function,

$$\Rightarrow \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} = 0 \qquad \text{[by definition of constant function]}$$

Now, $\hat{\nabla} f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k}\right) f$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k}$$

$\Rightarrow \quad \hat{\nabla} f = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k} = \hat{0}$$

as two vectors are equal, is its components are equal

$\Rightarrow \ \frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ \frac{\partial f}{\partial z} = 0$

$\Rightarrow f$ is independent of x, y and z

Hence f is a constant function.

Hence the proof.

Let us take some examples to calculate gradient of a scalar field.

Example 1 : If $f(x, y, z) = x^2y + y^2x + z^2$ find $\hat{\nabla} f$ at (1, 2, 3)

Solution : Given $f(x, y, z) = x^2y + y^2x + z^2$

We know that $\hat{\nabla} = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$

So $\hat{\nabla} f = \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y} + \hat{j}\frac{\partial f}{\partial z}\hat{k}\right)$ $(x^2y + y^2x + z^2)$

$= \frac{\partial f}{\partial x}(x^2y + y^2x + z^2)\ \hat{i} + \frac{\partial f}{\partial y}(x^2y + y^2x + z^2)\ \hat{j} + \frac{\partial f}{\partial z}(x^2y + y^2x + z^2)\ \hat{k}$

$\hat{\nabla} f \quad = (2xy + y^2)\ \hat{i} + (x^2 + 2xy)\ \hat{j} + 2z\ \hat{k}$

Now, taking x = 1, y = 2, z = 3 in above, we get

So $\left(\hat{\nabla} f\right)_{1,2,3} = (2\times1\times2 + (2)^2)\ \hat{i} + ((1)^2 + 2\times1\times2)\ \hat{j} + (2\times3)\ \hat{k}$

$\left(\hat{\nabla} f\right)_{1,2,3} = 8\ \hat{i} + 5\ \hat{j} + 6\ \hat{k}$

Example 2 : If $f = 2z^2 - x^3y$, find $\hat{\nabla} f$ and $\hat{a} \times \hat{\nabla} f$ where $\hat{a} = 2x^2\ \hat{i} - 3yz\ \hat{j} + xz^2\ \hat{k}$ at (1, -1, 1)

Solution : Given $f = 2z^2 - x^3y$

Now $\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$

Now, $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2z^2 - x^3y) = -3x^2y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2z^2 - x^3y) = -x^3$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(2z^2 - x^3y) = 4z$

$$\hat{\nabla} f = -3x^2y\,\hat{i} - x^3\,\hat{j} + 4z\,\hat{k}$$

Now $\hat{a} \times \hat{\nabla} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 4z \end{vmatrix}$

$$= \hat{i}\begin{vmatrix} -3yz & xz^2 \\ -x^3 & 4z \end{vmatrix} - \hat{j}\begin{vmatrix} 2x^2 & xz^2 \\ -3x^2y & 4z \end{vmatrix} + \hat{k}\begin{vmatrix} 2x^2 & -3yz \\ -3x^2y & -x^3 \end{vmatrix}$$

$$\Rightarrow \quad \hat{a}\times\hat{\nabla}f\ (-12yz^2 + x^4z^2)\,\hat{i} - (8x^2z + 3x^3yz^2)\,\hat{j} + (-2x^5 - 9x^2y^2z) + \hat{k}$$

Now $\hat{a}\times\hat{\nabla}f$ at x = 1, y = -1, z = 1

$$(\hat{a}\times\hat{\nabla}f)\left(_{1_11_1}\right) = \left[-2(-1)^2(1)^2+(1)^4(1)^2\right]\hat{i} - \left[8(1)^2 1+3(1)^3(-1)(1)^2\right]\hat{j}$$

$$+ \left[-2(1)^5-9(1)^2(1)\right]\hat{k}$$

$$= (12 + 1)\,\hat{i} - (8 - 3)\,\hat{j} + (-2-9)\,\hat{k}$$

$$\Rightarrow \quad (\hat{a}\times\hat{\nabla}f)_{(1,-1,1)} = 13\,\hat{i} - 5\,\hat{j} - 11\,\hat{k}.$$

Example 3: Find Unit vector normal to the surface $f\,(x_1y_1z)$

$= x^4 - 3xyz + z^2 + 1 = 0$ at point $(1_11_1\ 1)$.

Solution: Since we know that $\hat{\nabla}f$ is a vector normal (perpendicular) to the surface $f(x_1y_1z)$ = constant So, in order to find, unit vector normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$, we have to find vector $\hat{\nabla}f$ and its unit vector.

So, given $f = x^4 - 3xyz + z^2 + 1$

$$\hat{\nabla}f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$$

Now $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(x^4 - 3xyz + z^2 + 1) = -3xy$

$\frac{\partial}{\partial x}$ = x^4 - 3xyz

$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}$ (x^4 - 3xyz + z^2 + 1) = -3xy + 2z

$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}$ (x^4 - 3xyz + z^2 + 1) = -3xy + 2z

$\Rightarrow$ $\hat{\nabla} f$ = ($4x^3$ - 3yz) $\hat{i}$ + (-3xy) $\hat{j}$ + (-3xy + 2z) $\hat{k}$

Now $\hat{\nabla} f$ at point ($1_1 1_1$ 1) i.e. at x = 1_1 y = 1_1 z = 1

$\Rightarrow$ $\left(\hat{\nabla} f\right)_{1_1 1_1 1}$ = (4 - 3) $\hat{i}$ + (-3) $\hat{j}$ + (-3+2) $\hat{k}$

$\Rightarrow$ $\left(\hat{\nabla} f\right)_{1_1 1_1 1}$ = $\hat{i}$ - 3 $\hat{j}$ - $\hat{k}$

So, $\left(\hat{\nabla} f\right)_{1_1 1_1 1}$ = $\hat{i}$ - 3 $\hat{j}$ - $\hat{k}$ is the vector which normal to the surface x^4-3xy3 + 3^2 + 1.

Now, Unit vector normal to given surface is $\hat{n} = \frac{\hat{\nabla} f}{\left|\hat{\nabla} f\right|}$ $\left[\because \hat{x} = \frac{\hat{x}}{|\hat{x}|}\right]$

$$= \frac{\hat{i} - 3\hat{j} - \hat{k}}{\left(\hat{i}\right)^2 + (-3)^2 + (-1)^2} = \frac{\hat{i} - 3\hat{j} - \hat{k}}{\sqrt{1+9+1}} = \frac{1}{\sqrt{11}}\left(\hat{i} - 3\hat{j} - \hat{k}\right)$$

Hence Unit vector normal to given surface is $\hat{n} = \frac{1}{\sqrt{11}}\left(\hat{i} - 3\hat{j} - \hat{k}\right)$.

Example 4: Find unit vector normal to surface x^2-y^2+z = 2 at point (1, -1, 2)

Solution: Given surface is f(x, y, z) = x^2- y^2+ z - 2, as equation of surface is

f (x, y, z) = 0.

So $\hat{\nabla} f$ = $\frac{\partial}{\partial x}$ $\hat{i}$ + $\frac{\partial f}{\partial y}$ $\hat{j}$ + $\frac{\partial f}{\partial z}$ $\hat{k}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - y^2 + z - 2) = 2x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2 + z - 2) = -2y$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x^2 - y^2 + z - 2) = 1$$

$\therefore \quad \hat{\nabla} f = 2x\,\hat{i} - 2y\,\hat{j} + \hat{k}$

Now $\left(\hat{\nabla} f\right)$ at 1, -1, 2 i.e. at $x = 1_1$ $y = -1_1$ $z = 2$ is

$$\left(\hat{\nabla} f\right)_{(1_1-1_12)} = 2.1\,\hat{i} - 2(-1)\,\hat{j} + \hat{k}$$

$$\left(\hat{\nabla} f\right)_{(1_1-1_12)} = 2\,\hat{i} + 2\,\hat{j} + \hat{k}$$

Now Unit vector normal to surface $x^2 - y^2 + z = 2$ at point $(1_1 - 1_1\ 2)$ is

$$\hat{n} = \frac{\hat{\nabla} f}{\left|\hat{\nabla} f\right|}$$

$$= \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{(2)^2 + (2)^2 + 1}}$$

$$= \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4 + 4 + 1}}$$

$$= \frac{1}{\sqrt{9}}\,(2\hat{i} + 2\hat{j} + \hat{k})$$

$$\hat{n} = \frac{1}{3}$$

Example 5 : The height of a hill is given by $f = 50 - x^2y^2$. Calculate the maximum rate of change in the height of the hill at the point (1, 2). What is its direction?

Solution: Since we know that maximum rate of change of scalar field is given by the magnitude of $\hat{\nabla} f$ i.e. $\left|\hat{\nabla} f\right|$.

So, Here given $f = 50 - x^2y^2$

Now, $\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j}$

$\frac{\partial f}{\partial x}$ = -2xy^2

$\frac{\partial f}{\partial y}$ = -2x^{2}y

$\therefore$ $\hat{\nabla} f$ = 2xy^2 $\hat{i}$ - 2x^{2}y $\hat{j}$

$\hat{\nabla} f$ at point (1, 2) i.e. at x = 1, y = 2

$\left(\hat{\nabla} f\right)_{(1_1 2)}$ = -2.1.(2)^2 $\hat{i}$ - (2.1.2) $\hat{j}$

= - 8 $\hat{i}$ - 4 $\hat{j}$

Now, maximum rate of change in f = $\left|\hat{\nabla} f\right|$

= |- 8 $\hat{i}$ - 4 $\hat{j}$ |

= $\sqrt{(-8)^2 + (-4)^2}$

= $\sqrt{64+16}$

= $\sqrt{80}$

= $4\sqrt{5}$

$\therefore$ maximum rate of change in f is $4\sqrt{5}$

Also the direction in which maximum rate of change of height is along the gradient of f at (1, 2) and is given by $\left(\hat{\nabla} f\right)_{(1_1 2)}$ = - 8 $\hat{i}$ - 4 $\hat{j}$

Example 6: What is maximum rate of increase of f = x^2 + yz^2 at (1_1-1_1 3)?

Solution: Since f = x^2 + yz^2

Since maximum rate of increase of f is given by $\left|\hat{\nabla} f\right|$ so.

$\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$

$\hat{\nabla} f$ = 2x $\hat{i}$ + 3^2 $\hat{j}$ + 2yz $\hat{k}$

Now $\hat{\nabla} f$ at $(1_1\text{-}1_1\ 3)$ i.e. at x = 1_1 y - -1_1 z = 3 is

$$\left(\hat{\nabla} f\right)_{(1_1-1_1 3)} = 2x\,\hat{i} + 9\,\hat{j} + 6\,\hat{k}$$

Now maximum rate of increase of f at $(1_1\text{-}1_1\ 3)$ = $\left|\left(\hat{\nabla} f\right)_{(1_1-1_1 3)}\right|$

$= |2\,\hat{i} + 9\,\hat{j} - 6\,\hat{k}\,|$

$= \sqrt{(2)^2 + (9)^2} + (-6)^2$

$= \sqrt{4+81+36}$

$= \sqrt{121}$

= 11

Self Check Exercise - 1

Q.1 Find $\hat{\nabla} f$ where $f = 3x^2y - y^3z^2$ at $(1_1\ \text{-}2_1\ \text{-}1)$

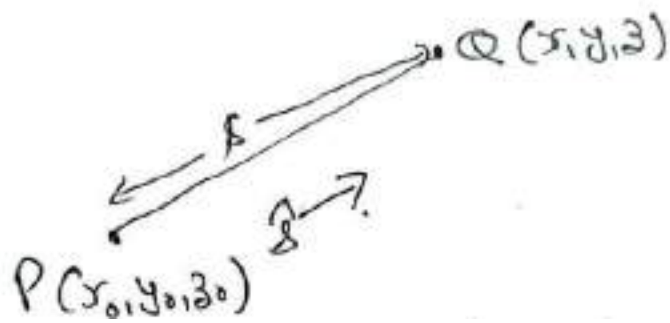
Q.2 Find normal to the surface $2xy^2 - 3xy - 4x = 7$ at $(1_1\ \text{-}1_1\ 2)$

Q.3 What is the greatest rate of increase of $f = xyz^2$ at point $(1_1\ 0_1\ 3)$

8.4 Directional Derivative:

Whenever we find the rate of change of f(x, y, z) with distance S, at a given point P(x, y, z) in the field, then this rate of change is known as directional derivative of a function with distance S.

Let P and Q be two neighboring point on a field. Such that P = P(x_0,y_0, z_0) and Q = Q(x, y, z) Let the starting point is P and in order to reach point Q the distance cover is s($\geq$,0) along the direction of unit vector $\hat{s}$.



Here $\hat{s}$ is unit vector in the direction in which the rate of change of the scalar field is to calculated.

So $\hat{s}$ = a $\hat{i}$ + b $\hat{j}$ + c $\hat{k}$ (1)

So the small displacement $\overrightarrow{PQ}$ = s $\hat{s}$ is written as:

$\hat{s}$ = (x - x_0) $\hat{i}$ + (y-y_0) $\hat{j}$ +)z-z_0) $\hat{k}$ (2)

also in term of unit vector $\hat{s}$ can be written as

$\hat{s}$ = sa $\hat{i}$ + sb $\hat{j}$ + sc $\hat{k}$ (3)

so from equ. (2) and (3)

sa = x - x_0, y - y_0 = sb, z-z_0 = sc

$\Rightarrow$ x = x_0 + sa, y s= y_0 + sb, z = z_0 + sc (4)

Equation (4) represent that the variables x, y, and z are all now the function of single variable s. The equations represented by equation (4) are known as parametric equation of line PQ which passes through the point ($x_{01}y_{01}z_0$) and (x, y, z) having parameter 8.

Since, x, y, z = x, y, z (s), So f(x, y, z) = f(s)

So directional derivative $\frac{df}{ds}$ can be calculated by using chain rule, as

$$\Rightarrow \quad \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Since, from equ. (4), x, y, z are only function of s, so

$$\frac{\partial x}{\partial s} = \frac{dx}{ds}, \ \frac{\partial y}{\partial s} = \frac{dy}{ds}, \ \frac{\partial z}{\partial s} = \frac{dz}{ds}$$

Hence above equation can be written as

$$\Rightarrow \quad \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s} \tag{5}$$

Also, from equ. (4), we have

$$\frac{dx}{ds} = a, \ \frac{dy}{ds} = b, \ \frac{\partial z}{\partial s} = c \tag{6}$$

Therefore, using (6) equ (5) becomes

$$\frac{df}{ds} = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z} \tag{7}$$

Since, we know that gradient of scalar field f is given by

$$\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$$

Now also $\hat{\nabla} f\ \ \hat{s}\ = \left(\frac{\partial f}{\partial x}\hat{i}\,\frac{\partial f}{\partial y}\,\hat{j}+\frac{\partial f}{\partial z}\hat{k}\right).\ \left(a\hat{i}+b\hat{j}+c\hat{k}\right)$

$$\hat{\nabla} f\ \ \hat{s}\ = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z} \qquad (8)$$

So from equ (7) and (8) we have

$$\frac{df}{ds} = \hat{\nabla} f\ .\ \hat{s} \qquad (9)$$

i.e. **directional derivative** of a scalar field at a point in a given direction is the scalar (dot) product of the gradient of the scalar field at that point and the unit vector along the given direction.

Note:

If the direction is specified by any given vector $\hat{s}$ then directional derivative will be

$$\frac{df}{ds} = \hat{\nabla} f\ .\ \frac{\hat{s}}{|\hat{s}|} \qquad (10)$$

* Directional derivative of a scalar field f is the projection of $\hat{\nabla} f$ on $\hat{s}$.

Now, Let us try following examples for calculating directional derivative of a scalar field.

Example 1: Find the directional derivative of f=xy + yz + zx in the direction of vector $\hat{i}+2\hat{j}+2\hat{k}$ at the point (1₁2₁0)

Solution: Given f(x, y, z) = xy + yz + zx

Now, $\frac{\partial f}{\partial x}$ = y + z

$\frac{\partial f}{\partial y}$ = x + z

$\frac{\partial f}{\partial z}$ = y + x

So $\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$

$\Rightarrow$ $\hat{\nabla} f$ = (y+z) $\hat{i}$ + (x+z) $\hat{j}$ + (y+x) $\hat{k}$

Now, $\hat{\nabla} f$ at point (1, 2, 0) i.e. x = 1, y = 2, z = 0 is

$$\left(\nabla f\right)_{(1,2,0)} = 2\hat{i} + \hat{j} + 3\hat{k}$$

Now, we have to find directional derivative in the direction of vector $\hat{s} = \hat{i} + 2\hat{j} + 2\hat{k}$

So, $$\hat{s} = \frac{\hat{s}}{|\hat{s}|} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (2)^2 + (2)^2}}$$

$$\Rightarrow \quad \hat{s} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

Since we know that directional derivative is

$$\frac{df}{ds} = \hat{\nabla} f \,.\, \hat{s}$$

$$= \left(2\hat{i} + \hat{j} + 3\hat{k}\right).\ \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\Rightarrow \quad \frac{df}{ds} = \frac{1}{3}(2 + 2 + 6) = \frac{10}{3}$$

Example 2: Find the directional derivative of f(x, y, z) = xy2 + yz2 at the point (2_1 -1_1 1) in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution: Given $f(x, y, z) = xy^2 + yz^2$

$$\frac{\partial f}{\partial x} = y^2$$

$$\frac{\partial f}{\partial y} = 2xy + z^3$$

$$\frac{\partial f}{\partial z} = 3yz$$

Now $$\hat{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\Rightarrow \quad \hat{\nabla} f = y^2\,\hat{i} + (2xy + z^3)\,\hat{j} + (3yz)\,\hat{k}$$

Now $$\left(\hat{\nabla} f\right)_{(2,-1,1)} = (-1)^2\,\hat{i} + (-3+1)\,\hat{j} + (-3)\,\hat{k}$$

$$= \hat{i} - 3j - 3\hat{k}$$

$\Rightarrow \qquad \left(\hat{\nabla} f\right)_{(2_1-1_11)} = \hat{i} - 3j - 3\hat{k}$

Also, given $\vec{s}\ \hat{i} + 2\hat{j} + 2\hat{k}$

$$\hat{s} = \frac{\hat{s}}{|\hat{s}|}$$

$$\hat{s} = \frac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{9}}$$

$\Rightarrow \qquad \hat{s} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

$\therefore$ directional derivative is

$$\frac{df}{ds} = \hat{\nabla} f \ . \ \hat{s}$$

$$= (\hat{i} - 3j - 3\hat{k}).\frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$= \frac{1}{3}[1 - 6 - 6]$$

$\Rightarrow \qquad \frac{df}{ds} = \frac{-11}{3}$

Example 3: Find the directional derivative for $f = x^2y^3z^4$ at (2, 3, -1) in the direction making equal angles with x, y, and z axis.

Solution: Given $f = x^2y^3z^4$

$$\frac{\partial f}{\partial x} = 2x\, y^3z^4$$

$$\frac{\partial f}{\partial y} = 3x^2y^2\, z^4$$

$$\frac{\partial f}{\partial z} = 4x^2y^3z^3$$

Now $\hat{\nabla} f = \frac{\partial f}{\partial x}\ \hat{i} + \frac{\partial f}{\partial y}\ \hat{j} + \frac{\partial f}{\partial z}\ \hat{k}$

$$\hat{\nabla} f = (2xy^3z^4)\ \hat{i} + (3x^2y^2 z^4)\ \hat{j} + (4x^2y^3z^3)\ \hat{k}$$

Now, $\left(\hat{\nabla} f\right)_{(2,3,-1)}$ = 108 $\hat{i}$ + 108 $\hat{j}$ + (-432) $\hat{k}$

$\Rightarrow$ $\left(\hat{\nabla} f\right)_{(2,3,-1)}$ = 108 $\hat{i}$ + 108 $\hat{j}$ - 432 $\hat{k}$

Since directional derivative is

$$\frac{df}{ds} = \left(\nabla f\right).\,\hat{s}$$

But Here vector is not given, we have to find the unit vector $\hat{s}$ which is in the direction making on equal angle (v) with x_1y and z axis. Here we will use the concept of direction cosines. So

$$\hat{s} = \cos\,\hat{i} + \cos\,\hat{j} + \cos\,\hat{k}$$

So $\quad \cos 2 \propto + \cos 2 \propto + \cos^2 \propto = 1$

$$\Rightarrow 3\cos^2 \propto = 1$$

$$= \cos^2 \propto = \frac{1}{3}$$

$$\Rightarrow \cos^2 \propto = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \hat{s} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

So the required directional derivative is

$$\frac{df}{ds} = \left(\nabla f\right).\,\hat{s}$$

$$= (108\,\hat{i} + 108\,\hat{j} - 432\,\hat{k})\,.\,\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$$

$$= \frac{1}{\sqrt{3}}(108+108-432)$$

$$\Rightarrow \frac{df}{ds} = \frac{1}{\sqrt{3}}(-216)$$

Example 4 : Find the directional derivative for $f = \frac{y}{x^2 + y^2}$ at (0, 1) in the direction making an angle 30^0 with positive x - axis.

Solution : Given $f = \dfrac{y}{x^2+y^2}$

$$\frac{\partial f}{\partial x} = \frac{\left(x^2+y^2\right).0 - y(2x)}{\left(x^2+y^2\right)^2} = \frac{-2xy}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\left(x^2+y^2\right)1 - y(2y)}{\left(x^2+y^2\right)^2} = \frac{x^2+y^2-y^2}{\left(x^2+y^2\right)^2} = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}$$

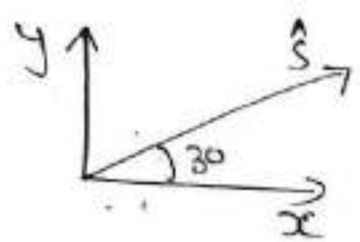
$$\frac{\partial f}{\partial z} = 0$$

So $\quad \hat{\nabla} f = \dfrac{\partial f}{\partial x}\,\hat{i} + \dfrac{\partial f}{\partial y}\,\hat{j} + \dfrac{\partial f}{\partial z}\,\hat{k}$

$$\Rightarrow \quad \hat{\nabla} f = \frac{-2xy}{\left(x^2+y^2\right)^2}\,\hat{i} + \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\,\hat{j}$$

Now $\quad \left(\hat{\nabla} f\right)_{(0\,1\,1)} = \dfrac{-1}{1}\,\hat{j} = -\,\hat{j}$

Since, we have to find the directional derivative along the direction making on angle 30^0 with +ve x axis.



So using the concept of direction cosine, the direction cosine of this vector are cos 30^0, cos 60^0

So $\quad \hat{s} = (\cos 30^0)\,\hat{i} + (\cos 60^0)\,\hat{j}$

$$\hat{s} = \frac{\sqrt{3}}{2}\,\hat{i} + \frac{1}{2}\,\hat{j} = \frac{1}{2}(\sqrt{3}\,\hat{i} + \hat{j})$$

So, the required direction is $\hat{s} = \dfrac{\sqrt{3}}{2}\,\hat{i} + \dfrac{1}{2}\,\hat{j}$

Hence directional derivative is

$$\frac{df}{ds} = \left(\hat{\nabla} f\right).\ \hat{s}$$

$$= -\ \hat{j}\ .\ \frac{1}{2}(\sqrt{3}\ \hat{i} + \hat{j})$$

$$= \frac{1}{2}(-1)$$

$$\frac{df}{ds} = .\frac{1}{2}$$

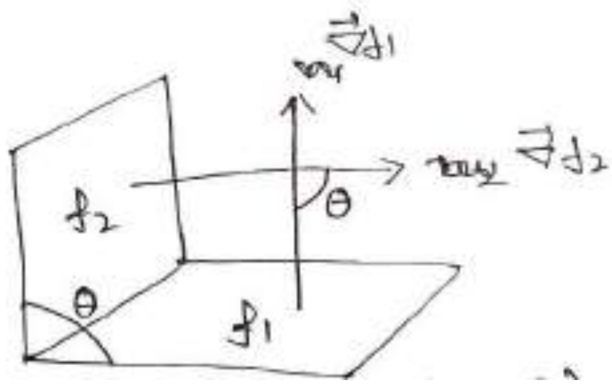
Self Check Exercise - 2

Q. 1 Find the directional derivative of $f(x, y, z) = x^2yz + 4xz^2$ at the point (1, -2, 1) in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Q. 2 Find the directional derivative of $f(x, y, z) = x^2 - 2y^2 + 4z^2$ at the point (1, -1, 1) in the direction $2\hat{i} + \hat{j} - 2\hat{k}$.

Q. 3 Find the directional derivative of $f = 2x^3 - 3yz$ at the point (2, 1, 3) in the direction parallel to a line whose direction cosines are proportional to 2, 1, 2.

8.5 Angle of intersection between to surfaces



Let f_1 (x, y, z) and f_2 (x, y, z) be two surfaces and n_1 and n_2 are normal to surfaces f_1 and f_2 respectively. Then the angle of intersection between two surfaces is equal to angel between the normals to the surfaces at the point of intersection. Since we know that normal to any surface is given by its gradient. So, $\hat{\nabla} f_1$ is the normal to the f_1 (x, y, z) and $\hat{\nabla} f_2$ is the normal to surface f_2 (x, y, z), so angle of intersection between two surfaces is angle between $\hat{\nabla} f_1$ and $\hat{\nabla} f_2$ at point of intersection and is given by

$$\text{Cos } \Theta = \frac{(\hat{\nabla} f_1).(\hat{\nabla} f_2)}{|\hat{\nabla} f_1||\hat{\nabla} f_2|}$$

Let us trey this concept, to find angle between two surfaces, using some examples.

Example 1 : Find angel of intersection between the surfaces $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 - z = 3$ at the point (2, -1, 2).

Solution : Given surface are $f_1 = x^2 + y^2 + z^2 = 9$ and

$f_2 = x^2 + y^2 - z = 3$

Now, we have to find $\hat{\nabla} f_1, \hat{\nabla} f_2$ and $|\hat{\nabla} f_1|, |\hat{\nabla} f_2|$

Now $\hat{\nabla} f_1 = \frac{\partial f_1}{\partial x}\hat{i} + \frac{\partial f_1}{\partial y}\hat{j} + \frac{\partial f_1}{\partial z}\hat{k}$

$\hat{\nabla} f_1 = 2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}$

$(\hat{\nabla} f_1)_{2,-1,2} = 4\,\hat{i} - 2\,\hat{j} + 4\,\hat{k}$

Now, $|\hat{\nabla} f_1| = \sqrt{16+4+16} = \sqrt{36} = 6$

Again $\hat{\nabla} f_2 = \frac{\partial f_2}{\partial x}\hat{i} + \frac{\partial f_2}{\partial y}\hat{j} + \frac{\partial f_2}{\partial z}\hat{k}$

$\hat{\nabla} f_2 = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$(\hat{\nabla} f_2)_{2,-1,2} = 4\,\hat{i} - 2\,\hat{j} - \hat{k}$

$|\hat{\nabla} f_2| = \sqrt{16+4+1} = \sqrt{21}$

If Θ be angle of intersection between two surfaces then

$$\text{Cos } \Theta = \frac{\hat{\nabla} f_1.\hat{\nabla} f_2}{|\hat{\nabla} f_1||\hat{\nabla} f_2|}$$

$$\Rightarrow \quad \text{Cos } \Theta = \frac{(4\hat{i} + 2\hat{j} + 4\hat{k}).(4\hat{i} - 2\hat{j} - \hat{k})}{6.\sqrt{21}}$$

$$= \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\text{Cos}\,\Theta = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \quad \Theta = \text{Cos}^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Hence angle between two given surface is $\text{Cos}^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.

Example 2 : Find angel between the surfaces $xy^2z - 3x - z^2 = 0$ and $3x^2 - y^2 + 2z = 1$ at the point (1, -2, 1).

Solution : Given surface are $f_1 = xy^2z - 3x - z^2 = 0$ and

$$f_2 = 3x^2 - y^2 + 2z = 1$$

We have to find $\hat{\nabla}f_1, \hat{\nabla}f_2$ and $|\hat{\nabla}f_1|, |\hat{\nabla}f_2|$

So $\quad \hat{\nabla}f_1 = \frac{\partial f_1}{\partial x}\,\hat{i} + \frac{\partial f_1}{\partial y}\,\hat{j} + \frac{\partial f_2}{\partial z}\,\hat{k}$

$$\hat{\nabla}f_1 = (y^2z - 3)\,\hat{i} + 2xyz\,\hat{j} + (xy^2 - 2z)\,\hat{k}$$

$$\left(\hat{\nabla}f_1\right)_{(1,-2,1)} = \hat{i} - 4\,\hat{j} + 2\,\hat{k}$$

$$|\hat{\nabla}f_1| = \sqrt{1+16+4} = \sqrt{21}$$

Again $\quad \hat{\nabla}f_2 = \frac{\partial f_2}{\partial x}\,\hat{i} + \frac{\partial f_2}{\partial y}\,\hat{j} + \frac{\partial f_2}{\partial z}\,\hat{k}$

$$\hat{\nabla}f_2 = 6x\,\hat{i} - 2y\,\hat{j} + \hat{k}$$

$$\left(\hat{\nabla}f_2\right)_{(1,-2,1)} = 6\,\hat{i} + 4\,\hat{j} + 2\,\hat{k}$$

Now $\quad |\hat{\nabla}f_2| = \sqrt{36+16+4} = \sqrt{56}$

Now $\quad \text{Cos}\,\Theta = \frac{\left(\hat{\nabla}f_1\right).\left(\hat{\nabla}f_2\right)}{|\hat{\nabla}f_1||\hat{\nabla}f_2|}$

$$= \frac{\left(\hat{i} - 4\hat{j} + 2\hat{k}\right).\left(6\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{21}\sqrt{56}}$$

$$= \frac{6-16+4}{\sqrt{21}\sqrt{56}} \quad = \frac{-6}{\sqrt{21}\sqrt{56}}$$

$$= \frac{-6}{2\sqrt{294}}$$

$$= \frac{-6}{2\sqrt{49\times 6}}$$

$$= \frac{-6}{14\sqrt{6}}$$

$$\Rightarrow \quad \text{Cos } \Theta = \frac{-3}{7\sqrt{6}}$$

$$\Theta = \text{Cos}^{-1}\left(\frac{-3}{7\sqrt{6}}\right)$$

$$\Theta = \text{Cos}^{-1}\left(\frac{-3}{7\sqrt{6}}\right) \qquad [\because \text{Cos}(-\Theta) = \text{Cos } \Theta]$$

Self Check Exercise - 3

Q. 1 Find the angle of intersection between surfaces $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z = 4$ at (4, -3, 2)

8.6 Summary:

Dear students, in this unit, we studied

1. Field is a function which describes a physical quantity at different points in space.
2. Scalar field is a function that assigns a unique scalar to every point in a given region.
3. Contour surface is the collection of those point for which value of scalar field is constant.
4. $\hat{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is known an operator known as 'del' operator.
5. $\hat{\nabla} f$ is gradient of scalar field and is given as $\hat{\nabla}\hat{f} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$
6. $\hat{\nabla}\hat{f}$ gives the normal (perpendicular) to the surface $\hat{f}$ (x, y, z) = constant.

7. $\left|\hat{\nabla}\hat{f}\right|$ gives the maximum rate of increase of scalar field $\hat{f}$.

8. The necessary and sufficient condition for a scalar filed f to be constant is

   $\hat{\nabla} f = 0$

9. Directional derivative of a scalar field at a point in a given direction is dot product of gradient of scalar field and the unit vector along that direction.

   $$\frac{df}{ds} = \hat{\nabla} f \cdot \frac{\hat{s}}{|\hat{s}|}$$

10. Directional derivative is a scalar quantity.

11. Directional derivative of scalar field f is the projection of $\hat{\nabla} f$ on $\hat{s}$.

12. The angle of intersection between two surface is equal to angle of intersection between their normals and is given by

   $$\text{Cos } \Theta = \frac{\left(\hat{\nabla}f_1\right).\left(\hat{\nabla}f_2\right)}{\left|\hat{\nabla}f_1\right|\left|\hat{\nabla}f_2\right|}$$

8.7 Glossary

1. **Projection Vector :**The projection of one vector over another vector is the length of the shadow of the given vector over another vector.

2. **Direction Cosine :**These are the cosine of angle made by given vector with x, y and z axis.

3. **Intersection :**The point where two vectors meets or cross each other.

8.8 Answers to Self Check Exercises

Self Check Exercise - 1

Q.1 -12 $\hat{i}$ - 9 $\hat{j}$ -16 $\hat{k}$

Q. 2 7 $\hat{i}$ - 3 $\hat{j}$ +8 $\hat{k}$

Q. 3 9

Self Check Exercise - 2

Q.1 $-{}^{13}\!/_{3}$

Q. 2 ${}^{8}\!/_{\sqrt{5}}$

Q. 3 11

Self Check Exercise - 2

Q.1 Θ = Cos-1 $\left(\sqrt{\frac{19}{29}}\right)$

8.9 References/Suggested Readings

1. R. Murray, S. Lipchitz, D. Spellman, Vector Calculus, Schaum's outlines:
2. S. Narayan, and P.K. Mittal, Vector Calculus, S Chand and Company Limited.
3. J.N. Sharma and A.R. Vasishtha, Vector Calculus, Krishna Prakashan Mandir.

8.10 Terminal Questions

Q. 1. Prove that $\hat{\nabla} f(r) = f^1(r) = \nabla r$ where $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Q. 2 Find $\hat{\nabla} f$ for $f = 3x^2y$

Q. 3 Find the unit normal to the surface $z = x^2 + y^2$ at (-1, 2, 5)

Q. 4 Find directional derivative of $f = (x^2 + y^2 + z^2)^{1/2}$ at the point (3, 1, 2) in the direction of the vector $yz\hat{i} + zx\hat{j} + xy\hat{k}$

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Unit - 9

Divergence of A Vector Field

Structure

9.1 Introduction

Dear student, in this unit we will study about one another operator application on vector point function, just like gradient. Since gradient is operated on scalar point function of scalar field. But when same operator del or ∇ is operated by means of dot product on vector point function, we get a new term known as divergence of vector. This is one of the important fundamental operator. Divergen generally describes the behavious of a vector field moving toward or moving away from a point. We will study in this unit how to apply divergence operator and its application along with its physical interpretation.

9.2 Learning Objectives:

After studying this unit, students will be able to

1. define vector point function and vector field.
2. define divergence of a vector.
3. give physical significance of divergence.
4. apply and solve divergence operator or a vector function.
5. Check what a given field is solenoidal or not.

9.3 Divergence of A Vector Point Function

Vector Point Function : A vector point function is defined as a function which assigns a vector to every point of a part of the region of space. If to every point (x, y, z) of a region X in space, there is assigned a vector $\vec{F} = \vec{F}$ (x, y, z), the $\vec{F}$ is called a vector point function and the function is represented as :-

$\vec{F}$ = F_1(x, y, z) $\hat{i}$ + F_2(x, y, z) $\hat{j}$ + F_3(x, y, z) $\hat{k}$

Divergence of a Vector Point Function :

The divergence of a vector field is a scalar field. The divergence is generally denoted by "div". The divergence of a vector field can be calculated by taking the scalar product of the vector operator (∇), applied to the vector field. i.e. ∇.F (x, y)

If $\vec{F}$ (x, y, z) is the vector field in 3-dimension i.e.

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Then $\nabla.\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . (F_1\hat{i} + F_2\hat{j} + F_3\hat{k})$ $\left[\nabla = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\right]$

$$= \hat{i} . \left(\frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}\right) + \hat{j} . \left(\frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}\right)$$

$$+ \hat{k} . \left(\frac{\partial F_1}{\partial x}\hat{i} + \frac{\partial F_2}{\partial y}\hat{j} + \frac{\partial F_3}{\partial z}\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \qquad \because [\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$$

$$\hat{i}.\hat{j} = \hat{j}.\hat{i} = \hat{i}.\hat{k} = \hat{k}.\hat{i} =$$

$$\hat{j}.\hat{k} = \hat{k}.\hat{j} = 0$$

$$\therefore \nabla.\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

This is the expression of divergence of $\vec{F}$ in component form.

Physical significance of Divergence

Divergence of vector quantity indicates how much the vector quality spreads out form the certain point.

Imagine a fluid with the vector field representing the velocity of fluid at each point in space. Divergence measure the net flow of fluid out of a given point. If fluid is intead flowing into that point, the divergence will be negative.

A point of region with positive divergence is often referred to as a source (of fluid or whatever the filed is describing) while a point or region with negative divergence is sink.

If div. v = 0. then the fluid entering and leaving is the same i.e. the fluid is incompressible and vector is called Solenoidal vector.

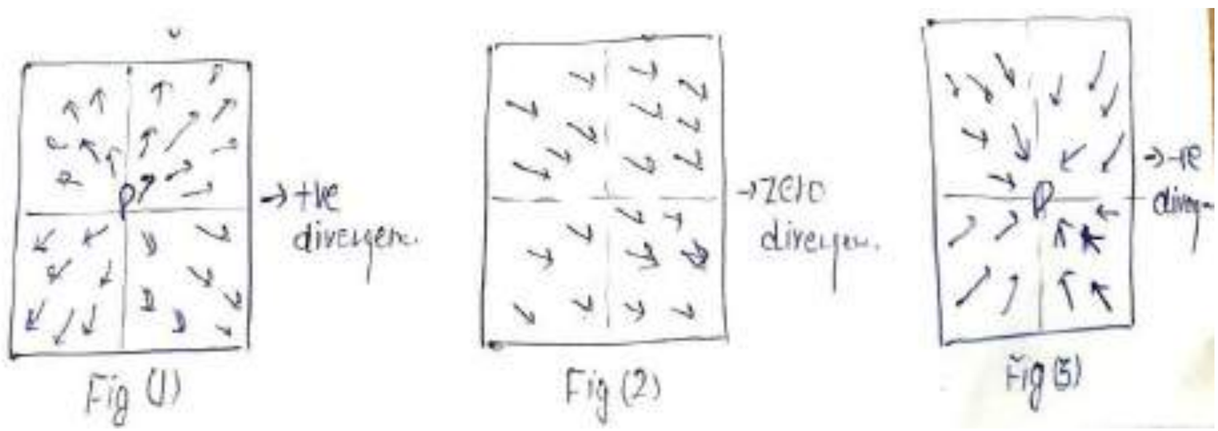
Solenoidal Vector:

A vector point function $\hat{F}$ is said to be solenoidal vector if its divergent is equal to zero i.e. div. $\vec{F}$ = 0 at all points of the function. For such a vector, there is no loss or gain of fluid.

$$\nabla \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0$$

Physical Meaning of the Divergence:

The divergence measures How much a vector field "Spreads out" or diverges from a given point. For example Fig (1) has positive divergence at P, Since the vectors of the vector field are all spreading as they move away from P. The figure (2) in the center has zero divergence everywhere Since the vectors are not spreading out at all This is easy to compute also, since the vector field is constant everywhere and the derivative of constant is zero. The fig (3) on the right has negative divergence since the vectors are coming closer together instead of spreading out.



Some Related Questions:

Let us try to apply divergence operator on some vector function field.

Example 1: Find the divergence of a vector $\hat{A} = 2x\hat{i} + 3y\hat{j} + 5z\hat{k}$.

Solution: By definition

$$\text{div. } \hat{F} = \nabla \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(5)$$

$$= 2 + 3 + 5$$

$$= 10$$

$\therefore$ div. $\vec{F}$ = 10

Example 2: Show that $\vec{A} = (3y^4z^2)\hat{i} + (4x^3z^2)\hat{j} - (3x^2y^2)\hat{k}$ is solenoidal.

Solution: Given $\hat{A}$ = $(3y^4z^2)\,\hat{i}$ + $(4x^3z^2)\,\hat{j}$ - $(3x^2y^2)\,\hat{k}$

By definition for Solenoidal

div. $\hat{F}$ = $\nabla.\vec{F}$ = 0

$\therefore$ $\Delta\,.\,\vec{A}$ = $\frac{\partial}{\partial x}(3y^4z^2)$ + $\frac{\partial}{\partial y}(4x^3z^2)$ + $\frac{\partial}{\partial z}(3x^2y^2)$

= 0 + 0 + 0

= 0

Here $\Delta\,.\,\vec{A}$ = 0

Thus, $\vec{A}$ is solenoidal

Example 3: Find the divergence of position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: Given $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

div. = ∇ = $\hat{i}\,\frac{\partial}{\partial x}$ + $\hat{j}\,\frac{\partial}{\partial y}$ + $\hat{k}\,\frac{\partial}{\partial z}$

$\nabla\,\vec{r} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\,(x\hat{i} + y\hat{j} + z\hat{k})$

$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$

= 1 + 1 + 1

= 3

$\therefore$ Divergence of position vector $\vec{r}$ = 3

Example 4: Show that div. $\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$, r is magnitude of $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution: We have r = $\left|\vec{r}\,\right| = \sqrt{x^2 + y^2 + z^2}$ $\because |\,\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ position vector|

Therefore $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial y} = \frac{z}{r}$

Hence, div. $\left(\frac{\vec{r}}{r}\right) = \nabla . \left(\frac{\vec{r}}{r}\right) = \nabla . \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r}\right)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right) . \left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$ [using product rule]

$$= \left(\frac{1}{r}-\frac{x}{r^2}.\frac{\partial r}{\partial x}\right) + \left(\frac{1}{r}-\frac{y}{r^2}.\frac{\partial r}{\partial y}\right) + \left(\frac{1}{r}-\frac{z}{r^2}.\frac{\partial r}{\partial z}\right)$$

$$= \left(\frac{1}{r}-\frac{x}{r^2}.\frac{x}{r}\right) + \left(\frac{1}{r}-\frac{y}{r^2}.\frac{y}{r}\right) + \left(\frac{1}{r}-\frac{z}{r^2}.\frac{z}{r}\right) \left[\because u\sin g\, the\, value\, of\, \frac{\partial r}{\partial x},\frac{\partial r}{\partial y},\frac{\partial r}{\partial z}\right]$$

$$= \frac{1}{r^3}\left\{\left(r^2-x^2\right)+\left(r^2-y^2\right)+\left(r^2-z^2\right)\right\}$$

$$= \frac{1}{r^3}\left\{\left(y^2+z^2\right)+\left(x^2+z^2\right)+\left(x^2+y^2\right)\right\}$$ [$\because$ $r^2 = x^2 + y^2 + z^2$]

$\Rightarrow$ div $\left(\frac{\hat{r}}{r}\right) = \frac{1}{r^3} . 2\,(x^2 + y^2 + z^2)$

$$= \frac{2r^2}{r^3}$$

$$= \frac{2}{r}$$

$\Rightarrow$ div $\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$

Example 5: If $\hat{e}$ be unit vector, show that div $(\hat{e} . \vec{r})\,\hat{e} = 1$.

Solution: Let $\hat{e} = e_1\hat{i} + e_2\hat{j} + e_3\hat{k}$., where $e_1^2 + e_2^2 + e_3^2 = 1$

and $\vec{r} = \left\{\left(e_1\hat{i}+e_2\hat{j}+e_3\hat{k}\right).\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\right\}\hat{e}$

$$= \left(e_1\hat{i}+e_2\hat{j}+e_3\hat{k}\right)\left(e_1x+e_2y+e_3z\right)$$

$= e_1\left(e_1x+e_2y+e_3z\right)\hat{i}\ + e_2\left(e_1x+e_2y+e_3z\right)\hat{j}\ + e_3\left(e_1x+e_2y+e_3z\right)\hat{k}$

$= \sum e_1\left(e_1x+e_2y+e_3z\right)\hat{i}$

$\therefore$ div $(\hat{e}\,.\,\hat{r}\,)\,\hat{e}$

$= \sum \frac{\partial}{\partial x} e_1\left(e_1x+e_2y+e_3z\right)$

$= \sum e_1^{\ 2}$

$= e_1^{\ 2} + e_2^{\ 2} + e_3^{\ 2}$

$= 1$

$\Rightarrow$ div $(\hat{e}\,.\,\hat{r}\,)\,\hat{e} = 1$

Hence proved

Example 6: If $\vec{A} = x^2z\,\hat{i} + yz^3\,\hat{j} - 3xy\,\hat{k}$ and $\vec{B} = y^2\,\hat{i} - yz\,\hat{j} + 2x\,\hat{k}$

Find (i) $\left(\nabla.\vec{A}\right)\vec{B}$ (ii) $\left(\nabla.\vec{B}\right)\vec{A}$.

Solution: (i) We have $\nabla.\vec{A} = \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(yz^3) + \frac{\partial}{\partial z}(-3xy)$

$= 2xz + z^3$

Hence $\left(\nabla.\vec{A}\right)\vec{B} = (2xz + z^3)\,(y^2\,\hat{i} - yz\,\hat{j} + 2x\,\hat{k})$

(ii) Here $\left(\nabla.\vec{B}\right) = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(2x)$

$= 0 - z + 0$

Hence $\left(\nabla.\vec{B}\right)\vec{A} = -Z\,(x^2z\,\hat{i} + yz^3\,\hat{j} - 3xy\,\hat{k})$

Example 7: Show that the vector $\vec{F} = 3y^4z\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$ is solenoidal.

Solution: Given that $\vec{F} = 3y^4z\,\hat{i} + 4x^3z^2\,\hat{j} - 3x^2y^2\,\hat{k}$

Therefore div $\vec{F} = \nabla.\vec{F}$

$$= \frac{\partial}{\partial x}(3y^4 z) + \frac{\partial}{\partial y}(4x^3z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$= 0$

This, shows that $\vec{F}$ is solenoldal.

Example 8: Find the value of constant a so that the vector $\vec{F} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

Solution: Given $\vec{F} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$

div. is a solenoidal field.

By definition of solenoldal vector we have div $\vec{F} = 0$

So, div $\vec{F} = \nabla.\vec{F}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left[(x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az)$$

$\nabla.\vec{F} = 1 + 1 + a = 2 + a$

Since $\nabla.\vec{F} = 0$

$\Rightarrow$ $2 + a = 0$

$\Rightarrow$ $a = -2$

Example 9: If $\hat{a}$ is a constant vector, then find div $(\hat{r} \times \hat{a})$.

Solution: Given $\hat{a}$ is a constant vector, also we know that $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is known as position vector.

Let $\hat{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are components of $\hat{a}$ in x, y and z direction respectively and are constant.

Now, $$\hat{r} \times \hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$= \hat{i}(a_3y - a_2z) - \hat{j}(a_3x - a_1z) + \hat{k}(a_2x - a_1y).$

Now div ($\hat{r} \times \hat{a}$) = $\hat{\nabla}$.($\hat{r} \times \hat{a}$)

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) . \left[(a_3 y - a_2 z)\hat{i} - (a_3 x - a_1 z)\hat{j} + (a_2 x - a_1 y)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(a_3 y - a_2 z) - \frac{\partial}{\partial y}(a_3 x - a_1 z) + \frac{\partial}{\partial z}(a_2 x - a_1 y)$$

= 0 - 0 + 0

div ($\hat{r} \times \hat{a}$) = 0

Self Check Exercise

Q.1 If $\hat{F}$ (x, y, z) = $\hat{e}\,\hat{i}$ + yz $\hat{j}$ - yz$^2\hat{k}$ then find the divergence of $\hat{F}$ at (0, 2, -1).

Q.2 Show that $\nabla . (\hat{a} \times \hat{r}) = 0$

Q.3 Show that $\nabla . \left\{\frac{F(r)}{r}\vec{r}\right\} = \frac{1}{r^2}\frac{d}{dr}(r^2 f)$ where f(r) is arbitrary differentiable function

Q.4 If $\vec{A}$ = x$^2\hat{i}$ + yz $\hat{j}$ - xz$\hat{k}$ and f = 3x^2 yz. Show that

(1) $\vec{A}\,(\nabla f) = (\vec{A}\,.\nabla)\,f$

Q.5 Find $\nabla . \nabla f$ at (1, 1, 1) where $f = 2x^3y^2z^4$

9.4 Summary:

Dear students, in this unit, we studied that

1. Divergence operates on a vector field but results in a scalar field.
2. Divergence of a vector field can be found by taking the scalar product of the vector operator $\hat{\nabla}$ applied to the vector field $\hat{F}$, That is, $\nabla . \hat{F}$ (x, y, z).
3. Divergence shows how the field behaves towards or away from a point.
4. Divergence of a vector field is the extend to which the vector field flux behaves like a source at a given point.
5. A vector field with zero divergence everywhere called solenhoidal. In this case any closed surface has no net flux across it.

6. The divergence of a vector field is often illustrated using the simple example of the velocity field of a fluid, a Liquid or gas.

9.5 Glossary

1. **Scalar Quantity :**A scalar quantity is described as a quantity that has only magnitude but no particular direction. For example volume, energy, speed, mass density and time.

2. **Vector Quantity :**Vector quantity is described as the quantity that has both magnitude and direction. For example-force, velocity.

