BA-IInd Year
Mathematics

Course Code: MATH309TH
Course Name: Integral Calculus

Integral Calculus

Units 1 to 20

Centre for Distance and Online Education (CDOE)
Himachal Pradesh University,
Gyan Path, Summer Hill, Shimla - 171005

Math 309th

Integral Calculus

CONTENTS

SYLLABUS

Himachal Pradesh University
B.A. with Mathematics
Syllabus and Examination Scheme

Course Code	MATH 309TH
Credits	4
Name of the Course	Integral Calculus
Type of the Course	Skill Enhancement Course
Assignments	Max. Marks:30
Yearly Based Examination	Max Marks: 70
	Maximum Times: 3 hrs.

Instructions

Instructions for Candidates: Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

SEC 3.3 : Integral Calculus

(In B.Sc/B.A. Mathematics this Course in Sec 1.3)

Unit-I

Integration by Partial fractions, Integration of rational and irrational functions. Properties of definite integrals.

Unit-II

Reduction Formulae, $\int \sin^n x dx$, $\int \cos^n x dx$, $\int e^{ax} \sin^x x dx$, $\int x^n (\log x)^m dx$, $\int x^m \sin nx dx$, $\int x^n \cos nx dx$, $\int \sin^m x \cos^n x dx$, $\int_0^{\pi/2} \sin^n x dx$ $\int_0^{\pi/2} \cos^n x dx$, $\int_0^{\pi/2} \sin^m \cos^n x dx$. Reduction by connecting two integrals (Smaller Index +1 Method)

Unit-III

Area and lengths of curves in the plane, volumes and surfaces of solids of revolution, Cartesian and parametric form

Unit-IV

Double and Triple integrals.

Books Recommended

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, Delhi, 2005.
2. H. Anton, I. Bivens and S. Davis, Calculus, Jhon Wiley and Sons (Asia) P.Ltd., 2002.

Unit - 1

Integration By Partial Fractions

Structure

1.1 Introduction

If the integrand (the expression after the integral sign) is in the form of an algebraic fraction and the integral cannot be evaluated by simple methods, the fraction needs to be expressed in partial fractions before integration takes place. The steps needed to decompose an algebraic fraction into its partial fractions results from a consideration of the reverse process-addition (or subtraction) consider the following addition of algebraic fractions:

$$\frac{1}{x+2} + \frac{5}{x+3} = \frac{(x+3)+5(x+2)}{(x+2)(x+3)}$$

$$= \frac{6x+13}{x^2+5x+6}$$

Here we want to go the other way around. That is, if we were to start with the expression.

$$\frac{6x+13}{x^2+5x+6}$$

and try to find the fractions whose sum gives this result, then the two fractions obtained, i.e. $\frac{1}{x+2}$ and $\frac{5}{x+3}$, we called the partial fractions of $\frac{6x+13}{x^2+5x+6}$. We decompose fractions into partial fractions like this because it makes certain integrals much easier to do.

Integration by partial fractions involves breaking down a complex fraction into simpler fractions, allowing us to integrate each component individually. This method is particularly useful when dealing with functions that cannot be easily integrated using other methods.

Partial fractions are especially valuable when dealing with improper rational functions, where the degree of the numerator is greater than or equal to the degree of the denominator. By decomposing the improper fraction into partial fractions, we can often transform the integral into a sum of integrals that can be evaluated using elementary techniques.

1.2 Learning Objectives

After studying this unit, you should be able to:-

- Define the type of functions whose integrals be evaluated by the method of partial fractions and able to solve different integrals by partial fractions.
- Discuss the method to evaluate $\int \frac{1}{ax^2+bx+c}$ dx, where 4ac - b2 > 0, a $\neq$ 0 and find the solution of this type of integral.
- Discuss the method to evaluate $\int \frac{\propto x+\beta}{ax^2+bx+c}$ dx, where $\propto \neq 0$ and 4 ac - $b^2 > 0$ and solve the questions of this type of integrals.

1.3 Kinds of Functions Whose Integrals Be Evaluated By the Method Of Partial Fractions

The functions of the type f(x) = $\frac{\phi(x)}{\psi(x)}$, where ϕ(x) and ψ(x) are polynomials.

If degree of ψ(x) is 0 i.e. ψ(x) is a constant, then f(x) is a polynomial in x and we have already done methods to evaluate such an integral.

But if degree of $\psi(x) \neq 0$

Also we assume here that ϕ(x) is not divisible by ψ(x)

If the degree of ϕ(x) is not less than that of ψ(x), we apply long division and obtain.

ϕ(x) = q(x) ψ(x) + r(x), where q(x) and r(x) are polynomials, the degree of r(x) is less than that Thus,

$$f(x) = \frac{\phi(x)}{\psi(x)} = q(x) + \frac{r(x)}{\psi(x)}$$

In orders to integrate f(x) it suffices to integrate $\frac{r(x)}{\psi(x)}$, Since we know how to integrate q(x).

Let us consider the following examples to clear the idea:-

Example 1: Evaluate the following integral:

$$\int \frac{dx}{x^2 - a^2} \ ; |x| > a$$

Sol: Let us split $\frac{1}{x^2 - a^2}$ into partial fractions.

$$\frac{1}{x^2 - a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} \qquad .....(1)$$

Or 1 = A (x + a) + B(x - a) [Taking L.C.M.]

Put x = a, 1 = A (a + a) + B (a - a)

= 2aA

Or A = $\frac{1}{2a}$

Put x = -a, 1 = A (- a + a) + B (-a - a)

Or 1 = 0 - 2aB

Or B = - $\frac{1}{2a}$

Putting the values of A and B in (1), we get

$$\frac{1}{x^2 - a^2} = \frac{1}{2a} . \frac{1}{x-a} - \frac{1}{2a} . \frac{1}{x+a}$$

$$= \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\therefore \quad \int \frac{1}{x^2 - a^2} dx = \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a}\right) dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{x-a} dx - \int \frac{1}{x+a} dx\right]$$

$= \frac{1}{2a}$ [log |x - a| - log |x + a|]

$= \frac{1}{2a}\left[\log\left|\frac{x-a}{x+a}\right|\right]$

$= \frac{1}{2a}$ log $\left|\frac{x-a}{x+a}\right|$

Example 2: Evaluate the following integral

$\int \frac{x-1}{x^3 - x^2 - 2x}$ dx

Sol: We factor the denominator $x^3 - x^2 - 2x$

Now $x^3 - x^2 - 2x = x (x^2 - x - 2) = x [x^2 - 2x + x - 2]$

x [x(x - 2) + 1 (x - 2)]

= x (x - 2) (x + 1)

$\therefore \quad \frac{x-1}{x^3 - x^2 - 2x} = \frac{x-1}{x(x-2)(x+1)}$

Splitting the R.H.S. into partial fractions, we get

$$\frac{x-1}{x^3 - x^2 - 2x} = \frac{x-1}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1} \quad .....(1)$$

Or x - 1 = A (x - 2) (x + 1) + B;0 (0 + 1) + c.0.(0 - 2)

Or -1 = -2A

Or A = $\frac{1}{2}$

Put x = 2; 2 - 1 = A[2 - 2) (2 + 1) + B.2 (2 + 1) + c.2(2 - 2)

Or 1 - 0 + 6B + 0

Or B = $\frac{1}{6}$

Put x = -1; -1 -1 = A(-1 -2) (-1 +1) + B (-1) (-1 +1) + c (-1) (-1 -2)

Or -2 = 3c

Or c = -$\frac{2}{3}$

Putting these values of A, B, C in (1), we get

$$\frac{x-1}{x^3-x^2-2x} = \frac{1/2}{x} + \frac{1/6}{x-2} + \frac{-2/3}{x+1}$$

$\therefore$ $$\int \frac{x-1}{x^3-x^2-2x}\,dx = \int \frac{1/2}{x}\,dx + \int \frac{1/6}{x-2} + \int \frac{-2/3}{x+1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{6}\int \frac{1}{x-2}\,dx - \frac{2}{3}\int \frac{1}{x+1}\,dx$$

$$= \frac{1}{2}\log|x| + \frac{1}{6}\log|x-2| - \frac{2}{3}\log|x+1|$$

Example 3: Evaluate $\int \frac{x^2+x+2}{x^2-1}\,dx$

Sol: Let I = $\int \frac{x^2+x+2}{x^2-1}\,dx$

Here power in the numerator is equal to the power of denominator.

$\therefore$ Let us divide the numerator by denominator and then form partial fractions

Factors of $x^2 - 1$ are (x - 1) (x + 1)

Now
$$\begin{array}{r|l} & \quad\quad\quad 1 \\ x^2-1 & x^2+x+2 \\ & \underline{-x^2 \quad\quad +1} \\ & \quad\quad x+3 \end{array}$$

$\therefore$ $$\frac{x^2+x+2}{x^2-1} = \frac{x^2+x+2}{(x+1)(x+1)} = 1 + \frac{A}{x-1} + \frac{B}{x+1} \quad .....(1)$$

Multiplying both sides of (1), by the L.C.M. = (x - 1) (x + 1), we have

$x^2 + x + 2 = (x - 1)(x + 1) + A(x + 1) + B(x - 2)$

Putting x = 1; 1 + 1 + 2 = A (1 + 1)

Or 2A = 4

$\therefore$ A = 2

Putting x = -1; 1 - 1 + 2 = B (-1 -1)

Or -2B = 2

$\therefore$ B = -1

Putting the values of A and B in (1), we have

$$\frac{x^2+x+2}{x^2-1} = 1 + \frac{2}{x-1} - \frac{1}{x+1}$$

$$\therefore \quad \int \frac{x^2+x+2}{x^2-1}\, dx = \int 1dx + 2\int \frac{1}{x-1}\, dx - \int \frac{1}{x+1}\, dx$$

= x + 2 log |x - 1) - log |x + 1|

Example 4: Evaluate the following integral

$$\int \frac{1}{x^4-1}\, dx, \; x > 1$$

Sol: Let $\quad I = \int \frac{1}{x^4-1}\, dx$

Put $\quad x^2 = t$

Differentiating w.r.t. x

$$2x = \frac{dt}{dx} \text{ or } 2x\, dx = dt$$

$$\therefore \quad x\, dx = \frac{1}{2}\, dt$$

$$\therefore \quad I = \int \frac{\frac{1}{2}dt}{t^2-1}$$

$$= \frac{1}{2}\int \frac{1}{t^2-1^2}\, dt$$

$$= \frac{1}{2}\,\frac{1}{2.1} \log\left|\frac{t-1}{t+1}\right|$$

$$\left[Since \int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|\right]$$

$$= \frac{1}{4} \log\left|\frac{x^2-1}{x^2+1}\right|$$

Example 5: Evaluate $\int \frac{x^2+3x-7}{(2x+3)(x+1)^2}\, dx$

Sol: Let f(x) = $\int \dfrac{x^2+3x-7}{(2x+3)(x+1)^2}$ dx

Let us split $\dfrac{x^2+3x-7}{(2x+3)(x+1)^2}$ into partial fractions

$$\frac{x^2+3x-7}{(2x+3)(x+1)^2} = \frac{A}{2x+3} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \qquad .....(1)$$

Or $x^2 - 3x - 7 = A(x+1)^2 + B(x+1)(2x+3) + C(2x+3)$ ......(2)

Put $x = {-3}/{2}$; $\frac{9}{4} + \frac{9}{2} - 7 = A\left(-\frac{3}{2}+1\right)^2 + B\left(-\frac{3}{2}+1\right)(-3+3) + C(-3+3)$

Or $\dfrac{9+18-28}{4} = A\,\dfrac{1}{4}$

Or $-\dfrac{1}{4} = A\,\dfrac{1}{4}$

Or A = -1

Put x = -1 ; $1 + 3 - 7 = A(-1+1)^2 + B(-1+1)(-2+3) + C(-2+3)$

Or -3 = C

Comparing coefficients of x^2 on both sides of (2), we get

1 = A + 2B

= -1 + 2B

Or 2B = 2

Or B = 1

Putting the values of A, B, C in (1), we get

$$\frac{x^2-3x-7}{(2x+3)(x+1)^2} = \frac{-1}{(2x+3)} + \frac{1}{x+1} + \frac{-3}{(x+1)^2}$$

$$\therefore \quad \int \frac{x^2-3x-7}{(2x+3)(x+1)^2}\,dx = -\int \frac{1}{2x+3}\,dx + \int \frac{1}{x+1}\,dx - 3\int (x+1)^{-2}\,dx$$

$$= -\frac{\log|2x+3|}{2} + \log|x+1| - 3\,\frac{(x+1)^{-2+1}}{-2+1}$$

$$= -\frac{1}{2}\log|2x+3| + \log|x+1| + 3(x+1)^{-1}$$

Self-Check Exercise-1

Q.1 Evaluate the following integral

$$\int \frac{dx}{a^2 - x^2}, \ a > x$$

Q.2 Evaluate $\int \frac{x-1}{(x-2)^2(x+3)} dx$

1.4 Method To Evaluate $\int \frac{1}{ax^2 + bx + c} dx$, where $4ac - b^2 > 0, a \neq 0$

$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ [Making the coefficient of x^2 as 1]

$\left(\text{Adding and Subtracting} \frac{b^2}{4a^2}\right)$ i.e. $\left(\frac{1}{2}\text{Coeff of x}\right)^2$

$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right]$ [Completing the square]

$= a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{1ac - b^2}{4a^2}\right]$ ......(1)

Put $\sqrt{\frac{4ac - b^2}{4a^2}} = q$ $\left[\sqrt{4ac - b^2} \text{ is real because } (4ac\text{-}b^2) \text{ is positive}\right]$

$\therefore$ $ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + q^2\right]$

$\therefore$ $\int \frac{1}{ax^2 + bx + c} dx = \frac{1}{a} \int \frac{1}{\left(x + \frac{b}{2a}\right)^2 + q^2} dx$

Let us reduce this integral to $\int \frac{1}{x^2 + 1} dx$

Set $x + \frac{b}{2a} = qt$

$\therefore$ $dx = q\,dt$

$\therefore \quad \int \frac{1}{ax^2+bx+c}$ dx = $\frac{1}{a} \int \frac{1}{q^2t^2+q^2}$ q dt

= $\frac{q}{aq^2} \int \frac{1}{t^2+1}$ dt

= $\frac{1}{aq}$ $\tan^{-1}$ t

where q = $\sqrt{\frac{4ac-b^2}{4a^2}}$ and t = $\frac{x+\frac{b}{2a}}{q}$

Let us consider the following example to clear the idea:

Example 6: Evaluate the following integral

$$\int \frac{dx}{4x^2+6x+9}$$

Sol: 4x2 + 6x + 9 = 4 $\left[x^2+\frac{3}{2}x+\frac{9}{4}\right]$

= 4$\left[\left(x+\frac{3}{4}\right)^2+\left(\frac{9}{4}-\frac{9}{16}\right)\right]$

= 4$\left[\left(x+\frac{3}{4}\right)^2+\frac{27}{16}\right]$

= 4$\left[\left(x+\frac{3}{4}\right)^2+\left(\sqrt{\frac{27}{16}}\right)^2\right]$

= 4$\left[\left(x+\frac{3}{4}\right)^2+q^2\right]$, where q = $\sqrt{\frac{27}{4}}$

$\therefore \quad$ I = $\int \frac{1}{4x^2+6x+9}$ 9x = $\int \frac{1}{4\left[\left(x+\frac{3}{4}\right)^2+q^2\right]}$ dx

Set $\quad$ x + $\frac{3}{4}$ = qt

$\therefore \quad$ dx = q dt

$\therefore$ I = $\frac{1}{4q^2}\int \frac{1}{t^2+1}$ q dt

= $\frac{1}{4q}\int \frac{1}{t^2+1}$ dt

= $\frac{1}{4q}$ $\tan^{-1}$ t

= $\frac{1}{4q}$ $\tan^{-1}\frac{x+3/4}{q}$

= $\frac{1}{\sqrt{27}}\tan^{-1}\left(\frac{x+3/4}{\frac{\sqrt{27}}{4}}\right)$

= $\frac{1}{\sqrt{27}}\tan^{-1}\left(\frac{4x+3}{\sqrt{27}}\right)$

Self-Check Exercise-2

Q.1 Evaluate the following integral

$$\int \frac{dx}{x^2-2x+5}$$

1.5 Method To Evaluate Integrals of The Type $\int \frac{\propto x+\beta}{ax^2+bx+c}$ dx, where $\propto \neq 0$ and $4ac - b^2 > 0$

$$\int \frac{\propto x+\beta}{ax^2+bx+c}\,dx = \int \frac{\frac{\propto}{2a}(2ax+b)+\left(\beta-\frac{b}{2a}\right)}{ax^2+bx+c}\,dx$$

[Making the numerator i.e. linear factor as a differential of the denominator i.e. quadratic]

$$= \int \frac{\frac{\propto}{2a}(2ax+b)}{ax^2+bx+c}\,dx + \int \frac{\beta-\frac{b\propto}{2a}}{ax^2+bx+c}\,dx$$

$$= \frac{\propto}{2a}\int \frac{2ax+b}{ax^2+bx+c}\,dx + \left(\beta-\frac{b\propto}{2a}\right)\int \frac{1}{ax^2+bx+c}\,dx$$

The first integral i.e. $\int \frac{2ax+b}{ax^2+bx+c}$ dx is of the type

$\int \frac{f'(x)}{f(x)}$ dx and = log |ax^2 + bx + c| $\left[\because \int \frac{f'(x)}{f(x)}dx = \log|f(x)|\right]$

The second integral i.e. $\int \frac{1}{ax^2+bx+c}$ dx can be evaluated by the method given in 1.3

Let us consider the following examples to clear the idea:

Example 7: Evaluate the following integral $\int \frac{dx}{2x^3+x}$

Sol: Let us factors 2x^3 + x

2x^3 + x = x (2x^2 + 1)

$\therefore \quad \frac{1}{2x^3+x} = \frac{1}{x(2x^2+1)} = \frac{A}{x} + \frac{Bx+c}{2x^2+1}$ .....(1)

Or 1 = A (2x^2 + 1) + (Bx + c)x ......(2)

[Taking L.C.M.]

Putting x = 0; 1 = A(0 + 1) + (B.0 + C) 0

Or 1 = A

Comparing coefficient of x^2 on both sides of (2), we get

1 = 2A + B

Or 1 = 2 + B

i.e. B = -1

Comparing coefficient of x on both sides of (2), we get

0 = c

Putting these values of A, B, C in (1) we get

$\frac{1}{2x^3+x} = \frac{1}{x} + \frac{-x+0}{2x^2+1}$

$= \frac{1}{x} - \frac{x}{2x^2+1}$

Or $\int \frac{1}{2x^3+x}$ dx - $\int \frac{1}{x}$ dx - $\int \frac{x}{2x^2+1}$ dx

= log |x| - $\frac{1}{4}\int \frac{4x}{2x^2+1}$ dx

$= \log|x| - \frac{1}{4}\log|2x^2 + 1|$

$= \log|x| - \frac{1}{4}\log(2x^2 + 1)$ [$\because$ $(2x^2 + 1) > 0$ for all real x]

Example 8: Evaluate the following integral

$$\int \frac{x^2 + x}{x^3 - x^2 + x - 1}dx$$

Sol: Let I = $\int \frac{x^2 + x}{x^3 - x^2 + x - 1}dx$

Now $x^3 - x^2 + x - 1 = x^2(x - 1) + 1(x - 1)$

$= (x - 1)(x^2 + 1)$

$\therefore \quad \frac{x^2 + x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$ .....(1)

Multiplying by L.C.M. $(x - 1)(x^2 + 1)$ we have

$x^2 + x = A(x^2 + 1) + (Bx + C)(x - 1)$

Put $x = 1$; $1 + 1 = A(1 + 1)$

Or $2A = 2$

Or $A = 1$

comparing coefficients of x^2,

$1 = A + B$

Or $1 = 1 + B$

$\therefore$ $B = 0$

comparing constant,

$0 = A - C$

Or $0 = 1 - C$

$\therefore$ $C = 1$

Putting these values of A, B, C in (1), we get

$$\frac{x^2 + x}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{1}{x^2+1}$$

$\therefore \quad \int \frac{x^2 + x}{(x-1)(x^2+1)}dx = \int \frac{1}{x-1}dx + \int \frac{1}{x^2+1}dx$

$= \log|x - 1| + \tan^{-1} x$

Example 9: Evaluate the following integral

$$\int \frac{x^5 - x^4 - 3x + 5}{(x^2+1)(x-1)^2}\,dx$$

Sol: The integrand $\frac{x^5 - x^4 - 3x + 5}{(x^2+1)(x-1)^2}$ is a fraction but not a proper fraction

Denominator = $(x^2 + 1)(x - 1)^2$

$= (x^2 + 1)(x^2 + 1 - 2x)$

$= x^4 + x^2 - 2x^3 + x^2 + 1 - 2x$

$= x^4 - 2x^3 + 2x^2 - 2x + 1$

Now we apply long division

$$x^4 - 2x^3 + 2x^2 - 2x + 1 \overline{)\; x^5 - x^4 \quad - 3x \quad + \quad 5 \;}\; x + 1$$

$$\begin{array}{l} x^5 - 2x^4 + 2x^3 - 2x^2 + x \\ - \quad + \quad - \quad + \quad - \\ \hline x^4 - 2x^3 + 2x^2 - 4x + 5 \\ x^4 - 2x^3 + 2x^2 - 2x + 1 \\ - \quad + \quad - \quad + \quad - \\ \hline \qquad\qquad\qquad -2x + 4 \end{array}$$

$\therefore \quad \frac{x^5 - x^4 - 3x + 5}{(x^2+1)(x-1)^2} = x + 1 + \frac{-2x+4}{(x^2+1)(x-1)^2}$

$\therefore \quad \int \frac{x^5 - x^4 - 3x + 5}{(x^2+1)(x-1)^2}\,dx = \int x\,dx + \int 1\,dx + \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$

$= \frac{x^2}{2} + x + \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$ ....(1)

Let us split $\frac{-2x+4}{(x-1)^2(x^2+1)}$ into partial fractions.

$\frac{-2x+4}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}$ ....(2)

Or $\quad -2x + 4 = A(x - 1)(x^2 + 1) + B(x^2 + 1) + (x + D)(x - 1)^2$ .....(3)

Put $\quad x = 1, -2 + 4 = A(1 - 1)(1 + 1) + B(1 + 1) + (C + D)(1 - 1)^2$

Or $2 = B.2$

Or $B = 1$

Comparing coefficients of x^3 in both sides of (3), we get

$$O = A + C \quad .....(4)$$

Comparing coefficients of x^2 in both sides of (3), we get

$$0 = -A + B + D - 2C \quad .....(5)$$

Comparing coefficients of x in both sides of (3), we get

$$-2 = A + C - 2D \quad ......(6)$$

Putting the value of (A + C) from (4) in (6), we get

$$-2 = -2D$$

Or $D = 1$

Putting D = 1 and B = 1 in (5), we have

$$0 = -A + 1 + 1 - 2C$$

Or $A + 2C = 2$

By (4) $A + C = 0$

Subtracting these two, we get C = 2

Putting C = 2 in (4), we get 0 = A + 2

Or $A = -2$

Putting these values of A, B, C, D in (2), we get

$$\frac{2x+4}{(x+1)^2(x^2+1)} = \frac{-2}{x-1}\frac{1}{(x-1)^2} + \frac{2x+1}{x^2+1}$$

$$\therefore \quad \int \frac{-2x+4}{(x+1)^2(x^2+1)}dx = \int \frac{-2}{x-1}dx + \int \frac{1}{(x-1)^2}dx + \int \frac{2x+1}{x^2+1}dx$$

$$= -2\int \frac{1}{x-1}dx + \int (x-1)^{-2}dx + \int \frac{2x}{x^2+1}dx + \int \frac{1}{x^2+1}dx$$

$$= -2\log|x-1| + \frac{(x-1)^{-1}}{-1} + \log|x^2+1| + \tan^{-1}x$$

Putting this value of $\int \frac{-2x+4}{(x+1)^2(x^2+1)}dx$ in (1), we have

$$\int \frac{x^5 - x^4 - 3x + 5}{(x+1)^2(x^2+1)}dx = \frac{x^2}{2} + x - 2\log|x-1| - \frac{1}{x-1}\log|x^2+1| + \tan^{-1}x$$

Example 10: Evaluate the following

$$\int \frac{1}{x(x^n + 1)} dx$$

Sol: Rule to integrate $\frac{1}{x(x^n + k)}$

Put $x^n = t$ and then resolve into partial fractions and integrate

Now let $I = \int \frac{1}{x(x^n + 1)} dx$

Set $x^n = t$ so that $nx^{n-1}\, dx = dt$

Or $dx = \frac{1}{n.x^{n-1}} dt$

$\therefore$ $I = \int \frac{1}{x(x^n + 1)} . \frac{1}{n\, x^{n-1}} dt$

$$= \frac{1}{n} \int \frac{1}{x^n (x^n + 1)} dt$$

$$= \frac{1}{n} \int \frac{1}{t(t + 1)} dt \qquad .....(1)$$

Let us now resolve $\frac{1}{t(t+1)}$ into partial fractions

$$\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1} \qquad .....(2)$$

Or 1 = A (t + 1) + Bt

Put t = 0, 1 = A (0 + 1) + B.0

Put t = 0, 1 = A (0 + 1) + B.0

Or A = 1

Put t = -1, 1 = A(-1 +1) + B (-1)

Or 1 = - B

Or B = - 1

Putting these values of A and B in (2), we get

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$$

$\therefore \quad \int \frac{1}{t(t+1)} dt = \int \frac{1}{t} dt - \int \frac{1}{t+1} dt$

$= \log |t| - \log |t + 1|$

$= \log \frac{|t|}{(t+1)}$

Putting this value of $\int \frac{1}{t(t+1)} dt$ in (1), we get

$I = \frac{1}{n} \log \frac{|t|}{(t+1)}$

$= \frac{1}{n} \log \left|\frac{t}{t+1}\right|$

Example 11: If F'(x) = $\frac{1}{x+x^{11}}$, x $\neq$ 0 then find F(x)

Sol: Now F'(x) = $\frac{1}{x+x^{11}}$, x $\neq 0$

$\therefore \quad F(x) = \int \frac{1}{x+x^{11}} dx$

$= \int \frac{1}{x\ (x^{10}+1)} dx \qquad \qquad ....(1)$

Put $\quad x^{10} = t$

Diff. w.r.t. x,

$10\ x^9 = \frac{dt}{dx}$

Or $\quad dx = \frac{dt}{10x^9}$

$\therefore$ (1) becomes

$F(x) = \int \frac{1}{x\ (x^{10}+1)} \frac{dt}{10x^9}$

$$= \frac{1}{10} \int \frac{1}{x^{10}(x^{10}+1)} \text{ dt}$$

$$= \frac{1}{10} \int \frac{1}{t(t+1)} \text{ dt}$$

$$= \frac{1}{10} \int \left(\frac{1}{t} - \frac{1}{t+1}\right) \text{dt}$$ [Resolving into partial fractions]

$$= \frac{1}{10} [\log |t| - \log |t + 1|]$$

$$= \frac{1}{10} \log \left|\frac{t}{t+1}\right|$$

$$= \frac{1}{10} \log \frac{x^{10}}{x^{10}+1}$$

Self-Check Exercise-3

Q.1 Given that

$$\frac{dI}{dy} = \frac{1}{y+y^3}, \text{ (y > 0)},$$

Q.2 Find I

Find a primitive of

$$\frac{x^2+1}{(x^2+2)(2x^2+1)}$$

Q.3 Evaluate

$$\int \frac{1}{x(x^5+1)} \text{ dx}$$

1.6 Summary:

We conclude this UNIT by Summarizing what we have covered in it.

1. Discussed the kinds of functions whose integrals be evaluated by the method of partial fractions. Examples given in Support of this.

2. Discussed in detail the method to evaluate integral of the type $\int \frac{1}{ax^2+bx+c}$ dx, where $4ac - b^2 > 0$, $a \neq 0$. Solved some examples to clarify this method

3. Discussed in detail the method to evaluate integrals of the type $\int \frac{\propto x+\beta}{ax^2+bx+c}$ dx, where $\propto \neq 0$ and 4ac - b^2 > 0. Some solved examples are given to clarify this method.

1.7 Glossary:

1. Integration by partial fractions involves breaking down a complex fraction into simpler fractions, allowing us to integrate each component individually.
2. Partial fractions are especially valuable when dealing with improper rational functions, where the degree of the numerator is greator than or equal to the degree of the denominator

1.8 Answer to Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\frac{1}{2a} \log \left|\frac{a+x}{a-x}\right|$

Ans. 2 $\frac{4}{25} \log \left|\frac{x-2}{x+3}\right| - \frac{1}{5}(x - 2)^{-1}$

Self-Check Exercise-2

Ans. 1 log y - $\frac{1}{2}$ log (1 + y^2)

Ans. 2 $\frac{1}{3}.\frac{1}{\sqrt{2}}$ are tax $\frac{x}{\sqrt{2}}$ + $\frac{1}{3}.\frac{1}{\sqrt{2}}$ are tan $\sqrt{2}$ x

Ans. 3 $\frac{1}{5} \log \left|\frac{x^5}{x^5+1}\right|$

1.9 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.

20.11 Terminal Questions

1. Evaluate the following integral

$$\int \frac{1}{(x-1)(2x-1)} dx, \; x > 1$$

2. Evaluate the following integral

$$\int \frac{x}{1-x^4} dx, \ x \neq \pm 1$$

3. Evaluate $\int \frac{dx}{x(x^2+1)^2}$, x > 0

4. Evaluate $\int \frac{dx}{3+2x+x^2}$

5. Evaluate $\int \frac{dx}{x^3+x^2+x}$

6. Evaluate $\int \frac{x}{x(x^4-1)} dx$

Unit - 2

Integration of Rational Functions

Structure

2.1 Introduction

Integration of rational functions is a fundamental concept in calculus that involves finding the antiderivative of a rational function. A rational function is the ratio of two polynomials, where both the numerator and the denominator are polynomial functions. The process of integrating rational functions is essential in various area of mathematics, science and engineering. Integration of rational functions involves decomposing a rational function into partial fractions. This process is helpful in simplifying complex rational functions, making them easier to integrate.

2.2 Learning Objectives

After studying this unit, you should be able to:-

- Define rational function.
- Find the standard result

$$\int \frac{dx}{x^2-a^2}, \ (x^2 > a^2) = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c$$

- Find the standard result

$$\int \frac{dx}{x^2-a^2}, \ (x^2 < a^2) = \frac{1}{2a} \log\left|\frac{x+a}{x-a}\right| + c$$

2.3 Rational Function

A function is said to be rational function if it is of the form $\frac{f(x)}{g(x)}$, where f(x) is non-zero.

For rational functions. A rational function is free from logarithmic terms, trigonometric and inverse trigonometric functions and the logarithmic functions.

The antiderivative of a rational function may or may not be a rational function.

Let f(x) = $\frac{1}{x}$, (x $\neq$ 0)

f(x) is a rational function of x.

But antiderivative of f(x) = $\int \frac{1}{x}$ dx = log |x|; which is a logarithmic function and therefore is not a rational function.

If the integrand is a rational function of the form $\frac{f(x)}{g(x)}$ and the degree of the numerator is equal or greater than that of the denominator, then before integration we divide the numerator by the denominator until the degree of the remainder is less than that of the denominator. A rational function, in which degree of numerator is less than that of denominator, is called a proper rational function.

2.4 Two Standard Results

Result 1:

$$\int \frac{dx}{x^2 - a^2}, \ (x^2 > a^2) = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + c$$

Result 2:

$$\int \frac{dx}{x^2 - a^2}, \ (x^2 < a^2) = \frac{1}{2a} \log \left| \frac{x+a}{x-a} \right| + c$$

Proof:

Result 1:

Let $\quad I = \int \frac{dx}{x^2 - a^2} dx$

$$= \int \left[\frac{1}{(x-a)(x+a)} \right] dx$$

$$= \int \left[\frac{1}{(x-a)(a+a)} + \frac{1}{(-a-a)(x+a)} \right] dx$$

$$= \int \left[\frac{1}{2a(x-a)} + \frac{1}{(-2a)(x+a)}\right] dx$$

$$= \frac{1}{2a} \int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx$$

$$= \frac{1}{2a} \left[\log|x-a| - \log|x+a|\right] + c$$

$$= \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$$

Result 2:

Let I = $\int \frac{1}{a^2 - x^2} dx$

$$= \int \frac{1}{(a-x)(a+x)} dx$$

$$= \int \left[\frac{1}{(a-x)(a+a)} + \frac{1}{(a+a)(a+x)}\right] dx$$

$$= \int \left[\frac{1}{2a(a-x)} + \frac{1}{2a(a+x)}\right] dx$$

$$= \frac{1}{2a} \int \left[\frac{1}{a-x} + \frac{1}{a+x}\right] dx$$

$$= \frac{1}{2a} \left[\frac{\log|a-x|}{-1} + \frac{\log|a+x|}{1}\right] + c$$

$$= \frac{1}{2a} [-\log|a-x| + \log|a+x|] + c$$

$$= \frac{1}{2a} \log \left|\frac{x+a}{x-a}\right| + c$$

Let us consider the following examples to clear the idea:-

Example 1: Find $\int \frac{2x+1}{18-4x-x^2} dx$

Sol. : Let I = $\int \frac{2x+1}{18-4x-x^2} dx$

$$= \int \frac{-(-4-2x)-3}{18-4x-x^2} dx$$

$$= - \int \frac{-4-2x}{18-4x-x^2} dx - 3 \int \frac{1}{18-4x-x^2} dx$$

$$= - \log \left|18-4x-x^2\right| - 3 \int \frac{1}{18-(x^2+4x)} dx$$

$$= - \log \left|18-4x-x^2\right| - 3 \int \frac{1}{22-(x^2+4x+4)} dx$$

$$= - \log \left|18-4x-x^2\right| - 3 \int \frac{1}{\left(\sqrt{22}\right)^2-(x+2)^2} dx$$

$$= - \log \left|18-4x-x^2\right| - 3 \frac{1}{2\sqrt{22}} \log \left|\frac{\sqrt{22}+x+2}{\sqrt{22}-x-2}\right| + c$$

$$= - \log \left|18-4x-x^2\right| - \frac{3}{2\sqrt{22}} \log \left|\frac{\sqrt{22}+x+2}{\sqrt{22}-x-2}\right| + c$$

Example 2: Evaluate $\int \frac{x+3}{x^2-2x-5} dx$

Sol: Let I = $\int \frac{x+3}{x^2-2x-5} dx$

$$= \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5} dx$$

$$= \frac{1}{2} \int \frac{2x-2}{x^2-2x-5} dx + 4 \int \frac{1}{x^2-2x-5} dx$$

$\therefore$ $I = \frac{1}{2} I_1 + 4 I_2$ .....(1)

where $I_1 = \int \frac{2x-2}{x^2-2x-5} dx$

$$= \log \left|x^2-2x-5\right| + c_1$$

and $\quad I_2 = \int \dfrac{1}{x^2-2x-5}\,dx$

$$= \int \frac{1}{(x^2-2x+1)-6}\,dx$$

$$= \int \frac{1}{(x-1)^2-(\sqrt{6})^2}\,dx$$

$$= \frac{1}{2\sqrt{6}} \log \left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + c$$

$\therefore \quad I_2 = \dfrac{1}{2\sqrt{6}} \log \left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + c$

$\therefore$ from (1), we get

$$I = \frac{1}{2} \log \left|x^2-2x-5\right| + \frac{2}{\sqrt{6}} \log \left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right| + c$$

Example 3: Find $\int \dfrac{x\,dx}{(x+1)(x+2)}$

Sol: Let $I = \int \left[\dfrac{x}{(x+1)(x+2)}\right]dx$

$$= \int \left[\frac{-1}{(x+1)(-1+2)} + \frac{-2}{(-2+1)(x+2)}\right]dx$$

$$= \int \left[\frac{-1}{x+1} + \frac{2}{x+2}\right]dx$$

$= -\log|x+1| + 2\log|x+2| + c$

Example 4: Evaluate $\int \dfrac{3x-1}{(x-1)(x-2)(x-3)}\,dx$

Sol: Let $I = \int \left[\dfrac{3x-1}{(x-1)(x-2)(x-3)}\right]dx$

$$= \int \left[\frac{3-1}{(x-1)(1-2)(1-3)} + \frac{6-1}{(2-1)(x-2)(2-3)} + \frac{9-1}{(3-1)(3-2)(x-3)}\right]dx$$

$$= \int \left[\frac{1}{x-1} - \frac{5}{x-2} + \frac{4}{x-3}\right] dx$$

$$= \int \frac{1}{x-1} dx - 5\int \frac{1}{x-2} dx + 4\int \frac{1}{x-3} dx$$

= log |x - 1| - 5 log |x - 2| + 4 |x - 3| + c

Example 5: Evaluate $\int \frac{2x}{x^2+3x+2} dx$

Sol: Let I = $\int \frac{2x}{x^2+3x+2} dx$

$$= \int \left[\frac{-2}{(x+1)(-1+2)} + \frac{-4}{(-2+1)(x+2)}\right] dx$$

$$= \int \left[\frac{-2}{x+1} + \frac{4|}{x+2}\right] dx$$

$$= -2 \int \frac{1}{x+1} dx + 4\int \frac{1}{x+2} dx$$

= -2 log |x + 1| + 4 log |x + 2| | c

Example 6: Find $\int \frac{1}{x-x^3} dx$

Sol: Let I = $\int \frac{1}{x-x^3} dx$

$$= \int \frac{1}{x(1-x^2)} dx$$

$$= \int \left[\frac{1}{x(1-x)(1+x)}\right] dx$$

$$= \int \left[\frac{1}{x(1-0)(1+0)} + \frac{1}{(1)(1-x)(1+1)} + \frac{1}{(-1)(1+1)(1+x)}\right] dx$$

$$= \int \left[\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right] dx$$

= log |x| + $\frac{1}{2}$ $\frac{\log|1-x|}{-1}$ - $\frac{1}{2}$ log |1 + x| + c

= log |x| $\frac{1}{2}$ log |1 - x| - $\frac{1}{2}$ log |1 + x| + c

Example 7: Find $\int \frac{x^3+x+1}{x^2-1}$ dx

Sol: Let I = $\int \frac{x^3+x+1}{x^2-1}$ dx .....(1)

Here degree of numerator is greater than the degree of denominator.

$$\therefore \quad \begin{array}{r|l} & x \\ \hline 1 & x^3+x+1 \\ & x^3-x \\ & -\quad + \\ \hline & 2x+1 \end{array}$$

$\therefore$ (1) becomes after division as

$$I = \int \left[x + \frac{2x+1}{x^2-1}\right] dx$$

$$= \int \left[x + \frac{2x+1}{(x-1)(x+1)}\right] dx$$

$$= \int \left[x + \frac{2x+1}{(x-1)(x+1)} + \frac{-2+1}{(-1-1)(x+1)}\right] dx$$

$$= \int \left[x + \frac{3}{2(x-1)} + \frac{1}{2(x+1)}\right] dx$$

$$= \frac{x^2}{2} + \frac{3}{2} \log |x - 1| + \frac{1}{2} \log |x + 1| + c$$

Example 8: Evaluate $\int \frac{2x^3+3}{x^2-x-2}$ dx

Sol: Let I = $\int \frac{2x^3+3}{x^2-x-2}$ dx .....(1)

Here degree of numerator is greater than the degree of denominator

$$
\begin{array}{r|l}
 & \quad 2x+2 \\
\cline{2-2}
x^2-x-2 & 2x^3+3 \\
 & 2x^3-2x^2-4x \\
 & -\quad+\quad\quad+ \\
\cline{2-2}
 & 2x^2+4x+3 \\
 & 2x^2-2x-4 \\
 & -\quad+\quad+ \\
\cline{2-2}
 & \quad\quad 6x+7
\end{array}
$$

$\therefore$

$\therefore$ (1) becomes after division as

$$I = \int \left[2x+2+\frac{6x+7}{x^2-x-2}\right]dx$$

$$= \int \left[2x+2+\frac{6x+7}{(x+1)(x-2)}\right]dx$$

$$= \int \left[2x+2+\frac{-6+7}{(x+1)(-1-2)}+\frac{12+7}{(2+1)(x-2)}\right]dx$$

$$= \int \left[2x+2-\frac{1}{3(x+1)}+\frac{19}{3(x-2)}\right]dx$$

$$= x^2 + 2x - \frac{1}{3} \log|x+1| + \frac{19}{3}\log|x-2| + c$$

Example 9: Evaluate

$$\int \frac{x}{1-x^4}\,dx$$

Sol: Let $I = \int \frac{x}{1-x^4}\,dx$

Put $x^2 = y$

$\therefore$ $2xdx = dy$

$\therefore$ $I = \frac{1}{2}\int \frac{dy}{1-y^2}$

$$= \frac{1}{2}.\frac{1}{2}\log\left|\frac{1+y}{1-y}\right| + c$$

$= \frac{1}{4} \log \left|\frac{1+x^2}{1-x^2}\right| + c$

Example 10: Find $\int \frac{dx}{x(x^4+1)}$

Sol: Let I = $\int \frac{dx}{x(x^4+1)}$

$= \int \frac{x^3}{x^4(x^4+1)} dx$

Put $x_4 = y$

$\therefore$ $4x^3 dx = dy$

$\Rightarrow$ $x^3 dx = \frac{1}{4} dy$

$\therefore$ $I = \frac{1}{4} \int \frac{1}{y(y+1)} dy$

$= \frac{1}{4} \int \left[\frac{1}{y(0+1)} + \frac{1}{(-1)(y+1)}\right] dy$

$= \frac{1}{4} \int \left[\frac{1}{y} - \frac{1}{y+1}\right] dy$

$= \frac{1}{4} [\log |y| - \log |y + 1|] + c$

$= \frac{1}{4} \log \left|\frac{y}{y+1}\right| + c$

$= \frac{1}{4} \log \left|\frac{x^4}{x^4+1}\right| + c$

Example 11: Find $\int \frac{x^3}{x^4+3x^2+2} dx$

Sol: Let I = $\int \frac{x^3}{x^4+3x^2+2} dx$

$$= \int \frac{x^2 . x\, dx}{x^4 + 3x^2 + 2}$$

Put $x^2 = t$, $\therefore$ $2x\,dx = dt$

$\Rightarrow$ $x\,dx = \frac{1}{2}\,dt$

$\therefore$ $I = \frac{1}{2} \int \frac{t\, dt}{t^2 + 36 + 2}$

$$= \frac{1}{2} \int \frac{t\, dt}{(t+1)(t+2)}$$

$$= \frac{1}{2} \int \left[\frac{-1}{(t+1)(t+2)} + \frac{-2}{(-2+1)(t+2)} \right] dt$$

$$= \frac{1}{2} \int \left[\frac{-1}{t+1} + \frac{2}{t+2} \right] dt$$

$$= \frac{1}{2} \int [- \log |t + 1| + 2 \log |t + 2|] + c$$

$$= - \frac{1}{2} \log |t + 1 + \log |t + 2| + c$$

$$= - \frac{1}{2} \log (x^2 + 1) + \log (x^2 + 2) + c$$

$$= - \frac{1}{2} \log (x^2 + 1) + \log (x^2 + 2) + c$$

Example 12: Evaluate $\int \frac{\sin x}{(1-\cos x)(2-\cos x)}\,dx$

Sol: Let $I = \int \frac{\sin x}{(1-\cos x)(2-\cos x)}\,dx$

Put $-\cos x = y$

$\therefore$ $\sin x\,dx = dy$

$\therefore$ $I = \int \frac{1}{(1+y)(2+y)}\,dy$

$$= \int \left[\frac{1}{(1+y)(2-1)} + \frac{1}{(1-2)(2+y)}\right] dy$$

$$= \int \left[\frac{1}{1+y} - \frac{1}{2+y}\right] dy$$

= log |1 + y| - log |2 + y| + c

$$= \log \left|\frac{1+y}{2+y}\right| + c$$

$$= \log \left|\frac{1-\cos x}{2-\cos x}\right| + c$$

Example 13: Evaluate $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$

Sol: Let I = $\int \frac{\sin 2x}{(1+\sin x)(2+\sin x)} dx$

$$= \int \frac{2\sin x \cos x}{(1+\sin x)(2+\sin x)} dx$$

$$= 2\int \frac{\sin x \cos x}{(1+\sin x)(2+\sin x)} dx$$

Put sin x = y,

∴ cos x dx = dy

∴ $I = 2\int \frac{y\,dy}{(1+y)(2+y)}$

$$= 2\int \left[\frac{y}{(y+2)(y+2)}\right] dy$$

$$= 2\int \left[\frac{-1}{(y+1)(-1+2)} + \frac{-2}{(-2+1)(y+2)}\right] dy$$

$$= 2\int \left[-\frac{1}{y+1} + \frac{2}{y+2}\right] dy$$

= 2 [- log |y + 1| + 2 log |y + 2| + c

= 2 [- log |sin x + 1| + 2 log |sin x + 2|] + c

Example 14: Evaluate $\int \dfrac{1}{\sin x(3+2\cos x)}$ dx

Sol: Let I = $\int \dfrac{1}{\sin x(3+2\cos x)}$ dx

$$= \int \frac{\sin x}{\sin^2 x(3+2\cos x)} dx$$

$$= \int \frac{\sin x dx}{(1-\cos^2 x)(3+2\cos x)}$$

$$= \int \frac{\sin x dx}{(1-\cos x)(1+\cos x)(3+2\cos x)}$$

Put cos x = t

$\therefore$ - sin x dx = dt

$\Rightarrow$ sin x dx = - dt

$\therefore$ I = $-\int \dfrac{dt}{(1-t)(1+t)(3+2t)}$

$$= -\int \left[\frac{1}{(1+1)(1+2)(3+2)} + \frac{1}{(1+1)(1+t)(3-2)} + \frac{1}{\left(1+\frac{3}{2}\right)\left(1-\frac{3}{2}\right)(3+2t)} \right] dt$$

$$= -\int \left[\frac{1}{10(1-t)} + \frac{1}{2(1+t)} - \frac{4}{5(3+2t)} \right] dt$$

$$= -\frac{1}{10}\int \frac{1}{1-t} dt - \frac{1}{2}\int \frac{1}{1-t} dt + \frac{4}{5}\int \frac{1}{3+2t} dt$$

$$= -\frac{1}{10}\frac{\log|1-t|}{-1} - \frac{1}{2}\log|1+t| + \frac{4}{5}\frac{\log|3+2t|}{2} + c$$

$$= -\frac{1}{10}\log|1-t| - \frac{1}{2}\log|1+t| + \frac{2}{5}\log|3+2t| + c$$

$$= \frac{1}{10}\log|1-\cos x| - \frac{1}{2}\log|1+\cos x| + \frac{2}{5}\log|3+2\cos x|\, x + c$$

Example 15: Evaluate $\int \dfrac{x^2+x}{x^4-9}$ dx

Sol: Let I = $\int \frac{x^2+x}{x^4-9}$ dx

= $\int \frac{x^3}{x^4-9}$ dx + $\int \frac{x}{x^4-9}$ dx

$\therefore$ I = $I_1 + I_2$ .....(1)

where $I_1 = \int \frac{x^2}{x^4-9}$ dx

= $\frac{1}{4} \int \frac{4x^3}{x^4-9}$ dx

= $\frac{1}{4}$ log $|x^4 - 9| + c_1$ .....(2)

and $I_2 = \int \frac{x}{x^4-9}$ dx

Put $x^2 = y$

$\therefore$ 2x dx = dy

$\Rightarrow$ xdx = $\frac{1}{2}$ dy

$\therefore$ $I_2 = \frac{1}{2} \int \frac{1}{y^2-(3)^2}$ dy

= $\frac{1}{2} \frac{1}{2\times 3}$ log $\left|\frac{y-3}{y+3}\right| + c_2$

= $\frac{1}{12}$ log $\left|\frac{x^2-3}{x^2+3}\right| + c_2$ .....(3)

From (1), (2), (3), we get

I = $\frac{1}{4}$ log $|x^4 - 9|$ + $\frac{1}{12}$ log$\left|\frac{x^2-3}{x^2+3}\right|$ + c

where $c = c_1 + c_2$ is constant.

Example 16: Find $\int \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ dx

Sol: Let I = $\int \dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$ dx

[Note:- If numerator and denominator both contain only even powers of x, then we put x^2 = y to receive integrand into partial fractions and then put y = x^2 and integrate]

Now integrand = $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$

$$= \frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} \text{ - where } y = x^2$$

$$= 1 + \frac{(-3+1)(-3+2)}{(y+3)(-3+4)} + \frac{(-4+1)(-4+2)}{(-4+3)(y+4)}$$

$$= 1 + \frac{2}{y+3} - \frac{6}{y+4}$$

$$= 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}$$

$\therefore$ $$I = \int \left[1+\frac{2}{x^2+3}-\frac{6}{x^2+4}\right] dx$$

$$= \int 1dx + 2\int \frac{1}{x^2+\left(\sqrt{3}\right)^2} dx - 6\int \frac{1}{x^2+\left(2\right)^2} dx$$

$$= x + 2.\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 6.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + c$$

Example 17: Find $\int \dfrac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

Sol: [Note:- If fractional powers of only one linear occur in the integrand, then we put linear = I^s, where s is the L.C.M. of the denominator of the fractional indices of the linear]

Let $\quad I = \int \dfrac{dx}{x^{\frac{1}{2}}+x^{\frac{1}{3}}}$

Here fractional powers of only one linear x occur in integrand.

The fractional indices of x are $\frac{1}{2}$, $\frac{1}{3}$

L.C.M. of the denominators 2 and 3 is 6.

$\therefore$ we put $x = t^6$

$\therefore$ $dx = 6t^5\, dt$

$\therefore$ $I = \dfrac{6t^5\, dt}{t^3 + t^2}$

$$= 6\int \frac{t^5}{t^3+t^2}\, dt$$

$$= 6\int \frac{t^3}{t+1}\, dt$$

Here degree of numerator is more than degree of denominator.

$\therefore$

$$
\begin{array}{rl}
 & t^2 - t + 1 \\
t+1 \big) & \overline{t^3 \qquad\qquad} \\
 & t^3 + t^2 \\
 & - \quad - \\
\hline
 & -t^2 \\
 & -t^2 - t \\
 & + \quad + \\
\hline
 & t \\
 & t + 1 \\
 & - \quad - \\
\hline
 & \qquad -1
\end{array}
$$

$\therefore$ $I = 6\displaystyle\int \frac{t^3}{t+1}\, dt = 6\int \left[t^2 - t + 1 - \frac{1}{t+1}\right] dt$

$$= 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] dt + c$$

$$= 6\left[\frac{x^{\frac{1}{2}}}{3} - \frac{x^{\frac{1}{3}}}{2} + x^{\frac{1}{6}} - \log|t^{\frac{1}{6}} + 1|\right] + c$$

$= 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6 \log |x^{\frac{1}{6}} + 1| + c$

Example 18: Evaluate $\int \frac{x^2+1}{(x-1)^2(x+3)}$ dx

Sol: Let I = $\int \frac{x^2+1}{(x-1)^2(x+3)}$ dx

Put $\frac{x^2+1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\therefore$ $x^2 + 1 = A (x - 1)^2 + B (x - 1) (x + 3) + c (x + 3)$ .....(1)

Put $x + 3 = 0$ or $x = -3$ in (1), we have

$9 + 1 = A(-3 -1)^2 + 0 + 0$

$\Rightarrow$ $10 = 10A$

$\Rightarrow$ $A = \frac{5}{8}$

Put $x - 1 = 0$ or $x = 1$ in (1), we have

$1 + 1 = 0 + 0 + c (c_1 + 3)$

$\Rightarrow$ $2 = 4C$

$\Rightarrow$ $C = \frac{1}{2}$

(1) can be written as

$x^2 + 1 \equiv A (x^2 - 2x + 1) + B (x^2 + 2x - 3) + C (x + 3)$ .....(2)

Equating coefficients of x^2 in (2), we get

$1 = A + B$

$\Rightarrow$ $1 = \frac{5}{8} + B$

$\Rightarrow$ $B = \frac{3}{8}$

$\therefore$ $\frac{x^2+1}{(x-1)^2(x+3)} \equiv \frac{5}{8(x+3)} + \frac{3}{8(x-1)} + \frac{1}{2(x-1)^2}$

$\therefore \quad I = \frac{5}{8}\int \frac{1}{x+3}dx + \frac{3}{8}\int \frac{1}{x-1}dx + \frac{1}{2}\int (x-1)^{-2}\,dx$

$= \frac{5}{8}\log|x+3| + \frac{3}{8}\log|x-1| + \frac{1}{2}\frac{(x-1)^{-1}}{-1} + c$

$\therefore \quad I = \frac{5}{8}\log|x+3| + \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + c$

Example 19: Evaluate $\int \frac{1}{1-x^3}dx$

Sol: Let $I = \int \frac{1}{1-x^3}dx$

$= \int \frac{1}{(1-x)(1+x+x^2)}dx$

Put $\quad \frac{1}{(1-x)(1+x+x^2)} \equiv \frac{A}{1-x} + \frac{Bx+C}{1+x+x^2}$

Multiplying both sides by $(1 - x)(1 + x + x^2)$, we get

$1 \equiv A(1 + x + x^2) + (Bx + C)(1 - x)$

$\therefore \quad 1 \equiv A(x^2 + x + 1) + Bx(1 - x) + C(1 - x) \quad$ .....(1)

Put $\quad 1 - x = 0 \quad$ or $\quad x = 1$ in (1), we get

$1 = A(1 + 1 + 1) + 0 + 0$

$\Rightarrow \quad A = \frac{1}{3}$

(1) can be written as

$1 \equiv A(x^2 + x + 1) + B(-x^2 + x) + C(-x + 1) \quad$ ....(2)

Equating coefficient of x^2 in (2), we have

$0 = A - B$

$\Rightarrow \quad 0 = \frac{1}{3} - B$

$\Rightarrow \quad B = \frac{1}{3}$

Equating coefficient of constant term in (2), we have

$1 = A + C$

$= \frac{1}{3} + C$

$\Rightarrow \quad C = \frac{2}{3}$

$$\therefore \quad \frac{1}{(1-x)(1+x+x^2)} \equiv \frac{\frac{1}{3}}{1-x} + \frac{\frac{x}{3}+\frac{2}{3}}{1+x+x^2}$$

$$\text{Or} \quad \frac{1}{1-x^3} \equiv \frac{1}{3(1-x)} + \frac{1}{3}\left(\frac{x+2}{1+x+x^2}\right)$$

$$\therefore \quad I = \frac{1}{3}\int \frac{1}{1-x}\,dx + \frac{1}{3}\int \frac{x+2}{1+x+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}\,dx + \frac{1}{3}\int \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{1+x+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}\,dx + \frac{1}{6}\int \frac{2x+1}{1+x+x^2}\,dx + \frac{1}{2}\int \frac{1}{1+x+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}\,dx + \frac{1}{6}\int \frac{2x+1}{1+x+x^2}\,dx + \frac{1}{2}\int \frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1-x}\,dx + \frac{1}{6}\int \frac{2x+1}{1+x+x^2}\,dx + \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$$

$$= \frac{1}{3}\,\frac{\log|1-x|}{-1} + \frac{1}{6}\log|1+x+x^2| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + c$$

$$\therefore \quad I = -\frac{1}{3}\log|1-x| + \frac{1}{6}\log|1+x+x^2| + \frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2x+1}{\sqrt{3}}\right] + c$$

Example 20: Find $\int \frac{x^3-1}{x^3+x}\,dx$

Sol: Let I = $\int \frac{x^3-1}{x^3+x}dx$

Here degree of numerator is equal to degree of denominator

$$\begin{array}{r|l} & 1 \\ x^3+1 & \overline{x^3-1} \\ \therefore & x^3+x \\ & -\quad - \\ \hline & -x-1 \end{array}$$

$\therefore$ I = $\int \left[1-\frac{x+1}{x^3+x}\right]dx$

$= \int 1dx - \int \frac{x+1}{x(x^2+1)}dx$ .....(1)

Put $\frac{x+1}{x(x^2+1)} \equiv \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\therefore$ $x + 1 \equiv A(x^2 + 1) + Bx^2 + Cx$ .....(2)

Put x = 0 in (2), we have

$0 + 1 = A(0 + 1) + 0 + 0$

$\Rightarrow$ A = 1

Equating coefficient of x^2 in (2), we get

$0 = A + B = 1 + B$

$\Rightarrow$ B = -1

Equating coefficient of x in (2), we get

$1 = C$

$\therefore$ $\frac{x+1}{x(x^2+1)} \equiv \frac{1}{x} + \frac{-x+1}{x^2+1}$

$\therefore$ from (1), we have

I = $\int 1dx - \int \left[\frac{1}{x}+\frac{-x+1}{x^2+1}\right]dx$

$= \int 1dx - \int \frac{1}{x}dx + \frac{1}{2}\int \frac{2x}{x^2+1}dx - \int \frac{1}{x^2+1}dx$

$\therefore$ I = x - log |x| + $\frac{1}{2}$ log (x^2 + 1) - $\tan^{-1}$ x + c

Example 21: Evaluate $\int \frac{x^2+1}{x^4-3x^2+1}$ dx

Sol: Let I = $\int \frac{x^2+1}{x^4-3x^2+1}$ dx

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{x^2-3+\frac{1}{x^2}} \, dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}-3}$$

Put x - $\frac{1}{x}$ = y, $\therefore$ $\left(1+\frac{1}{x^2}\right)$dx = dy

Also x^2 + $\frac{1}{x^2}$ - 2 = y^2 $\Rightarrow$ x^2 + $\frac{1}{x^2}$ = y^2 + 2

$$\therefore \quad I = \int \frac{dy}{y^2+2-3}$$

$$= \int \frac{1}{y^2-(1)^2} \, dy$$

$$= \frac{1}{2} \log \left|\frac{y-1}{y+1}\right| + c$$

$$= \frac{1}{2} \log \left|\frac{x-\frac{1}{x}-1}{x-\frac{1}{x}+1}\right| + c$$

$$= \frac{1}{2} \log \left|\frac{x^2-x-1}{x^2+x-1}\right| + c$$

Example 22: Evaluate $\int \sqrt{\tan x}$ dx

Sol. : Let $I = \int \sqrt{\tan x}\, dx$

Put $\sqrt{\tan x} = y$

$\therefore$ $\tan x = y^2$

$\therefore$ $\sec^2 x\, dx = 2y dy$

$\Rightarrow$ $(1 + \tan^2 x)dx = 2y dy$

$\Rightarrow$ $(1 + y^4)dx = 2y dy$

$\Rightarrow$ $dx = \frac{2y}{1+y^4} dy$

$\therefore$ $I = \int y.\frac{2y}{1+y^4} dy$

$= \int \frac{2y^2}{y^4+1} dy$

$= \int \frac{(y^2+1)+(y^2-1)}{y^4+1} dy$

$= \int \frac{y^2+1}{y^4+1} dy + \int \frac{y^2-1}{y^4+1} dy$

$\therefore$ $I = I_1 + I_2$ ....(1)

where $I_1 = \int \frac{y^2+1}{y^4+1} dy$

$= \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}} dy$

Put $y - \frac{1}{y} = t$, $\therefore \left(1+\frac{1}{y^2}\right) dy = dt$

Also $y^2 + \frac{1}{y^2} - 2 = t^2 \Rightarrow y^2 + \frac{1}{y^2} = t^2 + 2$

$\therefore$ $I_1 = \int \frac{dt}{t^2+2} = \int \frac{1}{t^2+\left(\sqrt{2}\right)^2} dt$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c_1$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\left(y - \frac{1}{y}\right)}{\sqrt{2}}\right] + c_1$$

$$\therefore \quad I_1 = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{y^2 - 1}{y\sqrt{2}}\right] + c_1 \qquad \text{....(2)}$$

$$I_2 = \int \frac{y^2 - 1}{y^4 + 1} dy = \int \frac{1 - \frac{1}{y^2}}{y^2 + \frac{1}{y^2}} dy$$

Put $\quad y + \frac{1}{y} = z, \quad \therefore \quad \left(1 - \frac{1}{y^2}\right) dy = dz$

Also $\quad y^2 + \frac{1}{y^2} + 2 = z^2 \Rightarrow y^2 + \frac{1}{y^2} = z^2 - 2$

$$\therefore \quad I_2 = \int \frac{dz}{z^2 - 2} = \int \frac{1}{z^2 - \left(\sqrt{2}\right)^2} dz$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{z - \sqrt{2}}{z + \sqrt{2}}\right| + c_2$$

$$= \frac{1}{2\sqrt{2}} \log\left|\frac{y + \frac{1}{y} - \sqrt{2}}{y + \frac{1}{y} + \sqrt{2}}\right| + c_2$$

$$\therefore \quad I_2 = \frac{1}{2\sqrt{2}} \log\left|\frac{y^2 - y\sqrt{2} + 1}{y^2 + y\sqrt{2} + 1}\right| + c_2 \qquad \text{.......(3)}$$

From (1), (2), (3), we get

$$I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{y^2 - 1}{y\sqrt{2}}\right] + \frac{1}{2\sqrt{2}} \log\left|\frac{y^2 - y\sqrt{2} + 1}{y^2 + y\sqrt{2} + 1}\right| + c$$

Example 23: Evaluate $\int \frac{dx}{\cos^4 x + \sin^4 x}$ dx

Sol: Let I = $\int \frac{dx}{\cos^4 x + \sin^4 x}$ dx

$$= \int \frac{\frac{1}{cos^4 x}}{1 + \frac{\sin^4 x}{cos^4 x}}$$

$$= \int \frac{\sec^4 x}{1 + \tan^4 x} \text{ dx}$$

$$= \int \frac{\sec^2 x.\sec^2 x}{1 + \tan^4 x} \text{ dx}$$

$$= \int \frac{1 + \tan^2 x}{1 + \tan^4 x} \sec^2 x \text{ dx}$$

Put tan x = y, $\therefore$ $\sec^2 x$ dx = dy

$\therefore$ I = $\int \frac{1 + y^2}{1 + y^4}$

$$= \int \frac{\frac{1}{y^2} + 1}{\frac{1}{y^2} + y^2} \text{ dy}$$

$$= \int \frac{\left(\frac{1}{y^2} + 1\right)}{\left(y - \frac{1}{y}\right)^2 + 2} \text{ dy}$$

Put y - $\frac{1}{y}$ = t

$\therefore$ $\left(1 + \frac{1}{y^2}\right)$ dy = dt

$\therefore$ I = $\int \frac{1}{t^2 + 2}$ dt

$$= \int \frac{1}{t^2 + \left(\sqrt{2}\right)^2} dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y - \frac{1}{y}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{y^2 - 1}{\sqrt{2}y}\right] + c$$

$$\therefore \quad I = \frac{1}{\sqrt{2}} \tan^{-1}\left[\frac{\tan^2 x - 1}{\sqrt{2}\tan x}\right] + c$$

Self-Check Exercise

Q.1 Evaluate $\int \frac{dx}{3x^2 + 13x - 10}$

Q.2 Evaluate $\int \frac{2x - 3}{(x^2 - 1)(2x + 3)} dx$

Q.3 Find $\int \frac{1 - x^2}{x(1 - 2x)} dx$

Q.4 Evaluate $\int \frac{e^2}{e^{2x} - 4} dx$

Q.5 Find $\int \frac{1}{x(x^4 - 1)} dx$

Q.6 Evaluate $\int \frac{\cos x}{(1 - \sin x)(2 - \sin x)} dx$

Q.7 Evaluate $\int \frac{1}{(x^2 + 1)(x^2 + 4)} dx$

Q.8 Evaluate $\int \frac{x^2}{x^4 + 1} dx$

2.5 Summary:

We conclude this UNIT by Summarizing what we have covered in it:-

1. Defined rational function
2. Find the two standard integral results i.e.

$$\int \frac{dx}{x^2 - a^2}, (x^2 > a^2) = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$$

and $$\int \frac{dx}{a^2 - x^2}, (x^2 < a^2) = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$$

3. Attempted some examples related to these two standard results.

2.6 Glossary:

1. A function is said to be rational function if it is of the form $\frac{f(x)}{g(x)}$, where f(x) and g(x) are two polynomials in x and g(x) is non-zero.
2. $\int \frac{dx}{x^2 - a^2}, (x^2 > a^2) = \frac{1}{2a} \log \left|\frac{x-a}{x+a}\right| + c$
3. $\int \frac{dx}{a^2 - x^2}, (x^2 < a^2) = \frac{1}{2a} \log \left|\frac{a+x}{a-x}\right| + c$

2.7 Answer to Self-Check Exercise

Self-Check Exercise

Ans. 1 $\frac{1}{17}\log \left|\frac{3x-2}{3x+15}\right| + c$

Ans. 2 $-\frac{1}{10}\log |x - 1| + \frac{5}{2}\log |x + 1| - \frac{12}{5}\log |2x + 3| + c$

Ans. 3 $\frac{x}{2} + \log |x| - \frac{3}{4} \log |1 - 2x| + c$

Ans. 4 $\frac{1}{4}\log \left|\frac{e^x - 2}{e^x + 2}\right| + c$

Ans. 5 $\frac{1}{4}\log\left|\frac{x^4 - 1}{x^4}\right| + c$

Ans. 6 $\log \left|\frac{2-\sin x}{1-\sin x}\right| + c$

Ans. 7 $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\left(\frac{x}{2}\right) + c$

Ans. 8 $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right) + \frac{1}{2\sqrt{2}}\log\left|\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}\right| + c$

2.8 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.
3. Shanti Narayan and P.K. Mittal, Integral Calculus, S. Chand Publishing, New Delhi, 2005.

2.9 Terminal Questions

1. Evaluate $\int \log(1 - x^2)\, dx$
2. Find $\int \frac{x^2}{)x-1)(x-2)(x-3)} dx$
3. Find $\int \frac{x^2+1}{x^2-5x+6} dx$
4. Evaluate $\int \frac{dx}{x(x^3+8)}$
5. Find $\int \frac{e^x}{(1+e^x)(2+e^x)} dx$
6. Evaluate $\int \frac{\sin x}{\sin 3x} dx$
7. Evaluate $\int \frac{2x^2+1}{x^2(x^2+4)} dx$
8. Evaluate $\int \frac{x^2-1}{x^4+x^2+1} dx$

Unit - 3

Integration Of Irrational Functions

Structure

3.1 Introduction

Irrational functions are mathematical functions that involve irrational expressions, such as square roots or fractional exponents. These functions can be challenging to work with due to the presence of irrational numbers, which cannot be expressed as exact ratios of integers. An irrational function typically takes the form of $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials, and at least one of them contains an irrational expression. Common examples of irrational functions include functions containing square roots, cube roots, or higher-order roots. When it comes to integrating irrational functions, the process can be more involved compared to integrating rational functions. There is no general formula or algorithm to integrate all irrational functions, as it depends on the specific form of the function. The integration of irrational functions requires a combination of algebraic manipulation, trigonometric or hyperbolic substitutions, and specialized techniques for handling square roots, higher-order roots, and fractional exponents. The choice of method depends on the specific form of the function being integrated.

3.2 Learning Objectives

After studying this unit, you should be able to:-

- Define irrational functions
- Discuss method to find the integrate

  $$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$$

- Discuss rule to evaluate $\int x^m(a + bx^n)^p$
- Solve examples related to these methods.

3.3 Method To Integrate $\int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

Let $I = \int \frac{1}{\sqrt{ax^2+bx+c}}\,dx$

Two cases arise

Case I. a is positive

$$I = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}}\,dx$$

$$= \frac{1}{\sqrt{a}}\int \frac{1}{\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)}\,dx$$

$$= \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}}\,dx$$

Two sub-case arise:

Sub-Case (i) $b^2 - 4ac > 0$

$\therefore$ I = $\frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{b^2-4ac}{4a^2}}}$ dx

= $\frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2-\left(\frac{\sqrt{b^2-4ac}}{2a^2}\right)^2}}$ dx

$=\frac{1}{\sqrt{a}}$ log $\left[x+\frac{b}{2a}+\sqrt{\left(x+\frac{b}{2a}\right)^2-\frac{\sqrt{b^2-4ac}}{4a^2}}\right]$ + C

= $\frac{1}{\sqrt{a}}$ log $\left[x+\frac{b}{2a}+\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\right]$+ C

Sub-case (ii) b^2 - 4ac < 0

$\therefore$ I = $\frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{\left(x+\frac{b}{2a}\right)^2+\left(\frac{\sqrt{4ac-b^2}}{2a}\right)^2}}$ dx

= $\frac{1}{\sqrt{a}}$ log$\left[x+\frac{b}{2a}+\sqrt{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}}\right]$ + C

$=\frac{1}{\sqrt{a}}$ log $\left[x+\frac{b}{2a}+\sqrt{x^2+\frac{b}{a}x+\frac{c}{a}}\right]$ + C

Case II a is negative

Let a = - A, where A is positive

$\therefore$ I = $\int \frac{1}{\sqrt{-Ax^2+bx+c}}$ dx

= $\frac{1}{\sqrt{A}}\int \frac{1}{\sqrt{-x^2+\frac{b}{A}x+\frac{c}{A}}}$

$$= \frac{1}{\sqrt{A}} \int \frac{1}{\sqrt{\frac{c}{A} - \left(x^2 - \frac{b}{A}x\right)}} \text{dx}$$

$$= \frac{1}{\sqrt{A}} \int \frac{1}{\sqrt{\left(\frac{b^2}{4a^2} + \frac{c}{A}\right) - \left(x^2 - \frac{b}{A}x + \frac{b^2}{4A^2}\right)}} \text{dx}$$

$$= \frac{1}{\sqrt{A}} \int \frac{1}{\sqrt{\frac{b^2 + 4Ac}{4A^2} - \left(x - \frac{b}{2A}\right)^2}} \text{dx}$$

$$= \frac{1}{\sqrt{A}} \int \frac{1}{\sqrt{\left(\frac{b^2 + 4Ac}{2A}\right)^2 - \left(x - \frac{b}{2A}\right)^2}} \text{dx}$$

$$= \frac{1}{\sqrt{A}} \sin^{-1} \left(\frac{x - \frac{b}{2A}}{\frac{\sqrt{b^2 + 4Ac}}{2A}} \right) + \text{C}$$

$$= \frac{1}{\sqrt{A}} \sin^{-1} \left(\frac{2Ax - b}{\sqrt{b^2 + 4Ac}} \right) + \text{C}$$

$$= \frac{1}{\sqrt{-a}} \sin^{-1} \left(\frac{-2ax - b}{\sqrt{b^2 + 4Ac}} \right) + \text{C}$$

Note 1. Rule to Integrate $\int \frac{1}{\sqrt{quadratic}}$ dx

(i) Take the numerical value of coefficient of x^2 outside the quadratic (to make coefficient of x^2 numerically = 1).

(ii) Complete the square in terms containing x.

(iii) Use the proper standard form from the following results:

(1) $\int \frac{1}{\sqrt{x^2 + a^2}}$ dx = log $\left| x + \sqrt{x^2 + a^2} \right|$ + c where a is positive.

$$= \sin h^{-1} \frac{x}{a} + c$$

(2) $\int \frac{1}{\sqrt{x^2+a^2}}$ dx = log $\left|x+\sqrt{x^2+a^2}\right|$ + c where a is positive.

= $\cosh^{-1}\frac{x}{a}$ + c

(3) $\int \frac{1}{\sqrt{a^2-x^2}}$ dx = $\sin^{-1}\frac{x}{a}$, where a is positive

Note 2. Rule to Integrate $\int \frac{1}{\sqrt{quadratic}}$ dx

(i) Take the numerical value of coefficient of x^2 outside the quadratic (to make coefficient of x^2 numerically = 1).

(ii) Complete the square in terms containing x.

(iii) Use the proper standard form from the following results:

(1) $\sqrt{x^2+a^2}$ dx = $\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|$ + c, where a is positive.

= $\frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}$ + c

(2) $\int \sqrt{x^2-a^2}$ dx = $\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right|$ + c, where a is positive.

= $\frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\frac{x}{a}$ + c

(3) $\int \sqrt{a^2-x^2}$ dx = $\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}$ + c, where a is positive.

Let us improve our understanding of these results by looking at some following examples:

Example 1: Evaluate $\int \frac{dx}{\sqrt{x^2+12x+11}}$

Sol: Let I = $\int \frac{dx}{\sqrt{x^2+12x+11}}$ dx

= $\int \frac{1}{\sqrt{(x^2+12x+36)-25}}$ dx

$$= \int \frac{1}{\sqrt{(x+6)^2-(5)^2}}\,dx$$

$$= \log\left|(x+6)+\sqrt{(x+6)^2-(5)^2}\right| + c$$

$$= \log\left|x+6+\sqrt{x^2+12x+11}\right| + c$$

Example 2: Integrate $\int \frac{dx}{\sqrt{2x^2+3x+4}}$

Sol: Let I = $\int \frac{dx}{\sqrt{2x^2+3x+4}}$ dx

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{x^2+\frac{3}{2}x+2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-\frac{9}{16}+2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}}\,dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{23}{4}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\log\left|\left(x+\frac{3}{4}\right)+\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}\right| + c$$

$$= \frac{1}{\sqrt{2}}\log\left|x+\frac{3}{4}+\sqrt{x^2+\frac{3}{2}x+2}\right| + c$$

Example 3: Integrate $\int \sqrt{2x^2-x-1}$ dx

Sol: Let I = $\int \sqrt{2x^2-x-1}$ dx

$$= \sqrt{2}\ \sqrt{x^2 - \frac{1}{2}x + \frac{1}{2}}\ dx$$

$$= \sqrt{2} \int \sqrt{\left(x^2 - \frac{x}{2} + \frac{1}{16}\right) - \frac{1}{2} - \frac{1}{16}}\ dx$$

$$= \sqrt{2} \int \sqrt{\left(x^2 - \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}\ dx$$

$$= \sqrt{2}\left[x + \frac{1}{4}\sqrt{\left(x - \frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2} - \frac{\left(\frac{3}{4}\right)^2}{2}\cosh^{-1}\left(\frac{x - \frac{1}{4}}{\frac{3}{4}}\right)\right] + c$$

$$= \sqrt{2}\left[x + \frac{1}{4}\sqrt{x^2 - \frac{x}{2} - \frac{1}{2}} - \frac{9}{32}\cosh^{-1}\left(\frac{4x-1}{3}\right)\right] + c$$

Example 4: Integrate $\int \frac{dx}{\sqrt{3+8x-3x^2}}$

Sol: Let I = $\int \frac{dx}{\sqrt{3+8x-3x^2}}$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1 + \frac{8x}{3} - x^2}}\ dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{1 - \left(x^2 - \frac{8}{3}x\right)}}\ dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(1 + \frac{16}{9}\right) - \left(x^2 - \frac{8}{3}x + \frac{16}{9}\right)}}\ dx$$

$$= \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\left(\frac{5}{3}\right)^2 - \left(x - \frac{4}{3}\right)^2}}\ dx$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left[\frac{x-\frac{4}{3}}{\frac{5}{3}}\right] + c$$

$$= \frac{1}{\sqrt{3}} \sin^{-1}\left[\frac{3x-4}{5}\right] + c$$

Example 5: Integrate

$$\int \sqrt{x^2-4x+2}\, dx$$

Sol: Let I = $\int \sqrt{x^2-4x+2}\, dx$

$$= \int \sqrt{(x^2-4x+4)-2}\, dx$$

$$= \int \sqrt{(x-2)^2-(\sqrt{2})^2}\, dx$$

$$= \frac{(x-2)\sqrt{(x-2)^2\left(\sqrt{2}\right)^2}}{2} - \frac{\left(\sqrt{2}\right)^2}{2} \log\left|(x-2)+\sqrt{(x-2)^2-(\sqrt{2})^2}\right| + c$$

$$= \frac{1}{2}(x-2)\sqrt{x^2-4x+4} - \log\left|(x-2)+\sqrt{(x^2-4x+2}\right| + c$$

Self-Check Exercise-1

Q.1 Integrate $\int \frac{dx}{\sqrt{3-x+x^2}}$

Q.2 Evaluate $\int \frac{1}{\sqrt{2x-x^2}}\, dx$

Q.3 Integrate $\int \frac{dx}{\sqrt{4+3x-2x^2}}$

Q.4 Evaluate $\int \sqrt{(x^2+6x-4}\, dx$

3.4 Method To Evaluate

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\, dx$$

Let I = $\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}$ dx .....(1)

Let linear = l(diff. Coeff. of quadratic) + m, i.e.

Let Ax + B = l(2a + b) + m .....(2)

Equating coeffs of x on both sides, we have

A = 2al

$\therefore$ I = $\frac{A}{2a}$;

Equating constant terms on both sides, we have

B = lb + m

$\therefore$ B = $\frac{A}{2a}$ b + m, $\therefore$ m = B = $\frac{Ab}{2a}$

$\therefore$ t = $\frac{A}{2a}$, m = B - $\frac{Ab}{2a}$ ....(3)

From (2) and (1),

I = $\int \frac{l(2ax+b)+m}{\sqrt{ax^2+bx+c}}$ dx

= $l \int \frac{2ax+b}{\sqrt{ax^2+bx+c}}$ dx + m$\int \frac{1}{\sqrt{ax^2+bx+c}}$ dx

= u_1 + mI_2 ......(4)

Where I_1 = $\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}$ dx, I_2 = $\int \frac{1}{\sqrt{ax^2+bx+c}}$ dx ......(5)

Now I_1 = $\int \frac{2ax+b}{\sqrt{ax^2+bx+c}}$ dx

= $\int \left(ax^2+bx+c\right)^{-\frac{1}{2}}$ (2ax + b) dx

= $\frac{(ax^2+bx+c)^{\frac{1}{2}}}{\frac{1}{2}}$

= $2\sqrt{ax^2+bx+c}$

From (5), $I_2 = \frac{1}{\sqrt{ax^2+bx+c}}$ dx, which can be integrated.

$\therefore$ from (4), substituting the values of I_1, I_2 found above and then substituting the values of l and m from (3), we get I.

Note. Important $\int (Ax + B)\sqrt{ax^2+bx+c}\, dx$ can also be integrated in the same manner by using "l,m" method.

Let us consider the following examples to clear the idea:

Example 6: Integrate $\int \frac{x^2+2}{\sqrt{x^2+2x-1}}\, dx$

Sol: Let I = $\frac{x^2+2}{\sqrt{x^2+2x-1}}$ dx

$$= \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x-1}}\, dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x-1}}\, dx + \int \frac{1}{\sqrt{x^2+2x-1}}\, dx$$

$$= \frac{1}{2}\int \left(x^2+2x-1\right)^{-\frac{1}{2}} (2x + 2)\, dx + \int \frac{1}{\sqrt{(x+1)^2-(\sqrt{2})^2}}\, dx$$

$$= \frac{1}{2}\, \frac{(x^2+2x-1)^{\frac{1}{2}}}{\frac{1}{2}} + \log\left|\frac{(x+1)+\sqrt{(x+1)^2-(\sqrt{2})^2}}{\sqrt{2}}\right| + c$$

$$= \sqrt{x^2+2x-1} + \log\left|\frac{(x+1)+\sqrt{x^2+2x-1}}{\sqrt{2}}\right| + c$$

Example 7: Integrate $\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\, dx$

Sol: Let I = $\int \frac{(x^2+x+1)+(x+2)}{\sqrt{x^2+x+1}}\, dx$

$$= \int \frac{x^2+x+1}{\sqrt{x^2+x+1}}\, dx + \int \frac{x+2}{\sqrt{x^2+x+1}}\, dx$$

$$= \int \sqrt{x^2+x+1}\,dx + \int \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \sqrt{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}\,dx + \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx + \frac{1}{2}\int \left(x^2+x+1\right)^{-\frac{1}{2}}\,dx + \frac{3}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$

$$= \frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\sinh^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]$$

$$+\frac{1}{2}\,\frac{(x^2+x+1)^{\frac{1}{2}}}{\frac{1}{2}} + \frac{3}{2}\sinh^{-1}\left[\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + c$$

$$= \frac{1}{4}(2x+1)\left(x^2+x+1\right) + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$+ \left(x^2+x+1\right) + \frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$$= \frac{(2x+5)\sqrt{x^2+x+1}}{4} + \frac{15}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

Example 8: Evaluate $\int (x-5)\sqrt{x^2+x}\,dx$

Sol: Let $I = \int (x-5)\sqrt{x^2+x}\,dx$

$$= \int \left(x^2+x\right)^{-\frac{1}{2}}\left\{\frac{1}{2}(2x+1)-\frac{11}{2}\right\}dx$$

$$= \frac{1}{2}\int \left(x^2+x\right)^{-\frac{1}{2}}(2x+1)\,dx - \frac{11}{2}\int \left(x^2+x\right)^{-\frac{1}{2}}dx$$

$$= \frac{1}{2}\,\frac{\left(x^2+x\right)^{-\frac{3}{2}}}{\frac{3}{2}} - \frac{11}{2}\int \left[\left(x^2+x+\frac{1}{4}\right)-\left(\frac{1}{4}\right)\right]^{\frac{1}{2}} dx$$

$$= \frac{1}{3}\left(x^2+x\right)^{\frac{3}{2}} - \frac{11}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}\, dx$$

$$= \frac{1}{3}\left(x^2+x\right)^{\frac{3}{2}} - \frac{11}{2}\left[\frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\log\left|\frac{\left(x+\frac{1}{2}\right)+\sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2}}{\frac{1}{2}}\right|\right] + c$$

$$= \frac{1}{3}\left(x^2+x\right)^{-\frac{3}{2}} - \frac{11}{8}(2x+1)\sqrt{x^2+x} + \frac{11}{16}\log\left|(2x+1)+2\sqrt{x^2+x}\right| + c$$

Self-Check Exercise-2

Q.1 Evaluate $\int \frac{x\,dx}{\sqrt{2x^2-6x+4}}$

Q. 2 Integrate $\int x\sqrt{1+x-x^2}\, dx$

3.5 Rule To Evaluate $\int x^m(a + b x^n)^p dx$, where p is a fraction = $\frac{r}{s}$ (say), r and s being integers and s positive

Compare the given integral with $\int x^m(a + bx^n)^p$, and find the values of m, n, p $\left(=\frac{r}{s}\right)$

Case I: If $\frac{m+1}{n}$ is a positive integer or 0, put $a + bx^n = z^s$

Case II: If $\frac{m+1}{n}$ is neither a positive integer nor o, then find $\frac{m+1}{n}$ + p.

If $\frac{m+1}{n}$ + p is a negative integer or o, take x^n outside the binomial $a + bx^n$ thus getting $a + bx^n = x^n (ax^n + b)$, and put the new binomial $ax^{-n} + b = zs$.

Let us consider the following example to clear the idea:-

Example 9: Evaluate $\int \frac{x^5}{\sqrt{x^2+1}}$ dx

Sol: Let I = $\int \frac{x^5}{\sqrt{x^2+1}}$ dx

= $\int x^5\left(1+x^2\right)^{-\frac{1}{2}}$ dx .....(1)

(i) Comparing $\int x^5\left(1+x^2\right)^{-\frac{1}{2}}$ dx with $\int x^m (a + bx^n)^p$ dx, we get

m = 5, n = 2, p = $\frac{-1}{2}$, a fraction = $\frac{r}{s}$, $\therefore$ s = 2 .....(2)

$\therefore$ $\frac{m+1}{n} = \frac{5+1}{2} = \frac{6}{2}$ = 3, which is a +ve integer.

(ii) Put $1 + x^2 = z^s = z^2$ .....(3)

$\therefore$ 2x dx = 2z dz or dx = $\frac{z}{x}$ dz

$\therefore$ from (1)

I = $\int$ xs $\left(z^2\right)^{-\frac{1}{2}} \frac{z}{x}$ dz

= $\int x^4 z^{-1}$ dz

= $\int x^4$ dz = $\int (z^2 - 1)^2$ dz [From (3), $x^2 = z^2 - 1$]

= $\int (z^4 - 2z^2 + 1)$dz = $\frac{z^5}{5} - 2\frac{z^3}{3} + z + c$ [$\therefore$ $x^4 = (z^2 - 1)^2$]

= $\frac{1}{5}\left(1+x^2\right)^{\frac{5}{2}} - \frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + \left(1+x^2\right)^{\frac{1}{2}}$ + c. $\left[\because z = \left(1+x^2\right)^{\frac{1}{2}}\right]$

Example 10: Evaluate $\int \frac{\sqrt{x^2+1}}{x^4}$ dx.

Sol: Let I = $\int \frac{\sqrt{x^2+1}}{x^4}$ dx

= $\int x^{-4}\left(1+x^2\right)^{\frac{1}{2}}$ dx ....(1)

(i) Comparing $\int x^{-4}\left(1+x^2\right)^{\frac{1}{2}} dx$ with $\int x^m (a + bx^n)^p dx$, we get,

m = -4, n = 2, p = $\frac{1}{2}$, a fraction = $\frac{r}{s}$, $\therefore$ s = 2 .....(2)

(ii) $\therefore \frac{m+1}{n} = \frac{-4+1}{2} = \frac{-3}{2}$, which is neither a +ve integer nor 0.

(iii) Now $\frac{m+1}{n} + p = \frac{-3}{2} + \frac{1}{2} = \frac{-2}{2} = -1$, which is a -ve integer.

Take x^2 outside the binomial $1 + x^2$, thus getting

$1 + x^2 = x^2(x^{-2} + 1)$

$\therefore$ From (1), I = $\int x^{-4}.x\left(x^{-2}+1\right)^{\frac{1}{2}} dx$

$= \int x^{-3}\left(x^{-2}+1\right)^{\frac{1}{2}} dx$ .....(3)

Put $x^2 + 1 = z^s = z^2$

$\therefore$ $-2x^{-3}dx = 2z\,dz$ or $x^{-3}dx = -z\,dz$. .....(4)

$\therefore$ from (3),

$I = \int z(-z)\,dz = -\int z^2dz = -\frac{z^3}{3} + c = -\frac{1}{3}\left(x^2+1\right)^{\frac{2}{3}} + c$

$= -\frac{1}{3}\left[\frac{1}{x^2}+1\right]^{\frac{3}{2}} + c$

Self-Check Exercise-3

Q.1 Evaluate $\int x^5\left(1+x^3\right)^{\frac{1}{2}} dx$

3.6 Rules For Some Other Important Types

I. Rule to integrate if $\sqrt{a^2-x^2}$ occurs in the integrand

Put x = a sin θ

II. Rule to integrate if $\sqrt{a^2-x^2}$ occurs in the integrand

Put x = a tan θ

III. Rule to integrate if $\sqrt{a^2 - x^2}$ occurs in the integrand

Put x = a sec θ

IV. Rule to evaluate $\int \frac{1}{\text{linear}\sqrt{\text{Linear}}} dx$

Put $\sqrt{\text{Linear}}$ = y

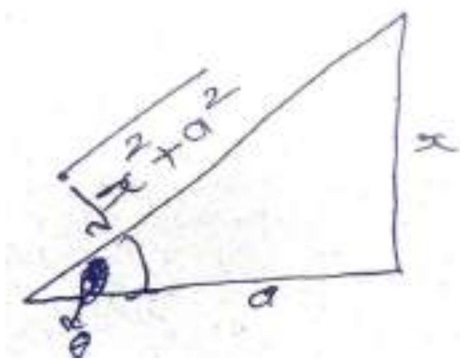
Let us consider the following examples to clear the idea:-

Example 9: Integrate $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$

Sol: Let I = $\int \frac{dx}{(a^2 + x^2)^{\frac{3}{2}}}$

Put x = a tan θ,

∴ dx = a sec^2 θ dθ



∴ I = $\int \frac{a\sec^2\theta d\theta}{(a^2 + a^2\tan^2\theta)^{\frac{3}{2}}}$

= $\frac{1}{a^2}\int \frac{1}{\sec\theta}$ dθ

= $\frac{1}{a^2}\int$ cos θ dθ

= $\frac{1}{a^2}$ sin θ + c

$= \dfrac{x}{a^2\sqrt{x^2+a^2}}$

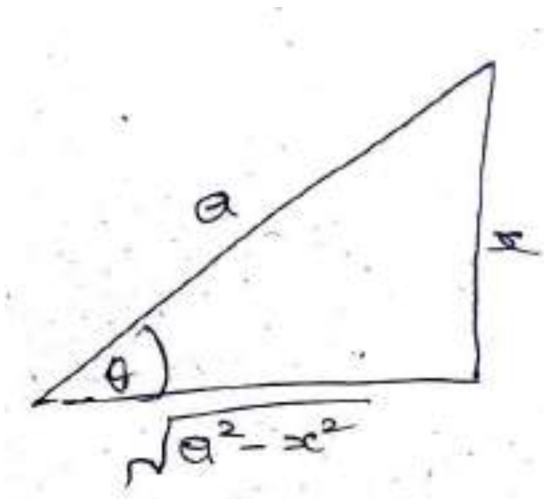
Example 10: Evaluate $\int \dfrac{dx}{(a^2-x^2)^{\frac{3}{2}}}$

Sol: Let I = $\int \dfrac{dx}{(a^2-x^2)^{\frac{3}{2}}}$

Put x = a sin θ

∴ dx = a cos θ dθ

∴ I = $\int \dfrac{a\cos\theta d\theta}{(a^2-x^2\sin^2\theta)^{\frac{3}{2}}}$



$= \int \dfrac{a\cos\theta d\theta}{a^3\cos^3\theta}$

$= \dfrac{1}{a^2}\int \dfrac{1}{\cos^2\theta}$ dθ

$= \dfrac{1}{a^2}\int$ $\sec^2$ θ dθ

$= \dfrac{1}{a^2}$ tan θ + c

$$= \frac{1}{a^2} \frac{x}{\sqrt{a^2 - x^2}} + c$$

Example 11: Integrate $\frac{dx}{(x^2-1)\sqrt{x-1}}$

Sol: [Note: In order to integrate $\int \frac{1}{\text{quadratic}\sqrt{\text{linear}}}$ dx, put $\sqrt{\text{linear}}$ = y]

Let I = $\int \frac{dx}{(x^2-1)\sqrt{x-1}}$

Put $\sqrt{x-1}$ = y

Or x - 1 = y^2

Or x = y^2 + 1

∴ dx = 2y dy

∴ I = $\int \frac{2ydy}{\left[\left(y^2+1\right)^2 - 1\right]y}$

$= 2\int \frac{dy}{y^4 + 2y^2}$

$= 2\int \frac{1}{y^2(y^2+2)dy}$

Now $\frac{1}{y^2(y^2+2)dy} = \frac{1}{t(t+2)}$, where t = y^2

$= \frac{1}{t(0+2)} + \frac{1}{(-2)(t+2)}$

$= \frac{1}{2t} - \frac{1}{2(t+2)}$

$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{t+2}\right]$

$= \frac{1}{2}\left[\frac{1}{y^2} - \frac{1}{y^2+2}\right]$, as t = y^2

$\therefore$ $\quad$ I = 2. $\frac{1}{2}\int \left(\frac{1}{y^2}-\frac{1}{y^2+2}\right)$ dy

$= \int \left[y^{-2}-\frac{1}{y^2+\left(\sqrt{2}\right)^2}\right]$dy

$= \frac{y^{-1}}{-1}$ - $\frac{1}{\sqrt{2}}$ tan^{-1} $\frac{y}{\sqrt{2}}$ + c

$= \frac{1}{y}$ = $\frac{1}{\sqrt{2}}$ tan^{-1} $\frac{y}{\sqrt{2}}$ + c

$= -\ \frac{1}{\sqrt{\text{x-1}}}$ - $\frac{1}{\sqrt{2}}$ tan^{-1} $\left(\frac{\sqrt{x-1}}{\sqrt{2}}\right)$ + c

$= -\left[\frac{1}{\sqrt{x-1}}+\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{\frac{x-1}{2}}\right)\right]$ + c

Self-Check Exercise-4

Q.1 $\quad$ Evaluate $\frac{dx}{(x^2-1)^{\frac{3}{2}}}$

Q.2 $\quad$ Integrate $\int \frac{dx}{(x+2)\sqrt{x+3}}$

3.7 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Defined irrational functions
2. Discussed the method to find the integral

$$\int \frac{1}{\sqrt{\text{ax}^2+bx+c}}\,\text{dx}$$

3. Discussed the method to find the integral

$$\int \frac{Ax+B}{\sqrt{\text{ax}^2+bx+c}}\,\text{dx}$$

4. Discussed the rule to evaluate $\int x^m(a + bx^n)^p\, dx$, where p is a fraction = $\frac{r}{s}$ (say), r and s being integers and s positive.

5. Solved some questions related to these methods.

3.8 Glossary

1. Irrational functions are mathematical functions that involve irrational expressions, such as square roots or fractional exponents.

2. An irrational function typically takes the form of $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomials, and at least one of these contains an irrational expression.

3.9 Answers To Self-Check Exercises

Self-Check Exercise-1

Ans. 1 $\log \left| x - \frac{1}{2} + \sqrt{x^2 - x + 3} \right| + c$

Ans. 2 $\text{Sin}^{-1}(x - 1) + c$

Ans. 3 $\frac{1}{\sqrt{2}} \sin^{-1} \frac{4x-3}{41} + c$

Ans. 4 $\frac{1}{2}(x + 3)\sqrt{x^2+6x-4} - \frac{13}{2} \log \left| x+3+\sqrt{x^2+6x-4} \right|$

Self-Check Exercise-2

Ans. 1 $\frac{1}{2}\sqrt{x^2+6x-4} + \frac{5}{2\sqrt{2}} \log \left| x - \frac{3}{2} + \sqrt{x^2-3x+2} \right| + c$

Ans. 2 $- \frac{1}{3}\left(1+x-x^2\right)^{\frac{3}{2}} + \frac{1}{8}(2x - 1)\sqrt{1-x-x^2} + \frac{5}{16} \sin^{-1}\left(\frac{2x-1}{\sqrt{5}}\right) + c$

Self-Check Exercise-3

Ans. 1 $\frac{2}{15}\left(1+x^3\right)^{\frac{5}{2}} - \frac{2}{9}\left(1+x^3\right)^{\frac{3}{2}} + c$

Self-Check Exercise-4

Ans. 1 $\frac{x}{\sqrt{x^2-1}} + c$

Ans. 2 log $\left|\frac{\sqrt{x+3}-1}{\sqrt{x+3+1}}\right|$ + c

3.10 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.

2. G.B. Thomas and R.L. Finney, Calculus, 9^{th} Ed., Pearson Education, 2005.

3.11 Terminal Questions

1. Integrate $\int \frac{dx}{\sqrt{2-4x+x^2}}$

2. Evaluate $\int \frac{1}{\sqrt{x}\sqrt{5-x}}$ dx

3. Evaluate $\int \frac{dx}{\sqrt{2x^2-2x+3}}$

4. Integrate $\int \frac{2x+1}{\sqrt{x^2+4x+3}}$ dx

5. Evaluate $\int$ (2x - 5) $\sqrt{x^2-4x+3}$ dx

6. Integrate $\int \frac{dx}{(2ax+x^2)^{\frac{3}{2}}}$

7. Integrate $\int \frac{dx}{(x-3)\sqrt{x+1}}$

8. Integrate $\int \frac{x+2}{(x^2+3x+3)\sqrt{x+1}}$ dx

Unit - 4

Definite Integrals-I

Structure

4.1 Introduction

Definite integrals are an essential concept in calculus that allow us to determine the exact accumulation of quantities over a given interval. They are closely related to ant derivatives and provide a powerful tool for calculating areas, calculating total change, and solving a wide range of mathematical problems.

The definite integral of a function f(x) over the interval [a, b] is denoted as $\int_a^b f$(x), where the symbol $(\int)$ represents the process of integration, while the dx at the end indicates that we are integrating with respect to the variable x. The limits of integration, a and b, determine the interval over which the integration is performed.

Geometrically, the definite integral represents the signed area between the graph of the function f(x) and the x-axis over the interval [a, b]. The term "signed" indicates that the area can be positive or negative depending on whether the function lies above or below the x-axis.

The process of finding the definite integral involves dividing the interval [a, b] into small subintervals and approximating the area under the curve using rectangles or other geometric shapes. As the number of subintervals increases, the approximation becomes more accurate, and in the limit, we obtain the exact value of the definite integral.

4.2 Learning Objectives

After studying this unit, you should be able to:-

- Define definite integral

Show definite integral as a limit of sum.

- Discuss some properties of definite integral i.e. integration is independent of the change of variable; if the limits of definite integral are interchanged, then its value changes by minus sign only.
- Find the integrals using properties of definite integral.

4.3 Definite Integral

Def: Let f be a function of x defined in the closed interval [a, b] and ϕ be another function, such that $\phi'(x) = f(x)$ for all x in the domain of f, then

$$\int_a^b f(x)dx = \left[\phi(x)\right]_a^b$$

$$= \phi(b) - \phi(a)$$

is called the definite integral of the function $f(x)$ over the interval [a, b] a and b are called the limits of integration, a being the lower limit and b be the upper limit.

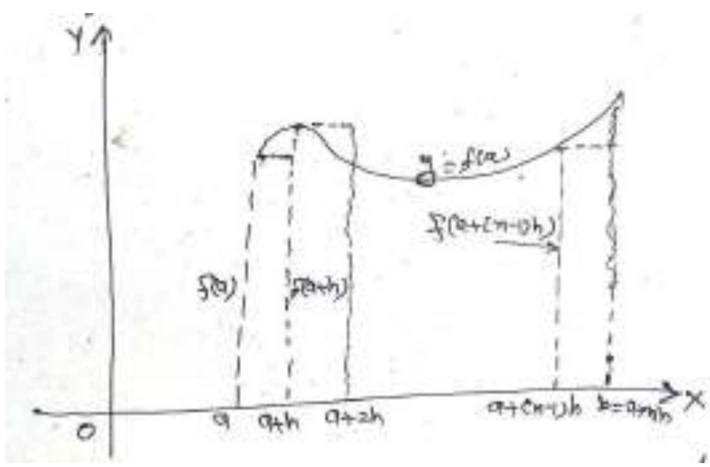
Geometrical Interpretation of Definite Integral

If $f(x) > 0$ for all $x \in [a, b]$, then $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the x-axis and the straight lines x = a and x = b

4.4 Definite Integral As A Limit Of Sum

Let f be a real valued non-negative continuous function defined on [a, b]. Then $\int_a^b f(x)dx = \underset{h\to 0}{Lim}\, h[f(a) + f(a+h) + \dots\dots + f(a + (n-1))h]$, where $h = \frac{b-a}{n}$

Proof: Let [a, b] be divided into x equal parts so that length of each subinterval is $h = \frac{b-a}{n}$



Let S_n denotes the sum of areas of n rectangles as shown in the figure each having width h and height $f(a)$, $f(a + h)$,......, $f(a + (n - 1) h)$.

Then Sn = h f(a) + hf(a + h) +......+ hf(a + (n-1)h)

= h [f(a) + f(a + h) +.......+ f(a + (n - 1)h)].

Now, if n increases, then number of rectangles increases, width of each rectangle decreases so that S_n gives a closer approximation to the area bounded by curve y = f(x), ordinates x = a, x = b and the x - axis.

Thus $\int_a^b f(x)dx = \underset{n\to\infty}{Lim}$ Sn = $\underset{n\to\infty}{Lim}$ h[f(a) + f(a + h) +......+ f(a + (n - 1)h]

= $\underset{h\to 0}{Lim}$ h[f(a) + f(a + h) +......+ f(a + (n-1)h)] $\left[h = \frac{b-a}{n} \to 0 \text{ as n} \to \infty\right]$

Remark: Sometimes it is convenient to divide [a, b] into subintervals [a, ar], [ar, ar^2], [ar^2, ar^3],......, [ar^{n-1}, ar^n] where $r^n = \frac{b}{a}$.

Then $\int_a^b f(x)dx$ = $\underset{n\to\infty}{Lim}$[(ar - a) f(a) + (ar^2 - ar) f (ar) +.......+ (ar^n - ar^{n-1}) f(ar^{n-1})]

= $\underset{n\to\infty}{Lim}$(r-1) [a$f$(a) + ar f(ar) +........+ ar^{n-1} f(ar^{n-1})]

= $\underset{n\to\infty}{Lim}$(r-1) [a$f$(a) + ar f(ar) +.......+ ar^{n-1} f(ar^{n-1})] $\left[\text{Since } r = \left(\frac{b}{a}\right)^{\frac{1}{n}} \to 1 \text{ as n} \to \infty\right]$

4.5 Properties Of Definite Integrals

Property I: $\int_a^b f(x)dx = \int_a^b f(z)dz$

i.e. The integration is independent of the change of variable.

Proof: Let $\int$ f(x) dx = ϕ(x) .....(1)

Now L.H.S. = $\int_a^b f(x)dx$ = $[\phi(x)]_a^b$ = ϕ (b) - ϕ (a)

R.H.S. = $[\phi(z)]_a^b$ [$\because$ from (1), putting x = z, dx = dz, $\int$ f(z)dx = ϕ(z)]

= ϕ (b) - ϕ (a)

$\therefore$ L.H.S. = R.H.S.

Hence the result

Property II: $\int_a^b f(x)dx$ = -$\int_b^a f(x)dx$

i.e. if the limits of definite integral are interchanged, then its value changes by minus sign only

Proof: Let $\int f(x)\,dx = \phi(x)$

Then L.H.S. = $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$

R.H.S. = $-\int_b^a f(x)dx = -[\phi(x)]_b^a = -[\phi(a) - \phi(b)] = \phi(b) - \phi(a)$

$\therefore$ L.H.S. = R.H.S.

Hence the result

Property III: $\int_a^b f(x)dx = \int_b^c f(x)dx + \int_c^b f(x)dx$

Proof: Let $\int f(x)\,dx = \phi(x)$

$\therefore$ L.H.S. = $\int_a^b f(x)dx = [\phi(x)]_a^b = \phi(b) - \phi(a)$

R.H.S. = $\int_a^c f(x)dx + \int_c^b f(x)dx = [\phi(x)]_a^c + [\phi(x)]_c^b$

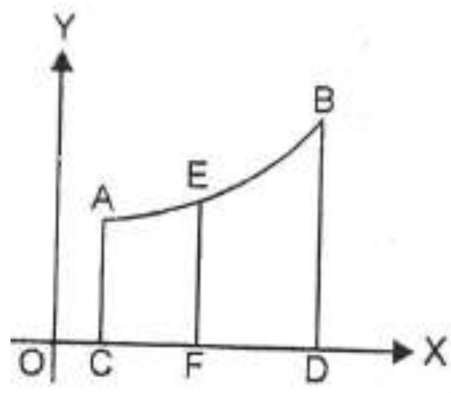
$= [\phi(c) - \phi(a)] + [\phi(b) - \phi(c)] = \phi(b) - (a)$

$\therefore$ L.H.S. = R.H.S.

Hence the result

Note Geometrical Illustration

Let AB be the curve y = f(x) and CA, DB the ordinates x = a, x = b.



Let FE be the ordinate x = c

Then the theorem

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

expresses that area ACDB = area ACFE + area EFDB, which is obvious.

Note. Generalization of theorem III

Prove that $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^e f(x)dx + ... + \int_k^b f(x)dx$

Proof: We have to prove that

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^e f(x)dx + ... + \int_k^b f(x)dx \qquad ......(1)$$

Let $\int f(x)\, dx = \phi(x)$

Then L.H.S. of (1) = $\left[\phi(x)\right]_a^b = \phi(b) - \phi(a)$

R.H.S. of (1) = $\left[\phi(x)\right]_a^c + \left[\phi(x)\right]_c^d + \left[\phi(x)\right]_d^e + ... + \left[\phi(x)\right]_k^b$

$= \phi(c) - \phi(a) + \phi(d) - \phi(c) + \phi(e) - \phi(d) + ... + \phi(b) - \phi(k)$

$= \phi(b) - \phi(a)$

$\therefore$ L.H.S. = R.H.S.

Property IV: $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Proof: Put x = a - z,

When x = 0, z = a; When x = a, z = 0

$$\therefore \quad \int_0^a f(x)dx = -\int_0^a f(a-z)dz = \int_0^a f(a-z)dz \qquad [\because \text{ of Theorem II}]$$

$$= \int_0^a f(a-x)dx \qquad [\because \text{ of Theorem I}]$$

$$\therefore \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

Note. Geometrical Illustration

Let AB be the curve y = f(x) .....(1)

and OA, O'B the ordinates x = 0, x = a

Let P(x, y) be any pt. on the curve and MP its ordinate.

Take O'O and O'B as the new axes and let (x',y') be the co-ordinates of P referred to them.

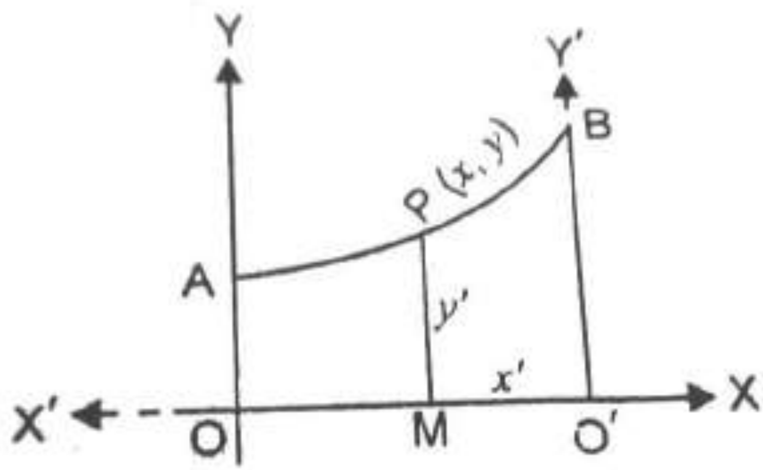
Then x = OM = OO' - MO' = a - x',

y = MP = y'

$\therefore$ from (1), the equation of the curve AB becomes y' = f(a - x') or, dropping deshes,

y = f(a - x) .....(2)

and the equations of the ordinates O'B, OA are x = 0, x = a



$\therefore$ from (1) and (2), the theorem

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

expresses that area AOO'B = area BAOO', which is obvious.

Note:- Prove that the rule to write down the value of $\int_0^{\pi/2} \cos^n x\,dx$, where n is a posite integer, is the same as that to write down the value of $\int_0^{\pi/2} \cos^n x\,dx$.

Proof: $\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x\right)$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \int_0^{\pi/2} \cos^n$ x dx

Property IV is useful to evaluate a definite integral without first finding the corresponding indefinite integrals which may be difficult or sometimes impossible to find.

Dear Students, further properties of definite integral be discussed in next UNIT and now let us improve our understanding of properties discussed above by looking at some of the following examples:-

Example 1: Show that

$$\int_a^b f(x)dx = \int_0^a f(a+b-x)dx$$

Sol: Put x = a + b - z

$\Rightarrow$ dx = - dz

When x = a, z = b

and when x = b, z = a

$$\therefore \quad \int_a^b f(x)dx = -\int_b^a f(a+b-z)dz$$

$$= \int_a^b f(a+b-z)dz \qquad \left[\text{By Property I i.e., } \int_a^b f(x)dx = \int_a^b f(z)dz\right]$$

Example 2: prove that

$$\int_a^b f(x)dx = \frac{b-a}{m-l}\int_l^m f\left(\frac{am-bl}{m-l}+\frac{b-a}{m-l}x\right)\text{dx}$$

Sol: Let I = $\int_a^b f(x)dx$

Put x = $\frac{am-bl}{m-l}+\frac{b-a}{m-l}$ t

$\therefore$ dn = $\frac{b-a}{m-l}$ dt

When x = a; a = $\frac{am-bl}{m-l}+\frac{b-a}{m-l}$ t

$\Rightarrow$ am - al = am - bl + (b - a)t

$\Rightarrow$ (b - a)l = (b - a)t

$\Rightarrow$ t = l

When x = b; we have

b = $\dfrac{am-bl}{m-l}+\dfrac{b-a}{m-l}$ t

$\Rightarrow$ bm - bl = am - bl + (b - a)t

$\Rightarrow$ (b - a)m = (b - a)t

$\Rightarrow$ t = m

$\therefore$ I = $\dfrac{b-a}{m-l}\int_l^m f\left(\dfrac{am-bl}{m-l}+\dfrac{b-a}{m-l}t\right)$dt

$\therefore$ $\int_a^b f(x)dx$ = $\dfrac{b-a}{m-l}\int_l^m f\left(\dfrac{am-bl}{m-l}+\dfrac{b-a}{m-l}x\right)$dx

$$\left[\text{By Property I i.e., } \int_a^b f(x)dx=\int_a^b f(z)dz\right]$$

Hence the result

Example 3: Evaluate

$$\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,\text{dx}$$

Sol: Let I = $\int_0^{\pi/2} \dfrac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}$ dx .....(1)

$$\left[\text{By Property IV i.e., } \int_0^a f(x)dx=\int_0^a f(a-x)dx\right]$$

$\therefore$ I = $\int_0^{\pi/2} \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}$ dx .....(2)

Adding (1) and (2), we get

2I = $\int_0^{\pi/2}\left[\dfrac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}+\dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\right]$dx

= $\int_0^{\pi/2} \dfrac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$ dx

= $\int_0^{\pi/2}$ 1.dx

$= [x]_0^{\pi/2}$

$= \dfrac{\pi}{2}$

$\Rightarrow \quad I = \dfrac{\pi}{4}$

Example 4: Evaluate $\int_0^{\pi/2} \dfrac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$

Sol: Let $I = \int_0^{\pi/2} \dfrac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$ ....(1)

Then $I = \int_0^{\pi/2} \dfrac{\sin^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)}\,dx$ $\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore \quad I = \int_0^{\pi/2} \dfrac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$ .....(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \left[\frac{\sin^3 x}{\sin^3 x + \cos^3 x} + \frac{\cos^3 x}{\cos^3 x + \cos^3 x}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x}\,dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$= [x]_0^{\pi/2}$$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore \quad 2I = \dfrac{\pi}{2}$

$\Rightarrow \quad I = \dfrac{\pi}{4}$

Example 5: Evaluate $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}$ dx

Sol: Let I = $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}$ dx .....(1)

Then I = $\int_0^{\pi/2} \frac{\sqrt{\cot x\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cot\left(\frac{\pi}{2}-x\right)}+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$ dx

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

∴ I = $\int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}$ dx .....(2)

Adding (1) and (2), we get

2I = $\int_0^{\pi/2} \left[\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}+\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\right]$ dx

= $\int_0^{\pi/2} \frac{\sqrt{\cot x}+\sqrt{\tan x}}{\sqrt{\cot x}+\sqrt{\tan x}}$ dx

= $\int_0^{\pi/2}$ 1 dx

= $[x]_0^{\pi/2}$

= $\frac{\pi}{2}$ - 0

= $\frac{\pi}{2}$

∴ 2I = $\frac{\pi}{2}$

⇒ I = $\frac{\pi}{4}$

Example 6: Evaluate $\int_0^{\pi/2} \dfrac{dx}{1+\cot x}$

Sol: Let I = $\int_0^{\pi/2} \dfrac{dx}{1+\cot x}$

= $\int_0^{\pi/2} \dfrac{dx}{1+\dfrac{\cos x}{\sin x}}$

= $\int_0^{\pi/2} \dfrac{\sin x}{\sin x+\cos x}$ dx .....(1)

$\therefore$ I = $\int_0^{\pi/2} \dfrac{\sin\left(\dfrac{\pi}{2}-x\right)}{\sin\left(\dfrac{\pi}{2}-x\right)+\cos\left(\dfrac{\pi}{2}-x\right)}$ dx

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore$ I = $\int_0^{\pi/2} \dfrac{\cos x}{\cos x+\sin x}$ dx .....(2)

Adding (1) and (2), we get

2I = $\int_0^{\pi/2} \dfrac{\sin x}{\sin x+\cos x} + \dfrac{\cos x}{\cos x+\sin x}$ dx

= $\int_0^{\pi/2} \dfrac{\cos x+\cos x}{\sin x+\cos x}$ dx

= $\int_0^{\pi/2}$ 1 dx

= $[x]_0^{\pi/2}$

= $\dfrac{\pi}{2}$ - 0

= $\dfrac{\pi}{2}$

$\therefore$ 2I = $\dfrac{\pi}{2}$

$\Rightarrow$ I = $\frac{\pi}{4}$

Example 7: Evaluate $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}$ dx

Sol: Let I = $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}$ dx .....(1)

= $\int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f\left[2a-(2a-x)\right]}$ dx

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

= $\int_0^{2a} \frac{f(2a-x)}{f(2a-x)+f(x)}$ dx

$\therefore$ I = $\int_0^{2a} \frac{f(2a-x)}{f(x)+f(2a-x)}$ dx ....(2)

Adding (1) and (2), we get

2I = $\int_0^{2a} \left[\frac{f(x)}{f(x)+f(2a-x)} + \frac{f(2a-x)}{f(x)+f(2a-x)}\right]$dx

= $\int_0^{2a} \frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}$ dx

= $\int_0^{2a}$ 1 dx

= $\left[x\right]_0^{2a}$

= 2a - 0

= 2a

$\therefore$ 2I = 2a

$\Rightarrow$ I = a

$\Rightarrow$ $\int_0^{2a} \frac{f(x)}{f(x)+f(2a-x)}$ dx = a

Example 8: If g(x) = $\int_0^{x} \cos^4 t\, dt$, then $g(x+\pi)$ is equal to $g(x) + g(\pi)$.

Sol: Here $g(x) = \int_0^x \cos^4 t\,dt$ ....(1)

$\therefore$ $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\,dt$

$= \int_0^x \cos^4 t\,dt + \int_\pi^{\pi+x} \cos^4 t\,dt$

$\therefore$ $g(x + \pi) = g(\pi) + f(x)$ .....(2)

where $f(x) = \int_\pi^{\pi+x} \cos^4 t\,dt$

Put $t = \pi + u$

$\therefore$ $dt = du$

when $t = \pi$; $\pi + u \Rightarrow u = 0$

when $t = \pi + x$; $\pi + x = \pi + u \Rightarrow u = x$

$f(x) = \int_0^x \cos^4 u\,du$ $[\because \cos^4(\pi + u) = \cos^4 u]$

$= \int_0^x \cos^4 t\,dt$ $\left[\because \int_a^b f(x)dx = \int_a^b f(t)dt\right]$

$\therefore$ $f(x) = g(x)$ [$\because$ of (1)]

$\therefore$ from (2), $g(x + \pi) = g(\pi) + g(x)$

Or $g(x + \pi) = g(x) + g(\pi)$

Self-Check Exercise

Q. 1 Prove that

$$\int_0^{2a} f(x)dx = \int_0^{2a} f(2a-x)dx$$

Q.2 Evaluate

$$\int_0^{\pi/2} \frac{dx}{1+\tan^3 x}$$

Q. 3 Evaluate

$$\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\,dx$$

Q. 4 Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\sqrt{\tan x}}\,dx$$

4.6 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Defined definite integral and discussed its geometrical interpretation.
2. Discussed in detailed definite integral as a limit of sum.
3. Discussed in detailed some properties of definite integral like the integration is independent of the change of variable; if the limits of definite integral are changed, then its value changes by minus sign only.
4. Discussed the property $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
5. Discussed and proved the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

4.8 Glossary

1. Let f be a function of x defined in the closed interval [a, b] and ϕ be another function, such that $\phi'(x) = f(x)$ for all x in the domain of f, then

   $$\int_a^b f(x)dx = [\phi(x)]_a^b$$

   $= \phi(b) = \phi(a)$

   is called the definite integral of the function $f(x)$ over the interval [a, b], a and b are called the limits of integration, a being the lower limit and b be the upper limit.

2. If f be a real valued non-negative continuous function defined on [a, b]. Then $\int_a^b f(x)dx = \lim_{h\to 0} h[f(a) + f(a + h) + ...+ f(a + (n-1))h]$, where $h = \frac{b-a}{n}$.

3. $\int_a^b f(x)dx = \int_a^b f(z)dz$

4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

6. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

4.8 Answers To Self-Check Exercises

Ans. 1 Result Verified i.e.

$$\int_0^{2a} f(x)dx = \int_0^{2a} f(2a-x)dx$$

Ans. 2 $\frac{\pi}{4}$

Ans. 3 $\frac{\pi}{4}$

Ans. 4 $\frac{\pi}{4}$

4.9 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.

4.10 Terminal Questions

1. If f is a continuous function on [a, b] prove that

   $\int_a^b f(x)dx = \int_{-b}^{-a} f(x)dx$, when f is even.

2. Show that

   $$\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{4}$$

3. Evaluate

   $$\int_0^{\pi/2} \frac{dx}{1+\tan x}$$

4. Evaluate

   $$\int_0^{\pi/2} \frac{dx}{1+\cot^3 x}$$

Unit - 5

Definite Integrals-II

Structure

5.1 Introduction

Definite integrals are an essential concept in calculus that allow us to calculate the accumulated change or total amount of a quantity over a given interval. Unlike indefinite integrals, which represent a family of functions, definite integrals produce a single numerical value. Properties of definite integrals help us to manipulate and evaluate integrals, making them powerful tools for solving a wide range of mathematical problems. The definite integral is a linear operator, which means it satisfies the properties of linearity. This property allows us to break down complicated integral into simpler ones and apply the integral to each term separately. The additively property of definite integrals states that the integral of a sum of functions is equal to the sum of the integral of the individual functions. This property allows us to split the interval of integration and evaluate the integral over each subinterval separately. It is particularly useful when dealing with piecewise-defined functions or functions with discontinuities. The reversal property of definite integrals states that changing the limits of integration changes the sign of the integral. This property allows us to reverse the direction of integration without changing the value of the integral. Further, the interval addition property states that if we split the interval of integration into multiple subintervals and integrate over each subinterval, the sum of these integrals is equal to the integral over the entire interval. All these properties we discussed in UNIT-4 and some further properties are discussed in this UNIT.

5.2 Learning Objectives

After studying this unit, you should be able to:-

- discuss the following properties of definite integral:-

(i) $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a - x)dx$

(ii) If f(2a - x) = f(x), then $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$

(iii) If f(x) = f(a + x), then

$$\int_0^{na} f(x)dx = n\int_0^{a} f(x)dx$$

(v) If f is an eoen function, then

$$\int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx$$

(vi) If f is an odd function, then

$$\int_0^{a} f(x)dx = 0$$

- Solve questions by using these properties.

5.3 Further Properties of Definite Integrals

Property V: $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx.$

Proof: We have $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{2a} f(x)\,dx$ .....(1)

Let $I^1 = \int_0^{2a} f(x)\,dx$

Put x = 2a - z, $\therefore$ dx = - dz

When x = a, z = a

When x = 2a, z = 0

$\therefore$ $I_1 = \int_a^0 f(2a - z)\,dz = \int_0^a f(2a - z)\,dz$ [$\because$ of Theorem II]

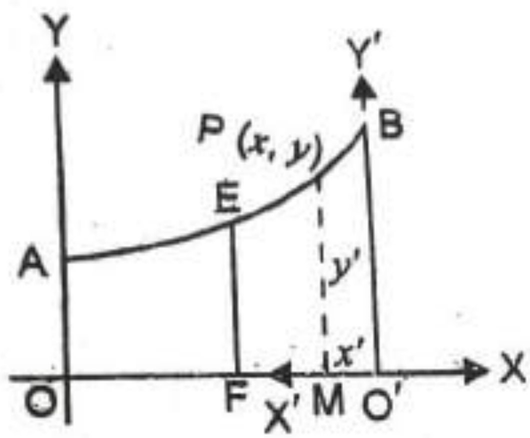
$= \int_0^a f(2a - x)\,dx$ [$\because$ of Theorem 1]

$\therefore$ $\int_0^{2a} f(x)\,dx = \int_0^a f(2a - x)\,dx$

$\therefore$ from (1), we get,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a - x)\,dx$$

Note. Geometrical Illustration



Let AB be the curve $y = f(x)$ ....(1)

and OA, O'B the ordinates $x = 0$, $x = 2a$

Let EF be the ordinate $x = a$

Let P(x, y) be any pt. on the curve EB and MP its ordinate.

For the area EFO'B take O'F and O'B as new axes and let (x',y') be the coordinates of P referred to them.

Then $x = OM = OO' - MO' = 2a - x'$,

$y = MP = y'$

$\therefore$ from (1), the equation of the curve EB becomes $y' = f(2a - x')$ or, dropping dashes, $y = f(2a - x)$ .....(2)

and the equation of the ordinates O'B, FE are $x = 0$, $x = a$.

$\therefore$ from (1) and (2), the theorem

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$$

expresses that area AOO'B = area AOFE + area BEFO, which is obvious.