3. **Vector Function :**A vector function is that which assigns a group of real variables to the vector. It is represented in general form as

   $V = V_1(P_1,P_2,P_3,.........,P_n)\,\hat{i} + V_2(P_1, P_2, P_3,..........P_n)\,\hat{j}$

   $+V_3(P_1, P_2, P_3,.........P_n)\,\hat{k}$

   Where $P_1, P_2,.......P_n$ can be considered real numbers

4. **Vector Field :** A vector field is a vector point function defined over some region. A vector field that is not dependent on time is called a steady state vector field or stationary.

   A vector field varying with time is represented as :

   $V = V_1\,(p, q, r, t) + V_2(p, q, r, t)\ \hat{j} + V_3(p, q, r, t)\,\hat{k}$

Examples : (i) Magnetic Field

(ii) Gravitational Field.

9.6 Answers to Self Check Exercises

Self Check Exercise - 1

Q.1 4

Q. 2 Verified

Q. 3 Verified

Q.4 Verified

Q.5 40

9.7 References/Suggested Readings

1. Vector Analysis by J.G. Chakraverty and P.R. Ghosh.

2. Analytical Geometry of two and Three Dimensions & Vector Analysis by R.M. Khan

9.8 Terminal Questions

Q. 1. Find div $\vec{F}$ for $\vec{F}$ (x, y, z) = (xy, 5-z^2, x^2 + y^2)

Q. 2 Show that ∇ . {F(r) $\hat{r}$ } = 3F(r) + r F(r)

Q. 3 Show that $\nabla\left|\nabla\frac{\vec{r}}{r}\right| = \frac{-2}{r^3}\vec{r}$

Q. 4 Show that the vector $\vec{F}$ = (x + 3y) $\hat{i}$ +(y + az) $\hat{j}$ +(x + az) $\hat{k}$ is a solenoidal if a = -2.

Q. 5 Compute the divergence of the vector field

$\vec{F}$ (x, y, z) = $x^2\hat{i}$ + 2z $\hat{j}$ - y $\hat{k}$

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Unit - 10

Curl of a vector Field

Structure

10.1 Introduction

Dear student, in last unit we studied about the divergence of a vector field in which we operate the del operator on a vector field using scalar product or dot product. In this unit we will study about how did operator is applied on a vector field using vector product or cross product. When del operator is used to vector product of a vector field then it is known as curl of a vector field. It this unit we will study about curl, its physical meaning and application.

10.2 Learning Objectives: After studying this unit, students will be able to

1. define curl of a vector field or function.
2. give physical meaning of curl of a vector.
3. evaluate the value of curl of a vector.
4. prove that a given field is irrotational field.

10.3 Curl of a Vector Point Function

Vector : Vector is a physical quantity that has magnitude and direction

Vector Point Function :

A vector point function is a function that assigns a vector to each point of some region of space. If to each point (x, y, z) of a region R in space there is assigned a vector $\vec{F} = \hat{F}$ (x, y, z) then $\vec{F}$ is called a vector point function. Such a function would have a representation

$$\vec{F} = f_1(x, y, z)\,\hat{i} + f_2(x, y, z)\,\hat{j} + f_3(x, y, z)\,\hat{k}.$$

Where f_1, f_2, f_3 are components of $\hat{F}$ along x, y, z direction respectively.

Curl of a Vector Point Function :

Curl of a vector field is a measure of its rotation at a particular position. Curl of vector shows how much the vector field rotates or circulates around that location. The curl is a vector quantity itself and is defined as the cross product of the del operator and the vector field. It is represented as,

$\nabla \times \vec{F}$, where $\vec{F}$ represents the vector field

if $\vec{F}$ = F_1(x, y, z) $\hat{i}$ + f_2(x, y, z) $\hat{j}$ + f_3(x, y, z) $\hat{k}$.

$$\text{then} \quad \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k} .$$

Properties of the Curl:

Curl of a vector field is another vector field.

The curl of a scalar field V, $\nabla \times V$ makes no sense as its is not defined.

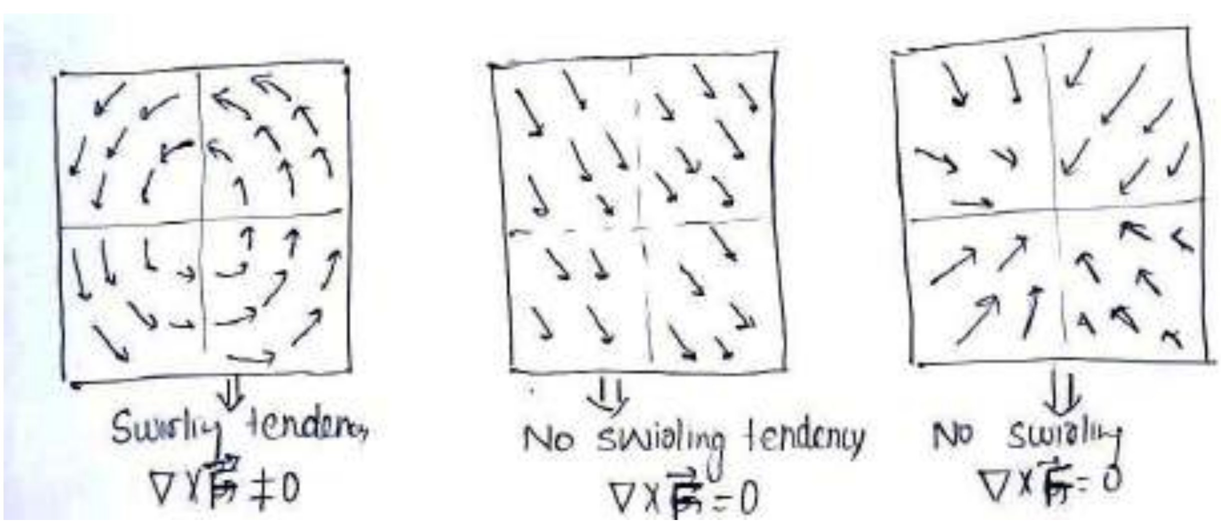
A vector field $\vec{F}$ is conservative if curl $\vec{F}$ = 0

or if curl $\hat{F}$ = 0 then vector field is said to be irrotational.

Physical Interpretation of the Curl :

The curl of a vector field measures the tendency for the vector field to swirl around. Imagine that the vector field represents the velocity vectors of water in a lake. If the vector field swirl water, it will tend to spin. The amount of spin will depend on how we orient the paddle. Thus, we should expect the curl to be vector valued.

The swirling tendency can be shown as in the figures given below.



Some Related Questions :

Let us try following examples to have more understanding of curl of a vector field.

Example 1: Find the curl of a vector field $\vec{G}$ (x, y, z) = (x^2, yz, xyz).

Solution: Given $\vec{G}$ (x, y, z) = (x^2, yz, xyz)

$\Rightarrow$ $G_1 = x^2$, $G_2 = yz$, $G_3 = xyz$

$$\therefore \quad \vec{\nabla} \times \vec{G} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} \qquad \therefore \hat{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right)\hat{i} - \left(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}\right)\hat{j} + \left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)\hat{k}$$

$$= \left(\frac{\partial}{\partial y}(xyz) - \frac{\partial}{\partial z}(yz)\right)\hat{i} - \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial z}(x^2)\right) + \left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(x^2)\right)\hat{k}$$

$$= (xz - y)\hat{i} - (yz)\hat{j} + 0\hat{k}$$

$\Rightarrow$ Curl $\vec{G} = (xz - y)\hat{i} - (yz)\hat{j}$

Example 2: Determine curl of the vector field:

$$\vec{T} = \frac{1}{r^2}\cos\theta\,\hat{a}\,r + r\sin\theta\cos\phi\,\hat{a}\,\theta + \cos\theta.\,\hat{a}\,\phi$$

Solution: If $\vec{A} = A\,r\,\hat{a}\,r + A\,\theta\,\hat{a}\,\theta + A\phi\,.\,\hat{a}\,\phi$

$$\text{the } \nabla \times \vec{A} = \begin{vmatrix} \hat{a}r & r\hat{a}\theta & r\sin\theta\,\hat{a}\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ Ar & rA\theta & r\sin\theta\,A\phi \end{vmatrix} \times \frac{1}{r^2 \sin\theta}$$

$$= \frac{1}{rSin\theta}\left((A\phi\ Sin\ \theta) - \frac{\partial A\theta}{\partial \phi}\right)\hat{a}\,r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial Ar}{\partial \phi} - \frac{\partial}{\partial r}(rA\phi)\right)\hat{a}\phi$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA\theta) - \frac{\partial Ar}{\partial \theta}\right)\hat{a}\phi$$

Given $\vec{T} = \dfrac{1}{r^2}\cos\theta\,\hat{a}\,r + r\sin\theta\cos\phi\,\hat{a}\,\theta + \cos\theta.\ \hat{a}\,\phi$

Then $\nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(T\phi\sin\theta)-\frac{\partial}{\partial\phi}(T\theta)\right]\hat{a}\,r +$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}Tr-\frac{\partial}{\partial r}(rT\phi)\right]\hat{a}\;\theta+\frac{1}{r}\left[\frac{\partial}{\partial\theta}(rT\theta)-\frac{\partial}{\partial\theta}(Tr)\right]\hat{a}\phi$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\cos\theta\sin\theta)-\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)\right]\hat{a}\,r +$$

$$\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{\cos\theta}{r^2}\right)-\frac{\partial}{\partial r}\left(r\cos\theta\right)\right]\hat{a}\,\theta+\frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\sin\theta\cos\phi)-\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{r^2}\right)\right]\hat{a}\phi$$

$$= \frac{1}{r\sin\theta}\left[\cos 2\theta+r\sin\theta\sin\phi\right]\hat{a}\,r \;+\frac{1}{r}\left[-\cos\theta\right]\hat{a}\theta$$

$$+ \frac{1}{r}\left[2r\sin\theta\cos\phi+\sin\phi\frac{\sin\theta}{r^2}\right]\hat{a}\phi$$

$$\therefore \qquad \nabla\times\vec{A} = \left[\frac{\cos 2\theta}{r\sin\theta}+\sin\phi\right]\hat{a}\,r \;-\; \frac{\cos\theta}{r}\;\hat{a}\theta \;+\; \left[2\cos\phi\frac{1}{r^3}\right]\sin\theta\;\hat{a}\phi$$

Example 3 : For a vector field A, show explicitly that $\nabla.\nabla\times\vec{A}$ = 0 that is the divergence of the curl of any vector field is zero.

Solution : Let $\vec{A}$ = A x$\hat{i}$ + Ay $\hat{j}$ + Az $\hat{k}$

$$\nabla\times\vec{A} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ Ax & Ay & Az\end{vmatrix}$$

$$= \left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right)\hat{i} \;-\; \left(\frac{\partial Az}{\partial x}-\frac{\partial Ax}{\partial z}\right)\hat{j} \;+\; \left(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y}\right)\hat{k}$$

$$\therefore \qquad \nabla.\nabla\times\vec{A} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left[\left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right)\hat{i}-\left(\frac{\partial Az}{\partial x}-\frac{\partial Ax}{\partial z}\right)\hat{j}+\left(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y}\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial Az}{\partial y}-\frac{\partial Ay}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial Az}{\partial x}-\frac{\partial Ax}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial Ay}{\partial x}-\frac{\partial Ax}{\partial y}\right)$$

$$= \frac{\partial^2 Az}{\partial x \partial y} - \frac{\partial^2 Ay}{\partial x \partial z} - \frac{\partial^2 Az}{\partial x \partial y} + \frac{\partial^2 Ax}{\partial y \partial z} + \frac{\partial^2 Ay}{\partial z \partial x} - \frac{\partial^2 Ax}{\partial z \partial y}$$

$= 0$

$\Rightarrow \quad \nabla . \nabla \times \vec{A} = 0$

$\therefore$ Divergence of the curl of any vector field is zero.

Example 4 : Given $\vec{F}$ = (xy - xz)$\hat{i}$ + 3x $\hat{j}$ + yz $\hat{k}$. Find curl $\vec{F}$ at origin (0, 0, 0) and at the point P = (1, 2, 3).

Solution : Curl $\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy - xz & 3x & yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(3x^2) \right] \hat{i} + \left[\frac{\partial}{\partial z}(xy - xz) - \frac{\partial}{\partial x}(yz) \right] \hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(xy - xz) \right] \hat{k}.$$

$= z\hat{i} + x\hat{j} + 5x\hat{k}$

At the point (0, 0, 0) Curl $\vec{F} = 0$

At the point (1, 2, 3) Curl $\vec{F} = 3\hat{i} - \hat{j} + 5\hat{k}$

Example 5 : Show that $\vec{u} = x^2\hat{i} + y^2\hat{j}$ is irriotational.

Solution : Given $\vec{u} = x^2\hat{i} + y^2\hat{j}$

$\therefore$ Curl $\vec{u} = \nabla \times \vec{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y & 0 \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^2) \right] \hat{i} + \left[\frac{\partial}{\partial z}(x^2 - \frac{\partial}{\partial x}(0) \right] \hat{j}$$

$$+ \left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2) \right] \hat{k}.$$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= 0$

$\Rightarrow$ Curl $\vec{u} = 0$

$\therefore$ $\vec{u}$ is irrotational.

Example 6 : Show that Curl $\left(\frac{\vec{r}}{r^m}\right) = \vec{O}$ where r =$|\vec{r}|$

Solution : We have $\nabla \times \vec{r}\left(\frac{\vec{r}}{r^m}\right) = \nabla \times \left(\frac{x}{r^m}\hat{i} + \frac{y}{r^m}\hat{j} + \frac{z}{r^m}\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^m} & \frac{y}{r^m} & \frac{z}{r^m} \end{vmatrix}$$

$$= \sum \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{z}{r^m}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^m}\right)\right\}$$

$$= \sum \hat{i}\left\{-\frac{mz}{r^{m+1}}\frac{\partial r}{\partial y} + \frac{my}{r^{m+1}}\frac{\partial r}{\partial z}\right\}$$

$$= \sum \hat{i}\left\{-\frac{mz}{r^{m+1}}\frac{y}{r} + \frac{my}{r^{m+1}}\frac{z}{r}\right\}$$

$$= \sum 0\hat{i} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$$= \vec{0}$$

Example 7 : If $\hat{e}$ be a unit vector, show that curl $\{(\hat{e} \times \vec{r}) \times \hat{e}\} = \vec{O}$

Solution : $|\hat{e} \times \vec{r}| \times \hat{e} = (\vec{r} \,.\, \hat{e}) - |\hat{e} \,.\, \hat{e}|\ \vec{r}$

$= (\vec{r} \,.\, \hat{e})\hat{e} - \vec{r}$ as $|\hat{e} \,.\, \hat{e} = |\hat{e}|^2 = 1$

Therefore, curl $\{(\hat{e} \times \vec{r}) \times \hat{e}\}$= curl $\{(\vec{r} \,.\, \hat{e})\hat{e} - \vec{r}\}$

$= \text{curl } (\vec{r} \,.\, \hat{e})\hat{e} - \text{curl } \vec{r}$

$= \text{curl } \vec{r}$

$= \vec{O}$

$\therefore \quad \text{curl}\,\{(\hat{e} \times \vec{r}) \times \hat{e}\} = \vec{O}$

Example 8 : Find curl of $\vec{F} = \hat{i}\, x \cos z + \hat{j}\, z\, y \log x + \hat{k}\, z^2$.

Solution : Given $\vec{F} = \hat{i}\, x \cos z + \hat{j}\, z\, y \log x + \hat{k}\, z^2$.

$$\text{so curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x\cos z & y\log x & z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial z} y \log x\right] - \hat{j}\left[\frac{\partial}{\partial x} z^2 - \frac{\partial}{\partial z} x \cos z\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x} y \log x - \frac{\partial}{\partial y} x \cos z\right].$$

$$= \hat{i}\,[0 - 0] - \hat{j}\,[0 + x \sin z] + \hat{k}\left(\frac{y}{x}\right)$$

$$\therefore \quad \text{curl } \vec{F} = - x \sin z\, \hat{j} + \frac{y}{x}\hat{k}$$

Example 9 : If $\vec{F} = (x+y+1)\,\hat{i} + \hat{j} + (-x-y)\hat{k}$. Then prove that $\vec{F}$. curl $\vec{F} = 0$

Solution : Given $\vec{F} = (x+y+1)\,\hat{i} + \hat{j} + (-x-y)\hat{k}$

$$\text{then curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-x-y) - \frac{\partial}{\partial z}(1)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-x-y) - \frac{\partial}{\partial z}(x+y+1)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right].$$

$$= \hat{i}\,[-1] - \hat{j}\,[-1] + \hat{k}\,(-1)]$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$\therefore$ curl $\vec{F}$ $= -\hat{i} + \hat{j} - \hat{k}$

Now, $\vec{F}$. curl $\vec{F}$ = [(x+y+1) $\hat{i}$ + $\hat{j}$ + (-x-y) $\hat{k}$] . ($-\hat{i} + \hat{j} - \hat{k}$)

= - (x+y+1) + 1.1 + (-x-y)

= - x-y-1 + 1 + x - y

= 0

$\Rightarrow$ $\vec{F}$. curl $\vec{F}$ = 0

Example 10 : If $\vec{F}$ = z $\hat{i}$ + x $\hat{j}$ + y $\hat{k}$ prove that curl curl $\vec{F}$ = $\vec{0}$

Solution : Given $\vec{F}$ = z $\hat{i}$ + x $\hat{j}$ + y $\hat{k}$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial y}{\partial y} - \frac{\partial x}{\partial z}\right] - \hat{j}\left[\frac{\partial y}{\partial x} - \frac{\partial z}{\partial z}\right]$$

$$+ \hat{k}\left[\frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}\right]$$

$$= \hat{i}\,(1) - \hat{j}\,(-1) + \hat{k}\,(1)$$

$\therefore$ curl $\vec{F}$ $= \hat{i} + \hat{j} + \hat{k}$

$$\text{Now, curl curl } \vec{F} = \nabla\times(\nabla\times\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & 1 & 1 \end{vmatrix}$$

= 0

[$\therefore$ 1 is constant and its derivation w.r.t. x, y, z is zero]

Example 11 : Find the constant a, b, c so that the vector $\vec{F}$ = (x + xy + az) $\hat{i}$ + (bx - 3y - 3) $\hat{j}$ + (4x + cy + 2z) $\hat{k}$ is irratational.

Solution :- Since $\vec{F}$ = (x + 2y + az) $\hat{i}$ + (bx - 3y - 3) $\hat{j}$ + (4x + cy + 2z) $\hat{k}$

Since, we know that a vector $\vec{F}$ is irratational if

curl $\vec{F} = \nabla\times\vec{F} = 0$

$$\therefore \quad \text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right] - \hat{j}\left[\frac{\partial}{\partial x}(4x-cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right].$$

$= \hat{i}$ [c+1] - $\hat{j}$ [4-a] + $\hat{k}$ [b-2]

So, curl $\vec{F} = 0$, For irrotational fields.

$\Rightarrow$ = (c+1) $\hat{i}$ - (4-a) $\hat{j}$ + (b-2) $\hat{k}$ = 0 $\hat{i}$ + 0 $\hat{j}$ + 0 $\hat{k}$

Using the concept of equality of two vectors, equating coefficients of = $\hat{i}$, $\hat{j}$ and $\hat{k}$.

c + 1 = 0 $\Rightarrow$ c = -1

-(4 - a) = 0 $\Rightarrow$ a = -1

b - 2 = 0 $\Rightarrow$ b = -2

So, a = 4, b = 2, c = -1

Self Check Exercise

1. Find the curl of a vector field F(x, y, z) = (2xy, $-yz^2$, xz)
2. Determine curl of the vector field

   Q = p Sin $\phi\, \hat{a}$ p + p^2z $\hat{a}\,\phi$ + z Cos $\phi\, \hat{a}$ z
3. Determine if the vector field $\vec{F}$ = $yz^2\hat{i}$ + (xz^2 +2) $\hat{j}$ + (2xyz - 1) $\hat{k}$ is irrotational.
4. Verify div (curl $\vec{F}$) = 0 For the vector $\vec{F}$ = $yz^2\hat{i}$ + xy $\hat{j}$ + yz $\hat{k}$
5. Show that (i) $\dfrac{x\hat{i}-y\hat{j}}{x+y} = \dfrac{1}{x+y}\hat{k}$

   (ii) $\nabla.(\vec{a}\times\vec{r}) = 0$

   (iii) $\nabla\times(\vec{a}\times\vec{r}) = 2\vec{a}$

6. Find λ, π and ν so that the vector

$$\vec{F} = (2x + 3y + \lambda z)\hat{i} + (\pi x + 2y + 3z)\hat{j} + (2x + \nu y + 3z)\hat{k} \text{ is irrotational.}$$

7. Show that $\vec{F} = \dfrac{-y\hat{i} + x\hat{j}}{x^2 + y^2}$ is irrotational.

10.4 Summary :

Dear students, in this unit we studied that

1. The curl of a vector field is a mathematical operation that measures the tendency of the field to rotate around a point. It gives the vector quantity representing the rotation of the field.
2. Mathematically, the curl of vector field $\vec{F}$ in three dimensional Cartesian coordinates is denoted by $\nabla \times \vec{F}$. It can be calculated using a determinant involving partial derivatives of the component of the vector field.
3. Physically, the curl represents the tendency of a vector field to circulate or form vortices around a point.
4. The curl operation is linear meaning that it satisfies the properties of linearity, scaling and addition. It also obeys the product rule and satisfies certain identities, such as the vector identity involving the cross product of gradients.
5. The curl of a vector field is zero if and only if the vector field is "irrotational" meaning that it does not exhibit any rotational motion. This typically occurs in regions where the vector field has no tendency to swirl or rotate.

10.5 Glossary

- **Scalar Field:** A scalar point function defined over some region is called a scalar field. A scalar field which is independent of time is called a stationary or steady state scalar field. A scalar field which varies with time would have the representation.

  $u = \phi(x, y, z, t)$.

- **Vector Field:** A vector point function defined over some region is called a vector field. A vector field which is independent of time is called a stationary or steady state vector field. A vector field that varies with time would have the representation.

  $F = f_1(x, y, z, t)\hat{i} + f_2(x, y, z, t)\hat{j} + f_3(x, y, z, t)\hat{k}$

Examples: Gravitational field of Earth, Magnetic field generated by magnet

$$\text{Del, } \nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

10.6 Answer to Self Check Exercise

1. $(3 - x)\hat{i} + (z - 2y)\hat{k}$

2. $\frac{-1}{p}$ (z sin ϕ + p^3) $\hat{a}$ p + (3pz - cos ϕ) $\hat{a}$ z

3. Irrotational i.e. curl $\vec{G}$ = 0.

4. Verified it using the concept of curl and divergence

5. Verified this using the concept of curl

6. λ = 2, r = 3, v = 3

7. $\vec{\nabla} \times \hat{F}$ = 0 so irrotation.

10.7 References/Suggested Readings

1. Vector Analysis by J.G. Chakravorty and P.R. Ghosh

2. Analytic Geometry of Two and Three Dimension & Vector Analysis by R.M. Khan.

10.8 Terminal Questions

1. Consider the vector field $\vec{H}$ (x, y, z) = (z, x, y). Find the curl of given vector field.

2. Find the curl of vector field F(x, y, z) = (e^x Cosy, e^y sin z, e^z cos x).

3. Determine curl of the vector field

   P = x2 yz $\hat{a}$ x + x z $\hat{a}$ z

4. Determine if $\vec{F} = x^2y\,\hat{i} + xyz\,\hat{j} - x^2y^2\,\hat{k}$ is an irrotational vector field

5. Determine if the vector field is conservative where

   $\vec{F} = 6x\,\hat{i} + (2y-y^2)\,\hat{j} + (6z - x^3)\,\hat{k}$

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Unit - 11

The Laplacian Operator

Structure

11.1 Introduction

Dear student, in this unit we will study one more operator of vector calculus, which is known as Laplacian operator. This operator is named after Pieue Simosn de Laplace. When we combine divergence after with gradient, we get Laplace operator. So, this is nothing but divergence of gradient of scalar function. In this unit we will study about this operator and how to apply Laplace operator on a function. On the basis of Laplace operator we will dyino a Harmonic function and Laplace equation.

11.2 Learning Objectives:

After studying this unit, students will be able to

1. define Laplace operator
2. define harmonic function
3. solve questions related to Laplace operator.

11.3 The Laplacian Operator $\left(\nabla^2\right)$

The Laplace operator or Laplacian is the divergence of the gradient of a function i.e. $\hat{\nabla}.\vec{\nabla}$ or ∇^2, where $\vec{\nabla}$ is the nabla or det operator, which is a vector operator, where as ∇^2 is a scalar operator.

(i) In two dimensional : (a) in Cartesian Coordinates': Let f be a scalar point function.

then, $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$

$\Rightarrow \quad \nabla .(\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

(b) In polar coordinates:

$$\nabla^2 f = \frac{1}{r}\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

where r represents the distance and θ is the angle.

(ii) In three dimensional:

(a) In Cartesian coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

(b) In cylindrical coordinates:

$$\nabla^2 f = \frac{1}{P}\frac{\partial}{\partial P}\left(P\frac{\partial f}{\partial P}\right) + \frac{1}{P^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Where P represents the radial distance, ϕ the azimuthal angle and z the height.

(c) In spherical coordinates:

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

where ϕ represents azimuthal angle and θ is zenith angle.

Harmonic Function

A function u(x, y) is known as harmonic function when it is twice continuously differentiable and also satisfies the below partial differential equation i.e. the Laplace equation.

∇^2u = uxx + uyy = 0

or

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

That means a function is called a harmonic function if it satisfies Laplace equation.

Some Related Questions:

To have more understanding of caplace operator us after following examples:

Example 1: Show that $\hat{\nabla} . \hat{\nabla} f$ = 40 at (1, 1, 1), where $f = 2x^3y^2z^4$.

Solution: We have $\hat{\nabla} . \hat{\nabla} f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) (2x^3y^2z^4)$$

$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\Rightarrow \qquad \nabla^2 f = |_{(1,1,1)} = 12(1) + 4(1) + 24(1)$$

Therefore $\hat{\nabla} . \hat{\nabla} f \,|_{(1,1,1)} = 40$

Example 2: Find the value of $\nabla^2 \left(\frac{x}{r^3}\right)$

Solution: We have $\nabla^2 \left(\frac{x}{r^3}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \left(\frac{x}{r^3}\right)$

$$= \frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) \qquad (1)$$

Now, $\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x}$

$$= \frac{1}{r^3} - \frac{3x}{r^4} . \frac{x}{r}$$

$$= \frac{1}{r^3} - \frac{3x^2}{r^5}$$

So, $\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)$

$$= \frac{-1}{r^4}\frac{\partial r}{\partial x} - \frac{6x}{r^5} + \frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$= \frac{-x}{r^5} - \frac{6x}{r^5} + 15\frac{x^3}{r^7}$$

$$= \frac{-7x}{r^5} + \frac{15x^3}{r^7}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{-3x}{r^5} + \frac{15xy^2}{r^7}$$

and $$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = \frac{-3x}{r^5} + \frac{15xz^2}{r^7}$$

Therefore, equation (1) becomes

$$\nabla^2\left(\frac{x}{r^3}\right) = \left(\frac{-7x}{r^5} + \frac{15x^3}{r^7}\right) + \left(\frac{-3x}{r^5} + \frac{15xy^2}{r^7}\right) + \left(\frac{-3x}{r^5} + \frac{15xz^2}{r^7}\right)$$

$$= \frac{-13x}{r^5} + \frac{15x\left(x^2+y^2+z^2\right)}{r^7}$$

$$= \frac{13x}{r^5} + \frac{15x}{r^5}$$

$$= \frac{2x}{r^5}$$

$$\therefore \boxed{\nabla^2\left(\frac{x}{r^3}\right) = \frac{2x}{r^5}}$$

Example 3: Find the Laplacian of the function V = $x^2 + y^2 + z^2$.

Solution: Given V = $x^2 + y^2 + z^2$.

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial x^2}(x^2 + y^2 + z^2) + \frac{\partial}{\partial y^2}(x^2 + y^2 + z^2) + \frac{\partial}{\partial z^2}(x^2 + y^2 + z^2)$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(2z)$$

= 2 + 2 + 2

= 6

$\therefore$ Laplacian of given function = 6

Example 4: If $\phi = 2x^3y^2z^4$ find $\nabla . \nabla \phi$ or $\nabla^2 \phi$.

Solution: Given $\phi = 2x^3y^2z^4$

Now, $$\nabla . \nabla \phi = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} . \left[\frac{\partial}{\partial x}\phi\hat{i} + \frac{\partial}{\partial z}\phi\hat{j} + \frac{\partial}{\partial z}\phi\hat{k}\right]$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\Rightarrow \quad \nabla^2 \phi = \frac{\partial^2}{\partial x^2}(2x^3y^2z^4) + \frac{\partial^2}{\partial y^2}(2x^3y^2z^4) + \frac{\partial^2}{\partial z^2}(2x^3y^2z^4)$$

$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$\Rightarrow \quad \nabla^2 \phi = 12xy^2z^4 + 4x^2z^4 + 24x^3y^2z^2.$$

Example: Show that $\nabla^2 u = 0$ for $u = x^2-y^2+4z$

Solution: Given $u = x^2-y^2+4z$

$$\text{Now} \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(x^2-y^2+4z) + \frac{\partial^2}{\partial y^2}(x^2-y^2+4z) + \frac{\partial^2}{\partial z^2}(x^2-y^2+4z)$$

$$= \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(4)$$

$$= 2 - 2 + 0$$

$$\Rightarrow \quad \nabla^2 u = 0$$

Self Check Exercise

1. Prove that $\nabla^2 \left(\frac{x}{r^2}\right) = \frac{-x}{r4}$

2. If $u = x^2 - y^2 + 4z$, show that u is an harmonic function i.e. $\nabla^2 u = 0$

3. If u is a scalar function of r and u', u'' denotes its 1st and 2nd derivatives w.r.t. 'r' then. $\nabla^2 u = u'' + \frac{2u'}{r}$

4. Find Laplacian of function

   $T(r, \theta, \phi) = r(\cos\theta + \sin\theta\cos\phi)$

11.4 Summary: Dear students, in this unit we studied that

1. Laplace operator is the divergence of the gradient of a function

2. ∇^2 is a scalar operator whereas $\hat{\nabla}$ is a vector operator.

3. The equation $\nabla^2 u = 0$ is known as Laplace equation, when u is a scalar function.

4. The function which satisfies the Laplace equation is known as harmonic function.

11.5 Glossary

- **Operator :** In mathematics, any symbol that indicates an operation to be performed. Examples are $\sqrt{x}$ (which indicates the square root is to be taken) and $\frac{d}{dx}$ (which indicates differentiation with respect to x is to be performed).

- **Laplace Equation:** An equation having the second order partial derivatives of the form

  $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

  where ∇^2 is Laplacian operator.

- **Divergence:** Divergence is a vector operator that operates on a vector field. The latter can be though as representing field represents a flow of a liquid or gas, where each vector in the vector field represents a velocity vector of the moving fluid.

  $$\nabla \cdot F = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z},\right] \cdot [f, g, h]$$

  $$= \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

11.6 Answers to Self Check Exercise

1. Prove using the concept of ∇^2

2. Verified using the concept of ∇^2.

3. Verified using the concept of ∇^2.

4. Hint : Use Laplace operator for spherical coordinates.

11.7 References/Suggested Readings

1. Vector Analysis by J.G. Chakravorty and P.R. Ghosh.

2. Analytic Geometry of two and three Dimension & Vector Analysis by R.M. Khan.

3. Vector Calculus by J.N. Sharma & A.R. Vaishtha.

11.8 Terminal Questions

1. Show that $\nabla^2 \left(\frac{x}{r^3} \right) = 0$

2. Find the Laplacian of f where $f = x^2 + y^3 + xy^2z$.

3. Verify that functions are harmonic.

   (i) $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{\frac{-1}{2}}$

   (ii) $f(x, y, z) = x^2 + xy + 2y^2 - 3z^2 + xyz$

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Unit - 12

Important Vector Identities

Structure

12.1 Introduction

Dear student, in previous units we studied about different vector operators like gradient divergence and curl, which are associated with multiplication of scalar function, and dot and cross product of vector functions, respectively. Here in this unit we will study about the vector identities we are obtained from sum and product of vector, for gradient, divergence and curl. These vector identities are helpful further in this course. In this unit we will also study the vector identities which are the outcomes of product of different operations.

12.2 Learning Objectives: After studying this unit students will be able to

1. define and apply gradient of sum and product of two vector
2. define and apply divergence of sum and product of two vector
3. define and apply curl of sum and product of two vector
4. define and apply the product of two operators an vector and scalar.

12.3 Sum And Product of Vector Under Operator Gradient Divergences And Curl of Sums

Identity: Grad (ϕ + ϕ) = Grad ϕ + Grad ϕ, where ϕ and if are sealdr functions.

$$\hat{\nabla}\ (\phi + \phi) = \left(\frac{\partial \hat{i}}{\partial x} + \frac{\partial \hat{j}}{\partial y} + \frac{\partial \hat{k}}{\partial z}\right)(\phi + \phi)$$

$$= \hat{i} \frac{\partial}{\partial x}(\phi + \phi) + \hat{j} \frac{\partial}{\partial y}(\phi + \phi) + \hat{k} \frac{\partial}{\partial z}(\phi + \phi)$$

$$= \hat{i} \left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x}\right) + \hat{j} \left(\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y}\right) + \hat{k} \left(\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial z}\right)$$

$$= \hat{i} \left(\frac{\partial \phi}{\partial x}\right) + \hat{j} \left(\frac{\partial \phi}{\partial y}\right) + \hat{k} \left(\frac{\partial \phi}{\partial z}\right) + \hat{i} \left(\frac{\partial \phi}{\partial x}\right) + \hat{j} \left(\frac{\partial \phi}{\partial y}\right) + \hat{k} \left(\frac{\partial \phi}{\partial z}\right)$$

$$= \left(\hat{i} + \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)\phi + \left(\hat{i} + \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right)\phi$$

$\Rightarrow$ Grad $(\phi + \phi)$ = grad ϕ + grad ϕ

or $\nabla(\phi + \phi) = \nabla \phi + \nabla \phi$.

Identity 2: Div $(\hat{u} + \hat{v})$ = Div $\hat{u}$ + Div $\hat{v}$ where $\hat{u}$ and $\hat{v}$ are vectors.

Since div $(\hat{u} + \hat{v}) = \left(\hat{i} + \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) . (\hat{u} + \hat{v})$

$$\Rightarrow \quad \hat{i} \frac{\partial}{\partial x} . (\hat{u} + \hat{v}) + \hat{j} \frac{\partial}{\partial y} (\hat{u} + \hat{v}) + \hat{k} \frac{\partial}{\partial z} (\hat{u} + \hat{v})$$

$$\Rightarrow \quad \hat{i} \left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial x}\right) + \hat{j} \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial y}\right) + \hat{k} \left(\frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{v}}{\partial z}\right)$$

$$= \hat{i} \left(\frac{\partial \hat{u}}{\partial x} + \hat{j} \frac{\partial \hat{u}}{\partial y} + \hat{k} \frac{\partial \hat{u}}{\partial z}\right) + \hat{i} \left(\frac{\partial \hat{v}}{\partial x}\right) + \hat{j} \left(\frac{\partial \hat{v}}{\partial y}\right) + \hat{k} \left(\frac{\partial \hat{v}}{\partial z}\right)$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) . \hat{u} + \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) . \hat{v}$$

div $(\hat{u} + \hat{v})$ = div $\hat{u}$ + div $\hat{v}$

or $\hat{\nabla} . (\hat{u} + \hat{v}) = \hat{\nabla}\, \hat{u} + \hat{\nabla}\, \hat{v}$.

Identity 3 Curl $(\hat{u} + \hat{v})$ = Curl $\hat{u}$ + Curl $\hat{v}$

Curl $(\hat{u} + \hat{v}) = \hat{\nabla} \times (\hat{u} + \hat{v})$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \times (\hat{u} + \hat{v})$$

$$= \hat{i} \times \frac{\partial}{\partial x} \left(\hat{u}+\hat{v}\right) + \hat{j} \times \frac{\partial}{\partial y} \left(\hat{u}+\hat{v}\right) + \hat{k} \times \frac{\partial}{\partial z} \left(\hat{u}+\hat{v}\right)$$

$$= \hat{i} \times \left(\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial x}\right) + \hat{j} \times \left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial y}\right) + \hat{k} \times \left(\frac{\partial \hat{u}}{\partial z} + \frac{\partial \hat{v}}{\partial z}\right)$$

$$= \hat{i} \times \left(\frac{\partial \hat{u}}{\partial x}\right) \times \hat{j} \times \left(\frac{\partial \hat{u}}{\partial y}\right) \times \hat{k} \times \left(\frac{\partial \hat{u}}{\partial z}\right) + \hat{i} \times \frac{\partial \hat{v}}{\partial x} + \hat{j} \times \frac{\partial \hat{v}}{\partial y} + \hat{k} \times \frac{\partial \hat{v}}{\partial z}$$

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \times \hat{u} + \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) \hat{v}$$

$= \hat{\nabla}$ and $\hat{\nabla} \times \hat{u} + \hat{\nabla} \times \hat{v}$

Curl $\left(\hat{u}+\hat{v}\right)$ = Curl $\hat{u}$ + Curl $\hat{v}$

Gradient, Divergence and Curl of Product

Now, we will represent gradient, divergence and curl of products of point functions Since we know that gradient is defined on scalar point functions and divergence and curl are defined on vector point function, we can state some of the identities as

Identity 4: Grad (ϕϕ) = ∇ (ϕϕ)

$$= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}\right) (\phi\phi)$$

$$= \hat{i} \times \frac{\partial}{\partial x} (\phi\phi) + \hat{j} \frac{\partial}{\partial y} (\phi\phi) + \hat{k} \frac{\partial}{\partial z} (\phi\phi) \qquad \text{[using product rule]}$$

$$= \hat{i} \left(\phi \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial x} \phi\right) + \hat{j} \left(\phi \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial y} \phi\right) + \hat{k} \left(\phi \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial z} \phi\right)$$

us.

$$= \left(\hat{i}\phi \frac{\partial \phi}{\partial x} + \hat{j}\phi \frac{\partial \phi}{\partial y} + \hat{k}\phi \frac{\partial \phi}{\partial z}\right) + \left(\hat{i} \frac{\partial \phi}{\partial x} \phi + \hat{j} \frac{\partial \phi}{\partial y} \phi + \hat{k} \frac{\partial \phi}{\partial z} \phi\right)$$

$$= \phi\left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}\right) + \phi\left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}\right)$$

$$= \phi \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}\right) \phi + \phi \left(\hat{i} \frac{\partial \phi}{\partial x} + \hat{j} \frac{\partial \phi}{\partial y} + \hat{k} \frac{\partial \phi}{\partial z}\right) \phi$$

$= \phi \nabla \phi + \phi \nabla \phi$

Hence. grad (ϕϕ) = ϕ grad ϕ + ϕ grad ϕ.

Identity 5: Grad $\left(\hat{u}+\hat{v}\right)$ = $\hat{u}$ × curl $\hat{v}$ + $\hat{v}$ × curl $\hat{u}$ + $\hat{u}$. ∇ $\hat{v}$ + $\hat{v}$. ∇ $\hat{u}$.

Since $\hat{u}$ and $\hat{v}$ vector and product of two vectors is a scalar so $\hat{u}\,.\,\hat{v}$ can be operated on gradient.

$$\text{Grad } \left(\hat{u}.\hat{v}\right) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\left(\hat{u}.\hat{v}\right).$$

$$= \hat{i}\ \frac{\partial}{\partial x}\left(\hat{u}.\hat{v}\right) + \hat{j}\ \frac{\partial}{\partial y}\left(\hat{u}.\hat{v}\right) + \hat{k}\ \frac{\partial}{\partial z}\left(\hat{u}.\hat{v}\right)$$

$$= \hat{i}\left(\hat{u}.\frac{\partial \hat{v}}{\partial z}+\frac{\partial \hat{u}}{\partial z}.\hat{v}\right) + \hat{j}\left(\hat{u}.\frac{\partial \hat{v}}{\partial y}+\frac{\partial \hat{u}}{\partial y}.\hat{v}\right) + \hat{k}\left(\hat{u}.\frac{\partial \hat{v}}{\partial z}+\frac{\partial \hat{u}}{\partial z}.\hat{v}\right)$$

$$\text{grad } \left(\hat{u}.\hat{v}\right) = \left(\frac{\partial \hat{u}}{\partial x}\hat{v}\hat{i}+\frac{\partial \hat{u}}{\partial y}.\hat{v}\hat{j}+\frac{\partial \hat{u}}{\partial z}\hat{v}\hat{k}\right) + \left(\frac{\partial \hat{u}}{\partial x}\hat{v}\hat{i}+\frac{\partial \hat{u}}{\partial y}.\hat{v}\hat{j}+\frac{\partial \hat{u}}{\partial z}\hat{v}\hat{k}\right) \qquad (1)$$

Taking $\hat{u} \times \left(\hat{u}.\frac{\partial \hat{v}}{\partial x}\right) = \left(\hat{u}.\frac{\partial \hat{v}}{\partial x}\right) - (\hat{u}\,.\,\hat{i}\,)\frac{\partial \hat{v}}{\partial x}$

$$\left[\because \hat{a}\times\left(\hat{b}\times\hat{c}\right)=\left(\hat{a}.\hat{c}\right)\hat{b}-\left(\hat{a}.\hat{b}\right)\hat{c}\right]$$

$$\Rightarrow \qquad \left(\hat{u}.\frac{\partial \hat{v}}{\partial x}\right)\hat{i} = \hat{u}\times\left(\hat{i}\times\frac{\partial \hat{v}}{\partial x}\right) + \left(\hat{u}.\hat{i}\right)\frac{\partial \hat{v}}{\partial x}$$

Similarly $\left(\hat{u}.\frac{\partial \hat{v}}{\partial y}\right)\hat{j} = \hat{u}\times\left(\hat{j}\times\frac{\partial \hat{v}}{\partial y}\right) + \left(\hat{u}.\hat{j}\right)\frac{\partial \hat{v}}{\partial y}$

and $\left(\hat{u}.\frac{\partial \hat{v}}{\partial z}\right)\hat{k} = \hat{u}\times\left(\hat{k}\times\frac{\partial \hat{v}}{\partial z}\right) + \left(\hat{u}.\hat{k}\right)\frac{\partial \hat{v}}{\partial z}$

$$\therefore \qquad \left(\hat{u}.\frac{\partial \hat{v}}{\partial x}\right)\hat{i} + \left(\hat{u}.\frac{\partial \hat{v}}{\partial y}\right)\hat{j} + \left(\hat{u}.\frac{\partial \hat{v}}{\partial z}\right)\hat{k} = \hat{u}\times\left[\hat{i}\times\frac{\partial \hat{v}}{\partial x}+\hat{j}\times\frac{\partial \hat{v}}{\partial y}+\hat{k}\times\frac{\partial \hat{v}}{\partial z}\right]$$

$$+\ \hat{u}\,.\,\hat{i}\ +\frac{\partial \hat{v}}{\partial x} + \hat{u}\,.\,\hat{j}\ \frac{\partial \hat{v}}{\partial y} + \hat{u}\,.\,\hat{k}\ \frac{\partial \hat{v}}{\partial z}$$

$$= \hat{u}\times \text{curl } \hat{v} + \hat{u}\,.\nabla\ \hat{v}$$

$$\because \hat{v} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\hat{v}$$

Curl $\hat{v}$ = $\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\hat{v}$

= $\hat{i}\times\frac{\partial\hat{v}}{\partial x}+\hat{j}\times\frac{\partial\hat{v}}{\partial y}+\hat{k}\times\frac{\partial\hat{v}}{\partial z}$

$$\left(\hat{u}\frac{\partial\hat{v}}{\partial x}+\hat{i}+\hat{u}.\frac{\partial\hat{v}}{\partial y}\hat{j}+\hat{u}.\frac{\partial\hat{v}}{\partial z}\hat{k}\right)= \hat{u}\times \text{curd}\,\hat{v} + \hat{u}.\nabla\,\hat{v} \quad (2)$$

Interchanging of $\hat{u}$ of $\hat{v}$ in (2)

$$\left(\hat{v}.\frac{\partial\hat{u}}{\partial x}\hat{i}+\hat{v}.\frac{\partial\hat{u}}{\partial y}\hat{j}+\hat{v}.\frac{\partial\hat{u}}{\partial z}\hat{k}\right) = \hat{v}\times \text{curd}\,\hat{u} + \hat{v}.\nabla\,\hat{u} \quad (3)$$

using (2) and (3), (1) becomes.

grad $(\hat{u}.\hat{v})$ = $\hat{u}$ × curl $\hat{v}$ + $\hat{u}.\nabla\,\hat{v}$ + $\hat{v}$ × curl $\hat{u}$ + $\hat{v}.\nabla\,\hat{u}$

or

grad $(\hat{u}.\hat{v})$ = $\hat{u}$ × curl $\hat{v}$ + $\hat{u}$. grad $\hat{v}$ + $\hat{v}$ × curl $\hat{u}$ + $\hat{v}$ .grad $\hat{u}$.

Identity 6: div (ϕ$\hat{u}$) = ϕ div $\hat{u}$ + $\hat{u}$. grad ϕ

where ϕ is scalar function and $\hat{u}$ is a vector function.

div (ϕ$\hat{u}$) = ∇ . (ϕ$\hat{u}$)

= $\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)$. (ϕ$\hat{u}$)

= $\hat{i}\ \frac{\partial}{\partial x}\phi\hat{u} + \hat{j}\ \frac{\partial}{\partial y}\phi\hat{u} + \hat{k}\ \frac{\partial}{\partial z}\phi\hat{u}$.

= $\hat{i}\left(\phi\frac{\partial\hat{u}}{\partial x}+\frac{\partial\phi}{\partial x}\hat{u}\right)+\hat{j}\left(\phi\frac{\partial\hat{u}}{\partial y}+\frac{\partial\phi}{\partial y}\hat{u}\right)+\hat{k}\left(\phi\frac{\partial\hat{u}}{\partial z}+\frac{\partial\phi}{\partial z}\hat{u}\right)$

= $\phi\left(\hat{i}\frac{\partial\hat{u}}{\partial x}+\hat{j}\frac{\partial\hat{u}}{\partial y}+\hat{k}\frac{\partial\hat{u}}{\partial z}\right)+\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).\hat{u}$

= $\phi\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\hat{u}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi.\ \hat{v}$

= ϕ div$\hat{u}$ + grad ϕ. $\hat{u}$

div (ϕ$\hat{u}$) = ϕ div $\hat{u}$ + $\hat{u}$. grad ϕ $[\because \hat{a}.\hat{b}=\hat{b}.\hat{a}]$

Identity 7: div ($\hat{u} \times \hat{v}$) = $\hat{v}$ curl $\hat{u}$ - $\hat{u}$. curl $\hat{v}$

div ($\hat{u} \times \hat{v}$) = $\hat{\nabla}.(\hat{u} \times \hat{v})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right).\,(\hat{u} \times \hat{v})$$

$$= \hat{i}\ \frac{\partial}{\partial x}(\hat{u} \times \hat{v}) + \hat{j}\ \frac{\partial}{\partial y}(\hat{u} \times \hat{v}) + \hat{k}\ \frac{\partial}{\partial z}(\hat{u} \times \hat{v})$$

$$= \hat{i}\left(\frac{\partial \hat{u}}{\partial x} \times \hat{v} + \hat{u}\frac{\partial \hat{v}}{\partial x} \right) + \hat{j}\left(\hat{u} \times \frac{\partial \hat{v}}{\partial y} + \frac{\partial \hat{u}}{\partial y} \times \hat{v} \right) + \hat{k}\left(\frac{\partial \hat{u}}{\partial y} \times \hat{v} + \hat{u}\frac{\partial \hat{v}}{\partial y} \right)$$ [using product rule]

div ($\hat{u} \times \hat{v}$) $= \left\{ \hat{i}\left(\frac{\partial \hat{u}}{\partial x} \times \hat{v} \right) + \hat{j}\left(\frac{\partial \hat{u}}{\partial y} \times \hat{v} \right) + \hat{k}\left(\frac{\partial \hat{u}}{\partial z} \times \hat{v} \right) \right\}$

$$\left\{ \hat{i}\left(\hat{u} \times \frac{\partial \hat{v}}{\partial x} \right) + \hat{j}\left(\hat{u} \times \frac{\partial \hat{v}}{\partial y} \right) + \hat{k}\left(\hat{u} \times \frac{\partial \hat{v}}{\partial z} \right) \right\}$$

Since $\hat{a} \times \hat{b}$ = -$\hat{b} \times \hat{a}$, so

div ($\hat{u} \times \hat{v}$) = $\left\{ \hat{i}\left(\frac{\partial \hat{u}}{\partial x} \times \hat{v} \right) + \hat{j}\left(\frac{\partial \hat{u}}{\partial y} \times \hat{v} \right) + \hat{k}\left(\frac{\partial \hat{u}}{\partial z} \times \hat{v} \right) \right\}$

$$= \left\{ \hat{i}\left(\frac{\partial \hat{v}}{\partial x} \times \hat{u} \right) + \hat{j}\left(\frac{\partial \hat{v}}{\partial y} \times \hat{u} \right) + \hat{k}\left(\frac{\partial \hat{v}}{\partial z} \times \hat{u} \right) \right\}$$

$$= \left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} \right).\,\hat{v} + \left(\hat{j} \times \frac{\partial \hat{u}}{\partial y} \right).\,\hat{v} + \left(\hat{k} \times \frac{\partial \hat{u}}{\partial z} \right).\,\hat{v}$$

$$\left\{ \left(\hat{i} \times \frac{\partial \hat{v}}{\partial x} \right).\hat{u} + \left(\hat{j} \times \frac{\partial \hat{v}}{\partial y} \right).\hat{u} + \left(\hat{k} \times \frac{\partial \hat{v}}{\partial z} \right).\hat{u} \right\}$$

$$= \left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} + \hat{j} \times \frac{\partial \hat{u}}{\partial y} + \hat{k} \times \frac{\partial \hat{u}}{\partial z} \right).\,\hat{v}$$ [using the property of scalar treble product]

$$- \left(\hat{i} \times \frac{\partial \hat{v}}{\partial x} + \hat{j} \times \frac{\partial \hat{v}}{\partial y} + \hat{k} \times \frac{\partial \hat{v}}{\partial z} \right).\,\hat{u}$$

div ($\hat{u} \times \hat{v}$) = curl div $\hat{u}\,.\,\hat{v}$ = curl div $\hat{v}\,.\,\hat{u}$

= $\hat{v}$. curl $\hat{u}$ - $\hat{u}$. curl $\hat{v}$ [$\because \hat{a}.\hat{b} = \hat{b}.\hat{a}$]

Identity 8: curl ($\phi\hat{u}$) = grad $\phi \times \hat{u}$ + ϕ curl $\hat{u}$

curl ($\phi\hat{u}$) = $\hat{\nabla} \times (\phi\hat{u})$

$$\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times \phi\hat{u}$$

$$= \hat{i} \times \frac{\partial}{\partial x}(\phi\hat{u}) + \hat{j}\times\frac{\partial}{\partial y}(\phi\hat{u}) + \hat{k}\ \frac{\partial}{\partial z}(\phi\hat{u})$$

$$= \hat{i} \times\left(\phi\frac{\partial\hat{u}}{\partial x}+\frac{\partial\phi}{\partial x}\hat{u}\right)+\hat{j}\times\left(\phi\frac{\partial\hat{u}}{\partial y}+\frac{\partial\phi}{\partial y}\hat{u}\right)+\hat{k}\times\left(\phi\frac{\partial\hat{u}}{\partial z}+\frac{\partial\phi}{\partial z}\hat{u}\right)$$

$$= \phi\left(\hat{i}\times\frac{\partial\hat{u}}{\partial x}+\hat{j}\times\frac{\partial\hat{u}}{\partial y}+\hat{k}\times\frac{\partial\hat{u}}{\partial z}\right)+\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\times\hat{u}$$

curl ($\phi\hat{u}$) = ϕ curl $\hat{u}$ + grad $\phi \times \hat{u}$

Identity 9 curl ($\hat{u}\times\hat{v}$) = $(\hat{v}\,.\nabla)\hat{u}$ - $(\hat{u}\,.\nabla)\hat{v}$ + $\hat{u}$ div $\hat{v}$ - $\hat{v}$ div $\hat{v}$

$$\text{curl }(\hat{u}\times\hat{v}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\hat{u}\times\hat{v})$$

$$= \hat{i}\times\frac{\partial}{\partial x}(\hat{u}\times\hat{v}) + \hat{j}\times\frac{\partial}{\partial y}(\hat{u}\times\hat{v}) + \hat{k}\times\frac{\partial}{\partial z}(\hat{u}\times\hat{v})$$

$$= \hat{i}\times\left[\left(\frac{\partial\hat{u}}{\partial x}\times\hat{v}\right)+\left(\hat{u}\times\frac{\partial\hat{v}}{\partial x}\right)\right]+\hat{j}\times\left[\left(\frac{\partial\hat{u}}{\partial y}\times\hat{v}\right)+\left(\hat{u}\times\frac{\partial\hat{v}}{\partial y}\right)\right]+\hat{k}\times\left[\left(\frac{\partial\hat{u}}{\partial z}\times\hat{v}\right)+\left(\hat{u}\times\frac{\partial\hat{v}}{\partial z}\right)\right]$$

$$= \left\{\hat{i}\times\left(\frac{\partial\hat{u}}{\partial x}\times\hat{v}\right)+\hat{j}\times\left(\frac{\partial\hat{u}}{\partial y}\times\hat{v}\right)+\hat{k}\times\left(\frac{\partial\hat{u}}{\partial z}\times\hat{v}\right)\right\} +$$

$$\left\{\hat{i}\times\left(\hat{u}\times\frac{\partial\hat{v}}{\partial x}\right)+\hat{j}\times\left(\hat{u}\times\frac{\partial\hat{v}}{\partial y}\right)+\hat{k}\times\left(\hat{u}\times\frac{\partial\hat{v}}{\partial z}\right)\right\}$$

using $\hat{a}\times\hat{b}\times\hat{c} = (\hat{a}\,.\hat{c})\hat{b} - (\hat{a}\,.\hat{b})\hat{c}$

$$\text{curl }(\hat{u}\times\hat{v}) = \left(\hat{i}.\hat{v}\right)\frac{\partial\hat{u}}{\partial x} - \left(\hat{i}.\frac{\partial\hat{u}}{\partial x}\right)\hat{v} + \left(\hat{j}.\hat{v}\right)\frac{\partial\hat{u}}{\partial y} - \left(\hat{j}.\frac{\partial\hat{u}}{\partial y}\right)\hat{v} + \left(\hat{k}.\hat{v}\right)\frac{\partial\hat{u}}{\partial z} - \left(\hat{k}.\frac{\partial\hat{u}}{\partial z}\right)\hat{v}$$

$$+ \left(\hat{i}.\frac{\partial\hat{v}}{\partial x}\right)\hat{u} - (\hat{i}\,.\hat{u})\frac{\partial\hat{v}}{\partial x} + \left(\hat{j}.\frac{\partial\hat{v}}{\partial y}\right)\hat{u} - \left(\hat{j}.\hat{v}\right)\frac{\partial\hat{v}}{\partial y} + \left(\hat{j}.\frac{\partial\hat{v}}{\partial y}\right)\hat{u} - \left(\hat{k}.\hat{v}\right)\frac{\partial\hat{v}}{\partial z}$$

$$= \left(\hat{i}.\hat{v}\right)\frac{\partial \hat{u}}{\partial x} + \left(\hat{j}.\hat{v}\right)\frac{\partial \hat{u}}{\partial y} + \left(\hat{k}.\hat{v}\right)\frac{\partial \hat{u}}{\partial z} - \left[\hat{i}.\frac{\partial \hat{u}}{\partial x} + \hat{j}.\frac{\partial \hat{u}}{\partial y} + \hat{k}.\frac{\partial \hat{u}}{\partial z}\right]\hat{v} + \left[\hat{i}.\frac{\partial \hat{v}}{\partial x} + \hat{j}.\frac{\partial \hat{v}}{\partial y} + \hat{k}.\frac{\partial \hat{v}}{\partial z}\right]\hat{u} - \left[\left(\hat{i}.\hat{u}\right)\frac{\partial \hat{v}}{\partial x} + \left(\hat{j}.\hat{u}\right)\frac{\partial \hat{v}}{\partial y} + \left(\hat{k}.\hat{u}\right)\frac{\partial \hat{v}}{\partial z}\right]$$

$= \hat{v}$ div $\hat{u}$ - $(\hat{v}.\nabla)\hat{u}$ + $\hat{u}$ div $\hat{v}$ - $(\hat{u}.\nabla)\hat{v}$

curl $(\hat{u}\times\hat{v}) = \hat{u}$ div $\hat{v}$ - $\hat{v}$ div $\hat{u}$ - $(\hat{v}.\nabla)\hat{u}$ - $(\hat{u}.\nabla)\vec{v}$

Now Let us try some examples based on these identities.

Example 1: If $\phi = x^2 + y^2 + z^2$ and $\hat{u} = x\hat{i} + y\hat{j} + z\hat{k}$

then show that div $(\phi u) = 5u$

Solution: Given $\phi = x^2 + y^2 + z^2$ and

$u = x\hat{i} + y\hat{j} + z\hat{k}$

Since div $(\phi\hat{u})$ = grad ϕ . $\hat{u}$ + ϕ div $\hat{u}$ (1)

Now, grad $\phi = \hat{\nabla}\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

grad $\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

div $\hat{u} = \hat{\nabla}\hat{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k})$$

$= 1 + 1 + 1$

div $\hat{u} = 3$

$\therefore$ from, (1)

div $(\phi\hat{u}) = (2x\hat{i} + 2y\hat{j} + 2z\hat{k}) . (x\hat{i} + y\hat{j} + z\hat{k}) + (x^2 + y^2 + z^2) \times 3$

$= 2x^2 + 2y^2 + 2z^2 + 3x^2 + 3y^2 + 3z^2$

$= 5(x^2 + y^2 + z^2)$

div $(\phi\hat{u}) = 5\phi$ $[\because x^2 + y^2 + z^2 = \phi]$

Example 2: Prove that

div $(\hat{a}\times\hat{u}) = \hat{a}$. curl $\hat{u}$ where $\hat{a}$ is a constant vector.

Solution: div $(\hat{a}\times\hat{u}) = \hat{u}$ curl $\hat{a}$ - $\hat{a}$, curl $\hat{u}$ [using identity div $(\hat{u}\times\hat{v})$]

Since $\hat{a}$ is a constant vector, so

curl $\hat{a}$ = 0

$\therefore$ div ($\hat{a} \times \hat{u}$) = - $\hat{a}$ curl $\hat{u}$.

Example: Show that curl ($\hat{a} \times \hat{u}$) = $\hat{a}$ div $\hat{u}$ - ($\hat{a} . \hat{\nabla}$) $\hat{u}$

where $\hat{a}$ is a constant vector.

Solution: Since curl ($\hat{a} \times \hat{u}$) = ($\hat{u} . \hat{\nabla}$) $\hat{a}$ - ($\hat{a} . \hat{\nabla}$) $\hat{u}$ + $\hat{a}$ div $\hat{u}$ - $\hat{u}$ div $\hat{a}$

Since $\hat{a}$ is a constant vector

So div $\hat{a}$ = 0 also ($\hat{u} . \hat{\nabla}$) $\hat{a}$ = 0

So curl ($\hat{a} \times \hat{u}$) = $\hat{a}$ div $\hat{u}$ - ($\hat{a} . \hat{\nabla}$) $\hat{u}$

Example 3: $\hat{\nabla} \times \left(\hat{a} \times \hat{r}\right)$ where $\hat{a}$ is constant vector

$\hat{a}$ = a ($\hat{i}$ + $\hat{j}$ + $\hat{k}$)

$\hat{r}$ = x $\hat{i}$ + y $\hat{j}$ + z $\hat{k}$

then $\hat{a} \times \hat{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix}$

= $\hat{i}$ a(z - y) - $\hat{j}$ a(z - x) + $\hat{k}$ a(y - x)

$\hat{a} \times \hat{r}$ = a $\left[\hat{i}\left(z-y\right)-\hat{j}a\left(z-x\right)+\hat{k}a\left(y-x\right)\right]$

$$\hat{\nabla} \times \left(\hat{a} \times \hat{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a\left(z-y\right) & -a\left(z-x\right) & a\left(y-x\right) \end{vmatrix}$$

$$= a\hat{i}\left[\left(\frac{\partial}{\partial y}\left(y-x\right)+\frac{\partial}{\partial z}\left(z-x\right)\right)-\hat{j}\left(\frac{\partial}{\partial x}\left(y-x\right)-\frac{\partial}{\partial z}\left(z-y\right)\right)+\hat{k}\left(\frac{-\partial}{\partial x}\left(z-x\right)+\frac{-\partial}{\partial y}\left(z-y\right)\right)\right]$$

= a $\left[i\left(1+1\right)-j\left(-1-1\right)+k\left(1+1\right)\right]$

= a (2 $\hat{i}$ + 2 $\hat{j}$ + 2 $\hat{k}$)

$\hat{\nabla} \times \left(\hat{a} \times \hat{r}\right)$ = 2a ($\hat{i}$ + $\hat{j}$ + $\hat{k}$)

So, $\hat{\nabla} \times \left(\hat{a}\times\hat{r}\right)= 2$ where $\hat{a}$ is constant vector

Example 4: Show that curl $\left(\dfrac{\hat{a}\times\hat{r}}{r^3}\right) = \dfrac{a}{r^3} + \dfrac{3\hat{r}}{r^5}\ \hat{a}.\hat{r}$

Where $\hat{a}$ is constant vector

Curl $\left(\dfrac{\hat{a}\times\hat{r}}{r^3}\right) = \hat{\nabla} \times \left(\dfrac{1}{r^3}\hat{a}\times\hat{r}\right)$

$= \nabla\left(\dfrac{1}{r^3}\right)\times\left(\hat{a}\times\hat{r}\right) + \dfrac{1}{r^3}\ \hat{\nabla}\times\left(\dfrac{1}{r^3}\hat{a}\times\hat{r}\right)$

Since $\nabla\ r^n = nr^{n-2}\ \vec{r}$

So $\nabla\ r^{-3} = -3r^{-3-2}\ r^{-1}$

and $\hat{\nabla} \times \left(\hat{a}\times\hat{r}\right)= 2\hat{a}$ (example 3)

So curl $\left(\dfrac{\hat{a}\times\hat{r}}{r^3}\right) = -3r^{-5}\ \hat{r}\times\left(\hat{a}\times\hat{r}\right) + \dfrac{1}{r^3}\times 2\hat{a}$

$= \dfrac{-3}{r^5}\left[\hat{r}\times\left(\hat{a}\times\hat{r}\right)+\dfrac{2\hat{a}}{r^3}\right]$

$= \dfrac{-3}{r^5}\ \left[\left(\hat{r}.\hat{r}\right)\hat{a}-\left(\hat{r}.\hat{a}\right)\hat{r}\right] + \dfrac{2\hat{a}}{r^3}$ [using $\hat{a}\times(\hat{b}\times\hat{c}) = (\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c}$]

$= \dfrac{-3}{r^5}\left[\hat{r}^2\hat{a}-\left(\hat{r}.\hat{a}\right)\hat{r}\right] + \dfrac{2\hat{a}}{r^3}$

$= \dfrac{3\hat{a}}{r^3} + \dfrac{3\hat{r}}{r^5}\left(\hat{r}.\hat{a}\right)+\dfrac{2\hat{a}}{r^3}$

Hence curl $\left(\dfrac{\hat{a}\times\hat{r}}{r^3}\right) = \dfrac{3\hat{r}}{r^5}\left(\hat{r}.\hat{a}\right) - \dfrac{\hat{a}}{r^3}$ [$\because\ \hat{r}\,.\,\hat{a} = \hat{a}\,.\,\hat{r}$]

Self Check Exercise

Q.1 If $\hat{a}$ and $\hat{b}$ are constant vector then show that div $\left\{\left(\hat{r}.\hat{a}\right)\times\hat{b}\right\} = -2\hat{b}.\hat{a}$

Q.2 If $\hat{a}$ and $\hat{b}$ are constant vector then curl $\left\{\left(\hat{r}\times\hat{a}\right)\times\hat{b}\right\} = \hat{b}\times\hat{a}$

12.4 Second Order Differential Operators

Since we know that gradient of a scalar function is a vector quantity. On taking divergence and curl of a grad operator, we get other operators of higher order. Similar concepts can be extend to divergence and curl operator. In this section we will learn such identities.

Identity 1: div (grad ϕ) = $\nabla . \nabla \phi = \nabla^2 \phi$

Since div (grad ϕ) = $\nabla . \nabla \phi$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \hat{i}\frac{\partial^2 \phi}{\partial x^2}+\hat{j}\frac{\partial^2 \phi}{\partial y^2}+\hat{k}\frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi \qquad \because \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} = \nabla^2$$

Identity 2: Curl (grad ϕ) = $\hat{0}$

Curl (grad ϕ) = $\hat{\nabla} \times (\hat{\nabla}\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \hat{i}\times\frac{\partial}{\partial x}\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)+\hat{j}\times\frac{\partial}{\partial y}\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$+ \hat{k}\times\frac{\partial}{\partial z}\left(\hat{i}\frac{\partial \phi}{\partial x}+\hat{j}\frac{\partial \phi}{\partial y}+\hat{k}\frac{\partial \phi}{\partial z}\right)$$

(curl/grad ϕ) = $\hat{i}\times\left(\hat{i}\frac{\partial^2 \phi}{\partial x^2}+\hat{j}\frac{\partial^2 \phi}{\partial x\partial y}+\hat{k}\frac{\partial^2 \phi}{\partial x\partial z}\right)+\hat{j}\times\left(\hat{i}\frac{\partial^2 \phi}{\partial z\partial y}+\hat{j}\frac{\partial^2 \phi}{\partial y^2}+\hat{k}\frac{\partial^2 \phi}{\partial y\partial z}\right)$

$$+ \hat{k}\times\left(\hat{i}\frac{\partial^2 \phi}{\partial z\partial x}+\hat{j}\frac{\partial^2 \phi}{\partial z\partial y}+\hat{k}\frac{\partial^2 \phi}{\partial z^2}\right)$$

Since $\hat{i}\times\hat{i} = 0$, $\hat{j}\times\hat{j} = 0$, $\hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = \hat{k}$, $\hat{i}\times\hat{k} = -\hat{j}$, $\hat{j}\times\hat{i} = -\hat{k}$ $\hat{j}\times\hat{k} = \hat{i}$ $\hat{k}\times\hat{i} = \hat{j}$, $\hat{k}\times\hat{j} = -\hat{i}$, so

curl (grad ϕ) = $\hat{k}\frac{\partial^2 \phi}{\partial x\partial y}-\hat{j}\frac{\partial^2 \phi}{\partial x\partial z}-\hat{k}\frac{\partial^2 \phi}{\partial x\partial y}+\hat{i}\frac{\partial^2 \phi}{\partial y\partial z}+\hat{j}\frac{\partial^2 \phi}{\partial z\partial x}-\hat{i}\frac{\partial^2 \phi}{\partial z\partial y} = 0$

⇒ curl (grad ϕ) = $\hat{0}$

Identity 3: div (curl $\hat{u}$) = $\hat{0}$

div (curl $\hat{u}$) = $\hat{\nabla}.(\hat{\nabla}\times\hat{u})$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\times\hat{u}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\left(\hat{i}\times\frac{\partial\hat{u}}{\partial x}+\hat{j}\times\frac{\partial\hat{u}}{\partial y}+\hat{k}\times\frac{\partial\hat{u}}{\partial z}\right)$$

$$\hat{i}\,.\frac{\partial}{\partial x}\left(\hat{i}\times\frac{\partial\hat{u}}{\partial x}+\hat{j}\times\frac{\partial\hat{u}}{\partial y}+\hat{k}\times\frac{\partial\hat{u}}{\partial z}\right)+\hat{j}\,\frac{\partial}{\partial y}\left(\hat{i}\times\frac{\partial\hat{u}}{\partial x}+\hat{j}\times\frac{\partial\hat{u}}{\partial y}+\hat{k}\times\frac{\partial\hat{u}}{\partial z}\right)$$

$$+\hat{k}\,.\frac{\partial}{\partial z}\left(\hat{i}\times\frac{\partial\hat{u}}{\partial x}+\hat{j}\times\frac{\partial\hat{u}}{\partial y}+\hat{k}\times\frac{\partial\hat{u}}{\partial z}\right)$$

$$= \hat{i}\,.\left(\hat{i}\times\frac{\partial^2\hat{u}}{\partial x^2}+\hat{j}\times\frac{\partial^2\hat{u}}{\partial x\partial y}+\hat{k}\times\frac{\partial^2\hat{u}}{\partial z}\right)+\hat{j}\,.\left(\hat{i}\times\frac{\partial^2\hat{u}}{\partial y\partial x^2}+\hat{j}\times\frac{\partial^2\hat{u}}{\partial y^2}+\hat{k}\times\frac{\partial^2\hat{u}}{\partial y\partial z}\right)$$

$$+\hat{k}\left(\hat{i}\times\frac{\partial^2\hat{u}}{\partial z\partial x}+\hat{j}\times\frac{\partial^2\hat{u}}{\partial z\partial y}+\hat{k}\times\frac{\partial^2\hat{u}}{\partial z^2}\right)$$

$$= \hat{i}\,.\hat{i}\times\frac{\partial^2\hat{u}}{\partial x^2}+\hat{i}\,.\,\hat{j}\times\frac{\partial^2\hat{u}}{\partial x\partial y}+\hat{i}\,.\hat{k}\times\frac{\partial^2\hat{u}}{\partial z\partial x}+\hat{j}\,.\hat{i}\times\frac{\partial^2\hat{u}}{\partial y\partial x}+\hat{j}\,.\,\hat{j}\,\frac{\partial^2\hat{u}}{\partial y^2}$$

$$+\hat{j}\,.\hat{k}\times\frac{\partial^2\hat{u}}{\partial y\partial z}+\hat{k}\,.\hat{i}\times\frac{\partial^2\hat{u}}{\partial z\partial x}+\hat{k}\,.\,\hat{j}\times\frac{\partial^2\hat{u}}{\partial z\partial y}+\hat{k}\,.\hat{k}\times\frac{\partial^2\hat{u}}{\partial z^2}$$

$\hat{a}.(\hat{b}\times\hat{c}) = \left(\hat{a}.\hat{b}\right).\hat{c}$ interchanging the scalar triple product in the terms

$$\hat{i}\,.\hat{i}\times\frac{\partial^2\hat{u}}{\partial x^2} = 0$$

$$\hat{j}\,.\,\hat{j}\times\frac{\partial^2\hat{u}}{\partial y^2} = 0$$

$$\hat{k}\,.\hat{k}\times\frac{\partial^2\hat{u}}{\partial y^2} = 0$$

We left with $\because$ scalar triple product of two equal vector is zero.

div (curl $\hat{u}$) = $\hat{k}\times\frac{\partial^2\hat{u}}{\partial x\partial y} - \hat{j}\times\frac{\partial^2\hat{u}}{\partial z\partial x} + -\hat{k}\times\frac{\partial^2\hat{u}}{\partial y\partial x} + \hat{j}\times\frac{\partial^2\hat{u}}{\partial z\partial x} - \hat{i}\times\frac{\partial^2\hat{u}}{\partial z\partial y}$

$= 0$

div (curl $\hat{u}$) = 0

Identity 4: grad (div $\hat{v}$) = curl curl $\hat{v}$ + $\nabla^2 \hat{v}$

or curl curl $\hat{u}$ = grad (div $\hat{u}$) - $\nabla^2 \hat{u}$

Solution: curl curl $\hat{v}$ = $\hat{\nabla} \times (\hat{\nabla} \times \hat{u})$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \times \hat{u}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} + \hat{j} \times \frac{\partial \hat{u}}{\partial y} + \hat{k} \times \frac{\partial \hat{u}}{\partial z} \right)$$

$$= \hat{i} \times \frac{\partial}{\partial x}\left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} + \hat{j}\frac{\partial \hat{u}}{\partial y} + \hat{k}\frac{\partial \hat{u}}{\partial z} \right) + \hat{j} \times \frac{\partial}{\partial y}\left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} + \hat{j} \times \frac{\partial \hat{u}}{\partial y} + \hat{k} \times \frac{\partial \hat{u}}{\partial z} \right)$$

$$+ \hat{k} \times \frac{\partial}{\partial z}\left(\hat{i} \times \frac{\partial \hat{u}}{\partial x} + \hat{j} \times \frac{\partial \hat{u}}{\partial y} + \hat{k} \times \frac{\partial \hat{u}}{\partial z} \right)$$

$$= \hat{i} \times \left(\hat{i} \times \frac{\partial^2 \hat{u}}{\partial x^2} + \hat{j} \times \frac{\partial^2 \hat{u}}{\partial x \partial y} + \hat{k} \times \frac{\partial^2 \hat{u}}{\partial x \partial z} \right) + \hat{j} \times \left(\hat{i} \times \frac{\partial^2 \hat{u}}{\partial x \partial y} + \hat{j} \times \frac{\partial^2 \hat{u}}{\partial y^2} + \hat{k} \times \frac{\partial^2 \hat{u}}{\partial z \partial y} \right)$$

$$+ \hat{k} \times \left(\hat{i} \times \frac{\partial^2 \hat{u}}{\partial z \partial x} + \hat{j} \times \frac{\partial^2 \hat{u}}{\partial z \partial y} + \hat{k} \times \frac{\partial^2 \hat{u}}{\partial z^2} \right)$$

$$= \hat{i} \times \hat{i} \times \frac{\partial^2 \hat{u}}{\partial x^2} + \hat{i} \times \hat{j} \times \frac{\partial^2 \hat{u}}{\partial x \partial y} + \hat{i} \times \hat{k} \times \frac{\partial^2 \hat{u}}{\partial x \partial z} + \hat{j} \times \hat{i} \times \frac{\partial^2 \hat{u}}{\partial x \partial y} + \hat{j} \,.\, \hat{j} \times \frac{\partial^2 \hat{u}}{\partial y^2}$$

$$+ \hat{j} \times \hat{k} \times \frac{\partial^2 \hat{u}}{\partial z \partial y} + \hat{k} \times \hat{i} \times \frac{\partial^2 \hat{u}}{\partial z \partial x} + \hat{k} \times \hat{j} \times \frac{\partial^2 \hat{u}}{\partial z \partial y} + \hat{k} \times \hat{k} \times \frac{\partial^2 \hat{u}}{\partial z^2}$$

$$= \left(\hat{i}.\frac{\partial^2 \hat{u}}{\partial x^2} \right)\hat{i} - (\hat{i}\,.\,\hat{i})\frac{\partial^2 \hat{u}}{\partial x^2} + \left(\hat{i}.\frac{\partial^2 \hat{u}}{\partial x \partial y} \right)\hat{j} - (\hat{i}\,.\,\hat{j})\frac{\partial^2 \hat{u}}{\partial x \partial y} + \left(\hat{i}.\frac{\partial^2 \hat{u}}{\partial x \partial z} \right)\hat{k} - (\hat{i}\,.\,\hat{k})\frac{\partial^2 \hat{u}}{\partial x \partial z}$$

$$+ \left(\hat{j}.\frac{\partial^2 \hat{u}}{\partial x \partial y} \right)\hat{i} - (\hat{j}\,.\,\hat{i})\frac{\partial^2 \hat{u}}{\partial x \partial y} + \left(\hat{j}.\frac{\partial^2 \hat{u}}{\partial y^2} \right)\hat{j} - (\hat{j}\,.\,\hat{j})\frac{\partial^2 \hat{u}}{\partial y^2} + \left(\hat{j}.\frac{\partial^2 \hat{u}}{\partial z \partial y} \right)\hat{k} - (\hat{j}\,.\,\hat{k})\frac{\partial^2 \hat{u}}{\partial z \partial y}$$

$$+ \left(\hat{k}.\frac{\partial^2 \hat{u}}{\partial z \partial x} \right)\hat{i} - (\hat{k}\,.\,\hat{i})\frac{\partial^2 \hat{u}}{\partial z \partial x} + \left(\hat{k}.\frac{\partial^2 \hat{u}}{\partial z \partial y} \right)\hat{j} - (\hat{k}\,.\,\hat{j})\frac{\partial^2 \hat{u}}{\partial z \wp y} + \left(\hat{k}.\frac{\partial^2 \hat{u}}{\partial z^2} \right)\hat{k} - (\hat{k}\,.\,\hat{k})\frac{\partial^2 \hat{u}}{\partial z^2}$$

Since $\hat{i}\,.\,\hat{i}$ = $\hat{j}\,.\,\hat{j}$ = $\hat{k}\,.\,\hat{k}$ = 0, and any other combination is zero i.e. $\hat{i}\,.\,\hat{j}$ = $\hat{i}\,.\,\hat{k}$ = $\hat{j}\,.\,\hat{i}$ = $\hat{j}\,.\,\hat{k}$ = $\hat{k}\,.\,\hat{i}$ = $\hat{k}\,.\,\hat{j}$ = 0

So we left with

$$\text{curl curl } \vec{u} = \left(\hat{i}.\frac{\partial^2\hat{u}}{\partial x^2}\right)\hat{i} - \frac{-\partial^2\hat{u}}{\partial x^2} + \left(\hat{i}.\frac{\partial^2\hat{u}}{\partial x\partial y}\right)\hat{j} - \left(\hat{i}.\frac{\partial^2\hat{u}}{\partial x\partial z}\right)\hat{k}$$

$$+ \left(\hat{j}.\frac{\partial^2\hat{u}}{\partial x\partial y}\right)\hat{i} + \left(\hat{j}.\frac{\partial^2\hat{u}}{\partial y^2}\right)\hat{j} - \frac{\partial^2\hat{u}}{\partial y^2} + \left(\hat{j}.\frac{\partial^2\hat{u}}{\partial z\partial y}\right)$$

$$\hat{k} + \left(\hat{k}.\frac{\partial^2\hat{u}}{\partial z\partial x}\right)\hat{i} + \left(\hat{k}.\frac{\partial^2\hat{u}}{\partial z\partial y}\right)\hat{j} + \left(\hat{k}.\frac{\partial^2\hat{u}}{\partial z^2}\right)\hat{k} - \frac{\partial^2\hat{u}}{\partial z^2}$$

$$= \hat{\nabla}\ \hat{\nabla}\,.\,\hat{u} - \left(\frac{\partial^2\hat{u}}{\partial x^2} + \frac{\partial^2\hat{u}}{\partial y^2} + \frac{\partial^2\hat{u}}{\partial z^2}\right)$$

$$\text{curl curl } \hat{u} = \hat{\nabla}\ \hat{\nabla}\,.\,\hat{u} - \nabla^2\,\hat{u}$$

Let us try following example.

Example 1: If $\hat{u} = x^2y\,\hat{i} + xz\,\hat{j} + xyz\,\hat{k}$

$$\text{then} \quad \text{curl } \hat{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & xy^3 \end{vmatrix}$$

$$\text{curl } \hat{u} = \left[\hat{i}\left(xz - x\right) - \hat{j}\left(xz - x\right)\hat{k}\left(xz - x^2\right)\right]$$

Now div curl $\hat{u} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right).\ \left[\hat{i}\left(xz - x\right) - yz\hat{k} + \left(z - x^2\right)\hat{k}\right]$

= (z - 1) - z + 1

div curl $\hat{u}$ = 0

Example 2: If ϕ = x^2y + 2xyz + z^2 show that curl grad ϕ = 0

Solution: Given ϕ = x^2y + 2xyz + z^2

grad ϕ = $\left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)$(x^2y + 2xyz + z2) grad ϕ (2xy + 2yz) $\hat{i}$ + (x^2 + 2xz) $\hat{j}$ + (2xy + 2z) $\hat{k}$

curl grad ϕ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+2yz & x^2+2xz & 2xy+2y \end{vmatrix}$

= $\hat{i}$ [2x - 2x]- $\hat{j}$ [2y - 2y] + $\hat{k}$ [2x - 2z - (2x - 2z)

= 0 - 0 + $\hat{k}$ (2x - 2z - 2x + 2z)

= 0 - 0 + $\hat{k}$ (2x - 2z - 2x + 2z)

= 0

⇒ curl grad ϕ = 0

Self Check Exercise 2

Q.1 If $\hat{A}$ = $x^2\hat{i}$ + $y^2\hat{j}$ + $z^2\hat{k}$

$\hat{B}$ = z$\hat{i}$ + zx $\hat{j}$ + xy$\hat{k}$

then show that grad div $\left(\hat{A}\times\hat{B}\right)$ = $\hat{0}$

Q.2 ∇^2 r^m = m(m + 1)r^{m-2}

Q.3 ∇^2 f(r) = $\dfrac{2}{r}$ f'(r) + r''(r)

12.5 Summary: Dear students in this unit, we studied that

1. Gradient, divergence and curl, all the three differential operators are distributive with respect to sum of vectors.
2. div grad (ϕϕ) = ϕ grad ϕ + p grad ϕ.
3. grad ($\hat{u}$. $\hat{v}$) = $\hat{u}$ × curl $\hat{v}$ + $\hat{v}$ × curl $\hat{u}$ + $\hat{u}$. ∇ $\hat{v}$ + $\hat{v}$ + $\hat{v}$. ∇ $\hat{u}$
4. div (ϕ$\hat{u}$) = ϕ div $\hat{u}$ + $\hat{u}$ + $\hat{u}$. grad ϕ
5. div ($\hat{u}$ × $\hat{v}$) = $\hat{v}$. curl $\hat{u}$ - $\hat{u}$ curl $\hat{v}$
6. curl (ϕ $\hat{u}$) = grad ϕ × $\hat{u}$ + ϕ curl $\hat{u}$
7. curl = ($\hat{v}$. ∇) $\hat{u}$ - ($\hat{u}$. ∇) $\hat{v}$ + ($\hat{u}$ div $\hat{v}$) - ($\hat{v}$ div $\hat{u}$)
8. div (grad ϕ) = ∇^2 ϕ
9. curl (grad ϕ) = 0

10. div (curl $\hat{u}$) = 0

11. grad (div $\hat{v}$) = curl curl $\hat{v}$ $+\nabla^2 \hat{v}$

12.6 Glossary:

Scalar & Vectors: Scalars are the quantities which have only magnitude where vectors are the quantities which has magnitude as well as direction.

Del (∇) = $\nabla\ \hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$. Mathematical operation which has no geometrical meaning.

12.7 Answers to Self Check Exercise

Self Check Exercise

Q.1 Use $\hat{a} \times \left(\hat{b}\times\hat{c}\right) = \left(\hat{a}.\hat{b}\right)\hat{c}$ - $\left(\hat{a}.\hat{c}\right)\hat{b}$

Q.2 $\nabla\left(\hat{b}.\hat{r}\right)$ = $\hat{b}$ ∇ . $\hat{r}$ = $\hat{b}$

$\nabla\left(\hat{b}.\hat{a}\right)$ = 0, curl $\hat{r}$ = 0 [use these concepts.]

Self Check Exercise-2

Q.1 Find $\hat{A}\times\hat{B}$, then apply $\hat{\nabla}\left(\hat{A}\times\hat{B}\right)$ which is equal to 0.

Q.2 Use the concept ∇^{2m} ∇ . (∇ r^m)

Q.3 Since $f = f(r)$ so $\frac{\partial f}{\partial x} = f'(r)\ \frac{\partial r}{\partial x}$ we this concept

12.8 References/Suggested Readings.

1. Vector Calculus by I.N. Sharma & A.R. Vasishtha
2. A Textbook of Vector Calculas by Shanti Narayan & P.K. Mittal.
3. Vector Calculus by P.C. Mathews.

12.9 Terminal Questions

1. Show that div $\hat{r}$ = $\frac{2}{r}$

2. div ($r^n \hat{r}$) = (n + 3) r^n.

*****

Unit - 13

Curvilinear Coordinates

Structure

13.1 Introduction

The curvilinear co-ordinates are the common name of different sets of coordinates other than Cartesian coordinates. In many problems of physics and applied mathematics it is usually necessary to write vector (quantities) equations in terms of suitable coordinates instead of Cartesian coordinates. First we develop the instead analysis in rectangular Cartesian coordinates to see the fundamental role played by the vector valued differential operator, ∇.

All objects of interests are constructed with the ∇ operator the gradient of the scalar field, the divergence of a vector field and curl of a vector field. Later we generalse the results to the more general settings, orthogonal curvilinear coordinate system and it will be matter of taking into account the scale factors h_1, h_2 and h_3. Curvilinear coordinate system are general ways of locating points in Euclidean space using coordinate functions that are invertible functions of usual xi Cartesian coordinates their utility arises in problems with obvious geometric symmetric such as cylindrical or spherical symmetry.

13.2 Learning Objectives: After studying this unit students will be able to:

1. Define curvilinear coordinates.
2. Transform orthogonal curvilinear coordinates
3. Define condition for orthogonality
4. Express unit vectors in curvilinear coordinates
5. Define grad, div and curl far curvilinear coordinate
6. Solve the questions related to curvilinear coordinates

13.3 Curvilinear Coordinates

Let us consider a three dimensional space, defined by three single valued functions say u_1, u_2 and u_3 along the three directions respectively.

A Point: Let P be a point in this space. This point can be represented mathematically by the function $P \equiv P(u_1, u_2, u_3)$.

Coordinate Surface: A coordinate surface is two dimensional place along which any two functions defining the position may change, while the third remains a constant. Thus $u_1 = c_1$, $u_2 = c_2$ and $u_3 = c_3$ define coordinates surfaces along the three directions. For the surface $u_1 = c_1$, the function u_1 is a constant, while the function u_2 and u_3 may vary. Similarly for the surface $u_2 = c_2$, the function u_2 is a constant equal to c_2 while the functions u_1 and u_3 may vary, while for the surfaces $u_3 = c_3$ the function u_3 is a constant equal to c_3, while the functions u_1 may vary.

Coordinate Lines: When two coordinate surfaces intersect each other, they form a line pointing along the third direction. This Line of intersection is called as the coordinate line. For a three dimensional space. We have three coordinate Lines, namely u_1, u_2 and u_3 formed by the intersections of the surfaces. (u_2 & u_3), (u_1 & u_3) and (u_1 & u_2) respectively.

Coordinates axes: Tangent drawn to the coordinate lines at the coordinate point P are called as coordinate axes. Thus for the point P we have a_1, a_2 and a_3 as coordinate axes. which are tangents to the coordinate lines u_1, u_2 and u_3 respectively as shown.

General Curvilinear Coordinates: If the relative orientation of the coordinate surfaces changes from point to point, then the coordinates u_1 u_2 and u_3 are called as general curvilinear coordinates.

Orthogonal Curvilinear Coordinates: If the three dimensional surfaces are mutually perpendicular at all points then the coordinates u_1 u_2 and u_3 are called as orthogonal curvilinear coordinates.

Transformation of co-ordinates: Let the rectangular Cartesian coordinates (x, y, z) of any point P in space be express in terms of three independent, single valued and continuously differentiable scalar point functions u_1, u_2, u_3 as follows:

$$x = \psi_1(u_1, u_2, u_3),\ y = \psi_2(u_1, u_2, u_3),\ z = \psi_3(u_1, u_2, u_3) \qquad (1)$$

Suppose that the Jacobian of x, y, z w.r.t. u_1, u_2, u_3 does not vanish.

i.e. $$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} \neq 0$$

then u_1, u_2, u_3 can be expressed in terms of x, y, z giving

$$u1 = \phi_1 (x, y, z),\ u_2 = \phi_2 (x, y, z),\ u_3 = \phi_3 (x, y, z) \qquad (2)$$

Due to the condition imposed on these functions, with each point P(x, y, z) in space, ∃ unique triad of numbers u_1, u_2, u_3 and to each such triad there is definite point in space.

The set (u_1, u_2, u_3) are called the curvilinear coordinate of P. The set of equations (1) and (2) defines a 'transformation of co-ordinates'.

For Example: Circular cylindrical coordinates (x_1, x_2, x_3) = r, q, z) x = r cos Q

y = r sin Q

z = z

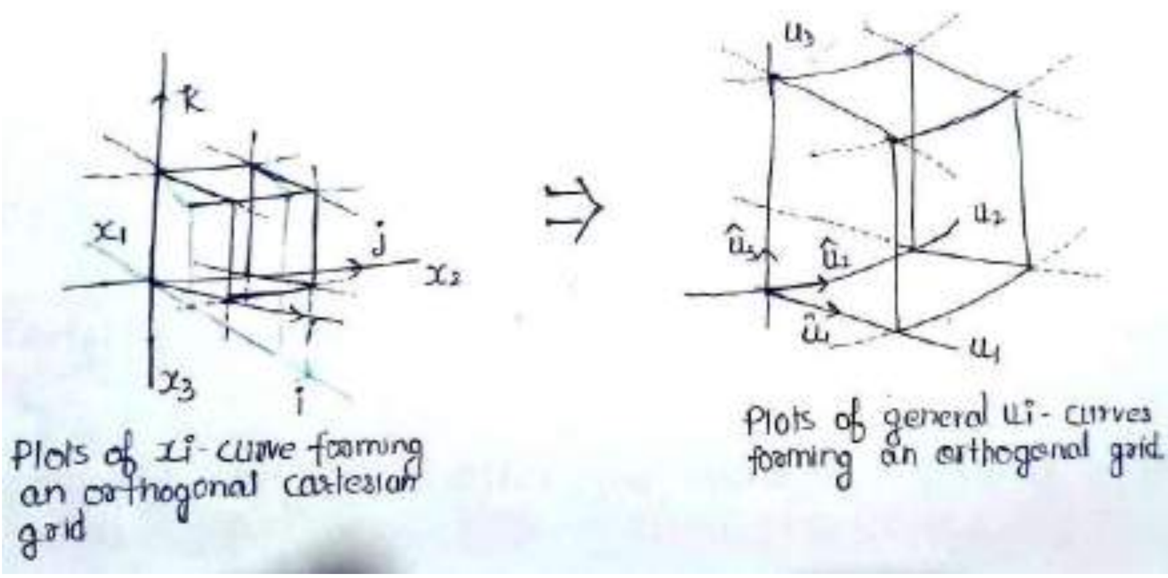
i.e. at any point P, x_1 curve is a straight line, x_2 curve is a circle and x_3 curve is a straight line.

i.e.

r = $\sqrt{x^2 + y^2}$

Q = tan + $\frac{y}{x}$

z = z



Plots of x_i-curve forming an orthogonal cartesian grid

Plots of general u_i-curves forming an orthogonal grid.

Transformation of Orthogonal Curvilinear Coordinates:

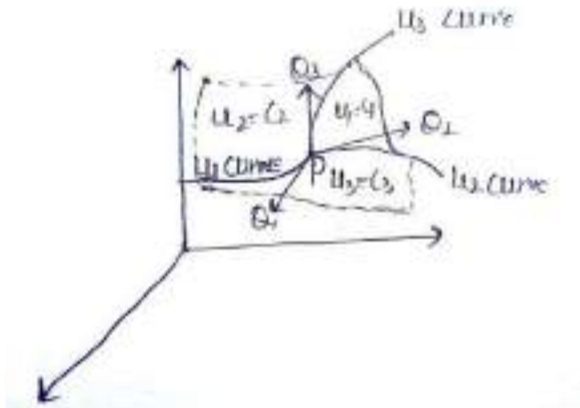
The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1, c_2, c_3 are constant are called co-ordinate surfaces and each pair of these surfaces intersect in curves called co-ordinate curves or lines. Thus,

(i) u_1 curve is given by $u_2 = c_2$, $u_3 = c_3$

(ii) u_2 curve is given by $u_3 = c_3$, $u_1 = c_1$

(iii) u_3 curve is given by $u_1 = c_1$, $u_2 = c_2$

The coordinate axes are determined by tangents PQ_1, PQ_2 and PQ_3 to the coordinate curve at the point P. If at every point P(x, y, z) the coordinate axes are initially perpendicular, then u_1, u_2 u_3 are called orthogonal curvilinear coordinal of P.



Condition for Orthogonality: In curvilinear coordinate system $u_1 = \phi_1(x, y, z)$, $u_2 = \phi_2(x, y, z)$, $u_3 = \phi_3(x, y, z)$ Solving these for x, y, z in terms of u_1, u_2, u_3. we have

$x = \psi_1(x, y, z)$, $y = \psi_2(x, y, z)$, $z = \psi_3(x, y, z)$

$$\therefore \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \psi_1(x, y, z), \hat{i} + \psi_2(x, y, z), \hat{j} + \psi_3(x, y, z)\hat{k}$$

$$= \vec{f}(u_1, u_2, u_3)$$

Co-ordinate curve through $u_2 = c_2$ and $u_3 = c_3$ (i.e. u1 curve) is

$$\vec{r} = f(u_1, c_2, c_3)$$

Tangent to this coordinate curve is parallel to the vector $\dfrac{\partial \vec{r}}{\partial u_1}$

Similarly, tangent to other two curves are parallel to the vectors $\dfrac{\partial \vec{r}}{\partial u_2}, \dfrac{\partial \vec{r}}{\partial u_3}$.

For orthogonal curvilinear co-ordinate system, these taken two at a time are perpendicular.

$$\frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} = 0, \quad \frac{\partial \vec{r}}{\partial u_2} \cdot \frac{\partial \vec{r}}{\partial u_3} = 0, \quad \frac{\partial \vec{r}}{\partial u_3} \cdot \frac{\partial \vec{r}}{\partial u_1} = 0$$

Unit Vectors in Curvilinear Coordinates:

Let the Cartesian coordinates of a point P be (x, y, z). Then the position vector of the point is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= x(u_1, u_2, u_3)\hat{i} + y(u_1, u_2, u_3)\hat{j} + z(u_1, u_2, u_3)\hat{k}$$

$= \vec{r}\,(u_1, u_2, u_3)$

At every point P we have a system of (u_1, u_2, u_3) curves is the tangent vector to the u_1 curve along which the coordinate (u_2, u_3) are constant. The vector

$$\hat{e}_1 = \frac{\frac{\partial \hat{r}}{\partial u_1}}{\left|\frac{\partial \hat{r}}{\partial u_1}\right|}$$

is therefore a unit vector along the tangent to the u_1 curve. We can write this equation also as

$$\frac{\partial \vec{r}}{\partial u_1} = h_1 \hat{e}_1 \ ; \ h_1 = \frac{\partial \vec{r}}{\partial u_1}$$

Similarly, if $\hat{e}_2$ and $\hat{e}_3$ are unit vectors to the (u_2, u_3) curves respectively, then

$$\frac{\partial \vec{r}}{\partial u_2} = h_2\ \hat{e}_2\ ,\ h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|$$

$$\frac{\partial \vec{r}}{\partial u_3} = h_3\ \hat{e}_3\ ,\ h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right|$$

The quantities h_1 are called scale factors

Since, $\nabla \vec{u}_1$ is normal to the u1 = constant surface at P.

$$\hat{E}_1 = \frac{\nabla \overrightarrow{u_1}}{\left|\nabla \overrightarrow{u_1}\right|}$$ is a unit vector normal to the surface.

Similarly, $\hat{E}_2 = \dfrac{\nabla \overrightarrow{u_2}}{\left|\nabla \overrightarrow{u_2}\right|}$, $\qquad \hat{E}_3 = \dfrac{\nabla \overrightarrow{u_3}}{\left|\nabla \overrightarrow{u_3}\right|}$

Thus, at each point of a Curvilinear coordinates there exist two sets of unit vectors, which are in general distinct from each other

The two sets are identical, if and only if, the curvilinear system is orthogonal.

Relation between $\hat{e}_1 = h_1 \nabla u_1$ **and** $\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 (\nabla u_2 \times \nabla u_3)$

$\hat{e}_2 = h_2 \nabla u_2$ and $\hat{e}_2 = h_3 h_1 (\nabla u_3 \times \nabla u_1)$

$\hat{e}_3 = h_3 \nabla u_3$ and $\hat{e}_3 = h_1 h_2 (\nabla u_1 \times \nabla u_2)$

Arc Length For Curvilinear Coordinates:

The Arc Length ds is the length of the infinitesimal vector dr

$(ds)^2 = dr.dr.$

In Cartesian coordinates

$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$

In Curvilinear coordinates, if we change all three coordinates ui by infinitesimal amounts dui, then we have

$$dr = \frac{\partial f}{\partial u_3} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3$$

$$= h_1\, du1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3$$

For the case of orthogonal curvilinear, because the basis vectors are orthonormal we have

$$(ds)^2 = h_1^2\, du_1^2 + h_2^2 + h_3^2\, du^2{}_3$$

For spherical polar, We showed that

$h_2 = 1$, $h\theta = r$ and $h\phi = r \sin \theta$

Therefore

$$(ds)^2 = (dr)^2 + r^2\, (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2$$

Element of Area and Volume For curvilinear coordinates:

Vector Area: If $u_1 \to u_1 + du_1$ then $r = r + dr$ where $dr_1 = h_1 e_1\, du_1$ and if $u_2 \to u_2 + du$, then $r \to r + dr_2$ where $dr_2 = h_2\, e_2\, du_2$.

On the surface of constant u_1 the vector area bounded by dr_2 and dr_3 is given by:

$$ds_1 = (dr_2 \times dr_3) = (h_2\, du_2\, \hat{e}_2) \times (h_3\, du_3\, \hat{e}_3) = h_2\, h_3\, du_2\, du_3\, \hat{e}_1$$

Since, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$ for orthogonal system.

Thus, ds_1 is a vector pointing in the direction of normal to the surfaces u_1 = constant its magnitude being the area of the small parallelogram with edges dr_2 and dr_3. Similarly, one can define ds_2 and ds_3.