Property VI: (i) If $f(2a - x) = f(x)$ then $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

(ii) If $f(2a - x) = -f(x)$, then $\int_0^{2a} f(x)\,dx = 0$

Proof: We have $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(2a - x)\,dx$ .....(1)

[$\because$ of Theorem V]

(i) If $f(2a - x) = f(x)$, then from (1), $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$

$\therefore \quad \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

(ii) If $f(2a - x) = f(x)$, then from (1), $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_0^{a} f(x)\,dx$

$\therefore \quad \int_0^{2a} f(x)\,dx = 0$

Cor. Changing a to $\frac{a}{2}$, we get

(i) if $f(a - x) = f(x)$, then $\int_0^{a} f(x)\,dx = 2\int_0^{\frac{a}{2}} f(x)\,dx$

(ii) if $f(a - x) = -f(x)$, then $\int_0^{a} f(x)\,dx = 0$

Property VII: If $f(x) = f(a + x)$, then $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$

Proof: We have $\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \ldots\ldots + \int_{(n-1)}^{na} f(x)\,dx$

Let $\quad I_1 = \int_a^{2a} f(x)\,dx$

Put $\quad x = a + z, \quad \therefore dx = dz$

When $x = a$, $z = 0$

When $x = 2a$, $z = a$

$\therefore \quad I_1 = \int_0^{a} f(a + z)\,dz = \int_0^{a} f(a + x)\,dx = \int_0^{a} f(x)\,dx \qquad [\because f(a + x) = f(x)]$

$\therefore \quad \int_0^{2a} f(x)\,dx = \int_0^{2a} f(x)\,dx \qquad .....(2)$

Again, let $I_2 = \int_{2a}^{3a} f(x)\,dx$

Put $x = a + z$, $\therefore\ dx = dz$

When $x = 2a$, $z = a$

When $x = 3a$, $z = 2a$

$$\therefore \quad I_2 = \int_{a}^{2a} f(a + z)\,dz$$

$$= \int_{a}^{2a} f(a + x)\,dx$$

$$= \int_{a}^{2a} f(x)\,dx \qquad [\because f(a + x) = f(x)]$$

$$\therefore \quad \int_{2a}^{3a} f(x)\,dx = \int_{0}^{a} f(x)\,dx \qquad [\because \text{ of } (2)]$$

and so on.

$\therefore$ from (1), we get

$$\int_{0}^{na} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx + \text{....... to n terms.}$$

$$\therefore \quad \int_{0}^{na} f(x)\,dx = n\int_{0}^{a} f(x)\,dx$$

Note. Geometrical Illustration

Let $P_0P_1P_2P_n$ be the curve $y = f(x)$ and OP_0, M_1P_1, M_2P_2,......., M_nP_n the ordinates $x = 0$, $x = a$, $x = 2a$,........., $x = na$.

$\because$ $f(x) = f(a + x)$ [Given]

$\therefore$ the corresponding ordinates shown dotted in the fig, are equal.

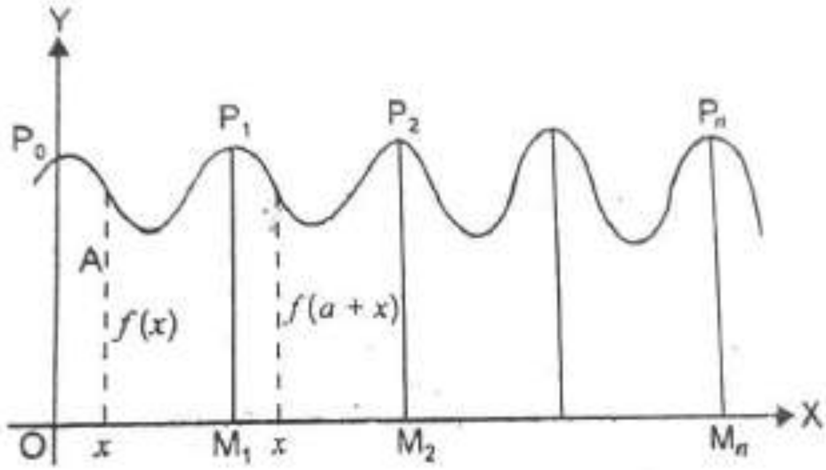
$\therefore$ the curve consists of the portion P_0P_1 from $x = 0$ to $x = a$.

$\therefore$ area $P_0OM_1P_1$ = area $P_1M_1M_2P_2$ = ...

$\therefore$ the theorem $\int_{0}^{na} f(x)\,dx = n\int_{0}^{a} f(x)\,dx$

expresses that area $P_0OM_nP_n$ = n. area $P_0OM_1P_1$

which is obvious.



Property VIII:

(i) If f(x) is an even function, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$

(ii) If f(x) is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

Proof: (i) we have $\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx$ .....(1)

Let $I_1 = \int_{-a}^{0} f(x)\, dx$

Put x = -y so that dx = -dy

When x = -a, y = a; When x = 0, y = 0

$\therefore$ $I_1 = \int_{a}^{0} f(-y)\, dy$

$= \int_{0}^{a} f(-y)\, dy$

$= \int_{0}^{a} f(-x)\, dx$

$= \int_{0}^{a} f(x)\, dx$ [$\because$ f(- x) = f(x) as f(x) is an even function]

$\therefore$ from (1), $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$

$\therefore$ $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(ii) We have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad ....(1)$$

Let $I_1 = \int_{-a}^{0} f(x)\,dx$, Put x = - y so that dx = -dy

When x = -a, y = a

When x = 0, y = 0

$\therefore$ $I_1 = \int_{a}^{0} f(-y)\,dy$

$= \int_{0}^{a} f(-y)\,dy$

$= \int_{0}^{a} f(-x)\,dx$

$= -\int_{0}^{a} f(x)\,dx$

[$\because$ $f(-x) = -f(x)$ as $f(x)$ is an odd function]

$\therefore$ from (1), $\int_{-a}^{a} f(x)\,dx = 0$

Property IX: $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a + b - x)\,dx$

Proof: Putting x = a + b - t

$\therefore$ dx = -dt

When x = a, t = b

When x = b, t = a, we get

$$\int_a^b f(a + b - t)\,(-dt)$$

$$= -\int_b^a f(a + b - t)\,dt$$

$$= \int_a^b f(a + b + t)\,dt$$

$$= \int_a^b f(a + b + x)\,dx$$

$$\Rightarrow \quad \int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$$

Let us improve our understanding of these results by looking at some of following examples:-

Example 1: Evaluate $\int_1^4 f(x)\,dx$, where

$$f(x) = \begin{cases} 4x+3 & if \quad 1 \le x \le 2 \\ 3x+5 & if \quad 2 \le x \le 4 \end{cases}$$

Sol: Here

$$f(x) = \begin{cases} 4x+3 & if \quad 1 \le x \le 2 \\ 3x+5 & if \quad 2 \le x \le 4 \end{cases}$$

Let I = $\int_1^4 f(x\,dx = \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx$

$$= \int_1^2 (4x + 3)dx + \int_2^4 (3x + 5)\,dx$$

$$= \left[4\frac{x^2}{2} + 3x\right]_1^2 + \left[\frac{x^2}{2} + 5x\right]_2^4$$

$$= \left[(8+6)-(2+3)\right]_1^2 + \left[\frac{3x^2}{2} + 5x\right]_2^4$$

$$= \left[(8+6)-(2+3)\right] + \left[(24+20)-(6+10)\right]$$

$$= (14 - 5) + (44 - 16)$$

$= 9 + 28 = 37$

Example 2: Evaluate

$$\int_0^3 f(x)\,dx, \text{ where } f(x) = \begin{cases} x & on \quad [0,1] \\ [x] & on \quad [1,3] \end{cases}$$

Sol: Here

$$f(x) = \begin{cases} x & on \quad 0 \le x \le 1 \\ [x] & on \quad 1 \le x \le 3 \end{cases}$$

Let $\quad I = \int_0^3 f(x) = \int_0^1 f(x)\,dx + \int_1^3 f(x)\,dx$

$$= \int_0^1 dx + \int_1^3 [x]\,dx$$

$$= \int_0^1 x\,dx + \int_1^2 [x]\,dx + \int_2^3 [x\}dx$$

[∵ [x] has discontinuities at x = 1, 2, 3]

$$= \int_0^1 x\,dx + [1]\,dx + \int_2^3 (2)dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + [x]_1^2 + 2[x]_2^3$$

$$= \left(\frac{1}{2} - 0\right) + (2 - 1) + 2(3 - 2)$$

$$= \frac{1}{2} + 1 + 2$$

$$= \frac{7}{2}$$

Example 3: Show that

$$\int_0^{\pi/2} \sin 2x \log \tan x\,dx = 0$$

Sol: Let $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$ .....(1)

$\therefore \quad I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2}-x\right) \log \tan \left(\frac{\pi}{2}-x\right) dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx$

$\Rightarrow \quad I = \int_0^{\pi/2} \sin 2x \log \left(\frac{1}{\tan x}\right) dx$

$= \int_0^{\pi/2} \sin 2x (\log 1 - \log \tan x) \, dx$

$= \int_0^{\pi/2} \sin 2x (0 - \log \tan x) \, dx$

$= - \int_0^{\pi/2} \sin 2x \log \tan x \, dx$

[$\because$ of (1)]

$= - I$

$\therefore \quad I = - I$

$\Rightarrow \quad 2I = 0$

$\Rightarrow \quad I = 0$

Hence the result

Example 4: Show that

$$\int_0^1 \log \left(\frac{1}{x}-1\right) dx = 0$$

Sol: Let $I = \int_0^1 \log \left(\frac{1}{x}-1\right) dx$

Put $\quad x = \cos^2 \theta$

$\therefore \quad dx = 2 \cos \theta (- \sin \theta) d\theta$

$= -2 \sin\theta \cos\theta \, d\theta$

$= -\sin 2\theta \, d\theta$

When $x = 0, \cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

When $x = 1, \cos^2\theta = 1 \Rightarrow \theta = 0$

$\therefore \quad I = -\int_{\pi/2}^{0} \log\left(\frac{1}{\cos^2\theta} - 1\right) \sin 2\theta \, d\theta$

$= \int_{0}^{\pi/2} \log\left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) \sin 2\theta \, d\theta$

$$\left[\because \int_b^a f(x)dx. = -\int_b^a f(x)dx\right]$$

$= \int_{0}^{\pi/2} \log\left(\frac{\sin^2\theta}{\cos^2\theta}\right) \sin 2\theta \, d\theta$

$= \int_{0}^{\pi/2} \log \tan^2\theta \sin 2\theta \, d\theta$

$= 2\int_{0}^{\pi/2} \log \tan\theta . \sin 2\theta \, d\theta$ .....(1)

$= 2\int_{0}^{\pi/2} \log \tan\left(\frac{\pi}{2} - \theta\right) \sin 2\left(\frac{\pi}{2} - \theta\right) d\theta$

$$\left[\because \int_b^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$= 2\int_{0}^{\pi/2} \log \cot\theta . \sin 2\theta \, d\theta$

$= 2\int_{0}^{\pi/2} \log\left(\frac{1}{\tan\theta}\right) \sin 2\theta \, d\theta$

$= 2 \int_0^{\pi/2}$ (log 1 - log tan θ) sin 2θ dθ

$\therefore$ $I = -2 \int_0^{\pi/2}$ log tan θ. sin 2θ dθ

= - I [$\because$ of (1)]

$\Rightarrow$ 2I = 0

$\Rightarrow$ I = 0

Example 5: Evaluate the following

$\int_0^{\pi} \sin^6 x \, dx$

Sol: Let $I = \int_0^{\pi} \sin^6 x \, dx$

$= 2 \int_0^{\pi} \sin^6 x \, dx$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ when } f(2a-x) = f(x) \text{and here } \sin^6(\pi - x) = \sin^6 x \right]$$

$= 2. \dfrac{5.3.1}{6.4.2} . \dfrac{\pi}{2}$

$= \dfrac{5\pi}{16}$

Example 6: Evaluate the following

$\int_0^{2\pi} \cos^{2x} x \, dx, x \in \mathrm{N}$

Sol: Let $I = \int_0^{2\pi} \cos^{2x} x \, dx$

$= 2 \int_0^{\pi} \cos^{2x} x \, dx$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ when } f(2a-x) = f(x) \text{ and here } \cos^{2n}(2\pi - x) = \cos^{2x} x\right]$$

$$= 2.2 \int_0^{\pi/2} \cos^{2x} x \, dx$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ when } f(2a-x) = f(x) \text{ and here } \cos^{2n}(\pi - x) = \cos^{2x} x\right]$$

$$= 4\,\frac{(2n-1)(2n-3).........3.1}{(2n)(2n-2).........4.2} \times \frac{\pi}{2}$$

$$= 2\pi\left[\frac{(2n-1)(2n-3).........3.1}{(2n)(2n-2).........4.2}\right]$$

Example 7: Evaluate

$$-\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx$$

Sol: Let $I = \displaystyle\int_{-\pi/4}^{\pi/4} \sin^2 x \, dx$

$$= 2 \int_0^{\pi/4} \sin^2 x \, dx$$

[$\because$ $\sin^2 x$ is an even function as $\sin^2(-x) = \sin^2 x$]

$$= \int_0^{\pi/4} \sin^2 x \, dx$$

$$= \int_0^{\pi/4} (1 - \cos 2x) \, dx$$

$$= \left[x - \frac{\sin 2x}{2}\right]_0^{\pi/4}$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right] - \left[0 - \frac{1}{2}\sin\sin 0\right]$$

$$= \left[\frac{\pi}{4} - \frac{1}{2}\right] - [0 - 0]$$

$$= \frac{\pi}{4} - \frac{1}{2}$$

Example 8: Show that

$$\int_0^{\pi/4} \log(1 + \tan x)\, dx = \frac{\pi}{8} \log 2$$

Sol: Let $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$ .....(1)

$$= \int_0^{\pi/4} \log\left[1 + \tan\left(\frac{\pi}{4} - x\right)\right] dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{1-\tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x + 1 - \tan x}{1+\tan x}\right] dx$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right) dx$$

$$= \int_0^{\pi/4} \log 2\, dx - \int_0^{\pi/4} \log(1 + \tan x)\, dx$$

$\therefore$ $I = \log 2 \int_0^{\pi/4} 1.dx - I$

[$\because$ of (1)]

$\Rightarrow$ 2I = log 2 $[x]_0^{\pi/4}$

= log 2 $\left[\frac{\pi}{4} - 0\right]$

= $\frac{\pi}{4}$ log 2

$\therefore$ I = $\frac{\pi}{8}$ log 2

Hence the result

Example 9: Show that

$$\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \pi \left[\frac{\pi}{2} - 1\right]$$

Sol: Let I = $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}$ dx ....(1)

= $\int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$ dx

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore$ I = $\int_0^{\pi} \left[\frac{(\pi - x)(-\tan x)}{-\sec x - \tan x}\right]$dx

= $\int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x}$ dx

= $\pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x}$ dx - $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}$ dx

= $\pi \int_0^{\pi} \frac{\tan x}{\sec x + \tan x} \times \frac{\sec x - \tan x}{\sec x - \tan x}$ dx - I

[$\because$ of (1)]

$\Rightarrow$ 2I = $\pi \int_0^{\pi} \frac{\tan x \sec x - \tan^2 x}{\sec^2 x - \tan^2 x}$ dx

$$= \pi \int_0^{\pi} \frac{\sec x \tan x - (\sec^2 x - 1)}{1} \, dx$$

$$= \pi \int_0^{\pi} (\sec x \tan x - \sec^2 x + 1) dx$$

$$= \pi \left[\sec x - \tan x + x\right]_0^{\pi}$$

$$= \pi \left[\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)\right]$$

$$= \pi \left[(-1 - 0 + \pi) - (1 - 0 + 0)\right]$$

$$= \pi (\pi - 2)$$

$$\Rightarrow \quad I = \pi\left(\frac{\pi}{2} - 1\right)$$

Example 10: Prove that

$$\int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} = \frac{\pi^2}{2a\sqrt{a^2 - 1}} \, dx$$

Sol. : Let $I = \int_0^{\pi} \frac{x\,dx}{a^2 - \cos^2 x} dx$

$$\therefore \quad I = \int_0^{\pi} \frac{\pi - x}{a^2 - \cos^2(\pi - x)} \, dx$$

$$\left[\because \int_0^{a} f(x)dx = \int_0^{a} f(a-x)dx\right]$$

$$= \int_0^{\pi} \frac{\pi - x}{a^2 - \cos^2 x} dx \qquad .....(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \left[\frac{x}{a^2 - \cos^2 x} + \frac{\pi - x}{a^2 - \cos^2 x}\right] dx$$

$$= \int_0^{\pi} \frac{\pi}{a^2 - \cos^2 x} dx$$

$\therefore \quad 2I = \pi \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x}$

$= \pi . 2 \int_0^{\pi} \frac{dx}{a^2 - \cos^2 x}$

$$\left[\because \int_0^{2a} f(x) dx = 2\int_0^{a} f(a-x) dx \text{ when } f(2a-x) = f(x) \text{and here} \frac{1}{a^2 - \cos^2(\pi - x)} = \frac{1}{a^2 - \cos^2 x} \right]$$

$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2(\cos^2 x + \sin^2 x) - \cos^2 x}$

$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \sin^2 x + (a^2 - 1)\cos^2 x}$

$\therefore \quad 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 \tan^2 x + (a^2 - 1)}$

Put $\tan x = t$

$\therefore \quad \sec^2 x \, dx = dt$

When $x = 0$, $t = \tan 0 = 0$

When $x = \frac{\pi}{2}$, $t = \tan \frac{\pi}{2} = \infty$

$\therefore \quad 2I = 2\pi \int_0^{\infty} \frac{dt}{a^2 t^2 + (a^2 - 1)}$

$= \frac{2\pi}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{\sqrt{a^2 - 1}}{a} \right)^2}$

$= \frac{2\pi}{a} \frac{1}{\frac{\sqrt{a^2 - 1}}{a}} \left[\tan^{-1} \left(\frac{t}{\frac{\sqrt{a^2 - 1}}{a}} \right) \right]_0^{\infty}$

$= \dfrac{2\pi}{a\sqrt{a^2-1}}$ [$\tan^{-1} \infty$ $\tan^{-1} 0$]

$= \dfrac{2\pi}{a\sqrt{a^2-1}} \left[\dfrac{\pi}{2} - 0\right]$

$\therefore$ $2I = \dfrac{\pi^2}{a\sqrt{a^2-1}}$

$\Rightarrow$ $I = \dfrac{\pi^2}{2a\sqrt{a^2-1}}$

Hence the result

Example 11: Find c such that

$\int_1^3 f(x)\,dx = 2f(c)$, where $f(x) = x^2 - 2x + 1$

Sol: Here $f(x) = x^2 - 2x + 1$

$\therefore$ $2 f(c) = 2 (c^2 - 2(+))$

Also $\int_1^3 f(x)\,dx = \int_1^3 (x^2 - 2x + 1)dx$

$= \left[\dfrac{x^3}{3} - \dfrac{2x^2}{2} + x\right]_1^3$

$= \left(\dfrac{27}{3} - \dfrac{18}{2} + 3\right) - \left(\dfrac{1}{3} - \dfrac{2}{2} + 1\right)$

$= (9 - 9 + 3) - \left(\dfrac{1}{3} - 1 + 1\right)$

$= 3 - \dfrac{1}{3}$

$= \dfrac{8}{3}$

We want to find c such that

$\int_1^3 f(x)\,dx = 2 f(c)$

$\therefore \quad \frac{8}{3} = 2\,(c^2 - 2c + 1)$

$\Rightarrow \quad c^2 - 2c + 1 = \frac{4}{3}$

Or $\quad (c - 1)^2 = \frac{4}{3}$

$\Rightarrow \quad c - 1 = \pm \frac{2}{\sqrt{3}}$

Or $\quad c = 1 + \frac{2}{\sqrt{3}},\ 1 - \frac{2}{\sqrt{3}}$

Or $\quad c = 1 + \frac{2}{\sqrt{3}} \qquad \left[\because c = 1 - \frac{2}{\sqrt{3}} [1, 3]\right]$

$\therefore \quad c = 1 + \frac{2}{\sqrt{3}}$

Example 12: Evaluate $\int_0^2 |x^2 + 2x - 3|\, dx$

Sol: Let $I = \int_0^2 |x^2 + 2x - 3|\, dx$

Now $\quad x^2 + 2x - 3 = (x + 3)(x + 1)$

For $\quad 0 \leq x \leq 1,\ x^2 + 2x - 3 \leq 0$

$\Rightarrow \quad |x^2 + 2x - 3| = -(x^2 + 2x - 3)$

and for $1 \leq x \leq 2,\ x^2 + 2x - 3 \geq 0$

$\Rightarrow \quad |x^2 + 2x - 3| = x^2 + 2x - 3$

$\therefore \quad I = \int_0^2 |x^2 + 2x - 3|dx = \int_0^1 |x^2 + 2x - 3|dx + \int_1^2 |x^2 + 2x - 3|dx$

$= \int_0^1 \left\{-(x^2 + 2x - 3)\right\} dx + \int_1^2 (x^2 + 2x - 3)dx$

$= -\int_0^1 (x^2 + 2x - 3)dx + \int_1^2 (x^2 + 2x - 3)dx$

$= - \left[\frac{x^3}{3}+x^2-3x\right]_0^1 + \left[\frac{x^3}{3}+x^2-3x\right]_1^2$

$= \left[\left(\frac{1}{3}+1-3\right)-(0+0-0)\right] + \left[\left(\frac{8}{3}+4-6\right)-\left(\frac{1}{3}+1-3\right)\right]$

$= \frac{5}{3} + \frac{2}{3} + \frac{5}{3}$

$= \frac{12}{3} = 4$

Example 13: show that

$$\int_{-1}^{1} \sqrt{|x|-x}\, dx = -\frac{2\sqrt{2}}{3}$$

Sol: Let I = $\int_{-1}^{1} \sqrt{|x|-x}\, dx$

$= \int_{-1}^{0} \sqrt{|x|-x}\, dx + \int_{0}^{1} \sqrt{|x|-x}\, dx$

$= \int_{-1}^{0} \sqrt{|x|-x}\, dx + \int_{0}^{1} \sqrt{|x|-x}\, dx$

$$\left[\because |x| = \begin{cases} -x, & x<0 \\ x, & x\geq 0 \end{cases}\right]$$

$= \int_{-1}^{0} (-2x)^{\frac{1}{2}}\, dx + 0$

$= \left[\frac{(-2x)^{3/2}}{(-2)\left(\frac{3}{2}\right)}\right]_{-1}^{0}$

$= \frac{1}{3}\left[(-2x)^{3/2}\right]_{-1}^{0}$

$= \frac{1}{3}\left[0-2^{3/2}\right]$

$= -\frac{1}{3} \times 2^{3/2}$

$= -\frac{1}{3} 2\sqrt{2}$

$= -\frac{2\sqrt{2}}{3}$

Example 14: If [x] stands for integral part of x, then show that

$$\int_{-1}^{1} [4x]\, dx = -4$$

Sol: Since [4x] has integral values at

$x = -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$\therefore$ [4x] is discontinuous at

$x = -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$\therefore$ $I = \int_{-1}^{1} [4x]\, dx$

$= \int_{-1}^{-3/4} [4x]\, dx + \int_{-3/4}^{-1/2} [4x]dx + \int_{-1/4}^{-1/4} [4x]\, dx + \int_{-1/4}^{0} [4x]dx + \int_{0}^{1/4} [4x]\, dx$

$+ \int_{1/4}^{1/2} [4x]dx + \int_{1/2}^{3/4} [4x]\, dx + \int_{3/4}^{1} [4x]dx$

$= \int_{-1}^{-3/4} [-4]\, dx + \int_{-3/4}^{-1/2} [-3]dx + \int_{-1/2}^{-1/4} (-2)dx + \int_{3/4}^{1} (-1)dx +$

$+ \int_{0}^{1/4} (0)\, dx + \int_{1/4}^{1/2} dx + \int_{1/2}^{3/4} 2dx + \int_{3/4}^{1} 3dx$

$= -4[x]_{-1}^{-3/4} - 3\, [x]_{-3/4}^{-1/2} - 2\, [x]_{-1/2}^{-1/4} - [x]_{-1/4}^{0} + 0$

+ $[x]_{1/4}^{1/2}$ + 2 $[x]_{1/2}^{3/4}$ + 3$[x]_{3/4}^{1}$

= 4 $\left[-\frac{3}{4}+1\right]$ - 3$\left[-\frac{1}{2}+\frac{3}{4}\right]$ - 2$\left[-\frac{1}{4}+\frac{1}{2}\right]$ - $\left[0+\frac{1}{4}\right]$ + 0

+ $\left[\frac{1}{2}-\frac{1}{4}\right]$ + 2$\left[\frac{3}{4}-\frac{1}{2}\right]$ + 3 $\left[1-\frac{3}{4}\right]$

= - 4$\left(\frac{1}{4}\right)$ - 3$\left(\frac{1}{4}\right)$ - 2$\left(\frac{1}{4}\right)$ - $\left(\frac{1}{4}\right)$ + $\left(\frac{1}{4}\right)$ + 2$\left(\frac{1}{4}\right)$ + 3$\left(\frac{1}{4}\right)$

= $\frac{1}{4}$ [-4 -3 -2 -1 +1 +2 +3]

= $\frac{1}{4}$ (- 4)

= -1

Self-Check Exercise

Q.1 Evaluate $\int_0^{\pi/2} \frac{\sin 8x \log(\cot x)}{\cos 2x}$ dx

Q.2 Evaluate $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x}+\sqrt{a-x}}$ dx

Q.3 Evaluate $\int_1^{4}$ f(x) dx, where f(x) = $\begin{cases} 8x+3, & 1 \le x \le 3 \\ x^3, & 3 \le x \le 4 \end{cases}$

Q.4 Evaluate $\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}$ dx

Q.5 Evaluate $\int_0^{\pi/2}$ (2 log sin x - log sin 2x)dx

Q.6 Evaluate $\int_0^{\pi} \frac{x \sin x}{1+\sin x}$ dx

Q.7 Evaluate $\int_0^{2}$ |x - 1| dx

Q.8 If [x] stands for integral part of x, then evaluate $\int_1^2 [2x]\, dx$

5.4 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Discussed and proved the following properties of definite integrals:-

(i) $\int_0^{2a} f(x)\, dx = \int_0^{a} f(x)\, dx + \int_0^{a} f(2a - x)\, dx$

(ii) If $f(2a - x) = f(x)$, then $\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$

(iii) If $f(2a - x) = - f(x)$, then $\int_0^{2a} f(x)\, dx = 0$

(iv) If $f(x) = f(a + x)$, then $\int_0^{na} f(x)\, dx = n\int_0^{a} f(x)\, dx$

(v) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\, dx = 2\int_0^{a} f(x)\, dx$$

(vi) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\, dx = 0$

2. Solved some questions using above stated properties.

5.5 Glossary

1. Properties of definite integrals help us to manipulate and evaluate integrals, making them powerful tool for solving a wide range of mathematical problems.
2. The definite integral is a linear operator, which means it satisfies the properties of linearity.

5.6 Answers To Self-Check Exercises

Ans. 1 0

Ans. 2 $\frac{0}{2}$

Ans. 3 $81\frac{3}{4}$

Ans. 4 0

Ans. 5 $-\frac{\pi}{2}$ log 2

Ans. 6 $\frac{\pi^2}{2}$ - π

Ans. 7 1

Ans. 8 $\frac{5}{2}$

5.7 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.
2. G.B. Thomas and R.L. Finney, Calculus, 9^{th} Ed., Pearson Education, 2005.

5.8 Terminal Questions

1. Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}$ dx

2. Evaluate $\int_1^3 f(x)$ dx, where $f(x) = \begin{cases} x^2-8, & 1 \le x \le 2 \\ -2x, & 2 \le x \le 3 \end{cases}$

3. Show that $\int_0^{\pi/2}$ log tan x dx = 0

4. Show that $\int_0^1$ x (1 - x)^x dx = $\frac{1}{(x+1)(x+2)}$

5. Evaluate $\int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x}$ dx

6. Evaluate $\int_{-\pi/2}^{\pi/2}$ |sin x| dx

7. If [x] stands for integral part of x, then show that

   $\int_1^2 [x^2]dx = 5 - \sqrt{2} - \sqrt{3}$

Unit - 6

Reduction Formulae For

$$\int \sin^n x\,dx,\ \int \cos^n x\,dx$$

Structure

6.1 Introduction

For certain integrals, both definite and indefinite, the function being integrated (that is, the "integrand") consists of a product of two functions, one of which involves an unspecified integer, say n. Using the method of integration by parts, it is sometimes possible to express such on integral in terms of a similar integral where n has been replaced by (n-1), or sometimes (n-2). The relationship between the two integrals is called a "reduction formula" and by repeated application of this formula, the original integral may be determined in terms of x. Hence, a reduction formula for an integral is a formula which connects the integral with one or two other integrals in which the integrand is of the same type but of lower degree or of lower order.

The repeated application of the reduction formula enables us to express the given integral into such a simple integral which can be evaluated by one of the methods done in lower class. Thus, reduction formulae play an important role in systematic integration. There are two important methods of obtaining reduction formulae:-

(i) the method of integration by parts, and

(ii) the method of connecting the integrals.

6.2 Learning Objectives

After studying this Unit, you should be able to:-

- Define reduction formula
- Find a reduction formula for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$
- Find a reduction formula for $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$
- Find a reduction formula for $\int \sec^n x\, dx$ and $\int \text{cosec}^n x\, dx$
- Solve questions related to these reduction formulae

6.3 Reduction Formulae For $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$

Sol: Method to find a reduction formula for $\int \sin^n x\, dx$

1. Write $\sin^n x$ as $\sin^{n-1}x. \sin x$
2. Integrate by parts taking $\sin^{n-1}x$ as first function.
3. Replace $\cos^2 x$ by $1-\sin^2 x$
4. Transpose to get terms of $\int \sin^n x\, dx$ on L.H.S.

The above method is also applicable for finding a reduction formula for $\int \cos^n x\, dx$. Now we proceed further to find the reduction formulae for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$.

I. Reduction formula for $\int \sin^n x\, dx$

Let $In = \int \sin^n x\, dx$

$= \int \sin^{n-1}x. \sin x\, dx$

[Note this step]

$= \sin^{n-1} x\,(- \cos x) - \int (n-1) \sin^{n-2} x \cos x\,(- \cos x)\, dx$

[Integrating by parts Here $f(x) = \sin^{n-1}x$, $g(x) = \sin x$]

$= - \sin^{n-1} x \cos x + (n-1)\int \sin^{n-2}x \cos^2 x\, dx$

$= - \sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\,(1 - \sin^2 x)\, dx$

[Note this step]

$= - \sin^{n-1}x \cos x + (n-1)\int \sin^{n-2}x\, dx - (n-1)\int \sin^n x\, dx$

$\therefore$ In = - $\sin^{n-1}x \cos x$ + (n-1) I_{n-2} - (n-1) In Transposing the last term on the L.H.S., we get

$\Rightarrow$ $[1 + (n-1)]\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$

$\Rightarrow$ $n.\, I_n = -\sin^{n-1} x \cos x + (n-1)\, I_{n-2}$

$$\Rightarrow \quad I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

which is the required reduction formula.

II. Reduction formula for $\int \cos^n x\, dx$

Let $I_n = \int \cos^n x\, dx$

$= \int \cos^{n-1}x \cos x\, dx$

[Note this step]

$= \cos^{n-1}x (\sin x) - \int (n-1) \cos^{n-2}x (-\sin x)(\sin x)\, dx$

[Integrating by parts. Here $f(x) = \cos^{n-1}x$, $g(x) = \cos x$]

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2x\, dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x (1-\cos^2x)\, dx$

$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\, dx - (n-1)\int \cos^2x\, dx$

$\therefore$ In = $\cos^{n-1}x \sin x$ + (n-1) I_{n-2} - (n-1) In Transposing the last term on the L.H.S., we get

$[1 + (n-1)\, I_n] = \cos^{n-1}x \sin x + (n-1)\, I_{n-2}$

$\Rightarrow$ $n.\, I_n = \cos^{n-1}x.\, \sin x + (n-1)\, I_{n-2}$

$$\Rightarrow \quad I_n = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2}$$

which is the required reduction formula

6.4 Reduction Formulae For $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$

Method to find a reduction formula for $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$

For finding a reduction formula for $\int \tan^n x\, dx$, we write $\tan^n x$ as $\tan^{n-2}x.\, \tan^2x$ and then change $\tan^2x$ to $(\sec^2x-1)$

Similarly for finding a reduction formula for $\int \cot^n x\, dx$, we write $\cot^n x$ as $\cot^{n-2}x.\, \cot^2x$ and then change $\cot^2x$ to $(\operatorname{cosec}^2 x-1)$

Now we proceed further to find the reduction formulae for $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$

I. Reduction formula for $\int \tan^n x\, dx$

Let $I_n = \int \tan^n x\, dx$

$= \int \tan^n x\, dx.\ \tan^2 x\, dx$

$= \int \tan^{n-2} x\, (\sec^2 x - 1)\, dx$

[$\because \tan^2 x = \sec^2 x - 1$]

$= \int \tan^{n-2} x \sec 2x\, dx - \int \tan^{n-2} x\, dx$

$= \int \tan^{n-2} x \sec^2 x\, dx - I_{n-2}$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

$$\left[\text{Using the formula} \left\{ f(x) \right\}^n f^1(x) dx = \frac{\left\{ f(x) \right\}^{n+1}}{n+1} \right]$$

$\therefore$ $I_n = \dfrac{1}{n-1} \tan^{n-1} x - I_{n-2}$

which is the required reduction formula

II. Reduction formula for $\int \cot^n x\, dx$

Let $I_n = \int \cot^n x\, dx$

$= \int \cot^{n-2} x \cot^2 x\, dx$

[Note this step]

$= \int \cot^{n-2} x\, (\mathrm{cosec}^2 x - 1)\, dx$ [$\because \cot^2 x = \mathrm{cosec}^2 x - 1$)

$= \int \cot^{n-2} x\, \mathrm{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx$

$= -\int \cot^{n-2} x\, (-\mathrm{cosec}^2 x) dx - I_{n-2}$

$= -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$

$$\left[\text{By using the formula n+1} \int \{ f(x) \}^n f^1(x)dx = \frac{\{ f(x) \}^{n+1}}{n+1} \right]$$

$\therefore$ $I_n = -\frac{1}{n-1} \cot^{n-1}x - I_{n-2}$

which is the required formula

6.5 Reduction Formulae For $\int \sec^n x \, dx$ or $\int \operatorname{cosec}^n x \, dx$

Method to find a reduction formula for $\int \sec^n x \, dx$ or $\int \operatorname{cosec}^n x \, dx$

For finding a reduction formula for $\int \sec^n x \, dx$, we write

$\sec^n x$ as $\sec^{n-2}x \sec^2 x$

Now We integrate by parts, taking

$f(x) = \sec^{n-2}x$ and $g(x) = \sec^2 x$

After the integration we change $\tan^2 x$ to $\sec^2 x-1$.

Similarly for $\int \cos^n x \, dx$

Now we proceed further to find the reduction formula for $\int \sec^n x \, dx$ and $\int \operatorname{cosec}^n x \, dx$

I. Reduction formula for $\int \sec^n x \, dx$

Let $I_n = \int \sec^n x \, dx$

$= \int \sec^{n-2}x \sec^2 x \, dx$

[Note this step]

Integrating by parts, taking $f(x) = \sec^{n-2}x$ and $g(x) = \sec^2 x$

$= \sec^{n-2}x \tan x - \int (n-2) \sec^{n-3}x \sec(x \tan x \tan x \, dx$

$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x \tan^2 x \, dx$

[Type is not same as of I_n]

$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x (\sec^2 x - 1) \, dx$

($\because$ $\tan^2 x = \sec^2 x-1$)

$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x\, (\sec^n x\, dx + (n-2) \int \sec^{n-2}x\, dx$

$= \sec^{n-2}x \tan x - (n-2)I_n + (n-2)\, I_{n-2}$

Transposing the second term on the L.H.S., we have

$\{1+(n-2)\}\, I_n = \sec^{n-2}x \tan x + (n-2)\, I_{n-2}$

Or $\quad I_n = \frac{1}{n-1}\sec^{n-2}x \tan x + \frac{n-2}{n-1} I_{n-2}$

which is the required reduction formula

II. Reduction formula for $\int \operatorname{cosec}^n x\, dx$

Let $\quad I_n = \int \operatorname{cosec}^{n-2}x\, dx$

$= \int \operatorname{cosec}^{n-2}x \operatorname{cosec}^2 x\, dx$

[Note this step]

Integrating by parts, taking $f(x) = \operatorname{cosec}^{n-2}x$ and $g(x) = \operatorname{cosec}^2 x$

$= \operatorname{cosec}^{n-2}x\,(-\cot x) - \int \{(n-2)\cos ec^{n-3}x.(-\cos ec\, x \cot x)\}(-\cot x)\, dx$

$= -\operatorname{cosec}^{n-2}x \cot x - (n-2) \int \operatorname{cosec}^{n-2}x \cot^2 x\, dx$

$= -\operatorname{cosec}^{n-2}x \cot x - (n-2) \int \operatorname{cosec}^{n-2}x\, (\operatorname{cosec}^2 x-1)\, dx$

$= -\operatorname{cosec}^{n-2}x. \cot x - (n-2) \int \operatorname{cosec}^n x\, dx + (n-2) \int \operatorname{cosec}^{n-2}x\, dx$

$= -\operatorname{cosec}^{n-2}x \cot x - (x-2)\, I_n + (n-2)\, I_{n-2}$

Transposing the second term on the L.H.S. we have

$[1 + (n-2)]\, I_n = -\operatorname{cosec}^{n-2}x \cot x + (x-2)\, I_{n-2}$

$\Rightarrow \quad (n-1)\, I_n = -\operatorname{cosec}^{n-2}x \cot x + (x-2)\, I_{n-2}$

$\Rightarrow \quad I_n = -\frac{1}{n-1}\operatorname{cosec}^{n-2}x \cot x + \frac{n-2}{n-1} I_{n-2}$

which is the required reduction formula

Let us do some examples to clear the idea:-

Example 1: obtain a reduction formula for

$\int \cos^n x\, dx$. Hence evaluate $\int \cos^6 x\, dx$

Sol: Do 6.2 (II)

The reduction formula for $\int$ $\cos^n x\,dx$ is

$$I_n = \frac{\cos^{n-1} x.\sin x}{n} + \frac{n-1}{n} I_{n-2} \qquad .....(1)$$

Put n = 6, we get

$$I_6 = \int \cos^6 x\,dx = \frac{\cos^{6-1} x.\sin x}{6} + \frac{6-1}{6} I_4$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6} I_4 \qquad .....(2)$$

Putting n = 4 in (1), we have

$$I_4 = \frac{\cos^{4-1} x.\sin x}{4} + \frac{4-1}{4} I_2$$

$$= \frac{1}{4}\cos^3 x \sin x + \frac{3}{4} I_2 \qquad ....(3)$$

Putting I_2 in (1), we have

$$I_2 = \frac{\cos^{2-1} x.\sin x}{2} + \frac{2-1}{2} I_0$$

$$= \frac{\cos x \sin x}{2} + \frac{1}{2} I_0 \qquad ....(4)$$

Also $\quad I0 = \int (\cos x)^0\,dx = \int 1\,dx = x \qquad .....(5)$

Putting values of $I_0$from (5) in (4), then of I_2 from (4) in (3) and then of I_4 from (3) in (2), we get

$$\int \cos^6 x\,dx = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left[\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\left\{\frac{\cos x \sin x}{2} + \frac{1}{2}x\right\}\right]$$

$$= \frac{1}{6}\cos^5 x \sin x + \frac{5}{24}\cos^3 x \sin x + \frac{15}{48}\cos x \sin x + \frac{15}{48} x$$

Example 2: Use reduction formula to integrate $\int \tan^4 x\,dx$

Sol: $\int \tan^n x\,dx = \int \tan^{n-2} x.\ \tan^2 x\,dx$

$$= \int \tan^2 x\,dx(\sec^2 x - 1)dx$$

$= \int \tan^{n-2}x \sec^2 x\, dx - \int \tan^{n-2}x\, dx$

$\therefore \quad \int \tan^n x = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2}x\, dx$ .....(1)

which is the required reduction formula

Putting n = 4 in (1), we get

$\int \tan^4 x\, dx = \dfrac{\tan^x x}{3} - \int \tan^2 x\, dx$

$= \dfrac{1}{3}\tan^3 x - \int (\sec^2 x - 1) dx$

$= \dfrac{1}{3}\tan^3 x - (\tan x - x)$

$= \dfrac{1}{3}\tan^3 x - \tan x + x$

Example 3: If $\int \tan^n x\, dx = A(x) + B(x) \int \tan^{n-2}x\, dx$,

get a formula connecting $\int \tan^m x\, dx$ with $\int \tan^{m+2}x\, dx$

Sol: Here $\int \tan^n x\, dx = A(x) + B(x) \int \tan^{n-2}x\, dx$ ....(1)

Now $\int \tan^n x\, dx = \int \tan^{n-2}x \tan^2 x\, dx$

$= \int \tan^{n-2}x (\sec^2 n-1)\, dx$

$= \int \tan^{n-2}x \sec^2 x\, dx - \int \tan^{n-2}x\, dx$

$\therefore \quad \int \tan^n x\, dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2}x\, dx$

comparing it with (1), we get

$A(x) = \dfrac{1}{n-1}\tan^{n-1}x, \; B(x) = -1$

$\therefore$ (1) becomes $\int \tan^n x\, dx = \dfrac{1}{n-1} \tan^{n-1}x - \int \tan^{n-2}x\, dx$

$\therefore \quad \int \tan^{n-2}x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n}x\,dx$

Replacing n by m+2, we get

$$\int \tan^{m}x\,dx = \frac{1}{m+1}\tan^{m+1}x - \int \tan^{m+2}x\,dx,$$

which is required formula

Example 4: If $U_n = \int_0^{\pi/4} \tan^n x\,dx$, n > 1,

Show that $U_n + U_{n-2} = \frac{1}{n-1}$

Deduce the value of U_3

Deduce the value of U_5

Sol: $\int \tan^n x\,dx = \int \tan^{n-2}x \tan^2 x\,dx$

$$= \int \tan^{n-2}x (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2}x \sec^2 x\,dx - \int \tan^{n-2}x\,dx$$

$$= \frac{(\tan x)^{n-1}}{n-1} - \int \tan^{n-2}x\,dx$$

$$\therefore \quad \int_0^{\pi/4} \tan^n x\,dx = \left[\frac{(\tan x)^{n-1}}{n-1}\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2}x\,dx$$

$$= \left[\frac{1}{n-1} - 0\right] - \int_0^{\pi/4} \tan^{n-2}x\,dx$$

$$\therefore \quad U_n = \frac{1}{n-1} - U_{n-2}$$

$$\Rightarrow \quad U_n + U_{n-2} = \frac{1}{n-1} \qquad .....(1)$$

Put n = 5, 3 in (1), we have

$$U_5 + U_3 = \frac{1}{4} \qquad .....(2)$$

$U_3 + U_1 = \frac{1}{2}$ ....(3)

Now $U_1 = \int_0^{\pi/4} \tan x \, dx$

$= \left[-\log(\cos x)\right]_0^{\pi/4}$

$= -\log\cos\frac{\pi}{4} + \log\cos 0$

$= -\log\frac{1}{\sqrt{2}} + \log 1$

$= -\left[\log 1 - \log\sqrt{2}\right] + \log 1$

$= \log\sqrt{2}$

$= \log 2^{\frac{1}{2}}$

$= \frac{1}{2}\log 2$

$\therefore$ from (3), $U_3 = \frac{1}{2} - U_1$

$= \frac{1}{2} - \frac{1}{2}\log 2$

From (2), $U_5 = \frac{1}{4} - U_3$

$= \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2$

$\therefore$ $U_5 = \frac{1}{2}\log 2 - \frac{1}{4}$

Example 5: Obtain a reduction formula for $\int \cot^n x \, dx$, n being a +ve integer. Hence evaluate $\int \cot^5 x \, dx$.

Sol: Let $I_n = \int \cot^n x \, dx$

$= \int \cot^{n-2}x \cot^2x \, dx$

$= \int \cot^{n-2}x (\operatorname{cosec}^2 x - 1)dx$

$= \int \cot^{n-2}x \operatorname{cosec}^2x \, dx - \int \cot^{n-2}x \, dx$

$\therefore \quad I_n = -\dfrac{\cot^{n-1} x}{n-1} - I_{n-2}$

Or $\quad I_n = -\dfrac{1}{n-1}\cot^{n-1}x - I_{n-2}$ ....(1)

which is the required reduction formula

Putting n = 5, 3 in (1), we get

$I_5 = -\dfrac{1}{4}\cot^4x - I_3$ ....(2)

and $\quad I_3 = -\dfrac{1}{2}\cot^2x - I_1$

$= -\dfrac{1}{2}\cot^2x - \int \cot x \, dx$

$= -\dfrac{1}{2}\cot^2x - \log|\sin x|$

$\therefore$ From (2), $I_5 = -\dfrac{1}{4}\cot^4x + \dfrac{1}{2}\cot^2x + \log|\sin x|$

Example 6: Obtain a reduction formula for $\int \sec^{2n+1} x \, dx$. Hence evaluate $\int \sec^5x \, dx$

Sol: Let $I_{2n+1} = \int \sec^{2n+1}x \, dx$

$= \int \sec^{2n-1}x \sec^2x \, dx$

$= \sec^{2n-1}x \tan x - \int \left\{(2n-1)\sec^{2n-2} x.\sec x \tan x\right\} \tan x \, dx$

[Integrating by parts]

$= \sec^{2n-1}x \tan x - (2n-1) \int \sec^{2n-1}x \tan^2x \, dx$

$= \sec^{2n-1}x \tan x -(2n-1) \int \sec^{2n+1}x (\sec^2x - 1)dx$

= $\sec^{2n-1}x \tan x - (2n-1) \int \sec^{2n+1}x\, dx + (2n-1) \int \sec^{2n-1}x\, dx$

$\therefore$ $I_{2n+1} = \sec^{2n-1}x \tan x - (2n-1)\, I_{2n+1} + (2n-1)\, I_{2n-1}$

$\Rightarrow$ $[1 + (2n-1)]\, I_{2n+1} = \sec^{2n-1}x \tan x + (2n-1)\, I_{2n-1}$

$\Rightarrow$ $2n.\, I_{2n+1} = \sec^{2n-1}x \tan x + (2n-1)\, I_{2n-1}$

$\Rightarrow$ $I_{2n+1} = \frac{1}{2n} \mathrm{se}\bar{c}x \tan x + \frac{2n-1}{2n} I_{2n-1}$

which is the required reduction formula. ....(1)

Putting n = 2, 1 in (1), we get

$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} I_3$ ....(2)

and $I_3 = \frac{1}{2} \sec x \tan x + \frac{1}{2} I_1$ .....(3)

Now $I_1 = \int \sec x\, dx = \log |\sec x + \tan x|$

$\therefore$ from (3),

$I_3 = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log |\sec x + \tan x|$

From (2), we have

$I_5 = \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \log |\sec x + \tan x|$

Self-Check Exercise

Q.1 Use reduction formula to integrate

$\int \tan^5 x\, dx$

Q.2 If $I_n = \int_0^{\pi/4} \tan^n \theta\, d\theta$, show that

$I_{n-1} + I_{n+1} = \frac{1}{n}$, hence evaluate $\int_0^a \frac{x^5}{(2a^2 - x^2)^3} dx$.

Q.3 Use reduction formula integrate $\int \cot^3 x\, dx$

Q.4 Obtain a reduction formula for $\int \sec^n x\, dx$. Hence evaluate $\int \sec^5 x\, dx$

6.6 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Defined reduction formula
2. Derived reduction formula for $\int \sin^n x\, dx$ and $\int \cos^n x\, dx$
3. Derived reduction formula for $\int \tan^n x\, dx$ and $\int \cot^n x\, dx$
4. Derived reduction formula for $\int \sec^n x\, dx$ and $\int \text{cosec}^n x\, dx$
5. Solved examples related to above stated reduction formulae.

6.7 Glossary

1. A reduction formula for an integral is a formula which connects the integral with one or two other integrals in which the integrand is of the same type but of lower degree or of lower order.
2. Two important methods of obtaining reduction formulae are:-
   (i) the method of integration by parts, and
   (ii) the method of connecting the integrals.

6.8 Answers To Self-Check Exercises

Ans. 1 $\frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \log|\cos x|$

Ans. 2 $\frac{1}{4}(2\log 2 - 1)$

Ans. 3 $-\frac{1}{2}\cot^2 x - \log|\sin x|$

Ans. 4 $I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$;

and $\frac{\sec^3 x \tan x}{4} + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

6.9 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.

6.10 Terminal Questions

1. Obtain a reduction formula for $\int \tan^n x\, dx$. Hence evaluate $\int \tan^3 x$.

2. If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, prove that $I_n + I_{n-2} = \frac{1}{n-1}$, n being a positive integer > 1. Hence evaluate I_5.

3. Obtain a reduction formula for $\int \cot^n x\, dx$, n being a +ve integer. Hence evaluate $\int \cot^4 x\, dx$.

4. Obtain a reduction formula for $\int \operatorname{cosec}^{2n+1} x\, dx$ and hence evaluate $\int \operatorname{cosec}^5 x\, dx$

Unit - 7

Reduction Formulae For

$$\int x^n (\log x)^m dx, \int e^{ax} \sin^n x \, dx$$

Structure

7.1 Introduction

Reduction formulae for integration are mathematical techniques used to evaluate indefinite integrals that involve repeated applications of integration by parts. These formulas allow us to express a given integral in terms of integrals that are simpler or have a lower degree. By reducing the complexity of the integrals, we can after solve them more easily. Reduction formulae are particularly useful when dealing with integrals involving powers of trigonometric functions, logarithmic functions or exponential functions, where integration by parts alone may not lead to a solution. By employing reduction formulae, we can often find closed-form solutions to these integrals.

7.2 Learning Objectives

After studying this unit, you should be able to:-

- Derive the reduction formula for $\int x^n e^{ax} dx$ and solve questions related to it.

- Derive the reduction formula for $\int x^n (\log x)^m \, dx$ and solve questions related to it.
- Derive the reduction formula for $\int e^{ax} \sin^n x \, dx$ and solve questions related to it.

7.3 Reduction Formula For $\int x^n e^{ax} dx$

Let $I_n = \int x^n e^{ax} \, dx$

$$= x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} \, dx$$

[Integration by parts by taking $f(x) = x^n$, $g(x) = e^{ax}$]

$$= \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$$

$$\therefore \quad I_n = \frac{1}{a} x^n e^{ax} - \frac{n}{a} I_{n-1}$$

which is the required reduction formula

Let us do some examples:-

Example 1: Obtain a reduction formula for

$\int x^n e^{-x} dx$. Hence evaluate $\int_0^\infty x^n e^{-x} dx$,

where n is a positive integer

Sol: Let $I_n = \int x^n e^{-x} \, dx$

$$= x^n \left(\frac{e^{-x}}{-1}\right) - \int (nx^{n-1}) \left(\frac{e^{-x}}{-1}\right) dx$$

[Integrating by parts by taking $f(x) = x^n$, $g(x) = e^{-x}$]

$$= - x^n e^{-x} + n \int x^{n-1} e^{-x} \, dx$$

$$\therefore \quad In = - x^n e^{-x} + n I_{n-1} \qquad ....(1)$$

which is the required reduction formula From (1)

$$\int x^n e^{-x} dx = -x^n e^{-x} + n \int x^{n-1} e^{-x} dx$$

$$\therefore \quad \int_0^\infty x^n e^{-x} dx = \left[-x^n e^{-x}\right]_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$$

$$= -\left[\underset{x\to\infty}{Lt}\frac{x^n}{e^x} - 0\right] + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$= -[0 - 0] + n\int_0^\infty x^{n-1} e^{-x}\, dx$$

$$\left[\because \underset{x\to\infty}{Lt}\frac{x^n}{e^x} = 0, n > 0\right]$$

$$\therefore \quad \int_0^\infty x^n e^{-x}\, dx = n\int_0^\infty x^{n-1} e^{-x}\, dx \qquad ....(2)$$

Changing n to n-1, we get

$$\int_0^\infty x^{n-1} e^{-x}\, dx = (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx \qquad .....(3)$$

From (2) and (3), we get

$$\int_0^\infty x^n e^{-x}\, dx = n(n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

Generalizing, we get

$$\int_0^\infty x^n e^{-x}\, dx = n(n-1)(n-2) .... 2.1\int_0^\infty x^{n-n} e^{-x}\, dx$$

$$= \lfloor \underline{n} \int_0^\infty e^{-x}\, dx$$

$$= \lfloor \underline{n} \left[\frac{e^{-x}}{-1}\right]_0^\infty$$

$$= -\lfloor \underline{n}\, [0 - 1] \qquad \left[\because \underset{x\to\infty}{Lt} e^{-x} = 0\right]$$

$$= \lfloor \underline{n}$$

$$\therefore \quad \int_0^\infty x^n e^{-x}\, dx = \lfloor \underline{n}$$

Example 2: Obtain a reduction formula for $\int x^n e^{2x}\, dx$. Hence evaluate $\int x^3 e^{2x}\, dx$.

Sol: Let $I_n = \int x^n e^{2x}\, dx$

$= x^n \dfrac{e^{2x}}{2} - \int n\, x^{n-1} \dfrac{e^{2x}}{2}\, dx$

[Integrating by parts]

$= \dfrac{1}{2} x^n e^{2x} - \dfrac{n}{2} \int x^{n-1} e^{2x}\, dx$

$\therefore \quad I_n = \dfrac{1}{2} x^n e^{2x} - \dfrac{n}{2} I_{n-1}$ ....(1)

which is the required reduction formula.

Putting n = 3, 2, 1 in (1), we get

$I_3 = \dfrac{1}{2} x^3 e^{2x} - \dfrac{3}{2} I_2$ .....(2)

$I_2 = \dfrac{1}{2} x^2 e^{2x} - \dfrac{2}{2} I_1$ .....(3)

$I_1 = \dfrac{1}{2} xe^{2x} - \dfrac{1}{2} I_0$ .....(4)

Now $I_0 = \int x^0 e^{2x}\, dx$

$= \int e^{2x}\, dx$

$= \dfrac{e^{2x}}{2}$

$\therefore$ from (4), $I_1 = \dfrac{1}{2} xe^{2x}, \dfrac{1}{4} e^{2x}$

From (3), $I_2 = \dfrac{1}{2} x^2 e^{2x} - \dfrac{1}{2} xe^{2x} + \dfrac{1}{4} e^{2x}$

From (2), $I_3 = \dfrac{1}{2} x^3 e^{2x} - \dfrac{3}{4} x^2 e^{2x} + \dfrac{3}{4} xe^{2x} - \dfrac{3}{8} e^{2x}$

$\therefore \quad \int x^3 e^{2x}\, dx = e^{2x} \left(\dfrac{x^3}{2} - \dfrac{3}{4} x^2 + \dfrac{3}{4} x - \dfrac{3}{8} \right)$

Self-Check Exercise-1

Q.1 Obtain a reduction formula for

$\int$ $x^n e^x$ dx. Hence evaluate I_4
Q.2 Evaluate $\int$ $x^5 e^x$ dx

7.4 Reduction Formula For $\int x^n (\log x)^m dx$

Let $I_m = \int x^n (\log x)^m dx$

$= \int (\log x)^m x^n dx$

$= (\log x)^m \frac{x^{n+1}}{n+1} - \int \left\{m(\log x)^{m-1}\frac{1}{x}\right\} \frac{x^{n+1}}{n+1} dx$

[Integrating by parts Here we take $f(x) = (\log x)^m$ and $g(x) = x^n$]

$= \frac{x^{n+1}}{n+1}(\log x)^m - \frac{m}{n+1}\int x^n (\log x)^{m-1} dx$

$\therefore \quad I_m = \frac{x^{n+1}}{n+1}(\log x)^m - \frac{m}{n+1} I_{m-1}$ ....(1)

which is the required reduction formula

Let us do some examples:-

Example 3: Obtain a reduction formula for $\int x^m (\log x)^n dx$, $x > 0$; m, n are natural numbers and hence evaluate $\int_0^1 x^4 (\log x)^3 dx$

Sol: Let $I_n = \int x^m (\log x)^n dx$

$= \int (\log x)^n . x^m dx$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \int \left\{n(\log x)^{n-1}\frac{1}{x}\right\} \frac{x^{m+1}}{m+1} dx$

[Integrating by parts]

$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1} dx$

$\therefore \quad I_n = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{n-1}$ ....(1)

which is the required reduction formula

Now

$$\int x^m (\log x)^n\, dx = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx$$

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = \left[\frac{x^{m+1}}{m+1}(\log x)^n\right]_0^1 \frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx$$

$$= \frac{1}{m+1}\left[0 - \underset{x\to 0}{Lt}\, x^{m+1}(\log x)^n\right] - \frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx$$

[$\because$ $x^{m+1}(\log x)^n = 0$ fir x = 1, m $\geq$ 0 and n is a positive integer]

$$= \frac{1}{m+1}[0 - 0] - \frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx$$

[$\because$ $\underset{x\to 0}{Lt}\, x^{m+1} (\log x)^n = 0$, m $\geq$ 0 and n is a positive integer]

$$\therefore \quad \int_0^1 x^m (\log x)^n\, dx = -\frac{n}{m+1}\int_0^1 x^m (\log x)^{n-1}\, dx$$

Putting m = 4, n = 3 in (2), we get .....(2)

$$\int_0^1 x^4 (\log x)^3\, dx = -\frac{3}{4+1}\int_0^1 x^4 (\log x)^2\, dx \qquad .....(3)$$

Putting m = 4, n = 2 in (2), we get

$$\int_0^1 x^4 (\log x)^2\, dx = -\frac{2}{4+1}\int_0^1 x^4 (\log x)^1\, dx \qquad ....(4)$$

Putting m = 4, n = 1 in (2), we get

$$\int_0^1 x^4 (\log x)^1\, dx = -\frac{1}{4+1}\int_0^1 x^4 (\log x)^0\, dx$$

$$= -\frac{1}{4+1}\int_0^1 x^4\, dx$$

$$= -\frac{1}{5}\left[\frac{x^5}{5}\right]_0^1$$

$$= -\frac{1}{25}[1 - 0]$$

$= -\ \frac{1}{25}$

$\therefore$ from (4), $\int_0^1 x^4 (\log x)^2\, dx = \left(-\frac{2}{5}\right) \times \left(-\frac{1}{25}\right) = \frac{2}{125}$

From (3), $\int_0^1 x^4 (\log x)^3\, dx = \left(-\frac{3}{5}\right) \times \left(\frac{2}{125}\right) = -\frac{6}{625}$

Example 4: Evaluate $\int_0^1 x^m (\log x)^n\, dx$

Sol: Let $I_n = \int x^m (\log x)^n\, dx$

$= \int (\log x)^n x^m\, dx$

$= (\log x)^n \frac{x^{m+1}}{m+1} - \int \left\{ n(\log x)^{n-1} \frac{1}{x} \right\} \frac{x^{m+1}}{m+1}\, dx$

[Integrating by parts]

$= \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1}\, dx$

$\therefore$ $I_n = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} I_{n-1}$ .....(1)

which is the required reduction formula

Now

$\int x^m (\log x)^n\, dx = \frac{x^{m+1}}{m+1} (\log x)^n - \frac{n}{m+1} \int x^m (\log x)^{n-1}\, dx$

$\therefore$ $\int_0^1 x^m (\log x)^n\, dx = \left[\frac{x^{m+1}}{m+1} (\log x)^n \right]_0^1 - \frac{n}{m+1} \int_0^1 x^m (\log x)^{n-1}\, dx$

$= \frac{1}{m+1} \left[0 - \underset{x \to 0}{Lt}\, x^{m+1} (\log x)^n \right] - \frac{n}{m+1} \int_0^1 x^m (\log x)^{n-1}\, dx$

[$\because$ $x^{m+1}(\log x)^n = 0$ for x = 1, m $\geq$ 0 and n is a positive integer]

$= \frac{1}{m+1} [0 - 0] - \frac{n}{m+1} \int_0^1 x^m (\log x)^{n-1}\, dx$

$[\because \underset{x \to 0}{Lt}\ x^{m+1} (\log x)^n = 0$, $m \geq 0$ and n is a positive integer]

$\therefore \quad \int_0^1 x^m (\log x)^n \, dx = -\frac{n}{m+1} \int_0^1 x^m (\log x)^{n-1} \, dx$

Let $\quad I_n^1 = \int_0^1 x^m (\log x)^n \, dx \quad$ ....(2)

$\therefore$ from (2), we have

$$I_n^1 = -\frac{n}{m+1} I_{n-1}^1$$

Changing n to n-1, n-2, ..., 2, 1, we get

$$I_{n-1}^1 = -\frac{n-1}{m+1} I_{n-2}^1$$

$$I_{n-2}^1 = -\frac{n-2}{m+1} I_{n-3}^1$$

....