For the case of spherical polars, if we vary Q and ϕ. Keeping r fixed then

$$dSr = (hQdQ\, \hat{e}\, Q) \times (hQdQ\, \hat{e}\, Q) = hQhQdQd\phi\, \hat{e}\, r$$

$$= r \,\mathrm{Sin}\, Q\, dQd\phi\, \hat{e}\, r$$

Similarly, for dSQ and $dS\phi$

Volume : The volume contained in the parallelopiped with edges dr_1, dr_2 and dr_3 is

$$dv = dr_1, dr_2 \times dr_3$$

$= (h_1\, du_1\, e_1) \,.\, (h_2\, du_2\, e_2) \times (h_3\, du_3\, e_3)$

$= h_1\, h_2\, h_3\, du_1\, du_2\, du_3$

because $e_1 . e_2 \times e_3 = 1$

For spherical polars we have

$dV = hr\; hQ\; h\phi\; dr\; dQ\; d\phi$

$= r^2 \sin Q\; dr\; dQ\; d\phi$

Gradient, Divergence and Curl for curvilinear coordinates:

Gradient: Let $u_1 \to u_1 + du_1$, $u_2 \to u_2 + du_2$, $u_3 \to u_3 + du_3$

By Taylor's theorem, we have

$$\delta f = \frac{\partial f}{\partial u_1} du_1 + \frac{\partial f}{\partial u_2} du_2 + \frac{\partial f}{\partial u_3} du_3$$

As, $dr = h_1 du_1\, \hat{e}_1 + h_2\, du_2\, \hat{e}_2 + h_3\, du_3\, \hat{e}_3$

Using orthogonality of basis vectors $\hat{e}_i . \hat{e}_j = \delta_{ij}$, we can write

$$\delta f = \left(\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{\partial f}{\partial u_3}\hat{e}_3 \right) . \left(\hat{e}_1 du_1 + \hat{e}_2\, du_2 + \hat{e}_3 du_3 \right)$$

$$= \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3 \right) . \left(h_1 \hat{e}_1 du_1 + h_2 \hat{e}_2\, du_2 + h_3\, \hat{e}_3 du_3 \right)$$

$$= \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3 \right) . dr$$

Compairing this result with equation

$$\delta f = \nabla f(r) . dr$$

We have

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial u_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial u_3}\hat{e}_3$$

$$= \sum_{i=1}^{3} \frac{1}{h_i}\frac{\partial f}{\partial u_1}\hat{e}_i$$

For spherical polars, we obtain:

$$\nabla f\,(r, \theta\; \phi) = \hat{e}_r \frac{\partial f}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} + \frac{\partial f}{\partial \phi}$$

For cylindrical polars, we obtain

$$\nabla f\ (p, \phi, z) = \hat{e}_p \frac{\partial f}{\partial p} + \hat{e}_\phi \frac{1}{p}\frac{\partial f}{\partial \phi} + \hat{e}_z \frac{\partial f}{\partial z} \text{ w}$$

Divergence (div $\vec{f}$ = $\nabla . \vec{f}$) in terms of orthogonal curvilinear coordinates:

If $\vec{f} = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$ is a vector function in orthogonal curvilinear coordinates.

$$\nabla \vec{f} = \nabla.(f_1 \hat{e}_1) + \nabla.(f_2 \hat{e}_2) + \nabla.(f_3 \hat{e}_3) \qquad \text{(A)}$$

We know,

$$\nabla \vec{u} - \frac{\hat{e}_1}{h_1}, \nabla \overrightarrow{u_2} = \frac{\hat{e}_2}{h_2}, \nabla \overrightarrow{u_3} = \frac{\hat{e}_3}{h_3}$$

Now, $\nabla u_2 \times \nabla u_3 = \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} = \frac{\hat{e}_1}{h_2 h_3}$

$\Rightarrow \quad \hat{e}_1 = h_2\ h_3\ (\nabla u_2 \times \nabla h_3)$

Similarly: $\hat{e}_2 = h_1\ h_3\ (\nabla u_1 \times \nabla u_3)$

$\hat{e}_3 = h_1\ h_2\ (\nabla u_1 \times \nabla u_2)$

$\therefore \quad \nabla.\ (f_1 \hat{e}_1) = \nabla.\ [f_1\ h_2\ h_3\ (\nabla u_2 \times \nabla u_3)] \quad (1)$

$\nabla.\ (f_2 \hat{e}_2) = \nabla.\ [f_2\ h_1\ h_3\ (\nabla u_1 \times \nabla u_3)] \quad (2)$

$\nabla.\ (f_3 \hat{e}_3) = \nabla.\ [f_3\ h_1\ h_2\ (\nabla u_1 \times \nabla u_2)] \quad (3)$

Also $\nabla.\ (\phi \vec{\psi}) = \nabla \phi .\ \vec{\psi} + \phi \nabla . \vec{\psi}$

$\Rightarrow \quad \nabla.\ [f_1\ h_2\ h_3\ (\nabla u_2 \times \nabla u_3)] = \nabla\ (f_1 h_2 h_3).\ (\nabla u_2 \times \nabla u_3)$

$f_1\ h_2\ h_3\ \nabla.\ (\nabla u_2 \times \nabla u_3)$

$= \nabla\ (f_1 h_2 h_3).\ (\nabla u_2 \times \nabla u_3)$

$$= \nabla(f_1\ h_2\ h_3).\left(\frac{\hat{e}_1}{h_2 h_3}\right)$$

$$= \frac{\hat{e}_1}{h_2 h_3}.\ \nabla(f_1\ h_2\ h_3)$$

$$\nabla u_2 = \frac{\hat{e}_2}{h_2}, \nabla u_3 = \frac{\hat{e}_3}{h_3}$$

$$\nabla u_2 \times \nabla u_3 = \frac{\hat{e}_2 \times \hat{e}_3}{h_2 h_3} = \frac{\hat{e}_1}{h_2 h_3}$$

$$\nabla.\ (\nabla u_2 \times \nabla u_3) = \nabla.\left(\frac{\hat{e}_1}{h_2 h_3}\right)$$

$$= \left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)$$

$$.\left(\frac{\hat{e}_1}{h_2 h_3}\right) = 0$$

$\Rightarrow$ $$\nabla.\left[f_1 h_2 h_3 \frac{\hat{e}_1}{h_2 h_3}\right] = \left(\frac{\hat{e}_1}{h_2 h_3}\right).\ \nabla(f_1\ h_2\ h_3)$$

$\Rightarrow$ $$\nabla.\ [f_1 \hat{e}_1] = \frac{\hat{e}_1}{h_2 h_3}.\ \nabla(f_1\ h_2\ h_3) \qquad (4)$$

Similarly: $$\nabla\ [f_2 \hat{e}_2] = \frac{\hat{e}_2}{h_3 h_1}.\ \nabla(f_2\ h_3\ h_1) \qquad (5)$$

$$\nabla.\ [f_3 \hat{e}_3] = \frac{\hat{e}_3}{h_1 h_2}.\ \nabla(f_3\ h_1\ h_2) \qquad (6)$$

Now (4) $\Rightarrow$ $$f_1 \hat{e}_1 = \frac{\hat{e}_1}{h_2 h_3}.\left(\frac{\hat{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial u_3}\right)(f_1\ h_2\ h_3)$$

$$= \frac{\hat{e}_1.\hat{e}_1}{h_2 h_3 h_1}\ \frac{\hat{e}_1.\hat{e}_1}{h_2 h_3 h_1}\ \frac{\partial}{\partial u_1}(f_1\ h_2\ h_3)$$

$$\nabla.\ f_1 \hat{e}_1 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(f_1\ h_2\ h_3) \qquad (7)$$

Similarly, $$\nabla.\ f_2 \hat{e}_2 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}(f_2\ h_3\ h_1) \qquad (8)$$

$$\nabla.\ f_3 \hat{e}_3 = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}(f_3\ h_1\ h_2) \qquad (9)$$

Putting (7), (8), (9) in (A) we get

$$\nabla.\vec{F} = \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_1}(f_1\ h_2\ h_3) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_2}(f_2\ h_3\ h_1) + \frac{1}{h_1 h_2 h_3}\frac{\partial}{\partial u_3}(f_3\ h_1\ h_2)$$

$\Rightarrow \qquad \nabla.\vec{F} = \dfrac{1}{h_1h_2h_3}\left[\dfrac{\partial}{\partial u_1}(f_1\mathrm{h}_2\mathrm{h}_3) + \dfrac{\partial}{\partial u_3}(f_3\mathrm{h}_1\mathrm{h}_2)\right]$

Which is the required result

Curl $\vec{F}$ = ∇×$\vec{F}$ in orthogonal curvilinear coordinates.

Let $\vec{F} = \vec{F}$ (u_1, u_2, u_3)

$\Rightarrow \qquad \vec{F} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 \qquad (1)$

Now, Curl $\vec{F} = \nabla\times\vec{F}$

$= \nabla\times [f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3]$

$\Rightarrow \qquad$ Curl $\vec{F} = \nabla\times f_1\hat{e}_1 + \nabla\times f_2\hat{e}_2 + \nabla\times f_3\hat{e}_3 \qquad$ (A)

Now $\nabla\times f_1\hat{e}_1 = \nabla\times (f_1\mathrm{h}_1\nabla\mathrm{u}) \qquad \because \left|\nabla u = \dfrac{\hat{e}_1}{h_1}\right|$

We know $\nabla\times (\phi\vec{A}) = \phi\,(\nabla\times\vec{A}) + \nabla\phi\times\vec{A}$

$\therefore\ \nabla\times f_1\hat{e}_1 = \nabla\times (f_1\mathrm{h}_1\nabla\mathrm{u}_1)$

$\Rightarrow \qquad \nabla\times f_1\hat{e}_1 = f_1\mathrm{h}_1 (\nabla\times\nabla\mathrm{u}_1) + \nabla (f_1\mathrm{h}_1)\times\nabla\mathrm{u}_1$

$\Rightarrow \qquad \nabla\times f_1\hat{e}_1 = \nabla(f_1\mathrm{h}_1)\times\nabla\mathrm{u} \qquad \because \left|\begin{array}{l}\nabla\times\nabla u_1 = Curl\ grad\ u_1 = 0\\ \Rightarrow f_1h_1\left(\nabla\times\nabla u_1\right) = 0\end{array}\right|$

$= \nabla(f_1\mathrm{h}_1) \times \dfrac{\hat{e}_1}{h_1}$

$= \left(\dfrac{\hat{e}_1}{h_1}\dfrac{\partial}{\partial u_1} + \dfrac{\hat{e}_2}{h_2}\dfrac{\partial}{\partial u_2} + \dfrac{\hat{e}_3}{h_3}\dfrac{\partial}{\partial u_3}\right) f_1\mathrm{h}_1 \times \dfrac{\hat{e}_1}{h_1}$

$= \dfrac{\hat{e}_1}{h_1}\dfrac{\partial}{\partial u_1}(f_1\mathrm{h}_1) \times \dfrac{\hat{e}_1}{h_1} + \dfrac{\hat{e}_2}{h_2}\dfrac{\partial}{\partial u_2}(f_1\mathrm{h}_1) \times \dfrac{\hat{e}_1}{h_1} + \dfrac{\hat{e}_3}{h_3}\dfrac{\partial}{\partial u_3}(f_1\mathrm{h}_1) \times \dfrac{\hat{e}_1}{h_1}$

$= 0 + \dfrac{\left(-\hat{e}_3\right)}{h_1h_2}\dfrac{\partial}{\partial u_2}(f_1\mathrm{h}_1) + \dfrac{\hat{e}_2}{h_3h_1}\dfrac{\partial}{\partial u_3}(f_1\mathrm{h}_1)$

$\Rightarrow \qquad \nabla\times f_1\hat{e}_1 = \dfrac{\hat{e}_2}{h_3h_1}\dfrac{\partial}{\partial u_3}(f_1\mathrm{h}_1) - \dfrac{\hat{e}_3}{h_1h_2}\dfrac{\partial}{\partial u_2}(f_1\mathrm{h}_1)$

Similarly: $\nabla \times f_2 \hat{e}_2 = \dfrac{\hat{e}_3}{h_1 h_2} \dfrac{\partial}{\partial u_1}(f_2 h_2) - \dfrac{\hat{e}_1}{h_2 h_3} \dfrac{\partial}{\partial u_3}(f_2 h_2)$

and $\nabla \times f_3 \hat{e}_3 = \dfrac{\hat{e}_1}{h_2 h_3} \dfrac{\partial}{\partial u_2}(f_3 h_3) - \dfrac{\hat{e}_2}{h_3 h_1} \dfrac{\partial}{\partial u_1}(f_3 h_3)$

$\therefore$ (A) $\Rightarrow$ $\nabla \times \vec{F} = \dfrac{\hat{e}_1}{h_2 h_3}\left(\dfrac{\partial}{\partial u_2}\left(f_3 h_3\right) - \dfrac{\partial}{\partial u_3}\left(f_2 h_2\right)\right) + \dfrac{\hat{e}_2}{h_3 h_1}\left[\dfrac{\partial}{\partial u_1}\left(f_1 \mathrm{h}_1\right)\right]$

$- \dfrac{\partial}{\partial u_1}(f_3 h_3) + \dfrac{\hat{e}_3}{h_1 h_2}\left(\dfrac{\partial}{\partial u_1}\left(f_2 h_2\right) - \dfrac{\partial}{\partial u_2}\left(f_1 h_1\right)\right)$

$$\Rightarrow \quad \nabla \times \vec{F} = \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ f_1 h_1 & f_2 h_2 & f_3 h_3 \end{vmatrix} \dfrac{1}{h_1 h_2 h_3}$$

$$= \dfrac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ f_1 h_1 & f_2 h_2 & f_3 h_3 \end{vmatrix}$$

Laplacian Operator ($\nabla^2 \phi$) in orthogonal curvilinear coordinates

We know $\nabla \psi = \left(\dfrac{\hat{e}_1}{h_1}\dfrac{\partial}{\partial u_1} + \dfrac{\hat{e}_2}{h_2}\dfrac{\partial}{\partial u_2} + \dfrac{\hat{e}_3}{h_3}\dfrac{\partial}{\partial u_3}\right)$

$$\nabla \psi = \dfrac{\hat{e}_1}{h_1}\dfrac{\partial \psi}{\partial u_1} + \dfrac{\hat{e}_2}{h_2}\dfrac{\partial \psi}{\partial u_2} + \dfrac{\hat{e}_3}{h_3}\dfrac{\partial \psi}{\partial u_3} \qquad (1)$$

Let $\nabla \psi = \vec{F}$

$$\nabla \psi = f_i \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3 \qquad (2)$$

Comparing (1) and (2)

$$f_1 = \dfrac{1}{h_1}\dfrac{\partial \phi}{\partial u_1},\ f_2 = \dfrac{1}{h_2}\dfrac{\partial \phi}{\partial u_2},\ f_3 = \dfrac{1}{h_3}\dfrac{\partial \phi}{\partial u_3}$$

Now $\nabla^2 \phi = \nabla . \nabla \phi = \nabla . \vec{F}$

$$\Rightarrow \qquad \nabla.\vec{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(f_1\mathrm{h}_2\mathrm{h}_3)+\frac{\partial}{\partial u_2}(f_2\mathrm{h}_1\mathrm{h}_3)\frac{\partial}{\partial u_3}(f_3\mathrm{h}_1\mathrm{h}_2)\right]$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\right)\mathrm{h}_2\mathrm{h}_3+\frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\right)\mathrm{h}_1\mathrm{h}_3\frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\right)\mathrm{h}_1\mathrm{h}_2)\right]$$

$$\Rightarrow \qquad \nabla^2\ \phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

$$\Rightarrow \qquad \nabla^2 = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\right)\frac{\partial}{\partial u_1}+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\right)\frac{\partial}{\partial u_2}+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\right)\right]$$

Some Related Questions

Example (1) if u = 2x + 3, v = y - 4, w = z + 2. Show that u, v, w are orthogonal?

Solution: Here u = 2x + 3, v = y - u, w = z + 2

$$\Rightarrow \qquad 2x = u - 3, \qquad y = v + u, \qquad z = w - 2$$

$$\Rightarrow \qquad x = \frac{4-3}{2}, \qquad y = v + u, \qquad z = w - 2 \qquad (1)$$

If $\vec{r}$ denotes the position vector of (x, y, z) then

$$\vec{r} = x\hat{i} + y\,\hat{j} + z\hat{k}$$

$$\Rightarrow \qquad \vec{r} = \left(\frac{4-3}{2}\right)\hat{i} + (v+u)\,\hat{j} + (w-2)\hat{k}$$

$$\therefore \qquad \frac{\partial\vec{r}}{\partial u} = \frac{1}{2}\,\hat{i}\,,\ \frac{\partial\vec{r}}{\partial v} = \hat{j}\,,\ \frac{\partial\vec{r}}{\partial w} = \hat{k}$$

Now, $$\frac{\partial\vec{r}}{\partial u}.\frac{\partial\vec{r}}{\partial v} = \frac{1}{2}\,\hat{i}\,.\,\hat{j} = 0$$

$$\frac{\partial\vec{r}}{\partial v}.\frac{\partial\vec{r}}{\partial w} = \hat{j}\,.\hat{k} = 0$$

$$\frac{\partial\vec{r}}{\partial w}.\frac{\partial\vec{r}}{\partial u} = \hat{k}\,.\frac{1}{2}\,\hat{i} = 0$$

$$\therefore \qquad \frac{\partial\vec{r}}{\partial u}.\frac{\partial\vec{r}}{\partial v} = \frac{\partial\vec{r}}{\partial v}.\frac{\partial\vec{r}}{\partial w} = \frac{\partial\vec{r}}{\partial w}.\frac{\partial\vec{r}}{\partial u} = 0$$

$\therefore$ $\vec{u}$, $\vec{v}$ and $\vec{w}$ are orthogonal

Example (2) If u = 3x + 2, v = y + 3, w = z - 2, Show that u, v, w are orthogonal and find $(ds)^2$. Also find h_1, h_2, h_3.

Solution: Here u = 3x + 2, v = y + 3, w = z - 2

$$x = \frac{u-2}{3},\ y = v - 3,\ z = w + 2 \qquad (1)$$

If $\vec{r}$ denotes the position vector of (x, y, z) then

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = \left(\frac{u-2}{3}\right)\hat{i} + (v-3)\hat{j} + (w+2)\hat{k}$$

$$\Rightarrow \quad \frac{\partial \vec{r}}{\partial u} = \frac{1}{3}\hat{i},\ \frac{\partial \vec{r}}{\partial v} = \hat{j},\ \frac{\partial \vec{r}}{\partial w} = \hat{k}$$

Now, $$\frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} = \frac{1}{3}\hat{i}\cdot\hat{j} = 0$$

$$\frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial w} = \hat{j}\cdot\hat{k} = 0$$

$$\frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial u} = \frac{1}{3}\hat{k}\cdot\hat{i} = 0$$

$$\Rightarrow \quad \frac{\partial \vec{r}}{\partial u}\cdot\frac{\partial \vec{r}}{\partial v} = \frac{\partial \vec{r}}{\partial v}\cdot\frac{\partial \vec{r}}{\partial w} = \frac{\partial \vec{r}}{\partial w}\cdot\frac{\partial \vec{r}}{\partial u} = 0$$

$\therefore$ u, v and w are orthogonal

Now, $$d\vec{r} = \frac{\partial \vec{r}}{\partial u}\,du + \frac{\partial \vec{r}}{\partial v}\,dv + \frac{\partial \vec{r}}{\partial w}\,dw.$$

$$d\vec{r} = \frac{1}{3}\hat{i}\,du + \hat{j}\,dv + \hat{k}\,dw$$

Also, $(ds)^2 = d\vec{r}\cdot d\vec{r}$

$$= \left(\frac{1}{3}\hat{i} + \hat{j}\,dv + \hat{k}\,dw\right)\cdot\left(\frac{1}{3}\hat{i}du + \hat{j}\,dv + \hat{k}\,dw\right)$$

$= \frac{1}{9}$ $(du)^2 + (dv)^2 + (dw)^2$

$= h_1^2\ (du)^2 + h_2^2\ (dv)^2 + h_3^2 (dw)^2$

Where $h_1 = \frac{1}{3}$, $h_2 = 1$, $h_3 = 1$

Example (3) Let $u_1 = xy$, $u_2 = \frac{x^2 + y^2}{2}$, $u_3 = z$. Show that u_1, u_2, u_3 are not orthogonal.

Solution: Here $u_1 = xy$ (1), $u_2 = \frac{x^2 + y^2}{2}$ (2), $u_3 = z$ (3)

Differentiating (1) and (2) partially w.r.t. u, we get

$$1 = x\frac{\partial y}{\partial u_1} + y\frac{\partial x}{\partial u_1} \qquad (4)$$

$$0 = \frac{1}{2}\left[2x\frac{\partial y}{\partial u_1} + y\frac{\partial x}{\partial u_1}\right] \Rightarrow 0 = x\frac{\partial x}{\partial u_1} + y\frac{\partial y}{\partial u_1} \quad (5)$$

Multiplying (4) by y and (5) by x, we gel

$$y = xy\frac{\partial y}{\partial u_1} + y = \frac{\partial x}{\partial u_1} \qquad (6)$$

and $$0 = x^2\frac{\partial x}{\partial u_1} + xy\frac{\partial y}{\partial u_1} \qquad (7)$$

Subtracting (7) from (6), we gel

$$y = (y^2 - x^2)\frac{\partial x}{\partial u_1}$$

$$\Rightarrow \qquad \frac{\partial x}{\partial u_1} = \frac{y}{y^2 - x^2} \qquad (8)$$

Similarly, Putting values of $\frac{\partial x}{\partial u_1}$ in (5), we gel

$$0 = \frac{xy}{y^2 - x^2} + y\frac{\partial y}{\partial u_1}$$

$$\Rightarrow \qquad \frac{-xy}{y^2-x^2} = y\,\frac{\partial y}{\partial u_1}$$

$$\Rightarrow \qquad \frac{\partial y}{\partial u_1} = \frac{-xy}{y^2-x^2} \Rightarrow \frac{\partial y}{\partial u_1} = \frac{x}{y^2-x^2} \qquad (9)$$

Again, differentiating (1) and (2) partially w.r.t. u_2, we gel.

$$0 = x\frac{\partial y}{\partial u_2} + y\frac{\partial x}{\partial u_2} \qquad (10)$$

$$1 = x\,\frac{\partial x}{\partial u_2} + y\,\frac{\partial y}{\partial u_2} \qquad (11)$$

Solving (10) and (11), we have

$$\frac{\partial x}{\partial u_2} = \frac{x}{y^2-x^2} \qquad (12)$$

and $$\frac{\partial y}{\partial u_2} = \frac{y}{y^2-x^2} \qquad (13)$$

Let $\vec{r}$ be the position vector of point (x, y, z) in space then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\frac{\partial \vec{r}}{\partial u_1} = \frac{\partial x}{\partial u_1}\hat{i} + \frac{\partial y}{\partial u_1}\hat{j} + \frac{\partial z}{\partial u_1}\hat{k}$$

and $$\frac{\partial \vec{r}}{\partial u_2} = \frac{\partial x}{\partial u_2}\hat{i} + \frac{\partial y}{\partial u_2}\hat{j} + \frac{\partial z}{\partial u_2}\hat{k}$$

Now, $$\frac{\partial \vec{r}}{\partial u_1}\cdot\frac{\partial \vec{r}}{\partial u_2} = \frac{\partial x}{\partial u_1}\cdot\frac{\partial x}{\partial u_2} + \frac{\partial y}{\partial u_1}\cdot\frac{\partial y}{\partial u_2} + \frac{\partial z}{\partial u_1}\cdot\frac{\partial z}{\partial u_2}$$

$$= \left(\frac{x}{y^2-x^2}\right)\left(\frac{y}{y^2-x^2}\right) + \left(\frac{x}{y^2-x^2}\right)\left(\frac{y}{y^2-x^2}\right) + 0$$

$$= \frac{-xy}{\left(y^2-x^2\right)^2} - \frac{xy}{\left(y^2-x^2\right)^2}$$

$$= \frac{-2xy}{\left(y^2-x^2\right)^2} = \frac{-2xy}{\left(y^2+x^2\right)^2 - 4x^2y^2}$$

$$= \frac{-2xy}{4u_2^2 - 4u_1^2} = \frac{u_1}{2\left(u_2^2 - u_1^2\right)} \neq 0$$

$\therefore$ $$\frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} \neq 0$$

$\therefore$ System of Coordinates u_1, u_2, u_3 are not orthogonal.

Example (4) if x = r cos θ, y = r sin θ, z = z

Solution : Here x = r cos θ, y = r sin θ, z = z

Squaring and adding first two equations, we get

$$x^2 + y^2 = r^2 \quad \Rightarrow \quad r = \sqrt{x^2 + y^2} \qquad (1)$$

$$\text{Also } \tan\theta = \frac{y}{x} \quad \Rightarrow \quad \theta = \tan \frac{y}{x} \qquad (2)$$

Now, $$\nabla^2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r \qquad (3)$$

From (1) $\frac{\partial r}{\partial u} = \frac{x}{r} = \cos\theta$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = 0$

Putting these in (3), we get $\nabla_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

$$\nabla\theta = \hat{i}\frac{\partial\theta}{\partial x} + \hat{j}\frac{\partial\theta}{\partial y} + \hat{k}\frac{\partial\theta}{\partial z} \qquad (4)$$

From (2) $\frac{\partial\theta}{\partial x} = -\frac{\sin\theta}{r}$, $\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$, $\frac{\partial\theta}{\partial z} = 0$

Substituting these in (4), we have

$$\nabla\theta \frac{1}{r} (\cos\theta\,\hat{j} - \sin\theta\,\hat{i})$$

Self Check Exercise

Q.1 Express $\nabla\phi$, $\nabla.A$, $\nabla\times A$, $\nabla^2\phi$ in cylindrical coordinates using curvilinear coordinate system.

Q.2 Write Laplace equation in parabolic cylindrical coordinate.

13.4 Summary: Dear students, in this unit, we studied

1. The general curvilinear coordinate are represented by u_1, u_2, u_3.

2. Condition for orthogonality for $u_1 = \phi_1 (x, y, z)$, $u_2 = \phi_2 (x, y, z)$ $u_3 = \phi_3 (x, y, z)$ is given by

$$\frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} = \frac{\partial \vec{r}}{\partial u_2} \cdot \frac{\partial \vec{r}}{\partial u_3} = \frac{\partial \vec{r}}{\partial u_3} \cdot \frac{\partial \vec{r}}{\partial u_4} = 0$$

3. Gradient, divergence and curl in curvilinear coordinate is given by

$$\nabla F = \frac{1}{hi} \frac{\partial f}{\partial u_i} \hat{e}i$$

$$\nabla . \vec{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (f_1 h_2 h_3) + \frac{\partial}{\partial u_2} (f_2 h_3 h_1) + \frac{\partial}{\partial u_3} (f_3 h_1 h_2) \right]$$

$$\nabla \times \vec{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ f_1 h_1 & f_2 h_2 & f_3 h_3 \end{vmatrix}$$

4. Laplacian in curvilinear coordinates

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \right) \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2} \left(\frac{h_1 h_3}{h_2} \right) \frac{\partial}{\partial u_2} + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \right) \frac{\partial}{\partial u_3} \right]$$

13.5 Glossary

1. **Del operator (∇):** $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$

   It is a Mathematical operator which has no geometrical meaning, we can operate it on scalar or vector field.

2. **Scale Factor:** Scale factor is ratio between the corresponding sides of similar figures.

3. **Coordinates:** A pair of numbers which are used to determine the position of a point.

13.6 Answers to Self Check Exercise

1. Verified using curvilinear coordinate system

2. Verified using curvilinear coordinate system.

13.7 References/Suggested Reading

1. Vector calculus by P.C. Mathews.

2. A Textbook of vector calculus by Shanti Narayan and P.K. Mittal

3. A Textbook of vector calculus by Anil Kumar Sharma.

13.8 Terminal Questions

1. Show that in any orthogonal curvilinear system

   (i) div curl $\vec{A}$ = 0

   (ii) curl grad ϕ = $\vec{0}$

2. Let $x = u_1 u_2 \cos u_3$, $y = u_1 u_2 \sin u_3$, $z = \frac{1}{2}(u_1^2 - u_2^2)$. Obtain h_1, h_2, h_3 and the unit vector $\hat{e}_1, \hat{e}_2, \hat{e}_3$. Determine grad ϕ, div $\vec{F}$, curl $\vec{F}$ and $\nabla^2\phi$ for the above system of coordinates.

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Unit - 14

Integration of Vector Function

Structure

14.1 Introduction

Dear student, in this unit you will learn to evaluate the integral of a vector functions and vector field. Since the integration is same as in ordinary calculus studied in previous classes but the integral of a vector function and field are different in the way in which the integrand is handed as well as in the physical meanings of the quantities obtained. So, in this unit you will learn how to integral vector functions of scalar variables in the form of indefinite and definite integral.

14.2 Learning Objectives: After studying this unit, students will be able to

1. determine the integral of a vector function with respect to a scalar.
2. determine definite integral.
3. determine indefinite integral.
4. apply the concept of vector integral in real life situations like finding position vector and velocity.

14.3 Integration of a Vector Function

Since we know that integration is the reverse process of differentiation. We also use this concept in integration of vector functions relative to a scalar.

Consider vector $\vec{F}$ which is a function of a scalar t i.e. $\vec{F} = \vec{F}$ (t) such that

$$\frac{d}{dt}\hat{F}(t) = \hat{f}(t) \qquad (1)$$

then integral of $\hat{f}$ (t) with respect to 't' is $\hat{F}$ (t) + $\hat{c}$, where $\hat{c}$ is constant of integration. Mathematically

$$\int \hat{f}(t) = \hat{F}\text{ (t) + c.}$$

Properties of Vector Integrals : Some properties related vector integrals are:

1. $$\int\left(\frac{d\dot{f}}{dt}.\vec{g}+\vec{f}\,\frac{d\dot{g}}{dt}\right)\text{dt} = \vec{f}\,.\,\vec{g}\ \text{+ c}$$

2. $$\int\left(\frac{d\dot{f}}{dt}\times\vec{g}+\vec{f}\times\frac{d\dot{g}}{dt}\right)\text{dt} = \vec{f}\times\vec{g}\ \text{+ c}$$

3. $$\int\left(2\vec{r}.\frac{d\vec{r}}{dt}\right)\text{dt} = \vec{r}^{\,2}\ \text{+ c}$$

4. $$\int\left(2\frac{d\vec{r}^{\,2}}{dt}.\frac{d\vec{r}}{dt^2}\right)\text{dt} = \left(\frac{d\vec{r}}{dt}\right)^2 \text{+ c}$$

5. $$\int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)\text{dt} = \vec{r}\times\frac{d\vec{r}}{dt} + \vec{c}$$

6. $$\int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)\text{s dt} = \vec{a}\times\vec{r}+\vec{c}\ \text{ where } \vec{a} \text{ is constant vector}$$

7. $$\int c\,\vec{r}\,dt = c\int\vec{r}\,dt$$

Indefinite Integral

Let $\vec{f}$ (t) and $\vec{F}$ (t) be two vector functions such that $\frac{d}{dt}\left\{\vec{F}(t)\right\} = \vec{f}$ (t). Then $\vec{F}$ (t) is called the indefinite integral of $\vec{f}$ (t) w.r.t. 't' and we write it as

$$\vec{f}\text{ (t) at = }\vec{F}\text{ (t) + }\vec{c}$$

Theorem: If vector function $\vec{F}$ is indefinite integral of $\vec{f}$ w.r.t. 't', then prove that $\vec{F}$ + $\vec{C}$ is also indefinite integral of $\vec{f}$, where $\vec{C}$ is a constant vector.

Proof: Here $\int \ddot{f}\,dt = \ddot{F}$

$$\Rightarrow \qquad \frac{d\vec{F}}{dt} = \vec{f} \qquad \text{(By definition)} \qquad (1)$$

Now, $\frac{d}{dt}\left(\vec{F}+\vec{c}\right) = \frac{d\vec{F}}{dt} + \frac{d\vec{c}}{dt}$

$$= \frac{d\vec{F}}{dt} \left\{\because \frac{d\vec{c}}{dt} = 0, \vec{c}\, is\, constant\right\}$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\vec{F}+\vec{c}\right) = \vec{f} \text{ (from (1))}$$

$$\Rightarrow \qquad \int \vec{f}\,dt = \vec{F} + \vec{c}$$

Note: 1. The constant of integration 'c' is scalar if integrand is scalar and vector if integrand is vector

2. If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, where f_1, f_2, f_3 are scalar functions of some variable 't' (say), then

$$\int \vec{f}\,dt = \hat{i}\int f_1 dt + \hat{j}\int f_2 dt + \hat{k}\int f_3 dt$$

Let us do some examples to have better understanding of vector integration.

Example 1: Evaluate $\int \vec{A}\times\frac{d^2\vec{A}}{dt^2}$ dt

Solution: Now, $\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right) = \vec{A}\times\frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right) + \frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt}$

$$= \vec{A}\times\frac{d^2\vec{A}}{dt^2} + \frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt}$$

$$= \vec{A}\times\frac{d^2\vec{A}}{dt^2} + 0 \qquad \left\{\because \frac{d\vec{A}}{dt}\times\frac{d\vec{A}}{dt} = 0\right\}$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\vec{A}\times\frac{d\vec{A}}{dt}\right) = \vec{A}\times\frac{d^2\vec{A}}{dt^2}$$

By using definition of indefinite integral, we have

$$\int\left(\vec{A}\times\frac{d^2\vec{A}}{dt^2}\right) dt = \vec{A}\times\frac{d\vec{A}}{dt} + \vec{c}$$

Example 2: If $\vec{f}$ (t) = (t - t²) $\hat{i}$ + 2t³ $\hat{j}$ - 3 $\hat{k}$, find $\int \vec{f}(t)dt$

Solution: $\int \vec{f}(t)dt = \int\left[\left(t-t^2\right)\hat{i}+2t^3\hat{j}-3\hat{k}\right]$ dt

$$= \int\left(t-t^2\right)\hat{i}dt+\int 2t^3\hat{j}at-\int 3\hat{k}dt$$

$$= \left[\int tdt-\int t^2dt\right]\hat{i} + 2\,\hat{j}\int t^3dt-3k\int dt$$

$$= \left(\frac{t^2}{2}-\frac{t^3}{3}\right)\hat{i} + 2\,\hat{j}\left(\frac{t^4}{4}\right) - 3\,\hat{k}\,(t) + \vec{c}$$

$$= \left(\frac{t^2}{2}-\frac{t^3}{3}\right)\hat{i} + \frac{t^4}{2}\,\hat{j} - 3t\,\hat{k} + \vec{c}$$

Example 3: Evaluate $\int\left[t\hat{i}+\left(t^2-2t\right)\hat{j}+\left(3t^2+3t^3\right)\hat{k}\right]$ dt

Solution: $\int\left[t\hat{i}+\left(t^2-2t\right)\hat{j}+\left(3t^2+3t^3\right)\hat{k}\right]$dt

$$= \int t\hat{i}\,dt+\int\left(t^2-2t\right)\hat{j}\,dt+\int\left(3t^2+3t^3\right)\hat{k}\ \mathrm{dt}$$

$$= \frac{t^2}{2}\,\hat{i} + \int t^2\hat{j}\,dt - \int 2t\,\hat{j}\,dt + \int 3t^2\,\hat{k}\,dt + \int 3t^3\,\hat{k}\,dt$$

$$= \frac{t^2}{2}\,\hat{i} + \frac{t^3}{3}\,\hat{j} - 2\frac{t^2}{2}\,\hat{j} + 3\frac{t^3}{3}\,\hat{k} + 3\frac{t^4}{3}\,\hat{k}$$

$$= \frac{t^2}{2}\,\hat{i} + \frac{t^3}{3}\,\hat{j} - t^2\,\hat{j} + t^3\,\hat{k} + \frac{3}{4}t^4\,\hat{k}$$

$$= \frac{t^2}{2}\,\hat{i} + \left(\frac{t^3}{3}-t^2\right)\hat{j} + \left(t^3+\frac{3}{4}t^4\right)\hat{k}$$

Example 4 Solve $\frac{d^2\vec{r}}{dt^2}$ = -k² $\vec{r}$

Solution: Given equation is $\frac{d^2\vec{r}}{dt^2}$ = -k² $\vec{r}$

$$\Rightarrow \qquad 2\frac{d^2\vec{r}}{dt^2} = -2\,k^2\,\vec{r}$$

$$\Rightarrow \qquad 2\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} = -2k^2\,\vec{r}\cdot\frac{d\vec{r}}{dt}$$

Integrating both sides

$$\Rightarrow \qquad \int 2\frac{d^2\bar{r}}{dt^2}\frac{d\bar{r}}{dt}dt = -\int 2k^2\bar{r}.\frac{d\bar{r}}{dt} + \bar{c} \qquad (1)$$

Now $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{d}{dt}\left[\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}\right] = \frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt^2} + \frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}$

$$= \frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt^2} = \frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} + \frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = 2\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt}$$

$$\Rightarrow \qquad \int 2\frac{d^2\vec{r}}{dt^2}\frac{d\vec{r}}{dt}\,dt = \left(\frac{d\vec{r}}{dt}\right)^2$$ (By definition of Indefinite integral)

and similarly $\int 2\vec{r}.\frac{d\vec{r}}{dt}\,dt = \vec{r}^{\,2}$

$$\therefore (1) \Rightarrow \left(\frac{d\vec{r}}{dt}\right)^2 = -k^2\,\vec{r}^{\,2} + \vec{c}$$

Example 5: The velocity of particle at time 't' is given by $\hat{v}$ (t) = sin t $\hat{i}$ - cox t $\hat{j}$ +t^2 $\hat{k}$. Find the position vector of particle $\vec{r}$ (t). Given that $\vec{r}$ (t = 0) = $\hat{i}$ + $\hat{j}$ + $\hat{k}$.

Solution: Using the definition of velocity, we know that

$$\hat{v}\text{ (t) = Velocity} = \frac{d\vec{r}^{(t)}}{dt}$$

$$\Rightarrow \qquad \vec{r}\text{ (t)} = \int \hat{v}(t)\,dt \qquad (1)$$

Given $\hat{v}$ (t) = sint $\hat{i}$ - cox t $\hat{j}$ + $t^2\hat{k}$

Therefore $\vec{r}$ (t) = $\int\left(\sin t\,\hat{i} - coxt\,\hat{j} + t^2\hat{k}\right)dt$

$$= \hat{i}\int \sin t\,dt - \hat{j}\int coxt\,dt + \hat{k}\int t^2 dt$$

$$\vec{r}\,(t) = -\,\text{coxt}\,\hat{i} - \text{sint}\,\hat{j} + \frac{t^3}{3}\hat{k} + \vec{c} \qquad (2)$$

In order to find the value of c, we will use the given initial condition i.e. $\vec{r}$ (t = 0) = $\hat{i} + \hat{j} + \hat{k}$.

So, at t = 0,

$$\vec{r}\,(t = 0) = \hat{i} + \hat{j} + \hat{k} = -\,\text{cox}\ 0\,\hat{i} - \sin 0\,\hat{j} + 0\hat{k} + \hat{c}$$

$$= \hat{i} + \hat{c}$$

$$\Rightarrow \qquad \hat{c} = 2\hat{i} + \hat{j} + \hat{k} \qquad (3)$$

Putting the value of $\hat{c}$ in (2) we get

$$\vec{r}\,(t) = -\,\text{cox t}\,\hat{i} - \text{sint}\,\hat{j} + \frac{t^3}{3}\hat{k} + 2\hat{i} + \hat{j} + \hat{k}$$

$$\vec{r}\,(t) = (2 - \text{cox t})\,\hat{i} + (1 - \text{sint})\,\hat{j} + \left(1 + \frac{t^3}{3}\right)\hat{k}$$

Example 6: The acceleration of a particle at any time 't' is given as $e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$. Find $\vec{v}$ (velocity) given that $\vec{v} = \hat{i} + \hat{j}$ when t = 0

Solution: We know Acceleration = $\frac{d\vec{v}}{dt}$ $e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$

$$\Rightarrow \qquad \frac{d\vec{v}}{dt} = e^t\hat{i} + e^{2t}\hat{j} + \hat{k}$$

Integrating both sides

$$\Rightarrow \qquad \vec{v} = \int\left(e^t\hat{i} + e^{2t}\hat{j} + \hat{k}\right)dt$$

$$= \int e^t dt + \hat{j}\int e^{2t}dt + \hat{k}\int dt$$

$$\vec{v} = \hat{i}\,e^t + \hat{j}\,e^{2t} + \hat{k}\,t + \vec{c} \qquad (1)$$

Given when t = 0, $\vec{v} = \hat{i} + \hat{j}$

$$(1) \qquad \Rightarrow \qquad \hat{i} + \frac{\hat{j}}{2} + \vec{c}$$

$$\Rightarrow \qquad \vec{c} = -\frac{1}{2}\hat{j} + \hat{j}$$

$$\Rightarrow \qquad \vec{c} = \frac{1}{2}\hat{j}$$

Putting value of $\vec{c}$ in (1), we get

$$\vec{v} = \hat{i}\, e^t + \hat{j}\,\frac{e^{2t}}{2} + \hat{k}\, t + \frac{1}{2}\hat{j}$$

$$\vec{v} = e^t \hat{i} + \frac{1}{2}(e^{2t} + 1)\hat{j} + t\hat{k}$$

Example 7: The acceleration of a particle at any time t > 0 is given by $\vec{a} = \frac{d\vec{v}}{dt}$ = (12 cos 2t)$\hat{i}$ - (8 sin 2t) $\hat{j}$ + (16t)$\hat{k}$. If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at t = 0, find $\vec{v}$ and $\vec{r}$ at any time t.

Solution: Here $\frac{d\vec{v}}{dt}$ = (12 cos 2t)$\hat{i}$ - (8 sin 2t) $\hat{j}$ + (16 t)$\hat{k}$

Integrating w.r.t. t, we get

$$\vec{v} = \hat{i}\int 12\cos 2t\, dt - \hat{j}\int 8\sin 2t\, dt + \hat{k}\int 16t\, dt$$

$$= \hat{i}\left[12\frac{\sin 2t}{2}\right] - \hat{j}\left(\frac{-8\cos 2t}{2}\right) + \hat{k}\left[16\frac{t^2}{2}\right] + \vec{c}$$

$$\vec{v} = (6\sin 2t)\,\hat{i} + 4(\cos 2t)\,\hat{j} + (8t^2)\hat{k} + \vec{c} \qquad (1)$$

Where $\vec{c}$ is constant of integration.

Given, At t = 0, $\vec{v} = \vec{0}$

From (1) $\qquad \vec{0} = 0 + 4\hat{j} + 0 + \vec{c}$

$$\Rightarrow \qquad \vec{c} = -4\hat{j}$$

$\therefore$ (1) $\Rightarrow \qquad \vec{v}$ = (6 sin 2t) $\hat{i}$ + (4 cos 2t) $\hat{j}$ +(8t^2)$\hat{k}$ - 4 $\hat{j}$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = (6\sin 2t)\,\hat{i} + (4\cos 2t)\,\hat{j} - 4\hat{j} + 8t^2\,\hat{k}$$

Again integrating w.r.t. t, we get

$$\vec{r} = \hat{i}\int 6\sin 2t\, dt + \hat{j}\int 4\cos 2t\, dt + \hat{k}\,8\int t^2\, dt - 4\hat{j}\int dt$$

$$= \hat{i}\left[-6\frac{\cos 2t}{2}\right] + \hat{j}\left(4\frac{\sin 2t}{2}\right) + 8\hat{k}\left[\frac{t^3}{3}\right] - 4\,\hat{j}\,t + \vec{a}$$

$$\vec{r} = (-3\cos 2t)\,\hat{i} + (2\sin 2t)\,\hat{j} + \frac{8t^3}{3}\,\hat{k} - 4t\,\hat{j} + \vec{d}$$

Where $\vec{d}$ is constant of integration

At t = 0, $\vec{r} = \vec{0}$ (Given)

$$0 = -3\hat{i} + \vec{0} + \vec{0} + \vec{0} + \vec{d}$$

$$\Rightarrow \quad \vec{d} = 3\hat{i}$$

from (2) $\vec{r} = -3\cos 2t\,\hat{i} + 2\sin 2t\,\hat{j} - 4t\,\hat{j} + \frac{8t^3}{3}\,\hat{k} + 3t$

$$\vec{r} = (3 - 3\cos 2t)\,\hat{i} + (2\sin 2t - 4t)\,\hat{j} + \frac{8t^3}{3}\,\hat{k}$$

Definite Integral

If $\frac{d\vec{F}}{dt} = \vec{f}$ for all values of 't' in the interval [a, b] then the definite integral between a and b is denoted as $\int_a^b \vec{f}$ dt and is defined as $\int_a^b \vec{f}$ dt = $\left[\vec{F}\right]_a^b = \vec{f}$ (b) - $\vec{f}$ ($\vec{a}$)

Example 8: If $\vec{r} = (t = t^2)\,\hat{i} + 2t^3\,\hat{j} - 3\hat{k}$, find $\int_2^3 \vec{r}\,dt$

Solution: $\int_2^3 \vec{r}\,dt = \int_2^3 \left[\left(t - t^2\right)\hat{i} + 2t^3\hat{j} - 3\hat{k}\right]$ dt

$$= \int_2^3 t\,\hat{i}\,dt - \int_2^3 t^2\hat{i}\,dt + \int_2^3 2t^3\hat{j}\,dt - \int_2^3 3\hat{k}\,dt$$

$$= \left[\frac{t^2}{2}\right]_2^3 \hat{i} - \left[\frac{t^3}{3}\right]_2^3 \hat{i} + 2\left[\frac{t^4}{4}\right]_2^3 \hat{j} - 3[t]_2^3\,\hat{k}$$

$$= \left[\frac{(3)^2}{2} - \frac{(2)^2}{2}\right]\hat{i} - \left[\frac{3^3}{3} - \frac{2^3}{3}\right]\hat{i} + 2\left[\frac{3^4}{4} - \frac{2^4}{4}\right]\hat{j} - 3[3-2]\hat{k}$$

$$= \left[\frac{9}{2}-\frac{4}{2}\right]\hat{i}-\left[9-\frac{8}{3}\right]_2^3\hat{i}+2\left[\frac{81}{4}-\frac{16}{4}\right]_2^3\hat{j}-3\hat{k}$$

$$= \frac{5}{2}\hat{i}-\left[\frac{27-8}{3}\right]\hat{i}+2\left(\frac{65}{4}\right)\hat{j}-3\hat{k}$$

$$= \frac{5}{2}\hat{i}-\left(\frac{19}{3}\right)\hat{i}+\frac{65}{2}\hat{j}-3\hat{k}$$

$$= \left(\frac{15-38}{6}\right)\hat{i}+\frac{65}{2}\hat{j}-3\hat{k}$$

$$= \frac{-23}{6}\hat{i}+\frac{65}{2}\hat{j}-3\hat{k}$$

$$= -\frac{23}{6}\hat{i}+\frac{65}{2}\hat{j}-3\hat{k}$$

Example 9: Evaluate $\int_1^2 \dot{a}\left(\dot{b}\times\dot{c}\right)$ at, where $\vec{a}$ = t $\hat{i}$ -3 $\hat{j}$ + 2t $\hat{k}$,$\vec{b}$ = $\hat{i}$ - 2 $\hat{j}$ + 2 $\hat{k}$, $\vec{c}$ = 3 $\hat{i}$ + t $\hat{j}$ + $\hat{k}$

Solution: $\vec{a}\ \left(\vec{b}\times\vec{c}\right) = \left[\vec{a}\ \vec{b}\ \vec{c}\right] = \begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & 1 \end{vmatrix}$

= t (-2 - 2t) + 3 (1 - 6) + 2t (t + 6)

= -2t - 2t^2 - 15 + 2t^2 + 12t

= -2t - 2t^2 - 15 + 2t^2 + 12t

= 10t - 15

$\Rightarrow \quad \vec{a}\left(\vec{b}\times\vec{c}\right)$ = 10t - 15

$\Rightarrow \quad \int_1^2 \dot{a}\left(\dot{b}\times\dot{c}\right)^{dt} \int_1^2 \left(10t-15\right)^{dt} = \int_1^2 10t\,dt - \int_1^2 15\,dt$

$= 10\left[\frac{t^2}{2}\right]_1^2 - 15\left[t\right]_1^2 = 10\left[\frac{2^2}{2}-\frac{1}{2}\right] - 15\,[2-1]$

$= 10\left[\frac{4}{2}-\frac{1}{2}\right] - 15 = 10\left[\frac{3}{2}\right] - 15 = 15 - 15 = 0$

Example 10: Evaluate $\int_3^4 \vec{r}.\frac{d\vec{r}}{dt}$ dt where $\vec{r}$ (3) = 3 $\hat{i}$ + 2 $\hat{j}$ + $\hat{k}$ and

$\vec{r}$ (4) = 2 $\hat{i}$ + 5 $\hat{j}$ + $\hat{k}$

Solution: We know, $\frac{d}{dt}\left(\vec{r}.\vec{r}\right) = \vec{r}\,.\,\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\,.\,\vec{r}$

$$= \vec{r}\,.\,\frac{d\vec{r}}{dt} + \vec{r}\,.\,\frac{d\vec{r}}{dt} \qquad \left\{\because \bar{a}.\,\bar{b} = \bar{b}.\bar{a}\right\}$$

$$= 2\vec{r}\,.\,\frac{d\vec{r}}{dt}$$

= $\hat{i}$ [-36t^2 + 18t^2] - $\hat{j}$ [-12t - 0] + $\hat{k}$ [12 - 0]

= -18t^2 $\hat{i}$ + 12t $\hat{j}$ +12 $\hat{k}$

$$\therefore \qquad \int_1^2 \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) \text{dt} = \int_1^2 \left(-18t^2\hat{i} + 12t\,\hat{j} + 12\hat{k}\right)\text{dt}$$

$$= \left[-18\frac{t^3}{3}\hat{i} + 12\frac{t^2}{2}\hat{j} + 12t\,\hat{k}\right]_1^2$$

$$= \left[-6t^3\,\hat{i} + 6t^2\,\hat{j} + 12t\,\hat{k}\right]_1^2$$

$$= \left[-6(2)^3\,\hat{i} + 6(2)^2\,\hat{j} + 12(2)\,\hat{k}\right]$$

$$- \left[-6(1)^3\,\hat{i} + 6(1)^2\,\hat{j} + 12(1)\,\hat{k}\right]$$

$$= \left(-48\hat{i} + 24\hat{j} + 24\hat{k}\right) - \left(-6\hat{i} + 6\hat{j} + 12\hat{k}\right)$$

= - 42 $\hat{i}$ +18 $\hat{j}$ +12 $\hat{k}$

Example 11: Evaluate $\int_0^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right)$ dt where $\vec{r}$ = t^3 $\hat{i}$ + 3t^2 $\hat{j}$ - 2t $\hat{k}$

Solution: Here $\vec{r}$ = t^3 $\hat{i}$ + 3t^2 $\hat{j}$ - 2t $\hat{k}$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = 3t^2\,\hat{i} + 6t\,\hat{j} - 2\,\hat{k}$$

$$\Rightarrow \qquad \vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^3 & 3t^2 & -2t \\ 3t^2 & 6t & -2 \end{vmatrix}$$

$= \hat{i}\,[-6t^2 + 12t^2] - \hat{j}\,[-2t^3 + 6t^3] + \hat{k}\,[6t^4 - 9t^4]$

$$\vec{r} \times \frac{d\vec{r}}{dt} = 6t^2\,\hat{i} - 4t^3\,\hat{j} - 3t^4\,\hat{k} \qquad (1)$$

Now, $\displaystyle\int_0^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = \int_0^3 \left(6t^2 i - 4t^3 j - 3t^4 k\right) dt$

$$= \left[\hat{i}\,6\left(\frac{t^3}{3}\right) - \hat{j}\,4\left(\frac{t^4}{4}\right) - 3\hat{k}\left(\frac{t^5}{5}\right)\right]_0^3$$

$$= \left[2t^3\hat{i} - t^4\hat{j} - \frac{3}{5}t^5\,\hat{k}\right]_0^3$$

$$\Rightarrow \qquad \frac{d}{dt}\left(\vec{r}.\vec{r}\right) = 2\,\vec{r}\,.\,\frac{d\vec{r}}{dt}$$

$$\Rightarrow \qquad \vec{r}\,.\,\frac{d\vec{r}}{dt} = \frac{1}{2}\,\frac{d}{dt}\left(\vec{r}.\vec{r}\right)$$

Integrating both sides w.r.t. t

$$\Rightarrow \qquad \int\left(\vec{r}.\frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}\int \frac{d}{dt}\left(\vec{r}.\vec{r}\right) dt + \vec{c}$$

$$\Rightarrow \qquad \int\left(\vec{r}.\frac{d\vec{r}}{dt}\right) dt = \frac{1}{2}\left(\dot{r}.\dot{r}\right)_{\vec{r}(3)}^{\vec{r}(4)} + \dot{c}$$

$$= \frac{1}{2}\left[\dot{r}(4).\dot{r}(4) - \bar{r}(3).\dot{r}(3)\right] + \dot{c}$$

$$= \frac{1}{2}\left[\left(2\hat{i} - 5\hat{j} + \hat{k}\right).\left(2\hat{i} - 5\hat{j} + \hat{k}\right) - \left(3\hat{i} + 2\hat{j} + \hat{k}\right).\left(3\hat{i} + 2\hat{j} + \hat{k}\right)\right]$$

$$= \frac{1}{2}\left[\left(4 - 25 + 1\right) - \left(9 + 4 + 1\right)\right]$$

$= \frac{1}{2}[30 - 14] = 8$

Example 12: If $\vec{r} = 2t\,\hat{i} + 3t^2\,\hat{j} - t^3\,\hat{k}$, then calculate $\int_1^2 \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) dt$

Solution: Here $\vec{r} = 2t\,\hat{i} + 3t^2\,\hat{j} - t^3\,\hat{k}$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = 2\frac{dt}{dt}\,\hat{i} + 3\frac{d}{dt}(t^2)\,\hat{j} - \frac{d}{dt}(t^3)\,\hat{k}$$

$$\frac{d\vec{r}}{dt} = 2\,\hat{i} + 3(2t)\,\hat{j} - (3t^2)\,\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = 2\,\hat{i} + 6t\,\hat{j} - 3t^2\,\hat{k}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(2\,\hat{i}) + \frac{d}{dt}(6t\,\hat{j}) - \frac{d}{dt}(3t^2\,\hat{k})$$

$$\Rightarrow \quad \frac{d^2\vec{r}}{dt^2} = 0 + 6\,\hat{j} - 6t\,\hat{k}$$

Now $\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6t & -3t^2 \\ 0 & 6 & -6t \end{vmatrix}$

$$= 2(3)^3\,\hat{i} - (3)^4\,\hat{j} - \frac{3}{5}(3)^5\,\hat{k}$$

$$= 54\,\hat{i} - 81\,\hat{j} - \frac{729}{5}\,\hat{k}$$

Example 13: Evaluate $\int \phi\, dr$ for $\phi = x^3y + 2y$ from (1, 1, 0) to (2, 4, 0) along the parabola

$y = x^2$, $z = 0$

Solution: Let $x = t$, so that $y = t^2$

$\therefore$ Parametric equations of the parabola are

$x = t$, $y = t^2$, $z = 0$ (1)

$\Rightarrow$ $dx = dt$, $dy = 2t\,dt$, $dz = 0$ (2)

$\vec{r} = xt\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$d\vec{r}$ = dx $\hat{i}$ + dy $\hat{j}$ + dz $\hat{k}$

$\Rightarrow$ $d\vec{r}$ = dt $\hat{i}$ + 2t dt $\hat{j}$ + 0 $\hat{k}$ (from 2)

$\Rightarrow$ $d\vec{r}$ = dt $\hat{i}$ + 2t dt $\hat{j}$

when x = 1, t = 1, when y = 1, t = 1

x = 2, t = 2 y = 4, t = 2

Value of a 't' along the parabola from (1, 1, 0) to (2, 4, 0) varies from 1 to 2

Now, $\int_c \phi\, d\vec{r} = \int_c \left(x^3y+2y\right)\left(dt\,\hat{i}+2t\,dt\,\hat{j}\right)$

$$= \int_c \left(t^5+2t^2\right)\left(dt\,\hat{i}+2t\,dt\,\hat{j}\right)$$

$$= \hat{i}\int_1^2 \left(t^5+2t^2\right)dt+\hat{j}\int_1^2\left(2t^6+4t^3\right)dt$$

$$= \hat{i}\left[\frac{t^6}{6}+\frac{2t^3}{3}\right]_1^2 + \hat{j}\left[\frac{2t^7}{7}+\frac{4t^4}{4}\right]_1^2$$

$$= \hat{i}\left[\frac{2^6}{6}+\frac{2}{3}(2)^3-\frac{1}{6}+\frac{2}{3}\right]+\hat{j}\left[\frac{2(2)^7}{7}+(2)^4-\frac{2}{7}-1\right]$$

$$= \frac{91}{6}\,\hat{i} + \frac{359}{7}\,\hat{j}$$

Self Check Exercise - 1

Q.1 Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}$, where $\vec{a}$ is constant vector. Also it is given that when t = 0, $\vec{r} = \vec{0}$ and $\frac{d\vec{r}}{dt} = \vec{u}$

Q.2 Solve $\frac{d^2\vec{r}}{dt^2} = \vec{a}$ t + $\vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors, given that when t = 0, $\vec{r} = \vec{0}$ and $\frac{d\vec{r}}{dt} = \vec{u}$

Q.3 If $\vec{r} \times d\vec{r} = \vec{0}$, show that $\vec{r}$ = constant

Q.4 The acceleration $\vec{a}$ of a particle at any time t $\geq$ 0 is given by $\vec{a}$ = e^t $\hat{i}$ - 6(t + 1) $\hat{j}$ + 3(sin t) $\hat{k}$

If the velocity $\vec{v}$ and displacement $\vec{r}$ are zero at t = 0, find $\vec{v}$ and $\vec{r}$ at any time.

Q.5 If $\vec{r}$ (t) = $5t^2 \hat{i}$ + t $\hat{j}$ - $t^3 \hat{k}$, prove that

$$\int_1^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

Q.6 Evaluate $\int_0^2 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ where $\vec{r} = 2t^2\hat{i} + t\hat{j} - 3t^3\hat{k}$

Q.7 The acceleration $\vec{a}$ of a particle at any time t > 0 is given by $\vec{a}$ = e^{-t} $\hat{i}$ - 6(t+1) $\hat{j}$ + 3(sin t) $\hat{k}$.

If the velocity $\vec{v}$ and displacement $\vec{x}$ are zero at t = 0, find $\vec{v}$ and $\vec{r}$ at any time.

Q.8 Evaluate $\int (x\,dy - y\,d\,x)$ around the circle $x^2 + y^2 = 1$

Q.9 Evaluate $\int \left[yzdx + (zx+1)dy + xydz\right]$, where c is a straight line joining the points (1, 0, 0) to (2, 1, 4)

14.4 Summary: In this unit we studied that

1. For a vector function $\hat{F}$ (t) = $f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ where f_1, f_2, f_3 are scalar function of some variable 't' then.

   $\int f(t)dt = i\int \hat{f}(t)dt = \hat{i}\int f_1 dt + \hat{j}\int f_2 dt + \hat{k}\int f_3 dt.$ is indefinite integral.

2. If the variable t is defined on interval [a, b] then the definite integral of vector function is defined as $\int_a^b \hat{f}(t)dt = \int_a^b f_1 dt + \hat{j}\int_a^b f_2 dt + \hat{k}\int_a^b f_3 dt$

3. Vector integration is helpful in real life situations in order to find displacement and velocity from given velocity and acceleration respectively.

14.5 Glossary

1. **Integrand:** The integrand is a mathematical expression that represents the function being integrated

2. **Parametric Equations:** Parametric equation is an equation where variable (usually x and y) are expressed in terms of third parameter usually expressed as t.

3. **Parabola:** It is a curve formed by the intersection of a cone with a plane parallel to straight line in its surface.

14.6 Answers to Self Check Exercise - 1

Q.1 $\vec{r} = \frac{1}{2}\vec{a}\, t^2 + t\,\vec{u}$

Q.2 $\vec{r} = \frac{\vec{a}}{6} t^3 + \frac{\vec{b}}{2} t^2 + t\vec{u}$

Q.3 $\vec{v} = -e^{-t}\hat{i} - (3t^2 + 6t)\hat{j} - 3(\cos t)\hat{k} + \hat{i} + 3\hat{k}$

$\vec{r}$ $(t+e^{-t}-1)\hat{i} - (t^3+3t^2)\hat{j} + (3t - 3\sin t)\hat{k}$

Q.4 Use the concept of vector integration for its proving

Q.5 $-42\hat{i} + 90\hat{j} - 6\hat{k}$

Q.6 $\vec{v} = -e^{-t}\hat{i} - (3t^2 + 6t)\hat{j} - 3(\cos t)\hat{k} + \hat{i} + 3\hat{k}$

$\vec{r}$ $(t + e^{-t}-1)\hat{i} - (t^3+3t^2)\hat{j} + (3t - 3\sin t)\hat{k}$

Q.7 2π

Q.8 9

14.7 References/Suggested Readings

1. R. Murray, S. Lipschutz, D. Spellman, Vector Analysis, Schaunts Outline.
2. S. Narayan and P.K. Mittal, Vector Calculas, Schand and Company Limited.
3. I.N. Sharma and A.R. Vasistha, Vector Calcula, Krishna Parkashan Mandir.

14.8 Terminal Question

Q.1 Given that $\vec{r}(t) = \begin{cases} 2\hat{i} - \hat{j} + 2\hat{k}, when\ t = 2 \\ 4\hat{i} - 2\hat{j} + 3\hat{k}, when\ t = 3 \end{cases}$

Show that $\int_2^3 \left(\vec{r}.\frac{d\vec{r}}{dt}\right) dt = 10$

Q.2 Solve $\frac{d^2\vec{r}}{dt^2} = \vec{a}\, t + \vec{b}$, given that both $\vec{r}$ and $\frac{d\vec{r}}{dt}$ vanish when t = 0

Q.3 If $\vec{r} \,.\, d\vec{r} = 0$, show that r = constant

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Unit - 15

Line Integral

Structure

15.1 Introduction

Dear student, in this unit you will learn about line integral. In previous unit we learn the vector integration with definite and indefinite integral. In the integral the path of integration is not a straight line but an arbitrary curve in space. For example of we want to find the workdone by a force in moving a particle along a curve from point A to point B.

15.2 Learning Objectives

After studying this unit, students will be able to

1. define line integral over a curve c
2. define line integral over a parametric curve c
3. evaluate line integral.
4. define and evaluate workdone by a force field.
5. define and evaluate circulation of a vector field.

Simple vector integration will not help. To solve such problem we need the line integral or curve integrals. In this unit we will learn about some basics used in line integral and learn how to evaluate the line integral.

15.3 Line Integral

Line integral is a generalization of the concept of definite integral. In definite integral $\int_a^b \hat{F} dt_1$ we integrate the function f(t) w.r.t. time t between the time interval t = a and t = b.

In line integral, we integrate the given function or field along a curve C, here the integrand will be a function defined at every point of given curve. Here the path of integration may be a straight line or curve in space or in a plane.

If $\hat{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ be the $\hat{F}$ in its component form and

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

then $$\int_c \hat{F}.d\vec{r} = \int_c \left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$$

$$= \int_c \left(F_x dx + F_y dy + F_z dz\right)$$

Note: If should be noted that F_x, F_y, F_z the components of $\hat{F}$ in x, y and z direction are functions of x, y and z, but the integral will be either over x or y or z. So, you must have to express each integral in terms of a single variable. If means, to evaluate, $\int_c F_x(x, y, z)$ dx we have to express y and z in terms of x so that F_x is a function of x only.

Note:- Line integral is an integration over one variable before moving further let us define some terms as:

Curve Closed: Let C be any curve in space with A as initial point and B as terminal point, and the curve is moving from A to B, if the initial and terminal point coincide, then the curve is known as closed curve.

Smooth Curve: A curve c is said to be smooth curve if the curve $\vec{r}$ curve if the curve $\vec{r}$(t) is continuously differentiable i.e. $\frac{d\vec{r}(t)}{dt}$ exists and is not equal to zero anywhere on c, and the direction of this derivative i.e. $\vec{r}$(t) is along the tangent to the curve at every point.

In other word, a curve is said to be smoth curve if it possesses a unique tangent at each of its points.

Piecewise Smoth Curve: A curve is said to be piece wise smoth if it is composed of a finite number of smoth curves.

Line Integral Using Parametric Representation

The parametric representation of the path of integration to define line integral of a vector function along the path is define as.

$$\int_c \hat{F}.d\vec{r} = \int_{t_1}^{t_2}\left[\hat{F}\left(\vec{r}(t)\right)\frac{d\left[\vec{r}(t)\right]}{dt}\right]dt$$

where $\hat{F}\left(\vec{r}(t)\right)$ is a vector function,

$\vec{r}(t)$ = x(t) $\hat{i}$ + y(t) $\hat{j}$ + z(t) $\hat{k}$ is the position vector function and t_1 and t_2 are the end points of the path.

So, that $\frac{d\vec{r}}{dt} = \frac{d}{dt}$ [x(t) $\hat{i}$ + y(t) $\hat{j}$ + z(t) $\hat{k}$]

$$= \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

If $\hat{F}$ (x, y, z) = F_x (x, y, z) $\hat{i}$ + F_y(x, y, z) $\hat{j}$ + F_z(x, y, z) $\hat{k}$ is the vector function in Cartesian coordinates then replacing x = x(t), y = y(t), z = (t), we can write the vector function as a function of the parameter t.

So, $\hat{F}\left(\vec{r}(t)\right)$ = F_x(t) $\hat{i}$ + F_y(t) $\hat{j}$ + F_z(t) $\hat{k}$

So, $$\int_c \hat{F}.d\vec{r} = \int_{t_1}^{t_2}\left[\hat{F}.\frac{d\vec{r}}{dt}\right]dt$$

$\Rightarrow$ $\int_c \hat{F}.d\vec{r} = \int_{t_1}^{t_2}\left[F_x\frac{dx}{dt}+F_y\frac{dy}{dt}+F_z\frac{dz}{dt}\right]dt$, is the line integral of vector field $\hat{F}$ (t)

on a curve c which has a parametric represent action $\vec{r}$ (t)

The term F_x(t) $\frac{dx(t)}{dt}$ + F_y(t) $\frac{dy(t)}{dt}$ + F_z(t) $\frac{dz(t)}{dt}$ is a scalar function of a single variable t.

Properties of Line Integral

The line integral of a vector fields $\hat{F}$ and G alonge a curve C has the following properties.

1. For a constant a,

$$\int_c a\hat{F}.d\hat{r} = a\int_c \hat{F}.d\hat{r}$$

2. $\int_c \left(\hat{F} + \hat{G}\right).d\hat{r} = \int_c \hat{F}.d\hat{r} + \int_c \hat{F}.d\hat{r}$

3. $\int_c \hat{F}.d\hat{r} = \int_{c_1} \hat{F}.d\hat{r} + \int_{c_2} \hat{F}.d\hat{r}$

where the curve c is made up of two curves c_1 and c_2 Now, let us try to understand line integral by using following examples:

Example 1: Evaluate $\int_c \hat{F}.d\hat{r}$, where $\hat{F}$ = 2xy $\hat{i}$ - y^2 $\hat{j}$ + and c is the curve in the xy plane given by y = x^2 from (0, 0) to (2, 4).

Solution: Here given $\hat{F}$ = 2xy $\hat{i}$ - y^2 $\hat{j}$, is in two dimensions and we, know that

$$\hat{r} = x\,\hat{i} + y\,\hat{j}$$

so $\quad d\hat{r} = dx\,\hat{i} + dy\,\hat{j}$.

$$\therefore \quad \int_c \hat{F}.d\vec{r} = \int_c \left(2xy\,dx - y^2 dy\right) \qquad (1)$$

Since the equation of given curve is y = x^2 from (0, 0) to (2, 4), so putting y - x^2, dy = 2xdx in (1), Since by this substitution integrand will now be the function of x only so laking the limit of x from 0 to 2, we have

$$\int_c \hat{F}.d\vec{r} = \int_0^2 \left[2x(x)^2 dx - (x^2)^2 2xdx\right]$$

$$= \int_0^2 \left[2x^3 dx - 2x^5 dx\right]$$

$$= \int_0^2 2x^3 dx - \int_0^2 2x^5 dx$$

$$= \left[\frac{2x^4}{4}\right]_0^2 - \left[\frac{2x^6}{6}\right]_0^2$$

$$= \frac{1}{2}\ (16 - 0) - \frac{1}{3}\ (64 - 0)$$

$$= 8 - \frac{64}{3}$$

$$\int_c \hat{F}.d\vec{r} = -\frac{40}{3}$$

Example 2: Evaluate $\int_c \hat{F}.d\vec{r}$ where

$\hat{F}$ = xy $\hat{i}$ + yz $\hat{j}$ + zx $\hat{k}$ and c is the curve $\vec{r}$ = t $\hat{i}$ + t^2 $\hat{j}$ + t^3 $\hat{k}$ from (0, 0, 0) to (2, 4, 8).

Solution: Given $\hat{F}$ = xy $\hat{i}$ + yz $\hat{j}$ + zx $\hat{k}$ (1)

and $\vec{r}$ = t $\hat{i}$ + t^2 $\hat{j}$ + t^3 $\hat{k}$ (2)

Since we known that, in general

$\vec{r}$ = x $\hat{i}$ + y $\hat{j}$ + z $\hat{k}$ (3)

Comparing (2) and (3)

x = t, y = t^2. z = t^3.

Putting the values of x, y and z in (1), we get

$\therefore$ $\hat{F}$ = t^3 $\hat{i}$ + t^j $\hat{j}$ + t^4 $\hat{k}$ (4)

Also, $\frac{d\vec{r}}{dt}$ = $\hat{i}$ + 2t $\hat{j}$ + 3t^2 $\hat{k}$ (5)

Since now the given integrant is converted into a single parameter 't' so the limit of integration must be dependent on t. We can find these values of t_1 and t_2 as.

Since x = t_1 $\Rightarrow$ $t_1 = 0$

y = t_1^2 $\Rightarrow$ $t_1^2 = 0$ $\Rightarrow$ $t_1 = 0$

z = t_1^3 $\Rightarrow$ $t_1^3 = 0$ $\Rightarrow$ $t_1 = 0$ $\left\{\because\ initialy\,((x_1 y_1 z) = (0,0,0)\,Given\right\}$

Again, x = t_2 $\Rightarrow$ $t_2 = 2$

y = t_2^3 $\Rightarrow$ $t_2^2 = 4$ $\Rightarrow$ $t_2 = 2$

z = t_2^3 $\Rightarrow$ $t_2^3 = 8$ $\Rightarrow$ $t_2 = 2$ $\left\{\because\ (x_1 y_1 z)\,at\ finial\ stage\,is(2_1 4_1 8)\right\}$

So the limit of t varies from t1 = 0 to t2 = 2 using line integral for parametric form we have

$$\int_c \hat{F}.d\vec{r} = \int_{t_1}^{t_2}\left[\hat{F}.\frac{d\hat{r}}{dt}\right]dt$$

$$= \int_0^2 \left[\left(t^3\hat{i} + t^5\hat{j} + t^4\hat{k}\right).\left(\hat{i} + 2t\,\hat{j} + 3t^2\hat{k}\right)\right]$$

$$= \int_0^2 \left[t^3\hat{i} + 2t^6\hat{j} + 3t^6\right]dt$$

$$= \int_0^2 \left[t^3 + 5t^6\right] dt$$

$$\int_c \hat{F}.d\vec{r} = \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_0^2$$

$$= \frac{1}{4}\ (16 - 0) + \frac{5}{7}\ (128 - 0)$$

$$= 4 + \frac{640}{7}$$

$$\int_c \hat{F}.d\vec{r} = \frac{668}{7}$$

Example 3 Evaluate $\int_c \hat{F}.d\vec{r}$ where

$$\vec{F} = c\left[\left(-3a\sin^2\theta\cos\theta\right)\hat{i} + a\left(2\sin\theta - 3\sin 3\theta\right)\hat{j} + b\left(\sin 2\theta\right)\hat{k}\right]$$

and the curve c is given by $\vec{r}$ = a cos θ $\hat{i}$ + a sin θ $\hat{j}$

+ bθ $\hat{k}$, $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

Solution: Here $\vec{F} = c\left[\left(-3a\sin^2\theta\cos\theta\right)\hat{i} + a\left(2\sin\theta - 3\sin 3\theta\right)\hat{j} + \left(b\sin 2\theta\right)\hat{k}\right]$ and $\vec{r}$ = a cos θ $\hat{i}$ + a sin θ $\hat{j}$ + bθ $\hat{k}$,

$$\Rightarrow \qquad \frac{d\vec{r}}{d\theta} = a\,(-\sin\theta)\,\hat{i} + a\cos\theta\,\hat{j} + b\,\hat{k}$$

$$\frac{d\vec{r}}{d\theta} = -\,a\,(-\sin\theta)\,\hat{i} + a\cos\theta\,\hat{j} + b\,\hat{k}$$

$$\therefore \qquad \int_c \hat{F}.d\vec{r} = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \vec{F}.\frac{d\vec{r}}{dt}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} c\left[\left(-3a\sin^2\theta\cos\theta\right)\hat{i} + a\left(2\sin\theta - 3\sin 3\theta\right)\hat{j} + \left(b\sin 2\theta\right)\hat{k}\right]$$

$$.\left[-a\sin\theta\hat{i}+a\cos\theta\hat{j}+b\hat{k}\right] \text{do}$$

$$c\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[3a^2\sin^3\theta\cos\theta+a\left(2\sin\theta-3\sin^3\theta\right)a\cos\theta+b^2\sin 2\theta\right] d\theta$$

$$c\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[3a^2\sin^3\theta\cos\theta+2a^2\sin\theta\cos\theta-3a^2\sin^3\theta\cos\theta+b^2\sin 2\theta\right] d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(a^2\sin 2\theta+b^2\sin 2\theta\right) \text{do} \qquad \{\because 2\sin\theta\cos\theta=\sin 2\theta\}$$

$$= c\,(a^2+b^2)\,c\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin 2\theta d\theta$$

$$= c\,(a^2+b^2)\left[-\frac{\cos 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{-c\left(a^2+b^2\right)}{2}\,(-1-0)$$

$$= \frac{c}{2}(a^2+b^2)$$

Example 4 Evaluate $\int_c \hat{F}.d\vec{r}$ where $\vec{F}$ = z $\hat{i}$ + x $\hat{j}$ + y $\hat{k}$ and c is the are of the curve $\vec{r}$ = (cos t) $\hat{i}$ + (sin t) $\hat{j}$ + t $\hat{k}$ from t = 0 to t = 2 π

Solution: Here $\vec{F}$ = z $\hat{i}$ + x $\hat{j}$ + y $\hat{k}$

$$\vec{r} = (\cos t)\,\hat{i} + (\sin t)\,\hat{j} + t\,\hat{k} \qquad (1)$$

$$\Rightarrow \qquad \frac{d\vec{r}}{dt} = (-\sin t)\,\hat{i} + \cos t\,\hat{j} + \hat{k}$$

Also $\vec{r}$ = x $\hat{i}$ + y $\hat{j}$ + z $\hat{k}$ (2)

From (1) and (2)

x = cos t, y = sin t, z = t

$\therefore$ $\vec{F}$ = t $\hat{i}$ + cos t $\hat{j}$ + sin t $\hat{k}$ (3)

$$\therefore \int_c \hat{F}.d\vec{r} = \int_0^{2\pi} \vec{F}.\frac{d\vec{r}}{dt}\,dt$$

$$= \int_0^{2\pi} \left(t\hat{i} + \cos t\hat{j} + \sin t\hat{k}\right).\left(-\sin t\,\hat{i} + \cos t\,\hat{j} + \hat{k}\right)dt$$

$$= \int_0^{2\pi} \left[-t\sin t + \cos^2 t + \sin t\right]dt$$

$$= \int_0^{2\pi} (-t\sin t)\,dt + \int_0^{2\pi} \cos^2 t\,dt + \int_0^{2\pi} \sin t\,dt$$

$$= \left[t\int_0^{2\pi} \sin t\,dt - \int_0^{2\pi}\left[\frac{dt}{dt}\int \sin t\,dt\right]dt\right] + \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt - \left(\cos t\right)_0^{2\pi}$$

$$= -\left[-t\cos t - \left(-\sin t\right)\right]_0^{2\pi} + \frac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} - [\cos 2\pi - \cos 0]$$

$$= [-2\pi\cos 2\pi + \sin 2\pi] + [0 + \sin 0] + \frac{1}{2}\left[2\pi + \frac{\sin 4\pi}{2} - 0 + \sin 0\right] - \left[\cos 2\pi - \cos 0\right]$$

$$= (2\pi - 0) + 0 + \pi + 0 - 1 + 1$$

$$= 3\pi$$

Example 5 Evaluate $\int_c \hat{F}.d\vec{r}$ where c is the curve in the xy plane, y = $2x^2$ from (0, 0) to (1. 2), $\vec{F}$ = 3xy $\hat{i}$ - y^2 $\hat{j}$

Solution: Here $\vec{F}$ = 3xy $\hat{i}$ + y^2 $\hat{j}$, $\vec{r}$ = x $\hat{i}$ + y $\hat{j}$

$\therefore$ d$\vec{r}$ = d x $\hat{i}$ + dy $\hat{j}$

The equation of curve is y = $2x^2$

$$\therefore \int_c \hat{F}.d\vec{r} = \int_c \left(3xy\hat{i} - y^2\hat{j}\right).\left(dx\hat{i} + dy\,\hat{j}\right)$$

$= \int_c \left(3xydx - y^2dy\right)$

$= \int_0^1 \left[3x\left(2x^2\right)dx - \left(2x^2\right)^2\left(4xdx\right)\right]$

$= \int_0^1 \left(6x^3 - 16x^5\right)$

$= 6\left(\frac{x^4}{4}\right)_0^1 - 16\left(\frac{x^6}{6}\right)_0^1$

$= 6\left(\frac{1}{4} - 0\right) - 16\left(\frac{1}{6} - 0\right)$

$= -\frac{7}{6}$

Example 6 A vector field is given by $\vec{f} = \sin y\,\hat{i} + x\,(1+\cos y)\,\hat{j}$.

Evaluate the line integral $\oint_c \vec{f}.d\vec{r}$ along the curve given by $x_2 + y_2 = a_2$, z = 0

Solution: Here $\vec{f} = \sin y\,\hat{i} + x\,(1+\cos y)\,\hat{j}$

The given curve is $x^2 + y^2 = a^2$, z = 0

The parametric equations of the curve are x = a cos t y = a sin t, z = 0

The Position vector of any point on the curve is

$\vec{r} = x\,\hat{i} + y\,\hat{j}$

$\therefore$ $d\vec{r} = dr\,\hat{i} + dy\,\hat{j}$

Now, $\oint_c \vec{f}.d\vec{r} = \oint_c \left[\sin y\hat{i} + x\left(1+\cos y\right)\hat{j}\right].\left[dx\hat{i} + dy\hat{j}\right]$

$= \int_c \sin ydx + x\left(1+\cos y\right)dy$

$= \int_c \sin y\,dx + x\,dy + x\,cos\,y\,dy$

$= \int_c \sin y\,dx + x\cos y\,dy + x\,dy$

$$= \int_c d\left(x \sin y\right) + \int_c x\,dy$$

$$= \int_0^{2\pi} d\left[a\cos t \sin\left(a \sin t\right) + \int_0^{2\pi} a\cos t a \cos t dt\right] \quad \{\because 't' \text{ varies from } 0 \text{ to } 2\pi \text{ on the circle}\}$$

$$= \left[a\cos t \sin\left(a\sin t\right)\right]_0^{2\pi} + a^2 \int_0^{2\pi} \cos^2 t\,dt$$

$$= 0 + a^2 \int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right) dt$$

$$= a^2 \left[\frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\left(2\pi + \frac{\sin 4\pi}{2} - 0 - 0\right)$$

$$= \frac{a^2}{2}(2\pi)$$

$$= \pi a^2$$

Example 7 Evaluate $\vec{f}.d\vec{r}$ from (0, 0, 0) to (3, 9, 0) along the curve $y = x^2$, $z = 0$, for $\vec{f} = 3x^2 \hat{i} + (2x - z)^2 \hat{k}$

Solution: Let x = t Do that $y = t^2$ $\quad \{\because y = x^2 (given)\}$

$\therefore$ along the parabola $y = x^2$, $z = 0$, we have

$x = t$, $y = t^2$, $z = 0$

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = t\hat{i} + t^2\hat{j} + 0$

$\Rightarrow$ $\vec{r} = t\hat{i} + t^2\hat{j}$

$\therefore$ $\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j}$

Now $\vec{f} = 3x^2\hat{i} + (x + z)^2\hat{j} + (2x - z)^2\hat{k}$

$= 3t^2\hat{i} + t^2\hat{j} + (2t - 0)\hat{k}$

$\vec{f} = 3t^2\,\hat{i} + t^2\,\hat{j} + 2t\,\hat{k}$

when $x = 0 \Rightarrow t = 0$

$x = 3 \Rightarrow t = 3$

when $y = 0 \Rightarrow t^2 = 0 \Rightarrow t = 0$

$y = 9 \Rightarrow t^2 = 9 \Rightarrow t = 3$

$\therefore$ Value of t along the curve from (0, 0, 0) to (3, 9, 0)

Varies from 0 to 3

$$\therefore \int_c \vec{f}.d\vec{r} = \int_c \vec{f}.\frac{d\vec{r}}{dt}\,dt = \int_0^3 \left(3t^2\hat{i} + t^2\hat{j}\right).\left(\hat{i} + 2\hat{j}\right)dt$$

$$= \int_0^3 \left[3t^2(1) + t^2.(2t)\right]dt$$

$$= \int_0^3 3t^2 dt + \int_0^3 2t^3 dt$$

$$= \left.\frac{t^3}{3}\right|_0^3 + \left.\frac{2t^4}{4}\right|_0^3$$

$$= \left(27 + \frac{81}{2}\right) - 0$$

$$= \frac{135}{2}$$

Self Check Exercise - 1

Q.1 Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$ and c is the portion of curve $\vec{r} = (a\cos t)\,\hat{i} + (b\sin t)\,\hat{j} + (ct)\,\hat{k}$ from $t = 0$ to $t = \frac{\pi}{2}$

Q.2 Evaluate $\int_c \vec{F}.d\vec{r}$ where $\vec{F} = xy\,\hat{i} + yz\,\hat{j} + zx\,\hat{k}$ and curve c is $\vec{r} = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$ s, t varies from - 1 to 1

Q.3	Evaluate $\int_c \vec{f}.d\vec{r}$ where $\vec{f}$ = xy $\hat{i}$ + 2yz $\hat{j}$ - 9z $\hat{k}$ and curve c is $\vec{r}$ = t $\hat{i}$ + t^2 $\hat{j}$ + t^3 $\hat{k}$, varies from 1 to 2.	
Q.4	Evaluate $\int_c \vec{f}.d\vec{r}$ where $\vec{f}$ = y $\hat{i}$ + $(x + z)^2$ $\hat{j}$ + $(x - z)^2$ $\hat{k}$ from (0, 0, 0) to (2, 4, 0) along (i) the parabola $y = x^2$, z = 0 (ii) the straight line y = 2x in xy plane	
Q.5	If $\vec{F}$ = (2x2 + y2) $\hat{i}$ + (3y - 4x) $\hat{j}$, evaluate $\int_c \vec{F}.d\vec{r}$ around the triangle ABC whose vertices are A (0, 0), B (2, 0) and c (2, 1)	

15.4 Workdone

Work done by a force:Let $\vec{F}$ = F_1 $\hat{i}$ + F_2 $\hat{j}$ + F_3 $\hat{k}$ be a force acting at P with position vector x $\hat{i}$ + y $\hat{j}$ + z $\hat{k}$

Then the work done by the force $\vec{F}$ in displacing a unit particle from A to B is defined as line integral from A to B

$$\therefore \quad \text{Work done} = \int_A^B \vec{F}.\hat{t}\, ds = \int_A^B \vec{F}.d\vec{r} = \int_A^B \left(F_1 dx + F_2 dy + F_3 dz\right)$$

Conservative Field :

A force $\vec{F}$ is said to be conservative if the work done by it in moving its point of application from a point A to B depends only on the points A and B and not upon the path joining A and B.

Q.1 Find the work done when a force $\vec{F}$ = $(x^2 - y^2 + x)$ $\hat{i}$ - (2xy + y) $\hat{j}$ moves a particle in xy plane from (0, 0) to (1, 1) along the parabola $y^2 = x$

Solution: Let C denote the are of the parabola $y^2 = x$ from the point (0, 0) to the point (1, 1). The parametric equations of the parabola $y^2 = x$ can be taken as $x = t^2$, y = t. At the point (0, 0), t = 0 and at the point (1, 1), t = 1

Now $\vec{F}$ = $(x^2 - y^2 + x)$ $\hat{i}$ - (2xy + y) $\hat{j}$

d$\vec{r}$ = dx $\hat{i}$ + dy $\hat{j}$ $\quad \left\{\because \vec{r} = x\hat{i} + y\hat{j}\right\}$

Work done = $\int_c \vec{F}.d\vec{r}$

$$= \int_{1} \left[\left(x^2 - y^2 + x \right) i - \left(2xy + y \right) j \right] . \left[dx\hat{i} + dy\hat{j} \right]$$

$$= \int_0^1 \left[\left(x^2 - y^2 + x \right) \frac{dx}{dt} - \left(2xy + y \right) \frac{dy}{dt} \right] dt$$

$$= \int_0^1 \left[\left(x^4 - t^2 + t^2 \right) \left(2t \right) - \left(2t^3 + t \right) (1) \right] dt$$

$$= \int_0^1 \left(2t^5 - 2t^3 - t \right) dt$$

$$= \left[2\frac{t^6}{6} - \frac{2t^4}{4} - \frac{t^2}{2} \right]_0^1$$

$$= -\frac{2}{3}$$

Q.2 Find the work done in moving a particle in a force field $\vec{f}$ = $3x^2\,\hat{i}$ + $(2xz - y)\,\hat{j}$ + $z\,\hat{k}$ along the line joining the points (0, 0, 0) to (2, 1, 3).

Solution: Here $\vec{f}$ = $3x^2\,\hat{i}$ + $(2xz - y)\,\hat{j}$ + $z\,\hat{k}$

The equation of line joining (0, 0, 0) to (2, 1, 3) is

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0}$$

$$\Rightarrow \quad \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

$\Rightarrow$ x = 2t, y = t, z = 3t are parametric equations of line $\therefore$ t varies from 0 to 1 along the curve from (0, 0, 0) to (2, 1, 3)

$$\therefore \quad \vec{f} = 3\left(2t\right)^2 \hat{i} + \left[2\left(2t\right)\left(3t\right) - t \right] \hat{j} + 3t\,\hat{k}$$

$$\vec{f} = 12t^2\,\hat{i} + (12t^2 - t)\,\hat{j} + 3t\,\hat{k}$$

Also $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$$\vec{r} = 2t\,\hat{i} + t\,\hat{j} + 3t\,\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = 2\,\hat{i} + \hat{j} + 3\,\hat{k}$$

Work done in moving particle in force field from (0, 0, 0) to (2, 1, 3) = $\int_c \vec{f}.d\vec{r}$

$$= \int_c \left(\overline{f.d\vec{r}}\right) dt$$

$$= \int_0^1 \left[12t^2 i + \left(2t^2 - t\right)\hat{j} + 3t\,\hat{k}\right].\left[2\hat{i} + \hat{j} + 3\hat{k}\right] dt$$

$$= \int_0^1 \left[24t^2 + \left(12t^2 - t\right) + 9t\right] \text{dt}$$

$$= \int_0^1 \left(36t^2 + 8t\right) dt$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= \left[12t^3 + 4t^2\right]_0^1$$

= (12 + 4) - 0

= 16

Q.3 Find the total work done in moving a particle in a force field given by $\vec{F}$ = 3xy $\hat{i}$ - 5z $\hat{j}$ + 10x $\hat{k}$ along the curve x = t^2 + 1, y = $2t^2$, z = t^3 from t = 1 to t = 2.

Solution: The equations of curve c are

x = t^2 + 1, y = $2t^2$, z = t^3

Now, $\vec{F}$ = 3xy $\hat{i}$ - 5z $\hat{j}$ + 10x $\hat{k}$

$\vec{r}$ = x $\hat{i}$ + y $\hat{j}$ + z $\hat{k}$

$\Rightarrow$ d$\vec{r}$ = dx $\hat{i}$ + dy $\hat{j}$ + dz $\hat{k}$

Work done = $\int_c \vec{F}.d\vec{r}$

$$= \int_c \left[3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}\right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \int_c 3xy\,dx - 5z\,dy + 10x\,dz$$

$$= \int_1^2 \left[3\left(t^2+1\right)\left(2t^2\right)\left(2t\,dt\right) - 5t^3 4tdt + 10\left(t^2+1\right)3t^2 dt\right]$$

$$= \int_1^2 \left[12t^3\left(t^2+1\right)dt - 20t^4 dt + 30t^2\left(t^2+1\right)dt\right]$$

$$= 12\int_1^2 \left(t^5+t^3\right)dt - 20\int_1^2 t^4 dt + 30\int_1^2 \left(t^4+t^2\right)dt$$

$$= 12\left(\frac{t^6}{6}+\frac{t^4}{4}\right)_1^2 - 20\left(\frac{t^5}{5}\right)_1^2 + 30\left(\frac{t^5}{5}+\frac{t^3}{3}\right)_1^2$$

= 171 - 124 + 256

= 303

Q.4 Find the work done in moving a particle once around a circle c in the xy plane, if the circle has centre at the Qrign and radius 2 and if the force field is given by $\vec{F}$ = (2x - y + 2z) $\hat{i}$ + (x + y - z) $\hat{j}$ + (3x - 2y - 5z) $\hat{k}$

Solution: The equation of circle with centre (0, 0) and radius = 2 is $x^2 + y^2 = 4$

Its parametric equations are x = 2 cos t, y = 2 sin t, z = 0 t varies from 0 to 2π

Now, $\vec{F}$ = (2x - y + 2z) $\hat{i}$ + (x + y - z) $\hat{j}$ + (3x - 2y - 5z) $\hat{k}$

$$\therefore \quad \int_c \vec{F}.d\vec{r}$$

$$- \int_c \left[\left(2x-y+2z\right)\hat{i} + (x+y-z)\,\hat{j} + (3x-2y-5z)\,\hat{k}\right].\left[dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right]$$

$$= \int_c \left(2x-y+2z\right)dx + \left(x+y-z\right)dy + (3x-2y-5z)dz$$

$$= \int_0^{2\pi} \left[2\left(2\cos t\right) - 2\sin t\right] + \left(-2\sin t\right)dt + \int_0^{2\pi} \left[2\cos t + 2\sin t\right](2\cos t)dt$$

$$+ \int_0^{2\pi} \left[3\left(2\cos t\right) - 2(2\sin t) - 5(0)\right]0$$

$$= \int_0^{2\pi} \left(4\cos t - 2\sin t\right)(-2\sin t)dt + \int_0^{2\pi} \left(4\cos^2 t + 4\sin t\cos t\right)dt$$

$$= \int_0^{2\pi} 8\cos t \sin t \, dt + \int_0^{2\pi} 4\sin^2 t \, dt + \int_0^{2\pi} 4\cos^2 t \, dt + 4\int_0^{2\pi} \sin t \cos t \, dt$$

$$= \int_0^{2\pi} \left[-8\cos t \sin t + 4(\sin^2 t + \cos^2 t) + 4\sin t \cos t\right] dt$$

$$= \int_0^{2\pi} \left[-4\sin t \cos t + 4\right] dt$$

$$= \int_0^{2\pi} \left[4 - 2(2\sin t \cos t)\right] dt \qquad \{\sin 2A = 2\sin A \cos A\}$$

$$= \int_0^{2\pi} (4 - 2\sin 2t) \, dt$$

$$= \left(4t + 2\frac{\cos 2t}{2}\right)_0^{2\pi}$$

$= (8\pi + \cos 4\pi) - (0 + \cos 0)$

$= 8\pi + 1 - 1$

$= 8\pi$

Q.5 Find the work done in moving a particle once around a circle 'c' in xy plane, if the circle has centre at the origin and radius 3 and the force field is given by

$$\vec{f} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$$

Solution: Here $\vec{f} = (2x - y + z)\,\hat{i} + (x + y - z^2)\,\hat{j} + (3x - 2y + 4z)\,\hat{k}$

The equation of the circle in xy plane is $x^2 + y^2 = 9$, $z = 0$

The parametric equations of the circle are

$x = 3\cos t$, $y = 3\sin t$, $z = 0$

$\therefore \quad \vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

$\vec{r} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} \qquad \{\because z = 0\}$

$\therefore \quad \dfrac{d\vec{r}}{dt} = -2\sin t\,\hat{i} + 3\cos t\,\hat{j}$

$\therefore \quad$ Work done $= \int_c \vec{f}.d\vec{r} = \int_c \left(\vec{f}.d\vec{r}\right) dt$

$$= \int_c \left[(2x - y + z)\hat{i} + \left(x + y - z^2\right)\hat{j} + (3x - 2y + 4z)\hat{k} \right] . \left[-3\sin t\,\hat{i} + 3\cos t\,\hat{j} \right] dt$$

$$= \int_c \left[(2x - y + z)(-3\sin t) + \left(x + y - z^2\right)(3\cos t) \right] dt$$

$$= \int_0^{2\pi} \left[2(+3\cos t)(-3\sin t) - (3\sin t)(-3\sin t) + 0 + (3\cos t)(3\cos t) + (3\sin t)(3\cos t) + 0 \right] dt$$

$$= \int_0^{2\pi} \left[-18\cos t \sin t + 9\sin^2 t + 9\cos^2 t + 9\sin t \cos t \right] dt$$

$$= \int_0^{2\pi} \left[-9\sin t \cos t + 9\left(\sin^2 t + \cos^2 t\right) \right] dt$$

$$= \int_0^{2\pi} \left[-9\sin t \cos t + 9 \right] dt$$

$$= \int_0^{2\pi} \frac{-9}{2} (2\sin t \cos t)\, dt + \int_0^{2\pi} 9\,dt$$

$$= \frac{-9}{2} \int_0^{2\pi} \sin 2t\, dt + -(t)_0^{2\pi}$$

$$= \frac{-9}{2} \left(\frac{-\cos 2t}{2} \right)_0^{2\pi} + 9(2\pi - 0)$$

$$= \frac{9}{4} \left[\cos 4\pi - \cos 0 \right] + 18\pi$$

$$= \frac{9}{4} \left[1 - 1 \right] + 18\pi$$

$= 18\pi$

Self Check Exercise - 2

Q.1 Find the work done in moving a particle in a field of force given by $\vec{f} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + 3\,\hat{k}$ along the line joining the points (0, 0, 0) and (3, 1, 4)

Q.2 Find the total work done in moving a particle in a force field given by $\vec{f} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$.