....

$$I_2^1 = -\frac{2}{m+1} I_1^1$$

$$I_1^1 = -\frac{1}{m+1} I_0^1$$

Now $\quad I_0^1 = \int_0^1 x^m (\log x)^0 \, dx$

$$= \int_0^1 x^m \, dx$$

$$= \left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$= \frac{1}{m+1} [1 - 0]$$

$$= \frac{1}{m+1}$$

Now

$$I_n^1 = \left(-\frac{n}{m+1}\right)\left(-\frac{n-1}{m+1}\right)\left(-\frac{n-2}{m+1}\right)......\left(\frac{-2}{m+1}\right)\left(\frac{-1}{m+1}\right)\left(\frac{1}{m+1}\right)$$

[From the above equation]

$= \frac{(-1)^n}{(m+1)^{n+1}}$ [n(n-1)(n-2).......2.1]

$\therefore \quad \int_0^1 x^m (\log x)^n \, dx = \frac{(-1)^n \underline{|n}}{(m+1)^{n+1}}$

Example 5: Find a reduction formula for $\int \frac{x^m}{(\log x)^n} dx$

Sol: $\int \frac{x^m}{(\log x)^n} dx = \int x^{m+1} \left\{(\log x)^{-n} \frac{1}{x}\right\} dx$

[Note this step]

$= x^{m+1} \frac{(\log x)^{-n+1}}{-n+1} - \int (m+1)\, x^m \frac{(\log x)^{-n+1}}{-n+1} dx$

$$\left[\text{Integrating by parts and using} \int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1}\right]$$

$= - \frac{1}{n-1} \frac{x^{m+1}}{(\log x)^{n-1}} + \frac{m+1}{n-1} \int \frac{x^m}{(\log x)^{n-1}} dx$

which is the required reduction formula

Self-Check Exercise-2

Q.1 Obtain a reduction formula for $\int x^m(\log x)^n dx$. Hence evaluate $\int x^m(\log x)^2 dx$

Q.2 Obtain a reduction formula for $I_{m,n} = \int x^m (\log x)^n dx$.

Hence evaluate $\int x^4 (\log x)^3 dx$

7.5 Reduction Formula For $\int e^{ax} \sin^n x \, dx$

Let $I_n = \int e^{ax} \sin^n x \, dx$

$= \int \sin^n x \, e^{ax} dx$

$$= \sin^n x \left(\frac{e^{ax}}{a}\right) - \int (n \sin^{n-1} x \cos x) \left(\frac{e^{ax}}{a}\right) dx$$

[Integrating by parts by taking $f(x) = \sin^n x$ and $g(x) = e^{ax}$]

$$= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a} \int (\sin^{n-1}) x \cos x)\, e^{ax}\, dx$$

$$= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a}\left[(\sin^{n-1} x \cos x)\left(\frac{e^{ax}}{a}\right) - \int \left\{n-1)\sin^{n-2} x \cos x \cos x + \sin^{n-1} x(-\sin x)\right\}\left(\frac{e^{ax}}{a}\right) dx\right]$$

$$= \frac{1}{a} e^{ax} \sin^n x - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1} x \cos x}{a} - \int \left\{n-1)\sin^{n-2} x \cos^2 x - \sin^n x\right\}\frac{e^{ax}}{a} dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1} x \cos x}{a} - \int \left\{n-1)\sin^{n-2} x(1-\sin^2 x) - \sin^n x\right\}\frac{e^{ax}}{a} dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1} x \cos x}{a} - \int \left\{n-1)\sin^{n-2} x - n\sin^n x\right\}\frac{e^{ax}}{a} dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a}\left[\frac{e^{ax}\sin^{n-1} x \cos x}{a} - \frac{n-1}{a}\int e^{ax}\sin^{n-2} x dx + \frac{n}{a}\int e^{ax}\sin^n x\, dx\right]$$

$$= \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2} e^{ax} \sin^{n-1} x \cos x + \frac{n(n-1)}{a^2}\int e^{ax} \sin^{n-2} x\, dx - \frac{n^2}{a^2}\int e^{ax} \sin^n x\, dx$$

$$\therefore \quad I_n = \frac{e^{ax}\sin^n x}{a} - \frac{n}{a^2} e^{ax} \sin^{n-1} x \cos x + \frac{n(n-1)}{a^2} I_{n-1} - \frac{n^2}{a^2} I_n$$

$$\Rightarrow \quad \left(1+\frac{n^2}{a^2}\right) I_n = \frac{ae^{ax}\sin^n x - ne^{ax}\sin^{n-1} x\cos x}{a^2} + \frac{n(n-1)}{a^2}$$

$$\Rightarrow \quad (a^2 + n^2)\, In = e^{ax} \sin^{n-1}x\, (a \sin x - n \cos x) + n(n - 1)\, I_{n-2}$$

$$\Rightarrow \quad I_n = \frac{e^{ax}\sin^{n-1} x(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

which is the required reduction formula

Let us do example to clarify it:-

Example 6: If $U_n = \int_0^\infty e^{-x} \sin^n x\, dx$, then prove that $U_n = \frac{n(n-1)}{n^2+1} U_{n-2}$

Hence evaluate $\int_0^{\infty} e^{-x} \sin^4 x \, dx$

Sol: Now $U_n = \int_0^{\infty} e\text{-}x \sin^n x \, dx$

$$= \left[\sin^n x \frac{e^{-x}}{-1}\right]_0^{\infty} - \int_0^{\infty} n(\sin^{n-1} x \cos x) \frac{e^{-x}}{-1} dx$$

$$= (0 - 0) + n\int_0^{\infty} (\sin^{n-1} x \cos x)e^{-x} dx$$

[$\because$ $e^{-x} \to 0$ as $x \to \infty$, $\sin^n 0 = 0$]

$$= n\left\{\left[(\sin^{n-1} x \cos x)\frac{e^{-x}}{-1}\right]_0^{\infty} - \int_0^{\infty}\left\{\sin^{n-1} x(-\sin x) + (n-1)(\sin^{n-2} x \cos x \cos x)\frac{e^{-x}}{-1}\right\}dx\right\}$$

$$= n(0 - 0) + n\int_0^{\infty} \{- \sin^n x + (n-1) \sin^{n-2} x (1-\sin^2 x)\}e^{-x}dx$$

$$= n\int_0^{\infty} \{- \sin^n x + (n-1) \sin^{n-2} x - (n-1) \sin^2 x\}e^{-x} dx$$

$$= n\int_0^{\infty} \{- n \sin^n x + (n-1) \sin^{n-2} x\}e^{-x} dx$$

$\therefore$ $U_n = - n^2 U_n + n (n-1) U_{n-2}$

Or $(1 + n^2) U_n = n(n-1) U_{n-2}$

$\therefore$ $(1 + n^2) U_n = n(n-1) U_{n-2}$ .....(1)

which is required reduction formula

Putting n = 4, 2 in (1), we get

$U_4 = \frac{12}{17} U_2$ ....(2)

and $U_2 = \frac{2}{5} U_0$ ....(3)

Now $U_0 = \int_0^{\infty} e^{-x} dx$

$= \left[\frac{e^{-x}}{-1}\right]_0^\infty$

= [0 + 1]

= 1

∴ from (3), $U_2 = \frac{2}{5}$

From (2), $U_4 = \frac{12}{17} \times \frac{2}{5}$

$= \frac{24}{85}$

∴ $\int_0^\infty e^{-x} \sin^4 x \, dx = \frac{24}{85}$

Self-Check Exercise-3

Q.1 Obtain a reduction formula for $\int e^{ax} \cos^n x \, dx$. Hence evaluate $\int e^{ax} \cos^4 x \, dx$

7.6 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Derived the reduction formula for $\int x^n a^{ax} dx$ and solved some questions related to it.

2. Derived the reduction formula for $\int x^n (\log x)^m dx$ and solved some questions related toit.

3. Derived the reduction formula for $\int e^{ax} \sin^n x \, dx$ and solved some questions related to it.

7.7 Glossary

1. Reduction formulae allow us to express a given integral in terms of integrals that are simpler or have a lower degree.

2. If $I_n = \int x^n e^{ax} dx$, then its reduction formula is

   $I_n = \frac{1}{a} x^n$ eax - $\frac{n}{a} I_{n-1}$

3. If $I_m = \frac{n}{a} x^n (\log x)^m dx$, then its reduction formula is

$$I_m = \frac{x^{n+1}}{n+1}(\log x)^m - \frac{m}{n+1} I_{m-1}$$

4. If $I_n = \int e^{ax} \sin^n x \, dx$, then its reduction formula is

$$I_n = \frac{e^{ax}\sin^{n-1} x(a\sin x - n\cos x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$$

7.8 Answers To Self-Check Exercises

Self-Check Exercise-1

Ans. 1 $I_n = x^n e^x - n I_{n-1}$

and $I4 = (x^4 - 4x^3 + 12x^2 - 24x + 24)e^x$

Ans. 2 $\int x^5 e^x dx = e^x(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120)$

Self-Check Exercise-2

Ans. 1 $I_n = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{n-1}$

and $\frac{x^{m+1}}{m+1}\left[(\log x)^2 - \frac{2\log x}{m+1} - \frac{2}{(m+1)^2}\right]$

Ans. 2 $I_{m,n} = \frac{x^{m+1}}{m+1}(\log x)^n - \frac{n}{m+1} I_{m,n-1}$

and $\frac{x^5}{625}\left[125(\log x)^3 - 75(\log x)^2 + 30\log x - 6\right]$

Self-Check Exercise-3

Ans. 1 $I_n = \frac{e^{ax}\cos^{n-1} x(a\cos x + n\sin x)}{a^2+n^2} + \frac{n(n-1)}{a^2+n^2} I_{n-2}$

and $\frac{e^{ax}\cos^3 x(a\cos x + 4\sin x)}{a^2+16}\left[\frac{e^{ax}\cos^3 x(a\cos x + 2\sin x)}{a^2+4} + \frac{2e^{ax}}{a(a^2+4)}\right]$

7.9 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, 2005.

2. H. Anton, I. Bivens and S. Dans, Calculus, John Wiley and Sons (Asia) P. Ltd. 2002.

7.10 Terminal Questions

1. Obtain a reduction formula for $\int x^n e^{ax} dx$. Hence evaluate I^3 and I^4

2. Obtain a reduction formula for $\int x^m (\log x)^n dx$, x > 0, m, n are natural numbers and hence evaluate $\int_0^1 x^2 (\log x)^3 dx$

3. If $I_n = \int_0^1 (\log x)^n dx$, prove that

   $I_n + n I_{n-1} = x (\log x)^n$

4. Find a reduction formula for

   $\int_0^{\pi} e^{-x} \sin^n x \, dx$

Unit - 8

Reduction Formulae For

$$\int x^m \sin nx\, dx, \int \sin^m x \cos^n x\, dx$$

Structure

8.1 Introduction

A reduction formula for an integral is a formula which connects the integral with one or two other integrals in which the integrand is of the same type but of lower degree or lower order. Reduction formulae, also known as recurrence relations, are mathematical formulas that express integrals or series in terms of simpler forms. They are widely used in calculus to simplify complex calculations and solve difficult integration problems. Reduction formulae enable us to express a given integral or series in a recursive manner, allowing us to break down complex expressions into simpler components. This technique is particularly useful when dealing with functions that do not have elementary ant derivatives or when closed-form solutions are not readily available.

8.2 Learning Objectives

After studying this unit, you should be able to:-

- Derive reduction formula for $\int x \sin^n x\, dx$ and $\int x \cos^n x\, dx$ and solve questions related to it.
- Derive reduction formula for $\int \cos^m x \sin nx\, dx$ and $\int \cos^m x \cos nx\, dx$
- Derive reduction formula for $\int \sin^m x \cos^n x\, dx$ and solve questions related to it.

8.3 Reduction Formula For $\int x \sin^n x\, dx$ And $\int x \cos^n x\, dx$

Reduction formula for $\int x \sin^n x\, dx$

Let $I_n = \int x \sin^n x\, dx$

$= \int (x \sin^{n-1} x) \sin x\, dx$

$= (x \sin^{n-1} x)(-\cos x) - \int [x\{(n-1)\sin^{n-2} x \cos x\} + \sin^{n-1} x.0](-\cos x)\, dx$

[Integrating by parts by taking $f(x) = x \sin^{n-1} x$ and $g(x) = \sin x$]

$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x \cos^2 x\, dx + \int \sin^{n-1} x \cos x\, dx$

$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x (1-\sin^2 x)\, dx + \int \sin^{n-1} x \cos x\, dx$

$= -x \sin^{n-1} x \cos x + (n-1)\int x \sin^{n-2} x\, dx - (n-1)\int x \sin^n x\, dx + \dfrac{\sin^n x}{n}$

$\Rightarrow \quad I_n = -x \sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n + \dfrac{\sin^n x}{n}$

$\Rightarrow \quad [1 + (n-1)] I_n = -x \sin^{n-1} x \cos x + \dfrac{\sin^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow \quad n.\, I_n = -x \sin^{n-1} x \cos x + \dfrac{\sin^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow \quad I_n = -\dfrac{x \sin^{n-1} x \cos x}{n} + \dfrac{\sin^n x}{x^2} + \dfrac{n-1}{n} I_{n-2}$

which is the required reduction formula.

Reduction formula for $\int x \cos^n x\, dx$

Let $I_n = \int x \cos^n x \, dx$

$= \int (x \cos^{n-1}x) \cos x \, dx$

$= (x \cos^{n-1}x)(\sin x) - \int \left[x\left\{(n-1)\cos^{n-2} x(-\sin x)\right\} + \cos^{n-1} x.1\right](\sin x)\, dx$

[Integrating by parts by taking $f(x) = x \cos^{n-1}x$ and $g(x) = \cos x$]

$= x \cos^{n-1} x \sin x + (n-1) \int x \cos^{n-2}x \sin^2 x \, dx - \int \cos^{n-1}x \sin x \, dx$

$= x \cos^{n-1} x \sin x + (n-1) \int x \cos^{n-2}x (1-\sin^2 x)\, dx + \int (\cos x)^{n-1} (-\sin x)\, dx$

$= \int \cos^{n-1} x \sin x + (n-1) \int x \cos^{n-2}x \, dx - (n-1) \int x \cos^n x \, dx + \dfrac{(\cos x)^n}{n}$

$\therefore \quad I_n = x \cos^{n-1}x \sin x + (n-1) I_{n-2} - (n-1) I_n + \dfrac{1}{n} \cos^n x$

$\Rightarrow \quad [1 + (n-1)] I_n = x \cos^{n-1}x \cos_{n-1}x \sin x + (n-1) I_{n-2} + \dfrac{\cos^n x}{n}$

$\Rightarrow \quad n. I_n = x \cos^{n-1}x \sin x + \dfrac{\cos^n x}{n} + (n-1) I_{n-2}$

$\Rightarrow \quad I_n = \dfrac{1}{n} x \cos^{n-1}x \sin x + \dfrac{1}{n^2} \cos^n x + \dfrac{n-1}{n} I_{n-2}$

which is the required reduction formula

8.4 Reduction Formula For $\int x^m \sin nx \, dx$ And $\int x^m \cos nx \, dx$

Reduction formula for $\int x^m \sin nx \, dx$

Let $I_{m,n} = \int x^m \sin nx \, dx$

$= x^m\left(-\dfrac{\cos n x}{n}\right) - \int m x^{m-1} \left(-\dfrac{\cos n x}{n}\right) dx$

[Integrating by parts by taking $f(x) = x^m$ and $g(x) = \sin nx$]

$= -\dfrac{x^m \cos n x}{n} + \dfrac{m}{n} \int x^{m-1} \cos nx \, dx$

$= -\dfrac{x^m \cos n x}{n} + \dfrac{m}{n} \left[x^{m-1}\left(\dfrac{\sin nx}{n}\right) - \int (m-1)x^{m-2}\left(\dfrac{\sin nx}{n}\right) dx\right]$

[Again integrating by parts]

$$= -\frac{x^m \cos nx}{n} + \frac{m}{n^2} x^{m-1} \sin nx - \frac{m(m-1)}{n^2} \int x^{m-2} \sin nx\, dx$$

$$\therefore \quad I_{m,n} = \frac{x^m \cos nx}{n} + \frac{m}{n^2} x^{m-1} \sin nx - \frac{m(m-1)}{n^2} I_{m-2,n}$$

which is the required reduction formula

Reduction formula for $\int x^m \cos nx\, dx$

Let $I_{m,n} = \int x^m \cos nx\, dx$

$$= x^m \left(\frac{\sin nx}{n}\right) - \int m x^{m-1} \left(\frac{\sin nx}{n}\right) dx$$

[Integrating by parts by taking $f(x) = x^m$ and $g(x) = \cos nx$]

$$= \frac{x^m \sin nx}{n} - \frac{m}{n} \int x^{m-1} \sin nx\, dx$$

$$= \frac{x^m \sin nx}{n} - \frac{m}{n}\left[x^{m-1}\left(-\frac{\cos nx}{n}\right) - \int (m-1)x^{m-2}\left(-\frac{\cos nx}{n}\right)dx\right]$$

[Again integrating by parts]

$$= \frac{x^m \sin nx}{n} + \frac{m}{n^2} x^{m-1} \cos nx - \frac{m(m-1)}{n^2} \int x^{m-2} \cos nx\, dx$$

$$\therefore \quad I_{m,n} = \frac{x^m \sin nx}{n} + \frac{m}{n^2} x^{m-1} \sin nx - \frac{m(m-1)}{n^2} I_{m-2,n}$$

which is the required reduction formula

8.5 Reduction formula for $\int \cos^m x \sin nx\, dx$

Let $I_{m,n} = \int \cos^m x \sin nx\, dx$

$$= \cos^m x \left(-\frac{\cos nx}{n}\right) - \int m \cos^{m-1} x\, (-\sin x) \left(-\frac{\cos nx}{n}\right) dx$$

[Integrating by parts by taking $f(x) = \cos^m x$ and $g(x) = \sin nx$]

$$\therefore \quad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} \int \cos^{m-1} x\, (\cos nx \sin x) \qquad ....(1)$$

Now $\sin (n-1)x = \sin nx \cos x - \cos nx \sin x$

$\Rightarrow$ cos nx sin x = sin nx cos x - sin (n-1)x

Substituting value of cos nx sin x in (1), we have

$$I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin (n-1)x]dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m} x \sin nx\, dx + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)\, x\, dx$$

$$\therefore \quad I_{m,n} = -\frac{\cos^m x \cos nx}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad \left(1+\frac{m}{n}\right) I_{m,n} = -\frac{\cos^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$$

$$\Rightarrow \quad (m + n)\, I_{m,n} = - \cos^m x \cos nx + m\, I_{m-1,\, n-1}$$

$$\Rightarrow \quad I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

which is the required reduction formula

Reduction formula for $\int \cos^m x \cos nx\, dx$

Let $I_{m,n} = \int \cos^m x \cos nx\, dx$

$$= \cos^m x \left(\frac{\sin nx}{n}\right) - \int m \cos^{m-1} x\,(- \sin x)\left(\frac{\sin nx}{n}\right)dx$$

[Integrating by parts by taking $f(x) = \cos^m x$ and $g(x) = \cos nx$]

$$\therefore \quad I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\,(\sin nx \sin x)dx \qquad \text{....(1)}$$

Now cos (n-1) x = cos nx cos x + sin nx sin x

$\Rightarrow$ sin nx sin x = cos (n-1) x - cos nx cos x

Substituting value of sin nx sin x in (1), we get

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x\,[\cos (n-1)\, x - \cos nx \cos x]dx$$

$$= \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{m-1} x \cos (n-1)\, x\, dx - \frac{m}{n}\int \cos^m x \cos nx\, dx$$

$\Rightarrow$ $I_{m,n} = \dfrac{\cos^m x \sin nx}{n} + \dfrac{m}{n} I_{m-1,n-1} - \dfrac{m}{n} I_{m,n}$

$\Rightarrow$ $\left(1+\dfrac{m}{n}\right) I_{m,n} = \dfrac{\cos^m x \sin nx}{n} + \dfrac{m}{n} I_{m-1,n-1}$

$\Rightarrow$ $(m + n) I_{m,n} = \cos^m x \sin nx + m I_{m-1,n-1}$

$\Rightarrow$ $I_{m,n} = \dfrac{\cos^m x \sin nx}{n} + \dfrac{m}{m+n} I_{m-1,n-1}$

which is the required reduction formula

8.6 Reduction Formula For

$\int \sin^m x \cos^n x \, dx$

(A) Let $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$= \int \sin^m x \cos^{n-1} x \cos x \, dx$

[Note this step]

$= \int \cos^{n-1} x (\sin^m x \cos x) dx$

$= \cos^{n-1} x \dfrac{\sin^{M+1} x}{m+1} - \int (n-1) \cos^{n-2} x (- \sin x) \dfrac{\sin^{M+1} x}{m+1} dx$

[Integration by parts by taking $f(x) = \cos^{n-1} x$ and $g(x) = \sin^m x \cos x$]

$= \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} \int \cos^{n-2} x \sin^{m+2} x \, dx$

$= \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \sin^2 x \, dx$

$= \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} \int \cos^{n-2} x \sin^m x (1-\cos^2 x) \, dx$

$= \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \sin^2 x \, dx - \dfrac{n-1}{m+1} \int \cos^n x \sin^m x \, dx$

$\therefore$ $I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} I_{m,n-2} - \dfrac{n-1}{m+1} I_{m,n}$

$\Rightarrow$ $\left(1+\dfrac{n-1}{m+1}\right) I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+1} I_{m,n-2}$

$\Rightarrow$ (m + n) $I_{m,n} = \sin^{m+1}x \cos^{n-1}x + (n-1)\, I_{m,n-2}$

$\Rightarrow$ $I_{m,n} = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+n} I_{m,n-2}$

$\Rightarrow$ $\int \sin^m x \cos^n x\, dx = \dfrac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \dfrac{n-1}{m+n} \int \sin^m x \cos^{n-2} x\, dx$

which is the required reduction formula

(B) Let $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$= \int \sin^{m-1} x \sin x \cos^n x\, dx$

$= -\int \sin^{m-1} x \{\cos^n x (-\sin x)\}\, dx$

$= -\left[\sin^{m-1} x \dfrac{\cos^{x+1} x}{n+1} - \int (m-1) \sin^{m-2} x\, cos\, x \dfrac{\cos^{x+1} x}{n+1}\right]$

[Integrating by parts by taking $f(x) = \sin^{m-1} x$ and $g(x) = \cos^n x (-\sin x)$]

$= -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} \int \sin^{m-2} x \cos^{x+2} x\, dx$

$= -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \cos^2 x\, dx$

$= -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} \int \sin^{m-2} x \cos^n x (1-\sin^2 x)\, dx$

$= -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} \int \sin^{m-2} x \cos^n x\, dx - \dfrac{m-1}{n+1} \int \sin^m x \cos^n x\, dx$

$\therefore$ $I_{m,n} = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} I_{m-2,n} - \dfrac{m-1}{n+1} I_{m-2,n}$

$\Rightarrow$ $\left(1 + \dfrac{m-1}{n+1}\right) I_{m,n} = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} I_{m-2,n}$

$\Rightarrow$ (m + n) $I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1)\, I_{m-2,n}$

$\Rightarrow$ $I_{m,n} = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} I_{m-2,n}$

$\Rightarrow$ $\int \sin^m x \cos^n x\, dx = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \dfrac{m-1}{n+1} \int \sin^{m-2} x \cos^n x\, dx$

which is the required reduction formula

Let us improve our understanding of these results by looking at some of following examples:-

Example 1: If $U_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$, and n > 1, prove that $U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$. Hence find U_5.

Sol: $U_n = \int_0^{\pi/2} \theta \sin^n \theta \, d\theta$

$$= \left[\theta \sin^{n-1}\theta(-\cos\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} \left[\theta\left\{(n-1)\sin^{n-2}\theta\cos\theta\right\} + \sin^{n-1}\theta.1\right](-\cos\theta)\, d\theta$$

[Integrating by parts by taking $f(x) = \theta \sin^{n-1}\theta$ and $g(x) = \sin\theta$]

$$= (0 - 0) + (x - 1) \int_0^{\pi/2} \theta \sin^{n-2}\theta \cos^2\theta \, d\theta + \int_0^{\pi/2} \sin^{n-1}\theta \cos\theta \, d\theta$$

$$= (n-1) \int_0^{\pi/2} \theta \sin^{n-2}\theta (1 - \sin^2\theta)\, d\theta + \int_0^{\pi/2} \sin^{n-1}\theta \cos\theta \, d\theta$$

$$= (n-1) \int_0^{\pi/2} \theta \sin^{n-2}\theta \, d\theta - (n-1) \int_0^{\pi/2} \theta \sin^n\theta \, d\theta + \left[\frac{\sin^n\theta}{x}\right]_0^{\pi/2}$$

$\therefore$ $U_n = (n-1) U_{n-2} - (n-1) U_n + \frac{1}{n}\left(\sin^n\frac{\pi}{2} - \sin^n 0\right)$

Or $[1 + (n-1)] U_n = (n-1) U_{n-1} + \frac{1}{n}(1 - 0)$

Or $n u_n = (n-1) U_{n-2} + \frac{1}{n}$

$\Rightarrow$ $U_n = \frac{n-1}{n} U_{n-2} + \frac{1}{n^2}$ .....(1)

Putting n = 5, 3 in (1), we get

$U_5 = \frac{4}{5} U_3 + \frac{1}{25}$ ....(2)

and $U_3 = \frac{2}{3} U_1 = \frac{1}{9}$ .....(3)

Now $U_1 = \int_0^{\pi/2} \theta \sin\theta \, d\theta$

$= \left[\theta(-\cos\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} 1.(-\cos\theta)\, d\theta$

$= -\left[\frac{\pi}{2}\cos\frac{\pi}{2} - 0\cos 0\right] + \int_0^{\pi/2} \cos\theta \, d\theta$

$= -[0 - 0] + \left[\sin\theta\right]_0^{\pi/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1 - 0$

$\therefore$ from (3), $U_3 = \frac{2}{3} + \frac{1}{9} = \frac{6+1}{9} = \frac{7}{9}$

From (2), $U_5 = \frac{4}{5} \times \frac{7}{9} + \frac{1}{25}$

$\therefore$ $U_5 = \frac{149}{225}$

Example 2: If $I_n = \int_0^{\pi/2} x^n \sin(2p+1)x\, dx$, prove that

$(2p+1)^2 I_n + n(n-1) I_{n-2} = (-1)^p n \left(\frac{\pi}{2}\right)^{n-1}$, n and p being positive integers

Hence or otherwise evaluate $\int_0^{\pi/2} x^4 \sin 3x\, dx$

Sol: $I_n = \int_0^{\pi/2} x^n \sin(2p+1)x\, dx$

$= \left[x^n\left\{-\frac{\cos(2p+1)x}{2p+1}\right\}\right]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1}\left\{-\frac{\cos(2p+1)x}{2p+1}\right\} dx$

$$= - \frac{1}{2p+1}\left[\left(\frac{\pi}{2}\right)^n \cos(2p+1)\frac{\pi}{2} - 0\right] + \frac{n}{2p+1}\int_0^{\pi/2} x^{n-1} \cos (2p+1)x \, dx$$

$$= - \frac{1}{2p+1}\left[\left(\frac{\pi}{2}\right)^n 0 - 0\right] + \frac{n}{2p+1}\int_0^{\pi/2} x^{n-1} \cos (2p+1)x \, dx$$

$$= \frac{n}{2p+1}\left\{\left[x^{n-1}\frac{\sin(2p+1)x}{2p+1}\right]_0^{\pi/2} - \int_0^{\pi/2} (n-1)x^{n-2}\frac{\sin(2p+1)x}{2p+1}dx\right\}$$

$$= \frac{n}{(2p+1)^2}\left[\left(\frac{\pi}{2}\right)^{n-1} \sin(2p+1)\frac{\pi}{2} - 0\right] - \frac{n(n-1)}{(2p+1)^2}\int_0^{\pi/2} x^{n-2}\sin (2p + 1)\, x \, dx$$

$$\therefore \quad I_n = \frac{n}{(2p+1)^2}\left[\left(\frac{\pi}{2}\right)^{n-1}(-1)^p\right] - \frac{n(n-1)}{(2p+1)^2} I_{n-2}$$

$$\left[\because \sin(2p+1)\frac{\pi}{2} = \sin\left(p\pi + \frac{\pi}{2}\right) = \cos p\pi = (-)^p\right]$$

$$\therefore \quad (2p + 1)^2 I_n = n\left(\frac{\pi}{2}\right)^{n-1}(-1)^p - n\,(n-1)\, I_{n-2}$$

$$\Rightarrow \quad (2p + 1)^2 I_n + n\,(n-1)\, I_{n-2} = (-1)^p\, n\left(\frac{\pi}{2}\right)^{n-1}$$

which is required result

Again

$$\int_0^{\pi/2} x^n \sin (2p + 1)\, dx = \frac{n}{(2p+1)^2}(-1)^p\left(\frac{\pi}{2}\right)^{n-1}$$

$$- \frac{n(n-1)}{(2p+1)^2}\int_0^{\pi/2} x^n \sin (2p + 1)x \, dx \qquad ....(1)$$

Putting n = 4, p = 1 in (1), we get

$$\int_0^{\pi/2} x^n \sin 3x \, dx = \frac{4}{(2+1)^2}\int_0^{\pi/2} x^2 \sin 3x \, dx$$

$\therefore \quad \int_0^{\pi/2} x^4 \sin 3x \, dx = \frac{\pi^3}{18} - \frac{4}{3} \int_0^{\pi/2} x^2 \sin 3x \, dx \qquad \text{....(2)}$

Putting n = 2, p = 1 in (1), we get

$$\int_0^{\pi/2} x^2 \sin 3x \, dx = -\frac{2}{9} \times \frac{\pi}{2} - \frac{2}{9} \int_0^{\pi/2} x^2 \sin 3x \, dx$$

$$= -\frac{\pi}{9} - \frac{2}{9}\left[-\frac{\cos 3x}{3}\right]_0^{\pi/2}$$

$$= -\frac{\pi}{9} + \frac{2}{27}\left[\cos\frac{3\pi}{2} - \cos 0\right]$$

$$= -\frac{\pi}{9} + \frac{2}{27}(0 - 1)$$

$$= -\frac{\pi}{9} - \frac{2}{27}$$

$\therefore$ from (2), we get

$$\int_0^{\pi/2} x^4 \sin 3x \, dx = -\frac{\pi^3}{18} - \frac{4}{3}\left[-\frac{\pi}{9} - \frac{2}{27}\right]$$

$$\therefore \quad \int_0^{\pi/2} x^4 \sin 3x \, dx = -\frac{\pi^3}{18} + \frac{4\pi}{27} + \frac{8}{81}$$

Example 3: Prove that $\int_0^{\pi/2} x^n \sin 3x \, dx$

$= \frac{\pi}{2^{n+1}}$, where n is a positive integer

Sol: Let $I_n = \int_0^{\pi/2} x^n \sin 3x \, dx$

$$= \left[\cos^n x \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} \left\{n \cos^{n-1} x(-\sin x)\right\} \frac{\sin nx}{n} \, dx$$

$$= \frac{1}{n}(0 - 0) + \int_0^{\pi/2} \cos^{n-1} x\,(\sin nx \sin x)\,dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x\,[\cos (n-1)x - \cos nx \cos x]\,dx$$

[$\because$ cos (n-1)x = cos nx cos x + $\therefore$ sin nx sin x = cos (n-1) x - cos nx cos x]

$$= \int_0^{\pi/2} \cos^{n-1} x \cos (n-1)x\,dx - \int_0^{\pi/2} \cos^{n} x \cos nx\,dx$$

$\therefore \quad I_n = I_{n-1} - I_n$

$\Rightarrow \quad 2\,I_n = I_{n-1}$

$\Rightarrow \quad I_n = \frac{1}{2} I_{n-1}$

$\Rightarrow \quad 2\,I_n = I_{n-1}$

Similarly

$$\left.\begin{aligned} \Rightarrow \quad I_n &= \frac{1}{2} I_{n-1} \\ I_{n-1} &= \frac{1}{2} I_{n-2} (\text{changingn to n} - 1) \\ I_{n-2} &= \frac{1}{2} I_{n-3} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ I_1 &= \frac{1}{2} I_0 \end{aligned}\right\} \quad \ldots\ldots.(A)$$

$$\therefore \quad \int_0^{\pi/2} \cos^{n} x \cos nx = \frac{\pi}{2^{n+1}}$$

Example 4: Use a suitable reduction formula to evaluate

$$\int_0^{\pi/2} \cos^{3} x \sin 3x\,dx$$

Sol: Let $I_{m,n} = \int \cos^{m} x \sin nx\,dx$

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int \{m \cos^{m-1}x\,(-\sin x)\}\left(-\frac{\cos nx}{n}\right)dx$$

[Integrating by parts]

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x\,(\cos nx \sin x)\,dx$$

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin (n-1) x]$$

[$\because$ sin (n-1)x = sin nx cos x - cos nx sin x $\Rightarrow$ cos nx sin x = sin nx cos x - sin (n-1)x]

$$= -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n}\int \cos^m \sin nx\,dx + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)\,x\,dx$$

$\therefore$ $$I_{m,n} = -\frac{1}{n}\cos^m x \cos nx - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

$\Rightarrow$ $$\left(1+\frac{m}{n}\right)I_{m,n} = -\frac{1}{n}\cos^m x \cos nx + \frac{m}{n} I_{m-1,n-1}$$

$\Rightarrow$ $$(m + n)\, I_{m,n} = -\cos^m x \cos nx + M\, I_{m-1,n-1}$$

$\Rightarrow$ $$I_{m,n} = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

$\Rightarrow$ $$\int \cos^m x \sin nx\,dx = -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \sin (n-1)\,x\,dx \qquad ....(1)$$

Put m = 3, n = 3 in (1), we have

$$\int \cos^3 x \sin 3x\,dx = -\frac{\cos^3 x \cos 3x}{3+3} + \frac{3}{3+3}\int \cos^2 x \sin 2x\,dx$$

$\therefore$ $$\int_0^{\pi/2} \cos^3 x \sin 3x\,dx = -\frac{1}{6}\left[\cos^3 x \cos 3x\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} \cos^2 x \sin 2x\,dx$$

$$= -\frac{1}{6}[0 - 1] + \frac{1}{2}\int_0^{\pi/2} \cos^2 x \sin 2x\,dx$$

$\therefore$ $$\int_0^{\pi/2} \cos^3 x \sin 3x\,dx = \frac{1}{6} + \frac{1}{2}\int_0^{\pi/2} \cos^2 x \sin 2x\,dx \qquad ....(2)$$

Put m = 2, n = 2 in (1), we get

$$\int \cos^2 x \sin 2x \, dx = -\frac{\cos^2 x \cos 2x}{2+2} + \frac{2}{2+2}\int \cos x \sin x \, dx$$

$$\therefore \quad \int_0^{\pi/2} \cos^2 x \sin 2x \, dx = -\frac{1}{4}\left[\cos^2 x \cos 2x\right]_0^{\pi/2} + \frac{1}{2} \times \frac{1}{2}\int_0^{\pi/2} (2 \sin x \cos x) \, dx$$

$$= -\frac{1}{4}[0 - 1] + \frac{1}{4}\int_0^{\pi/2} \sin 2x \, dx$$

$$= \frac{1}{4} + \frac{1}{4}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{4} - \frac{1}{8}[\cos \pi - \cos 0]$$

$$= \frac{1}{4} - \frac{1}{8}[-1 - 1]$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= \frac{1}{2}$$

$\therefore$ from (2), we get

$$\int_0^{\pi/2} \cos^3 x \sin 3x \, dx = \frac{1}{6} + \frac{1}{2}\left(\frac{1}{2}\right)$$

$$= \frac{1}{6} + \frac{1}{4}$$

$$= \frac{2+3}{12}$$

$$= \frac{5}{12}$$

Example 5: If $f(p, q) = \int_0^{\pi/2} \cos^p x \sin qx \, dx$,

show that

$(p + q) f(p, q) - p f(p-1, q-1) = 1$

Sol: $f(p, q) = \int_0^{\pi/2} \cos^p x \sin qx \, dx$

$$= \left[\cos^p x\left(-\frac{\cos qx}{q}\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \{p \cos^{p-1} x (-\sin x)\} \left(-\frac{\cos qx}{q}\right) dx$$

[Integrating by parts]

$$= -\frac{1}{q}[0 - 1] - \frac{p}{q}\int_0^{\pi/2} \cos^{p-1} x (\cos qx . \sin x) dx$$

$$= \frac{1}{q} - \frac{p}{q}\int_0^{\pi/2} \cos^{p-1} x [\sin qx \cos x - \sin (q-1)x] dx$$

[$\because$ sin (q-1)x = sin qx cos x - cos qx sin x $\Rightarrow$ cos qx sin x = sin qx cos x - sin (q-1)x]

$$= \frac{1}{q} - \frac{p}{q}\int_0^{\pi/2} \cos^{p} x \sin qx \, dx + \frac{p}{q}\int_0^{\pi/2} \cos^{p-1} x \sin (q-1)x \, dx$$

$\therefore \quad f(p, q) = \frac{1}{q} - \frac{p}{q} f(p, q) + \frac{p}{q} f(p-1, q-1)$

$\Rightarrow \quad \left(1+\frac{p}{q}\right) f(p, q) = \frac{1}{q} + \frac{p}{q} f(p-1, q-1)$

$\Rightarrow \quad (p + q) f(p, q) = 1 + p f(p - 1, q - 1)$

$\Rightarrow \quad (p + q) f(p, q) - p f(p - 1, q - 1) = 1$

Self-Check Exercise

Q.1 Obtain a reduction formula for $\int x^n \sin x \, dx$ and apply it to evaluate $\int_0^{\pi/2} x^3 \sin x \, dx$

Q.2 If $U_n = \int_0^{\pi/2} x^n \sin x \, dx$, and $n > 1$, prove that

$U_n + n(n-1) U_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ Hence evaluate U_5

Q.3 If $I_{m,n} = \int_0^{\pi/2} \cos^m x \cos nx\, dx$, Show that $I_{m,n} = \frac{m}{m+n} I_{m-1,n-1}$. Hence evaluate $\int_0^{\pi/2} \cos^5 x \cos 3x\, dx$

8.7 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Derived reduction formula for $\int x \sin^n x\, dx$ and $\int x \cos^n x\, dx$
2. Derived reduction formula for $\int x^m \sin nx\, dx$ and $\int x^m \cos nx\, dx$
3. Derived reduction formula for $\int \cos^m x \sin nx\, dx$ and $\int \cos^m x \cos nx\, dx$
4. Derived reduction formula for $\int \sin^m x \cos^n x\, dx$
5. Solved some questions related to above stated reduction formulae.

8.8 Glossary

1. A reduction formula for an integral is a formula which connects the integral with one or two other integrals in which the integrand is of he same type but of lower degree or lower order.
2. If $I_n = \int x \sin^n x dx$, then its reduction formula is

$$I_n = \frac{x \sin^{n-1} x \cos x}{n} + \frac{\sin^n x}{n^2} + \frac{n-1}{n} I_{n-1}$$

3. If $I_{m,n} = \int x^m \sin nx\, dx$, then its reduction formula is

$$I_{m,n} = \frac{x^m \cos nx}{n} + \frac{m}{n^2} x^{m-1} \sin nx - \frac{m(m-1)}{n^2} I_{m-2,n}$$

4. If $I_{m,n} = \int \cos^m \sin nx\, dx$, then its reduction formula is

$$I_{m,n} = \frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$$

5. If $I_{m,n} = \int \sin^m x \cos^n x dx$, then its reduction formula is

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$$

8.9 Answers To Self-Check Exercises

Self-Check Exercise-1

Ans. 1 $\int$ xn sin x dx = $-x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x dx$

and $\dfrac{3\pi^4}{4} - 6$

Ans. 2 $\dfrac{5\pi^4}{16} - 15\pi^2 + 120$

Ans. 3 $I_{53} = \dfrac{5\pi}{64}$

8.10 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.
2. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.

8.11 Terminal Questions

1. If $u_n = \int_0^{\pi/2} x \cos nx dx$, x > 0, prove that $u_n = \left(\dfrac{n-1}{n}\right) u_{n-2} = \dfrac{1}{n^2}$

   Hence evaluate u_5.

2. Find a reduction formula for $\int x^4 \cos x dx$

3. Find the reduction formula for $\int \cos^x x \cos n x dx$ and hence find the value of

   $\int_0^{\pi/2} \cos^3 x \cos 2x dx$

4. Use a suitable reduction formula to evaluate $\int_0^{\pi/2} \cos^3 x \cos 3x dx$

5. If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx dx$ show that $I_{m,n} = \dfrac{1}{m+n} + \dfrac{m}{m+n} I_{m-1,\, n-1}$

   Hence evaluate $I_{5,\,3}$.

Unit - 9

Smaller Index +1 Method to Connect

$$\int x^m (a+bx^n)^p \, dx$$

Structure

9.1 Introduction

Reduction by connecting two integrals, also known as the Smaller Index +1 method, is a technique used in integral calculus to simplify and evaluate certain types of integrals. this method is particularly useful when dealing with integrals involving rational functions or when faced with complex or difficult-to-evaluate integrals. The basic idea behind this method is to express a given integral a sum or difference of two integrals, where one of the integrals has a smaller index (power) than the original integral, and the other integral has an index that is one greater than the original integral. By doing so, we can create a recurrence relation that allows us to simplify and evaluate the integral iteratively. By applying the reduction by connecting two integrals method iteratively, we can simplify and evaluate integrals that would otherwise be challenging or impossible to solve directly. This technique is a powerful tool in integral calculus and is widely used to solve a variety of integration problems.

9.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss rule of "Smaller Index +1"
- Discuss rule of "Smaller Index +1" to connect integral $\int x^m (a+bx^n)^p \, dx$ with a given integral of the same type.

- Solve questions by connecting two integrals by using smaller Index +1 method.

9.3 Rule of "Smaller Index +1" to Connect $\int x^m (a+bx^n)^p dx$ with a Given Integral of the Same Type

(i) Let P = $x^{\lambda+1}$ $(a+b\ x^n)^{u+1}$, where λ is the smaller of the two indices of x, and u is the smaller of the two indices of $(a+b\ x^n)$ in the two expression whose integrals are to be connected.

(ii) Find $\frac{dp}{dx}$ and express it as a linear function of the two expressions whose integrals are to be connected.

(iii) Integrate both sides w.r.t. x, transpose, and solve for the integral given to be connected.

Let us improve our understanding of this method by looking at some following examples:-

Example 1: Connect $\int x^m (a+bx^n)^p dx$

Sol: We have to connect $\int x^m (a+bx^n)^p dx$

with $\int x^m (a+bx^n)^{p-1} dx$

(i) Let P = x^{m+1} $(a + bx^n)^{p-1+1}$

[Rule of "smaller index + 1"]

= x^{m+1} $(a + b\ x^n)^p$

(ii) Then $\frac{dp}{dx}$ = $(m+1)\ x^m\ (a+b\ x^n)^p + x^{m+1}.\ p\ (a+b\ x^n)^{p-1}.\ bnx^{n-1}$

= $(m+1)\ x^m\ (a+bx^n)^p + pbnx^{m+n}\ (a+bx^n)^{p-1}$

= $(m+1)\ x^m\ (a+bx^m)^p + pbnx^m x^n.\ (a+bx^n)^{p-1}$

= $(m+1)\ x^m\ (a+bx^n)^p + pnx^m\left(\overline{a+bx^n}-a\right).\ (a+bx^n)^{p-1}$

[Note this step]

= $(m+1)\ x^m\ (a+bx^n)^p + pn\ x^m\ (a+bx^n)^p - pn\ a\ x^m\ (a+bx^n)^{p-1}$

= $(m+1 + pn)\ x^m\ (a+bx^n)^p - pn\ a\ x^m\ (a+bx^n)^{p-1}$

(iii) Integrate both sides w.r.t. x, we have

P = $(m+1+pn)\int x^m (a+bx^n)^p dx$ - pna $\int x^m (a+bx^n)^{p-1} dx$

Transposing, we get

$(m+1 + pn)\int x^m (a+bx^n)^p dx$ = P + pna $\int x^m (a+bx^n)^{p-1} dx$

$\therefore \quad \int x^m(a+bx^n)^p\,dx = \dfrac{P}{m+1+pn} + \dfrac{pna}{m+1+pn}\int x^m(a+bx^n)^{p-1}dx$

$= \dfrac{x^{m+1}(a+bx^n)^p}{m+1+pn} + \dfrac{pna}{m+1+pn}\int x^m(a+bx^n)^{p-1}dx$

[Substituting the value of P]

which is the required formula

Example 2: If X = a+bx^n, then prove that

(i) $\int x^{m-1}X^p dx = \dfrac{x^m X^p}{m+pn} + \dfrac{pna}{m+pn}\int x^{m-1}X^{p-1}dx$

(ii) $\int x^{m-1}X^p dx = \dfrac{x^m X^{p+1}}{am} - \dfrac{b(m+np+n)}{am}\int x^{m+n-1}X^p dx$

Sol: (i) We have to connect $\int x^{m-1}X^p dx$ with $\int x^{m-1}X^{p-1}dx$ where X = a+bx^n

Let P = x^{m-1+1} X^{p-1+1} = x^m X^p = x^m(a+bx^n)^p

$\therefore \quad \dfrac{dp}{dx}$ = mx^{m-1} (a+bx^n)^p + x^m.p (a+bx^n)$^{p-1}$.bnx^{n-1}

= m x^{m-1} (a+bx^n)^p + pn x^{m-1} (a+bx^n)$^{p-1}$.bx^n

= mx^{m-1} (a+bx^n)^p + pnx^{m-1} (a+bx^n)$^{p-1}$$\left(\overline{a+bx^n}-a\right)$

= mx^{m-1} (a+bx^n)^p + pn x^{m-1} (a+bx^n)^p - pnx^{m-1} (a+bx^n)$^{p-1}$

$\therefore \quad \dfrac{dP}{dx}$ = (m+p^n) x^{m-1} (a+bx^n)^p - pan x^{m-1} (a+bx^n)$^{p-1}$

Integrating both sides w.r.t. x, we have

P = (m+pn), $\int x^{m-1}(a+bx^n)^p\,dx$ - pan $\int x^{m-1}(a+bx^n)^{p-1}dx$

$\therefore$ xm (a+bx^n)^p = (m+pn) $\int x^{m-1}(a+bx^n)^p\,dx$ - pan $\int x^{m-1}(a+bx^n)^{p-1}dx$

$\Rightarrow$ (m+pn) $\int x^{m-1}(a+bx^n)^p\,dx$ = x^m X^p + pan $\int x^{m-1}(a+bx^n)^{p-1}dx$

$\Rightarrow \quad \int x^{m-1}(a+bx^n)^p\,dx = \dfrac{x^m X^p}{m+pn} + \dfrac{pna}{m+pn}\int x^{m-1}(a+bx^n)^{p-1}dx$

Hence the result

(ii) We have to connect $\int x^{m-1}(a+bx^n)^p\,dx$ with $\int x^{m+n-1}x^p dx$ where X = a+bx^n

Let $\quad P = x^{m-1+1} X^{p+1}$

$= x^m X^{p+1}$

$= x^m (a+bx^n)^{p+1}$

$\therefore \quad \frac{dP}{dx} = m\, x^{m-1} (a+bx^n)^{p+1} + x^m (p+1) (a+bx^n)^p\, nbx^{n-1}$

$= mx^{m-1} (a+bx^n)^p (a+bx^n) + (p+1)\, nbx^{m+n-1} (a+bx^n)^p$

$= am\, x^{m-1} (a+bx^n)^p + bm\, x^{m+n-1} (a+bx^n)^p + (p+1)\, nbx^{m+n-1} (a+bx^n)^p$

$= am\, x^{m-1} X^p + bm\, x^{m+n-1} X^p + (p+1)\, nb\, x^{m+n-1} X^p$

$\therefore \quad \frac{dP}{dx} = am\, x^{m-1} X^p + (bm+pnb+nb)\, x^{m+n-1} X^p$

Integrating both sides w.r.t. x, we get

$P = am \int x^{m-n} x^p dx + b\,(m+pn+n) \int x^{m-n-1} x^p dx$

$\therefore \quad am \int x^{m-1} X^p dx = P - b(m+pn+n) \int x^{m-n-1} x^p dx$

$\Rightarrow \quad \int x^{m-1} X^p dx = \frac{x^m X^{p+1}}{am} - \frac{b(m+pn+n)}{am} \int x^{m-n-1} x^p dx$

Hence the result

Example 3: Find reduction formula for $\int \frac{x^m dx}{(x^3-1)^{1/3}}$ and obtain the value of i.e. $\int x^8 (x^3-1)^{-1/3} dx$

Sol. : We connect $\int \frac{x^m dx}{(x^3-1)^{1/3}}$ i.e. $\int x^m (x^3-1)^{-1/3} dx$

with $\quad \int \frac{x^{m-3}}{(x^3-1)^{1/3}} dx$ i.e. $\int x^{m-3} (x^3-1)^{-1/3} dx$

Let $\quad P = x^{m-3+1} \left(x^3-1\right)^{-\frac{1}{3}+1}$

$= x^{m-2} \left(x^3-1\right)^{\frac{2}{3}}$

$\therefore \quad \frac{dP}{dx} = (m-2)\, x^{m-3} \left(x^3-1\right)^{\frac{2}{3}} + x^{m-2}\, \frac{2}{3}\, (x^3-1)^{-1/3} (3x^2)$

$$= (m-2)\, x^{m-3}\, (x^3 - 1)\, (x^3-1)^{-1/3} + 2x^m\, (x^3-1)^{-1/3}$$

$$= (m-2)\, x^m\, (x^3-1)^{-1/3} - (m-2)\, x^{m-2}\, (x^3-1)^{-1/3} + 2x^m (x^3-1)^{-1/3}$$

$$\therefore \quad \frac{dP}{dx} = m\, x^m\, (x^3-1)^{-1/3} - (m-2)\, x^{m-3} (x^3-1)^{-1/3}$$

Integrating both sides w.r.t. x, we get

$$P = m\int x^m (x^3-1)^{-1/3} dx - (m-2) \int x^{m-3} (x^3-1)^{-1/3} dx$$

$$\therefore \quad m\int x^m (x^3-1)^{-1/3} dx = P + (m-2)\int x^{m-3}(x^3-1)^{-1/3} dx$$

Or $$\int x^m (x^3-1)^{-1/3} dx = \frac{x^{m-2}(x^3-1)^{\frac{2}{3}}}{m} + \frac{m-2}{m} \int x^{m-3}(x^3-1)^{-1/3} dx$$

Or $$\int \frac{x^m}{(x^3-1)^{1/3}} dx = \frac{1}{m}\, x^{m-2} \left(x^3-1\right)^{\frac{2}{3}} + \frac{m-2}{m} \int \frac{x^{m-3}}{(x^3-1)^{1/3}} dx$$

which is the required reduction formula ....(2)

Putting m = 8 in (1), we get

$$\int \frac{x^8}{(x^3-1)^{1/3}} dx = \frac{1}{8}\, x^6 \left(x^3-1\right)^{\frac{2}{3}} + \frac{3}{4} \int \frac{x^5}{(x^3-1)^{1/3}} dx$$

Putting m = 5 in (1), we get

$$\int \frac{x^5}{(x^3-1)^{1/3}} dx = \frac{1}{5} x^3 \left(x^3-1\right)^{\frac{2}{3}} + \frac{3}{5} \int \frac{x^2}{(x^3-1)^{1/3}} dx$$

$$= \frac{1}{5} x^3 \left(x^3-1\right)^{\frac{2}{3}} + \frac{1}{5} \int (x^3-1)^{-1/3} (3x^2)\, dx$$

$$= \frac{1}{5} x^3 \left(x^3-1\right)^{\frac{2}{3}} + \frac{1}{5} \frac{(x^3-1)^{2/3}}{2/3}$$

$$\therefore \quad \int \frac{x^5}{(x^3-1)^{1/3}} dx = \frac{1}{5} x^3 \left(x^3-1\right)^{\frac{2}{3}} + \frac{3}{10} \left(x^3-1\right)^{\frac{2}{3}}$$

$\therefore$ from (2), we get

$$\int \frac{x^8}{(x^3-1)^{1/3}}\,dx = \frac{1}{8}x^6\left(x^3-1\right)^{\frac{2}{3}} + \frac{3}{20}x^3\left(x^3-1\right)^{\frac{2}{3}} + \frac{9}{40}\left(x^3-1\right)^{\frac{2}{3}}$$

$$= \left(x^3-1\right)^{\frac{2}{3}}\left[\frac{x^6}{8}+\frac{3x^3}{20}+\frac{9}{40}\right]$$

Example 4: Show that

$$\int \frac{dx}{(x^2+1)^n} = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(x^2+1)^{n-1}}$$

Sol: We have to connect

$$\int \frac{dx}{(x^2+1)^n} \text{ i.e. } \int x^0(x^2+1)^{-n}\,dx$$

with $\int \frac{dx}{(x^2+1)^{n-1}}$ i.e. $\int x^0(x^2+1)^{-n+1}dx$

Let $P = x^{0+1}(x^2+1)^{-n+1}$

$= x(x^2+1)^{-n+1}$

$\therefore$ $P = 1(x^2+1)^{-n+1} + x(-n+1)(x^2+1)^{-n}(2x)$

$= (x^2+1)^{-n+1} - 2(n-1)(x^2+1)^{-n}x^2$

$= (x^2+1)^{-n+1} - 2(n-1)(x^2+1)^{-n}\left(\overline{x^2+1}-1\right)$

$= (x^2+1)^{-n+1} - 2(n-1)(x^2+1)^{-n+1} + 2(n-1)(x^2+1)^{-n}$

$\therefore$ $\frac{dP}{dx} = -(2n-3)(x^2+1)^{-n+1} + 2(n-1)(x^2+1)^{-n}$

Integrating both sides w.r.t. x, we get

$$P = (2n-3)\int (x^2+1)^{-n+1}dx + 2(n-1)\int (x^2+1)^{-n}dx$$

$\therefore$ $2(n-1)\int \frac{1}{(x^2+1)^n}\,dx = P + (2n-3)\int \frac{1}{(x^2+1)^{n-1}}\,dx$

Or $\int \frac{1}{(x^2+1)^n}\,dx = \frac{x}{2(n-1)(x^2+1)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{1}{(x^2+1)^{n-1}}\,dx$

Hence the result

Example 5: If $U_n = \int x^n\sqrt{a^2-x^2}\,dx$, prove that

$$U_n = \frac{x^{n-1}(a^2-x^2)^{\frac{3}{2}}}{n+2} + \frac{n-1}{n+2} a^2 U_{n-2}$$

Sol: We have to connect $U_n = \int x^n (a^2-x^2)^{1/2}$ dx

with $U_n = \int x^{n-2}(a^2-x^2)^{1/2}$ dx

Let P = x^{n-2+1} $(a^2-x^2)^{\frac{1}{2}+1}$

$= x^{n-1}$ $(a^2-x^2)^{\frac{3}{2}}$

$$\therefore \quad \frac{dp}{dx} = (n-1)x^{n-2}(a^2-x^2)^{\frac{3}{2}} + x^{n-2}\frac{3}{2}(a^2-x^2)^{\frac{1}{2}}(-2x)$$

$$= (n-1)x^{n-2}(a^2-x^2)^{\frac{3}{2}} - 3x^n(a^2-x^2)^{\frac{1}{2}}$$

$$= (n-1)x^{n-2}(a^2-x^2)(a^2-x^2)^{\frac{1}{2}} - 3x^n(a^2-x^2)^{\frac{1}{2}}$$

$$= (n-1)a^2x^{n-2}(a^2-x^2)^{\frac{1}{2}} - (n-1)x^n(a^2-x^2)^{\frac{1}{2}} - 3x^n(a^2-x^2)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dp}{dx} = (n-1)a^2x^{n-2}(a^2-x^2)^{\frac{1}{2}} - (n+2)x^n(a^2-x^2)^{\frac{1}{2}}$$

Integrating both sides, we get

$$P = (n-1)a^2\int x^{n-2}(a^2-x^2)^{\frac{1}{2}}dx - (n+2)\int x^n(a^2-x^2)^{\frac{1}{2}}$$

$\Rightarrow$ P = (n -1)a^2 U_{n-2} - (n+2) U_n

$\Rightarrow$ (n +2) U_n = -P+(n - 1)a^2 U_{n-2}

$$\Rightarrow \quad U_n = -\frac{x^{n+1}(a^2-x^2)^{1/2}}{n+2} + \frac{n-1}{n+2} \text{ a2 Un-2}$$

Example 6 : Calculate the value of $\int_0^{2a} x^m\sqrt{2ax-x^2}$ dx (m being a positive integer) by the use of reduction formula.