Q.3 Find the work done by $\vec{f} = 3x^2\,\hat{i} + (2xz - y)\,\hat{j} + z\,\hat{k}$ over the $\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$, $0 \le t \le 1$ from (0, 0, 0) to (1, 1, 1)

Q.4 Find the total work done by the force represented by $\vec{f} = 3xy\,\hat{i} - y\,\hat{j} + 2zx\,\hat{k}$ in moving a particle round the circle $x^2 + y^2 = 4$

15.5 Circulation:

Circulation - If c is a closed curve, then the line integral of $\vec{F}$ along c is called the circle lation of $\vec{F}$ along c

$\therefore$ Circulation of $\vec{F}$ along c = $\int_c \vec{F}.d\vec{r}$

$$= \oint_c \overline{\left(F_1dx + F_2dy + F_3dz\right)}$$

Where $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$, $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$,

Q.6 Find the circulation of $\vec{F}$ round the curve c where $\vec{F} = y\,\hat{i} + z\,\hat{j} + x\,\hat{k}$ and c is the circle $x^2 + y^2 = 1$, $z = 0$

Solution: The equation of C in xy plane is $x^2 + y^2 = 1$ Its parametric equations are $x = \cos\theta$, $y = \sin\theta$, θ varies from 0 to 2π

$$\vec{r} = x\,\hat{i} + y\,\hat{j} \Rightarrow d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\Rightarrow d\vec{r} = (-\sin\theta\, d\theta)\,\hat{i} + (\cos\theta\, d\theta)\,\hat{j}$$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_c \left(y\hat{i} + z\hat{j} + x\hat{k}\right).\left(-\sin\theta\, d\theta\,\hat{i} + \cos\theta\, d\theta\,\hat{j}\right)$$

$$= \int_0^{2\pi} -y\sin\theta\, d\theta + z\cos\theta\, d\theta$$

$$= \int_0^{2\pi} \sin\theta\sin\theta\, d\theta \qquad \{y = \sin\theta\}$$

$$= \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

Q.7 Find the circulation of $\vec{f}$ round the curve c, where $\vec{f} = \hat{i}\, e^x \sin y + \hat{j}\, e^x \cos y$ and c is a rectangle whose vertices are (0, 0), (1, 0), $\left(1, \frac{\pi}{2}\right)$

Solution: Here $\vec{f} = \hat{i}\, e^x \sin y + \hat{j}\, e^x \cos y$

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$\therefore \quad d\vec{r} = xx\,\hat{i} + xy\,\hat{j}$

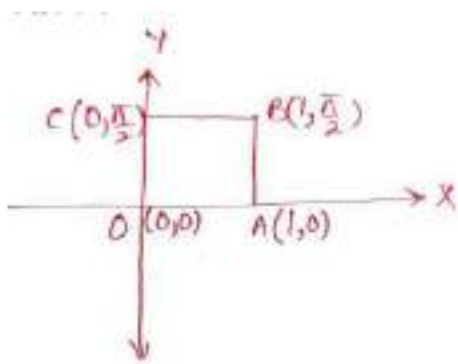
Now, circulation of $\vec{f}$ round c = $\oint_c \vec{f}.d\vec{r}$

$$= \oint_c \left(\hat{i}e^x \sin y + \hat{j}e^x \cos y\right).\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

$$= \oint_c \left(e^x \sin y\, dx + e^x \cos dy\right)$$

$$= \int_{OA} e^x \sin y\, dx + e^x \cos dy$$

$$= \int_{AB} e^x \sin y\, dx + e^x \cos dy$$



$$= \int_{AB} e^x \sin y\, dx + e^x \cos dy + \int_{co} e^x \sin y\, dx + e^x \cos dy \qquad (1)$$

Along 0A, y = 0, y = 0, $\Rightarrow$ dy = 0 and x varies from 0 to 1

Along AB, x = 1 $\Rightarrow$ dx = 0 and y varies from 0 to $\frac{\pi}{2}$

Along BC, $y = \frac{\pi}{2} \Rightarrow dy = 0$ and x varies from 1 to 0

Along co, $x = 0 \Rightarrow dx = 0$ and y varies from $\frac{\pi}{2}$ to 0

$\therefore$ (1) $\Rightarrow$ circulation of $\vec{f}$ round c = $\int_0^1 (0dx + 0) + \int_0^{\frac{\pi}{2}} e\cos y\, dy + \int_1^0 e^x \sin\frac{\pi}{2} dx + \int_{\frac{\pi}{2}}^0 \cos y\, dy$

$= 0 + [e\sin y]_0^{\frac{\pi}{2}} + [e^x]_1^0 + [\sin y]_{\frac{\pi}{2}}^0$

$= 0 + e\left[\sin\frac{\pi}{2} - \sin 0\right] + (e^0 - e^1) + \left(\sin 0 - \sin\frac{\pi}{2}\right)$

$= 0 + e + (1 - e) + (0 - 1)$

$= e + 1 - e - 1$

$= 0$

Self Check Exercise - 3

Q.1 Calculate circulation of a vector field

$\hat{F} = x\,\hat{i} + (3x^2 + y)\,\hat{j}$ around a circle $x^2 + y^2 = 4$

15.6 Summary:

Dear students in this unit, we studies

1. Line integral is a generalization of definite integral where path of integration is along a curve.

2. Mathematically line integral is given by

$$\int_c \hat{F}.d\hat{r} = \int_c \left(F_x dx + F_y dy + F_z dz\right)$$

using $\hat{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$

$d\hat{r} = d_x\,\hat{i} + d_y\,\hat{j} + d_z\,\hat{k}$

3. Line integral using parametric representation is given by

$$\int_c \hat{F}.d\hat{r} = \int_{t_1}^{t_2} \left[\hat{F}.\frac{d\hat{r}}{dt}\right] dt.$$

4. The workdone by a force field $\hat{F}$ in moving in object along a path c between the points P and Q is given by W = $\int_c \hat{F}.d\hat{r}$.

5. When the integration is done over a closed curve then $\oint_c \hat{F}.d\hat{r}$ is known as circulation of the vector field $\hat{F}$ around the closed curve c.

15.7 Glossary

Displacement - net change in location of a moving body

Differentiation - Instanteous rate of change of a function with respect to one of its variable.

Integration - The process of finding a function from its derivative.

Line integral - Integration along a line or curve

15.8 Answer to Self Check Exercises

Self Check Exercise - 1

Ans. 1: 0

Ans. 2: $\frac{10}{7}$

Ans. 3: - $\frac{11763}{28}$

Ans. 4: (i) $\frac{32}{3}$

(ii) $\frac{28}{3}$

Ans. 5: $\frac{-14}{3}$

Self Check Exercise - 2

Ans. 1: $\frac{93}{2}$

Ans. 2: 303

Ans. 3: $\frac{2}{3}$

Ans. 4: 0

Self Check Exercise - 3

Q.1 0

15.9 Suggested Readings

1. R. Murray, S. Lipschulz, D. Spellman, Vector Analysis, Schaunils Outline.
2. S. Narayan and P.K. Mittal, Vector Calculas, Schand and Company Limited.
3. J.N. Sharma and A.R. Vasistha, Vector Calcula, Krishna Prakashan Mandir.

15.10 Terminal Questions

Q.1 Evaluate $\oint \vec{r} \times d\vec{r}$ along the circle C represented by $x^2 + y^2 = a^2$, z = 0

Q.2 If $\vec{F}$ = 2y $\hat{i}$ + j $\hat{j}$ + x $\hat{k}$, evaluate $\int_c \vec{F}.d\vec{r}$ = along the curve x = cos t, y = sin t,

z = 2 cos t from t = 0 to t = $\frac{\overline{\wedge}}{2}$

Q.3 If $\vec{F}$ = $(3x^2 + 6y)$ $\hat{i}$ - (14yz) $\hat{j}$ + $(20xz^2)$ $\hat{k}$, evaluate $\int_c \vec{F}.d\vec{r}$ where c is the straight line joining (0, 0, 0) to (1, 1, 1)

Q.4 Calculate $\int_c \left[\left(x^2 + y^2\right)\hat{i} + \left(x^2 - y^2\right)\hat{j}\right]$. d$\vec{r}$ where c is the curve

(i) $y^2 = x$, joining (0, 0) to (1, 1)

(ii) consisting of two lines joining (0, 0) to (1, 0) and (1, 0) to (1, 1).

Q.5 Find the work done in moving a particle in the field $\vec{F}$ = (3x - 4y + 2z) $\hat{i}$

+ $(4x + 2y - 3z^2)$ $\hat{j}$ + $(2xz-4y^2+z)$ $\hat{k}$

along one round the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, z = 0

Q.6 If $\vec{F}$ = x^2 $\hat{i}$ + xy $\hat{j}$. Evaluate $\int_c \vec{F}.d\vec{r}$ from (0, 0) to (1, 0) along the parabola

y = $\sqrt{x}$

Q.7 Calculate the circulation of the vector field $\hat{F}$ = y^2 $\hat{i}$ + xy $\hat{j}$ around the closed path along the parabola y = $2x^2$ from (0, 0) to (1, 2) and basic from (1, 2) to (0, 0) along the straight line y = 2x.

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Unit - 16

Surface Integral

Structure

16.1 Introduction

Dear student, in this unit we will extend the idea of line integral i.e. single variable integral to double integral i.e. integral to be calculated for two variables. Double integral are integration of function of two variables and the region of integration are on the coordinate planes. Also in this unit we will study about surface integral of a vector field, where the integration is over a two dimensional surface in space. Again surface integral are generalization of double integral.

16.2 Learning Objectives: After studying this unit students will be able to

1. To find the parametric representation of a cylinder, a cone and a sphere.
2. To describe the surface integral of a scalar-valued function over a parametric surface.
3. To use surface integral to calculate the area of given surface.
4. To describe the surface integral of vector field.
5. To solve the questions related to surface integral.

16.3 Surface Integral: Any integral which is to be evaluated over a surface is known as surface integral. Before studying much about surface integral we will discuss the double integral, as surface integral is a generalization of double integral. Double integral can be used to find the area of a region and volume of a solid.

Double Integral:

Let us try following examples to have an idea of double integral.

Example 1: Determine the area of the region R on xy plane bounded by the curves y = x + 2 and y = x^2 by evaluating double integral.

Solution: Since the area of region R can be evaluated by, area = $\iint_R dxdy$

Here R is the region bounded by the curves y = x + 2 and y = x^2.

To evaluate double integral, we have to find the limits of integration for the variables x and y in the region R. For this we have to solve given system of equations.

y = x^2 and y = x + 2

These equations gives us

$x^2 = x + 2$

$\Rightarrow$ $x^2 - x - 2 = 0$

$\Rightarrow$ $x^2 - 2x + x - 2 = 0$

$\Rightarrow$ $x(x - 2) + 1(x - 2) = 0$

$\Rightarrow$ $(x - 2)(x + 1) = 0$

$\Rightarrow$ $x = 2, x = -1$

Therefore, x varies from -1 to 2 and y varies from x^2 to x + 2 because for x $\rightarrow$ (-1 to 2) $x^2 \le x + 2$

So $x^2 \le y \le x + 2$

Therefore Area = $\int_{-1}^{2}\int_{x^2}^{x+2}(dy)dx$

$$= \int_{-1}^{2}[y]_{x^2}^{x+2}\,dx$$

$$= \int_{-1}^{2}\left(x+2-x^2\right)dx$$

$$= \left[\frac{x^2}{2}+2x\frac{-x^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2}+4-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right]$$

$\Rightarrow$ Area $= \frac{20}{6} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$

Example 2: Fine the volume of the solid below the surface f(x, y) = 4 + cox x + cox y, above the region R on the xy plane bounded by the curve x = 0, x = π, y = 0 and y = π using double integral.

Solution: Since volume of the solid is given by

$$\text{Volume} = V = \int_x \int_y f(x, y)\, dy\, dx$$

Since Here f(x, y) = 4 + cox x + cox y and region is bounded by o $\leq$ x $\leq$ π so,

$$V = \int_0^\pi \int_0^\pi \left(4 + cox\,x + cox\,y\right) dy\, dx$$

Integrating over y, we get

$$V = \int_0^\pi \left(4y + y\,cox\,x + \sin y\right)_0^\pi dx$$

$$V = \int_0^\pi \left[4y + \pi\,cox\,x + \sin\pi - 0(0)\right] dx$$

$$= \int_0^\pi 4\pi + \pi cox\,x\, dx$$

Now, integrating over x_1 we get

$$V = \left[4\pi x + \pi \sin x\right]_0^\pi$$

$$= 4\pi^2 + \pi \sin\pi$$

$$\Rightarrow \quad V = 4\pi^2$$

16.4 Surface Integral:

The integral evaluated over a surface is known as surface integral. Consider a surface S having definite area. Let $f(x_1 y_1 z)$ be a single valued function defined on the surface S. Om subdividing the area S into n parts Δs_1, Δs_2 .... Δs_n. Let $P_k(x_k^1 y_k^1 z_k)$ be an arbitrary point on each Δs_k. Then the sum $\sum_{k=1}^{n} f\left(p_k\right)\delta s_k$, on which, on taking limit as k$\to\infty$ such that $\delta s_k \to 0$. If this limit exist, is called the surface integral of $f(x_1 y_1 z)$ over s and is denoted by $\int\int_s f\left(x_1 y_1 z\right) ds = \int_s fds.$

Surface Integral in Term of Flux Across the Surface

In order to find the surface integral, we use the concept of flux across the surface.

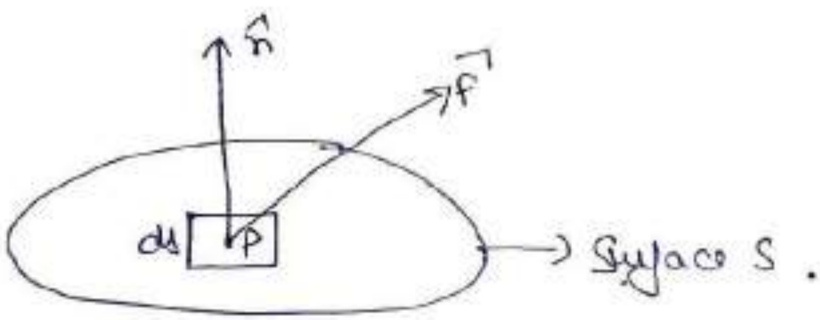
The word flux has meaning flow. This term is easier to understand in the field of fluid flow.

Flux:-

The amount of fluid that fluid that flows through any area in unit time is known as flux.

How, we will learn that how the flux of any vector field can be written as a surface integral.

Let $\vec{F}(x_1 y_1 z)$ be a vector function of position which is defined and continuous over smoth surface S. Let P be any point on that surface and $\hat{n}$ be the unit vector at point P in the direction of outward drawn normal to surface s.



Then the normal component of vector function is given by $\vec{F}.\hat{n}.$ and the integral of normal component of vector function over the surface is known as flux.

So flux = $\iint_s \vec{F}.\hat{n}.ds$ (1)

If $d\hat{s}$ be a area vector which has magnitude ds and is in the direction of $\hat{n}$. Then

$$\hat{n} = \frac{d\hat{s}}{ds} \qquad \hat{n} = \frac{\hat{n}}{|\vec{n}|}$$

$\Rightarrow$ $d\hat{s} = \hat{n}\, d\hat{s}$ (2)

So flux can be written as

$$\iint_s \hat{F}.\hat{n}\, ds = \iint_s \hat{F}.\overrightarrow{ds} \qquad (3)$$

Let $\hat{n}$ makes an angle α ,β, γ with x, y and z axis respectively. If l, m, n are direction-cosine of $\hat{n}$ then l = cox α , m = cox β and n = cox γ.

Then $\hat{n}$ = cox α $\hat{i}$ + cox β $\hat{j}$ + cox γ $\hat{k}$

If $\hat{F}\left(x_1 y_1 z\right)$ = F_1 $\hat{i}$ + F_2 $\hat{j}$ + F_3 $\hat{k}$

Then $\hat{F}.\hat{n}$ = (F_1 $\hat{i}$ + F_2 $\hat{j}$ + F_3 $\hat{k}$). (cox α $\hat{i}$ + cox β $\hat{j}$ + cox γ $\hat{k}$)

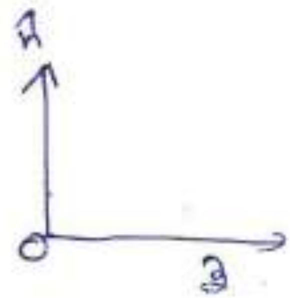
= F_1cox α $\hat{i}$ + F_2cox β $\hat{j}$ + cox γ $\hat{k}$

So $$\int\int_s \hat{F}.\hat{n}\, ds = \iint_s (F_1 \cos\alpha + F_2 \cos\beta + F3 \cos\gamma)\text{ ds.}$$

Let the surface s is such that line perpendicular to the x, y plane meets the surface s in not more than one point. If γ is the angle which $\hat{n}$ makes with z axis at point P surface S. If ds is the small element of area s at point P.

Cox γ ds = dx dy

$$\text{ds} = \frac{dxdy}{cox\gamma}$$



$$= \frac{dxdy}{\left|\hat{n}.\hat{k}\right|}$$ where $\hat{k}$ is the vector along z-axis

$$\therefore \quad \int\int \hat{F}.\hat{n}\, ds = \int\int_R \hat{F}.\hat{n} \frac{dxdy}{\left|\hat{n}.\hat{k}\right|}$$

Where R be the orthogonal projection of s on the xy plane

Evaluation of A surface Integral

Step 1: If $\hat{F}$ = $\hat{F}$ (x, y, z) be the given vector field and s is the surface then find normal to the given surface which is given by $\hat{n}$ = ∇S = grad $\hat{s}$

Step 2: Find normal to the surface which is given by $\hat{n} = \frac{\nabla S}{\left|\nabla S\right|}$

Step 3: Find the value of $\hat{F}$. $\vec{n}$

Step 4: Evaluate the integral $\iint_s \hat{F}.\hat{n}\, ds = \iint_R \hat{F}.\hat{n} \frac{dxdy}{\left|\hat{n}.\hat{k}\right|}$

Let us do some example to have batter understanding of surface integral.

Example 1: Evaluate $\iint_s \phi\hat{n}\, ds$ where $\phi = \frac{3}{8}$xyz and s is the surface of cylinder $x^2 + y^2 = 16$ included in first octant between z = 0 and 3 = 5.

Solution: Since the given surface is $x^2 + y^2 - 16 = 0 = S$

So normal to the surface is given by gradient of S.

So $\hat{n} = \hat{\nabla}s = \left(\frac{\partial\hat{i}}{\partial x} + \frac{\partial\hat{j}}{\partial y} + \frac{\partial\hat{k}}{\partial z}\right)(x^2 + y^2 - 16)$

$\hat{\nabla}s = 2x\,\hat{i} + 2y\,\hat{j}$

So, unit normal to the surface is

$$\hat{n} = \frac{\hat{n}}{|\hat{n}|} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(4x)^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{16}}$$

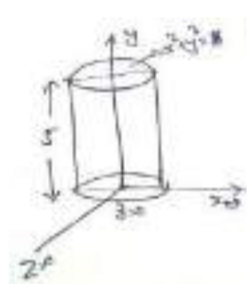
$$= \frac{2x\hat{i} + 2y\hat{j}}{2 \times 4} = \frac{x\hat{i} + y\hat{j}}{4}$$

$$\hat{n} = \frac{1}{4}(x\,\hat{i} + y\,\hat{j})$$

As the surface S is perpendicular to xy plane, so the projection is on xz plane, we have

$$ds = \frac{dxdz}{\left|\hat{n}.\hat{j}\right|}$$

$$\hat{n}.\hat{j} = \left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right).\hat{j} = \frac{1}{4}y$$



If R is the region of projection of surface S on xy plane, then region R is bonded by x = 0 and x = 4 and z = 0 and z = 5

Now, $\iint_s \phi\hat{n}\,ds = \iint_s \frac{3}{8}\left(\frac{1}{4}x\hat{i}+\frac{1}{4}y\hat{j}\right)\frac{dxdz}{|\hat{n}.\hat{j}|}$

$$= \int_{3=0}^{5}\int_{x=0}^{4}\frac{3}{32}\left(x^2yz\hat{i}+xy^2z\hat{j}\right)\frac{dxdz.b}{\frac{1}{4}y}$$

$$= \int_0^5\int_0^4\frac{3\times 4}{32}\left(x^2yz\hat{i}+xy^2z\hat{j}\right)dxdz$$

$$\iint_s \phi\hat{n}ds = \frac{3}{8}\int_0^5\int_0^4\left(x^2z\hat{i}+xyz\hat{j}\right)dxdz$$

Since x2 + y2 = 16 $\Rightarrow$ y2 = 16 - x2 $\Rightarrow$ $y^2\sqrt{16-x^2}$

$$= \frac{3}{8}\int_0^5\int_0^4\left(x^2z\hat{i}+xz\sqrt{16-x^2}\,\hat{j}\right)dxdz \qquad \because x^2 = 16 - y^2$$

Since $\sqrt{16-x^2}$ xdx = $\int\left(16-x^2\right)^{\frac{1}{2}}xdx$

$$= \frac{-1}{2}\int\left(16-x^2\right)^{\frac{1}{2}}(-2x)dx \qquad \int\sqrt{a^2-x^2}\,dx$$

$$\int\sqrt{16-x^2}\,xdx = \frac{1}{2}\frac{\left(16-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \iint_s \phi\hat{n}ds = \frac{3}{8}\int_0^5\left[\frac{x^3z}{3}\hat{i}-\frac{z}{2}\frac{\left(16-x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\hat{j}\right]_0^4 dz$$

$$= \frac{3}{8}\int_0^5\left[\frac{64}{3}z\hat{i}+\frac{64z}{3}\hat{j}\right]dz$$

$$= \frac{1}{8}\int_0^5\left(64z\hat{i}+64z\hat{j}\right)dz$$

$$= \frac{1}{8}\left[\frac{64z^2}{2}\hat{i} + \frac{64z^2}{2}\hat{j}\right]_0^5$$

$$= \frac{64}{8}\left[\frac{z^2}{2}\hat{i} + \frac{z^2}{2}\hat{j}\right]_0^5$$

$$= \frac{8}{2}\,[25\,\hat{i} + 25\,\hat{j}\,]$$

$$\int\int_s \phi\hat{n}ds = 100\,\hat{i} + 100\,\hat{j}$$

Example 2: Consider the hemisphere $x^2 + y^2 + (z - 2)^2 = 9$, $2 \leq z \leq 5$ and the vector field $\vec{F} = x\,\hat{i} + y\,\hat{j} + (z - 2)\,\hat{k}$. Find the value of $\|\vec{F}.\hat{n}$ ds over the hemisphere with $\hat{n}$ denoting the unit outword normal vector.

Solution: The unit vector normal to the surface will be given by

$$\hat{n} = \frac{\hat{\nabla}\phi}{|\nabla\phi|}$$

$$\phi = x^2 + y^2 + (z - 2)^2 = 9$$

$$\nabla\phi = 2x\,\hat{i} + 2y\,\hat{j} + 2(z - 2)\,\hat{k} \qquad \because \hat{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\therefore \quad \hat{n} = \frac{2\left(x\hat{i} + y\hat{j} + (z-2)\hat{k}\right)}{\sqrt{(2x)^2 + (2y)^2 + 2(z-2)^2}}$$

$$\hat{n} = \frac{\left(x\hat{i} + y\hat{j} + (z-2)\hat{k}\right)}{x\sqrt{x^2 + y^2 + (z-2)^2}}$$

$$\Rightarrow \quad \hat{n} = \frac{x\hat{i} + y\hat{j} + (z-2)\hat{k}}{\sqrt{x^2 + y^2 + (z-2)^2}}$$

$$\vec{F}.\hat{n} = \left[x\hat{i} + y\hat{j} + (z-2)\hat{k}\right].\left[\frac{x\hat{i} + y\hat{j} + (z-2)\hat{k}}{\sqrt{x^2 + y^2 + (z-2)^2}}\right]$$

$$= \frac{x^2 + y^2 + (z-2)^2}{\sqrt{x^2 + y^2 + (z-2)^2}}$$

$$= \frac{9}{\sqrt{9}} = \frac{\cancel{9}}{\cancel{3}} = 3$$

Thus. $\int\int\left(\vec{F}.\hat{n}\right)dx\,dy = \int\int 3\,dx\,dy$

$= 3\int\int dx\,dy$

= 3 (Area of Hemisphere) $\qquad \because \left|Surface area of hemisphere = 2\pi r^2 r = 3\right|$

= 3 × 2π $(3)^2$

= 3 × 2π × 9

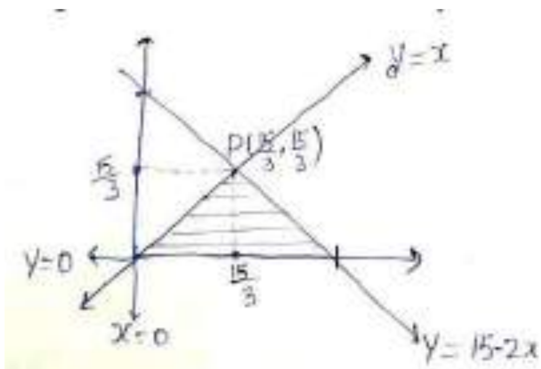
= 54π

$\therefore \quad \iint \vec{F}.\hat{n}$ ds = 54π

Example 3: Find the area of the triangle bounded by y = 0, x = y and y = 15 - 2x.

Solution: We know Area of triangle A = $\iint dx\,dy$

Now, plating the graph for the given data



$\therefore$ Coordinate of point P $\therefore$

As x = y and y = 15 - 2x

Substituting x = y in y = 15 - 2x we have

y = 15 - 2y

3y = 15

$\Rightarrow \quad \boxed{y = \frac{15}{3}}$

$\Rightarrow \quad \boxed{x = \frac{15}{3}}$

$$\because |x = y|$$

$\therefore$ P $\left(\frac{15}{3}, \frac{15}{3}\right)$

Now, A = $\iint dx\,dy$

$$= \int_{y=0}^{y=\frac{15}{3}} \int_{x=y}^{x=\frac{15-y}{2}} dx\,dy$$

$$= \int_{y=0}^{y=\frac{15}{3}} [x]_{x=y}^{x=\frac{15-y}{2}} dy$$

$$= \int_{y=0}^{y=\frac{15}{3}} \left[\frac{15-y}{2} - y\right] dy$$

$$= \left[\frac{15-y}{2} - \frac{y^2}{4} - \frac{y^2}{2}\right]_0^{\frac{15}{3}}$$

$$= \frac{75}{4}$$

$\therefore$ Area of triangle $\frac{75}{4}$ unit square

Example 4: Calculate the surface integral of the scalar field f(x, y, z) = $x^2 + y^2$ over the surface of the cylinder $x^2 + y^2 = 1$ for $0 \leq z \leq 3$.

Solution: Given f(x, y, z) = $x^2 + y^2$

and surface $x^2 + y^2 = 1$ i.e. surface of cylinder on βarameterize the given surface we get

x = cos θ, y = sin θ, z = 1

Thus, surface element ds = dz dθ

and f(x, y, z) = $x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1$

Therefore $\iint_s f(x, y, z) ds = \int_0^{2\pi}\int_0^3 1\,dz\,d\theta$ $\qquad \left| \begin{array}{l} \theta : 0 \to 2\pi \\ z : 0 \to 3 \end{array} \right|$

$$= \int_0^{2\pi} z \int_0^3 d\theta$$

$$= \int_0^{2\pi} 3\,d\theta$$

$$= 3\,[\theta]_0^{2\pi}$$

$= 3(2\pi)$

$= 6\pi$

$$\therefore \quad \iint_s f(x,y,z)\,ds = 6\pi$$

Example 5: Evaluate $\iint_s \vec{f}.\hat{n}$ ds, $\vec{f}$ = z $\hat{i}$ + x $\hat{j}$ + $3y^2z$ $\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the 1st octant between z = 0 and z = 5.

Solution: The equation of surface S is $x^2 + y^2 - 16 = 0$

$\therefore \quad \nabla S = \nabla(x^2 + y^2 - 16) = 2x\,\hat{i} + 2y\,\hat{j}$

$\hat{n}$ = unit vector normal to surface S at any point (x, y, z) = $\dfrac{\nabla s}{|\nabla s|}$

$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}}$$

$$= \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{x^2 + y^2}}$$

$$= \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{16}} \qquad \because |x^2 + y^2 = 16|$$

$$= \frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}$$

Also, $\vec{F} = z\hat{i} + x\hat{j} + 2y^2 = \hat{k}$

$$\therefore \quad \vec{F}.\hat{n} = \left(z\hat{i} + x\hat{j} + 3y^2z\hat{k}\right).\left(\frac{1}{4}x\hat{i} + \frac{1}{4}y\hat{j}\right)$$

$$= \frac{xz}{4} + \frac{xy}{4}$$

As surface S is perpendicular to xy plane, therefore we do not take any projection on xy plane. Taking projection on xz plane, we have.

ds = $\dfrac{dx\,dy}{|\hat{n}.\hat{j}|}$

If R is the region of projection of surface S on xz plane then region R is bounded by x = 0 to x = 4 and z = 0 to z = 5

Now, $\iint_s \vec{F}.\hat{n}\,ds = \iint_R \vec{F}.\hat{n}\,\dfrac{dx\,dy}{|\hat{n}.\hat{j}|}$

$$\Rightarrow \qquad \iint_s \vec{F}.\hat{n}\,ds = \iint_R \left(\frac{xz}{4}+\frac{xy}{4}\right).\frac{dx\,dz}{\frac{1}{4}y} \qquad \because \left|\,|n.j| = \frac{1}{4}y\right|$$

$$= \iint_R \left(\frac{xz+xy}{y}\right)\text{dxdz}$$

$$= \iint_R \left(\frac{xz}{y}+x\right)\text{dxdz}$$

$$= \int_{z=0}^{z=5}\int_{x=0}^{x=4}\left(\frac{xz}{\sqrt{16-x^2}}+x\right)\text{dxdz}$$

$$= \int_0^5\int_0^4\left(\frac{\frac{-1}{2}z(-2x)}{\sqrt{16-x^2}}+x\right)\text{dxdz}$$

$$= \int_0^5\left[\left(\frac{-1}{2}z\right)\frac{\sqrt{16-x^2}}{\frac{1}{2}}+\frac{x^2}{2}\right]_0^4 dz$$

$$= \int_0^5 (4z+8)\,dz$$

$$= \left[\frac{4z^2}{2}+8z\right]_0^5$$

$$= \left[2z^2+8z\right]_0^5$$

= (50 + 40) - 0

= 90

Example 6: Let S be the portion of the plane z = 2x + 2y - 100 which lies inside the cylinder $x_2 + y_2 = 1$. If the surface area of S is $a\pi$, then the value of a is equal to....... .

Solution: S is the portion of the plane z = 2x + 2y - 100 which lies inside the cylinder $x_2 + y_2 = 1$

zx = 2, zy = 2

Then surface area of S = $\iint_s \sqrt{1+zx^2+zy^2}\, dxdy$

$$= \iint_s \sqrt{1+(2)^2+(2)^2}\, dxdy$$

$$= \iint_s 3\, dxdy$$

Now, the cylinder is $x^2 + y^2 = 1$

Let x = cos θ, y = sin θ

Then, surface area of S = $\int_0^{2\pi}\int_0^1 3r\, dr\, d\theta$

$$= 3\int_0^{2\pi} \frac{r^2}{2}\Big|_0^1\, d\theta$$

$$= \frac{3}{2}\int_0^{2\pi} d\theta$$

$$= \frac{3}{2}[\theta]_0^{2\pi}$$

Hence [a = 3] = 3π

Some Other

Methods to evaluate surface integral:- Surface integral mainly depend upon how the surface is given to us. There are essentially two separate method to evaluate surface integral:-

(1) When surface S is given by z = z(x, y). In this case

$$\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}.dA$$

(2) When surface S is given by parameterization i.e.

$\vec{r}$ (u, v) = x (u, v) = x (u, v) $\hat{i}$ + y (u, v) $\hat{j}$ + z (u, v) $\hat{k}$

In these cases surface integral is

$$\iint_S F(x,y,z)\,ds = \iint_D F\left(\dot{r}(u,v)\right)\left\|\dot{r}u + \dot{r}v\right\| dA$$

Example 7: Evaluate the surface integral $\iint_s x^2 ds$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution: Given S is unit sphere ; $x^2 + y^2 + z^2 = 1$

Parametric representation is given as

$r(\phi\ \theta) = \sin\phi \cos\theta\ \hat{i} + \sin\phi \sin\theta\ \hat{j} + \cos\phi\ \hat{k}$

Where $x = \sin\phi \cos\theta$

$y = \sin\phi \sin\theta$

$z = \cos\phi$

and $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$

$\therefore |r\phi \times r\theta| = \sin\phi$

$$\Rightarrow \quad \iint_s x^2 ds = \iint_D (\sin\phi\cos\theta)^2 \left|r\phi + r\theta\right| dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\cos^2\theta d\phi d\theta$$

$$= \int_0^{2\pi} \cos^2\theta \int_0^{\pi}\left(\sin^3\phi\, d\phi\right) d\theta$$

$$= \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta \int_0^{\pi}\left(\sin\phi - \sin\phi\cos^2\phi\right) d\phi$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi}$$

$$= \frac{4\pi}{3}$$

$$\therefore \quad \iint_s x^2 ds = \frac{4\pi}{3}$$

Example 8: $\iint_s xyz\, ds$, in surface S which is part of the plane where z = 1 + 2x + 3y, which lies above the rectangle [0,3 ; 0,3].

Solution: Given $\iint_s xyz\, ds$, z = 1 + 2x + 3y, Rectangle [0,3 ; 0,2]

$$\Rightarrow \qquad \iint_s xyz\, ds = \int_0^3 \int_0^2 \left[xy(1+2x+3y)\sqrt{(2)^2+(3)^2+(1)^2} \right] dydx$$

$$= \int_0^3 \sqrt{14} \left[\frac{xy^2}{2} + \frac{2x^2y^2}{2} + \frac{3xy^3}{3} \right]_0^2 dx$$

$$= \sqrt{14} \int_0^3 \left(2x + 4x^2 + 8x\right) dx$$

$$= \sqrt{14} \left[\frac{2x^2}{2} + \frac{4x^3}{3} + \frac{8x^2}{2} \right]_0^3$$

$= \sqrt{14}$ (9 + 36 + 36)

$= \sqrt{14} \times 81$

$= 9^2 \sqrt{14}$

$\therefore$ Surface integral for given function is $9^2 \sqrt{14}$

Example 9: Evaluate $\iint_s 6xy\, ds$ where S is the portion of the plane x + y + z = 1 that lies in the 1st octant and is in front of yz plane.

Solution: Given x + y + z = 1

Since, we are looking for the portion of the plane that lies in yz plane. thus, x = g (y, z)

i.e. x = 1 - y - z = g (y, z)

Here ranges of y and z are

$0 \leq y \leq 1,\ 0 \leq z \leq 1 - y$

Now, $$\iint_s F(x,y,z)\, ds = \iint_D f\left(g(y,z), y, z\right)\sqrt{1+\left(\frac{\partial g}{\partial y}\right)^2+\left(\frac{\partial g}{\partial z}\right)^2}\, .dA$$

$$\Rightarrow \qquad \iint_s 6xy\, ds = \iint_D 6(1-y-z)\, y \sqrt{1+(-1)^2+(-1)^2}\, dA$$

$$\Rightarrow \qquad \iint_s 6xy\, ds = \sqrt{3} \iint_D 6\left(y - y^2 - zy\right) dA$$

$$= 6\sqrt{3}\int_0^1\int_0^{1-y}(y - y^2 - zy)dz\,dy$$

$$= 6\sqrt{3}\int_0^1\left[yz - zy^2 - \frac{z^2}{2}y\right]_0^{1-y} dy$$

$$= 6\sqrt{3}\int_0^1\left[\frac{1}{2}y - y^2 + \frac{1}{2}y^3\right]dy$$

$$= 6\sqrt{3}\int_0^1\left[\frac{1}{4}y^2 - \frac{1}{3}y^3\frac{1}{8}y^4\right]_0^1$$

$$= \frac{\sqrt{3}}{4}$$

$$\Rightarrow \qquad \iint_s 6xy\,ds = \frac{\sqrt{3}}{4}$$

Example 10: Evaluate $\iint_s y\,ds$ where S is the portion of the cylinder $x^2 + y^2 = 3$ that lies between z = 0 and z = 6.

Solution: Given $x^2 + y^2 = 3$ equation of cylinder.

After parametrization above eq^n become

$\vec{r}$ (z, θ) = $\sqrt{3}$ cos θ $\hat{i}$ + $\sqrt{3}$ sin θ $\hat{j}$ + z $\hat{k}$

s.t. $0 \leq z \leq G$, $0 \leq \theta \leq 2\pi$

Now, $\vec{r}$z (z, θ) = $\hat{k}$

$\vec{r}\theta$ (z, θ) = - $\sqrt{3}$ sin θ $\hat{i}$ + $\sqrt{3}$ cos θ $\hat{j}$

$$\therefore \qquad \left(\vec{r}\times\vec{r}\,\theta\right) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -\sqrt{3}\sin\theta & \sqrt{3}\cos\theta & 0 \end{vmatrix}$$

= $\sqrt{3}$ cos θ $\hat{i}$ + $\sqrt{3}$ sin θ $\hat{j}$

$$\left(\vec{r}\times\vec{r}\,\theta\right) = \sqrt{3}$$

Therefore $\Rightarrow \qquad \iint_s y\,ds = \iint_D \sqrt{3}\sin\theta\left(\sqrt{3}\right)dA$

$$= 3\int_0^{2\pi}\int_0^6 \sin\theta dz\, d\theta$$

$$= 3\int_0^{2\pi} 6\sin\theta\, d\theta$$

$$= -18 \cos \theta\Big|_0^{2\pi}$$

$$= 0$$

$$\Rightarrow \qquad \iint_s y\, ds = 0$$

Example 11: Find $\iint_s 40y\, ds$ where S is the portion of y = $3x^2 + 3z^2$ that lies behind y = 6.

Solution: Given $3x^2 + 3z^2 + y$ lies behind y = 6

To find D put y = 6 in above equation

$\therefore \qquad \Rightarrow \qquad 3x^2 + 3z^2 = 6$

$x^2 + 3z^2 = 2$

Thus, D is a disk $x^2 + 3z^2 \leq 2$

and $\qquad y = g(x, z) = 3x^2 + 3z^2$

Therefore, we have

$$\iint_s F(x,y,z)\, ds = \iint_D F\left(x, g(x,z), z\right)\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + 1 + \left(\frac{\partial g}{\partial z}\right)^2}\, dA$$

$$\iint_s 40y\, ds = \iint_D 40\left(3x^2 + 3z^2\right)\sqrt{\left(6x\right)^2 + 1 + \left(6z\right)^2}\, dA$$

$$= \iint_D 120\left(x^2 + z^2\right)\sqrt{36\left(x^2 + z^2\right) + 1}\, dA$$

Put x = r cos θ, y = r sin θ, $x^2 + z^2 = r^2$

Here D is a disk i.e. $x^2 + z^2 \leq 2$ Thus, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{2}$

$$\Rightarrow \qquad \iint_s 40y\, ds = \int_0^{2\pi}\int_0^{\sqrt{2}} 120r^2\sqrt{36r^2 + 1}(r)\, dr\, d\theta$$

Put $u = 36r^2 + 1$, $du = 72r\, dr \Rightarrow \frac{1}{72}$ du = r dr and $r^2 = \frac{1}{36}$ (u - 1)

$$\Rightarrow \quad \iint_s 40y\,ds = \int_0^{2\pi}\int_1^{73} 120\left(\frac{1}{72}\right)\left(\frac{1}{36}\right)(u-1)u^{\frac{1}{2}}du\,d\theta$$

$$= \int_0^{2\pi}\int_1^{73} \frac{5}{108}\left(u^{3/2} - u^{1/2}\right)du\,d\theta$$

$$= \int_0^{2\pi} \frac{5}{108}\left[\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right)\right]_1^{73} d\theta$$

$$= \frac{5\pi}{54}\left[\frac{2}{5}(73)^{5/2} - \frac{2}{3}(73)^{3/2} + \frac{4}{15}\right] = 5176.8958$$

16.5 Summary

- A surface integral is like a line integral in one higher dimension.
- The domain of the integration of a surface integral is a surface in a plane or space, rather than a alive in a plane or space.
- The integrand of a surface integral can be scalar function or a vector field.
- If S is a surface then area of S is given by $\iint_s ds$.
- Surfaces are parameterized just as curve can be parameterized in general, surface must be parameterized with two parameters.
- The parametric domain of parameterization is the set of points in the uv plane that can be substituted into $\vec{r}$. Where parameterization of surface is

$\vec{r}$ (u, v) = $\langle x(u,v), y(u,v), z(u,v)\rangle$

16.6 Glossary

- **Scalar Field:** A scalar field associates a scalar value to every point in a space possibly physical space. The scalar may either be a mathematical number or physical quantity.
- **Scalar Function:** Scalar functions are functions that yields scalar quantities when they map points and numbers in catteries space e.g. (f(x) = 2x, f(x, y) = 5x + 2y - 1
- **Vector Field:** A vector field is a set of vectors assigned to each point in a space region. Vector fields are frequently used to describe the speed and direction of moving fluid in a space.

- **Vector Function:** A vector function is a function where doman is subset of the real numbers and range is a vector e.g. r(t) = (t, - t^2 + 5) = t $\hat{i}$ + (-t^2 + 5) $\hat{j}$
- **Parameterization:** To parametrize means to express in terms of parameters. Parameterization is a mathematical process consisting of expressing the state of a system, a process or model as a function of some independent quantities called parameters e.g. y = x^2 sin (x) ; x $\in$ [-1, 2]
- Let S(t) = $\left\langle \begin{matrix} x = t \\ y = t^2 \sin(t); t \in [-1, 2] \end{matrix} \right\rangle$
- **Parameter domain:** The parameter domain of the parameterization is the set of points in the uv plane that can be substituted into $\vec{r}$. Where parameterization of surface is
- $\vec{r}$ (u,v) = $\langle x(u,v), y(u,v), z(u,v) \rangle$
- **Projection:** Projection is a linear transformation from vector space to itself.
- **Octant:** It is one of the eight divisions of a Euclidean three dimensional coordinate system.

Self Check Exercise

1. Evaluate $\iint_s zds$ where S is the upper half of the sphere of radius 2.
2. Evaluate $\iint_s (y+z)ds$ where S is the surface whose side is the cylinder x^2 + y^2 = 3, whose bottom is the disk x^2 + y^2 $\leq$ 3 in the xy - plane and whose top is the plane z = 4 - y.
3. Find $\iint_s 2y\,ds$ where S is the portion of y^2 + z^2 = 4 between x = 0 and x = 3 - z.
4. Find the integral $\iint_s xz\,ds$ where S is the portion of the sphere of radius 3 with x $\leq$ 0, y $\geq$ 0 and z $\geq$.
5. Evaluate $\iint_s \hat{F}\,ds$ where $\hat{F}$ = 4x $\hat{i}$ - 2y^2 $\hat{j}$ + z^2 $\hat{k}$ and S is the surface bounded by the region x^2 + y^2 = 4, z = 0 and z = 3.
6. Calculate the surface integral of $\hat{F}$ = (yz_1 x z x y) over the surface of plane x + y + z = 1 in first octant.
7. Calculate the surface integral of f(x,y,z) = x+y+z over the planr z = 1 in the region bounded by x = , y = 0 and x + y = 1.

16.7 Answers to Self Check Exercise

1. 8π
2. $\frac{\pi}{2}\left(29\sqrt{3}+24\sqrt{2}\right)$
3. 0
4. -27
5. 84π
6. $\frac{\sqrt{3}}{6}$
7. $\frac{3}{4}$

16.8 References/Suggested Readings

1. Vector Calculus by P.C. Mathews.
2. Differential and Integral Calculus by N. Piskunov.
3. Calculus of Several Variables by Springer.

16.9 Terminal Questions

1. Evaluate $\iint_s \left(z+3y-x^2\right)ds$ where S is the portion of Z = 2 - 3y + x^2 that lies over the triangle in the xy plane with vertices (0, 0), (2, 0) and (2, -4)
2. Find the value of surface integral for F = 2y where S is the portion of $y^2 + z^2 = 4$ between x = 0 and x = 3 - z.
3. Evaluate $\iint_s \left(x-z\right)ds$ where S is the surface of the solid bounded by $x^2 + y^2 = 4$, z = x - 3 and z = x + 2. Note that all three surfaces of solid are included in S.

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Unit - 17

Volume Integral

Structure

17.1 Introduction

Dear student, in the unit, we will study about the volume integral. It is just like the surface integral which is integral in two dimensions, here we will extend that idea to three dimensions. It is a special case of multiple integral. Volume integral plays an important role in various fields of sciences. In this unit we will study about volume integral and some of its applications.

17.2 Learning Objectives

After studying this unit students will be able to

1. define volume integral.
2. apply the technique to evaluate volume integral of given problem.

17.3 Volume Integral

Volume Integral of a Scalar Function

Let f(x, y, z) be a single-valued function defined and continuous over a there dimensional region V enclosed by a surface S. We divide V into n parts (we assume them as cubes) with arbitrary small volume ΔV_k (k = 1, 2,.........n). Let Pk (x_k, y_k, z_k) be an arbitrary in volume V_k. Then the volume integral of $\hat{f}$ overs volume V is denoted by $\iiint_v f\,dv$ and is given as

$$\iiint_v f(x,y,z)\,dv = \lim_{\Delta v_k \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k \text{ as } n\to\infty.$$

Volume integral $\iiint_v f\,dv$ can be written as $\int_v f\,dv$.

Volume Integral of A vector Function

Let $\hat{F}$ (x, y, z) = F_x (x, y, z) $\hat{i}$ + F_y (x, y, z) $\hat{j}$ + F_z (x, y, z) $\hat{k}$ be a single valued vector field, defined and continuous on a region V enclosed by a surface S in a rectangular carlegian three dimensional space. Then the volume integral of $\hat{F}$ over volume V is defined as $\iiint_v f\,dv =$

$\hat{i}\int_v f_x(x,y,z)dv + \hat{j}\int_v f_y(x,y,z)dv + \hat{k}\int_v f_z(x,y,z)dv$.

Steps to Evaluate the Volume Integral

1. The volume integral $\int_v f\,dv = \iiint_v f\,dv$, where dv is the elementary volume can be expressed as dx dy dz.

   So $\int_v f\,dv = \iiint_v f\,dv = \iiint_v f\,d_x d_y d_z$.

2. Now, to find the limiting values of the variables x, y, and z. Let us understand it by an example.

   If V is the volume bounded by the co-ordinate plane (0, 0, 0) and the plane

   2x + 2y + z = 4 then to find volume integral.

- For finding the limiting or maximum and minimum value of x, we put y and z = 0 in the given plane, so that we get 2x = 4 $\Rightarrow$ x = 2

  So $0 \le x \le 2$

- To find the limiting value of y, we put z = 0, and express y as a function of x i.e. y = y(x).

  So we get 2x + 2y = 4

  2y = 4 - 2x

  $\Rightarrow$ y = 2 - x.

  So $0 \le y \le 2 - x$

- To find the limiting values of z, we express z as a function of x and y i.e. z = z (x, y), so here we get z = 4 - 2x - 2y.

$$\iiint_v f\,dv = \int_x\int_y\int_z f\,dx\,dy\,dz$$

$$\iiint_v f\,dv = \int_{x=0}^{2}\left(\int_{y=0}^{y=x-2}\left(\int_{z=0}^{z=4-2x-2y} f\,dz\right)dy\right)dx.$$

3. White solving $\int\limits_{z=0}^{z=4-2x-2y} f\,dz$, we fueat both x and y as constant.

4. White solving $\int\limits_{y=0}^{y=x-2} f\,dy$, x is taken as constant.

To have more understanding of volume integral Let us try following examples.

Example 1: Evaluate $\int\limits_v f\,dv$ where f = 15 xy and V is the unit cube given by

$0 \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$.

Solution: $\int\limits_v f\,dv = \int\limits_v 15xy\,dv$

$$= \iiint\limits_v 15xy\,dx\,dy\,dz.$$

$$\int\limits_v 15xy\,dv = 15\int\limits_{x=0}^{1}\left(\int\limits_{y=0}^{1}\left(\int\limits_{z=0}^{1} xy\,dz\right)dy\right)dx$$

$$= 15\int\limits_{x=0}^{1}\int\limits_{y=0}^{1}\left[xyz\right]_0^1 dy\,dx$$

$$= 15\int\limits_{x=0}^{1}\left(\int\limits_{y=0}^{1} xy\,dy\right)dx$$

$$= 15\int\limits_{x=0}^{1}\left[\frac{xy^2}{2}\right]_0^1 dx$$

$$= 15\int\limits_{x=0}^{1}\frac{x}{2}\,dx$$

$$= \frac{15}{2}\left[\frac{x^2}{2}\right]_0^1$$

$$\int\limits_v 15xy\,dv = \frac{15}{4}$$

Example 2: Evaluate $\iiint\limits_v (2x+y)\,dv$, where V is the closed region bounded by the cylinder

$z = 4 - x^2$ and the planes x = 0, y = 0, x = 2, y = 2 and z = 0.

Solution: $\iiint\limits_{v}(2x+y)\,dv = \iiint\limits_{v}(2x+y)\,dx\,dy\,dz$

$$\Rightarrow \quad = \int_{x=0}^{2}\left(\int_{y=0}^{2}\left(\int_{z=0}^{3=4-x^2}(2x+y)\,dz\right)dy\right)dx$$

$$= \int_{x=0}^{2}\left(\int_{y=0}^{2}\left[(2x+y)z\right]_0^{4-x^2}dy\right)dx \text{ [integrating for z]}$$

$$= \int_{x=0}^{2}\left(\int_{y=0}^{2}(2x+y)(4-x^2)\,dy\right)dx$$

$$= \int_{x=0}^{2}\left(\int_{y=0}^{2}8x+4y-2x^3-yx^2\right)dy\,dx$$

$$= \int_{x=0}^{2}\left[8xy+2y^2-2x^3y-\frac{y^2x^2}{2}\right]_0^2 dx$$

$$= \int_{x=0}^{2}\left[16x+8-4x^3-2x^2\right]dx$$

$$= \left[\frac{16x^2}{2}+8x-\frac{4x^4}{4}-\frac{2x^3}{3}\right]_0^2$$

$$= 32 + 16 - 16 - \frac{16}{3}$$

$$= 32 - \frac{16}{3}$$

$$\iiint\limits_{v}(2x+y)\,dv = \frac{80}{3}$$

Example 3: Evaluate the following integral $\iiint\limits_{B}8xyz\,dv$;

$2 \le x \le 3 \qquad 1 \le y \le 2,\ 0 \le z \le 1$

Solution: Given F = 8 xyz

Therefore $\iiint\limits_B 8xyz\,dv = \int\limits_1^2\int\limits_2^3\int\limits_0^1 8xyz\,dz\,dx\,dy$

$$= \int\limits_1^2\int\limits_2^3 4xyz^2\,|_0^1\;dx\,dy \qquad \left[\because \int zdz = \frac{z^2}{2}\right]$$

$$= \int\limits_1^2\int\limits_2^3 4xy\,dx\,dy \qquad \left[\because \int xdxs = \frac{x^2}{2}\right]$$

$$= \int\limits_1^2 2x^2y\,|_2^3\;dy$$

$$= \int\limits_1^2 10y\,dy$$

$= 5y^2] = \left[5(2)^2 - 5(1)^2\right]$

= 15

$\Rightarrow \quad \iiint\limits_B 8xyz\,dv = 15$

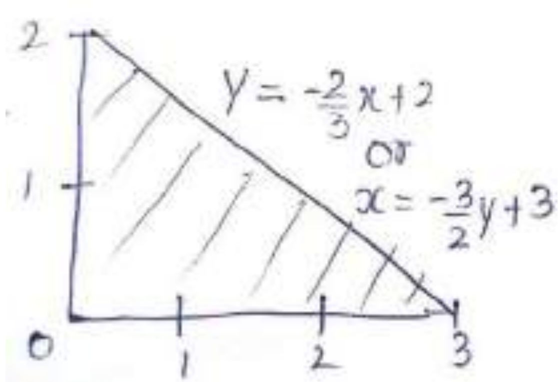
Example 4: $\iiint\limits_E 2x\,dv$ evaluate integral where E is the region under the plane 2x + 3y + z = 6 that lies in the 1st octant.