Sol. : We have to find a reduction formula for $\int x^m\sqrt{2ax-x^2}$ dx

Now $\int x^m\sqrt{2ax-x^2}$ dx = $\int x^{m+\frac{1}{2}}(2a-x)^{1/2}$

[Form $\int x^m (a+bx^n)^p$ dx]

Let us connect $\int x^{m+\frac{1}{2}}(2a-x)^{1/2}$ dx with $\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}$ dx

Let P = $x^{m+\frac{1}{2}+1}(2a-x)^{\frac{1}{2}+1}$ [Rule of "smaller index +1" method]

= $x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}}$

$$\therefore \quad \frac{dp}{dx} = \left(m+\frac{1}{2}\right)x^{m-\frac{1}{2}}(2a-x)^{\frac{3}{2}} + x^{m+\frac{1}{2}}.\frac{3}{2}(2a-x)^{\frac{3}{2}}(-1)$$

$$= \left(m+\frac{1}{2}\right)x^{m-\frac{1}{2}}(2a-x)(2a-x)^{\frac{1}{2}} - \frac{3}{2}x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - \left(m+\frac{1}{2}\right)x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - \frac{3}{2}x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - (m+2)x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

Integrating both sides w.r.t. x, we get

P = $(2m+1)a\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx - (m+2)\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx$

By transposing, we get

$$(2m+1)\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx = -P + (2m+1)\ a\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx$$

or $$\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx = -\frac{P}{m+2} + \frac{(2m+1)a}{m+2}\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx$$

or $$\int \frac{x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}}{x^{1/2}}\,dx = -\frac{x^{m+\frac{1}{2}}(2ax-x^2)^{3/2}}{(m+2)x^{3/2}} +$$

$$+ \frac{(2m+1)a}{m+2}\int \frac{x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}}{x^{1/2}}\,dx$$

[$\because$ 2a - x = $\frac{2ax-x^2}{x}$]

$\therefore$ $\int x^m\sqrt{2ax-x^2}$ dx = - $\dfrac{x^{m-1}(2ax-x^2)^{3/2}}{(m+2)}$ + $\dfrac{(2m+1)a}{m+2}$ $\int x^{m-1}\sqrt{2ax-x^2}$ dx

which is the required reduction formula

Now

$$\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = \left[\frac{x^{m-1}(2ax-x^2)^{3/2}}{(m+2)}\right]_0^{2a} + \frac{(2m+1)a}{m+2}\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}\,dx$$

$$= \frac{(2m+1)a}{m+2}\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}\,dx \qquad ....(1)$$

Changing m to m-1, we get

$$\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}\,dx = \frac{(2m+1)a}{m+2}\int_0^{2a} x^{m-2}\sqrt{2ax-x^2}\,dx \qquad ...(2)$$

From (1) and (2), we get

$$\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = \frac{(2m+1)(2m-1)}{(m+2)(m+1)}.a^2\int_0^{2a} x^{m-2}\sqrt{2ax-x^2}\,dx$$

Proceeding in this way, we get

$$\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = \frac{(2m+1)(2m-1)....\text{to m factors}}{(m+2)(m+1)....\text{to m factors}}.a^m\ .\int_0^{2a} x^{m-m}\sqrt{2ax-x^2}\,dx$$

[2m + 1, 2m - 1, ... is an A.P., $\therefore$ mth term = 2m + 1 + (m - 1) (-2) = 3,

and m + 1, m - 1, ... is an A.P., $\therefore$ mth term = m + 2 + (m - 1) (-1) = 3]

$$\frac{(2m+1)(2m-1)..........3}{(m+2)(m+1)..........3}\ a^m\int_0^{2a}\sqrt{2ax-x^2}\,dx \qquad ....(3)$$

Now

$$\int_0^{2a}\sqrt{2ax-x^2}\,dx = \int_0^{2a}\sqrt{-(x^2-2ax)}$$

$$= \int_0^{2a}\sqrt{-(x^2-2ax+a^2)+a^2}$$

$$= \int_0^{2a}\sqrt{a^2-(x-a)^2}\left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x-1}{a}\right]_0^{2a}$$

$$= \frac{a^2}{2}\sin^{-1} - \left[\frac{a^2}{2}\sin^{-1}(-1)\right]$$

$$= \frac{a^2}{2} \cdot \frac{\pi}{2} - \left[\frac{a^2}{2}\left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{a^2}{2} \pi$$

$\therefore$ From (30, we get

$$\int_0^{2a} x^m \sqrt{2ax - x^2} \, dx = \frac{(2m+1)(2m-1)..........3}{(m+2)(m+1)..........3} a^m \frac{a^2}{2} \pi$$

$$= \frac{(2m+1)(2m-1)..........3}{(m+2)(m+1)..........3} a^{m+2} \frac{\pi}{2}$$

Example 7 : If $I_n = x^n \sqrt{a-x}$ prove that

$$(2x+3) I_n = -2xn(a-x)^{\frac{2}{3}} + 2an \; I_{n-1}$$

Hence evaluate $\int_0^a x^2 \sqrt{ax - x^2} \, dx$

Sol. : Hence $I_n = \int x^2 \sqrt{a-x} \, dx$, $I_{n-1} = \int x^{n-1} \sqrt{a-x} \, dx$

we have to connect

$$\int x^n (a-x)^{\frac{1}{2}} \text{ with } \int x^{n-1} (a-x)^{\frac{1}{2}} dx$$

Let $P = x^{n-1+1}(a-x)^{\frac{1}{2}+1}$

$$= x^n (a-x)^{\frac{3}{2}}$$

$$\therefore \quad \frac{dP}{dx} = nx^{n-1}(a-x)^{\frac{3}{2}} + x^n \frac{3}{2}(a-x)^{\frac{1}{2}}(-1)$$

$$= nx^{n-1}(a-x)(a-x)^{\frac{1}{2}} + \frac{3}{2} x^n (a-x)^{\frac{1}{2}}$$

$$= na\, x^{n-1}(a-x)^{\frac{1}{2}} - nx^n (a-x)^{\frac{1}{2}} - \frac{3}{2} x^n (a-x)^{\frac{1}{2}}$$

$$\therefore \quad \frac{dP}{dx} = na\, x^{n-1}(a-x)^{\frac{1}{2}} - \left(n + \frac{3}{2}\right) x^n (a-x)^{\frac{1}{2}}$$

Integrating both sides w.r.t. x

P = na $\int x^{n-1}(a-x)^{\frac{1}{2}}$ dx $\frac{2n+3}{2}\int x^{n}(a-x)^{\frac{1}{2}}$ dx

$\therefore$ $\frac{2n+3}{2}\int x^n\sqrt{a-x}$ = - P + na $\int x^{n-1}\sqrt{a-x}$ dx

$\Rightarrow$ (2n + 3) $\int x^n\sqrt{a-x}$ dx = $-2x^n(a-x)^{\frac{3}{2}}+2an\int x^{n-1}\sqrt{a-x}$ dx ...(1)

$\Rightarrow$ (2n + 3) I_n = $-2x^n(a-x)^{\frac{3}{2}}+2an$ I_{n-1}

From (1), we get

$$\int x^n\sqrt{a-x}\,dx = -\frac{2}{2n+3}x^n(a-x)^{\frac{3}{2}}+\frac{2an}{2n+3}\int x^{n-1}\sqrt{a-x}\,dx$$

$\therefore$ $$\int_0^a x^n\sqrt{a-x}\,dx = -\frac{2}{2n+3}\left[x^n(a-x)^{3/2}\right]_0^a+\frac{2an}{2n+3}\int_0^a x^{n-1}\sqrt{a-x}\,dx$$

$$= -\frac{2}{2n+3}(0-0)+\frac{2an}{2n+3}\int_0^a x^{n-1}\sqrt{a-x}\,dx$$

or $$\int_0^a x^n\sqrt{a-x}\,dx = -\frac{2an}{2n+3}\int_0^a x^{n-1}\sqrt{a-x}\,dx \quad ...(2)$$

Putting n = $\frac{5}{2}$ in (2), we get

$$\int_0^a x^{\frac{5}{2}}\sqrt{a-x}\,dx = -\frac{2a\times\frac{2}{5}}{2\times\frac{2}{5}+3}\int_0^a x^{3/2}\sqrt{a-x}\,dx$$

$\therefore$ $$\int_0^a x^{\frac{5}{2}}\sqrt{a-x}\,dx = -\frac{5}{8}a\int_0^a x^{3/2}\sqrt{a-x}\,dx \quad ...(3)$$

Putting n = $\frac{3}{2}$ in (2), we get

$$\int_0^a x^{\frac{3}{2}}\sqrt{a-x}\,dx = -\frac{2a\times\frac{3}{2}}{2\times\frac{3}{2}+3}\int_0^a x^{1/2}\sqrt{a-x}\,dx$$

$$= \frac{a}{2}\int_0^a\sqrt{ax-x^2}\,dx$$

$$= \frac{a}{2} \int_0^a \sqrt{\frac{a^2}{4} - \left(x^2 - ax + \frac{a^2}{4}\right)} \text{ dx}$$

$$= \frac{a}{2} \left[\frac{\left(x-\frac{a}{2}\right)\sqrt{\left(\frac{a}{2}\right)^2 - \left(x-\frac{a}{2}\right)^2}}{2} \frac{\left(x-\frac{a}{2}\right)^2}{2} \sin^{-1}\left\{\frac{x-\frac{a}{2}}{\frac{a}{2}}\right\} \right]_0^a \text{ dx}$$

$$= \frac{a}{2} \left[\left\{0 + \frac{a^2}{8}\sin^{-1}(1)\right\} - \left\{0 + \frac{a^2}{8}\sin^{-1}(-1)\right\} \right]$$

$$= \frac{a}{2} \left[\frac{a^2}{8} \times \frac{\pi}{2} - \frac{a^2}{8}\left(-\frac{\pi}{2}\right) \right]$$

$$= \frac{a}{2} \times \frac{\pi a^2}{8}$$

$$= \frac{\pi a^3}{16}$$

$\therefore$ From(3), we get

$$\int_0^a x^{\frac{5}{2}} \sqrt{a-x} \text{ dx} = \frac{5a}{8} \times \frac{\pi a^3}{16}$$

$$\therefore \quad \int_0^a x^2 \sqrt{a-x^2} \text{ dx} = \frac{5\pi a^4}{128}$$

Self - Check Exercise

Q. 1 If x = a + b x^n, then prove that

$$\int x^{m-1} x^p \text{ dx} = - \frac{x^m X^{p+1}}{na(p+1)} + \frac{m+pn+n}{na(p+1)} \int x^{m-1} x^{p+1} \text{ dx}$$

Q. 2 Connect $\int x^{m-1}(a+bx^n)^p$ dx with

$\int x^{m-n-1}(a+bx^n)^p$ dx and hence evaluate $\int \frac{x^8}{(1-x^3)^{1/3}}$ dx

Q. 3 If $I_{m,n} = \int \frac{x^m}{(x^2+1)^n}$ dx, then

= $(n-1)\, I_{m,n} = -x^{m-1} (x^2 + 1)^{-(n-1)} + (m - 1)\, I_{m-2,\, n-1}$

Q. 4 Obtain a reduction formula for

$\int \frac{dx}{(a^2+x^2)^n}$, where n is a positive integer.

9.4 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Discussed "Smaller Index +1 method".
2. Discussed the rule of "Smaller Index +1" to connect $\int x^m (a + bx^n)^p\, dx$ with a given integral of the same type.
3. Solved questions related to reduction by connecting two integrals by using smaller Index +1 method.

9.5 Glossary

1. Reduction by connecting two integrals, also known as the Smaller Index + 1 method.
2. Basic idea behind Smaller Index + 1 method is to express a given integral as a sum or difference of two integrals, where one of the integrals has a smaller index (power) than the original integral, and the other integral has an index that is one greater than the original integral.

9.9 Answers To Self-Check Exercise

Self-Check Exercise

Ans. 1 By connecting $\int x^{m-1} x^p\, dx$ with

$\int x^{m-1} X^{p+1}$ dx, we get the result.

Ans. 2 $\int x^{m-1}(a+bx^n)^p\,dx = \frac{x^{m-n}(a+bx^n)^{p+1}}{b(m+pn)} - \frac{a(m-n)}{b(m+pn)} \int x^{m-n-1}(a+bx^n)^p\,dx$

and $-\left(1-x^3\right)^{2/3}\left[\frac{x^6}{8}+\frac{3}{20}x^3+\frac{9}{40}\right]$

Ans. 3 By connecting $I_{m,n} = \int \frac{x^m}{(x^2+1)^n}\,dx$

with $\quad I_{m-2,n-1} = \int \frac{x^{m-2}}{(x^2+1)^{n-1}}\,dx$, we get the result.

Ans. 4 $\int \frac{1}{(a^2+x^2)^n}\,dx = \frac{x}{2a^2(n-1)(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} \int \frac{dx}{(a^2+x^2)^{n-1}}$

9.7 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.
2. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.

9.8 Terminal Questions

1. What are the integrals with which $\int x^{m-1}(a+bx^n)^p\,dx$ can be connected by reduction formulae?
2. If X = a + bx^n, then prove that

$$\int x^{m-1}X^p dx = \int \frac{x^{m-n}X^{p+1}}{bn(p+1)} - \frac{m-n}{bn(p+1)} \int x^{m-1}X^{p+1}dx$$

3. Prove that

$$\int (a^2+x^2)^{\frac{2n+1}{2}} = \frac{x}{2n+2}(a^2+x^2)^{\frac{2n+1}{2}} + \frac{2n+1}{2}a^2 \int (a^2+x^2)^{\frac{2n-1}{2}}\,dx$$

4. Find $\int (a^2+x^2)^{\frac{5}{2}}dx$

Unit - 10

Smaller Inex+1 Method

To connect $\int \sin^p x \cos^q x dx$

And Reduction Formulae

For $\int_0^{\pi/2} \sin^n x dx$, $\int_0^{\pi/2} \sin^p x \cos^q x\, dx$

Structure

10.1 Introduction

The "smaller index + 1 rule" is a technique used in integral calculus to connect a given integral with another integral of the same type. This rule is particularly useful when the given integral is difficult to evaluate directly but can be related to a simpler integral. The rule states that if we have an integral of the form $\int f(x)dx$ and we can rewrite as $\int g'(x)dx$, where g(x is a function, then we can establish a connection between the two integrals by applying the "smaller index + 1 rule". This rule is often employed when dealing with definite integrals, where the limits of integration are specified. The rule states that if you have an integral with a certain index or variable, and you want to connect it to an integral with the same type but a different index, you can simply increment the index by + 1. By applying this rule, you can establish a relationship between the integrals and use it to simplify calculations our solve problems more efficiently.

10.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss rule of "Smaller Index +1" to connect $\int \sin^p x \cos^q x dx$ with a given integral of the same type.
- Solve questions of integrals of this type by smaller Index +1 method.
- Evaluate integral of the type $\int_0^{\pi/2} \sin^n x dx$, where n is a positive integer.
- Evaluate integral of the type $\int_0^{\pi/2} \cos^n x dx$, where n is a positive integer.
- Evaluate integral of the type $\int \sin^p x \cos^q x dx$, where p and q are positive integers.
- Solve questions related to these types of integrals.

10.3 Rule of "Smaller Index +1" to Connect $\int \sin^p x \cos^q x dx$ with a Given Integral of the Same Type

(i) Let P = $x^{\lambda+1}$ x $\cos^{u+1}$x, where λ is the smaller of the two indices of sin x, and u is the smaller of the two indices of cos x in the two expression whose integrals are to be connected.

(ii) Find $\frac{dp}{dx}$ and express it as a linear function of the two expressions whose integrals are to be connected.

(iii) Integrate both sides w.r.t. x, transpose, and solve for the integral given to be connected.

Let us improve our understanding of this rule by looking at some following examples:-

Example 1: Connect $\int \sin^p x \cos^q x dx$ with

$$\int \sin^{p-2} x \cos^q x dx$$

Sol: We have to connect

$\int \sin^p x \cos^q x dx$ with $\int \sin^{p-2} x \cos^q x dx$

Let P = $\int \sin^{p-2+1} x \cos^{q+1} x$

[Rule of "smaller index+1"]

= $\sin^{p-1}$ x $\cos^{q+1}$ x

$\therefore$ $\frac{dp}{dx}$ = (p-1) $\sin^{p-2}$x cos x $\cos^{q+1}$x + $\sin^{p-1}$x. (q+1)

Or $\frac{dp}{dx}$ = (p-1) $\sin^{p-2}$x $\cos^{q+2}$ x - (q+1) $\sin^{p}$ x $\cos^{q}$ x

= (p-1) $\sin^{p-2}$ x $\cos^q$ x $\cos^2$x - (q+1) $\sin^p$ x $\cos^q$x

= (p-1) $\sin^{p-2}$x $\cos^q$x (1-$\sin^2$x) - (q+1)

= (p-1) $\sin^{p-2}$x $\cos^q$x - (p-1) $\sin^p$x $\cos^q$x - (q+1) $\sin^p$x $\cos^q$x

= (p-1) $\sin^{p-2}$x $\cos^q$x - (p+q) $\sin^p$x $\cos^q$x

Integrating both sides w.r.t. x, we get

P = (p-1) $\int \sin^{p-2} x \cos^q x dx$ - (p+q) $\int \sin^p x \cos^q x dx$

By transposing, we get

(p+q) $\int \sin^p x \cos^q x dx$ = -P + (p-1) $\int \sin^{p-2} x \cos^q x dx$

$$\therefore \quad \int \sin^p x \cos^q x dx = - \frac{P}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx$$

$$= - \frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x dx$$

which is the required reduction formula,

Example 2: If $I_{m,n} = \int \sin^m x \cos^n x dx$, then prove that

$$I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}$$

Hence evaluate $\int \frac{dx}{\cos x \sin^4 x}$

Sol: We have to connect $I_{m,n} = \int \sin^m x \cos^n x dx$

with $I_{m+2,n} = \int \sin^{m+2} x \cos^n x dx$

Let $P = \sin^{m+1}x \cos^{n+1}x$

$\therefore \quad \frac{dp}{dx} = (m+1) \sin^m x \cos x \cos^{n+1}x + \sin^{m+1}x (n+1) \cos^n x (-\sin x)$

$= (m+1) \sin^m x \cos^n x \cos^2 x - (n+1) \sin^{m+2}x \cos^n x$

$= (m+1) \sin^m x \cos^n x (1-\sin^2 x) - (n+1) \sin^{m+2}x . \cos^n x$

$= (m+1) \sin^m x \cos^n x - (m+1) \sin^{m+2}x \cos^n x - (n+1) \sin^{m+2}x \cos^n x$

$\therefore \quad \frac{dp}{dx} = (m+1) \sin^m x \cos^n x - (m+n+2) \sin^{m+2}x \cos^n x$

Integrating both sides w.r.t. x, we get

$P = (m+1) \int \sin^m x \cos^n x dx - (m+n+2)\int \sin^{m+2} x \cos^n x dx$

$\therefore \quad (m+1) \int \sin^m x \cos^n x dx = P + (m+n+2)\int \sin^{m+2} x \cos^n x dx$

$\therefore \quad \int \sin^m x \cos^n x dx = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x dx$

$\therefore \quad I_{m,n} = \frac{\sin^{m+1} x \cos^{n+1} x}{m+1} + \frac{m+n+2}{m+1} I_{m+2,n}$

which is the required reduction formula.

To evaluate $\int \frac{dx}{\cos x \sin^4 x}$

Putting m = -4, n = -1 in (1), we get

$I_{-4,-1} = \frac{(\sin x)^{-3} \cos^0 x}{-3} + \frac{-4-1+2}{-3} I_{-2,-1}$

i.e. $I_{-4,-1} = -\frac{1}{3\sin^3 x} + I_{-2,-1}$ .....(2)

Now put m = -2, n = -1 in (1), we have

$I_{-2,-1} = \frac{(\sin x)^{-1} \cos^0 x}{-1} + \frac{-2-1+2}{-2+1} I_{0,-1}$

$= -\frac{1}{\sin x} + \int \sin^0 x (\cos x)^{-1} dx$

$$= -\frac{1}{\sin x} + \int \frac{1}{\cos x}\,dx$$

$$= -\frac{1}{\sin x} + \int \sec x dx$$

$\therefore$ $\quad I_{-2,-1} = -\frac{1}{\sin x} + \log|\sec x + \tan x|$

$\therefore$ $\quad$ from (2), we get

$$\int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{3\sin^3 x} - \frac{1}{\sin x} + \log|\sec x + \tan x|$$

Example 3: Use suitable reduction formula to evaluate

$$\int \frac{\sin^4 x}{\cos^2 x}\,dx$$

Sol: We will show that

$$\int \sin^p x \cos^q x dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x \cos^{q+2} x dx$$

Let $\quad P = \sin^{p-2+1}x \cos^{q+1}x$

$= \sin^{p-1}x \cos^{q+1}x$

$\therefore$ $\quad \frac{dp}{dx} = (p-1) \sin^{p-2}x \cos x \cos^{q+1}x + \sin^{p-1}x\, (q+1) \cos^{q}x\, (-\sin x)$

Or $\quad \frac{dp}{dx} = (p-1) \sin^{p-2}x \cos^{q+2}x - (q+1) \cos^{q}x \cos^{q} x$

Integrating both sides w.r.t. x, we get

$$P = (p-1) \int \sin^{p-2} x \cos^{q+2} x dx - (q+1) \int \sin^p x \cos^q x dx$$

$\therefore$ $\quad (q+1) \int \sin^p x \cos^q x dx = -P + (p-1) \int \sin^{p-2} x \cos^{q+2} x dx$

$\Rightarrow$ $\quad \int \sin^p x \cos^q x dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{q+1}\int \sin^{p-2} x \cos^{q+2} x dx$

Putting p = 4, q = -2, we get

$$\int \frac{\sin^4 x}{\cos^2 x}\,dx = -\frac{\sin^3 x (\cos x)^{-1}}{-2+1} + \frac{4-1}{-2+1}\int \sin^2 x (\cos x)^0 dx$$

$= \dfrac{\sin^3 x}{\cos\ x} - \dfrac{3}{2}\int 2\sin^2 x dx$

$= \dfrac{\sin^3 x}{\cos\ x} - \dfrac{3}{2}\int (1-\cos 2x)dx$

$= \dfrac{\sin^3 x}{\cos\ x} - \dfrac{3}{2}\left(x - \dfrac{\sin 2x}{2}\right)$

$= \dfrac{\sin^3 x}{\cos\ x} - \dfrac{3}{2}x + \dfrac{3}{4}\sin 2x$

Example 4: prove that

$$\int \frac{dx}{\sin^p x\cos^q x} = \frac{1}{(q-1)\sin^{p-1} x\cos^{q-1} x} + \frac{p+q-2}{q-1}\int \frac{dx}{\sin^p x\cos^{q-2} x}$$

and have evaluate $\dfrac{dx}{\sin x\cos^4 x}$

Sol: We have to connect $\displaystyle\int \frac{dx}{\sin^p x\cos^q x}$

with $\displaystyle\int \frac{dx}{\sin^p x\cos^{q-2} x}$

Let P = $\sin^{-p+1}x \cos^{-q+1}x$

$\therefore$ $\dfrac{dp}{dx}$ = (-p + 1) $\sin^{-p}x$. cosx. $\cos^{-q+1}x$ + (-q + 1) $\cos^{-q}x$ (-sin x) $\sin^{-p+1}x$

= (-p + 1) $\sin^{-p}x \cos^{-q+2}x$ + (q-1) $\cos^{-q}x \sin^{-p}x$. $\sin^2 x$

= (-p + 1) $\sin^{-p}x \cos^{-q+2}x$ + (q-1) $\cos^{-q}x \sin^{-p}x$ (1-$\cos^2 x$)

= (-p + 1) $\sin^{-p}x \cos^{-q+2}x$ + (q-1) $\cos^{-q}x \sin^{-p}x$ - (q-1) $\sin^{-p}x \cos^{-q+2}x$

$\therefore$ $\dfrac{dp}{dx}$ = (-p + 2) $\sin^{-p} x \cos^{-q+2} x$ + (q-1) $\sin^{-p}x \cos^{-q} x$

Integrating both sides w.r.t. x, we get

P = (-p - q + 2) $\int \sin^{-p} x\cos^{-q+2} xdx$ + (q-1) $\int \sin^{-p} x\cos^{-q} xdx$

$\therefore$ (q-1) $\displaystyle\int \frac{dx}{\sin^p x\cos^q x}$ = P + (p + q - 2) $\displaystyle\int \frac{dx}{\sin^p x\cos^{q-2} x}$

$\therefore$ $\displaystyle\int \frac{dx}{\sin^p x\cos^q x} = \frac{dx}{(q-1)\sin^{p-1} x\cos^{q-1} x} + \frac{(p+q-2)}{q-1}\int \frac{dx}{\sin^p x\cos^{q-2} x}$ .....(1)

which is the required reduction formula

Putting p = 1, q = 4 in (1), we get

$$\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{3\sin^3 x} + \int \frac{dx}{\sin x \cos^2 x} \qquad .....(2)$$

Putting p = 1, q = 2 in (1), we get

$$\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \int \frac{dx}{\sin x} = \frac{1}{\cos x} + \int \cos ec\, x\, dx$$

$\therefore$ $\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x}$ - log |cosec x + cot x|

$\therefore$ from (2), we get $\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{3\sin^3 x} + \frac{1}{\cos x}$ - log |cosec x + cot x|

Self-check Exercise-1

Q. 1 If $I_{m,n} = \int \sin^m x \cos^n x dx$, Show that $I_{m,n} = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-1}$ and hence evaluate $\int \sin^2 x \cos^4 x dx$

Q. 2 Use suitable reduction formula for evaluate $\int \sin^3 x \cos^2 x$

10.4 Evaluate $\int_0^{\pi/2} \sin^n x dx$ where n is A Positive Integer

Let us connect $\int \sin^n x dx$ with $\int \sin^{n-2} x dx$

i.e. $\int \sin^n x \cos^0 dx$ with $\int \sin^{n-2} x \cos^0 dx$

Let P = $\sin^{n-2+1} x \cos^{0+1} x$ [Rule of "smaller index +1"]

= $\sin^{n-1} x \cos x$

$\therefore$ $\frac{dp}{dx}$ = (n - 1) $\sin^{n-2} x \cos x \cos x + \sin^{n-1} x (-\sin x)$

= (n - 1) $\sin^{n-2} x \cos^2 x - \sin^n x$

= (n - 1) $\sin^{n-2} x (1 - \sin^2 x)$ -sin x

= (n - 1) $\sin^{n-2} x$ - (x - 1) $\sin^n x - \sin^n x$

= (n - 1) $\sin^{n-2} x$ - n $\sin^n x$

Integrating both sides w.r.t. x, we get

$$P = (n-1)\int \sin^{n-2} x\,dx - n \int \sin^{n} x\,dx$$

Transposing,

$$n\int \sin^{n} x\,dx = -P + (n-1)\int \sin^{n-2} x\,dx$$

$$\therefore \quad \int \sin^{n} x\,dx = -\frac{P}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$= -\frac{\sin^{n-1} x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\therefore \quad \int_0^{\pi/2} \sin^{n} x\,dx = \left[-\frac{\sin^{n-1} x\cos x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$\therefore \quad \int_0^{\pi/2} \sin^{n} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \qquad \text{....(1)}$$

[$\because$ when x = $\frac{\pi}{2}$, $\sin^{n-1}$ x cos x = 0, and when x = 0]

if n is a +ve integer > 1]

Changing n to n-2,

$$\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-3}{n-2}\int_0^{\pi/2} \sin^{n-4} x\,dx$$

Substituting this value of $\int_0^{\pi/2} \sin^{n-2} x\,dx$ in (1), we have

$$\int_0^{\pi/2} \sin^{n} x\,dx = \frac{(n-1)(n-3)}{n(n-2)}\int_0^{\pi/2} \sin^{n-4} x\,dx \qquad \text{...(2)}$$

Generalizing from (1) and (2),

Case I If n is a positive odd integer,

$$\int_0^{\pi/2} \sin^{n} x\,dx = \frac{(n-1)(n-3).......2}{n(n-2)........3}\int_0^{\pi/2} \sin^{1} x\,dx$$

$$= \frac{(n-1)(n-3).......2}{n(n-2)........3} \qquad \left[\because \int_0^{\pi/2} \sin^{1} x\,dx = \left[-\cos x\right]_0^{\pi/2} = -[0-1] = 1\right]$$

Case II : If n is a positive even integer,

$$\int_0^{\pi/2} \sin^n x dx = \frac{(n-1)(n-3).......1}{n(n-2)........2} \int_0^{\pi/2} \sin^0 x dx$$

$$= \frac{(n-1)(n-3).......1}{n(n-2)........2} \int_0^{\pi/2} \sin^1 x dx$$

$$= \frac{(n-1)(n-3).......1}{n(n-2)........2} [x]_0^{\pi/2}$$

$$= \frac{(n-1)(n-3).......1}{n(n-2)........2} \frac{\pi}{2}$$

Note : Working rule to write down the value of

$\int_0^{\pi/2} \sin^n x dx$, where n is a positive integer.

$\int_0^{\pi/2} \sin^n x dx = \dfrac{(n-1)\times \text{ go on diminishing by 2}}{n \times \text{ go on diminishing by 2}} \dfrac{\pi}{2}$ only if n is a positive even integer (otherwise no. $\frac{\pi}{2}$), each series of factors being continued so long as the factors are positive (i.e. omitting zero and negative factors)

10.5 Evaluate $\int_0^{\pi/2} \sin^n x dx$, where n Is A Positive Integer

Let us connect $\int \cos^n x \, dx$ with $\int \cos^{n-2} x \, dx$

i.e. $\int \sin^0 x \cos^n x dx$ with $\int \sin^0 x \cos^{n-2} x dx$

Let P = $\sin^{0+1} x \cos^{n-2+1} x$ [Rule of "smaller index +1"]

= $\sin x \cos^{n-1} x$

$\therefore \quad \dfrac{dp}{dx}$ = $\cos x \cos^{n-1} x + \sin x (n-1) \cos^{n-2} x (-\sin x)$

= $\cos^n x - (n-1) \sin^2 x \cos^{n-2} x$

= $\cos^n x - (n-1) (1 - \cos^2 x) \cos^{n-2} x$

= $\cos^n x - (n-1) \cos^{n-2} x + (n - 1) \cos^n x$

= $\cos^n x - (n-1) \cos^{n-2} x$

Integrating both sides w.r.t x, we have w.r.t. x, we have

$$P = n \int \cos^4 x\,dx - (n-1)\int \cos^{n-2} x\,dx$$

By transposing, we get

$$n \int \cos^4 x\,dx = P + (n-1)\int \cos^{n-2} x\,dx$$

$$\therefore \quad \int \cos^4 x\,dx = \frac{P}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\text{or} \quad \int \cos^4 x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\therefore \quad \int_0^{\pi/2} \cos^n x\;dx = \left[\frac{\sin x \cos^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\;dx$$

[Now where x = $\frac{\pi}{2}$, sin x $\cos^{n-1}$ x = 0, if n is a +ve integer > 1

and when x = 0, sin x $\cos^{n-1}$ x = 0]

$$\therefore \quad \int_0^{\pi/2} \cos^n x\;dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\;dx \qquad ...(1)$$

Changing n to n-2, we have

$$\int_0^{\pi/2} \cos^{n-2} x\;dx = \frac{n-3}{n-2}\int_0^{\pi/2} \cos^{n-4} x\;dx$$

Substituting this value of $\int_0^{\pi/2} \cos^{n-2} x\;dx$ in (1), we get

$$\int_0^{\pi/2} \cos^n x\;dx \; \frac{(n-1)(n-3)}{n(n-2)} \int_0^{\pi/2} \cos^{n-4} x\;dx \qquad ...(2)$$

Generalizing from (1) and (2), we get

Case I : If n is a positive odd integer,

$$\int_0^{\pi/2} \cos^n x\;dx \; \frac{(n-1)(n-3).....2}{n(n-2)......3} \int_0^{\pi/2} \cos^1 x\;dx$$

[Now $\int_0^{\pi/2} \cos^1 x\;dx = [\sin x]_0^{\pi/2}$ = 1 - 0 = 1]

$$= \frac{(n-1)(n-3).....2}{n(n-2)......3}$$

Case II : If n is a +ve even integer,

$$\int_0^{\pi/2} \cos^n x \, dx \ \frac{(n-1)(n-3).....1}{n(n-2)......2} \int_0^{\pi/2} \cos^0 x \, dx$$

[Now $\int_0^{\pi/2} \cos^0 x \, dx = \int_0^{\pi/2} 1 \, dx = \left[x\right]_0^{\pi/2} = \pi/2$]

$$\therefore \qquad \int_0^{\pi/2} \cos^n x \, dx \ \frac{(n-1)(n-3).....1}{n(n-2)......2} . \frac{\pi}{2}$$

Working rule to write down the value of $\int_0^{\pi/2} \cos^n x \, dx$, where n is a positive integer :

Same as the rule to write down the value of $\int_0^{\pi/2} \sin^n x \, dx$

Let us consider the following examples to clear the idea :-

Example 5 : Write down the value of $\int_0^{\pi/2} \sin^8 \theta \, d\theta$

Sol. : $\int_0^{\pi/2} \sin^8 \theta \, d\theta$

Compare with $\int_0^{\pi/2} \sin^n \theta \, d\theta$, here n = 8

$$\therefore \qquad \int_0^{\pi/2} \sin^8 \theta \, d\theta = \frac{7.5.3.1}{8.6.4.2} . \frac{\pi}{2}$$

[$\frac{\pi}{2}$ ($\because$ n = 8, a + +ve even integer)]

[By using the formula $\frac{(n-1)\times \text{ go on diminishing by 2}}{n \times \text{ go on diminishing by 2}}$

$\frac{\pi}{2}$ only if n is a +ve even integer]

$\therefore \quad \int_0^{\pi/2} \sin^8 \theta \, d\theta = \frac{35}{256} \pi$

Example 6 : Evaluate $\int_0^{\pi/2} \sin^3 \theta \, d\theta$

Sol. : $\int_0^{\pi/2} \sin^3 \theta \, d\theta = \frac{2}{3}$

Example 7 : Show that $\int_0^{\pi/6} \sin^7 3x \, dx = \frac{16}{105}$

Sol. : Let I = $\int_0^{\pi/6} \sin^7 3x \, dx$

Put 3x = θ, ∴ 3dx = dθ

$\Rightarrow \quad dx = \frac{1}{3} d\theta$

When x = 0, θ = 0

When $x = \frac{\pi}{6}$, $\theta = \frac{\pi}{2}$

$\therefore \quad I = \frac{1}{3} \int_0^{\pi/2} \sin^7 \theta \, d\theta$

$= \frac{1}{3} \cdot \frac{7.5.3}{6.4.2}$

$= \frac{16}{105}$

Example 8 : Write down the value of $\int_0^{\pi/2} \sin^9 \theta \, d\theta$

Sol. : $\int_0^{\pi/2} \sin^9 \theta \, d\theta = \frac{8.6.4.2}{9.7.5.3} = \frac{128}{315}$

Example 9 : Show that $\int_0^{\infty} \frac{dx}{(1+x^2)^5} = \frac{7.5.3.1}{8.6.4.2} \cdot \frac{\pi}{2}$

Sol. : Let I = $\int_0^\infty \frac{dx}{(1+x^2)^5}$

Put x = tan θ

$\therefore$ dx = $\sec^2 \theta \, d\theta$

When x = 0, tan θ = 0 $\Rightarrow$ θ = 0

When x = ∞, tan θ = ∞ $\Rightarrow$ θ = $\pi/2$

$$\therefore \qquad I = \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{(1+\tan^2 \theta)^5}$$

$$= \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{\sin^{10} \theta}$$

$$= \int_0^{\pi/2} \cos^8 \theta \, d\theta$$

$$= \frac{7.5.3.1}{8.6.4.2} \cdot \frac{\pi}{2}$$

Self-Check Exercise-2

Q. 1 Evaluate $\int_0^{\pi/2} \sin^4 x \, dx$

Q. 2 Show that $\int_0^a \frac{x^6}{\sqrt{a^2+x^2}}$ dx = $\frac{5\pi a^6}{32}$

10.6 Evaluate $\int_0^{\pi/2} \sin^p x \cos^q x$, Where p And q Are Positive Integers

Let us connect $\int \sin^p x \cos^q x \, dx$ with $\int \sin^{p-2} x \cos^q x \, dx$

Let P = $\sin^{p-2+1} x \cos^{q+1} x$

[Rule of "smaller index +1"]

= $\sin^{p-2+1} x \cos^{q+1} x$

Then $\frac{dP}{dx}$ = (p - 1) $\sin^{p-2} x \cos x \cos^{q+1} x + \sin^{p-1} x \, (q + 1) \cos^{q+1} x \, (-\sin x)$

$= (p - 1) \sin^{p-2} x \cos^{q+2} x - (q +1) \sin^p x \cos^q x$

$= (p - 1) \sin^{p-2} x \cos^q x + \cos^2 x - (q + 1) \sin^p x \cos^2 x$

$= (p - 1) \sin^{p-2} x \cos^q x (1 - \sin^2 x) - (q + 1) \sin^p x \cos^2 x$

$= (p - 1) \sin^{p-2} x \cos^q x - (p - 1) \sin^p x \cos^q x - (q + 1) \sin^p x \cos^2 x$

$= (p - 1) \sin^{p-2} x \cos^q x - (p + q) \sin^p x \cos^q x$

Integrating both sides w.r.t. x, we have

$$P = (p - 1) \int \sin^{p-2} x \cos^q x \, dx - (p + q) \int \sin^p x \cos^q x \, dx$$

By transposing, we get

$$(p + q) \int \sin^p x \cos^q x \, dx = - P + (p - 1) \int \sin^{p-2} x \cos^q x \, dx$$

or $$\int \sin^p x \cos^q x \, dx = - \frac{\sec^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q} \int \sin^{p-2} x \cos^q x \, dx$$

$\therefore$ $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \left[-\frac{\sec^{p-1} x \cos^{q+1} x}{p+q} \right]_0^{\pi/2} + \frac{p-1}{p+q} \int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx$$

[when x = $\pi/2$, $\sin^{p-1} x \cos^{q+1} x = 0$

($\because$ q is a +ve integer) and when x = 0,

$\sin^{p-1} x \cos^{q+1} x = 0$, if p is a +ve integer > 1]

$\therefore$ $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{p-1}{p+q} \int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx \qquad ...(1)$$

Changing p to p-2 in (1), we get

$$\int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx = \frac{p-3}{p+q-2} \int_0^{\pi/2} \sin^{p-4} x \cos^q x \, dx$$

Substituting this value of $\int_0^{\pi/2} \sin^{p-2} x \cos^q x \, dx$ in (1), we have

$$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{(p-1)(p-2)}{(p+q)(p+q-2)} \int_0^{\pi/2} \sin^{p-4} x \cos^q x \, dx \qquad ...(2)$$

Generalizing from (1) and (2), we get

Case I : If p is a +ve odd integer,

$$\int_0^{\pi/2} \sin^p x \cos^q x \ dx = \frac{(p-1)(p-3).....2}{(p+q)(p+q-2).....(q+3)} \int_0^{\pi/2} \sin^1 x \cos^q x \, dx$$

But $\int_0^{\pi/2} \sin^1 x \cos^q x \, dx$ = - $\int_0^{\pi/2} \cos^q x(-\sin x) \, dx$

$$= \left[-\frac{\cos^{q+1} x}{q+1} \right]_0^{\pi/2}$$

$$= \frac{1}{q+1} \ [0 - 1]$$

$$= \frac{1}{q+1}$$

$$\therefore \quad \int_0^{\pi/2} \sin^p x \cos^q x \ dx = \frac{(p-1)(p-3).....2}{(p+q)(p+q-2).....(q+3)} \cdot \frac{1}{q+1}$$

Case II : If p is a +ve even integer,

$$\int_0^{\pi/2} \sin^p x \cos^q x \ dx = \frac{(p-1)(p-3).....1}{(p+q)(p+q-2).....(q+2)} \int_0^{\pi/2} \sin^2 x \cos^q x \, dx$$

$$= \frac{(p-1)(p-3).....1}{(p+q)(p+q-2).....(q+2)} \int_0^{\pi/2} \cos^q dx \qquad ...(3)$$

Sub-case (i) If q is a +ve odd integer, then from (3),

$$\int_0^{\pi/2} \sin^p x \cos^q x \ dx = \frac{(p-1)(p-3).....1}{(p+q)(p+q-2).....(q+2)} \cdot \frac{(q-1)(q-3).......2}{q(q-2)............3}$$

Sub-case (ii) If q is a +ve even integer, then from (3), we have

$$\int_0^{\pi/2} \sin^p x \cos^q x \ dx = \frac{(p-1)(p-3).....1}{(p+q)(p+q-2).....(q+2)} \cdot \frac{(q-1)(q-3).......1}{q(q-2)............2} \cdot \frac{\pi}{2}$$

Working rule to write down the value of $\int_0^{\pi/2} \sin^p x \cos^q x \ dx$

where p, q are positive integers

$$\frac{(p-1)\times \text{ go on diminishing by } 2\times(p-1)\times \text{ go on diminishing by } 2}{(p+q)\ \times\ \text{go on diminishing by } 2}\cdot\frac{\pi}{2}$$

Only if both p and q are positive even integers (otherwise no $\frac{\pi}{2}$), each series of factors being continued so long as the factors are positive (i.e. omitting zero and negative factors)

Let us consider the following examples to clear the idea :

Example 10 : Write down the value of $\int_0^{\pi/2} \sin^5\theta\cos^6\theta\ d\theta$

Sol. : $\int_0^{\pi/2} \sin^5\theta\cos^6\theta\ d\theta$

[Compare with $\int_0^{\pi/2} \sin^p\theta\cos^2 x\ d\theta$, here p = 5, q = 6

$= \frac{4.2\times 5.3.1}{11.9.7.5.3.1}$ [no. $\frac{\pi}{2}$, $\because$ p(=5), q(=6) are not both +ve even integers]

$$\left[\frac{(p-1)\times \text{ go on diminishing by } 2\times(p-1)\times \text{ go on diminishing by } 2}{(p+q)\ \times\ \text{go on diminishing by } 2}\ \text{-}\ \frac{\pi}{2}\right.$$

only if both p and q are +ve even integers]

$= \frac{8}{693}$

Example 11 : Write down the value of $\int_0^{\pi/2} \sin^8 x\cos^4 x\,dx$

Sol. : $\int_0^{\pi/2} \sin^8 x\cos^4 x\,dx = \frac{7.5.3.1\times 3.1}{12.10.8.6.4.2}\cdot\frac{\pi}{2}$

$= \frac{7\pi}{2048}$

Example 12 : Find the value of $\int_0^{\pi/2} \sin^{2m}\theta\cos^{2n}\theta\,d\theta$, for positive integral values of m and n.

Sol. :- Let I = $\int_0^{\pi/2} \sin^{2m}\theta \cos^{2n}\theta \, d\theta = \dfrac{(2m-1)(2m-3)..........1\times(2n-1)(2n-3)..........1}{(2m+2n)(2m+2n-2)..............2} \cdot \dfrac{\pi}{2}$

[$\because$ 2m, 2n are both positive even integers)

Example 13 : Evaluate $\int_0^{2a} x^3\sqrt{2ax+x^2}\,dx$

So. : Let I = $\int_0^{2a} x^3\sqrt{2ax+x^2}\,dx$

Put x = 2a $\sin^2\theta$,

$\therefore$ dx = 4a sin θ cos θ dθ

When x = 0, then 2a $\sin^2\theta$ = 0 ⇒ sin θ = 0 ⇒ θ = 0

When x = 2a, then 2a $\sin^2\theta$ = 2a ⇒ sin θ = 0 ⇒ θ = $\dfrac{\pi}{2}$

$\therefore$ I = $\int_0^{\pi/2} (2a\sin^2\theta)^3 \sqrt{2a\sin^2\theta - 4a^2\sin^4\theta}$ 4a sin θ cos θ d θ

= $\int_0^{\pi/2} 8a^3\sin^6\theta.2a\sin\theta\cos\theta$. 4a sin θ cos θ d θ

= $64a^5 \int_0^{\pi/2} \sin^8\theta\cos^2\theta d\theta$

= $64a^5 \dfrac{7.5.3.1}{10.8.6.4.2} \cdot \dfrac{\pi}{2}$

= $\dfrac{7\pi a^5}{8}$

Self-Check Exercise-3

Q. 1 Write down the value of $\int_0^{\pi/2} \sin^4\theta\cos^6\theta d\theta$

Q. 2 Write down the value of $\int_0^{\pi/2} \sin^5\theta\cos^4 x\,dx$

Q. 3 Evaluate $\int_0^a x^2(a^2+x^2)^{3/2}\,dx$

10.7 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Discussed rule of "Smaller Index +1" to connect $\int \sin^p x \cos^q x\, dx$ with a given integral of the same type and solved questions related to this type of integral.
2. Evaluated integral of the type $\int_0^{\pi/2}$ $\sin^n$ x dx, where n is a positive integer.
3. Evaluated integral of the type $\int_0^{\pi/2}$ cosn x dx, where n is a positive integer.
4. Evaluated integral of the type $\int_0^{\pi/2}$ sin p x $\cos^q$ x dx, where p and q are positive integers
5. Solved questions related to above stated integrals.

10.8 Glossary

1. The "smaller index + 1 rule" is a technique used in integral calculus to connect a given integral with another integral of the some type.
2. $\int_0^{\pi/2}$ $\sin^n$ x dx = $\dfrac{(n-1)\times \text{go on diminishing by 2}}{\text{n} \times \text{go on diminishing by 2}}\ \dfrac{\pi}{2}$ only

   If n is a positive even integer (otherwise on $\frac{\pi}{2}$), each series of factors being continued so long as the factors are positive (i.e., omitting zero and negative factors).

10.9 Answers To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 Connect $\int$ $\sin^m$ x $\cos^n$ x dx with

$\int$ sim^m x $\cos^{n-2}$x dx, we get the required reduction formula

and $\int$ $\sin^2$ x $\cos^4$ x dx = $\dfrac{\sin^3 x\cos^3 x}{6}$ + $\dfrac{\sin^3 x\cos x}{8}$ + $\dfrac{1}{16}\left(x-\dfrac{\sin 2x}{2}\right)$

Ans. 2 Connect $\int$ $\sin^m x \cos^n x\, dx$ with $\int$ $\sin^{m-2} x \cos^n x\, dx$, we get the reduction formula as

$$\int \text{sim}^m x \cos^n x\, dx = \frac{\sin^{m-1} x \cos^{n+1} x}{\text{m+n}}$$

$$+ \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x\, dx$$

and $\int \sin^3 x \cos^2 x\, dx = -\frac{1}{5}\sin^2 x \cos^3 x - \frac{2}{15}\cos^3 x$

Self-Check Exercise-2

Ans. 1 $\frac{3\pi}{16}$

Ans. 2 By substituting x = a sin θ, we get the required result

Self-Check Exercise-3

Ans. 1 $\frac{3\pi}{5/2}$

Ans. 2 $\frac{8}{315}$

Ans. 3 $\frac{\pi 9^6}{32}$

10.10 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.

10.11 Terminal Questions

1. If $I_{m,n} = \int$ sinm x cosn x dx, then show that

$$I_{m,n} = \frac{\sin^{m-1} x \cos^{n+1} x}{\text{m+1}} + \frac{m+n+2}{m+1} I_{m+2,n}$$

Hence evaluate

2. Show that $\int \sin^p x \cos^q x\, dx = \frac{\sin^{p+1} x \cos^{q-1} x}{\text{p+1}}$

$+ \frac{q-1}{p+1} \int$ $\sin^{p+2} x \cos^{q-2} x \, dx$

3. Show that

$$\int_0^a \frac{x^4}{\sqrt{a^2-x^2}} dx = \frac{3\pi a}{16}$$

4. Show that

$$\int_0^\infty \frac{dx}{(1+x^2)^{n+\frac{1}{2}}} = \frac{(2n-2)(2n-4).......2}{(2n-1)(2n-3).......3}, \ n \in N$$

5. Write down the value of

$$\int_0^{\pi/2} \cos^3 x \sin^2 x \, dx$$

6. Show that $\int_0^{\pi/2} \sin^{2m} \theta \cos^{2m-1} \theta \, d\theta$

$= \frac{(2m-2)(2m-4).......4.2}{(4m-1)(4m-3).......(2m+1)}$, m being a positive integer > 1

7. Evaluate $\int_0^1 x^4 \left(1-x^2\right)^{3/2} dx$

Unit - 11

Length of An Arc of A Plane Curve

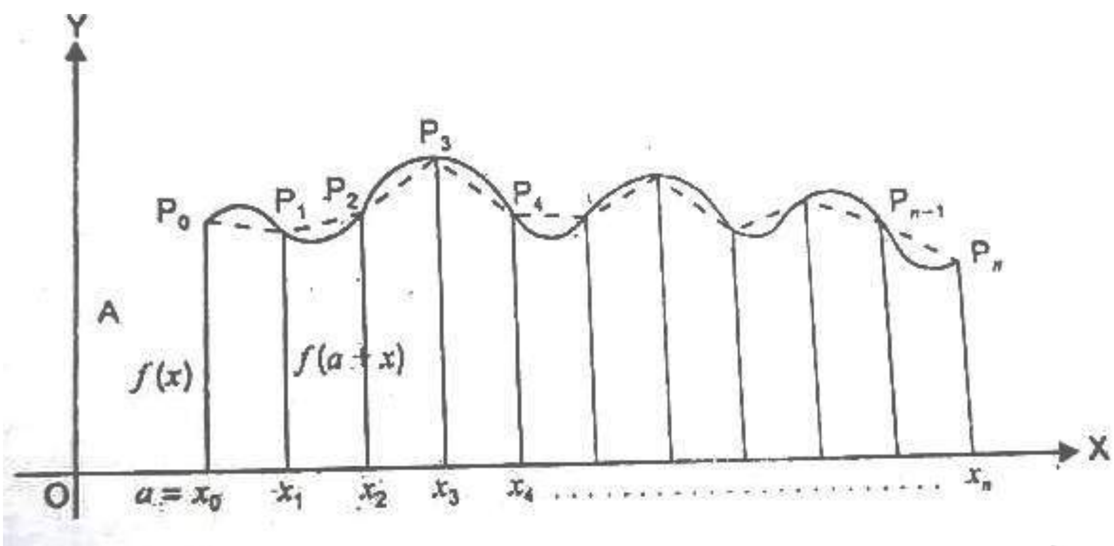
With Cartesian Equations

Structure

11.1 Introduction

Dear Students, you are already familiar with the idea of a curve. Here we are to find the length of a curve.

Let AB the curve defined by continuous function y = f(x) on [a, b]



Let P = {a = x_0, x_1, x_2,......x_{n-1}, x_n = b} be a partition of [a, b] into n equal parts, each of length h, where h = $\frac{b-a}{\mathrm{n}}$.

Let P_i (x_i, $f(x_i)$) be a point on the curve AB. Then P_0 = A, P_n = B. Now P_0, P_1, P_2,......, P_n are consecutive points on the curve. Join each pair P_{i+1}, P_i of consecutive points by line segments and we get broken lines P_0P_1,.....,$P_{n-1}P_n$. When n is very large, the line segment $P_{i-1}P_i$ will be the approximate part of the curve which corresponds to the sub-interval [x_{i-1}, x_i].

∴ as n → ∞, the broken line approaches the curve.

Let L_n denote the sum of lengths of segments of broken lines, then

$$L_n = \sum_{i=1}^{n} | P_{i-1}P_i |$$

If $\underset{n\to\infty}{Lt}$ L_n exists, then it is called length of the curve and is denoted by L. The number L, if it exists, is unique.

Let AB the curve defined by continuous function y = f(x) on [a, b]

Let P = {a = x_0, x_1, x_2,.....x_{n-1}, x_n = b} be a partition of [a, b] into n equal parts, each of length h, where h = $\frac{b-a}{\mathrm{n}}$

Let P_i(i, $f(x_i)$) be a point on the curve AB. Then P_0 = A, P_n = B. Now P_0, P_1, P_2,.....,P_n are consecutive points on the curve. Join each pair P_{i-1}, P_i of consecutive points by line segments and we get broken lines P_0P_1,......,P_{n-1}Pn. When n is very large, the line segment $P_{i-1}P_i$ will be the approximate part of the curve which corresponds to the sub-interval [x_{i-1}, x_i]

∴ as n → ∞, the broken line approaches the curve.

Let L_n denote the sum of lengths of segments of broken lines, then

$$L_n = \sum_{i=1}^{n} | P_{i-1}P_i |$$

If $\underset{n\to\infty}{Lt}$ L_n exists, then it is called length of the curve and is denoted by L. The number L, if it exists, is unique.

11.2 Learning Objectives

After studying this unit, you should be able to:-

- Define curve, rectifiable curve, rectification.
- Derive are formula for Cartesian equation.

- Do questions related to length of the curves.

11.3 Some Definitions

Curve: Let f be a continuous function on [a, b]. Then the graph of f on [a, b] i.e. [(x), f(x)): a $\leq$ x $\leq$ b} is called a curve.

Rectifiable Curve: A continuous curve, which has length, is called rectifiable.

Rectification: The process of finding the length of an are of a curve between two given points is called rectification.

11.4 ARC Formula for Cartesian

Equation : If c is a curve defined by y = f(x), where f has a continuous derivative f'(x) on [a, b], then the length of the curve c is given by $\int_a^b \sqrt{1+\{f^1(x)\}^2}$ dx

Proof: Let P = $\{a=x_0, x_1, x_2, ..., x_{i-1}, x_i, ..., x_{n-1}, x_n = b\}$ be a partition of [a, b] in n equal parts each of length h so that nh = b - a. Then $P_{i-1} \Leftrightarrow (x_{i-1}, f(x_{i-1}))$ and P_i $(x_i, f(x_i))$ are two consecutive points on the curve C. (Refer figure of 11-1)

Now $|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + \{f(x_i) - f(x_{i-1}\}^2}$

$\therefore$ $|P_{i-1}P_i| = |x_i - x_{i-1}| \sqrt{1+\left\{\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}}\right\}^2}$ ....(1)

$\because$ f is continuous and differentiable in [a, b]

$\therefore$ f is continuous and differentiable in $|x_{i-1}, x_i|$

$\therefore$ by Lagrange's Mean Value Theorem, there exists a point

$c_i \in (x_{i-1}, x_i)$ such that $\frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$ ....(2)

From (1) and (2), we get, $|P_{i-1}P_i| = |x_i - x_{i-1}| \sqrt{1+\{f'(c_i)\}^2}$

Or $|P_{i-1}P_i| = h\sqrt{1+\{f'(c_i)\}^2}$ $\left[\because |x_i - x_{i-1}| = h\right]$

Now length of arc = $\underset{n\to\infty}{Lt} L_n$

where $L_n = \sum_{i=1}^{n} |P_{i-1}P_i|$

$$= \sum_{i=1}^{n} h \sqrt{1+\{f'(c_i)\}^2}$$

$\therefore$ $$\underset{n\to\infty}{Lt}\ L_n = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} h \sqrt{1+\{f'(c_i)\}^2}$$

$$= \underset{n\to\infty}{Lt} \sum_{i=1}^{n} h \sqrt{1+\{f'(c_i)\}^2}$$

$$\left\{\begin{array}{l} \because n \to \infty \Rightarrow h \to 0 \Rightarrow \text{lenght of } [x_{i-1}, x_i] \to 0 \\ \Rightarrow x_{i-1} \to x_i \Rightarrow c_i \quad \Rightarrow x_i \text{ as } x_{i-1} < c < x_i \\ \Rightarrow f'(c_i) \to f'(x_i) \text{ as f' is continuous at } \mathrm{x}_i \end{array}\right\}$$

$$= \underset{n\to\infty}{Lt} \sum_{i=1}^{n} h \sqrt{1+\{f'(a+ih)\}^2}$$

$\therefore$ $$\underset{n\to\infty}{Lt}\ L_n = \underset{n\to\infty}{Lt} \sum_{i=1}^{n} h\, F(a + ih) \qquad \text{....(3)}$$

where $F(x) = \sqrt{1+\{f'(x)\}^2}$

Now $f'(x)$ is continuous on [a, b] and therefore $\sqrt{1+\{f'(x)\}^2}$ is continuous and hence integrable on [a, b]

$\therefore$ $$\underset{n\to\infty}{Lt} \sum_{i=1}^{n} h\, F(a + ih) = \int_a^b F(x)\, dx \qquad \text{[By def. of integral as a limit of a sum]}$$

$\therefore$ from (3), we get,

$$\underset{n\to\infty}{Lt}\ L_n = \int_a^b F(x)\, dx$$

$$= \int_a^b \sqrt{1+\{f'(x)\}^2}\, dx$$

$\therefore$ $\underset{n\to\infty}{Lt}\ L_n$ exists and hence

Length of the curve C = $\int_a^b \sqrt{1+\{f'(x)\}^2}\, dx$

Note i. $\because \quad f'(x) = \frac{dy}{dx}$

$\therefore$ length of curve C = $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

Note 2: If C is a curve defined by x = f (y) where f has a continuous derivative f' '(y) on [a, b], then the length of the curve C is given by

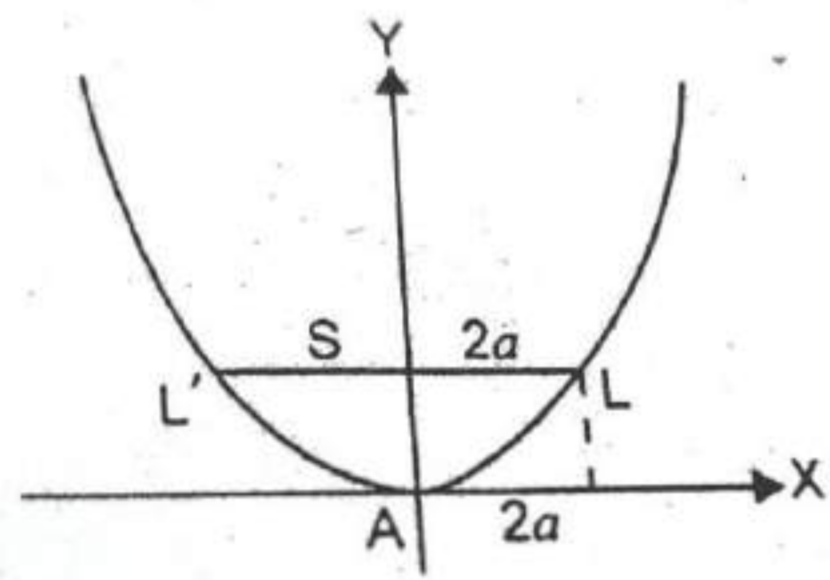
$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dy.$$

Let us consider the following examples to clear the idea:

Example 1: find the length of the arc of the parabola $x^2 = 4\,a\,y$ extending from teh vertex to one extremity of the latus rectum.

Sol: The equation of the parabola is $x^2 = 4\,a\,y$.

Let A be the vertex and L one extremity of the latus rectum.



Now $y = \frac{x^2}{4a}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{4a}(2x) = \frac{x}{2a}$

$\therefore$ arc AL = $\int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ x [$\because$ at A, x = 0; and at L, x = 2a]

= $\int_0^{2a} \sqrt{1+\frac{x^2}{4a^2}}$ dx

= $\frac{1}{2a}\int_0^{2a} \sqrt{4a^2+x^2}$ dx

= $\frac{1}{2a}\int_0^{2a} \sqrt{(2a)^2+x^2}$ dx

= $\frac{1}{2a}\left[\frac{x\sqrt{(2a)^2+x^2}}{2}+\frac{(2a)^2}{2}\sinh^{-1}\frac{x}{2a}\right]_0^{2a}$

= $\frac{1}{4a}$ [2a.2$\sqrt{2}$ a + 4a^2 sinhh^{-1} 1 - 0 (0 + 4a2.0)]

= $\frac{4a^2}{4a}$ [$\sqrt{2}$ + sinh^{-2} 1]

= a$\left[\sqrt{2}+\log\left(1+\sqrt{1}+(1)^2\right)\right]$

$\left[\because \sin^{-1}x = \log\left(x+\sqrt{1+x^2}\right)\right]$

= a$\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$

Example 2: Find the length of the boundary of the region bounded by the curve y = $\frac{1}{2}$x^2 = 1 and the lines y = x, x = 0 and x = 2.

Sol: The equation of the curve is y = $\frac{1}{2}$x^2 + 1 ....(1)

Or 2y = x^2 + 2

Or x^2 = 2y - 2

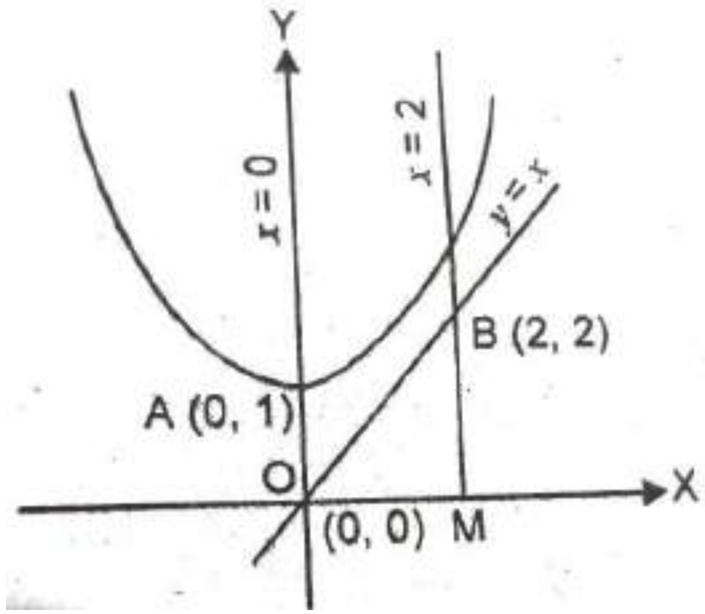
Or x^2 = 2(y - 1)

which represents an upward parabola with vertex at A(0, 1). Its rough sketch is given in the figure. The line x = 2 meets the parabola in C(2, 3) and the line y = x meels the line x = 2 in B(2, 2)

Now |OA| = 1

$|OB| = \sqrt{(2-0)^2 + (2-0)^2}$

$= \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$



$|BC| = \sqrt{(2-2)^2 + (3-2)^2} = 1$

From (1), $\frac{dy}{dx} = \frac{1}{2}.2x = x$

$\therefore$ length of arc AC = $\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^2 \sqrt{1+x^2}\, dx$

$= \left[\frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\sinh^{-1} x\right]_0^2$

$= \left(\frac{2\sqrt{5}}{2} + \frac{1}{2}\sinh^{-1} 2\right) - \left(0 + \frac{1}{2}\sinh^{-1} 0\right)$

$= \left(\sqrt{5} + \frac{1}{2}\sinh^{-1} 2\right) - \left(0 + \frac{1}{2} 0\right)$

$= \sqrt{5} + \frac{1}{2} \sinh^{-1} 2$

$\therefore$ Required length = |0A| + |0B| + |BC| + length of arc AC

$1 + 2\sqrt{2} + 1 + \sqrt{5} + \frac{1}{2} \sinh^{-1} 2$

$= 2 + 2\sqrt{2} + \sqrt{5} + \frac{1}{2} \sinh^{-1} 2$

Example 3: find the length of the arc of the curve

$y = \frac{1}{3} \left(x^2 + 2\right)^{3/2}$ from x = 0 to x = 3

Sol: The equation of the curve is

$y = \frac{1}{3} \left(x^2 + 2\right)^{3/2}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{3} \times \frac{3}{2} \left(x^2 + 2\right)^{1/2} (2x)$

$= x\sqrt{x^2 + 2}$

Required length of arc = $\int_0^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \int_0^3 \sqrt{1 + x^2(x^2 + 2)}\, dx$

$= \int_0^3 \sqrt{x^4 + 2x^2 + 1}\, dx$

$= \int_0^3 \sqrt{(x^2 + 1)^2}\, dx$

$= \int_0^3 (x^2 + 1)\, dx$

$= \left[\frac{x^3}{3} + x\right]_0^x$

$= \left(\frac{27}{3} + 3\right) - (0 + 0)$

$= 9 + 3$

$= 12$ units

Example 4: Find the length of the curve

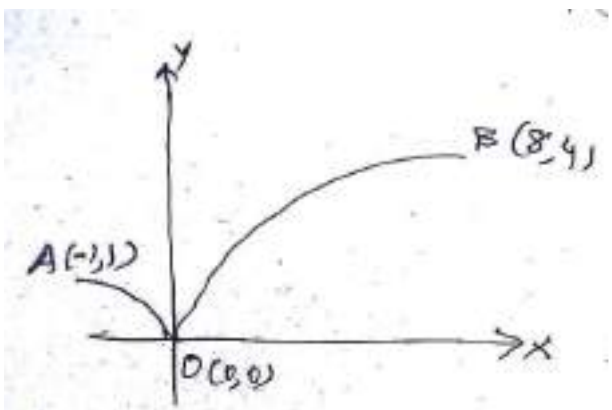
$y = x^{2/3}$ from $x = -1$ to $x = 8$

Sol: The equation of the curve is $y = x^{2/3}$

$\therefore \quad \frac{dy}{dx} = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$, which does not exist at

$x = 0$

$\therefore \quad$ We find $\frac{dy}{dx}$



Now $\quad y = x^{2/3}$

$\Rightarrow \quad y^3 = x^2$

$\Rightarrow \quad x = \pm y^{3/2}$

$\therefore \quad$ curve has two branches

$x = - y^{3/2}$ and $x = y^{3/2}$

Now $\quad \frac{dy}{dx} = + \frac{3}{2} y^{1/2}$

Required length $= \int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy + \int_0^4 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$$= \int_0^1 \left(1+\frac{9}{4}y\right)^{1/2} dy + \int_0^4 \left(1+\frac{9}{4}y\right)^{1/2} dy$$

$$= \left[\frac{\left(1+\frac{9y}{4}\right)^{3/2}}{\left(\frac{9}{4}\right)\left(\frac{3}{2}\right)}\right]_0^1 + \left[\frac{\left(1+\frac{9y}{4}\right)^{3/2}}{\left(\frac{9}{4}\right)\left(\frac{3}{2}\right)}\right]_0^4$$

$$= \frac{8}{27}\left[\left(1+\frac{9y}{4}\right)^{3/2}\right]_0^1 + \frac{8}{27}\left[\left(1+\frac{9y}{4}\right)^{3/2}\right]_0^4$$

$$= \frac{8}{27}\left[\left(1+\frac{9}{4}\right)^{3/2} - 1\right] + \frac{8}{27}\left[(1+9)^{3/2} - 1\right]$$

$$= \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right] + \frac{8}{27}\left[(10)^{3/2} - 1\right]$$

$$= \frac{8}{27}\left[\frac{13\sqrt{13}}{8} + 10\sqrt{10} - 2\right]$$

Example 5: Show that the length of an arc of the curve $x^2 = a^2\left(1-e^{y/a}\right)$ measured from (0, 0) to (x, y) is a log $\left|\frac{a+x}{a-x}\right|$ - x

Sol: The equation of the curve is

$$x^2 = a^2\left(1-e^{y/a}\right)$$

Or $\quad x^2 = a^2 - a^2 e^{y/a}$

Or $\quad a^2 e^{y/a} = a^2 - x^2$

Or $\quad e^{y/a} = \frac{a^2 - x^2}{a^2}$

Or $\quad \frac{y}{a} = \log\left(\frac{a^2 - x^2}{a^2}\right)$

Or $\quad y = a[\log(a^2 - x^2) = \log a^2]$

Or $\quad \frac{dy}{dx} = a\left[\frac{-2x}{a^2-x^2}\right]$

$$= -\frac{-2x}{a^2-x^2}$$

Required length of arc = $\int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_0^x \sqrt{1+\frac{4a^2x^2}{(a^2-x^2)^2}}\, dx$$

$$= \int_0^x \frac{\sqrt{(a^2-x^2)^2+4a^2x^2}}{a^2-x^2}\, dx$$

$$= \int_0^x \frac{a^2+x^2}{a^2-x^2}\, dx$$

$$= \int_0^x \left[1+\frac{2a^2}{a^2-x^2}\right] dx$$

$$= \left[-x+2a^2\frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]_0^x$$

$$= \left[-x+a\log\left|\frac{a+x}{a-x}\right|\right] - \left[-0+a\log\left|\frac{a}{a}\right|\right]$$

$$= -x + a\log\left|\frac{a+x}{a-x}\right|$$

$$= a\log\left|\frac{a+x}{a-x}\right| - x$$

Example 6: Find the length of the arc of the parabola $y^2 - 4y + 2x = 0$ which lies in the first quadrant.

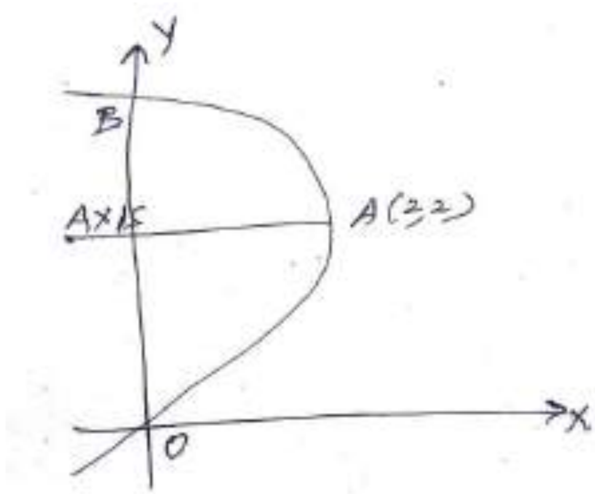
Sol: The equation of the parabola is

$y^2 - 4y + 2x = 0 \qquad$ ....(1)

Or $\quad y^2 - 4y = -2x$

Or $\quad y^2 - 4y + 4 = -2x + 4$

Or $(y - 2)^2 = - 2 (x - 2)$, which is left handed parabola with vertex at A(2, 2). Its rough shetch is given in the figure.



Differentiating (1) w.r.t. y, we get

$$2y - 4 + 2 \frac{dx}{dy} = 0$$

$$\therefore \quad \frac{dx}{dy} = 2 - y$$

Now length of arc OAB

$$= 2 \text{ (length of arc OA)}$$

$$= 2\int_0^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$= 2\int_0^2 \sqrt{1+(2-y)^2}\, dy$$

$$= -2\int_0^2 \sqrt{1+t^2}\, dt \qquad \begin{bmatrix} Put & 2-y & = & t \\ \therefore & -dy & = & dt \\ or & dy & = & -dt \end{bmatrix}$$

$$= 2\int_0^2 \sqrt{1+t^2}\, dt$$

$$= 2\left[\frac{t\sqrt{1+t^2}}{2}+\frac{1}{2}\sinh^{-1}t\right]_0^2$$

$$= 2\left[\left(\frac{2\sqrt{5}}{2}+\frac{1}{2}\sinh^{-1}2\right)-\left(0+\frac{1}{2}\sinh^{-1}0\right)\right]$$

$$= 2\left[\left(\sqrt{5}+\frac{1}{2}\sinh^{-1}2-(0+0)\right)\right]$$

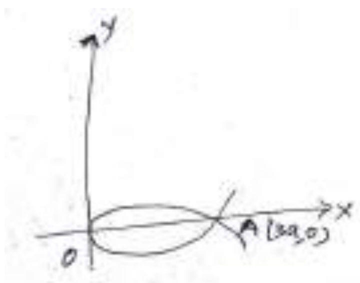
$$= \left(2\sqrt{5}+\sinh^{-1}2\right)\text{units}$$

Example 7: Find the length of a loop of the curve $9ay^2 = x(x - 3a)^2$, $a > 0$

Sol: The equation of a curve is

$$9ay^2 = x(x - 3a)^2$$

A rough sketch of the curve is shown in the figure.



From given equation,

$$y^2 = \frac{x(x-3a)^2}{9a}$$

$$\therefore \quad y = \frac{\sqrt{x}\,|x-3a|}{3\sqrt{a}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{3\sqrt{a}}\left[\sqrt{x}\,\frac{x-3a}{|x-3a|}+|x-3a|\frac{1}{2\sqrt{x}}\right]$$

$$=\frac{1}{3\sqrt{a}}\left[\frac{2x(x-3a)+(x-3a)^2}{2\sqrt{x}\,|x-3a|}\right]$$

$$= \frac{1}{\sqrt{a}}\left[\frac{(x-3a)(x-a)}{2\sqrt{x}\,|x-3a|}\right]$$

$\therefore$ length of loop = $2\int_0^{3a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}$ dx

$= 2\int_0^{3a} \sqrt{1+\frac{(x-a)^2}{4ax}}$ dx

$= \frac{2}{2\sqrt{a}}\int_0^{3a} \sqrt{\frac{4ax+(x-a)^2}{x}}$ dx

$= \frac{1}{\sqrt{a}}\int_0^{3a} \sqrt{\frac{(x+a)^2}{x}}$ dx

$= \frac{1}{\sqrt{a}}\int_0^{3a} \frac{x+a}{\sqrt{x}}$ dx

$= \frac{1}{\sqrt{a}}\int_0^{3a} \left(x^{1/2}+ax^{-1/2}\right)$ dx

$= \frac{1}{\sqrt{a}}\left[\frac{x^{3/2}}{3/2}+a\frac{x^{1/2}}{1/2}\right]_0^{3a}$

$= \frac{2}{\sqrt{a}}\left[\frac{1}{3}x^{3/2}+ax^{1/2}\right]_0^{3a}$

$= \frac{2}{\sqrt{a}}\left[\left\{\frac{1}{3}(3a)^{3/2}+a(3a)^{1/2}\right\}-\{0+0\}\right]$

$= \frac{2}{\sqrt{a}}\left[\frac{1}{3}(3a)\sqrt{3a}+a\sqrt{3a}\right]$

$= \frac{2\sqrt{3a}}{\sqrt{a}}$ [a + a]

$= 4\sqrt{3}$ a

Self-Check Exercise

Q. 1 Find the length of the arc of the curve

y = $(x+1)^{3/2}$ on [3, 8]

Q. 2	Find the length of the arc of the curve y = log $\left(\frac{e^x - 1}{e^x + 1}\right)$ from x = 1 to x = 2
Q. 3	Use integrals to compute perimeter of a circle of radius a
Q.4	Find the length of the arc of the semi-cubical parabola $ay^2 = x^3$ from the cusp (where x = 0) to any point.

11.5 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Defined curve, rectifiable curve, rectification.
2. Derived the arc formula i.e. derived the formula for calculating length of the curve for Cartesian equation.
3. Solved questions to find the length of the curve for Cartesian equation.

11.6 Glossary

1. Let f be a continuous function on [a, b]. Then the graph of f on [a, b] i.e. {(x, f(x)): a $\leq$ x $\leq$ b} is called a curve.
2. A continuous curve, which has length, is called rectifiable.
3. The process of finding the length of on arc of a curve between two given points is called rectification.
4. If c is a curve defined by y = f(x), where f has a continuous derivative f '(x) on [a, b], then the length of the curve c is given by

$$\int_a^b \sqrt{1+\{f'(x)\}^2}\, dx$$

11.7 Answers To Self-Check Exercise

Ans. 1 $\left[17\sqrt{17} - 16\sqrt{2}\right]$ units

Ans. 2 log $\left(e + \frac{1}{e}\right)$

Ans. 3 2a π

Ans. 4 $\frac{1}{27\sqrt{a}}\left[(9x+4a)^{3/2} - (4a)^{3/2}\right]$

11.8 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.

2. G.B. Thomas and R.L. Finney, *Calculus*, 9^{th} Ed., Pearson Education, 2005.

11.9 Terminal Questions

1. Find the length of the arc of the curve

   $y = \log(\cos x)$ on $0 \le x \le \frac{\pi}{4}$

2. Find the length of the arc of the curve

   $2y = x^2$ from $x = a$ to $x = b$

3. Find the length of the arc of the parabola $y^2 = 4ax$ from the vertex to on extremity of the latus rectum.

4. Show that the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$ is a $\left(\log_e 2 + \frac{15}{16}\right)$

5. Find the perimeter of the loop of the curve

   $9ay^2 = (x - 2a)(x - 5a)^2, a > 0$

Unit - 12

Length of an Arc of a Plane Curve

With Parametric Equations

Structure

12.1 Introduction

In mathematics, the concept of the length of a curve in the plane refers to the measurement of the distance along the curve between the two points. When a curve is defined by parametric equations, the coordinates of points on the curve are expressed as functions of a parameter Let's consider a curve defined by the parametric equations:-

$x = f(t)$; $y = g(t)$, where $f(t)$ and $g(t)$ are continuous functions that define the relationship between the parameter t and the coordinates (x, y) on the curve. The parameter t usually represents time or some other independent variable. To find the length of the curve between two points, say from t = a to t = b we can approximate the curve by dividing it into small segments and summing the lengths of those segments. As the segments get smaller, the approximation becomes more accurate

12.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss and derive arc formula for parametric equations.
- Do questions related to finding the length of the curve for parametric equations.

12.3 Arc Formula For Parametric Equations

If C is a curve defined by parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ and $f'(t) \neq 0 \ \forall \ t \in [\alpha,\beta]$, then length L of curve C is given by

$$L = \int_{\propto}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Proof: Here x = f(t), y = g(t)

$$\therefore \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

Also x = f (t) $\Rightarrow$ dx = f' (t) dt

Now L = Length of curve C

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_\alpha^\beta \sqrt{1 + \left\{\frac{g'(t)}{f'(t)}\right\}^2}\, .f'(t)\, dt$$

$$= \int_\alpha^\beta \sqrt{\{f'(t)\}^2 + \{g'(t)\}^2}\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Let us consider the following examples to clear the idea:-

Example 1: find the distance travelled between t = 0 to t = $\frac{\pi}{2}$ by a particle P(x, y) whose position at time t is given by x = a (cos t + t sin t), y = a (sn t - t cos t)

Sol: The position of the particle at time t is given by

x = a(cos t + t sin t), y = a(sin t - t cos t)

$$\therefore \qquad \frac{dx}{dt} = a\,[- \sin t + t.\cos t + \sin t.1] = \alpha\, t \cos t$$

$$\text{and} \qquad \frac{dy}{dt} = a\,[\cos t - t(- \sin t) - \cos t.1] = \alpha\, t \sin t$$

$$\therefore \qquad \text{required distance} = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{a^2t^2\cos^2 t + a^2t^2\sin^2 t}\ dt$$

$$= \alpha \int_0^{\pi/2} t\sqrt{\cos^2 t + \sin^2 t}\ dt$$

$$= \alpha \int_0^{\pi/2} t\ dt$$

$$= \alpha \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a}{2}\left[\left(\frac{\pi}{2}\right)^2 - (0)^2\right]$$

$$= \frac{a\pi^2}{8}$$

Example 2: Find the distance travelled by the particle P(x, y) between t = 0 and t = 4 if its position at time t is given by

$$x = \frac{1}{2}\ t^2,\ y = \frac{1}{3}\left(2t+1\right)^{3/2}$$

Sol: The position of the particle at time t is given by

$$= \frac{1}{2}\ t^2,\ y = \frac{1}{3}\left(2t+1\right)^{3/2}$$

$$\therefore \quad \frac{dx}{dt} = \frac{1}{2}(2t),\ \frac{dy}{dt} = \frac{1}{3}\left(\frac{3}{2}\right)\left(2t+1\right)^{1/2} 2$$

$$\therefore \quad \frac{dx}{dt} = t,\ \frac{dy}{dt} = \left(2t+1\right)^{1/2}$$

$$\therefore \quad \text{Required distance} = \int_0^4 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$= \int_0^4 \sqrt{t^2 + (2t+1)}\ dt$$

$$= \int_0^4 \sqrt{(t+1)^2}\, dt$$

$$= \int_0^4 (t + 1)\, dt$$

$$= \left[\frac{t^2}{2} + t\right]_0^4$$

$$= \left(\frac{16}{2} + 4\right) - (0 + 0)$$

$$= 12$$

Example 3: Find the length of the curve

$x = \cos^3 t$, $y = \sin^3 t$ on $[0, 2\pi]$

Sol: The parametric equations of the curve are

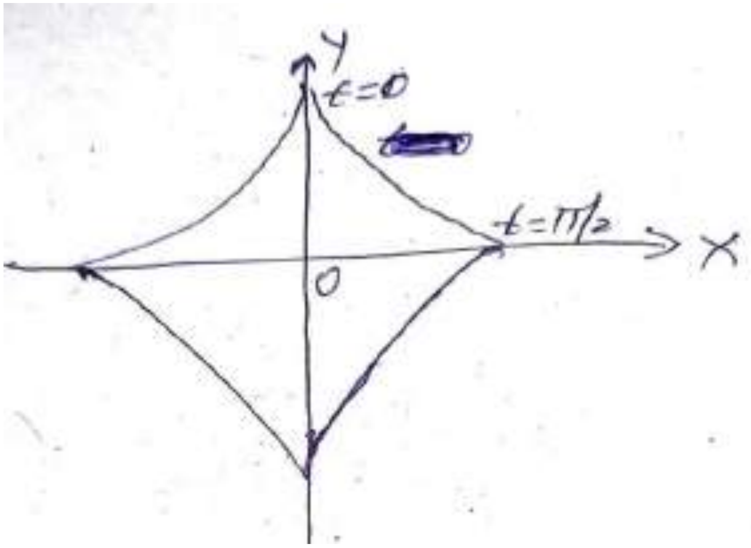
$x = \cos^3 t$, $y = \sin^3 t$

A rough sketch of the curve is shown in the figure. The curve is symmetrical about both the axes.

Now

$$\frac{dx}{dt} = - 3 \cos^2 t \sin t$$

$$\frac{dy}{dt} = 3 \sin^2 t \cos t$$



Length of the curve

$$= 4\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

$$\left[\because \text{ in the first quadrant, t varies from 0 to } \frac{\pi}{2}\right]$$

$$= 4\int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\ dt$$

$$= 4.3\int_0^{\pi/2} \sin t \cos t \sqrt{\cos^2 t + \sin^2 t}\ dt$$

$$= 12\int_0^{\pi/2} \sin t \cos t\ dt$$

$$= 6\int_0^{\pi/2} \sin 2t\ dt$$

$$= 6\left[\frac{-\cos 2t}{2}\right]_0^{\pi/2}$$

$$= -3\ [\cos 2t]_0^{\pi/2}$$

$= - 3\ [\cos \pi - \cos 0]$

$= - 3\ [-1 -1]$

$= 6$

Example 4: Find the entire length of the curve

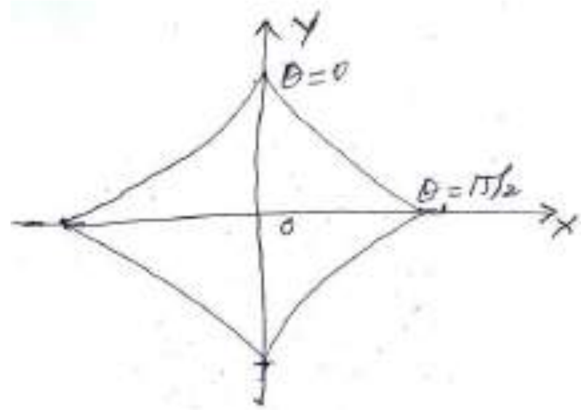
$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

Sol: The equation of the curve is

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

The parametric equation of the curve are

$x = a \cos^3 \theta,\ y = b \sin^3 \theta$



A rough sketch of the curve is shown in the figure. The curve is symmetrical about both the axes.

Now $\dfrac{dx}{d\theta}$ = - 3θ cos² θ sin θ

and $\dfrac{dy}{d\theta}$ = 3b sin² θ cos θ

Length of the curve = $4\int_0^{\pi/2} \sqrt{\left(\dfrac{dx}{d\theta}\right)^2+\left(\dfrac{dy}{d\theta}\right)^2}\, d\theta$

$\left[\because \text{ in the first quadrant, } \theta \text{ varies from 0 to } \dfrac{\pi}{2}\right]$

$= 4\int_0^{\pi/2} \sqrt{9a^2\cos^4\theta\sin^2\theta+9b^2\sin^4\theta\cos^2\theta}\, d\theta$

$= 12\int_0^{\pi/2} (a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}\sin\theta\cos\theta\, d\theta$

Put $a^2\cos^2\theta + b^2\sin^2\theta = t$

$\therefore$ $(-2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\, d\theta = dt$

Or $2(b^2 - a^2)\sin\theta\cos\theta\, d\theta = dt$

$\therefore$ $\sin\theta\cos\theta\, d\theta = \dfrac{1}{2(b^2-a^2)}\, dt$

When $\theta = 0$, $t = a^2\cos^2 0 + b^2\sin^2 0 = a^2$

When $\theta = \frac{\pi}{2}$, t = $a^2 \cos^2 \frac{\pi}{2} + b^2 \sin^2 \frac{\pi}{2} = b^2$

$\therefore$ length of the arc

$$= \frac{12}{2(b^2 - a^2)} \int_0^{b^2} t^{1/2}\, dt$$

$$\frac{6}{b^2 - a^2} \left[\frac{t^{3/2}}{\frac{3}{2}} \right]_{a^2}^{b^2}$$

$$= \frac{4}{b^2 - a^2} \left[t^{3/2} \right]_{a^2}^{b^2}$$

$$= \frac{4}{b^2 - a^2} \left[(b^2)^{3/2} - (a^2)^{3/2} \right]$$

$$= \frac{4}{b^2 - a^2} [b^3 - a^3]$$

$$= \frac{4}{(b-a)(b+a)} \left[(b-a)(b^2 + a^2 + ba) \right]$$

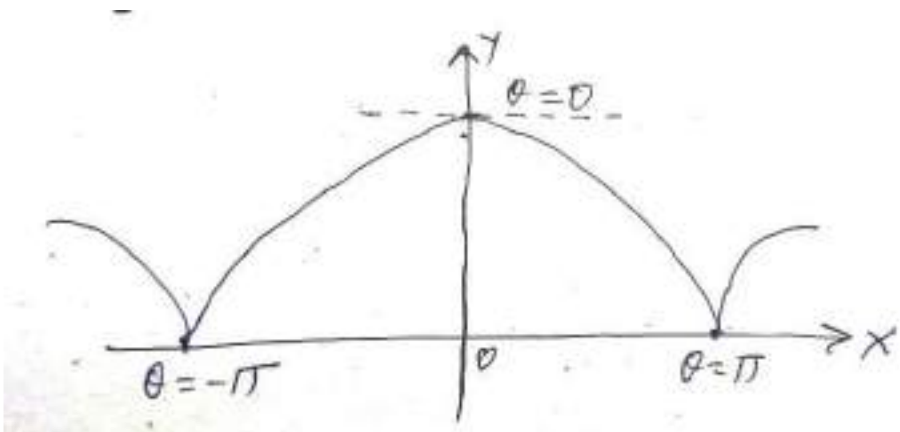
$$= \frac{4(a^2 + b^2 + ba)}{a+b}$$

Example 5: Find the length of an arc of the cycloid whose equations are

x = a (θ + sin θ), y = a (1 + cos θ)

Sol: The parametric equations of the cycloid are

x = a (θ + sin θ), y = a (1 + cos θ)



One arc of the cycloid is shown in the figure. The cycloid is symmetrical bout the line through the point where θ = 0 and perpencticular to x-axis.

Now $\frac{dx}{d\theta}$ = a (1 + cos θ)

= a.2 $\cos^2 \frac{\theta}{2}$

= 2a $\cos^2 \frac{\theta}{2}$

and $\frac{dy}{d\theta}$ = - a sin θ

= a. 2 sin $\frac{\theta}{2}$ cos $\frac{\theta}{2}$

= - 2a sin $\frac{\theta}{2}$ cos $\frac{\theta}{2}$

Length of one arc for the cycloid

$$= 2\int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2\cos^4\frac{\theta}{2} + 4a^2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta$$

$$= 4a\int_0^{\pi} \sqrt{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\, d\theta$$

$= 4a \int_0^{\pi} \cos \frac{\theta}{2} \, d\theta$

$= 4a \left[\frac{\sin \frac{\theta}{2}}{\frac{1}{2}} \right]$

$= 8a \left[\sin \frac{\theta}{2} \right]_0^{\pi}$

$= 8a \left[\sin \frac{\pi}{2} - \sin 0 \right]$

= 8a [1 - 0]

= 8a

Example 6: Find the length of the arc of the curve

$x = e^{\theta} \sin \theta$, $y = e^{\theta} \cos \theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$

Sol: The parametric equations of the curve are

$x = e^{\theta} \sin \theta$, $y = e^{\theta} \cos \theta$

$\therefore \quad \frac{dx}{d\theta} = e^{\theta} \cos \theta + e^{\theta} \sin \theta$

$= e^{\theta} (\cos \theta + \sin \theta)$

and $\quad \frac{dy}{d\theta} = e^{\theta} (- \sin \theta) + e^{\theta} \cos \theta$

$= e^{\theta} (\cos \theta - \sin \theta)$

Now Required length of arc = $\int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} \, d\theta$

$= \int_0^{\pi/2} \sqrt{e^{2\theta}(\cos\theta + \sin\theta)^2 + e^{2\theta})\cos\theta - \sin\theta)^2} \, d\theta$

$= \int_0^{\pi/2} e^{\theta} \sqrt{2(\cos^2\theta + \sin^2\theta)} \, d\theta$

$$= \int_0^{\pi/2} e^{\theta}\sqrt{2}\, d\theta$$

$$= \sqrt{2}\int_0^{\pi/2} e^{\theta}\, d\theta$$

$$= \sqrt{2}\left[e^{\theta}\right]_0^{\pi/2}$$

$$= \sqrt{2}\left[e^{\pi/2} - e^{0}\right]$$

$$= \sqrt{2}\left[e^{\pi/2} - 1\right]$$

Example 7: Find the length of loop of the curve

$$x = t^2,\ y = t - \frac{1}{3}t^3$$

Sol: The equations of the curve are

$$x = t^2,\ y = t - \frac{1}{3}t^3 \qquad \text{......(1)}$$

$\therefore \qquad \dfrac{dx}{dt} = 2t,\ \dfrac{dy}{dt} = 1 - t^2$

Now for tracing the curve, we find the Cartesian equation of the curve.

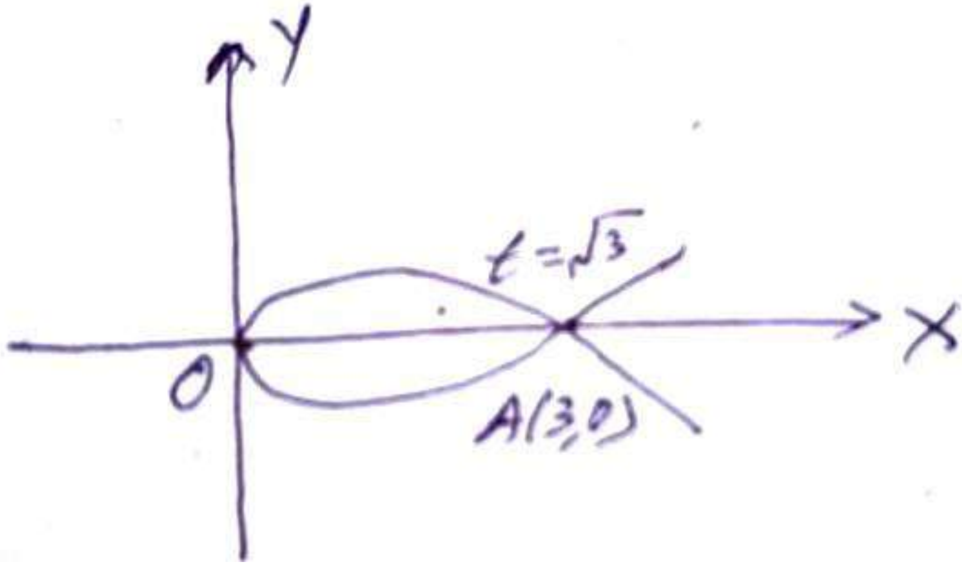
From (1), $y^2 = \dfrac{1}{9}t^6 - \dfrac{2}{3}t^4$

Or $\qquad y^2 = x + \dfrac{1}{9}x^3 - \dfrac{2}{3}x^2 \qquad [\because\ x = t^2]$

Or $\qquad 9y^2 = 9x + x^3 - 6x^2$

Or $\qquad 9y^2 = x(x^2 - 6x + 9)$

Or $\qquad 9y^2 = x(x - 3)^2$



A rough sketch of this curve is shown in the figure. Curve is symmetrical about x-axis and for upper half of the loop, x varies from 0 to 3 and therefore t varies from 0 to $\sqrt{3}$

$\therefore$ length of loop = $2\int_0^{\sqrt{3}} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$$2\int_0^{\sqrt{3}} \sqrt{4t^2+(1-t^2)^2}\,dt$$

$$= 2\int_0^{\sqrt{3}} \sqrt{(1+t^2)^2}\,dt$$

$$= 2\int_0^{\sqrt{3}} (1+t^2)\,dt$$

$$= 2\left[t+\frac{t^3}{3}\right]_0^{\sqrt{3}}$$

$$= 2\left[\left(\sqrt{3}+\frac{3\sqrt{3}}{3}\right)-(0+0)\right]$$

$$= 2\left[\sqrt{3}+\sqrt{3}\right]$$

$= 2\left(2\sqrt{3}\right)$

$= 4\sqrt{3}$

Self-Check Exercise

Q. 1 The position of a particle at time t is given x = $\frac{1}{3}\left(2t+3\right)^{3/2}$, y = $\frac{1}{2}$ $t^2 + t$

Find the distance moved by it between t = 0 to t = 3

Q. 2 Find the length of the curve $x = t^3$, $y = 2t^2$ on [0, 1]

Q. 3 Find the length of one arc of the cycloid

$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

Q.4 Find the length of the arc of the curve

$x = e^{\theta}\left(\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}\right)$

$y = e^{\theta}\left(\cos\frac{\theta}{2} - 2\sin\frac{\theta}{2}\right)$

measured from $\theta = 0$ to $\theta = \pi$

12.4 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Derived the arc formula i.e. to calculate the length of the curve for parametric equations.
2. Solved questions related to finding the length of the curve for parametric equations.

12.5 Glossary

1. The concept of the length of a curve in the plane refers to the measurement of the distance along the curve between the two points.
2. If c is a curve defined by parametric equations $x = f(t)$, $y = g(t)$, $\propto \leq t \leq \beta$ and $f'(t)$ 0 $\forall$ t $\in [\propto, \beta]$, then length L of curve c is given by

   $$L = \int_{\propto}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

12.6 Answers To Self-Check Exercise

Ans. 1 $\frac{21}{2}$

Ans. 2 $\frac{61}{27}$

Ans. 3 8 a

Ans. 4 $\frac{5}{2}\,(e^{\pi} - 1)$

12.7 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.

2. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.

12.8 Terminal Questions

1. The position of a particle at any time t is given by

$$x = \frac{1}{3}(2t + 3),\ y = \frac{1}{2}t^2 + 1$$

Find the distance moved by it between t = 1 to t = 3

2. Find the whose length of the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

3. Show that the length of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ measured from (0, a) to the point (x, y) is given by $\frac{3}{2}\,3\sqrt{ax^2}$

4. Prove that the length of the arc of the curve

$x = a \sin 2\theta\ (1 + \cos 2\theta)$,

$y = a \cos 2\theta\ (1 - \cos 2\theta)$

measured from (0, 0) to (x, y) is equal to $\frac{4}{3}$ a sin 3θ

5. Show that the length (s) of an arc of the curve

$x \sin\theta + y \cos\theta = f'(\theta)$

$x \cos\theta - y \sin\theta = f''(\theta)$

is given by $s = f(\theta) + f''(\theta) + c$,

where c is a constant of integration.

Unit - 13

Area of Curves in the Plane

Structure

13.1 Introduction

Quadrature, also known as finding the area of curves in the plane, is a mathematical concept that deals with determining the area enclosed by a curve or a set of curves. It is a fundamental topic in integral calculus and has wide-ranging application in various fields, including physics, engineering and geometry. In modern mathematics, the area under a curve is calculated using definite integral. Given a function $f(x)$ defined ever a certain interval [a, b], the area under the curve and above the x-axis can be found by evaluating the definite integrals of $f(x)$ from a to b. This integral represents the signed area, considering the portions above and below the x-axis separately. The process of finding the area under a curve can be divided into several steps the identity the curve, determine the interval [a, b] over which you want to find the area, formulating the definite integral that represents the area under the curve and then using integration techniques, such as antiderivatives, substitution or integration by parts, to evaluate the definite integral. This unit, we will try to find the area under a curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$.

13.2 Learning Objectives

After studying this unit, you should be able to:-

- define quadrature.
- find the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x=b$.

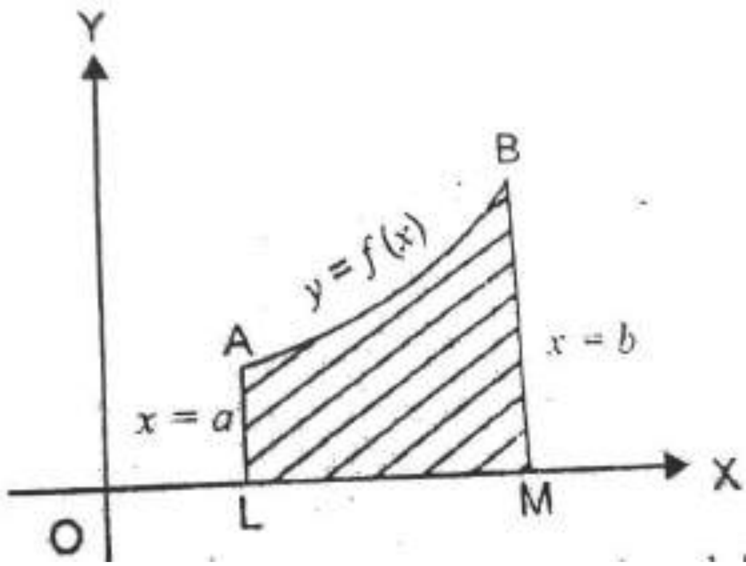
- and the area bounded by the curves y = f9x), y = g(x) and the ordinates x = a and x = b.
- solve questions related to finding the area as stated above.
- derive the area formulae for parametric equation and solve questions related to it.

13.3 Some Relevant Theorem

We know that if

f be a non=negative continuous function defined on [a, b]. Then the area bounded by the curve y = f(x), the x-axis and the ordinates

x = a, x = b is $\int_a^b f(x)dx = \int_a^b y\ dx$



Theorem 1 : Let f be a non-positive continuous function defined on [a, b]. Then the area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is $\int_a^b |f(x)|\, dx$

Proof : Since f(x) is a non-positive continuous function on [a, b].

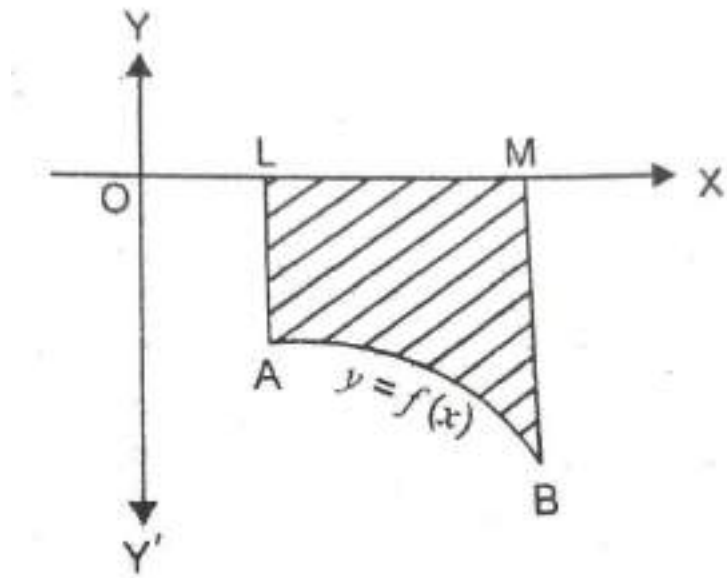
$\therefore$ $f(x) \leq 0\ \forall\ x \in [a, b]$

$\Rightarrow$ $-f(x) \geq 0\ \forall\ x \in [a, b]$

$\Rightarrow$ $-f$(x) is a non-negative continuous function defined on [a, b].

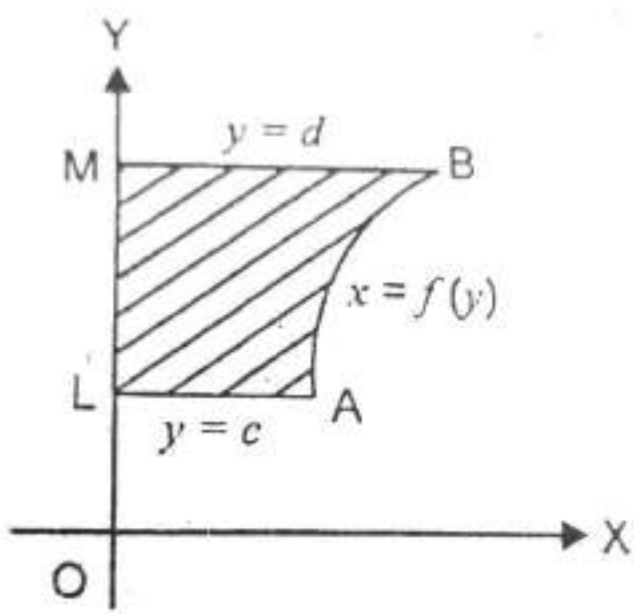
$\therefore$ area of region bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is

$\int_a^b \{ f(x) |\} dx$ or $\int_a^b | f(x) | dx$ [$\because$ $|f(x)| = -f(x)$ if $f(x) \leq 0$]



Note 1. Combining the above two results, we get the result :

If y = f (x) be a single-valued continuous function for a < x < b, then the area of the region bounded by x = a, x = b, the x-axis and the curve y = f(x) is given by $\int_a^b | f(x) | dx$



Note 2. If x = f (y) be a single-valued continuous function for c < y < d, then the area of the region bounded by y = c, y = d, the y-axis and the curve x = f (y) is given $\int_c^d | f(y) | \, dy$

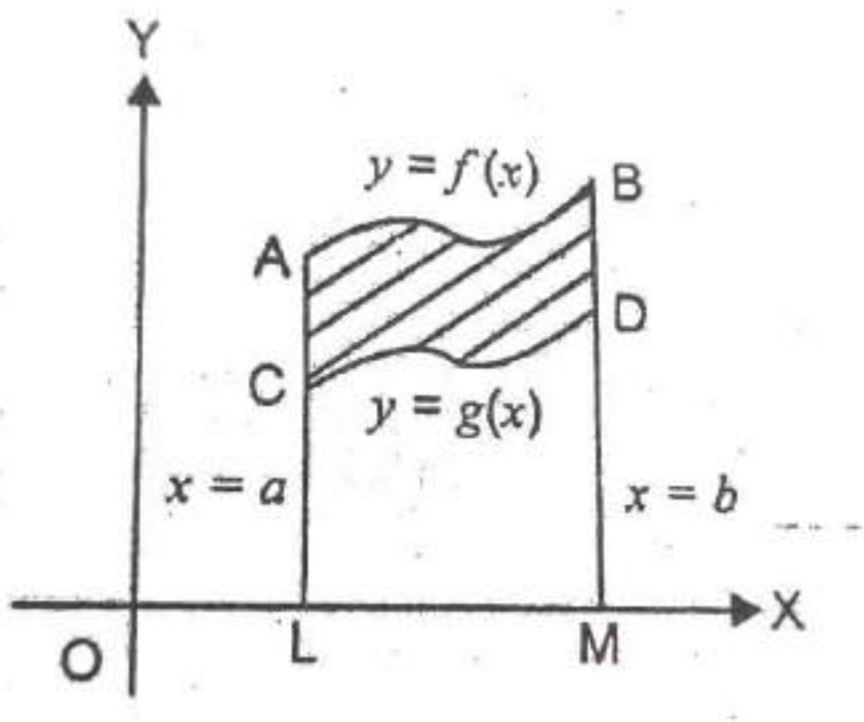
Theorem 2 : If f and g be two continuous functions defined on [a, b] such that g (x) $\leq f$ (x) on [a, b], then the area bounded by the curve y = f (x), y = g (x) and the ordinates x = a, x = b is

$$\int_a^b \{ f(x) - g(x) \} \, dx$$

Proof : Three cases arise :

Case I. Both f(x) and g (x) are non-negative on [a, b] i.e.,

f (x) $\geq$ 0, g (x) $\geq$ 0 $\forall$ x $\in$ [a, b]



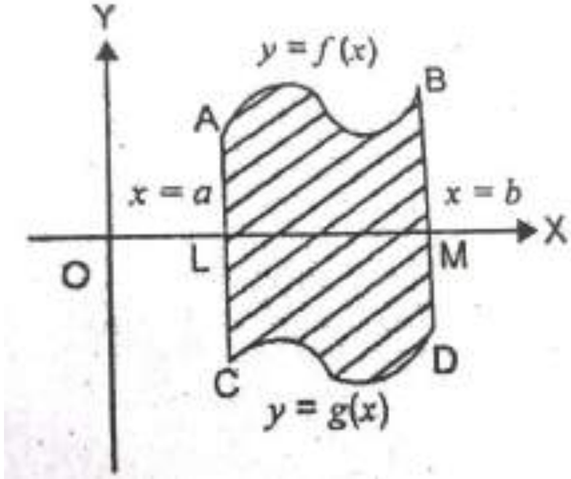
Required area = ACDB

= area ALMB - area CLMD

$$\int_a^b f(x) \, dx - \int_a^b g(x) \, dx$$

$$\int_a^b f(x) - g(x) \, dx$$

Case II : f (x) $\geq$ 0 and g (x) $\leq$ 0 $\forall \in$ [a, b]



Required area = area ACDB = area ALMB + area LCDM

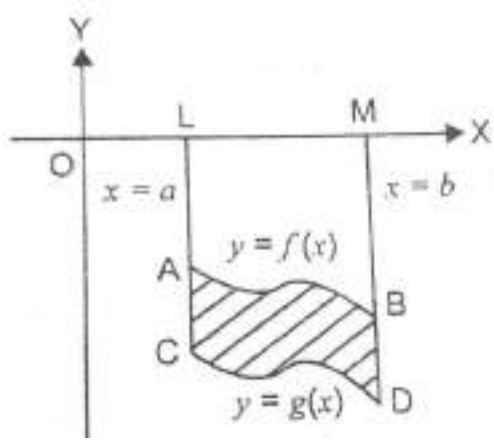
$$\int_a^b f(x)\, dx + \int_a^b |g(x)|\, dx$$

$$\int_a^b f(x)\, dx + \int_a^b \{-g(x)\}\, dx$$

$$= \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b \{f(x) - g(x)\}\, dx$$

Case III. f (x) $\leq$ 0, g (x) $\leq$ 0 $\forall$ x $\in$ [a, b]



Required area = area ACDB = area LCDM - area LABM

$$= \int_a^b |g(x)|\, dx - \int_a^b |f(x)|\, dx$$

$$= \int_a^b \{-g(x)\}\, dx - \int_a^b \{-f(x)\}\, dx$$

$$= -\int_a^b g(x)\, dx + \int_a^b f(x)\, dx$$

$$= \int_a^b \{f(x) - g(x)\}\, dx$$

Note. For finding the area, first draw the graph of curve.

Let us consider the following examples to the idea:

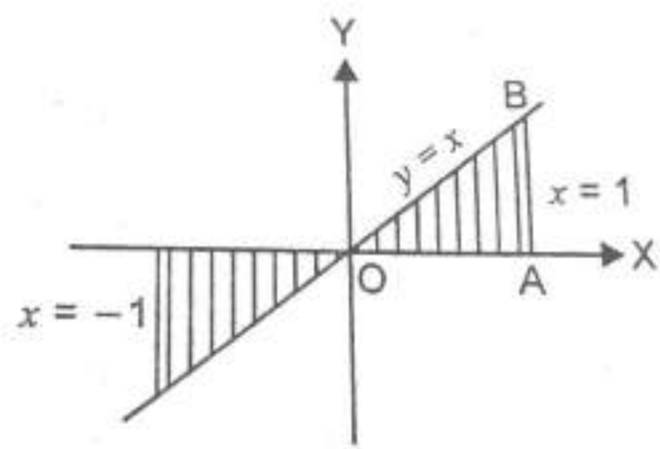
Example 1: Find the area of the region enclosed by the curve $y = x^2$ and the lines $x = 0$, $y = 0$ and $x = 2$

Sol: The equation of the curve is $y = x^2$

$\therefore$ required area $= \int_0^2 y\, dx = \int_0^2 x^2\, dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$ sq. units.

Example 2: Find the area bounded by the lines $y = x$, $x = -1$ and $x = 1$.

Sol: The equations of the lines are $y = x$, $x = -1$, $x = 1$



Required area = 2 (area OAB)

$$= 2\int_0^1 y\, dx$$

$$= 2\int_0^1 x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2\left[\frac{1}{2} - 0\right] = 1$$

Example 3: Find the area of the curve $y^2 = -4y + 2x = 0$ and y-axis

Sol: The equation of the curve is $y^2 = -4y + 2x = 0$ .....(1)

Or $y^2 - 4y + 4 = -2x + 4$

Or $(y - 2)^2 = -2(x - 2)$

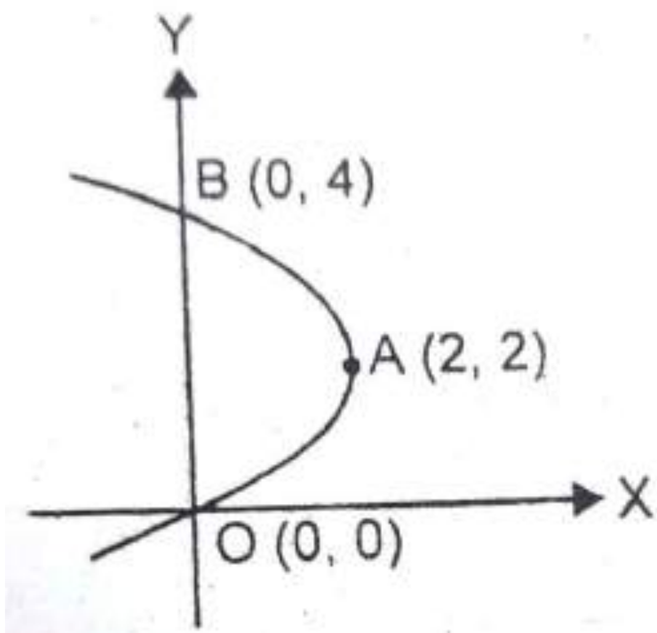
which is a left-handed parabola with vertex at (2, 2)

Putting x = 0 in (1), we get

$y^2 - 4y = 0$ or $y(y - 4) = 0$

∴ y = 0, 4

∴ curve meets y-axis in O(0, 0), B(0, 4)



Required area = 2(area OAM) = $2\int_0^2 \frac{1}{2}(4y - y^2)\,dy$

$$= \int_0^2 (4y - y^2)\, dy = \left[2y^2 - \frac{y^3}{3}\right]_0^2 = \left[8 - \frac{8}{3}\right] - (0 - 0) = \frac{16}{3}$$

Example 4: Find the area above the x-axis included between the curves

$y^2 = 2ax - x^2$ and $y^2 = ax$

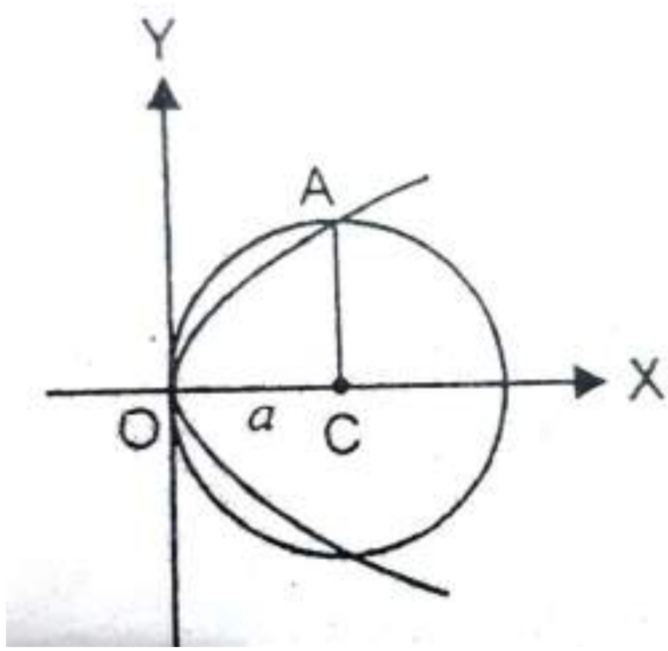
Sol: The equations of the curves are

$y^2 = 2ax - x^2$ or $x^2 + y^2 - 2ax = 0$ ....(1)

which is a circle whose centre is (a, 0) and redius a;

and $y^2 = ax$

which is a parabola. .....(2)



To find the points of intersection, substituting the value of y^2 from (2) in (1),

$x^2 + ax - 2ax = 0,$

Or $x^2 - ax = 0,$

Or $x(x - a) = 0$

$\therefore$ x = 0, a which are the limits of integration.

$\therefore$ required area = $\int_0^a \{f(x) - g(x)\}\, dx$

$$= \int_0^a \left(\sqrt{2ax - x^2} - \sqrt{a}\sqrt{x}\right) dx \qquad .....(3)$$

Now $\int_0^a \sqrt{-\left(2ax - x^2\right)}\, dx$

$$= \int_0^a \sqrt{-\left(x^2 - 2ax + a^2\right) + a^2}\, dx$$

$$= \int_0^a \sqrt{a^2 - \left(x - a^2\right)}$$

$$= \left[\frac{(x-a)\sqrt{a^2 - (x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a}\right]_0^a$$

$$= \frac{1}{2}\left[0.a + a^2(0) - \left\{(-a)0 + a^2\left(-\frac{\pi}{2}\right)\right\}\right] = \frac{\pi a^2}{4}$$

Also $\int_0^a \sqrt{a}\sqrt{x}\, dx$

$$= \sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= \frac{2}{3}\sqrt{a}\left(a^{\frac{3}{2}} - 0\right) = \frac{2}{3} a^2$$

$\therefore$ from (3), required area = $\frac{\pi a^2}{4} - \frac{2}{3}a^2 = a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$

Note: Take a = 1

$\therefore$ area enclosed between two parabolas $y^2 = 2x - x^2$ and $y^2 = x$ is $\left(\frac{\pi}{4} - \frac{2}{3}\right)$

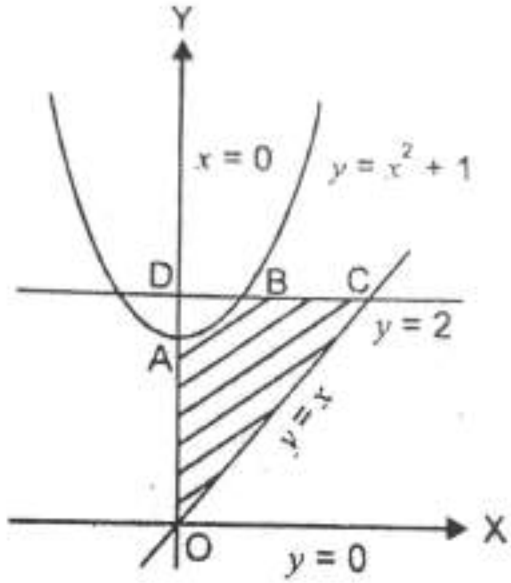
Example 5: Find the area of the region bounded by the curve:

$y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$

Sol: The equations of the curves are

$y = x^2 + 1$, $y = x$, $x = 0$ and $y = 2$

Solving $y = x^2 + 1$, and $x = 0$ we get, A as (0 ,1)



Required area - shaded area

= Area of ΔODC - area of region ADBA ....(1)

Now area of $\Delta ODC = \int_0^2 x\,dy$

(Area bounded by $y = x$, $y = 0$, $y = 2$)

$$= \int_0^2 y\,dx$$

$$= \left[\frac{y^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left[y^2\right]_0^2$$

$$= \frac{1}{2}[4 - 0) = 2$$

Area of region ADBA $= \int_1^2 x\,dy$ (Area bounded by $x^2 + 1 = y$, $y = 1$, $y = 2$)

$= \int_1^2 \sqrt{y-1}\, dy$

$= \int_1^2 (y-1)^{\frac{1}{2}} dy$

[$\because$ $x^2 + 1 = y \Rightarrow x = \sqrt{y-1}$ as x > 0 for the region]

$= \left[\dfrac{(y-1)^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2$

$= \dfrac{2}{3} \left[(y-1)^{\frac{3}{2}} \right]_1^2$

$= \dfrac{2}{3} [1 - 0] = \dfrac{2}{3}$

$\therefore$ from (1), required area = $2 - \dfrac{2}{3} = \dfrac{4}{3}$

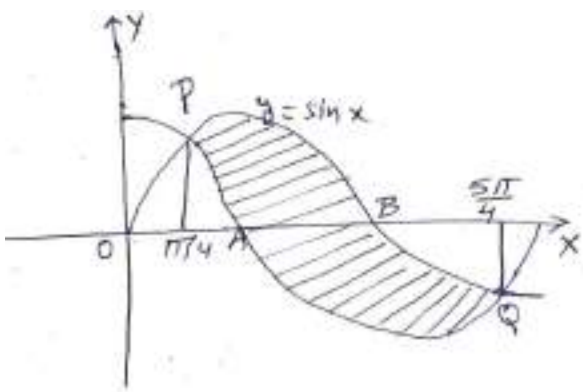
Example 6: Find the area of the region bounded by the curves

y = sin x, y = cos x, x = $\dfrac{\pi}{4}$ and x = $\dfrac{5\pi}{4}$

Sol: The equations of the curves are

y = sin x .....(1)

y = cos x .....(2)



Required area is shaded in the figure

Required area = $\int_{\pi/4}^{5\pi/4}$ (sin x - cos x) dx

$$\left[\because of \int_a^b \{f(x) - g(x)\} dx\right]$$

$$= [-\cos x - \sin x]_{\pi/2}^{5\pi/4}$$

$$= \left(-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{4}{\sqrt{2}}$$

$$\left[\text{As } \cos\frac{5\pi}{4} = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}} \; and \sin\frac{5\pi}{4} = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

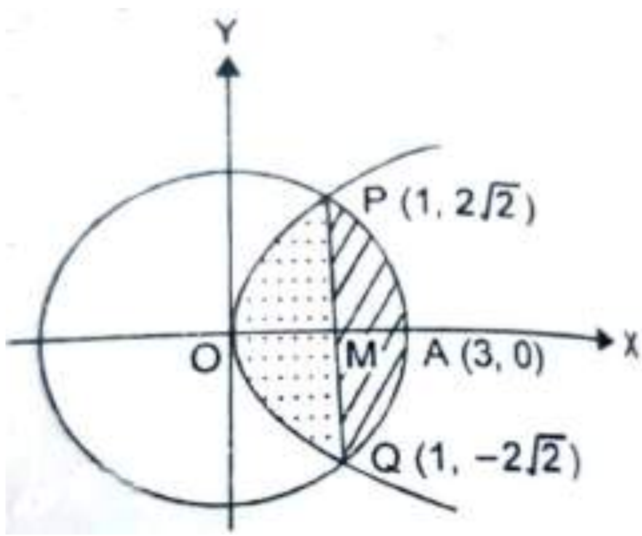
$$= 2\sqrt{2}$$

Example 7: Find the area of the smaller region enclosed by the curves $y^2 = 8x$ and $x^2 + y^2 = 9$

Sol: The equations of curve

$y^2 = 8x$ .....(1)

and $x^2 + y^2 = a$ .....(2)



From (1) and (2), we have

$x^2 + 8x = 9$

$\therefore \quad x^2 + 8x - 9 = 0$

$\Rightarrow \quad (x + 9)(x - 1) = 0$

$\Rightarrow \quad x = 1, -9$

$\Rightarrow \quad x = 1$

[$\because$ x = -9 $\Rightarrow y^2 = -72$, which gives, imaginary values of y]

$\therefore \quad y^2 = 8$

$\Rightarrow \quad y = \pm\ 2\sqrt{2}$

$\therefore$ curves (1) and (2) intersect in points P $\left(1, 2\sqrt{2}\right)$ and Q $\left(1, -2\sqrt{2}\right)$

Required Area = 2 (area OAPO)

= 2 (area OMPO + area MAPM)

$$= 2\left[\int_0^1 2\sqrt{2}\sqrt{x}dx + 2\int_1^3 \sqrt{9-x^2}dx\right]$$

$$\left[\because \text{in first quadrant, the equations of parabola and circle are } y = 2\sqrt{2}\sqrt{x}, y = \sqrt{9-x^2}\right]$$

$$= 4\sqrt{2}\int_0^1 x^{\frac{1}{2}} dx + 2\int_1^3 \sqrt{(3)^2 - x^2}\ dx$$

$$= 4\sqrt{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 + 2\left[\frac{x\sqrt{9-x^2}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_1^3$$

$$= \frac{8\sqrt{2}}{3}\left[x^{\frac{3}{2}}\right]_0^1 + 2\left[\left(0 + \frac{9}{2}\sin^{-1}1\right) - \left\{\frac{\sqrt{8}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{8\sqrt{2}}{3}[1 - 0] + 2\left[\frac{9}{2}\times\frac{\pi}{2} - \frac{2\sqrt{2}}{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{8\sqrt{2}}{3} + \frac{9\pi}{2} - \frac{2\sqrt{2}}{2} - 9\sin^{-1}\left(\frac{1}{3}\right)$$

$= \frac{2\sqrt{2}}{3} + \frac{9\pi}{2} - 9 \sin^{-1}\left(\frac{1}{3}\right)$

Example 8: Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$

Sol: The equations of two circles are

$x^2 + y^2 = 1$

and $(x - 1)^2 + y^2 = 1$

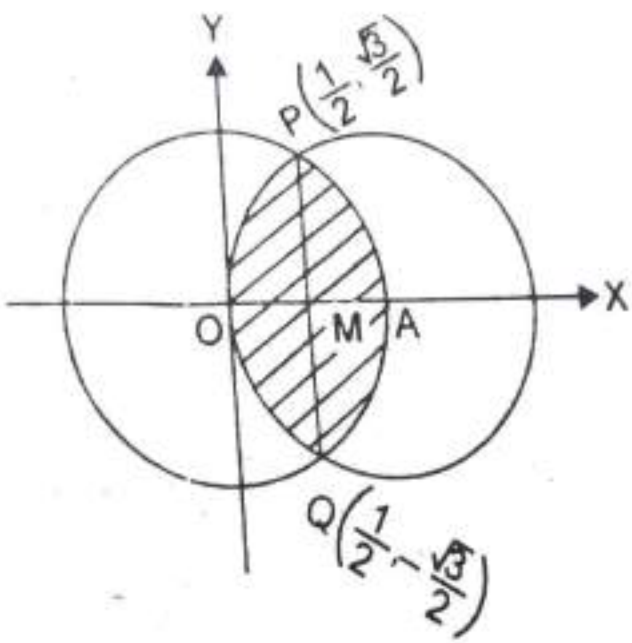
Centre of circle (1) is O (0, 0) and redius OA = 1

Centre of circle (2) is A (1, 0) and redius AO = 1

Subtracting (1) from (2), we get.

$- 2x + 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}$



Putting $x = \frac{1}{2}$ in (1), we get,

$\frac{1}{4} + y^2 = 1 \quad \Rightarrow \quad y^2 = \frac{3}{4}$

$\Rightarrow$ y = + $\frac{\sqrt{3}}{2}$

$\therefore$ points of intersection of circles (1) and (2) are P $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and Q $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

Required area = Area of region OQAP = 2 (area of region OMAP)

= (area of region OMPO + area of region MAPM)

$$= 2\left[\int_0^{\frac{1}{2}}\sqrt{1-(x-1)^2}\,dx+\int_{\frac{1}{2}}^{1}\sqrt{1-x^2}\,dx\right] \qquad \left[\because OM=\frac{1}{2}(OA)=\frac{1}{2}\right]$$

$$= 2\left[\frac{(x-1)\sqrt{1-(x-1)^2}}{2}+\frac{1}{2}\sin^{-1}(x-1)\right]_0^{\frac{1}{2}} + 2\left[\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right]_{\frac{1}{2}}^{1}$$

$$= 2\left[\left\{\frac{\left(\frac{1}{2}-1\right)\sqrt{1-\left(\frac{1}{2}-1\right)^2}}{2}+\frac{1}{2}\sin^{-1}\left(\frac{1}{2}-1\right)\right\}-\left\{\frac{(0-1)\sqrt{1-(0-1)^2}}{2}+\frac{1}{2}\sin^{-1}(0-1)\right\}\right] + 2$$

$$\left[\left(0+\frac{1}{2}\sin^{-1}1\right)-\frac{\frac{1}{2}\sqrt{1-\frac{1}{4}}}{2}+\frac{1}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= 2\left[\frac{-\frac{1}{2}\times\frac{\sqrt{3}}{2}}{2}+\frac{1}{2}\sin^{-1}\left(-\frac{1}{2}\right)-0-\frac{1}{2}\sin^{-1}(-1)\right] + 2\left[\frac{1}{2}\sin^{-1}1-\frac{\frac{1}{2}\times\frac{\sqrt{3}}{2}}{2}-\frac{1}{2}\sin^{-1}\frac{1}{2}\right]$$

$$= -\frac{\sqrt{3}}{4} + \left(\frac{-\pi}{6}\right) - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\pi}{6}$$

$$= \left(-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\pi}{6}\right) - 2\times\frac{\sqrt{3}}{4} = \left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)\text{ sq. units.}$$

Self-Check Exercise-1

Q.1 Find the area of the region bounded by $y = x^2 - x + 2$, x-axis and the lines $x = 0$, $x = 3$

Q.2 Find the area bounded by the parabola

$y = 2x - x^2$ and the x-axis.

Q.3 Find the area between the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = 2\pi$.