Solution: Given E is the region under the plane 2x + 3y + z = 6 that lies in the 1st octant

$\Rightarrow$ We are above the plane z = 0

Thus $0 \leq z \leq 6 - 2x - 3y$

So, the region D is the xy plane is the triangle with verities at (0, 0), (3, 0), and (0, 2) shown above



Thus, we have

$0 \le x \le 3$ or $0 \le x \le -\frac{3}{2}y + 3$

$0 \le y \le \frac{-2}{3}x + 2$ $\quad 0 \le y \le 2$

Therefore $\iiint\limits_E 2x\,dv = \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dA$

$= \iint\limits_D \left[\int_0^{6-2x-3y} 2x\,dz\right] dx\,dy \qquad \because \left[\int_0^{6-2x-3y} dz = [z]_0^{6-2x-3y} = 6-2x-3y\right]$

$= \int_0^3 \int_0^{\frac{-2}{3}x+2} 2x(6-2x-3y)\,dy\,dx$

$= \int_0^3 12xu - 4x^2y - 3xy^2 \int_0^{\frac{-2}{3}x+2} dx$

$= \int_0^3 \left(\frac{4}{3}x^3 - 8x^2 + 12x\right) dx$

$= \frac{1}{3}x^4 - \frac{8}{3}x^3 + 6x^2 \Big|_0^3$

= 9

$\iiint\limits_E 2x\,dv$ = 9

Example 5: Evaluate $\iiint\limits_E (3-4x)\,dv$ where E is the region below z = 4 - xy and above the region in the xy plane defined by $0 \le x \le 2$, $0 \le y \le 1$.

Solution : Given limits for x and y

$0 \le x \le 2$, $0 \le y \le 1$

Here E is the region below z = 4 - xy

$\therefore \quad 0 \le z \le 4 - xy$.

Therefore $\iiint\limits_E (3-4x)\,dv = \int_0^2 \int_0^1 \int_0^{4-xy} (3-4x)\,dz\,dy\,dx$

$$= \int_0^2 \int_0^1 (3-4x)(4-xy)\,dydx$$

$$= \int_0^2 \int_0^1 \left(4x^2y - 3xy - 16x + 12\right) dydx$$

$$= \int_0^2 \left(2x^2y^2 - \frac{3}{2}xy^2 - 16xy + 12y\right)_0^1 dx$$

$$= \int_0^2 \left(12 - \frac{35}{2}x + 2x^2\right) dx$$

$$= 12x = \frac{35}{4}x^2 + \frac{2}{3}x^3\Big|_0^2$$

$$= \frac{-17}{3}$$

$$\iiint_E (3-4x)\,dv = \frac{-17}{3}$$

Example 6: Evaluate $\iiint_E (12y-8x)\,dv$ where E is the region behind y = 10 - 2z and in front of the region in the xz - plane bounded by z = 2x, z - 5 and x = 0.

Solution: Given Limits for y is $0 \leq y \leq 10 - 2z$

Limits for x and z are

$0 \leq x \leq \frac{5}{2}$ or $0 \leq z \leq 5$

$2x \leq z \leq 5$ $\quad$ $0 \leq x \leq \frac{1}{2}z$

$$\therefore \quad \iiint_E (12y-8x)\,dv = \int_0^5 \int_0^{\frac{1}{2}z} \int_0^{10-2z} (12y-8x)\,dy\,dx\,dz$$

$$= \int_0^5 \int_0^{\frac{1}{2}z} \left(6y^2 - 8xy\right)\Big|_0^{10-2z} dxdz$$

$$= \int_0^5 \int_0^{\frac{1}{2}z} 6(10-2z)^2 - 8x(10-2z)\,dxdz$$

$$= \int_0^5 \left[6(10-2z)^2 x - 4x^2(10-2z)\right]\Big|_0^{\frac{1}{2}z} dz$$

$$= \int_0^5 \left(14z^3 - 13z^2 + 300z\right) dz$$

$$= \frac{7}{2}z^4 - \frac{130}{3}z^3 + 150z^2 \Big|_0^5$$

$$= \frac{3125}{6}$$

$$\Rightarrow \qquad \iiint_E (12y - 8x)\, dv = \frac{3125}{6}$$

Example 7: Evaluate $\iiint_E yz\, dv$ Where E is the region bounded by x = $2y^2 + 2z^2 - 5$ and the plane x = 1.

Solution: Given Limits for x; 2y2 + 2z2 - 5 $\leq$ x $\leq$ 1

Therefore $\iiint_E yz\, dv = \iint_D \left[\int_{2y^2+2z^2-5}^{1} yz\, dx\right] dA$

$$= \iint_D xyz \Big|_{2y^2+2z^2-5}^{1} dA$$

$$= \iint_D \left[1 - \left(2y^2 + 2z^2 - 5\right)\right] yz\, dA$$

$$= \iint_D \left[6 - 2\left(y^2 + z^2\right)\right] yz\, dA$$

Since, $2y^2 + 2z^2 - 5 = x$

For x = 1

$\Rightarrow$ $2y^2 + 2z^2 - 5 = 1$

$\Rightarrow$ $y^2 + z^2 = 3$

Thus D is a disk $y^2 + z^2 \leq 3$

s + y = r sin θ, z = r cos θ, $y^2 + z^2 = r^2$

Here Limits on θ and r ϕ are

$0 \leq \theta \leq 2\pi$, $0 \leq r \leq \sqrt{3}$

Therefore, $\iiint_E yz\,dv = \iint_D \left[6-2\left(y^2+z^2\right)\right] yz\,dA$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left[6-2r^2\right](r\sin\theta)(r\cos\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \left[6r^3-2r^5\right]\sin\theta\cos\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{3}{2}r^4-\frac{1}{3}r^6\right]\sin\theta\cos\theta\Big|_0^{\sqrt{3}}\,d\theta$$

$$= \int_0^{2\pi}\frac{9}{2}\sin\theta\cos\theta d\theta$$

$$= \int_0^{2\pi}\frac{9}{4}\sin\theta 2\theta d\theta$$

$$= \left[\frac{-9}{8}\cos 2\theta\right]_0^{2\pi}$$

$= 0$

$\therefore \qquad \iiint_E yz\,dv = 0$

17.4 Summary

Students in this unit we studied

1. Volume integral is a special care of multiple integral in three dimensions.
2. Volume integral of scalar function is given by $\iiint_v fdv$.
3. Volume integral of vector function is given by

$$\int_v \hat{F}\,dv = \hat{i}\int_v F_x\,dv + \hat{j}\int_v F_y\,dv + \hat{k}\int_v F_z\,dv.$$

Self Check Exercise

1. Evaluate $\int_2^3\int_{-1}^4\int_1^0 4x^2y - z^3dz\,dy\,dx$.
2. Use a triple integral to determine the volume of the region that is below

$z = 8-x^2 - y^2$ above $z = \sqrt{4x^2 + 4y^2}$ and inside $x^2 + y^2 = 4$
3. Evaluate $\iiint\limits_E 6z^2 dv$ where E is the region below 4x + y + 2z = 10m the 1st octant.
4. Find the volume of the solid of revolution generated when the finite region R that lies between $y = 4 - x^2$ and $y = x + 2$ is revolved about x axis.

17.5 Glossary

- **Multiple integration:** Volume integrals frequently involves integrating a function over two or three dimensions, requiring an understanding of double and triple integrals.
- **Divergence Theorem:** This theorem connects the flow (or divergence) of a vector field across a surface to the behaviour of the vector field inside the volume bounded by the surface. It is pivotal in simplifying complex 3D volume integrals.
- **Coordinate Systems:** Different coordinate systems (such as Cartesian, cylindrical and spherical) are used depends on the symmetry of the problem, impacting how the volume integral is set up and solved.

  (i) Cartesian coordinates (x, y, z)

  (ii) Cylindrical coordinates (r, θ, z)

  (iii) Spherical coordinate (r, θ, ϕ)

Integral for integral.	(i) **Definite:** An integral that gives fixed value for a curve within the two given limits.
	(ii) **Indefinite:** An integral not having upper and lower limit i.e. no fixed value

17.8 Terminal Questions:

1. Determine the volume of the region that lies behind the plane x + y + z = 8 and in front of the region in the yz plane that is bounded by $z = \frac{3}{2}\sqrt{y}$ and $z = \frac{3}{4}y$.

2. Evaluate $\int_0^1 \int_0^{z^2} \int_0^3 y\cos\left(z^5\right) dx\, dy\, dz$

3. Use a triple integral to determine the volume of the region below z = 4 - xy and above the region in the xy - plane defined by $0 \leq x \leq 2$, $\leq y \leq 1$.

4. Find the volume of the solid of revolution generated when the finite region R that lies between y = $\sqrt{x}$ and y = x^4 is revolved about the y axis.

17.6 Answers to Self Check Exercise

1. $\frac{-755}{4}$

2. $\frac{104\pi}{3}$

3. $\frac{625}{2}$

4. $\frac{108\pi}{5}$

17.7 References/Suggested Readings

1. Calculus of several Variable by Springer
2. A Textbook of Vector Calcula, by Shanti Narayan
3. Mathematical Analys by Apostol.

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Unit - 18

Gauss Divergence Theorem

Structure

18.1 Introduction

Dear student, in this unit we will study about the important theorem of vector calques. In previous units we studied about surface and volume integrals. In this unit we will study about the relationship between surface and volume integral which is given in term of a theorem known as Gauss Divergence Theorem. Gauss divergence theorem used to solve difficult surface integral by transforming it into an easier triple integral and vice versa.

18.2 Learning Objectives

After studying this unit students will be able to:

1. State Gauss divergence Theorem.
2. Apply Gauss divergence Theorem to find the surface integral inform of volume integral.
3. Establish relations between surface and volume integral.

18.3 Divergence Theorem

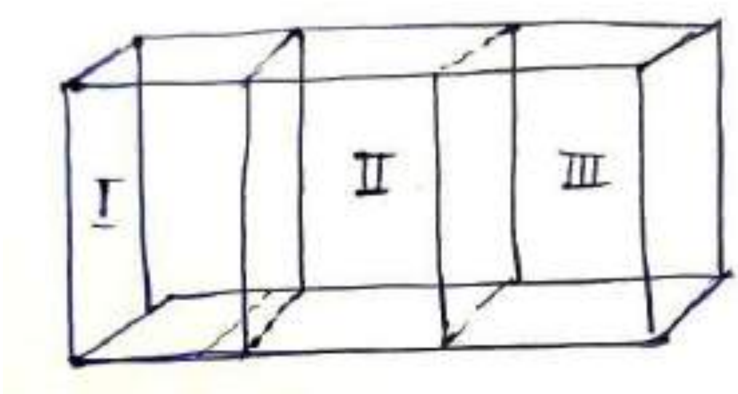
Statement: The Gauss Divergence theorem states that the vector's outward flux through a closed surface is equal to the volume integral of the divergence over the area within the surface.

OR

The Divergence theorem states that the surface integral of the normal component of a vector point function "F" over a closed surface "S" is equal to the volume integral of the divergence of $\vec{F}$ taken over the volume "V" enclosed by the surface S. Thus, the divergence theorem is symbolically denoted as :

$$\iiint_V \nabla \vec{F} dv = \iint_F \vec{F}.\hat{n}\, ds$$

Proof: Consider a surface S that surrounds a volume V. Let Vector A represents the vector field in the specified region. Let this volume be compressed of many elementary volumes in the form of parallelepiped.



Consider the dth parallelepiped, which has a volume ΔVj and is bounded by a surface Sj with an area vector Sj. The surface integral of vector A over surface Sj is denoted by $\phi_s \phi \vec{A}.\overrightarrow{ds_j}$

Here $\phi_s \phi \vec{A}.\overrightarrow{ds_j} = \sum \phi_{s_j} \phi \ddot{A}.d\bar{s}_j$ (1)

Multiply and divide R.H.S. F (1) by ΔVi, we obtain

$$\phi_s \phi \dot{A}.d\dot{s} = \sum \frac{1}{\Delta Vi} \left(\phi_{si} \phi \vec{A}.\overrightarrow{ds} \right) \Delta \text{Vi}$$

Now, suppose the volume of surface S is divided into infinite elementary volumes such that ΔVi $\rightarrow$ 0

$$\phi_s \phi \ \vec{A} \ . \ \overrightarrow{ds} = \lim_{\Delta Vi \to 0} \sum \frac{1}{\Delta Vi} \left(\phi_{si} \phi \vec{A}.\overrightarrow{ds} \right) \Delta Vi \quad (2)$$

Now,

$$\lim_{\Delta Vi \to 0} \left(\frac{1}{\Delta Vi} \left(\phi_{si} \phi \overline{A}.\overline{ds} \right) \right) = \left(\vec{V}.\vec{A} \right)$$

Therefore equ^n (2) becomes

$$\phi \phi \vec{A}.\overrightarrow{ds} = \sum \left(\vec{\nabla}.\vec{A} \right) \ \Delta \text{Vi} \quad (3)$$

We know that Δ Vi $\to$0. Thus $\sum \Delta Vi$ will become the integral over volume V.

$$\phi\phi \vec{A}.\vec{ds} = \iiint_V \left(\vec{\nabla}.\vec{A}\right) dV$$

Hence proved

Notes- s(1) Since $\iint_S \hat{A}.ds = \iint_S \hat{A}.\hat{n}ds$

So, above theorem can be written as

$$\iint_S \hat{A}.\hat{ds} = \iint_S \hat{A}.\hat{n}ds = \iiint_V \left(\vec{\nabla}.\vec{A}\right) dV$$

(2) If $\hat{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

then $\vec{V}.\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$

Let $\propto$, β, γ be the angle which outward drawn unit normal $\hat{n}$ makes with positive directions of axes and if cos $\propto$, cosβ, and cos γ are direction cosines of $\hat{n}$. then

$\hat{n}$ = (cos $\propto$) $\hat{i}$ + (cosβ) $\hat{j}$ + (cos γ) $\hat{k}$

$\hat{A}$. $\hat{n}$ = $\left(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}\right)$. (cos $\propto \hat{i}$ + cosβ $\hat{j}$ + cos $\hat{j}$ γ)

$\hat{A}$. $\hat{n}$ = A_1cos $\propto$ + A_2cosβ + A_3cos γ

Then by divergence theorem

$$\iiint_V \vec{\nabla}.\vec{A} \text{ dV} = \iint_S \hat{A}.\hat{ds} = \iint_S \hat{A}.\hat{n}ds$$

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \text{dxdydz} = \iint_S (A_1\cos\propto + A_2\cos\beta + A_3\cos\gamma)ds$$

$$\iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) \text{dxdydz} = \iint_S A_1dydz + A_2dzdx + A_3dxdy$$

Let us by following questions to have better understanding of the concept.

Example 1 : If $\hat{A}$ = ∇vand ∇^2vs = -4πP show that

$\int \hat{A}$. $\hat{n}$ ds = -4$\pi\int$ P dv

Solution : Since we know that Gauss's theorem gives

$$\int \hat{A} \cdot \hat{n} \; ds = \int \left(\vec{\nabla}.\vec{A}\right) dv$$

Given $\hat{A} = \nabla V$, so

$$\int \nabla V. \hat{n} \; ds \int \vec{\nabla}.(\nabla V) \; dv$$

$$\int \hat{A}.\hat{n} \; ds = \int \nabla^2 v$$

$$= \int -4\pi P \; dv$$

$$= -4\pi \int P \; dv$$

$\Rightarrow \quad \int \hat{A}.\hat{n} \; ds = -4\pi \int P \; dv$

Example 2 : Prove that $\int \frac{1}{r^2} \hat{r} \cdot \hat{n} \; ds = \int \frac{1}{r^2} \; dv$

Solution : L.H.S. $\Rightarrow \int \frac{1}{r^2} \hat{r} \cdot \hat{n} \; ds$

Using Gauss' s Theorem $\int_S \hat{A}.\hat{n} \; ds = \int_V \vec{\nabla} \cdot A dv$

Here $A = \frac{1}{r^2} \hat{r}$

$$\text{So } \int \frac{1}{r^2} \hat{r} \cdot \hat{n} \; ds = \int_V \left(\vec{\nabla}.\frac{1}{r^2}\hat{r}\right)$$

$$= \int \left[\nabla.\left(\frac{1}{r^2}\right)\hat{r} + \frac{1}{r^2}\nabla.\hat{r}\right] dv$$

Since $\nabla r^n = nr^{n-2} \Rightarrow \nabla r^2 = -2r^{-4}\hat{r}$

and $\nabla.\hat{r} = 3$

$$\text{So } \int \frac{1}{r^2} \hat{r} \cdot \hat{n} \; ds = \int_V \left[-2r^{-4}\vec{r}.\hat{r} + \frac{1}{r^2}.3\right] dv$$

$$= \int \left[\frac{-2}{r^4} r^2 + \frac{1}{r^2}.3\right] dv$$

$$= \int \left(\frac{-2}{r^2} + \frac{3}{r^2}\right) dv$$

$\Rightarrow \quad \int \frac{1}{r^2}\,\hat{r}\,.\,\hat{n}\;ds = \int \frac{1}{r^2}\;dv$ = R.H.S.

Example 3 : If S is any closed surface enclosing a volume V and $\vec{A}$ = x $\hat{i}$ + 2y $\hat{j}$ + 7z $\hat{k}$ then show that $\iint_S \hat{A}.\hat{n}ds$ = 10V.

Solution : Given $\vec{A}$ = x $\hat{i}$ + 2y $\hat{j}$ + 7z $\hat{k}$

Also we know that $\iint_S \hat{A}.\hat{n}ds = \iiint_V \left(\vec{\nabla}.\vec{A}\right)$dv

$$\Rightarrow \quad \iint_S \hat{A}.\hat{n}ds = \iiint_V \left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right).(x\,\hat{i} + 2y\,\hat{j} + 7z\,\hat{k})dv$$

$$= \iiint_V \left[1+2+7\right] dv$$

$$= \iiint_V \quad = 10dv$$

$$\Rightarrow \quad \iint_S \hat{A}.\hat{n}ds = 10\ v$$

Example 4 : Evaluate $\iint_S \hat{A}.\hat{n}ds$ when $\vec{A}$ = 4xy $\hat{i}$ + yz $\hat{j}$ - xz $\hat{k}$ and S is the surface on the cube bounded by the planes x = 0, x = 2, y = 0, y = 2, z = 1, z = 2

Solution : Given $\vec{A}$ = 4xy $\hat{i}$ + yz $\hat{j}$ - xz $\hat{k}$

Also we know that $\iint_S \hat{A}.\hat{n}ds = \iiint_V \left(\vec{\nabla}.\vec{A}\right)$dv

$$\text{So } \iint_S \hat{A}.\hat{n}ds = \iiint_V \left[\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right).\left(4xy\hat{i}+yz\hat{j}-xz\hat{k}\right)\right]dv$$

$$= \iiint_V \left[\frac{\partial 4xy}{\partial x}+\frac{\partial yz}{\partial y}+\frac{\partial(-xz)}{\partial z}\right] dv$$

$$= \iiint_V \;[4y + z - x]\; dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{2}\int_{z=0}^{2} (4y + z - x)\; dz\; dy\; dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} \left[4yz + \frac{z^2}{2} - xz\right]_0^2 \text{dy dx}$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} [8y + 2 - 2x] \text{ dy dx}$$

$$= \int_{x=0}^{2} \left[\frac{8y^2}{2} + 2y - 2xy\right]_0^2 \text{dx}$$

$$= \int_{x=0}^{2} (16 + 4 - 4x) \text{ dx}$$

$$= \int_{x=0}^{2} (20 - 4x) \text{ dx}$$

$$= \left[20x - \frac{4x^2}{2}\right]_0^2$$

= (40 - 8)

= 32

Hence $\iint_S \hat{A}.\hat{n}ds$ = 32

Example 5 : Evaluate $\int_S$ F.ds where F = 4x $\hat{i}$ - 2y^2 $\hat{j}$ + z^2 $\hat{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$, z = 0 and z = 3.

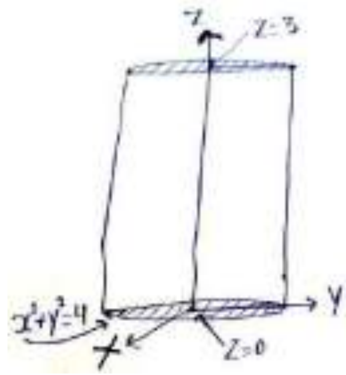
Solution : By divegence theorem

$$\int_S \text{F.ds} = \iiint_V \text{div. F dv}$$

$$= \iiint_V \left[\frac{\partial(4x)}{\partial x} + \frac{\partial(-2y^2)}{\partial y} + \frac{\partial(z^2)}{\partial z}\right] \text{dv}$$

$$= \iiint_V (4 - 4y + 2z) \text{ dx dy dz}$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3} \iiint_V (4 - 4y + 2z) \text{ dz dy dx}$$



$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + z^2\right]_0^3 \text{dy dx}$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9) \text{ dy dx}$$

$$= \int_{-2}^{2} \left[21y - 6y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \text{ dx}$$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2} \text{ dx}$$

$$= 84 \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\frac{\sin^{-1} x}{2}\right]_0^2$$

$$= 84\pi$$

Example 6 : Find $\iint_S$ F.Nds

where F(x, y, z) = $y^2 \hat{i}$ + $e^x(1 - \cos(x^2 + z^2)) \hat{j}$ + $(x + z) \hat{k}$ and S is the unit sphere centered at the point (1, 4, 6) with outwardly pointing normal vector?

Solution : We have

div F. = 0 + 0 + 1 = 1 $\therefore$ | as div F = $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Now, triple integral of the function 1 is the volume 0 | solid i.e. $\iiint_V$ 1.dv volume of solid

Since, the solid is a sphere of radius 1

Thus we get $\iiint_V 1.dv = \frac{4}{3}\pi(1) = \frac{4}{3}\pi$

By, Gauss's Divergence Theorem

$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

$\therefore \quad \iint_S F.N\, ds = \frac{4}{3}\pi$

Which is the required solution

Example 7 : Verify divergence theorem for F = $(x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Solution : Div. $\vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$

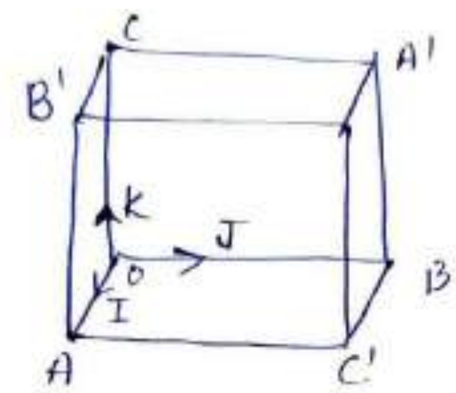
$= 2(x + y + z)$

$\therefore \quad \int_R \text{div. } F\, dv = 2\int_0^c \int_0^b \int_0^a (x + y + z)\, dx\, dy\, dz$

$= 2\int_0^c dz \int_0^b dy \left(\frac{a^2}{2} + ya + za\right)$

$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$

$= abc(a + b + c)$ ...(1)



Also, $\int_S F.N\, ds = \int_{S_1} F.N\, ds + \int_{S_2} F.N\, ds + \ldots + \int_{S_6} F.N\, ds$

Where S_1 is the face OAC'B, S_2 the face CB' P', S_2 the face OBA'C, S_4 the face AC'PB', S_5 the face OCB'A and S_6 the face BAP'C

Now , $\int_S$ F.N ds = $\int_{S_1}$ F.(-K) ds = $\int_0^b \int_0^a$ (O - xy) dx dy = $\frac{a^2b^2}{4}$

$\int_{S_2}$ F.N ds = $\int_{S_2}$ F.(-K) ds = $\int_0^b \int_0^a$ (C^2 - xy) dx dy = abc^2 $\frac{a^2b^2}{4}$

Similarly, $\int_{S_3}$ F.N ds = $\frac{b^2c^2}{4}$

$\int_{S_4}$ F.N ds = a^2bc - $\frac{b^2c^2}{4}$

$\int_{S_5}$ F.N ds = $\frac{c^2a^2}{4}$ and

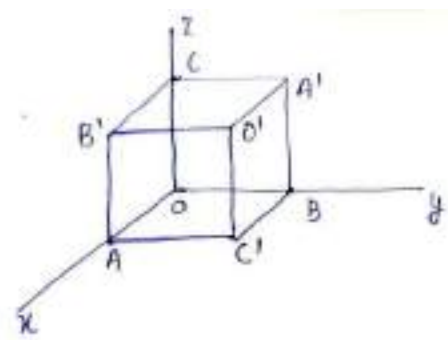
$\int_{S_6}$ F.N ds = ab^2c - $\frac{c^2a^2}{4}$

Thus, $\int_S$ F.N ds = abc (a + b +c) (2)

From (1) and (2) Gauss Divergence Theorem's verified.

Example 8 : Verify Divergence theorem for $\vec{F}$ = 3zx $\hat{i}$ - $2y^2$ $\hat{j}$ + yz $\hat{k}$ over the unit cube [0, 1; 0, 1, 0, 1]?

Solution : The cube is shown in the figure given below



Since, the cube has six faces, we have

$$\int_S \vec{F} \cdot \hat{n}\, ds = \int_{S_1} \vec{F} \cdot \hat{n}\, ds + \int_{S_2} \vec{F} \cdot \hat{n}\, ds + \int_{S_3} \vec{F} \cdot \hat{n}\, ds + \int_{S_4} \vec{F} \cdot \hat{n}\, ds + \int_{S_5} \vec{F} \cdot \hat{n}\, ds + \int_{S_6} \vec{F} \cdot \hat{n}\, ds$$

Where the surface S_1 is the face AC'OB' of the given cube

S_2 OBA'C, S_3 BA'O'C', S_4 OCB'A', S_5 O'A'CB', S_6 OAC'B

Here the each of surfaces S_1 & S_2 is parallel to yz plane. To each of these surfaces, the unit normal acts along the direction of x axis in positive direction for S_1 and in negative direction for S_2. Now, if we project S_1 on yz plane, the projection will be the surface S_2. Needless to say, the projection of S_2 on yz plane is S_2 itself.

Likewise, if we project S_3 or S_4 on zx plae, we have the projection S_4. Also, the projection of either of S_5 & S_6 on xy plane is S_6.

So, in every occasion, when we project either of the faces of the given cube on the coordinate plane parallel to it, we will obtain a square of side unity.

On the surfaces S_1, i.e. on the face AC'O'B' : x = 1 & $\hat{n} = \hat{i}$

Therefore, $\iint\limits_{S_1} \vec{F} \cdot \hat{n}\, ds = \iint\limits_{S_2} (3zx\, \hat{i} - 2y^2\, \hat{j} + yz\, \hat{k}) \cdot \hat{i}\, ds$

$$= \iint\limits_{S_2} 3z\, ds$$

$$= \int_{V=0}^{1} \int_{Z=0}^{1} 3z\, dy\, dz$$

$$= \frac{3}{2}$$

On the surface S_2 : x = 0, $\hat{n} = -\hat{i}$

Therefore, $\iint\limits_{S_2} \vec{F} \cdot \hat{n}\, ds = -\iint\limits_{S_2} (-2z^2\, \hat{j} + yz\, \hat{k}) \cdot \hat{i}\, ds = 0$

On the surface S_3 : y = 1, $\hat{n} = \hat{j}$

Therefore, $\iint\limits_{S_3} \vec{F} \cdot \hat{n}\, ds = +\iint\limits_{S_3} (3xz\, \hat{i} - 2\, \hat{j} + z\, \hat{k}) \cdot \hat{j}\, ds = -\int_{x=0}^{1} \int_{Z=0}^{1} dx\, dz = 2$

On the surface S_4 : y = 0, $\hat{n} = -\hat{j}$

Therefore, $\iint\limits_{S_4} \vec{F} \cdot \hat{n}\, ds = -\iint\limits_{S_4} (3xz\, \hat{i}) \cdot \hat{j}\, ds = 0$

On the surfaces S_5 : z = 1 & $\hat{n} = \hat{k}$

Therefore, $\iint\limits_{S_5} \vec{F} \cdot \hat{n}\, ds = \iint\limits_{S_5} (3zx\, \hat{i} - 2y^2\, \hat{j} + yz\, \hat{k}) \cdot \hat{k}\, ds$

$$= \iint_{S_5} y \, ds$$

$$= \int_{x=0}^{1} \int_{y=0}^{1} y \, dx \, dy$$

$$= \frac{1}{2}$$

On the surfaces S_6 : $z = 0$ & $\hat{n} = -\hat{k}$

Therefore, $\iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \iint_{S_6} (-2y^2 \, \hat{j}) \cdot \hat{k} \, ds = 0$

Hence (1) gives $\iint_{S} \vec{F} \cdot \hat{n} \, ds = \frac{3}{2} + 0 + (-2) + 0 + \frac{1}{2} + 0 = 0$

Again, div $\vec{F} = \frac{\partial}{\partial x}(3xz) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(yz) = 3z - 3y$

Let V be the volume enclosed by the surface S.

Then, $\int$ div $\vec{F}$ dV $= \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{Z=0}^{z=1} 3(x-y) \, dx \, dy \, dz$

$$= 3 \int_{x=0}^{x=1} \int_{y=0}^{y=1} \left[\frac{Z^2}{2} - yz \right]_{Z=0}^{Z=1} dy \, dz$$

$$= 3 \int_{x=0}^{x=1} \int_{y=0}^{y=1} \left(\frac{1}{2} - y \right) dy \, dx$$

$$= 3 \int_{x=0}^{x=1} \left[\frac{y}{2} - \frac{y^2}{2} \right]_{y=0}^{y=1} dx$$

$$= 3 \int_{x=0}^{x=1} \left(\frac{1}{2} - \frac{1}{2} \right) dx$$

$$= 0$$

Thus, we have $\int_{S} \vec{F} \cdot \hat{n} \, ds = \int$ div $\vec{F}$ dV

Hence, Gauss theorem is verified

Example 9 : Use Divergence Theorem to show that $\int_V \nabla\times\vec{b}\ dV = \int_S \hat{n}\times\vec{b}\ ds$

Solution : Let $\vec{c}$ be any arbitrary constant vector

Apply divergence theorem on $(\vec{b}\times\vec{c})$ to have

$$\int_S (\vec{b}\times\vec{c}).\ \hat{n}\ ds = \int_V \nabla.(\vec{b}\times\vec{c})\ dV \qquad ...(1)$$

Now, $\int_S (\vec{b}\times\vec{c}).\ \hat{n}\ ds = \int_S \vec{c}\ .\ (\hat{n}\times\vec{b})\ ds = \vec{c}\ .\int_S \hat{n}\times\vec{b}\ ds \qquad ...(2)$

as $\vec{c}$ is constant vector, it comes out from integration sign

Also, $\int_V \nabla.(\vec{b}\times\vec{c})\ dV = \int_V (\vec{c}\ .\ \{\nabla.\times\vec{b}\} - \vec{b}\ \{\nabla\times\vec{c}\})\ dV$

$= \int_V (\vec{c}\ .\ \{\nabla.\times\vec{b}\}\ dV$ $\qquad$ |as $\vec{c}$ is constant vector thus $\nabla\times\vec{c} = \vec{0}$

$= \vec{c}\ .\int_V \nabla\times\vec{b}\ dV \qquad ...(3) \quad |\vec{c}$ is constant vector

By (2) & (3), (1) imply, $\vec{c}\int_V \nabla\times\vec{b}\ dV = \vec{c}\ .\int_S \hat{n}\times\vec{b}\ ds$

Since, above holds for constant vector $\vec{c}$, we must have

$$\int_V \nabla\times\vec{b}\ dV = \int_S \hat{n}\times\vec{b}\ ds$$

Example 10 : Use Gauss theorem to evaluate $\int_S x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\ .\ \hat{n}\ ds.$

Where S the closed surface consisting of the cyllnder

$x^2 + y^2 = 1$, bounded by the plane z = 0 & z = 1

Solution : Let $\vec{F} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$

Then, $\nabla\vec{F} = \dfrac{\partial x}{\partial x} + \dfrac{\partial y}{\partial y} + \dfrac{\partial z}{\partial z} = 3$

Let V be the closed region bounded by the given surface S. Then by Gauss Theorem, we have,

$$\int_S \vec{F}\ .\ \hat{n}\ ds = \int_V (\nabla.\vec{F})\ dV = \int_V 3dV \qquad ...(1)$$

Let us switch to cylinderical polar coordinate system by applying the transformation.

$x = r \cos Q$, $y = r \sin Q$, $Z = z$

The Jacobian of the transformation is $J = \dfrac{\partial(x, y, z)}{\partial(r, Q, z)}$

Let in (r, Q, z) space, V' is the region into which V is mapped by the above transformation. Then (r, Q, z) space, V' is the rectangle [0, 1; 0, 2π; 0, 1]

Hence (1) gives

$$\int_S \vec{F} \cdot \hat{n}\ ds = \int_V 3\ dV = 3\int_V |J|\ dV' = 3\left(\int_{r=0}^{r=1} \int_{Q=0}^{2\pi} \int_{Z=0}^{z=1} r dr dQ dz\right)$$

$$= 3\pi$$

Example 11 : Given a function $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$ in three dimension cartesian space. Find value of surface integral

$$\oiint_S \hat{n} . \nabla \phi\ ds$$

Where S is the surface of a sphere of a unit radius and $\hat{n}$ is the outward unit normal vector on S.

Solution : Given $\phi = \frac{1}{2}(x^2 + y^2 + z^2)$

S = Surface of sphere, V = Volume of sphere $= \frac{4}{3}\pi r^3$, r = Radius of sphere = 1

$$\nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\nabla\phi = \frac{1}{2}(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\nabla\phi = x\hat{i} + y\hat{j} + z\hat{k}$$

Using Gauss Theorem

$$\oiint_S \hat{n} . \nabla \phi\ ds = \iiint_V \nabla . \nabla\phi\ dV$$

$$= \iiint_V \nabla . (x\hat{i} + y\hat{j} + z\hat{k})\ dV$$

$$= \iiint_V (1 + 1 + 1)\ dV$$

$$= 3 \iiint_V dV$$

$$= \cancel{3} \times \frac{4}{\cancel{3}} \pi r^3 \qquad \therefore \ |r = 1$$

$$= 4\pi (1)^3$$

$$= 4\pi$$

$$\therefore \quad \oiint \hat{n} . \nabla \phi \ ds = 4\pi$$

Self Check Exercise

1. Use Divergence Theorem to evaluate $\iint (xy^2z\,\hat{i} + z^2x\,\hat{j} - x^2y\,\hat{k})\,d\vec{s}$ over the sphere $x^2 + y^2 + z^2 = 1$ lying in the 1st octant and bounded by coordinate plane.

2. If $\nabla f = \vec{F}$ and $\nabla 2f = 0$, show that $\int_V \vec{F}^{\,2}\,dv = \int_S f\vec{F} . \hat{n}\,ds$, where S is closed surface enclosing the region V.

3. Use Gauss Theorem to evaluate $\int_S (x\,\hat{i} + y\,\hat{j} + z\,\hat{k}) . \hat{n}\,ds$, where S the closed surface consisting of the cones $x^2 + y^2 = Z^2$, bounded by the planes Z = 1

4. Compute $\iint_S \vec{F} .d\vec{s}$ where $\vec{F}$ =(x2 + 4y, 4y - tan z, z + y) and $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le 1$.

5. Evaluate $\iint_S \vec{F} . \hat{n}\,ds$ with the help of Gauss Theorem for

   $\vec{F} = 6z\,\hat{i} + (2x + y)\,\hat{j} - x\,\hat{k}$ taken over the region S bounded by the surface of cylinder $x^2 + z^2 = 9$ included between x = 0, y = 0, z = 0 and y = 8.

18.4 Summary

Dear students, this unit we studied that -

1. The divergence theorem relates a surface integral across closed surface S to a triple integral over the solid enclosed by S.
2. The divergence theorem can be used to transform difficult flux integral into an easier triple integral and vice versa.
3. The divergence theorem is a higher dimensional version of the flux form of Green's theorem.

4. The divergence theorem can be used derive Gauss Law, a fundamental law in electrostatics.
5. Gauss Divergence theorem deals with 3-D solid bounded by closed curve.
6. Mathematicaly, divergence theorem is denoted by

$$\iiint_V \nabla.\vec{F} \text{ dv} = \iint_S \vec{F}.\text{d}\vec{s}$$

18.5 Glossary

- **Divergence :** Divergence of a vector field $\vec{F}$ is denoted by div. $\vec{F}$ or $\nabla.\vec{F}$ defined as the vector operation which results in the scalar field calculating the rate of change of flux.

  If $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

  then $\nabla\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

- **Surface integral :** Surface integral is the generalization of double integral. In surface integral we integrate a surface in 2D or 3D to calculate the area approximation of all points present on the surface.
- **Volume integral :** Volume integral refer to the integral that extends through a 3-dimensional space, giving the total value of the function throughout the given region. Thus, it simply means to calculate the volume of three dimensional object.
- **Cartesian Equivalent of Divergence Theorem :**

  Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

  $\therefore$ div $\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

  $\therefore$ Divergence Theorem can be written as

$$\iiint\left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\right) \text{dx dy dz} = \iint (F_1 \text{ dy dz} + F_2 \text{ dz dx} + F_3 \text{ dx dy})$$

18.6 Answers to Self Check Exercise

1. $\frac{1}{18}$
2.
3. π
4. 6
5. 18π

18.7 References/Suggested Readings

1. Vector Analysis by J.G. Chakravorty & J.G. Ghosh.
2. Analytic Geometry of Two and Three Dimensions & Vector Analysis by R.M. Khan.
3. A Textbook of Vector Calculus by Shanti Narayan, P.K. Mittal.

18.3 Terminal Questions

1. Verify Divergence Theorem for $\vec{F} = x^3 \hat{i} + x^2y \hat{j} - xyz \hat{k}$. Over the cube

   [-1, 1; -1,1; -1, 1]

2. Use Divergence Theorem of evaluate $\iint_S (x^2 \hat{i} - y^2 \hat{j} - z \hat{k}) . \hat{n}\, ds$, where S is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$ bounded by the plane z = 0.

3. Use Gauss Theorem to evaluate $\int_S (x^2 \hat{i} + y^2 \hat{j} - z \hat{k}) . \hat{n}\, ds$, where S the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ bounded by the plans z = b and

   z = c (b < c)

4. Use Gauss Divergence Theorem to evaluate $\int_S (x^2 \hat{i} + zx \hat{j} - zy \hat{k}) . \hat{n}\, ds$, where S the closed surface consisting of four planes

   x = 0, y = 0, z = 0 & x + y + z = 1

5. Compute $\iint_S \vec{F} . d\vec{s}$ where $\vec{F} = (x + y + z, y^2, x^3 + z^3)$ and

   $0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq z \leq 2$.

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Unit - 19

Green Theorem

Structure

18.1 Introduction

Dear student, in this unit we will study about one another theorem of vector calculus known as Green Theorem. Just like Gauss Theorem Green Theorem also represents the relation between two types of integral. The relation between like integral and surface integral is given by Green's theorem. If we are given with a line integral, we can convert it into surface integral and vice versa. In this unit we will learn how to use Green theorem in such situations.

19.2 Learning Objectives

After studying this unit, students will be able to:

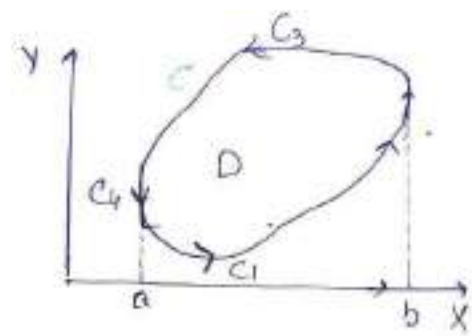
1. State Green Theorem.
2. define relationship between line and surface integral.
3. Apply Green Theorem in numerical problems.

19.3 Green's Theorem In Plane

Statement: Let C be the positively oriented, smooth and simple. Closed curve in a plane and D be the region bounded by the C. If L and M are the functions of (x, y) defined on the open region, containing D and have continuous partial derivatives then the Green theorem is stated as

$$\oint_c (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\, dy$$

Where the path integral is transversed counterclockwise along with C.



Proof : The given diagram has D region

$D = \{(x, y); a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$

Here, g_1 and g_2 are continuous function on [a, b]

Now $\iint_D \frac{\partial L}{\partial y}\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial L}{\partial y}(x, y)\, dy\, dx$

$$= \int_a^b \{L(x, g_2(x)) - (x, g_1(x))\}\, dx \qquad ...(1)$$

Now, calculate the line integral $\oint_c Ldx$. From the diagra C is written as C_1, C_2, C_3, C_4.

With C_1 :- $\int_{c_1} L(x, y)\, dx = \int_a^b L(x, g_1(x))\, dx$

With C_3 :- $\int_{c_3} L(x, y)\, dx = -\int_{-c_3} L(x, y)\, dx = -\int_a^b L(x, g_2(x))\, dx$

Therefore, C_3 goes in the negative direction from b to a.

Now, C_2 and C_4

$$\int_{c_4} L(x, y)\, dx = \int_{c_2} L(x, y)\, dx = 0$$

Therefore, we have

$$\oint_c Ldx = \int_{c_1} L(x, y)\, dx + \int_{c_2} L(x, y)\, dx + \int_{c_3} L(x, y)\, dx + \int_{c_4} L(x, y)\, dx$$

$$\Rightarrow \quad \oint_c Ldx = \int_a^b L(x, g_1(x))\, dx - \int_a^b L(x, g_2(x))\, dx \qquad ...(2)$$

From (1) and 2), we get

$$\oint_c L\,dx = \iint_D \frac{\partial L}{\partial y}\, dx\, dy$$

Similarly, we have $\oint_c M\, dy = \iint_D \frac{\partial M}{\partial x}\, dx\, dy$

Ultimately, we get

$$\oint_c (Ldx + Mdy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx\, dy$$

Hence proved

Vector form of Green Theorem in two dimensions

Let $\vec{F} = L\,\hat{i} + M\,\hat{j}$

and $\vec{r} = x\,\hat{i} + y\,\hat{j}$

then $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$\vec{F} \cdot d\vec{r} = (L\,\hat{i} + M\,\hat{j}) \cdot (dx\,\hat{i} + dy\,\hat{j})$$

$$= Ldx + Mdy$$

Also Curl $\vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ L & M & O \end{vmatrix}$

$$\hat{i}\left(\frac{-\partial M}{\partial z} \right) - \hat{j}\left(\frac{\partial L}{\partial z} \right) + \hat{k}\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right)$$

Since we are deleting with two dimensions

So $\frac{\partial M}{\partial z} = \frac{\partial L}{\partial z} = 0$

So, Curl $\vec{F} = \vec{\nabla} \times \vec{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) \hat{k}$

$$\text{Curl } \vec{F} \cdot \hat{k} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

So, Green Theorem is given by

$$\oint_c Ldx + Mdy = \iint_s \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dxdy$$

$$= \oint_c \hat{F}.d\hat{r} = \iint curl\, \hat{F}.\hat{K}\, dxdy$$

$$= \oint_c \hat{F}.d\hat{r} = \iint_R curl\, \hat{F}.\hat{K}\, dR$$

dR = dxdy and $\hat{K}$ is unit vector perpendicular to xy plane.

How to Apply Green Theorem

1. From the given problem, write the value of L and M.
2. Find $\frac{\partial L}{\partial y}$ and $\frac{\partial M}{\partial x}$
3. Find $\frac{\partial M}{\partial x}$ - $\frac{\partial L}{\partial y}$
3. Apply it into green's theorem i.e.

$$\oint_c (Ldx + Mdy) = \iint_s \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dxdy$$

- There are two types of questions (i) To verify green's theorem (2) To evaluate the given integral using Green Theorem.

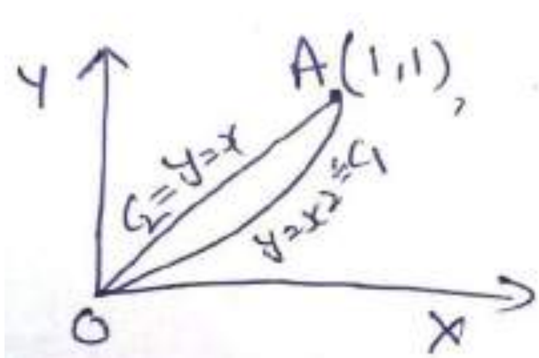
Let us try to do some example to verify Green's Theorem.

Example 1: Verify Green theorem in plane for

$\oint_c \left[\left(xy + y^2 \right) dx + x^2 dy \right]$ Where c is the closed curve of region bounded by

y = x and y = x^2

Solution: Since we know that Green theorem gives



$$\oint_c (Ldx + Mdy) = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dxdy$$

We have to verify this Firstly to evaluate the line integral along curve C.

L.H.S.

$$\oint_c Ldx + Mdy$$

The curve y = x and y = x^2, interest at origin (0, 0) and the point A (1, 1).

(1) Along the curve y = x^2, dy = 2xdx

$$\oint_{c_1} Ldx + Mdy = \oint_{c_1} \left[\left(xy + y^2 \right) dx + x^2 dy \right]$$

$$\oint_{c_1} \left[xx^2 + \left(x^2 \right)^2 \right] dx + x^2 2xdx \right]$$

$$= \oint_{c_1} \left[x^3 + x^4 + 2x^3 \right] dx$$

$$= \oint_{c_1} \left[x^3 + x^4 x^3 \right] dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \left(\frac{3}{4} + \frac{1}{5} - 0 \right) = \frac{19}{20}$$

Now, along the curve y = x. dy = dx

$$\oint_{c_2} Ldx + Mdy = \oint_{c_2} \left(xy + y^2 \right) dx + x^2 dy$$

$$= \oint_{c_2} \left[\left(x(x) + x^2 \right) dx + x^2 dx \right]$$

$$= \oint_{c_2} \left(x^2 + x^2 x^2 \right) dx$$

$$= \int_{01}^{0} 3x^2 dx$$

$$= \left[3\frac{x^3}{3}\right]_{01}^{0}$$

= [0 - 1]

= -1

So $\oint_c Ldx + Mdy = \oint_{c_1} Ldx + Mdy + \oint_{c_2} Ldx + Mdy$

$$= \frac{19}{20} - 1$$

$$\oint_c Ldx + Mdy = \frac{-1}{20} = \text{L.H.S.}$$

Now R.H.S. $\iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$

Here L = xy + y^2 M = x^2

$\frac{\partial L}{\partial y}$ = x + 2y $\frac{\partial M}{\partial x}$ = 2x

$$\therefore \quad \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy = \iint_R (2x - x + 2y)\, dxdy$$

$$= \iint_R (2x - x - 2y)\, dxdy$$

$$= \iint_R (x - 2y)\, dxdy$$

$$= \int_0^1 \int_{y=x^2}^{x} \left((x - 2y)\, dy\right) dx$$

$$= \int_0^1 \left[xy - y^2\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x^2 - x^2\right) - \left(x^3 - x^4\right)\right] dx$$

$$= \int_0^1 \left[x^4 - x^3\right] dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{5} - \frac{1}{4}$$

$$= \frac{-1}{20}$$

$$\Rightarrow \qquad \iint_R \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = \frac{-1}{20} = \text{R.H.S.}$$

Since L.H.S. = R.H.S, so Green Theorem is Verified.