Q.4 compute the area bounded by the curves $y = \sqrt{x}$ and $y = x^2$

Q.5 Find the area common to the circle x2 + y2 = 4 and the ellipse $x^2 + 4y^2 = 9$

13.4 Area Formulae For Parametric Equations

The area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points where $t = a$, $t = b$ is $\int_a^b y \frac{dx}{dt} dt$

Proof: The parametric equations of the curve are

$x = f(t)$, $y = \phi(t)$

Let $x = \propto$ when $t = a$ and $x = \beta$ when $t = b$

$\therefore$ Required Area $= \int_{\propto}^{\beta} y\,dx$

$$= \int_a^b \frac{dx}{dt} dt$$

Note: Similarly the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the y-axis and the abscissas at the points where $t = 0$, $t = b$ is $\int_a^b \frac{dy}{dt} dt$

Let us consider the following examples to clear the idea:-

Example 9: Find the area of the curve

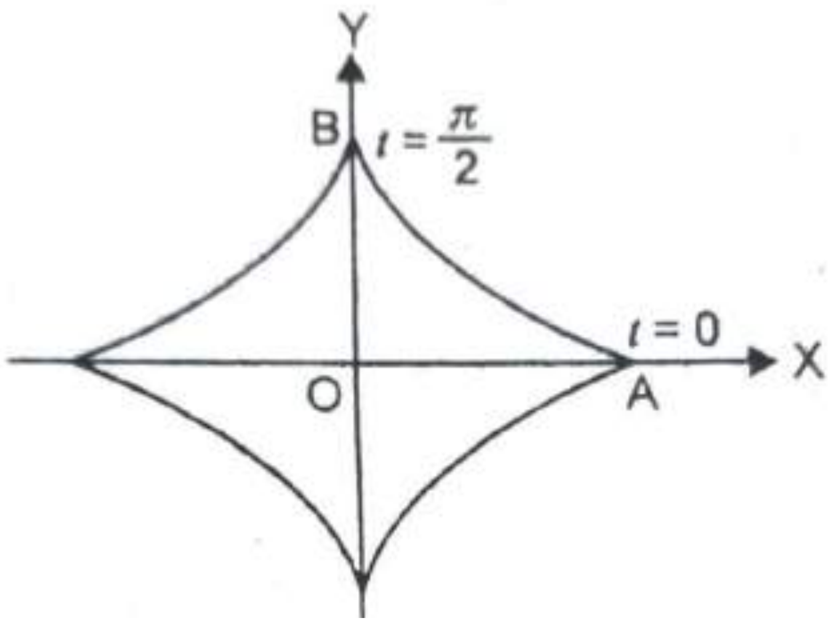
$x = a \cos^3 t$,

$y = b \sin^3 t$

Sol: The parametric equations of the curve are

$$\therefore \quad \left.\begin{aligned} & x = a\cos^3 t,\ y = b\sin^3 t \\ & \frac{dx}{dt} = -3a\cos^2 t\sin t \end{aligned}\right\} \qquad \text{....(1)}$$

The curve is symmetrical about both area, and in the first quadrant, t varies from 0 to $\frac{\pi}{2}$.



$\therefore$ area of the curve = $4\int_0^{\pi/2} y \frac{dx}{dt}\, dt$

$= 4\int_0^{\pi/2} b\sin^3 t\,[-3a\cos^2 t\sin t]\, dt$

$= -12ab\int_0^{\pi/2} \sin^4 t\cos^2 t\, dt$

$= -12ab\ \frac{3.1.1}{6.4.2} . \frac{\pi}{2} = -\frac{3}{8}\pi ab$

$= \frac{3}{8}\pi ab$ (in magnitude)

Note: If we apply the formula $4\int_{0}^{\pi/2}\left(x\frac{dy}{dt}\right)$dt, we get positive area.

Example 10: Find the area included between one of the cycloid

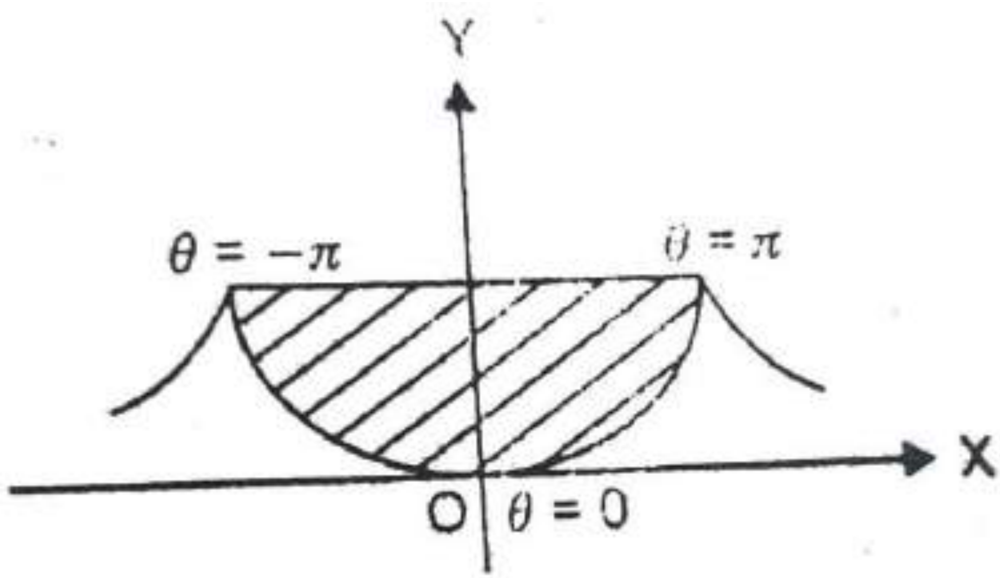
$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

Sol: The parametric equations of the cycloid are

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = a\sin\theta$$

A rough sketch of the sketch of the curve is given in the figure.



Required arc = $2\int_{0}^{\pi}\left(x\frac{dy}{d\theta}\right)d\theta$

$$= 2\int_{0}^{\pi} a(\theta + \sin\theta)\, a\sin\theta d\theta$$

$$= 2a^2\int_{0}^{\pi} (\theta\sin\theta + \sin^2\theta)\, d\theta$$

$$= 2a^2\int_{0}^{\pi} \theta\sin\theta\, d\theta + a^2\, 2a^2\int_{0}^{\pi} (2\sin^2\theta)\, d\theta$$

$$= 2a^2 \left\{\theta\left[-\cos\theta\right]_0^\pi \int_0^\pi 1.(-\cos\theta)d\theta\right\} + a^2 \int_0^\pi (1 - \cos 2\theta)\, d\theta$$

$$= 2a^2 \left\{(-\pi\cos\pi + 0) + \int_0^\pi \cos\theta\, d\theta\right\} + a^2 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^\pi$$

$$= 2a^2 \left\{\pi + \left[\sin\theta\right]_0^\pi\right\} + a^2\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 + \frac{1}{2}\sin 0\right)\right]$$

$$= 2a^2 \{\pi + \sin\pi - \sin 0\} + a^2 \left[(\pi - 0) - (0 + 0)\right]$$

$$= 2a^2 [\pi + 0 - 0] + a^2 [\pi]$$

$$= 2\pi a^2 + \pi a^2 = 3\pi a^2$$

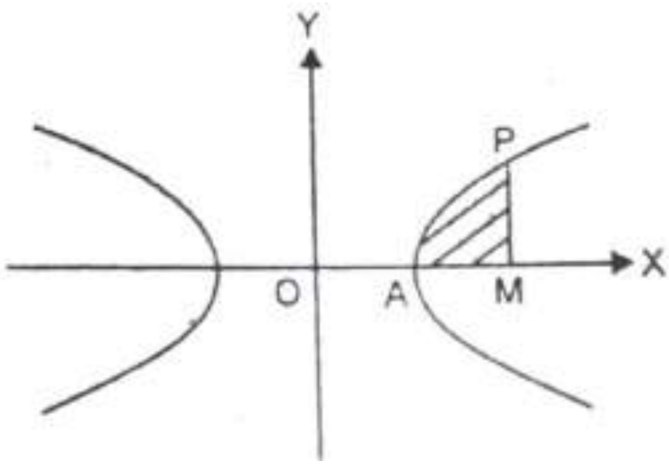
Example 11: Prove that the area bounded by the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the x-axis and the ordinate from P(a cosh θ, b sinh θ) is $\frac{1}{4}$ ab (sinh 2θ - 2θ)

Sol: The equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

its parametric equations are

$x = a\cosh\theta$, $y = b\sinh\theta$

A rough sketch of the curve is given in the figure.



We require area AMP. At A, $\theta = 0$.

Now $\frac{dx}{d\theta} = a\sinh\theta$

Required area = $\int_0^\theta \left(y \frac{dx}{d\theta} \right) d\theta$

$= \int_0^\theta$ b sinh θ, a sinh θ dθ

$= ab \int_0^\theta \sinh^2 \theta \, d\theta$

$= \frac{1}{2} ab \int_0^\theta 2 \sinh^2 \theta \, d\theta$

$= \frac{1}{2} ab \int_0^\theta (\cosh 2\theta - 1) \, d\theta$

$= \frac{ab}{2} \left[\frac{\sinh 2\theta}{2} - \theta \right]_0^\theta$

$= \frac{ab}{2} \left[\left(\frac{1}{2} \sinh 2\theta - \theta \right) - \left(\frac{1}{2} \sinh 0 - 0 \right) \right]$

$= \frac{ab}{2} \left[\left(\frac{1}{2} \sinh 2\theta - \theta \right) - (0 - 0) \right]$

Example 12: Prove that the area bounded by the cissoid

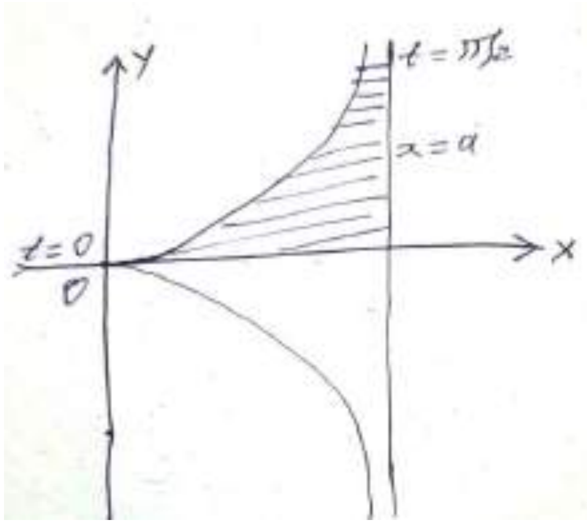
$x = a \sin^2 t$, $y = \frac{a \sin^3 t}{\cos t}$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$ and its asymptote is $\frac{3\pi a^2}{4}$

Sol: The parametric equations of the curve are

$x = a \sin^2 t$, $y = \frac{a \sin^3 t}{\cos t}$

where $-\frac{\pi}{2} < t < \frac{\pi}{2}$

$\therefore \quad \frac{dx}{dt} = 2a \sin t \cos t$



The curve is symmetrical about x-axis and $0 \leq t < \frac{\pi}{2}$

$\therefore$ Required Area = $2\int_{0}^{\pi/2} y\frac{dx}{dt}\,dt$

$$= 2\int_{0}^{\pi/2} \frac{a\sin^3 t}{\cos t}\, 2a \sin t \cos t\, dt$$

$$= 4a^2 \int_{0}^{\pi/2} \sin^4 t\, dt$$

$$= 4a^2 . \frac{3.1}{4.2} \times \frac{\pi}{2}$$

$$= \frac{3\pi a^2}{4}$$

Hence the result

Self-check Exercise-2

Q.1 Find the total area of the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1, \text{ where } a > 0, b > 0$$

Q.2 Find the area included between one arc of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ and the x-axis.

13.5 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Defined quadrature
2. Derived the formula for finding the area bounded by the curve y = f(x), the x-axis and the ordinates x = a and x = b.
3. Derived the formula for finding the area bounded by the curves y = f(x), y = g(x) and the ordinates x = a and x = b
4. Derived the area formulae for parametric equation
5. Solved questions related to finding the area as stated above

13.6 Glossary

1. Quadrature, also known as finding the area of curves in the plane, is a mathematical concept that deals with determining the area enclosed by a curve or a set of curves.
2. If f be a non-positive continuous function defined on [a, b] Then the area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is $\int_a^b |f(x)|\, dx$

13.7 Answers To Self-Check Exercise

Ans. 1 $10\frac{1}{2}$

Ans. 2 $\frac{4}{3}$

Ans. 3 4

Ans. 4 $\frac{1}{3}$

Ans. 5 $4\pi + 9 \sin^{-1}\left(\sqrt{\frac{7}{27}}\right) - 8 \sin^{-1}\left(\sqrt{\frac{7}{12}}\right)$

Self-Check Exercise-2

Ans.1 $\frac{3}{8}\pi ab$ (M magnitude)

Ans.2 $3\pi a^2$

13.8 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.

2. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.

13.9 Terminal Questions

1. Find the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the ordinates x = c, x = d and the x-axis. Deduce the area of the whole ellipse
2. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis
3. Find the area included between the curve

   $y^2 = (x + 1)^3$ and the y-axis.
4. Find the area of the region enclosed by the curves $y = x$, $y = x^2 + 1$, $x = 0$ and $x = 2$
5. prove that the area between the curve $y^2 (a + x) = (a - x)^3$ and its asymptote is three times the area of the circle whose radius is a.
6. Using the parametric equations, prove that the area of the ellipse is πab.
7. Find the area of the loop of the curve

   $x = a(1 - t^2)$, $y = at (1 - t^2)$

Unit - 14

Volumes Of Solids Of Revolution

Structure

14.1 Introduction

Solids of revolution are three dimensional objects that are formed by rotating a two-dimensional shape around on axis. The resulting solid has a symmetrical structure and is often characterized by its volume. The volume of a solid of revolution can be calculated using various mathematical techniques, such as the disk method, the shell method, and the method of cylindrical shells. To understand the concept of volumes of solids of revolution, let us consider a simple example. Imagine a function $f(x)$ defined on a closed interval [a, b]. If we rotate the graph of this function around the x-axis, it will generate a solid skape. The volume of this solid can be determined by integrating the cross-sectional areas of the infinitesimally thin slices obtained by slicing the solid perpendicular to the axis of rotation.

14.2 Learning Objectives

After studying this unit, you should be able to:-

- Define solid of revolution, surface of revolution and axis of revolution.
- Discuss the volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$
- Discuss the volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a$ and $x = b$.

- Discuss volume formulae for parametric equations.
- Solve questions related to finding the volume of the solid as stated above.

14.3 Some Relevant Theorem

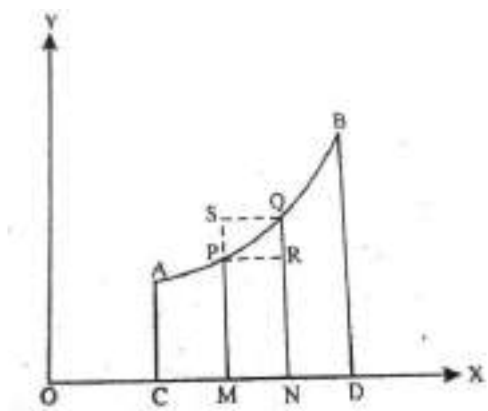
If a plane area is revolved about a fixed straight line in its own plane, then the body so generated by the plane area is called the solid of revolution and surface generated by the boundary of the plane is called the surface of revolution. The fixed line about which the plane area rotates is called the axis of revolution.

14.4 Volume of the Solid Generated By Revolution About The x-axis

The volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is $\int_a^b \pi y^2\, dx$.

Proof: Let AB be the curve $y = f(x)$ and CA, DB be the ordinate $x = a$, $x = b$ respectively.

Let P(x, y) be any point on the curve AB. From P, draw PM $\perp$ OY so that OM = x, MP = y.



Let V denote the volume of the solid generated by the revolution about x-axis of the area ACMP. Clearly V is a function x.

Let Q($x + \delta x$, $y + \delta y$) be a point on the curve in the neighborhood of P. Then the volume of the solid generated by the revolution about x-axis of the area ACNQ will be $V + \delta V$, so that the volume of the solid generated by the revolution about x-axis of the area PMNQ is δV.

Complete the rectangle PRQS.

Then the volume of the solid generated by the revolution about the x-axis of the area of PMNQ lies between the right circular cylinders generated by the rectangles PMNR and SMNQ i.e.,

δV lies between $\pi y^2 \delta x$ and $\pi (y + \delta y)^2 \delta x$

Or $\quad \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y + \delta y)^2$

Let Q → P so that $\delta x \to 0$, $\delta y \to 0$

$\therefore$ $\underset{\delta x \to 0}{Lt} \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\underset{\delta x \to 0}{Lt} \pi(y + \delta y)^2$

$\therefore$ $\dfrac{dV}{dx}$ lies between πy^2 and πy^2

$\therefore$ $\dfrac{dV}{dx} = \pi y^2$

$\therefore$ $\int_a^b \pi y^2\, dx = \int_a^b \dfrac{dV}{dx}$

$= [V]_a^b$

= (Volume V when x = b) - (Volume V when x = a)

= Volume generated by the area ACDB - 0

= Volume generated by the area ACDB

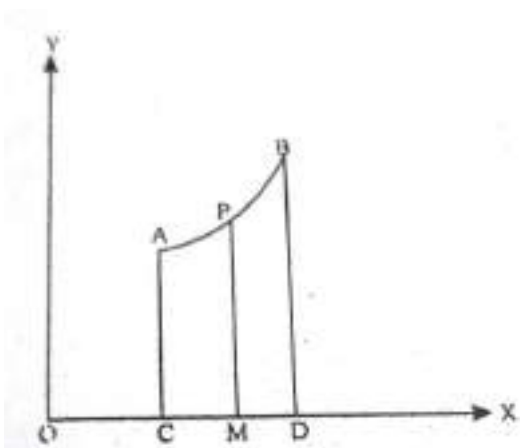
$\therefore$ Volume of the solid generated by the area ACDB about the x-axis is $\int_a^b \pi y^2\, dx$

Note 1: Revolution about y-axis

The volume of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis and abscissae y = a, y = b is $\int_a^b \pi x^2\, dy$

This result can be proved by the students.

Note 2: Revolution about any axis



1. Take any point P(x, y) on the curve.
2. Draw PM $\perp$ on the line CD about which the curve is to be revolved and find PM.
3. Find the distance OM of the foot of perpendicular from a fixed point O (say) on the line and take its differential.
4. Then V = $\int_{OC}^{OD} \pi\, (PM)^2\, d\,(OM)$ gives the required volume.

Note 3: Volume between two solids

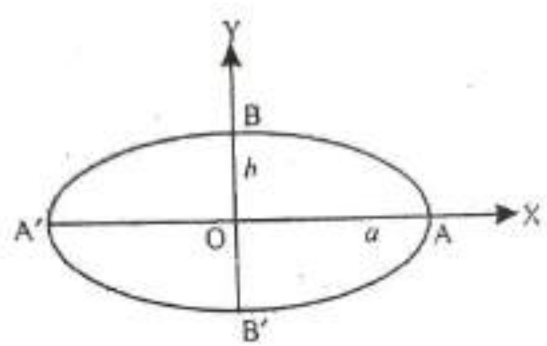
The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curves y = f(x), y = g(x), and the ordinates x = a, x = b is $\int_a^b \pi\left(y_1^2 - y_2^2\right)dx$, where y_1 is the y of the upper curve and y_2 that of the lower curve.

Let us improve our understanding of these results by looking at some following examples:-

Example 1: Find the volume of the solid generated by revolving the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ about the x-axis.

Sol: The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

$$\therefore \quad \frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \quad \Rightarrow \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2) \qquad ....(1)$$



Required volume = 2 $\times$ Volume generated by arc BA about x-axis.

$$= 2\int_0^a \pi y^2\, dx$$

$$= 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\, dx \qquad [\because of\ (1)]$$

$$= 2\pi \frac{b^2}{a^2} \int_0^a (a^2 - x^2)\, dx$$

$$= \frac{2\pi b^2}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a$$

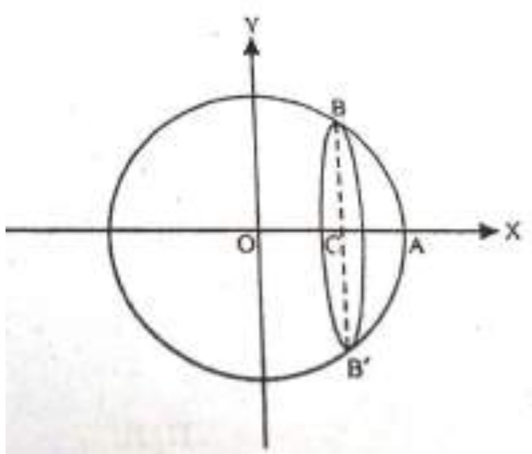
$$= \frac{2\pi b^2}{a^2} \left[\left(a^3 - \frac{x^3}{3} \right) - (0-0) \right]$$

$$= \frac{2\pi b^2}{a^2} \times \frac{2a^3}{3}$$

$$= \frac{4\pi ab^3}{3}$$

Example 2: A segment in cut off from a sphere of radius a by a plane at a distance $\frac{1}{2}$ a from the centre. Show that the volume of the segment is $\frac{5}{32}$ of the volume of the sphere.

Sol: The required volume of the segment is generated by revolving the area ABCA of the circle $x^2 + y^2 = a^2$ about the x-axis and for the arc BA, x varies from $\frac{a}{2}$ to a.



Required volume = $\int_{\frac{a}{2}}^{a} \pi y^2 dx$,

where $x^2 + y^2 = a^2$

$$= \pi \int_{\frac{a}{2}}^{a} (a^2 - x^2)\, dx$$

$$= \pi \left[a^2 x - \frac{1}{3} x^3 \right]_{\frac{a}{2}}^{a}$$

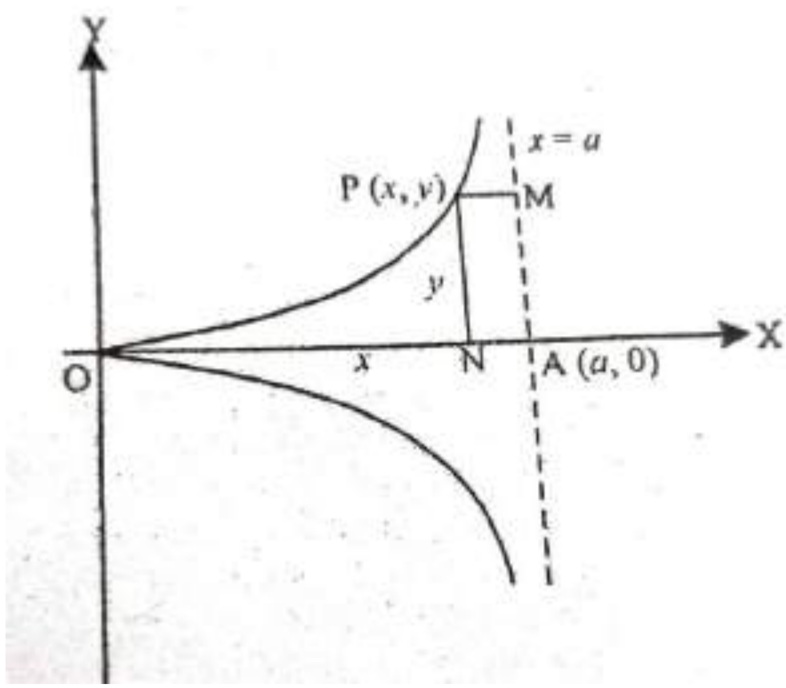
$$= \pi \left[\frac{5a^3}{24} \right]$$

$$= \frac{5}{32} \left[\frac{4}{3} \pi a^3 \right] = \frac{5}{32} \quad \text{[volume of the sphere of radius a]}$$

Example 3: Show that the volume of the solid generated by the revolution of the curve $(a - x) y^2 = a^2 x$ about its asymptote is $\frac{1}{2} \pi^2 a^3$.

Sol: The equation of curve is $y^2(a - x) = a^2 x$

The shape of the curve is as shown in the figure.



Its asymptote is $x = a$

Let (P(x, y) be any point on the curve.

From P, draw PN $\perp$ x-axis and PM $\perp$ asymptote.

Required volume = $2\int_0^\infty \pi\,(PM)^2.\,d(AM)$

$= 2\int_0^\infty \pi\,(a - x)^2\,dy$

$= 2\pi\int_0^\infty \left[a-\frac{ay^2}{a^2+y^2}\right]^2 dy \qquad \left[\because x=\frac{ay^2}{a^2+y^2}, from(1)\right]$

$= 2\pi\int_0^\infty \frac{a^6}{(a^2+y^2)}\,dy$

$= 2\pi a^6\int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\,d\theta}{(a^2+y^2\tan^2\theta)^2} \qquad \begin{array}{|lcl} Put\ y & = & a\tan\theta \\ \therefore\ dy & = & a\sec^2\theta d\theta \end{array}$

$= 2\pi a^6\int_0^{\frac{\pi}{2}} \frac{a\sec^2\theta\,d\theta}{a^4\sec^4\theta}$

$= 2\pi a^3\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$

$= 2\pi a^3.\ \frac{1}{2}\times\frac{\pi}{2}$

$= \frac{1}{2}\pi^2 a^3.$

Example 4: Find the volume of the solid generated by revolving the part of the parabola $y^2 = 4ax$ between vertex and the latus rectum about tangent at the vertex.

Sol: Given parabola is $y^2 = 4ax$

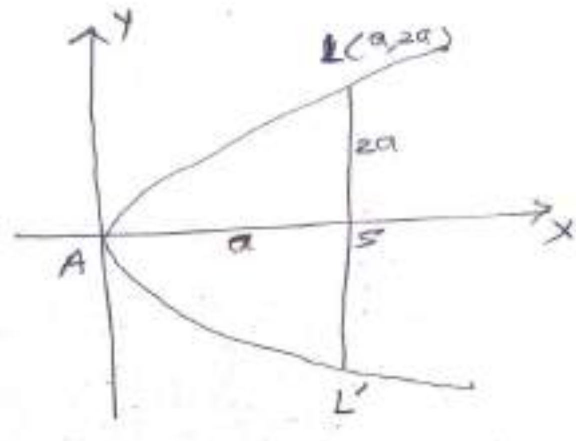
$\therefore \qquad x = \frac{y^2}{49} \qquad .....(1)$

Let LSL' be the latus rectum

$\therefore$ Required volume

$= 2 \times$ volume generated by the arc AL about y-axs

$= 2\int_0^{2a} \pi x^2\,dy$



$$= 2\pi \int_0^{2a} \frac{y^4}{16a^2}\, dy \qquad [\because \text{ of } (1)]$$

$$= \frac{\pi}{8a^2} \int_0^{2a} y^4\, dy$$

$$= \frac{\pi}{8a^2} \left[\frac{y^5}{5}\right]_0^{2a}$$

$$= \frac{\pi}{40a^2} \left[y^5\right]_0^{2a}$$

$$= \frac{\pi}{40a^2} \left[(2a)^5 - 0\right]$$

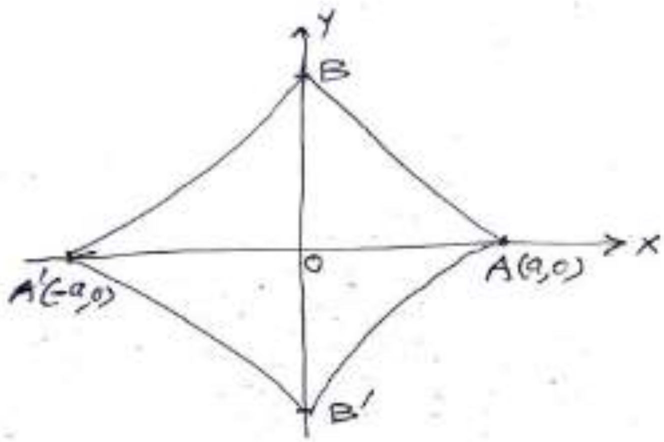
$$= \frac{\pi}{40a^2} \times 32a^5$$

$$= \frac{4}{5}\, \pi a^3$$

Example 5: Find the volume of the spindle shaped solid generated by revolving the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

Sol: The equation of the astriod is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

The required volume is generated by revolving the area ABA'OA about y-axis = 2 $\times$ volume generated by revolving the area ABOA about y-axis = $2\int_0^a \pi y^2\, dx$



$$= 2\pi \int_0^a \left[a^{\frac{2}{3}} - x^{\frac{2}{3}} \right]^3 dx$$

$$= 2\pi \int_0^{\frac{\pi}{2}} a^2 \cos^6 \theta \ \ 3a \sin^2\theta \cos \theta \, d\theta \qquad \text{[Put } x = a \sin^3 \theta]$$

$$= 6\pi a^3 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^7\theta \, d\theta$$

$$= 6\pi a^3 \frac{6.4.2.1}{9.7.5.3.1}$$

$$= \frac{32\pi a^3}{105}$$

Example 6: The area of the parabola $y^2 = 4ax$ lying between the vertex and the latus rectum is revolved about the x-axis. Find the volume generated.

Sol: The equation of parabola is $y^2 = 4ax$

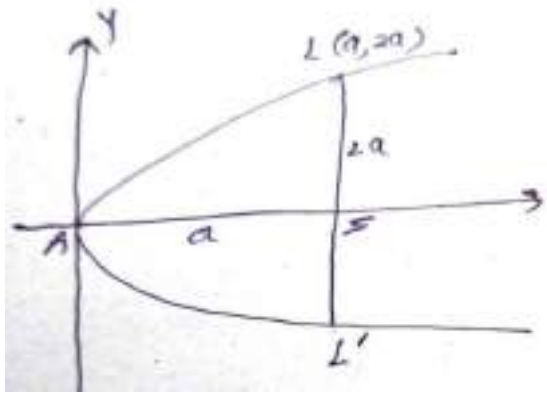
Let A be the vertex and LSL' be the latest rectum

$\therefore$ Required volume

= Volume generated by the area ASL about x-axis

$$= \int_0^a \pi y^2 \, dx$$

$$= \int_0^a \pi.4ax \, dx$$



$$= 4a\pi \int_0^a x\, dx$$

$$= 4a\pi \left[\frac{x^2}{2} \right]_a^a$$

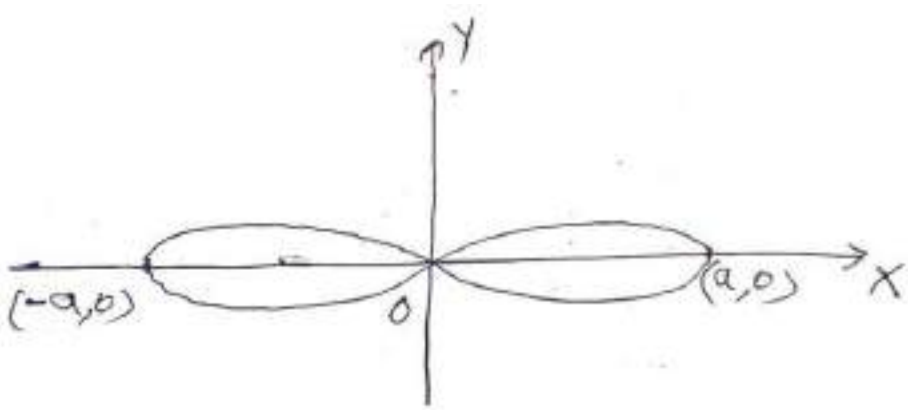
$$= 4a\pi \left[\frac{a^2}{2} - 0 \right]$$

$$= 2\pi a^3$$

Example 7: A loop of the curve $a^2y^2 = x^2 (a^2 - x^2)$ is rotated about the x-axis. Find the volume generated.

Sol: The equation of the curve is $a^2y^2 = x^2 (a^2 - x^2)$

A rough sketch of the curve is shown in the figure.



The curve is symmetrical about both the axes.

Required Volume = $\int_0^a \pi y^2\, dx$

$$= \frac{\pi}{a^2}\int_0^a \text{ x2 (a2 - x2) dx}$$

$$= \frac{\pi}{a^2}\int_0^a \text{ (a2x2 - x4) dx}$$

$$= \frac{\pi}{a^2}\left[\frac{a^2x^3}{3}-\frac{x^5}{5}\right]_0^a$$

$$= \frac{\pi}{a^2}\left[\left(\frac{a^2}{3}\times a^3-\frac{a^5}{5}\right)-(0-0)\right]$$

$$= \frac{\pi}{a^2}\left[\frac{a^5}{3}-\frac{a^5}{5}\right]$$

$$= \frac{\pi}{a^2}\times\frac{2a^5}{15}$$

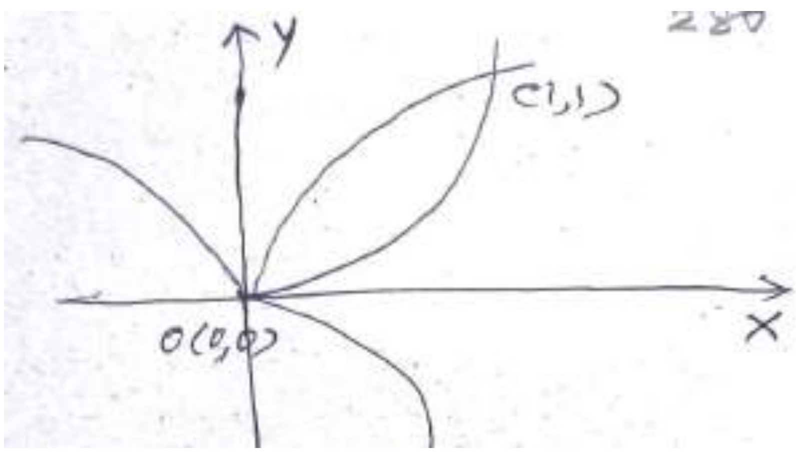
$$= \frac{2\pi a^3}{15}$$

Example 8: Find the volume generated by revolving about the x-axis, the area common to the two parabolas $y^2 = x$ and $x^2 = y$

Sol: The equations of the curves are

$y^2 = x$ ....(1)

and $x^2 = y$ .....(2)



The curve (1) is symmetrical about x-axis and curve (2) is symmetrical about y-axis.

From (1) and (2), we get

$x^4 = x$

Or $x^4 - x = 0$

Or $x(x^3 - 1) = 0$

$\Rightarrow$ $x = 0, 1$

$\therefore$ $y = 0, 1$

$\therefore$ two curves intersect in (0, 0) and (1, 1)

$$\text{Required Volume} = \pi\int_0^1 \left(y_1^2 - y_2^2\right)dx$$

$$= \pi\int_0^1 (x - x^4)\,dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\left(\frac{1}{2} - \frac{1}{5}\right) - (0 - 0)\right]$$

$$= \pi\left[\frac{5-2}{10}\right] = \frac{3\pi}{10}$$

Self-Check Exercise-1

Q.1 Find the volume generated by revolving the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ about its major axis.

Q.2 Find the volume generated by rotating about the y-axis the area bounded by the co-ordinate axes and the graph of the curve $y = \cos x$ from $x = 0$ to $x = \pi/2$.

Q.3 A loop of the curve $y^2 = x^2(1 - x^2)$ is rotated about the x-axis. Find the volume generated.

14.5 Volume Formulae for Parametric Equations

The volume of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the points where t = a, t = b is $\int_a^b \pi y^2 \frac{dx}{dt}\,dt$.

The parametric equations of the curve are x = f(t), y = ϕ(t).

Volume = $\int \pi y^2\, dx = \int \pi y^2 \frac{dx}{dt}\, dt$

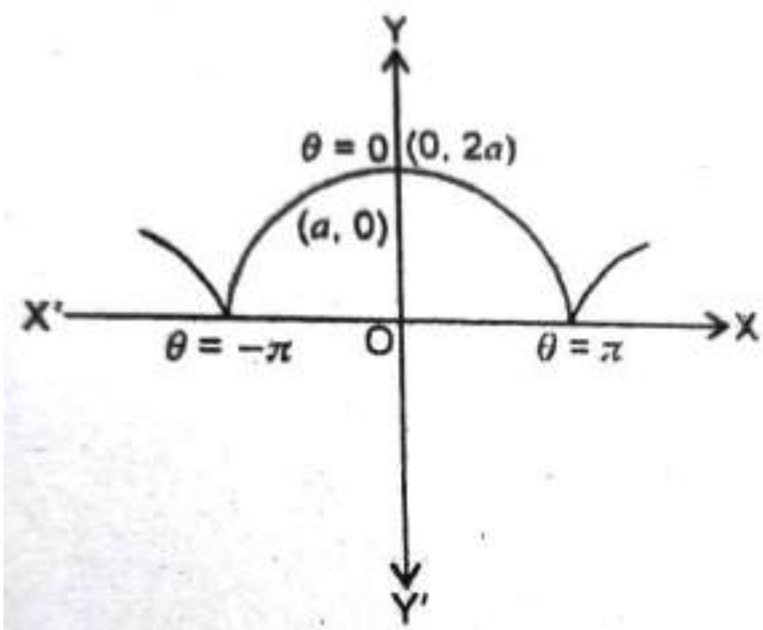
$\therefore$ required volume = $\int_a^b \pi y^2 \frac{dx}{dt}\, dt$

Similarly, the volume of the solid generated by the revolution, about the y-axis, of the area bounded by the curve x = f(t), y = ϕ(t), the y-axis and the abscissae at the points where t = a, t = b is = $\int_a^b \pi x^2 \frac{dy}{dt}\, dt$.

Let us improve our understanding of these results by looking at some following examples:-

Example 9: Find the volume of the solid obtained by revolving one arc of the cycloid x = a(θ + sin θ) and y = a (1 + cos θ) about x-axis.

Sol: The equations of the given curve are x = a(θ + sin θ), y = a(1 + cos θ)



Required volume = $\int_0^{2\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$

$= \pi \int a^2 (1 + \cos\theta)^2\, a\, (1 + \cos\theta)\, d\theta$

$= \pi a^3 \int_0^{2\pi} (1 + \cos\theta)^3\, d\theta$

$$= 2\pi a^3 \int_0^{2\pi} \left(2\cos^2\frac{\theta}{2}\right)^3 d\theta$$

$$= 16\pi a^3 \int_0^{\pi} \cos^6\frac{\theta}{2} d\theta$$

Put $\frac{\theta}{2} = t$, $\therefore d\theta = 2\,dt$

When $\theta = 0$, $t = 0$

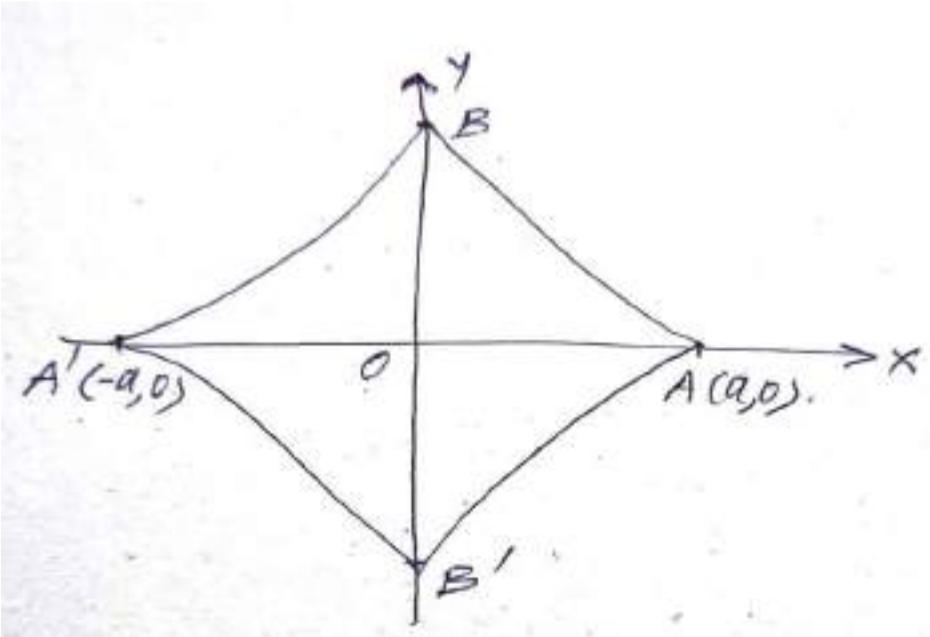
When $\theta = \pi$, $t = \frac{\pi}{2}$

$\therefore$ Required volume $= 16\pi a^3 \int_0^{\frac{\pi}{2}} \cos^6 t.\ 2dt$

$$= 32\,\pi a^3 \frac{5.3.1}{6.4.2}\,\frac{\pi}{2}$$

$$= 5\pi^2 a^3$$

Example 10: Find the volume of the solid generated by the revolution of the tractrix $x = a\cos t + \frac{1}{2} a \log \tan^2 \frac{t}{2}$ $y = a \sin t$, about its asymptote.



Sol: Here the asymptote is x-axis and for the curve from A' to B and B to A, t varies from 0 to $\pi/2$ and $\pi/2$ to π.

Volume = 2 $\times$ Volume generated by the area A'B OA'

$$= 2\int_0^{\frac{\pi}{2}} \pi y^2 \frac{dx}{dt}\, dt \qquad ....(1)$$

Also $\frac{dx}{dt} = -a \sin t + a \frac{1}{\tan\frac{t}{2}} \sec^2 \frac{t}{2} . \frac{1}{2}$

$$= -a \sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

Or $\frac{dx}{dt} = -a \sin t + \frac{a}{\sin t}$

$$= \frac{a}{\sin t}(1 - \sin^2 t)$$

$$= \frac{a\cos^2 t}{\sin t}$$

$\therefore$ from (1),

The required volume $= 2\int_0^{\frac{\pi}{2}} \pi a^2 \sin^2 t \frac{a\cos^2 t}{\sin t}$

$$= 2\pi a^3 \int_0^{\frac{\pi}{2}} \sin t \cos^2 t\, dt$$

$$= 2\pi a^3 \frac{1}{3.1}$$

$$= \frac{2}{3}\pi a^3$$

Self-Check Exercise-2

Q.1 Find the volume generated by revolving one arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about its base.

Q.2	Find the volume generated by the revolution of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ about the x-axis.

14.6 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Defined Solid of revolution, surface of revolution and axis of revolution.
2. Derived the formula for finding the volume of the solid generated by the revolution about the x-axis of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.
3. Derived the formula for finding the volume of the solid generated by the revolution about the x-axis of the area bounded by the curves $y = f(x)$, $y = g(x)$ and the ordinates $x = a$ and $x = b$.
4. Solved questions to finding the volume of the solid as stated above.

14.7 Glossary

1. When a plane area is revolved about a fixed straight line in its own plane, then the body so generated by the plane area is called the solid of revolution.
2. Surface generated by the boundary of the plane is called the surface of revolution.
3. The fixed line about which the plane area rotates is called the axis of revolution.

14.8 Answers To Self-Check Exercise

Ans. 1 48π

Ans. 2 $\pi(\pi - 2)$ (numerically)

Ans. 3 $\frac{2\pi}{15}$

Self-Check Exercise-2

Ans. 1 $5\pi^2 a^3$

Ans. 2 $\frac{3\pi}{4}$

14.9 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, 2005.
2. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.

14.10 Terminal Questions

1. Find the surface of the solid generated by revolving the arc of the parabola $y^2 = 4ax$ bounded by its latus rectum about x-axis.
2. Find the volume of the paraboloid generated by the revolution of the parabola $y^2 = 4ax$ about the x-axis from $x = 0$ to $x = h$.
3. Find the volume of the right circular cone with radius of the base as r and height h.
4. Show that the volume of the solid obtained by revolving the area included between the curves $y^2 = x^3$ and $x^2 = y^3$ about the x-axis is $\frac{5}{28}\pi$.
5. Find the volume of the reel formed by the revolution of the cycloid $x = a(t + \sin t)$, $y = a(1 - \cos t)$ about the tangent at the vertex.

Unit - 15

Surfaces of Solids of Revolution

Structure

15.1 Introduction

Surfaces of solids of revolution are an important concept in calculus and geometry. When a curve is rotated around a specific axis, it generates a three-dimensional solid object. The surface of this solid object is known as the surface of revolution. To understand surfaces of solids of revolution, let us consider a simple example. Imagine you have a curve, such as line segment, a parabola, or any other smooth curve, and you rotate it around a straight line called the axis of revolution. The resulting solid is called a solid of revolution. For example, rotating a line segment around its mid-point generates a three-dimensional object known at a cylinder. The surface of revolution refers specifically to the outer boundary or the skin of the solid of revolution. It is the curved surface that encloses the solid and can be visualized as the shape obtained by sweeping the curve around the axis. The surface of revolution can take various forms depending on the shape of the curve and the axis of revolution.

15.2 Learning Objectives

After studying this unit, you should be able to:-

- Define surface of revolution
- Discuss surface of the solid generated by the revolution about the x-axis.
- Discuss surface formula for parametric equations.
- Solve questions related to surface of the solid as stated above.

15.3 Surface of the Solid Generated by the Revolution About the x-Axis

The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve y = f(x), the x-axis and the ordinates x = a, x = b is $\int_{x=a}^{x=b} 2\pi\, y\, ds$

where s is the length of the arc of the curve measured from a fixed point on it to any point (x, y)

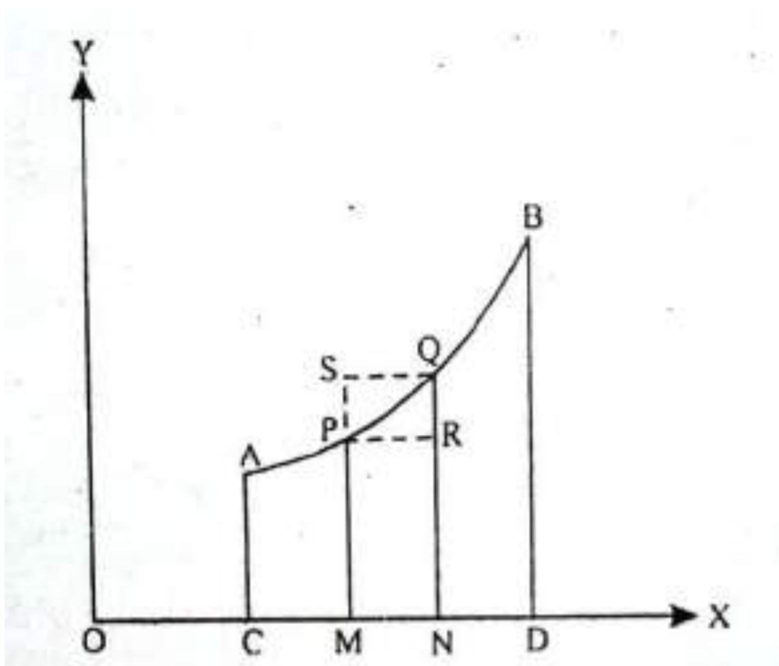
Proof: Let AB be the curve y = f(x) and CA, DB be the ordinates x = a, x = b respectively.

Let P(x, y) be any point on the curve AB. From P, draw PM $\perp$ x-axis so that OM = x, MP = y.

Let S denote the curved surface of the solid generated by the revolution about x-axis of the area ACMP. Clearly S is a function of s.

Let Q(x + δx, y + δy) be a point on the curve in the neighbourhood of P such that arc PQ = δs. Then the curved surface of the solid of revolution of the area PMNQ about the x-aix is δS.

Complete the paralielogram PRQS.



The area of the curved surface generated by the arc PQ lies between the areas of curved surfaces of the cylinders whose base radii are PM and NQ.

i.e., δS lies between $2\pi y\, \delta s$ and $2\pi (y + \delta y)\, \delta s$

$\therefore$ $\frac{\delta S}{\delta s}$ lies between $2\pi y$ and $2\pi (y + \delta y)$

Taking limits as Q → P i.e. as $\delta x \to 0$, $\delta y \to$, $\delta s \to 0$,

$$\frac{dS}{ds} \text{ lies between } 2\pi y \text{ and } 2\pi y$$

$$\therefore \quad \frac{dS}{ds} = 2\pi y$$

$$\therefore \quad \int_{x=a}^{x=b} 2\pi y \, ds = \int_{x=a}^{x=b} \frac{dS}{ds} ds = \left[S\right]_{x=a}^{x=b}$$

= (Value of S when x = b) - (Value of S when x = a)

= area of the surface area of the solid generated by the revolution of area ACDB - 0

∴ surface area of the solid generated by the revolution of area ACDB x = b

$$= \int_{x=a}^{x=b} 2\pi \, y \, ds$$

Note 1: ∴ $S = \int_{x=a}^{x=b} 2\pi y \, ds = \int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} dx$, where $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Note 2: Revolution about y-axis

The curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve x = f(y), the y-axis and the abscissae y = a, y = b is $\int_{y=a}^{y=b} 2\pi x \, ds$.

Let us improve our understanding of these results by looking at some following examples:-

Example 1: Find the area of the surface formed by the revolution of y2 = 4ax about the x-axis by the arc from the vertex to one end of the latus rectum.

Sol: The equation of parabola is $y^2 = 4ax$

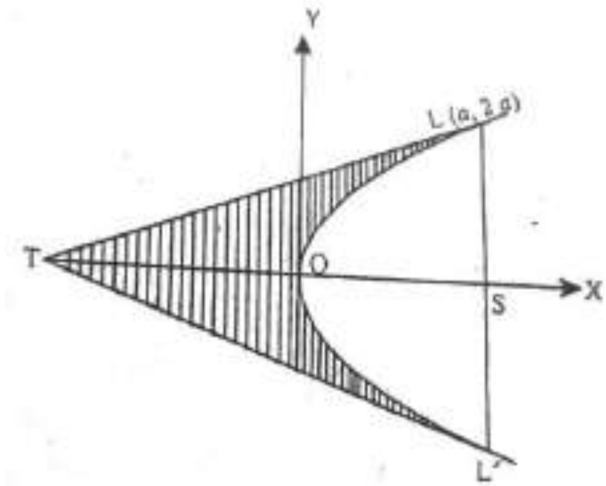
Or $\quad y = 2\sqrt{a}\sqrt{x} \quad$ ....(1)

$$\therefore \quad \frac{dy}{dx} = 2\sqrt{a} \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{a}}{\sqrt{x}}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{a}{x}}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\frac{x+a}{x}} \quad \text{....(2)}$$

Let LSL' be the latus rectum. For the portion AL, x varies from 0 to a.



$\therefore$ required surface = $\int_0^a 2\pi y \frac{ds}{dx} dx$

$= 2\pi \int_0^a 2\sqrt{a}\,\sqrt{x}\,\sqrt{\frac{x+a}{x}}\,dx$

$= 4\pi\sqrt{a} \int_0^a (x+a)^{\frac{1}{2}}\,dx$

$= 4a\sqrt{x}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$

$= \frac{8}{3}\pi\sqrt{a}\left[(x+a)^{\frac{3}{2}}\right]_0^a$

$= \frac{8}{3}\pi\sqrt{a}\left[(2a)^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$

$= \frac{8}{3}\pi\sqrt{a}\left[2\sqrt{2}a^{\frac{3}{2}} - a^{\frac{3}{2}}\right]$

$= \frac{8}{3}\pi\sqrt{a}\,.\,a^{\frac{3}{2}}\left(2\sqrt{2}-1\right)$

$= \frac{8}{3}\pi a^2\left(2\sqrt{2}-1\right)$

Example 2: Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.

Sol: The equation of ellipse is $x^2 + 4y^2 = 16$

Or $4y^2 = 16 - x^2$

$\therefore \quad y = \frac{1}{2}\sqrt{16-x^2} \qquad \text{....(1)}$

$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\frac{-2x}{2\sqrt{16-x^2}}$

$= -\frac{x}{2\sqrt{16-x^2}}$

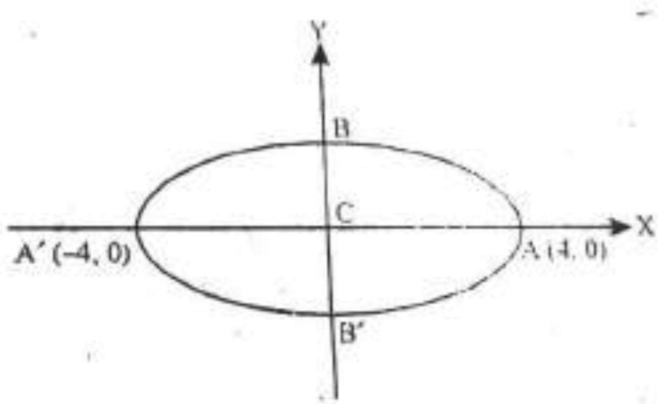
Now $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\frac{x^2}{4(16-x^2)}}$

$= \sqrt{\frac{64-4x^2+x^2}{4(16-x^2)}}$

$= \sqrt{\frac{64-3x^2}{4(16-x^2)}}$

The ellipse meets x-axis where y = 0



$\therefore$ putting y = 0 in $x^2 + 4y^2 = 16$, we get,

$x^2 = 16$ or x = -4, 4

$\therefore$ for upper half of the ellipse in first quadrant, x varies from 0 to 4.

$\therefore$ required surface = $2\int_0^4 2\pi y \frac{ds}{dx} dx$

$$= 4\pi\int_0^4 \frac{\sqrt{16-x^2}}{2} \frac{\sqrt{64-3x^2}}{4(16-x^2)} dx$$

$$= \pi\int_0^4 \sqrt{64-3x^2}\, dx$$

$$= \pi\sqrt{3}\int_0^4 \sqrt{\frac{64}{3}-x^2 dx}$$

$$= \pi\sqrt{3}\int_0^4 \sqrt{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2 dx}$$

$$= \pi\sqrt{3}\left[\frac{x}{2}\sqrt{\frac{64}{3}-x^2} + \frac{64}{3}\sin^{-1}\left(\frac{x}{\frac{8}{\sqrt{3}}}\right)\right]_0^4$$

$$= \pi\sqrt{3}\left[2\sqrt{\frac{64}{3}-16} + \frac{64}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi\sqrt{3}\left[2.\frac{4}{\sqrt{3}} + \frac{64}{3}.\frac{\pi}{3}\right]$$

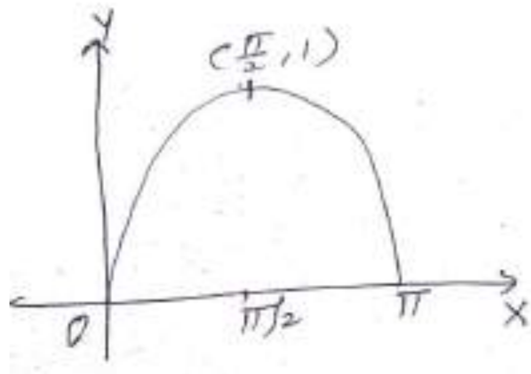
$$= 8\pi\left[1+\frac{4\pi}{3\sqrt{3}}\right]$$

Example 3: Show that the surface of the solid obtained by revolving the arc of the curve y = sin x from x = 0 to x = π about x-axis is $2\pi\left[\sqrt{2}+\log(1+\sqrt{2})\right]$

Sol: The equation of the given curve is

y = sin x

$\therefore \quad \frac{dy}{dx}$ = cos x



Now $\quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\cos^2 x}$

Required surface = $\int_0^{\pi} 2\pi \, y \, \frac{ds}{dx} dx$

$= \int_0^{\pi} 2\pi \sin x \sqrt{1+\cos^2 x} \, dx$

$= 2\pi \int_0^{\pi} \sin x \sqrt{1+\cos^2 x} \, dx$

Put cos x = t, $\therefore$ - sin x dx = dt

i.e. sin x dx = - dt

When x = 0, t = cos 0 i.e. t = 1

When x = π, t = cos i.e. t = -1

$\therefore$ Required surface = $2\pi \int_1^{-1} \sqrt{1+t^2} \, (-dt)$

$= 2\pi \int_1^{-1} \sqrt{1+t^2} \, dt$

$= 2\pi . 2 \int_0^{1} \sqrt{1+t^2} \, dt$

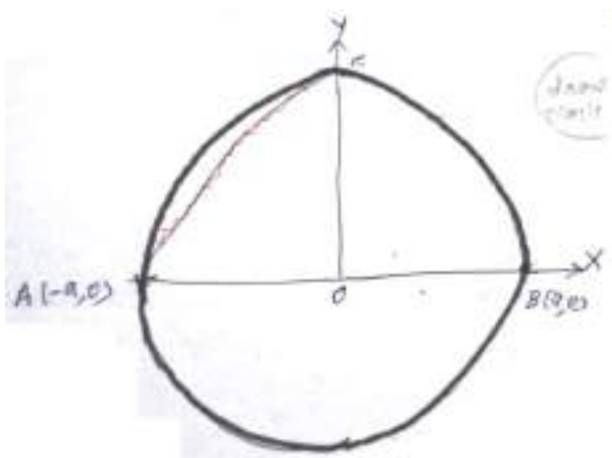
$$= 4\pi\left[\frac{t\sqrt{1+t^2}}{2}+\frac{1}{2}\log\left(1+\sqrt{1+t^2}\right)\right]_0^1$$

$$= 4\pi\left[\frac{1\sqrt{1+1}}{2}+\frac{1}{2}\log\left(1+\sqrt{1+1}\right)-0-\frac{1}{2}\log 1\right]$$

$$= 2\pi\left[\sqrt{2}+\log(1+\sqrt{2})\right]$$

Example 4: Find the surface of a sphere of radius a.

Sol: We known that the sphere is generated by the revolution of a semi-circle ACB about its diameter AB.



Let the equation of the circle be

$$x^2 + y^2 = a^2$$

$\therefore$ $$y = \sqrt{a^2-x^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{-2x}{2\sqrt{a^2-x^2}}$$

$\therefore$ $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= \sqrt{1+\frac{x^2}{a^2-x^2}}$$

$$= \sqrt{\frac{a^2-x^2+x^2}{a^2-x^2}}$$

$$= \frac{a}{\sqrt{a^2 - x^2}}$$

$\therefore$ Require surface $= \int_{-a}^{a} 2\pi y \frac{ds}{dx}\ dx$

$$= 2\pi \int_{-a}^{a} \sqrt{a^2 - x^2}\ .\ \frac{a}{\sqrt{a^2 - x^2}}\ dx$$

$$= 2\pi a \int_{-a}^{a} 1\ dx$$

$$= 2\pi a \left[x\right]_{-a}^{a}$$

= 2πz 0a + a)

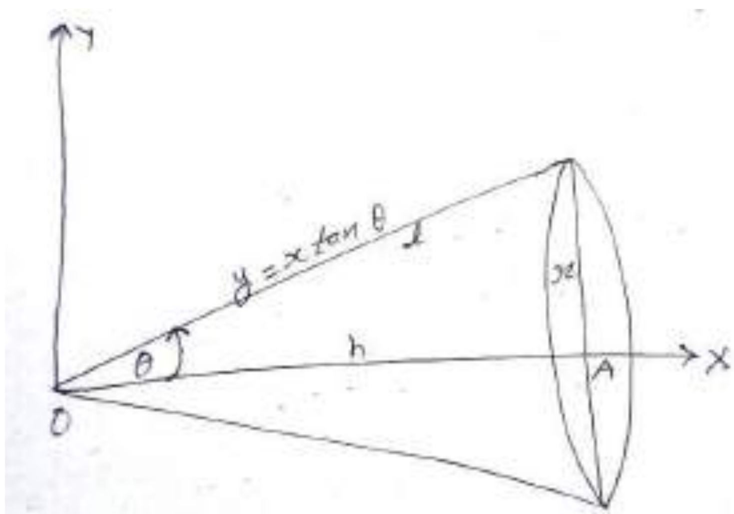
$= 4\pi a^2$

Example 5 : Find the surface of the right circular cone formed by the revolution of right-angled triangle about a side which contains the right angel.

Sol. : Let 0 be the vertex of the cone, 0A, h be the height of the cone, AB = r be the radius of the base and OB = l be the slant side

$\therefore$ $\tan\theta = \frac{r}{h}$ ....(1)

Where θ is the semi-vertical angle.



Also $\sec\theta = \frac{l}{h}$ ...(2)

The equation of OB is

$y = x \tan \theta$ ...(3)

$\therefore \quad \frac{dy}{dx} = \tan \theta$

Now $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\tan^2\theta}$

$= \sec \theta$

Required surface area $= \int_0^h 2\pi \; y \frac{ds}{dx}$ dx

$= 2\pi \int_0^h x \tan\theta \sec\theta \; dx$

$= 2\pi \tan\theta \sec \theta \int_0^h x \; dx$

$= 2\pi \tan\theta \sec \theta \left[\frac{x^2}{2}\right]_0^h$

$= 2\pi \tan\theta \sec \theta \left[\frac{h^2}{2} - 0\right]$

$= \pi h^2 \sec \theta \tan \theta$

$= \pi$ h2 $\frac{\ell}{h} \cdot \frac{r}{h}$ [$\because$ of (1), (2)]

$= \pi$ r l

Self-check Exercise-1

Q. 1 Find the curved surface of the solid generated by the revolution about the x-axis of the area bounded by the parabola $y^2 = 4ax$, the ordinate $x = 3a$ and the x-axis.