Example 2: Verify Green's Theorem in the plane for

$\oint_c \left[\left(3x^2 - 8y^2\right) dx + \left(4y - 6xy\right) dy \right]$ Where c is the boundary of the region defined by x = 0, y = 0, x + y = 1

Solution: Since by Green's Theorem, we have

$$\oint_c L dx + M dy = \iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

Here $L = 3x^2 - 8y^2$ $\quad M = 4y - 6xy$

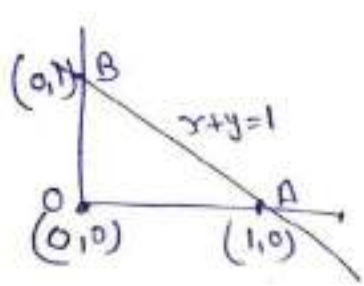
$\frac{\partial L}{\partial y} = -16y \qquad \frac{\partial M}{\partial x} = -6y$

Here the curve c is the boundary of the region defined by x = 0, y = 0, x + y = 1

x + y = 1

y = 1 - x

x	1	0
y	0	1



The curve c is the triangle from OA, AB and BO. having vertices (0, 0), (1, 0) and (0, 1). So

Curve C = OA + AB + BO

$\Rightarrow$ $C = C_1 + C_2 + C_3$

along C_1 i.e. OA, $y = 0 \Rightarrow dy = 0$, $x = 0$ to 1

Along C_2 i.e. AB, $x + y = 1$

$\Rightarrow$ $x = 1 - y$

$dx = -dy$ and y varies from 0 to 1

Along C_3 i.e. BO, y varies from 1 to 0 and $x = 0 \Rightarrow dx = 0$

$$\oint_c Ldx + Mdy = \oint_{c_1} Ldx + Mdy = \oint_{c_2} Ldx + Mdy = \oint_{c_3} Ldx + Mdy$$

Now $$\oint_{c_1} Ldx + Mdy = \int_{C_1}^{R} \left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy$$

As $y = 0$ & $x = 0$ to $x = 1$

$$= \int_0^1 3x^2 dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1$$

$$\int_{C_1} Ldx + Mdy = 1$$

Now $$\int_{C_2} Ldx + Mdy = \int_{c_2} \left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy$$

in C_2 $y = 0$ to $y = 1$ and $x = 0$, $x = 1 - y$, $dx = -dy$

$$= \int_0^1 \left[\left(3(1-y)^2 + 8y^2\right)(-dy) + \left(4y - 6(1-y)y\right)dy\right]$$

$$= \int_0^1 \left[-3 - 3y^2 + 8y^2 + 4y - 6y + 6y^2 + 6y\right]dy$$

$$= \int_0^1 \left(11y^2 + 4y - 3\right)dy$$

$$= \left[\frac{11y^3}{3} + \frac{4y^2}{2} - 3y\right]_0^1$$

$$= \frac{11}{3} + \frac{4}{2} - 3$$

$$= \frac{11+12-18}{6}$$

$$= \frac{34-18}{6} = \frac{16}{6} = \frac{8}{3}$$

Again $\int_{C_3} Ldx + Mdy = \int_{c_3} \left(3x^2 - 8y^2\right)dx + \left(4y - 6xy\right)dy$

Along C_3 y varies from 1 to 0 and x = 0, dx = 0

$$= \int_1^0 \left(-8y^2\right)0 + 4y\,dy$$

$$= \int_{01}^0 4y\,dy$$

$$= \left[\frac{4y^2}{2}\right]_{01}^0$$

= (0 - 2) = -2

$\therefore \quad \oint_c Ldx + Mdy = 1 + \frac{8}{3} - 2$

$$\frac{3+8-6}{3} = \frac{11-6}{3} = \frac{5}{3}$$

Now taking $\iint_R \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dxdy$

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \left(-6y + 16y\right)dxdy$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5\int_0^1 \left[\left(1-x^2\right)-0\right]dx$$

$$= 5\int_0^1 \left(1-x^2-2x\right)dx$$

$$= 5\left[\frac{x+x^3}{3}-\frac{2x^2}{2}\right]_0^1$$

$$= 5\left[1+\tfrac{1}{3}-1\right]$$

$$= 5\left(\frac{1}{3}\right)$$

$$= \frac{5}{3} = \text{L.H.S.}$$

So L.H.S. = R.H.S. i.e.

$$\oint_c Ldx+Mdy \quad \iint_R \left(\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}\right)dxdy.$$

Hence Green's Theorem is verified

Some Related Questions

Let us try following examples to have the understanding of Green's theorem to evaluate the value.

Example 3: Calculate the Line integral $\oint_C x^2ydx+\left(y-3\right)dy$ where "C" is a rectangle and its vertices are (1, 1), (4, 1), (4, 5), (1, 5)?

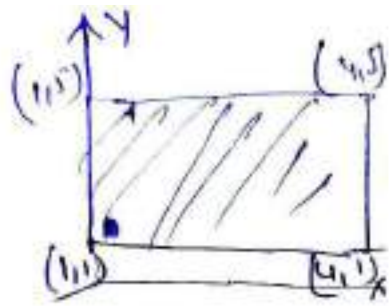
Solution: Let F (x, y) = $\left[L\left(x,y\right)M\left(x,y\right)\right]$. Where L and M are the two functions.

Here F (x, y) = $\left[x^2y,\left(y-3\right)\right]$

Then $\frac{\partial M}{\partial x}$ = 0

$\frac{\partial L}{\partial y}$ = x^2

Such that $\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}$ = $-x^2$



Let "D" is the rectangular region enclosed by the curve "C" By Green's Theorem

$$\oint_C x^2 y dx + (y-3) dy = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dA$$

$$= \iint_D x^2 dA$$

[As the value of x varies from 1 to 4
and value of y varies from 1 to 5]

$$= \int_1^5 \int_1^4 -x^2 dx dy$$

$$= \int_1^5 -21 dy$$

= - 84

The Line integral of given function is - 84

Example 2: Let S be the triangle with vertices (0, 0), (1, 0) and (0, 3) oriented clockwise. Calculate the flux of $\vec{F}$ (x, y) = $\langle P(x,y), Q(x,y) \rangle$ = $\langle x^2 + e^y, x + y \rangle$ across S.

Solution: Let D be the region enclosed by S

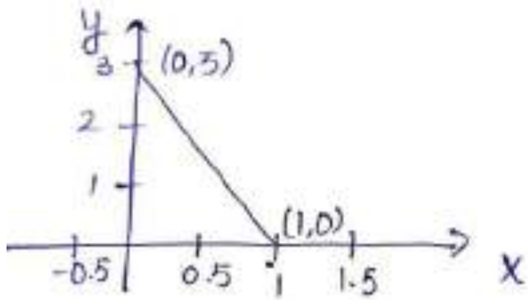
Given $\vec{F}$ (x, y) = $\langle P(x,y), Q(x,y) \rangle$

= $\langle x^2 + e^y, x + y \rangle$

Vertices = (0, 0), (1, 0) and (0, 3) of triangle.

Green's Theorem applies only to simple closed curves oriented counterclockwise but we still apply the theorem because

$\oint_C \vec{F}.\vec{N} ds$ = - $\oint_{-S} \vec{F}.\vec{N} ds$ and -S is oriented counterclockwise.



By Green's theorem, the flux is

$$\oint_C \vec{F}.\vec{N}ds \qquad = \oint_{-S} \vec{F}.\vec{N}ds$$

$$= \iint_D \left(Px + Qy\right) dA$$

Here Px = 2x, Qy = 1, Px + Qy = 2x + 1

and top edge of the triangle is the line y = -3x + 3. Therefore y values run from y = 0 to y = -3x + 3. Thus, we have.

$$\oint_C \vec{F}.\vec{N}ds = -\iint_D (2x+1)dA$$

$$= -\int_0^1 \int_0^{-3x+3} (2x+1)dydx$$

$$= -\int_0^1 (2x+1)(-3x+3)\, dx$$

$$= -\int_0^1 \left(-6x^2 + 3x + 3\right) dx$$

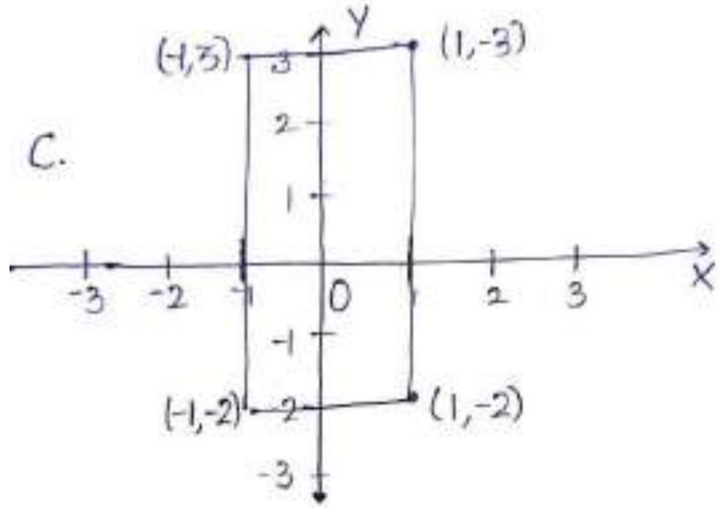
$$= -\left[-2x^3 + \frac{3x^2}{2} + 3x\right]_0^1$$

$$= \frac{-5}{2}$$

$$\therefore \qquad \oint_C \vec{F}.\vec{N}ds = \frac{-5}{2}$$

Example 5: Water flows from a spring located at the origin the velocity of the water is modeled by vector field ∇(x, y) = (5x + y, x + 3y) m/sec. Find the amount of water per seconds that flows across the rectangle with vertices (-1, -2), (1, -2), (1, 3) and (-1, 3) oriented counterclockwise.

Solution: Let C represents the given rectangle and let D be the rectangular region enclosed by C.



To find the amount of water flowing across C, we calculate Flux $\int_C \vec{V}.d\vec{r}$

Let P(x, y) = 5x + y and Q (x, y) = x + 3y

So that $\vec{V}$ = (P, Q)

Then, $P_x = 5$ and $Q_y = 3$.

By Green's Theorem

$\int_C \vec{V}.d\vec{r}$ = $\iint_D (Px + Qy)\, dA$

= $\iint_D 8\, dA$

= $8\iint_D dA$

= 8 (area of D i.e. area of rectangle)

= 8 × 10 $\quad \left| \begin{array}{l} Length = 5, Width = 2 \\ Area = L \times b = 5 \times 2 = 10 unit \end{array} \right|$

Therefore, the water flux is 80 m^2/sec.

Example 6: Calculate the integral $\oint_{\partial D} \left(\sin x - \frac{y^3}{3} \right) dx + \left(\frac{y^3}{3} + \sin y \right) dy$ Where D is the annulus given by the polar inequalities $1 \le r \le 2$, $0 \le \theta \le 2\pi$.

Solution : Although D is not simply connected, we can use the extended form of Green's theorem to calculate the integral. Since, the integration occurs over an annulus, we convert to polar coordination.

$$\therefore \quad \oint_{\partial D}\left(\sin x-\frac{y^3}{3}\right)dx+\left(\frac{y^3}{3}+\sin y\right)dy = \iint_D (Qx-Py)\,dA$$

$$= \iint_D \left(x^2+y^2\right)dA$$

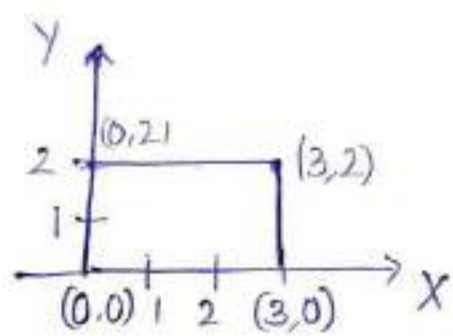
$$= \int_0^{2\pi}\int_1^2 r^3\,dr\,d\theta$$

$$= \int_0^{2\pi}\frac{15}{4}\,d\theta$$

$$= \frac{15\pi}{2}$$

$$\therefore \quad \oint_{\partial D}\left(\sin x-\frac{y^3}{3}\right)dx+\left(\frac{y^3}{3}+\sin y\right)dy = \frac{15\pi}{2}$$

Example 7: Consider a fluid flow represented by the vector field F = (y^2, x^2). We want to find the circulation of the fluid around a rectangular path defined by the coordinates (0, 0), (3, 0), (3, 2) and (0, 2)



Solution: Using Green's Theorem, we identity

P = y^2, Q = x^2

The formula implies

$$\oint_C (Pdx+Qdy) = \iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

$$= \int_0^3 \int_0^2 (2x - 2y)\, dydx$$

$$= \int_0^3 \left[2xy - y^2\right]_0^2 dx$$

$$= \int_0^3 (4x - 4) dx$$

$$= \left[\frac{4x^2}{2} - 4x\right]_0^3$$

= 2 (3)2 - 4(3)

= 6

Example 8: Evaluate the integral $\oint_C \left(y^2 dx + x^2 dy\right)$ where C is the boundary of the upper half of the unit disk, transverse counter clock.

Solution: According to Green's Theorem

$$\oint_C \left(y^2 dx + x^2 dy\right) = \iint_D (2x - 2y)\, dxdy$$

Where D is the upper half of the unit disk

Thus, $\displaystyle\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (2i - 2y)\, dydx = \int_{-1}^{1} \left(2xy - y^2\right)\Big|_0^{\sqrt{1-x^2}} dx$

$$= \int_{-1}^{1} \left(2x\sqrt{1-x^2} - \left(1 - x^2\right)\right) dx$$

$$= 0 - \int_{-1}^{1} \left(1 - x^2\right) dx$$

$$= -\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1}$$

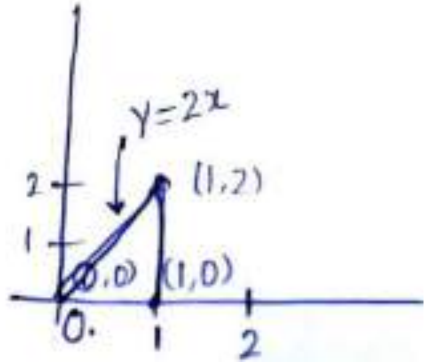
$$= -2 + \frac{2}{3}$$

$$= \frac{-4}{3}$$

$$\therefore \qquad \oint_C \left(y^2dx + x^2dy\right) = \frac{-4}{3}$$

Example 9: Use Green's Theorem to evaluate $\oint_c xydx + x^2y^3$ dy where is the triangle with vertices (0, 0), (1, 0), (1, 2) with positive orientation.

Solution: Given P = xy, Q = x2 y3 and 0 < x < 1, 0 < y < 2x



So, using Green's Theorem

$$\oint_c xydx + x^2y^3 \text{ dy} = \iint_D \left(2xy^3 - x\right)dA$$

$$= \int_0^1\int_0^{2x} \left(2xy^3 - x\right)dA$$

$$= \int_0^1 \left(\frac{1}{2}xy^4 - xy\right)\Bigg|_0^{2x} dx$$

$$= \int_0^1 \left(8x^5 - 2x^2\right)dx$$

$$= \left[\frac{4x^6}{3} - 2x^2\right]_0^1$$

$$= \frac{2}{3}$$

Self Check Exercise

Q.1 Calculate the work done on a particle by a force field F (x, y) = (y + sin x, e^y - x). The particle transverse the circle $x^2 + y^2 = 4$ in an anticlockwise direction the start and end point are (2, 0)

Q.2	Calculate the integral $\oint_C \sin\left(x^2\right)dx+\left(3x-y\right)dy$ where C is a right triangle with vertices (-1, 2), (4, 2) and (4, 5) oriented counterclockwise	
Q.3.	Calculate the flux of $\vec{F}(x,y)=\left\langle x^3,y^3\right\rangle$ across a unit circle oriented counterclockwise.	
Q.4	Calculate integral $\oint_{\partial D}\vec{F}.d\vec{r}$, where D is the annulus given by the polar inequalities 2 < r < 5, 0 < θ < 2π and F(x, y) = $\left\langle x^3,5x+e^y\sin y\right\rangle$.	
Q.5	Evaluate $\oint_C y^3dx-x^3dy$ where C are the two circles of radius 2 and radius centered at the origin with positive orientation.	
Q.6	Verify Green Theorem in plane for $\oint_C\left(3x^2-8y^2\right)dx$ + (4x - 6xy) dx where C is the boundary of the region bounded by y = $\sqrt{x}$ and y = x^2.	

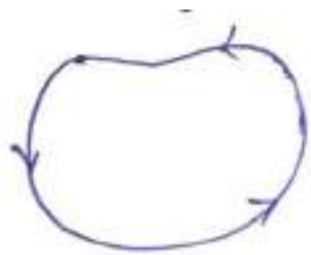
19.4 Summary:

Dear students, in this unit we studied that Green's Theorem converts a line integral to a double integral over microscopic circulation in a region.

1. It is applicable only over closed paths.
2. It is used to calculate the vector fields in a two dimensional space
3. It is also used to calculate the area and tangent vector of a boundary oriented in an anticlockwise direction.
4. Green's Theorem relates the integral over a connected region to an integral over the boundary of the region. Thus, it is a version of the Fundamental Theorem of Calculus in one higher dimension.
5. Green's Theorem comes in two forms : a circulation form and flux form. In the circulation form the integrand is $\vec{F}.\vec{T}$. in the flux form, the integrand is $\vec{F}.\vec{N}$.
6. Green's Theorem can be used to transform a difficult line integral into an easier double integral or to transform a difficult double integral into an easier line integral.
7. A vector field is source free if it has a stream function. The flux of a source free vector field across a closed curve is zero, just as the circulation of a conservative vector field across a closed curve is zero.

19.5 Glossary

- **Line integral:** A line integral is integral in which the function to be integrated is determined along a curve in the coordinate system. The function which is to be integrated may be either a scalar field or vector field.
- **Surface Integral:** Surface integral is the generalization of double integral. In surface integral we integrate a surface in 2D or 3D to calculate the are a approximation of all points present on the surface.
- **Closed Paths:** In a graph, a path is defined as being closed if it starts and ends in the same vertex. e.g.



- **Vector Field:** A function of a space whose value at each point is a vector quantity.
- **Flux:** Flux is a vector quantity, describing the magnitude and direction of the flow of a substance or property.

19.6 Answers to Self Check Exercise

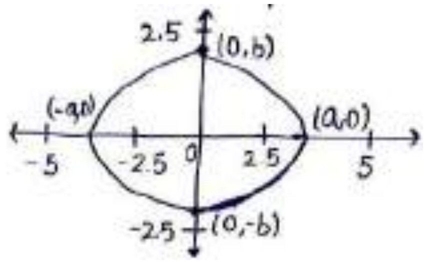
1. -8π
2. $\frac{45}{2}$
3. $\frac{3\pi}{2}$
4. 105π
5. $\frac{-45\pi}{2}$
6. Do fame as in example.

19.7 References/Suggested Readings

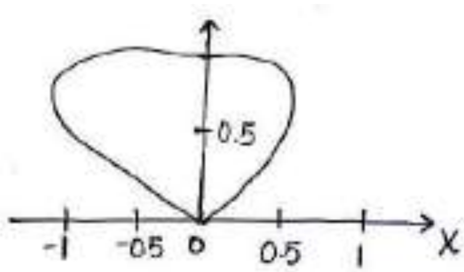
1. Mathematical Analysis by Apostol
2. A Textbook of Vector Calculas by Shanti Narayan
3. Vector Calculas by Steven G. Krantz and Hardd Parks.

19.8 Terminal Questions

1. Calculate the area enclosed by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1.



2. Find the area of the region enclosed by the curve with parameterization r (t) = $\langle \sin t \cos t, \sin t \rangle$, 0 $\leq$ t $\leq$ π.



3. Solve $\int_C y^3 dx - x^3 dy$ where C is the circle of radius 2 centered at origin.

4. Using Green's Theorem evaluate $\int_C \left(x^2 y dx + x^2 dy\right)$ where C is the boundary described counter clockwise of the triangle with vertices (0, 0), (1, 0), (1, 1).

5. Use Green's Theorem to evaluate the line integral $\int_C y^3 dz + 33xy^2 dy$ where C is the path from (0, 0) to (1, 1) along the graph of y = x^3 and from (1, 1) to (0, 0) along the graph y = x.

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Unit - 20

Stoke's Theorem

Structure

20.1 Introduction

Dear student, in this unit we will study about one another important theorem of vector calculus known as Stoke's Theorem. This theorem also relate the line integral and surface integral. In this theorem the integral of curl of the vector field over some surface is related to the line integral of the vector field around the boundary of the surface. In this we will try to learn this theorem, how to verify Stock's Theorem for a given vector field as well as try to evaluate the integral using Stoke's Theorem.

20.2 Learning Objectives

After studying this unit, students will be able to

1. define Stoke's Theorem.
2. relate line and surface integral
3. verify Stoke's Theorem for a given vector field.
4. evaluate the integral using Stoke's Theorem.

20.3 Stoke's Theorem

Statement: Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also, let $\int_C \vec{F}.d\vec{r} = \iint_S curl\,\vec{F}.d\vec{s}$

Proof: Let us assume that S has a smooth parametrization $\vec{r}$ = (s, t), such that A corresponds to region R and B corresponds to boundary C of R. We Stoke's Theorem for the surface S and continuously differentiable vector field $\vec{F}$ by expressing the integrals on both sides of the theorem in terms of s and t and using Green's Theorem.

Now, $\int_B \overline{F}.\overline{dr} = \int_C \overline{F}.\frac{\partial \overline{r}}{\partial s} ds + \overline{F}\frac{\partial \overline{r}}{\partial t} dt$

Let $\vec{G}$ = (G_1, G_2) on st plane then

$G_1 = \vec{F}.\frac{\partial \vec{r}}{\partial s}$, $G_2 = \vec{F}\frac{\partial \vec{r}}{\partial t}$

$$\int_B \overline{F}.\overline{dr} = \int_C \vec{G}.d\vec{s} \qquad ....(1)$$

Where $\vec{s}$ is the position vector of a point.

Now, $\int_A Curl\,\overline{F}.\overline{dA} = \int_R Curl\,\vec{F}.\frac{\partial \vec{r}}{\partial t}$ ds dt

Hence $\int_A Curl\,\overline{F}.\overline{dA} = \int_R \left(\frac{\partial G_1}{\partial t} - \frac{\partial G_1}{\partial t}\right) ds\,dt$ ...(2)

Using Green's theorem on (1) and (2) we get that R.H.S. of (1) and (2) is equal, thus we have.

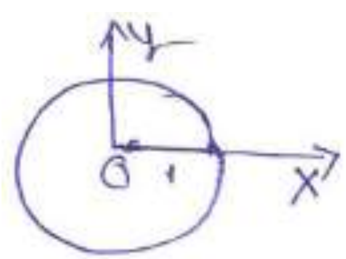
$$\int_A Curl\,\overline{F}.d\overline{A} = \int_B \overline{F}.\overline{dr}$$

Hence proved

Let us try some questions related to Stoke's Theorem.

Example 1: Verify Stoke's Theorem for the function $\hat{F} = z\hat{i} + x\hat{j} + y\hat{k}$ where curve is the unit circle in the xy plane bounding the semi-sphere z = $\sqrt{1-x^2-y^2}$

Solution: Here the curve C is of the surface S which is a unit circle i.e. circle having radius 1 in xy plane. Since we are talking about xy plane so z = 0 and equ. of unit circle is $x^2 + y^2 = 1$.



The parametric equation of circle are x = cox t, y = sin t, z = 0 and t varies from 0 to 2π.

Since stoke's theorem is given as

$$\oint \hat{F}.d\hat{r} = \iint_S curl\hat{F}.d\hat{s} = \iint_S curl\hat{F}.\hat{n}\,ds$$

Here $\hat{F} = z\hat{i} + x\hat{j} + y\hat{k}$ and $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$ so $d\hat{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

L.H.S. $\oint_C \hat{F}.d\hat{r} = \oint_C \left(z\hat{i} + x\hat{j} + y\hat{k}\right).\left(dx\hat{i} + dy\hat{j} + dz\hat{k}\right)$

$$\oint_C \hat{F}.d\hat{r} = \oint_C zdx + xdy + ydz$$

Since z = 0 so dz = 0

$\therefore$ $\oint_C \hat{F}.d\hat{r} = \oint_C x\,dy$, as x = cox t, y = sin t $\Rightarrow$ dy = cox t dt

$$= \int_0^{2\pi} cox\,t cox\,t\,dt$$

$$= \int_0^{2\pi} cox^2\,dt$$

$$= \int_0^{2\pi} \frac{\left(1 + cox^{2t}\right)dt}{2}$$

$$= \frac{1}{2}\int_0^{2\pi} \left(1 + cox^{2t}\right)dt$$

$$= \frac{1}{2}\left[t + \frac{\sin^{2\pi}}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[2\pi + \frac{\sin 4\pi}{2} + \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\,[2\pi + 0] = \pi$$

$\Rightarrow$ $\oint_C \hat{F}.d\hat{r} = \pi$

Now R.H.S. $\iint_S curl\hat{F}.\hat{n}\,ds$

Curl $\hat{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$

= $\hat{i}\left(\frac{\partial y}{\partial y}-\frac{\partial x}{\partial z}\right)-\hat{j}\left(\frac{\partial y}{\partial x}-\frac{\partial z}{\partial z}\right)+\hat{k}\left(\frac{\partial x}{\partial x}-\frac{\partial z}{\partial y}\right)$

Curl $\hat{F}$ = $\hat{i}$ + $\hat{j}$ + $\hat{k}$

Since here the surface is the plane region bounded by a circle

So $\iint_S curl\hat{F}.\hat{n}\, ds$ = $\iint_S curl\hat{F}.\hat{k}\, ds$

= $\iint_S \left(\hat{i}+\hat{j}+\hat{k}\right).\hat{k}\, ds$

= $\iint_S ds$ = area of unit circle

= $\pi r^2 = \pi(1)^2 = \pi$

Since L.H.S. = R.H.S

So Stoke's Theorem is Verified.

Example 2: Verify Stoke's Theorem for $\hat{F}$ = $(x^2 + y^2)\,\hat{i}$ - $2xy\,\hat{j}$ taken round the rectangles bounded by x = ± a, y = 0, y = 6.

Solution : Since stoke theorem is given by

$\int \hat{F}.d\hat{r}$ = $\iint_S$ curl $\hat{F}\,.\,\hat{n}$ ds = $\iint_S$ curl $\hat{F}\,.\,\hat{k}$ ds

Here $\hat{F}$ = $(x^2 + y^2)\,\hat{i}$ - $2xy\,\hat{j}$

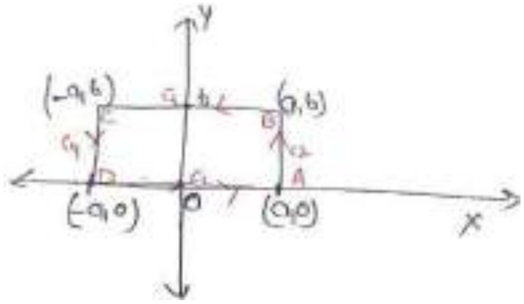
curl $\hat{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$

= $\hat{i}\left(0-0\right)-\hat{j}\left(0-0\right)+\hat{k}\left(\frac{\partial(-2xy)}{\partial x}-\frac{\partial(x^2+y^2)}{\partial y}\right)$

= $\hat{k}$ (-2y - 2y) = -4y $\hat{k}$

Curl $\hat{F}$ = -4y$\hat{k}$

Here the region is



x varies from (-9, 9)

y varies from (0, 6)

$$\therefore \iint_S \text{curl } \hat{F} \cdot \hat{n} \text{ ds} = \iint_S \text{curl } \hat{F} \cdot \hat{k} \text{ ds}$$

$$= \int_0^b \int_{-a}^a (-4y\hat{k}).\hat{k} \text{ ds}$$

$$= \int_0^b \int_{-a}^a -4y \text{ dx dy}$$

$$= \int_0^b \left[-4yx\right]_{-0}^9 \text{dy}$$

$$= -4\int_0^b [ya - y(-a)] \text{ dy}$$

$$= -4\int_0^b 2ya \text{ dy}$$

$$= -8a\int_0^b y \text{ dy}$$

$$= -8a \left[\frac{y^2}{2}\right]_0^b$$

$$= -8a \left(\frac{b^2}{2}\right)$$

$$= \frac{-8ab^2}{2}$$

$$= -4ab^2$$

$$\Rightarrow \iint_S \text{curl } \hat{F} \cdot \hat{n} \, ds = -4ab^2 \qquad \ldots(1)$$

Now $\oint_c \hat{F} \cdot d\hat{r} = \oint_c [(x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}] \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

$$= \oint_c [(x^2 + y^2)\,dx - 2xy\,dy]$$

$$= \oint_c [(x^2 + y^2)\,dx - 2xy\,dy]$$

$$= \oint_{c_1} [(x^2 + y^2)\,dx - 2xy\,dy] + \oint_{c_2} [(x^2 + y^2)\,dx - 2xy\,dy] + \oint_{c_3} [(x^2 + y^2)\,dx - 2xy\,dy]$$

$$+ \oint_{c_4} [(x^2 + y^2)\,dx - 2xy\,dy]$$

Along C_1 $\quad y = a,\ x \rightarrow -a$ to a

Along C_2 $\quad x = a,\ y \rightarrow a$ to b

$dx = 0$

Along C_3 $\quad y = b,\ x = a$ to $-a$

$dy = 0$

Along C_4 $\quad x = a,\ y = b$ to a

$dx = 0$

$$\oint_c \hat{F} \cdot d\hat{r} = \int_{-a}^{a} x^2 dx + \int_0^b a^2 dx - 2ay\,dy + \int_a^{-a} (x^2 + b^2)dx - 2xb\,dy$$

$$+ \int_b^0 (-a^2) + y^2)dx - 2(-a)y\,dy$$

$$= \int_{-a}^{a} x^2 dx + \int_0^b (-2ay)dy + \int_a^{-a} (x^2 + b^2)dx + \int_b^0 2ay\,dy$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} + \left[\frac{-2ay^2}{2}\right]_0^b + \left[\frac{x^2}{2} + b^2 x\right]_a^{-a} + \left[\frac{2ay^2}{2}\right]_b^0$$

$$= \frac{a^3}{3} + \frac{a^3}{3} + (-ab^2) + \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2a + (-ab^2)$$

$$= -ab^2 - ab^2 - b^2a - ab^2$$

$$\oint_c \hat{F} \cdot d\hat{r} = -4ab^2 \qquad \text{...(2)}$$

From (1) and (2)

$$\oint_c \hat{F} \cdot d\hat{r} = \iint_S \text{curl } \hat{F} \cdot \hat{n} \, ds$$

Hence Stokes theorem is verified

Example 3 : Evaluate $\oint_c$ (x+2y) dx + (x+3y) dy using stoke's theorem where C is the unit circle.

Solution : Here $\hat{F}$ = (x+2y) $\hat{i}$ + (x+3y) $\hat{j}$

$$d\hat{r} = dx\,\hat{i} + dy\,\hat{j}$$

Such that $\oint_c \hat{F} \cdot d\hat{r} = \oint_c (x+2y)\,dx + (x+3y)\,dy$

Since Stokes' theorem gives us

$$\oint_c \hat{F} \cdot d\hat{r} = \iint_S \text{curl } \hat{F} \cdot \hat{n} \, ds$$

$$\text{So, Curl } \hat{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y & x+3y & 0 \end{vmatrix}$$

$$= \hat{i}\,(0-0) - \hat{j}\,(0-0) + \hat{k}\,(1-2)$$

$$= -\hat{k}$$

$$\therefore \quad \iint_S \text{Curl } \hat{F} \cdot \hat{n} \, ds = \iint_S \text{Curl } \hat{F} \cdot \hat{k} \, ds$$

$$= \iint_S -\hat{k} \cdot \hat{k} \, ds$$

$$= \iint_S ds.$$

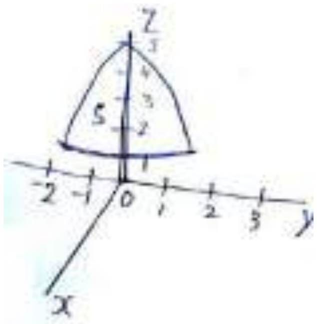
Here S is the surface of unit circle

=[-S]

$\therefore \quad \oiint \hat{F} \cdot d\hat{r} = -(\pi r^2)$ and of circle = πr^2

$= -\pi$ as $x^2 + y^2 = r^2$ for unit circle r=1

Example 3: Use Stoke's Theorem to evaluate $\iint_S Curl\vec{F}.\vec{ds}$ where $\vec{F} = z^2\,\hat{i} - 3xy\,\hat{j} + x^3y^3\,\hat{k}$ and S is the part of $z = 5 - x^2 - y^2$ above the plane z = 1. Assume that S is oriental upwards.



Solution: In this case the boundary curve C will be where the surface interests the plane z = 1. Assume that S is oriental upwards.

$\therefore \quad 1 = 5 - x^2 - y^2$

$\Rightarrow \quad x^2 + y^2 = 4$ at z = 1

So, the boundary curve will be the circle of radius 2 that is in the plane z = 1. The parameterization of this curve is

$\vec{r}(t) = 2\cos t\,\hat{i} + 2\sin t\,\hat{j} + \hat{k},\ 0 \le t \le 2\pi$

The first two components gives the circle and the third component make sure that it is in the plane z = 1.

Using Stoke's Theorem we can write the surface integral as the following line integral.

$$\iint_S Curl\vec{F}.\vec{ds} = \int_C \vec{F}.\vec{dr} = \int_0^{2\pi} \vec{F}\left(\vec{r}(t)\right).\dot{r}(t)dt$$

Here $\vec{F}\left(\vec{r}(t)\right) = (1)^2\,\hat{i} - 3\,(2\cos t)\,(2\sin t)\,\hat{j} + (2\cos t)^3\,(2\sin t)\,\hat{k}$

$= \hat{i} - 12\cos t\sin t\,\hat{j} + 64\cos^3 t\sin^3 t\,\hat{k}$

and $\vec{r}(t) = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$

$\Rightarrow \quad \vec{F}\left(\vec{r}(t)\right).\ \vec{r}\,'(t) = -2\sin t - 24\sin t\cos^2 t$

Therefore, we have

$$\iint_S Curl\vec{F}.\vec{ds} = \int_0^{2\pi} \left(-2\sin t - 24\sin t\cos^2 t\right)dt$$

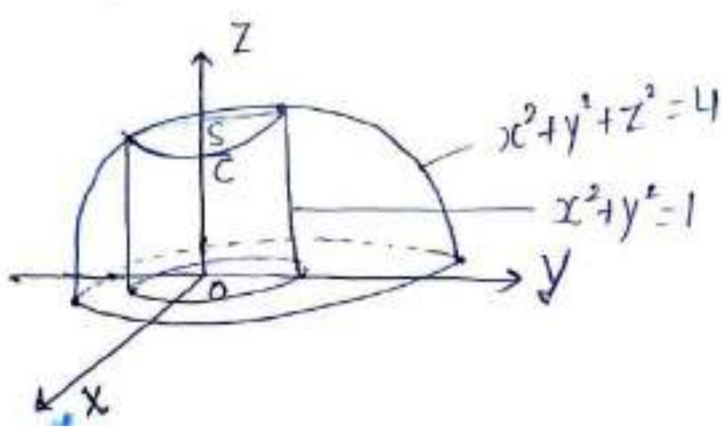
$$= \left[2\cos t + 8\cos^3 t\right]_0^{2\pi}$$

$$= 0$$

Example 4: Using Stoke's Theorem evaluate $\iint_S Curl\vec{F}.\vec{ds}$ where $\vec{F}$ = xz $\hat{i}$ + yz $\hat{j}$ +xy $\hat{k}$ such that S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy plane.

Solution: Given, Equation of Sphere : $x^2 + y^2 + z^2 = 4$ ...(1)

Equation of Cylinder : $x^2 + y^2 = 1$ ...(2)



Subtracting (2) from (1) we get

$Z^2 = 3$

Z = $\sqrt{3}$ (Since Z is positive)

Now, the circle C will be : $x^2 + y^2 = 1$, z = $\sqrt{3}$

The vector form of C is given by

r (t) = cost $\hat{i}$ + sin t $\hat{j}$ + $\sqrt{3}$ $\hat{k}$; $0 \leq t \leq 2\pi$

Thus, $\vec{r}'(t)$ = -sin t $\hat{i}$ + cos t $\hat{j}$

Let us write $F(r(t))$ as

$F(r(t)) = \sqrt{3}$ cost $\hat{i}$ + $\sqrt{3}$ sin t $\hat{j}$ + cost sin t $\hat{k}$

$$\iint_S Curl\vec{F}.\vec{ds} = \int_C \vec{F}.\vec{dr}$$

$$= \int_0^{2\pi} F\left(r(t)\right).r'(t)dt$$

$$= \int_0^{2\pi} \left(-\sqrt{3}\cos t\sin t + \sqrt{3}\sin t\cos t\right)dt$$

$$= \sqrt{3}\int_0^{2\pi} 0\,dt$$

$$= 0$$

Example 5: Let us consider a vector field $\vec{F}$ given by $\vec{F}$ = y $\hat{i}$ - x $\hat{j}$ + $yx^3\,\hat{k}$ and let S be the portion of the sphere of radius 4 with z $\geq$ 0 and the upward orientation. Use Stoke's Theorem to evaluate $\oint_C \vec{F}.\vec{ds}$.

Solution: Given $\vec{F}$ = y $\hat{i}$ - x $\hat{j}$ + $yx^3\,\hat{k}$

Here C is the circle of radus 4 at (z = 0)

Now, curl. $\vec{F} = \nabla \times \vec{F}$ = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & yx^3 \end{vmatrix}$

$$= \left(\frac{\partial}{\partial y}\left(yx^3\right) - \frac{\partial}{\partial z}(-x)\right)\hat{i} = \left(\frac{\partial}{\partial x}\left(yx^3\right) - \frac{\partial}{\partial z}(y)\right)\hat{j} + \left(\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(y)\right)\hat{k}$$

$= x^3\,\hat{i} - 3yx^2\,\hat{j} - 2\,\hat{k}$

Since S is the upper hemisphere $\vec{ds}$ will be $\hat{k}$. ds where d$\vec{s}$ is the area element of the hemisphere.

$$\Rightarrow \qquad \iint_S \left(x^3 i - 3yx^2 j - 2\hat{k}\right).\hat{k}.ds = -\iint_S 2\hat{k}.\hat{k}\,ds$$

$$= -2\iint_S ds$$

Thus, we have according to Stoke's Theorem

$$\oint_C \vec{F}.\vec{dr} = \iint_S Curl\vec{F}.\vec{ds}$$

$$= -2\iint_S ds \qquad \because |area of hemisphere = 2\pi r^2|$$

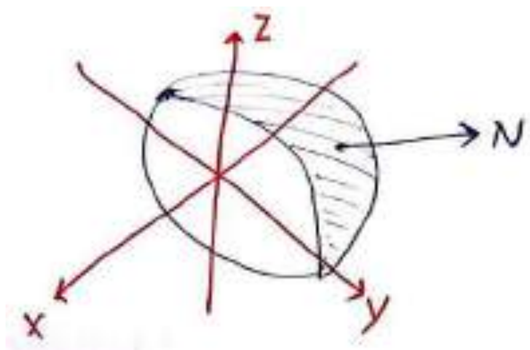
$$= -2\,(2\pi r^2) \qquad \because |r = 4|$$

$$= -\left(2\times 2\pi(4)^2\right)$$

$$= -64\pi$$

$$\therefore \quad \oint_C \vec{F}.\vec{dr} = -64\pi$$

Example 6: Verify that Stoke's Theorem is true for vector field $\vec{F}$ (x, y) = $\langle -z, x, 0\rangle$ and surface S, where S is the hemisphere, oriented outward with parameterization $\vec{r}\,(\theta, \phi)$ = $\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi$ as shown.



Solution: Let C be the boundary of S. Note that C is a circle of radius 1, centered at the origin, sitting in plane y = 0. This circle has parameterization $\langle \cos t, 0, \sin t >\rangle$, $0 < t < 2\pi$, the equation for scalar surface integrals.

$$\int_C \vec{F}.\vec{dr} = \int_0^{2\pi} \langle -\sin t, \cos t, 0\rangle.\langle -\sin t, 0, \cos t\rangle\, dt$$

$$= \int_0^{2\pi} \sin^2 t\, dt$$

$$= \pi$$

By the equation for vector line integrals,

$$\iint_S Curl\vec{F}.\vec{ds} = \iint_D curl\bar{F}\left(\dot{r}(\phi,\theta)\right).\left(\dot{t}\phi \times \bar{t}\theta\right) dA$$

$$= \iint_D \langle 0,-1,1\rangle . \langle c\,os\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi\rangle dA$$

$$= \int_0^{\pi}\int_0^{\pi}\left(\sin\phi\cos\phi - \sin\theta\sin^2\phi\right)$$

$$= \frac{\pi}{2}\int_0^{\pi}\sin\theta d\theta$$

$$= \pi$$

Therefore, we have verified Stoke's Theorem for this example.

Example 7: Calculate surface integral $\iint_S Curl\vec{F}.\vec{ds}$, where S is the surface, oriented outward and $\vec{F}$ = (z, 2xy, x + y).

Solution: According to stokes theorem

$$\iint_S Curl\vec{F}.\vec{ds} = \int_C \vec{F}.\vec{dr}$$

Here boundary c of the surface is merely a single circle with radius. Where C has parameterization (cost, sint, 1), $0 \leq t \leq 2\pi$

$$\Rightarrow \qquad \iint_S Curl\vec{F}.\vec{ds} = \int_C \vec{F}.\vec{dr}$$

$$= \int_0^{2\pi}\left(1, 2\sin t\cos t, \cos t + \sin t\right).\left(-\sin t, \cos t, 0\right)dt$$

$$= \int_0^{2\pi}\left(-\sin t + 2\sin t + \cos^2 t\right)dt$$

$$= \left[\cos t - \frac{2\cos^3 t}{3}\right]_0^{2\pi}$$

$$= \cos(2\pi) - \frac{2\cos^3(2\pi)}{3} - \left(\cos(0) - \frac{2\cos^3(0)}{3}\right)$$

$$= 0$$

$$\therefore \qquad \iint_S Curl\vec{F}.\vec{ds} = 0$$

Example 8: Calculate the integral $\int_C \vec{F}.\vec{dr}$ where $\vec{F}$ = (xy, $x^2 + y^2 + z^2$, yz) and C is the boundary of the parallelogram with vertices (0, 0, 1), (0, 1, 0), (2, 0, -1) and (2, 1, -2)?

Solution: As we know

$$\int_C \vec{F}.\vec{dr} = \iint_S Curl\vec{F}.\vec{ds}$$

Let S denote the surface of parallelogram. Note that S is the portion of the graph of Z = 1 - x - y for (x, y) varying over the rectangular region with vertices (0, 0), (0, 1), (2, 0) and (2, 1) in the xy plane. Therefore a parameterization of S is (x, y, 1 - x, 1 - x - y), $0 \le x \le 2$, $0 \le y \le 1$. The curl of $\vec{F}$ is (-z, 0, x)

$$\therefore \quad \int_C \vec{F}.\vec{dr} = \iint_S Curl\vec{F}.\vec{ds}$$

$$= \int_0^2\int_0^1 curl\ddot{F}(x,y).\left(\dot{t}x \times \dot{t}y\right)dydx$$

$$= \int_0^2\int_0^1 \left(-(1-x-y),0,x\right).\left((1,0,-1)\times(0,1,-1)\right)dydx$$

$$= \int_0^2\int_0^1 \left(x+y-1,0,x\right).(1,1,1)\,dydx$$

$$= \int_0^2\int_0^1 (2x+y-1)\,dy\,dx$$

= 3

Example 9: Prove that $\oint_C \vec{r}.\vec{dr}$ = 0

Solution: By Stoke's Theorem

$$\int \ddot{F}.\bar{dr} = \int \bar{n}.curl\bar{F}\,ds$$

Put $\vec{F} = \vec{r}$

$$\therefore \quad \int \vec{r}.\vec{dr} = \int \bar{n}.curl\bar{r}\,ds$$

$$= \int \vec{n}.\vec{0}\,ds \qquad \because \left|curl\,\vec{n} = 0\right|$$

= 0

$\vec{r}.\vec{dr}$ = 0

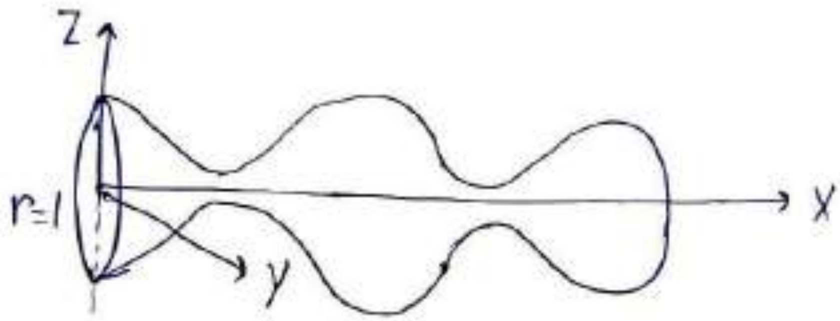
Hence proved

Self Check Exercise

Q.1 Use Stoke's Theorem to evaluate $\int_C \vec{F}.\vec{dr}$ where $\vec{F}$ = $z^2\,\hat{i} - y^2\,\hat{j} + x\,\hat{k}$ and C is the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1) with counter clockwise rotation.

Q.2 Verify that Stoke's Theorem is true for vector field $\vec{F}$ (x, y, z) = $\langle y, x, -z \rangle$ and surface S, where S is the upwardly oriented portion of the graph of f(x, y) = x^2 y over a triangle in the xy plane with vertices (0, 0), (2, 0) and (0, 2).

Q. 3 Use Stoke's Theorem to calculate surface integral $\iint_S Curl\vec{F}.\vec{ds}$ where $\vec{F}$ (x, y, z) and S is the surface as shown in figure



Q.4 Use Stoke's Theorem to calculate line integral $\int_C \vec{F}.\vec{dr}$ where $\vec{F}$ (x, y, z) and C is the boundary of a triangle with vertices (0, 0, 1) (3, 0, -2) and (0, 1, 2).

Q.5 Calculate the curl of electric field $\vec{F}$ if the corresponding magnetic field is $\vec{B}$(t) = (tx, ty, -2tz), $0 \leq t < \infty$.

20.4 Summary:

In this unit we studied that

1. Stoke's Theorem is a powerful tool for converting complex surface integral to more manageable line integral in multivariable calculas.
2. The Stoke's Theorem is applied to study rotation and curl in fluid flow. If can be used to analyze circulation and vorticity in fluids which are very useful in aerodynamics and weather systems.
3. This theorem is higher dimensional analog to Fundamental Theorem of calculas relating derivatives to integral
4. It can be applied to any surface that is smooth and has a well defined boundary.

5. Stoke's Theorem can be interpreted as a way to relate the rotation of a fluid within a surface to the flow along the boundary of the surface.

6. Thus, this theorem relates a flux integral over a surface to a line integral around the boundary of the surface. It transform a difficult surface integral into an easier line integral on a difficult line integral into an easier surface integral.

20.5 Glossary

- **Curl:** Curl of a vector field is obtained by taking the vector product of the vector operator applied to the vector field F(x, y, z)

  Curl $\vec{F}$ (x, y, z) = ∇ × $\vec{F}$ (x, y, z)

  $$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

- **Line integral:** A line integral is integral in which the function to be integrated is determined along a curve in the coordinate system. The function which is to be integrated may be either a scalar field or vector field.
- **Surface integral:** Surface integral is the generalization of the double integral. In surface integral we integrate a surface in 2D or 3D to calculate the area approximation of all points present on the surface.
- **Flux:** The rate of flow of fluids, particles or energy across a given surface or area.
- **Boundary:** The line which marks the limit of an area
- **Dimension:** A measurable extent of a particular kind such as length, depth or height.

20.6 Answers to Self Check Exercise

1. $\frac{-1}{6}$
2. $\frac{-136}{45}$
3. -π
4. $\frac{3}{2}$
5. curl $\vec{E}$ = (x, y, -2z)

20.7 References/Suggested Readings

1. A Textbook of Vector Calculus by Shanti Narayan
2. A Textbook of Vector Calculus by Anil Kumar Sharma
3. A Textbook of Vector Calculus by Dr. Us Negi and Dr. K.C. Petwal

20.8 Terminal Questions

1. Use Stoke's Theorem to evaluate $\iint_S \left(Curl\vec{F}\right).\hat{n}ds$ when $\vec{F}$ = $z^2\,\hat{i}$ + $y^2\,\hat{j}$ + xy $\hat{k}$ and C is the triangle defined by (1, 0, 0), (0;1, 0) and (0, 0, 2).

2. Verify the Stoke's Theorem for the vector field $\vec{F}$ = y $\hat{i}$ + 2z $\hat{j}$ + $x^2\,\hat{k}$ and surface S, where S is the parabola Z = 4-x^2 - y^2.

3. Compute $\oint_C x^2 z dx + 2xdy - y^3 dz$, where C is the unit circle $x^2 + y^2 = 1$ oriented counterclockwise.

4. Consider the vector field F = (z sin (x), yz, $x^2 + y^2$) and let S be the upper hemisphere of the sphere $x^2 + y^2 + z^2 = a^2$ with radius a and centered at the origin. Then verify stoke theorem.

5. Paroboloid surface let's take a vector field F = $\left(xy, e^z, z\cos(y)\right)$ and consider S to be the surface of the paraboloid z = 1 - x^2 - y^2 capped by the plane z = 0. Then evaluate $\oint_C F.dr$.

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