Q. 2 Show that the surface of the spherical zone contained between two parallel planes is 2π ah where a is the radius of the sphere and h the distance between the planes.

15.4 Surface formula for parametric Equations

The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve x = f(t), y = ϕ(t), the x-axis and the ordinates at the point where t = a, t = b is

$$\int_a^b 2iTy\,\frac{ds}{dt}\,dt, \text{ where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Revolution about any axis

The curved surface of the solid generated by the revolution, about an axis CD, of the area bounded by a curve AE, the axis CD and the perpendiculars AC, BD an the axis, is $\int 2\pi\ PM\ ds$,

where PM is the perpendicular from any point P of the curve on the axis and arc AP = s, the limits of integration being the values of the independent variable at the ends of the revolving arc.

Let us prove our understanding of these results by looking at some following examples:-

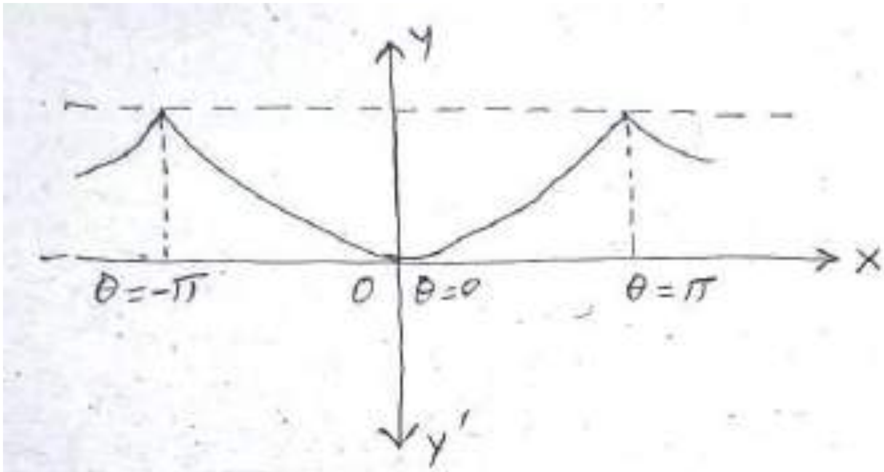
Example 6: Show that the ratio of the surface formed by the rotation of the arc of the cycloid x = a (θ + sin θ), y = a (1 + cos θ) between two consecutive cusps about the axis of x to the area enclosed by the cycloid end the axis of x is $\frac{64}{a}$

Sol. : The equation of given cycloid are

x = a (θ + sin θ), y = a (1 + cos θ)

$\frac{dx}{d\theta}$ = a(1 + cos θ)

$\frac{dy}{d\theta}$ = a sin θ



$\therefore \qquad \dfrac{ds}{d\theta} = \sqrt{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2}$

$= \sqrt{a^2(1+\cos\theta)^2 + (-a\sin\theta)^2}$

$= a\sqrt{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= a\sqrt{1+(\sin^2\theta + \cos^2\theta) + 2\cos\theta}$

$= a\sqrt{2+2\cos\theta}$

$= a\sqrt{2+(1+\cos\theta)}$

$= a\sqrt{2.2\cos^2\dfrac{\theta}{2}}$

$= 2\, a \cos\dfrac{\theta}{2}$

Required surface $= 2\int_0^{\pi} 2\pi y \dfrac{ds}{d\theta}\, d\theta$

$= 2\int_0^{\pi} 2\pi a\ (1+\cos\theta)\ 2a\cos\dfrac{\theta}{2}\, d\theta$

$= 8\pi\, a^2 \int_0^{\pi} 2\cos^2\dfrac{\theta}{2}\cos\dfrac{\theta}{2}\, d\theta$

$$= 16\pi\, a^2 \int_0^{\pi} \cos^3 \frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2}$ = t, $\therefore$ dθ = 2 dt

when θ = 0, t = 0

when θ = π, t = $\frac{\pi}{2}$

Required surface $= 16\pi\, a^2 \int_0^{\pi/2} \cos^3 t \,.\, 2\, dt$

$$= 32\pi\, a^2 \int_0^{\pi/2} \cos^3 t \,.\, dt$$

$$= 32\pi\, a^2 \,.\, \frac{2}{3}$$

$$= \frac{64}{3}\pi\, a^2$$

Area enclosed by the cycloid and the axis of x

$$= \int_{\theta=0}^{\pi} y \frac{dx}{d\theta}$$

$$= 2\int_0^{\pi} a\,(1 + \cos\theta)\, a \,.\, (1 + \cos\theta)\, d\theta$$

$$= 2a^2 \int_0^{\pi} 2\cos^2\frac{\theta}{2} \,.\, 2\cos^2\frac{\theta}{2}\, d\theta$$

$$= 8a^2 \int_0^{\pi} \cos^4 \,.\, \frac{\theta}{2}\, d\theta$$

Put $\frac{\theta}{2}$ = t, $\therefore$ dθ = 2 dt

when θ = 0, t = 0

when θ = π, t = $\frac{\pi}{2}$

$\therefore$ area enclosed by the cycloid and the axis of x

$$= 8\,a^2 \int_0^{\pi/2} \cos^4 t \,.\, 2dt$$

$$= 16\,a^2 \int_0^{\pi/2} \cos^4 t\, dt$$

$$= 16\,a^2 \,.\, \frac{3.1}{4.2}\,\frac{\pi}{2}$$

$$= 3\pi a^2$$

$\therefore$ ratio of surface and area $\dfrac{\frac{64}{3}\pi a^2}{3\pi a^2}$

$$= \frac{64}{9}$$

Example 7 : Find the surface of the solid generated by the revolution y the curve

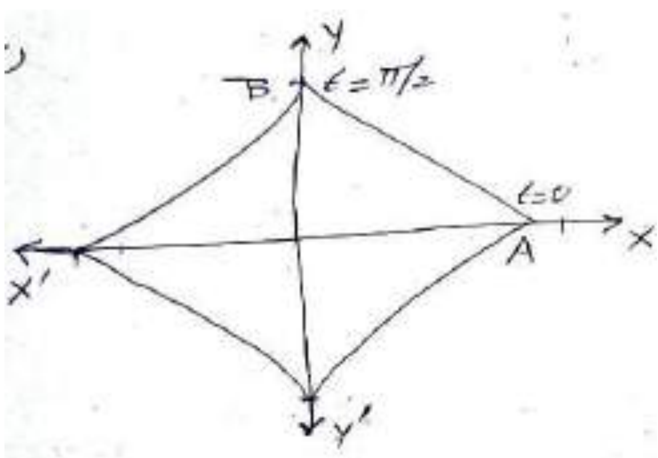
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

Or

x = a cos^3 t, y = a sin^3 t about the axis of x.

Sol. : The equation of the given curve are

x = a cos^3 t, y = a sin^3 t



$\therefore \qquad \dfrac{dx}{dt}$ = a . 3 cos^2 t (- sin t)

= - 3a sin t $\cos^2$ t

$\frac{dy}{dt}$ = a . 3 $\sin^2$ t cos t

= 3a $\sin^2$ t cos t

$\therefore$ $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{9a^2 \sin^2 t \cos^4 t + 9a^2 \sin^4 t \cos^2 t}$

= 3a sin t cos t $\sqrt{\cos^2 t + \sin^2 t}$

= 3a sin t cos t

Required surface = $\int_0^{\pi} 2\pi y \ \frac{ds}{dt}$ dt

= $\int_0^{\pi} 2\pi a$ sin3 t . 3a sin t cos t dt

= 12π a^2 $\int_0^{\pi/2} \sin^4 t \cos t \ dt$

= 12π a^2 $\left[\frac{\sin^5 t}{5}\right]_0^{\pi/2}$

$= \frac{12\pi a^2}{5} \left[\sin^5 \frac{\pi}{2} - \sin^5 0\right]$

$= \frac{12\pi a^2}{5}$ (1 - 0)

$= \frac{12}{5}\pi a^2$

Self-check Exercise - 2

Q. 1 Find the surface area of the solid generated by revolving once complete arch of the cycloid x = a (θ - sin θ), y = a(1 - cos θ) about

(i) the x - axis, and (ii) the line y = 2a

15.5 Summary

We conclude this Unit by summarizing what we have covered in it:-

1. Defined surface of revolution.

2. Derived the formula for finding the curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve y = f(x), the x-axis and the ordinates x = a and x = b.

3. Derived the formula for finding the curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curves x = f(t), y = ϕ(t), the x-axis and the ordinates at the points where x = a and x = b.

4. Solved questions related to finding the surface of the solid.

15.6 Glossary

1. When a curve is rotated around a specific axis, it generated a three-dimensional solid object. The surface of this solid object is known as the surface of revolution.

2. The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve y = (x), the x-axis and the ordinates x = a, x = b is $\int_{x=a}^{x=b} 2\pi y\, ds$, where is the length of the arc of the curve measured from a fixed point on it to any point (x, y).

3. The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve x = f(t), y = ϕ(t), the x-axis and the ordinates at the point where t = a, t = b is

   $\int_a^b 2\pi y \frac{ds}{dt}$ dt, where $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

15.7 Answers To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\frac{56\pi a^2}{3}$

Ans. 2 2π a h

Self-Check Exercise-2

Ans. 1 (i) $\frac{64\pi a^2}{3}$

(ii) $\frac{32\pi a^2}{3}$

15.8 References/Suggested Readings

1. H. Anton, I. Bivens and S. Dans, *Calculus*, John Wiley and Sons (Asia) P. Ltd. 2002.
2. G.B. Thomas and R.L. Finney, *Calculus*, 9^{th} Ed., Pearson Education, 2005.

15.9 Terminal Questions

1. Show that the surface of a belt of the paraboloid formed by the revolution, about the x-axis, of the parabola $y^2 = 4ax$ is

   $$\frac{8\pi}{3}\ \sqrt{a}\left[(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}\right]$$

2. The part of the parabola $y^2 = 4ax$ cut off by the lotus rectum revolves about the tangent at the vertex. Find the curved surface of the real thus formed.

3. Prove that the surface of the solid obtained by revolving the ellipse $b^2x^2 + a^2 y^2 = a^2 b^2$ about axis of x is

   2π ab $\left[\sqrt{1-e^2}+\frac{1}{e}\sin^{-1}e\right]$, e being the eccentricity of the ellipse.

4. Prove that the surface generated by the revolution of the tractrix

   $x = a \cos t + \frac{1}{2}\ a \log \tan^2 \frac{t}{2}$

   $y = a \sin t$

   about its asymptote is equal to the surface of a sphere of radius a.

Unit - 16

Repeated Integral Over A Rectangle and Region 'A'

Structure

16.1 Introduction

A about integral is a mathematical tool used to compute the signed area or volume under a two-dimensional surface or solid in three-dimensional space. It extends the concept of a single integral from one dimension to two dimension. To understand double integrals, let us start by considering a function of two variables f(x, y) defined over a region in the xy-plane. The double integral of f (x, y) over the region R is denoted as $\iint_R f(x, y)\ dA$, where dA represents an infinitesimal element of area in the xy-plane. It is defined as the product of the differential dx and dy, which represent infinitesimal changes in x and y, respectively. The region R is typically described by specifying its boundaries or inequalities.

The double integral computes the sum of the function values f(x, y) over each infinitesimal area element within the region R. This process involves dividing the region R into small sub regions and approximating the function values over those sub regions. As the size of the sub regions approaches zero, the approximation becomes more accurate, and the sum approaches the exact value of the double integral.

16.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss repeated (or iterated) integral over a rectangle.
- Discuss repeated (or iterated) integral over a region 'A'.
- Discuss double integral over a rectangle.
- Define and discuss refinement of a partition
- Solve questions related repeated integral ever a rectangle, a region 'A'.

16.3 Repeated (or iterated) Integral over a Rectangle

Let f(x, y) be a continuous function of x and y defined on the rectangle A where

A = {(x, y) : a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d}

The sides of this rectangle are taken parallel to axes.

For any fixed x $\in$ [a, b], the function g(y) = f (x, y) is a continuous function of y on [c, d]

$\therefore \quad \int_c^d g(y)dy$ exists [$\because$ g (y) = f (x, y) is a continuous function of y]

Then the integral $\int_c^d g(y)dy$ defines a function of x. Let this function be denoted by F(x).

$$\therefore \quad F(x) = \int_c^d g(y)dy = \int_c^d f(x,y) \quad \forall \text{ x in } [a, b]$$

We will show that F is continuous on [a, b],

Since f is continuous on A

$\therefore$ f is uniformly continuous on A.

$\therefore$ given ε 0, however small, there exists δ > 0 such that

$$|f(x, y) - f(x_1,y_1)| < \frac{\varepsilon}{d-c+1} \qquad ...(1)$$

for $|f(x, y) - (x_1,y_1)| < \delta$ and $(x, y), (x_1,y_1) \in A$

$\therefore$ for x, $x_1 \exists$ [a, b], $|x_1 - x_1| < \delta$, we have

$$|F(x) - F(x_1)| = \left|\int_c^d f(x,y)dy - \int_c^d f(x_1,y)dy\right|$$

$$= \left| \int_c^d \{ f(x, y) - f(x_1, y) \} dy \right|$$

$$\leq \int_c^d | f(x, y) - f(x_1, y) | dy \qquad \left[\because \left| \int f \right| \leq \int | f | \right]$$

$$< \int_c^d \frac{\varepsilon}{d - c + 1} \, dy$$

$$= \frac{\varepsilon}{d - c + 1} \int_c^d dy$$

$$= \frac{\varepsilon}{d - c + 1} (d - c)$$

$$< \varepsilon \qquad \left[\because \frac{d - c}{d - c + 1} < \in \right]$$

$\therefore$ $|F(x) - F(x_1)| < \varepsilon$ for $|x - x_1| < \delta$, $x, x_1 \in [a, b]$

$\therefore$ F is continuous on [a, b] and

$\therefore$ $I = \int_a^b F(x) \, dx = \int_a^b \left(\int_c^d f(x, y) dy \right) dx$ exists.

This integral I is called a repeated integral and is obtained by integrating $f(x, y)$ over [c, d] treating it as a function of y (regarding x as a constant) and then integrating the resulting function of x over the interval [a, b].

Note 1. Integrating $f(x, y)$ w.r.t. x first and y later, we can defined another repeated integral.

$$J = \int_c^d \left(\int_a^b f(x, y) dx \right) dy$$

Note 2. It is possible that one of the repeated integrals exists, but the other is not even defined.

Note 3. It is also possible that both the repeated integrals exists but are unequal.

The following examples will illustrate the idea more clearly :-

Example 1. Verify that $\int_1^2 \left(\int_3^4 \left(x y + e^y \right) dx \right) dy = \int_5^4 \left(\int_1^2 \left(x y + e^y \right) dy \right) dx$

Sol. : $\int_1^2 \left(\int_3^4 \left(x y + e^y \right) dx \right) dy = \int_1^2 \left[\frac{x^2}{y} y + x \, e^y \right]_{x=3}^{x=4} dy$

$$= \int_1^2 \left\{\left(8y+4e^y\right)-\left(\frac{9}{2}y+3e^y\right)\right\} \text{ dy}$$

$$= \int_1^2 \left(\frac{7}{3}y+e^y\right)\text{dy}$$

$$= \left[\frac{7}{3}y^2+e^y\right]_1^2$$

$$= \left(\frac{28}{4}+e^2\right) - \left(\frac{7}{4}+e\right)$$

$$= \frac{21}{4} + e^2 - e$$

Also $\displaystyle\int_3^4\left(\int_1^2\left(x\,y+e^y\right)dy\right)\text{dx}$

$$= \int_3^4\left[\frac{xy^2}{2}\; e^y\right]_{y=1}^{y=2} \text{ dx}$$

$$= \int_3^4 \left\{\left(2x+e^2\right)-\left(\frac{x}{2}+e\right)\right\} \text{ dx}$$

$$= \int_3^4 \left(\frac{3x}{3}+e^2-e^y\right) \text{ dx}$$

$$= \left[\frac{3x^2}{4}+(e^2-e)x\right]_3^4$$

$$= [12 + 4\,(e^2 - e)] - \left[\frac{27}{4}+3(e^2-e)\right]$$

$$\frac{21}{4} + e^2 - e$$

$$\therefore \qquad \int_1^2\left(\int_3^4\left(x\,y+e^y\right)\right)\text{dy} = \int_3^4\left(\int_1^2\left(x\,y+e^y\right)dy\right)\text{dy}$$

Example 2 : Verify

$$\int_3^4\int_1^2 \left(x^2+y^2\right)dy\ \text{dx} = \int_1^2\int_3^4 \left(x^2+y^2\right)dx\ \text{dy}$$

Sol. : L.H.S. $= \int_3^4\int_1^2 \left(x^2+y^2\right)dy\,\text{dx}$

$$= \int_3^4 \left[\int_1^2 \left(x^2+y^2\right)dy\right]\text{dx}$$

$$= \int_3^4 \left[x^2y+\frac{y^3}{3}\right]_1^2 \text{dx}$$

$$= \int_3^4 \left[\left(2x^2+\frac{8}{3}\right)-\left(x^2+\frac{1}{3}\right)\right]\text{dx}$$

$$= \int_3^4 \left(x^2+\frac{7}{3}\right)\text{dx}$$

$$= \left[\frac{x^3}{3}+\frac{7}{3}x\right]_3^4$$

$$= \left(\frac{64}{3}+\frac{28}{3}\right) - \left(\frac{27}{3}+\frac{21}{3}\right)$$

$$= \frac{92}{3} - \frac{48}{3} = \frac{44}{3}$$

R.H.S. $= \int_1^2\int_3^4 \left(x^2+y^2\right)dx\,\text{dy}$

$$= \int_1^2 \left[\int_3^4 \left(x^2+y^2\right)dx\right]\text{dy}$$

$$= \int_1^2 \left[\frac{x^3}{3}+xy^2\right]_3^4 \text{dy}$$

$$= \int_1^2 \left[\left(\frac{64}{3}+4y^2\right)-\left(\frac{27}{3}+3y^2\right)\right]\text{dy}$$

$$= \int_1^2 \left(\frac{37}{3} + y^2\right) dy = \left[\frac{37}{3} y + \frac{y^3}{3}\right]_1^2$$

$$= \left[\frac{37}{3} \times 2 + \frac{8}{3}\right] - \left[\frac{37}{3} + \frac{1}{3}\right]$$

$$= \frac{82}{3} - \frac{38}{3} = \frac{44}{3} \qquad ...(2)$$

From (1 and (2)

L.H.S. = R.H.S.

Self-check Exercise-1

Q. 1 Let $f(x, y) = x + y$ be defined in the rectangle

$A = \{(x, y) : 3 \leq x \leq 4, 1 \leq y \leq 2\}$

the show that

$$\int_1^2 \left(\int_3^4 f(x, y) dx \right) dy = \int_3^4 \left(\int_1^2 f(x, y) dy \right) dx$$

16.4 Repeated (or iterated) integral over a Region 'A'

Just as we have defined repeated integral over a rectangle A in xy-plane, we defined repeated integral over a particular region A defined as

Case I. When $A = \{(x, y) : \psi_1(x) = y < \psi_2 (x); a \leq x \leq b\}$

Let $f(x, y)$ be a continuous function, defined over the region A, where $\psi 1(x)$ and $\psi 2(x)$ are continuous in [a, b], then the corresponding repeated integral $\int_a^b \left(\int_{\psi_1(x)}^{\psi_2(x)} f(x, y) dy \right)$ dx exists.

Case II when $A = \{(x, y) ; \phi_1 (y) \leq x \leq \phi_2 (y) \leq x \leq d\}$

Let $f(x, y)$ be a continuous function defined over the region A. where $\phi_1(y)$ and $\phi_2(y)$ are continuous in [c, d], then the corresponding repeated integral.

$$\int_c^d \left(\int_{\phi_1(y)}^{\phi_2(2)} f(x,)dx \right) dy \text{ exists}$$

e.g., consider $f(x, y)$ to be a continuous function defined on a region A bounded by a circle with centre at origin and radius 'r' i.e., $x^2 + y^2 = r^2$

$\therefore$ $A = \{(x, y); x^2 + y^2 < r^2\}$

$= \left\{(x, y) : -\sqrt{r^2 - x^2} \le y \le \sqrt{r^2 - x^2}, -r \le x \le r\right\}$

$= \left\{(x, y); -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2}, -r \le y \le r\right\}$

The two repeated (or iterated) integrals of f(x, y) over A are

$$\int_{-r}^{r} \int_{\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} f(x, y)\, dy\, dx \text{ and } \int_{r}^{r} f(x, y)\, dx\, dy$$

In the subsequent work we shall prove and verify that if the function is continuous over A, then these two repeated integrals are equal.

16.5 Double Integral Over a Rectangle

Let A = {(x, y) : a $\le$ x $\le$ b, c $\le$ y $\le$ d} be a rectangle in the xy-plane and f: A $\to$ R be a bounded function of two variables x and y.

$\therefore$ $\in$ a positive real number M such that $|f(x, y)| \le M$ for all (x, y) $\in$ A.

Let $P_1 = \{a = x_0, x_1, x_2,.....,x_m = b\}$ be a partition of [a, b]

and $P_2 = \{c = y_0, y_1, y_2,....., y_n = b\}$ be a partition of [c, d]

$\therefore$ $P = P_1 \times P_2$ is a partition of A into sub-rectangles

$A_{ij} = \{(x, y): x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$

where $1 \le i \le m$ and $1 \le j \le n$.

Let $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y)$ and $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y)$

$\therefore$ $\Delta_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$

Let $L(P, f) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,\Delta_{ij}$ and $U(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\,\Delta_{ij}$

Here L(P,f) and U(P,f) are called the lower sum and upper sum corresponding to the partition P.

$\because$ $|f(x, y)| \le M \quad \forall\ (x, y) \in A$

$\therefore$ $-M \le f(x, y) \le M \quad \forall\ (x, y) \in A$

$\Rightarrow$ $-M \le m_{ij} \le M_{ij} \le M \quad \forall_{i,j}$

$\Rightarrow$ $-M.\Delta_{ij} \le m_{ij}.\Delta_{ij} \le M_{ij}.\Delta_{ij} \le M.\Delta_{ij}$

$\Rightarrow$ $-M \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} \le \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,\Delta_{ij} < \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\,\Delta_{ij} \le M \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij}$

$\Rightarrow$ $-M(b - a)(d - c) \le L(P, f) \le U(P, f) \le M(b - a)(d - c)$

$$\left[\because \sum_{i=1}^{m}\sum_{j=1}^{n}\Delta_{ij} = \text{Area of rectangle A} = (b-a)(d-c)\right]$$

Hence the set {L (P,f) : P is a partition of A} is bounded above and has the least upper bound L, say; and the set {U(P,f): P is a partion of A} is bounded below and has the greatest lower bound U, say

If L = U, then we say that f is integer able over A or the double integral of f over A exists. The common value is denoted by one of the following expressions.

$$\int_A f \quad \text{or} \quad \iint_A f(x, y)\, dx\, dy \quad \text{or} \quad \iint_{\substack{a\le x\le b \\ c\le y\le d}} f(x,y)dxdy \quad \text{or} \quad \iint_{\substack{c\le y\le d \\ a\le x\le b}} f(x,y)dydx$$

Let us consider the following examples to clear the idea:-

Example 1: Let A be the rectangle given by A = {(x, y): 1 $\le$ x $\le$ 2, 3, $\le$ y $\le$ 4}

Let f : A $\to$ R be defined as f(x, y) = 1 $\forall$ (x, y) $\in$ A

Show that $\int_A f=1$ by using the definition

Sol: Here f(x, y) = 1 $\forall$ (x, y) $\in$ A

where A = f(x, y): 1 $\le$ x $\le$ 2, 3 $\le$, $\le$ 4}

Let $P_1 = \{1 = x_0, x_1, x_2,....., x_m = 2\}$ be any partition of [1, 2]

and $P_2 = \{3 = y_1, y_2,....., y_n = 4\}$ that of [3, 4]

$\therefore$ $P = P_1 \times P_2$ is a partition of the rectangle A into sub-rectangle

$A_{ij} = \{(x, y): x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$ where $1 \le i \le m$ and $1 \le j \le n$.

Now $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y) = 1$

and $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y) = 1$

Let Δ_{ij} be area of rectangle A_{ij}

$$\therefore \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} = (2 - 1)(4 - 3) = 1 \times 1 = 1$$

$$\therefore \quad L(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\Delta_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} 1.\Delta_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} = 1$$

$$\text{and} \quad U(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\Delta_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} 1.\Delta_{ij} = 1$$

$\therefore$ L = l.u.b. L (P,f) = 1

U = g.l.b. $U(P,f) = 1$

$\therefore$ $L = U = 1$

$\Rightarrow$ f is integrable over A

$\therefore$ $\iint\limits_A f = 1$

Example 2: If A is the region given by A = {(x, y); $1 \le x \le 2$, $3 \le y \le 4$}

and if $f : A \to R$ is defined by $f(x, y) = 3$, $\forall (x, y) \in A$

Evaluate $\iint\limits_A f(x, y)\, dx\, dy$ starting from definition of double integral.

Sol: Here $f(x, y) = 3$ $\forall (x, y) \in A$ where A = ((x, y): $1 \le x \le 2$, $3 \le y \le 4$} is a rectangle.

Let $P_1 = \{1 = x_0, x_1, x_2, ..., x_m = 2\}$ be any partition of [1, 2]

and $P_2 = \{3 = y_0, y_1, y_2, ..., y_n = 4\}$ that of [3, 4]

$\therefore$ $P = P_1 \times P_2$ is a partition of the rectangle A into sub-rectangles

$A_{ij} = \{(x, y) : x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$

where $1 \le i \le m$ and $1 \le j \le n$

Now $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y) = 3$

and $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y) = 3$

Let Δ_{ij} be area of rectangle A_{ij}.

$\therefore$ $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta^{ij} = (2 - 1)(4 - 3) = 1 \times 1 = 1$

$\therefore$ $L(P, f) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\Delta_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} 3.\Delta_{ij} = 3\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} = 3(1) = 3$

and $U(P, f) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}.\Delta_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} 3.\Delta_{ij} = 3$

$\therefore$ L = l.u.b. $L(P, f) = 3$

and U = g.l.b $U(P, f) = 3$

$\Rightarrow$ f is integrable over A.

$\therefore$ $\iint\limits_A f(x, y)\, dx\, dy = 3$

Example 3: consider the region A = {(x, y): $1 \le x \le 2$, $3 \le y \le 4$}

Let f : A → R be defined as f(x, y) = 2 ∀ (x y ∈ A)

Evaluate $\iint\limits_A f(x, y)\, dx\, dy$ starting from definition of double integral.

Sol: Do line above example. Ans. = 2

Example 4: consider the region A = {(x, y): 1 ≤ x ≤ 2, 3 ≤ y ≤ 4}

Let f : A → R be defined by $f(x, y) = \begin{cases} 1, if & \text{x is rationl} \\ -1, if & \text{x is irrational} \end{cases}$

Show that f is not integrable over the region A.

Sol: Here $f(x, y) = \begin{cases} 1, if & \text{x is rationl} \\ -1, if & \text{x is irrational} \end{cases}$

and A = {(x, y): 1 ≤ x ≤ 2, 3 ≤ y ≤ 4}

Let $P_1 = \{1 = x_0, x_1, x_2,...., x_m = 2\}$ be any partition of [1, 2]

and $P_2 = \{3 = y_0, y_1, y_2,......, y_n = 4\}$ be any partition of [3, 4]

∴ $P = P_1 \times P_2$ is a partition of the rectangle A into sub-rectangles

$A_{ij} = \{(x, y) : x_{i-1} \le x \le x_i, y_{j-1} \le y \le y_j\}$

where $1 \le i \le m$ and $1 \le j \le n$

Now $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y) = -1$

and $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y) = 1$

Let Δ_{ij} be area of rectangle A_{ij}.

∴ $\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta^{ij} = (2 - 1)(4 - 3) = 1 \times 1 = 1$

∴ $L(P, f) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\Delta_{ij}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} (-1).\Delta_{ij}$

$= -\sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} = -(1) = -1$

and $U(P, f) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}.\Delta_{ij}$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} (1)\ \Delta_{ij}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta_{ij} = 1$$

$\therefore$ L = l.u.b. L(P, f) = -1 and U = g.l.b U(P, f) = 1

$\therefore$ $L \neq U$

$\Rightarrow$ f is not integrable over the region A.

Self-Check Exercise-2

Q.1 Let A be the rectangle given by A = {(x, y) : $1 \leq x \leq 2$, $3 \leq y \leq 4$}

Let $f : A \to R$ be defined as $f(x, y) = 1$ $\forall$ (x, y) $\in$ A show that $\int_A f$ = 1 by using the definition.

16.6 Refinement of a Partition

Let A = {(x, y) : $a \leq x \leq b$, $c \leq y \leq d$} be a rectangle in R^2

Let $P_1 = \{a_0 = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of [a, b]

and $P_2 = \{c = y_0, y_1, y_2, \ldots, y_n = d\}$ be any partition of [c, d]

$\therefore$ $P = P_1 \times P_2$ is a partitions of [a, b]

Let P_1', P_2' be partition of [a, b], [c, d] respectively such that

$P_1' \supset P_1$, $P_2' \supset P_2$. Then the partition

$P' = P_1' \times P_2'$ of A is said to be a refinement of $P = P_1 \times P_2$

We also say that P' is finer than P.

Art 1: Let P and P' be two partitions of rectangle A and $f : A \to R$ be a bounded function. If P' is finer than P, then U (P, f) $\geq$ U (P', f) and L (P, f) $\leq$ L (P', f)Proof: Same as done is lower class.

Cor. $L \leq U$

Art 2: Let A be a rectangle and $f : A \to R$ be bounded. Then f is integrable over A iff for every $\varepsilon > 0$, there exists a partition P of A such that U(P,f) - L(P,f) < ε

Proof: (i) Assume that U(P,f) - (P,f) < ε

$\therefore$ U (P,f) < L(P,f) + ε

$\Rightarrow$ $U \leq$ U((P,f) < L (P,f) + $\varepsilon \leq L + \varepsilon$ [$\because$ L(P,f) $\leq$ L and U (P,f) $\geq$ U]

$\therefore$ $U < L + \varepsilon$ ....(1)

This result is true for all ε

$\therefore$ U $\leq$ L .....(2)

Also L $\leq$ U ....(3)

From (2) and (3), we get,

L = U

$\Rightarrow$ f is integrable over A.

(ii) Assume that f is integrable over A

$\therefore$ L = U

Now L = l.u.b L (P,f)

$\therefore$ for a given ε > 0, there exists a partition P_1 of A such that

$$L(P_1,f) > L - \frac{\varepsilon}{2}$$

Again as U = g.l.b. U (P,f)

$\therefore$ for a given ε > 0, there exists a partition P_2 of A such that

$$U(P_2, f) > U + \frac{\varepsilon}{2}$$

Let P be any partition finer than both P_1 and P_2, then

$$L(P,f) \geq L(P_1,f) \geq L - \frac{\varepsilon}{2} = U - \frac{\varepsilon}{2} \qquad [\because L = U]$$

and $U(P,f) \leq U(P_2,f) < U + \frac{\varepsilon}{2}$

$\therefore$ $-L(P,f) < -U + \frac{\varepsilon}{2}$ and $U(P,f) < U + \frac{\varepsilon}{2}$

$\therefore$ $U(P,f) - L(P,f) < \varepsilon$

Art 3: Let f be continuous on A = {(x, y) : a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d}

Then f is integrable over A i.e. $\iint_A f(x, y)\,dx\,dy$ exists.

Proof: Since f is continuous on rectangle A

$\therefore$ f is uniformly continuous on A.

$\therefore$ given ε > 0 however small, there exists $\delta_1, \delta_2 > 0$ such that

$$\left| f(x,y) - f(\xi,\eta) \right| \quad \frac{\varepsilon}{2(b-a)1)(d-c+1)} \qquad ....(1)$$

for all (x, y), $(\xi, \eta) \in$ A and x - $\xi < \delta_1$ < y - $\eta < \delta_2$

Let P$_1$ = {a = x_0, x_1,....x_n = b} be a partition of [a, b] such that

$x_i - x_{i-1} < \delta_1 \ \forall_i = 1,2,...,m$

and P$_2$ = {c = y_0, y_1,y_2,....,y_n = b} be a partition of [c, d] such that

$y_i - y_{i-1} < \delta_1 \ \forall_i = 1,2,3,.....,n$

Let P = P$_1 \times$ P$_2$ be the corresponding partition of A dividing it into sub-rectangles

A_{ij} = ((x, y): $x_{i-1} \leq x \leq x_i - y_{i-1} \leq y \leq y_j$}

Let Δ_{ij} be area of rectangle A_{ij}

$$\therefore \quad \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij} = (b - a)(d - c)$$

Let $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y) = f(x', y')$, say

and $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y) = f(x'', y'')$, say

$$\therefore \quad U(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}.\Delta_{ij}.L(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,\Delta_{ij}$$

$$\therefore \quad U(P,f) - L(P,f) = \sum_{i=1}^{m}\sum_{j=1}^{n} (M_{ij} - m_{ij})\,\Delta_{ij} \qquad ....(2)$$

Now $(M_{ij} - m_{ij}) = M_{ij} - m_{ij}$

$$\therefore \quad |M_{ij} - m_{ij}| < \frac{\varepsilon}{(b-a+1)(d-c+1)} \qquad .....(3)$$

[$\because$ of (1)]

$\therefore$ from (2), we get,

$$U(P,f) - L(P,f) < \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{\varepsilon}{(b-a+1)(d-c+1)}\Delta_{ij}$$

$$= \frac{\varepsilon}{(b-a+1)(d-c+1)} \sum_{i=1}^{m}\sum_{j=1}^{n} \Delta_{ij}$$

$$= \frac{\varepsilon}{(b-a+1)(d-c+1)}(b - a)(d - c)$$

$< \varepsilon$ $\left[\because \frac{b-a}{b-a+1} < 1 \frac{d-c}{d-c+1} < 1\right]$

$\therefore$ U (P,f) - L L (P,f) < ε

$\Rightarrow$ f is integrable over A.

$\Rightarrow$ $\iint_A f(x, y)\, dx\, dy$ exists.

Note. If (x, y) is continuous in A where

A = {(x, y); 1 $\leq$ x $\leq$ 2, 1 $\leq$ y $\leq$ 2}. Then show that $\iint_A f(x, y)\, dx\, dy$ exists.

Exactly same as above article. Here replace a by 1, b by 2, c by 1 and d by 2.

Art 4: Let A = {(x, y); a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d}. Let f: A $\rightarrow$ R be continuous then

$$\iint_A f(x, y)\, dx\, dy = \int_a^b \left(\int_c^d f(x,y)dy\right) dx = \int_c^d \left(\int_a^b f(x,y)dx\right) dy.$$

Proof: Here A = {(x, y); a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d}

$\because$ f is continuous on A.

$\therefore$ $\iint_A f(x, y)\, dx\, dy = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$ and $\int_c^d \left(\int_a^b f(x,y)dx\right) dy$ exists.

Let $P_1 = \{a = x_0, x_1, \ldots. x_n = b\}$ be any partition of [a, b]

and $P_2 = \{c = y_0, y_1, y_2, \ldots., y_n = b\}$ be any partition of [c, d]

$\therefore$ $P = P_1 \times P_2$ is a partition of A into sub-rectangles

$A_{ij} = ((x, y): x_{i-1} \leq x \leq x_i - y_{i-1} \leq y \leq y_j\}$

where i = 1,2,.....,m j = 1, 2,...., n.

Let $m_{ij} = \underset{(x,y)\in A_{ij}}{Inf} f(x, y)$, $M_{ij} = \underset{(x,y)\in A_{ij}}{Sup} f(x, y)$

Let Δ_{ij} be area of rectangle A_{ij}

$\therefore$ $\Delta_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$

Now $m_{ij} < f(x,y) < M_{ij}$ $\forall$ (x, y) $\in A_{ij}$

$\therefore$ $\int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} m_{ij} dx\right) dy \leq \int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} f(x,y) dx\right) dy \leq \int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} M_{ij} dx\right) dy$

$\Rightarrow \quad m_{ij} (x_i - x_{i-1}) (y_j - y_{j-1}) \leq \int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} f(x, y)dx \right) dy \leq M_{ij} (x_i - x_{i-1}) (y_j - y_{j-1})$

$\Rightarrow \quad M_{ij} A_{ij} \leq \int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} f(x, y)dx \right) dy \leq M_{ij} (x_i - x_{i-1}) (y_j - y_{j-1})$

$\Rightarrow \quad \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} A_{ij} \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{y_{i-1}}^{y_i} \left(\int_{x_{i-1}}^{x_i} f(x, y)dx \right) dy \leq \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} A_{ij}$

$\Rightarrow \quad L (P,f) \leq \sum_{i=1}^{m} \int_{c}^{d} \left(\int_{x_{i-1}}^{x_i} f(x, y)dx \right) dy \leq U (P,f)$

$\Rightarrow \quad L (P,f) \leq \int_{c}^{d} \left(\sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x, y)dx \right) dy \leq U (P,f)$

$\Rightarrow \quad L (P,f) \leq \int_{c}^{d} \left(\int_{a}^{b} f(x, y)dx \right) dy \leq U (P,f)$ for every partition P.

Since $\iint_A f(x, y)\, dx\, dy$ exists

$\therefore \quad L = U = \iint_A f(x, y)\, dx\, dy$

$\therefore$ above result can be written as $\iint_A f(x, y)\, dx\, dy \leq J \leq \iint_A f(x, y)\, dx\, dy$

$\therefore \quad J = \iint f(x, y)\, dx\, dy$

$\Rightarrow \quad \iint_A f(x, y)\, dx\, dy = \int_{c}^{d} \left(\int_{a}^{b} f(x, y)dx \right) dy$

Similarly $\iint_A f(x, y)\, dx\, dy = \int_{a}^{b} \left(\int_{c}^{d} f(x, y)dy \right) dx$

$\therefore$ we have

$$\iint_A f(x, y)\, dx\, dy = \int_{a}^{b} \left(\int_{c}^{d} f(x, y)dy \right) dx = \iint_A f(x, y)\, dx\, dy = \int_{c}^{d} \left(\int_{a}^{b} f(x, y)dx \right) dy$$

Note: If $f : A \rightarrow R$ is continuous, then I = J and double integral = repeated integral.

Cor: Let A = {x, y): a $\le$ x $\le$ b, c $\le$ y $\le$ d} Let f : A $\rightarrow$ R be continuous.

Let f(x, y) = F(x) G(y) $\forall$(x, y) $\in$ A, then

$$\iint_A f(x, y)\, dx\, dy = \int_a^b F(x)\, dx \int_c^d G(y)\, dy$$

Proof: We have

$$\iint_A f(x, y)\, dx\, dy = \int_c^d \left(\int_a^b f(x, y) dx \right) dy = \int_c^d \int_a^b F(x)G(y) dx\, dy$$

$$= \int_c^d \left(G(y) \int_a^b F(x) dx \right) dy = \int_a^b F(x)\, dx \int_c^d G(y)\, dy.$$

16.7 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Discussed repeated (or iterated) integral over a rectangle and illustrated some examples in this support.
2. Discussed repeated (or iterated) integral over a region 'A'
3. Discussed double integral over a rectangle and given some solved examples related to it.
4. Discussed in detail refinement of a partition.

16.8 Glossary

1. If f(x, y) is a function of two variables defined over a region in the xy-plane, then the double integral of f(x, y) over the region R is denoted as $\iint_R f(x, y)\, dA$, where dA represents on infinitesimal element of area in the xy-plane.
2. If I = $\int_a^b \left(\int_c^d f(x, y) dy \right) dx$, then this integral I is called a repeated integral and is obtained by integrating f(x, y) over [c, d] treating it as a function of y (regarding x as a constant) and then integrating the resulting function of x over the interval [a, b]

16.9 Answer To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\int_1^2 \left(\int_3^4 f(x, y) dx \right) dy = \int_3^4 \left(\int_1^2 f(x, y) dx \right) dy = 5$

Self-Check Exercise-2

Ans. 1 L (P, f) = $\sum_{i=1}^{m}\sum_{j=1}^{n}$ $m_{ij}\,\Delta_{ij}$ = 1

and U (P, f) = $\sum_{i=1}^{m}\sum_{j=1}^{n}$ $M_{ij}\,\Delta_{ij}$ = 1

16.10 References/Suggested Readings

1. H. Anton, I. Bivens and S. Davis, Calculus, John Wiley and Sons (Asia) P. Ltd, 2002
2. G.B. Thomas and R.L. Finney, Calculus, 9th Ed., Pearson Education, Delhi, 2005

16.11 Terminal Questions

1. Let f(x, y) be defined as

   f(x, y) = 1 in $1 \leq x \leq 2$; $\frac{1}{2} \leq y \leq \frac{3}{2}$

   show that I = 1, J = 1

2. Let A = {(x, y): $2 \leq x \leq 3$, $3 \leq y \leq 4$}

   Let A → R be defined by

   $$f(x, y) = \begin{cases} 1, if & \text{x is rationl} \\ 0, if & \text{x is irrational} \end{cases}$$

   Show that $\iint_A f(x, y)\,dx\,dy$ does not exist.

3. Consider the region A = {(x, y) $1 \leq x \leq 2$, $3 \leq y \leq 4$}

   Let f : A → R be defined as f(x, y) = 2 $\forall$ (x, y) $\in$ A

   Evaluate $\iint_A f(x, y)\,dx\,dy$ starting from definition of double integral.

Unit - 17

Double Integral Over a General Region and Properties of The Double Integral

Structure

16.1 Introduction

Double integrals are a fundamental concept in calculus, specifically in multivariable calculus. They extend the idea of single-variables over a region in the plane. A double integral allow us to calculate the signed volume under a surface or the total accumulated value of a function over a two-dimensional region. Double integrals satisfies the properties of linearity. In double integrals, under certain conditions, the order of integration can be reversed. This property is known as Fubini's theorem and allows us to evaluate the double integral by iterated integration with respect to one variable at a time. The value of a double integral is independent of the path taken to obtain it. As long as the region of integration remains the same, the value of the integral will be unaffected by the specific choice of the order of integration or the coordinate system.

17.2 Learning Objectives

After studying this unit, you should be able to:-

- Define double integral over a general region

- Discuss different properties of double integral
- Solve questions related to double integral by using properties of double integral
- Define and discuss double integral in polar coordinates
- Solve questions of double integral in polar coordinates

17.3 Double Integral Over a General Region

Let f : B → R be a bounded function where B is a bounded subset of R^2. Let A be any rectangle containing B.

We define a function F : A → R as F(x, y) = $\begin{cases} f(x,y), if\ (x,y) \in B \\ 0, \quad if\,(x,y) \in A - B \end{cases}$

Then the function f is said to be integrable over B if the function F is integrable over A and in this case

$$\iint_B f(x, y)\, dx\, dy = \iint_A F(x, y)\, dx\, dy$$

Note: The result is independent of the choice of rectangle A.

17.4 Properties of the Double Integral

Property I: Let c be a non-zero number. Let f : B → R be a bounded function where B is a bounded sub-set of R^2. Then $\iint_B c f$ exists iff $\iint_B c f = c\iint_B f$.

Property II: Let A = {(x, y): a $\leq$ x $\leq$ b, c $\leq$ y $\leq$ d}

and B = {(x, y): b $\leq$ x $\leq$ e, c $\leq$ y $\leq$ d}

Let f be continuous on A ∪ B. Then $\iint_{A\cup B} f = \iint_A f + \iint_B f$

Property III: f : B → R is integrable and if f(x, y) $\geq$ 0 ∀ (x, y) ∈ B, then $\iint_B f \geq 0$.

Property IV: If f and g are integrable over B, then $f \pm g$ are also integrable over B and $\iint_B (f \pm g) = \iint_B f \pm \iint_B g$.

Property V: If f and g are integrable over B and if

f(x, y) $\geq$ g (x, y) ∀ (x, y) ∈ B, then $\iint_B f \geq \iint_B g$.

Property VI: Let ψ_1, ψ_2 be two continuous functions defined on [a, b] such that

$\psi_1(x) \leq \psi_2(x)$ ∀ x ∈ [a, b]

Let $A = \{(x, y): a \leq x \leq b, \psi_1(x) \leq y \leq \psi_2(x)\}$

Let $f: A \to R$ be continuous, then

$$\iint_A f(x, y)\, dx\, dy \text{ and } \int_a^b \left(\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)dy \right) dx \text{ exist and are equal.}$$

Property VII: Let ψ_1, ψ_2 be two continuous functions defined on [a, b] such that

$\psi_1(x) \leq \psi_2(x) \ \forall\ x \in [a, b]$

Let $A = \{(x, y): a \leq x \leq b, \psi_1(x) \leq y \leq \psi_2(x)\}$

Let $f: A \to R$ be continuous, then

$$\iint_A f(x, y)\, dx\, dy \text{ and } \int_a^b \left(\int_{\psi_1(x)}^{\psi_2(x)} f(x,y)dy \right) dx \text{ exist and are equal.}$$

Property VIII: If $f : B \to R$ is integrable on B, then the function $|f|$ is integrable over B and

$$\left| \iint_B f(x,y)dxdy \right| < \iint_B |f(x, y)|\, dx\, dy$$

Proof: Let A be a rectangle containing B.

Let F : A → R be deined by

$$F(x, y) = \begin{cases} f(x,y) & \forall (x,y) \in B \\ 0 & \forall (x,y) \in A - B \end{cases}$$

∵ f is integrable over B

∴ F is integrable over A.

∴ given $\varepsilon > 0$, however small, these exists a partition P' of A such that

$U(P',F) - L(P',F) < \varepsilon$ ....(1)

Now $$U(P', |F|) - L(P',|F|) = \sum_{i,j} \begin{bmatrix} S\text{up}.|F|(x,y) - Inf.|F|(x,y) \\ (x,y) \in A_{ij} \quad (x,y) \in A_{ij} \end{bmatrix} \Delta_{ij}$$

$$\leq \sum_{i,j} \begin{bmatrix} S\text{up}.F(x,y) - Inf.F(x,y) \\ (x,y) \in A_{ij} \quad (x,y) \in A_{ij} \end{bmatrix} \Delta_{ij}$$

$= U(P', F) - L(P', F)$ [∵ of (1)]

∴ $|F|$ is integrable over A ⇒ $|f|$ is integrable over B.

Now for every partition P of A,

$$U(P, f) = \sum_{i,j} \begin{bmatrix} \operatorname{Sup}. F(x,y) \\ (x,y) \in A_{ij} \end{bmatrix} \Delta_{ij} \le \sum_{i,j} \begin{bmatrix} \operatorname{Sup}. |F(x,y)| \\ (x,y) \in A_{ij} \end{bmatrix} \Delta_{ij} = U(P, |F|)$$

$$\therefore \quad \left| \iint_B f \right| = \left| \iint_A F \right| = |\operatorname{In}f.\ U(P, f)|$$

$$\le \operatorname{In}f.\ U(P, |F|) = \iint_A |F| = \iint_B |f|$$

$$\therefore \quad \left| \iint_B f \right| \le \iint_B |f|$$

Let us improve our understanding of these results by looking at some following examples.

Example 1: If A is a rectangle given by A = {(x, y) : $1 \le x \le 2$, $3 \le y \le 4$}

and if f : A → R is defined by f(x, y) = 1 ∀ (x, y) ∈ A, show that $\int_A f = 1$

Sol: Here f(x, y) = 1 ∀ (x, y) ∈ A

where A = {(x, y) : $1 \le x \le 2$, $3 \le y \le 4$}

Since f(x, y) is a continuous function of x and y

$$\therefore \quad \int_A f = \int_1^2 \left(\int_3^4 f(x,y)dy \right) dx = \int_1^2 \left(\int_3^4 1 dy \right) dx = \int_1^2 [y]_3^4 dx$$

$$= \int_1^2 1\, dx = [x]_1^2 = 2 - 1 = 1$$

Example 2: Evaluate $\iint\limits_{\substack{0 \le x \le 1 \\ 0 \le y \le 1}} (x^2 + y^2)\, dx\, dy$

Sol: Here f (x, y) = $x^2 + y^2$ is continuous over A where

A = {(x, y): $0 \le x \le 1$, $0 \le y \le 1$}

$$\therefore \quad \iint\limits_{\substack{0 \le x \le 1 \\ 0 \le y \le 1}} (x^2 + y^2)\, dx\, dy = \int_0^1 \left(\int_0^1 (x^2 + y^2) dx \right) dy$$

$$= \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_0^1 dy$$

$$= \int_0^1 \left(\frac{1}{3} + y^2\right) dy$$

$$= \left[\frac{1}{3}y + \frac{y^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} + \frac{1}{3}\right) - (0 + 0) = \frac{2}{3}$$

Example 3: Evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2)\, dy\, dx$

Sol: Let I = $\int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} (x^2 + y^2) dy\right] dx = \int_0^{2a} \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{2ax-x^2}} dx$

$$= \int_0^{2a} \left[x^2\sqrt{2ax - x^2} + \frac{1}{3}(2ax - x^2)^{\frac{3}{2}}\right] dx$$

$$= \int_0^{2a} \left[x^{\frac{5}{2}}\sqrt{2a - x} + \frac{1}{3}x^{\frac{3}{2}}(2a - x)^{\frac{3}{2}}\right] dx$$

$$= \int_0^{2a} \frac{x^{\frac{3}{2}}\sqrt{2a - x}}{3} [3x + (2a - x)]\, dx$$

$$= \int_0^{2a} \frac{x^{\frac{3}{2}}}{3} \sqrt{2a - x}\, (2x + 2a)\, dx$$

Put $x = 2a \sin^2 \theta$, $\therefore$ $dx = 2a.2 \sin \theta \cos \theta\, d\theta$

When $x = 1$, $\theta = \frac{\pi}{2}$

When $x = 0$, $\theta = 0$

$\therefore$ $I = \int_0^{\frac{\pi}{2}} \frac{1}{3}\left(2a\sin^2\theta\right)^{\frac{3}{2}} \sqrt{2a} \cos \theta.2\, (2a \sin^2 \theta + a).4a \sin \theta \cos \theta\, d\theta$

$$= \frac{32a^4}{3} \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^2 \theta\, (2 \sin^2 \theta + 1)\, d\theta$$

$$= \frac{32a^4}{3} \int_0^{\frac{\pi}{2}} (2 \sin^6 \theta \cos^2 \theta + \sin^4 \theta \cos^2 \theta)\, d\theta$$

$$= \frac{32a^4}{3} \left[2.\frac{5.3.1.1}{8.6.4.2}.\frac{\pi}{2} + \frac{3.1.1}{6.4.1}.\frac{\pi}{2} \right]$$

$$= \frac{32a^4}{3} \frac{\pi}{2} \left[\frac{15}{192} + \frac{3}{48} \right]$$

$$= \frac{32a^4}{3} . \frac{\pi}{2} . \frac{27}{192}$$

$$= \frac{3\pi a^4}{4}$$

Example 4: Evaluate the integral $\iint_D x^2 \cos (x^2 + xy)\, dx\, dy$ where D is region in R^2 bounded by sides of a triangle whose vertices are (0, 0), (1, 0), (0, 1)

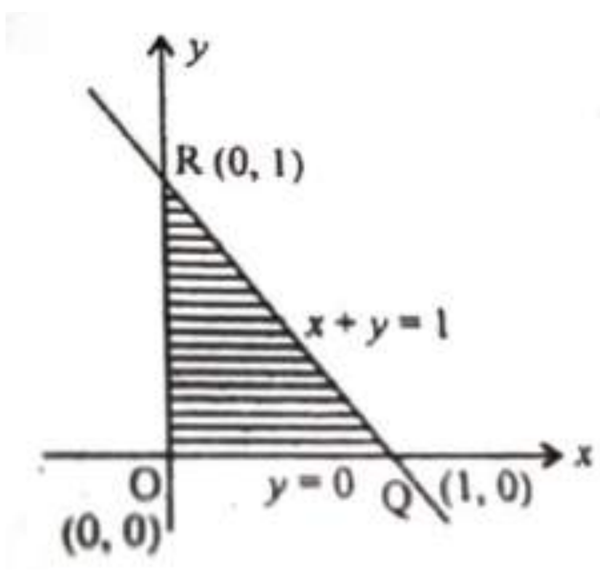
Sol: The vertices of triangle OQR are (1, 0) (1, 0), (0, 1)

$\therefore$ equation of line QR is

$$\frac{x}{1} + \frac{y}{1} = 1 \text{ or } x + y = 1$$

$\therefore$ D = {(x, y) : $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$}

where D is the region.



$\therefore \quad \iint_D x^2 \cos(x^2 + xy)\, dx\, dy$

$$= \int_0^1 x^2 \left(\int_0^{1-x} \cos(x^2 + xy) dy \right) dx$$

$$= \int_0^1 x^2 \left(\frac{\sin(x^2 + xy)}{x} \right)_0^{1-x} dx$$

$$= \int_0^1 x \left(\sin\left(x^2 + x(1-x) - \sin(x^2 + x(0)) \right) \right) dx$$

$$= \int_0^1 x \sin x dx - \frac{1}{2} \int_0^1 (2x) \sin x^2\, dx$$

$$= \left[x(-\cos x) \right]_0^1 - \int_0^1 1.(-\cos x)\, dx - \frac{1}{2} \left[-\cos x^2 \right]_0^1$$

$$= -\cos 1 + \left[\sin x \right]_0^1 + \frac{1}{2} (\cos 1 - \cos 0)$$

$$= -\cos 1 + \sin 1 - 0 + \frac{1}{2} \cos 1 - \frac{1}{2}$$

$$= \sin 1 - \frac{1}{2} \cos 1 - \frac{1}{2}$$

Example 5: Evaluate the double integral $\iint_D (4 - x^2 - y^2)\, dy\, dx$ if the region D is bounded by the straight lines x = 0, x = 1, y = 0 and y = $\frac{3}{2}$.

Sol: Here $f(x, y) = 4 - x^2 - y^2$ is continuous over D,

where D = $\left\{ (x, y) : 0 \le x \le 1, 0 \le y \le \frac{3}{2} \right\}$

$$\therefore \quad \iint_D (4 - x^2 - y^2)\, dy\, dx = \int_0^1 \left(\int_0^{3/2} (4 - x^2 - y^2) dy \right) dx$$

$$= \int_0^1 \left[4y - x^2 y - \frac{y^3}{3} \right]_0^{3/2} dx$$

$$= \int_0^1 \left(6 - \frac{3}{2}x^2 - \frac{9}{8}\right) dx$$

$$= \int_0^1 \left(\frac{39}{8} - \frac{3}{2}x^2\right) dx$$

$$= \left[\frac{39}{8}x - \frac{x^3}{2}\right]_0^1$$

$$= \left(\frac{39}{8} - \frac{1}{2}\right) - (0 - 0)$$

$$= \frac{39 - 4}{8} = \frac{35}{8}$$

Example 6: Show that

$$2 < \iint_{\substack{1 \le x \le 2 \\ 1 \le y \le 2}} (x^2 + y^2)\, dx\, dy < 8$$

Or

Let $A = \{(x, y): 1 \le x \le 2, 1 \le y \le 2\}$, show that

$$2 \le \iint_A f(x, y)\, dx\, dy \le 8$$

where $f(x, y) = x^2 + y^2$

Sol: Here $A = \{(x, y): 1 \le x \le 2, 1 \le y \le 2\}$

The function $f(x, y) = x^2 + y^2$ is continuous on A.

$\therefore$ Given integral $\iint_A (x^2 + y^2)\, dx\, dy$ exists

Now $1 \le x \le 2, 1 \le y \le 2 \Rightarrow 1 \le x^2 \le 4, 1 \le y^2 \le 4$

$\Rightarrow 1 + 1 \le x^2 + y^2 \le 4 + 4 \Rightarrow 2 \le x^2 + y^2 \le 8$

$$\therefore \iint_A 2\, dx\, dy \le \iint_A (x^2 + y^2)\, dx\, dy \le \iint_A 8\, dx\, dy$$

$$\Rightarrow 2\iint_A 1\, dx\, dy \le \iint_A (x^2 + y^2)\, dx\, dy \le 8\iint_A 1\, dx\, dy$$

$$\Rightarrow 2(1) \le \iint_A (x^2 + y^2)\, dx\, dy \le 8(1)$$

$$\left[\because \iint_A 1 dxdy = \int_1^2\left(\int_1^2 1\, dx\right)dy = \int_1^2 [x]_1^2\, dy = \int_1^2 1 dy = [y]_1^2 = 2-1=1\right]$$

$\Rightarrow$ $2 \leq \iint_A (x^2 + y^2)\, dx\, dy \leq 8$

Example 7: Evaluate $\iint x^2y^2\, dx\, dy$ over the region $x^2 + y^2 \leq 1$

Sol: Here $x^2 + y^2 = 1$

$\therefore$ $x^2 \leq 1$ and $y^2 \leq 1 - x^2$

Or $x^2 \leq 1$ and $y^2 \leq \left(\sqrt{1-x^2}\right)^2$

$\therefore$ $-1 \leq x \leq 1$ and $-\sqrt{1-x^2} < y < \sqrt{1-x^2}$

[$\because$ $x^2 \leq t^2 \Rightarrow -t \leq x \leq t$]

Now $f(x, y) = x^2 y^2$ is continuous over the region A given by $x^2 + y^2 \leq 1$

$\therefore$ $\iint_A x^2y^2\, dx\, dy = \int_1^{-1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\, dy\right)dx$

$= \int_1^{-1}\left[\frac{x^2y^3}{3}\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$

$= \int_1^{-1}\frac{x^2}{3}\left[\left(\sqrt{1-x^2}\right)^3 - \left(-\sqrt{1-x^2}\right)^3\right]dx$

$= \int_1^{-1}\frac{x^2}{3}\left[(1-x^2)^{3/2} + (1-x^2)^{3/2}\right]dx$

$= \frac{2}{3}x^2(1-x^2)^{3/2}\, dx$

$= \frac{2}{3}\, 2\int_0^1 x^2(1-x^2)^{3/2}\, dx$

[$\because$ $x^2 (1 - x^2)$ is an even function of x]

Put $x = \sin\theta$, $\therefore$ $dx = \cos\theta\, d\theta$

when $x = 0$, $\sin\theta = 0 \Rightarrow \theta = 0$

when $x = 1$, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$\therefore \quad \iint_A x^2 y^2 \, dx \, dy = \frac{4}{3}\int_0^{\pi/2} \sin^2\theta \, (1-\sin^2\theta)^{3/2} \cos\theta \, d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$$

$$= \frac{4}{3} \; \frac{1.3.1}{3.6.4.2} \; \frac{\pi}{2}$$

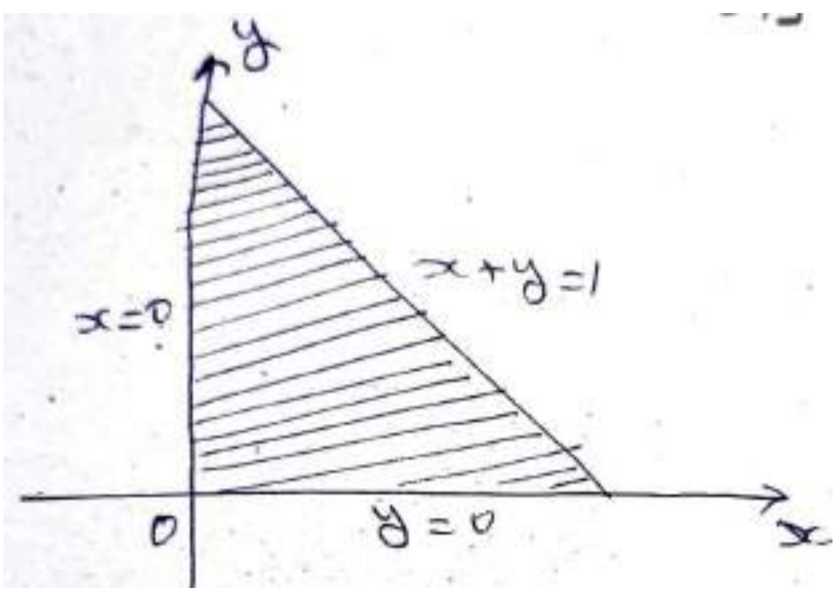
$$= \frac{\pi}{4}$$

Example 8: Evaluate $\iint e^{2x+3y} \, dx \, dy$ over the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$

Sol: Here $A = \{(x, y): 0 \le x \le 1, 0 \le y \le 1 - x\}$

$$\therefore \quad \iint_A e^{2x+3y} \, dx \, dy = \int_0^1 \left(\int_0^{1-x} e^{2x+3y} \, dy\right) dx$$

$$= \int_0^1 \left[\frac{e^{2x+3y}}{3}\right]_0^{1-x} dx$$



$$= \frac{1}{3}\int_0^1 (e^{3-x} - e^{2x})\, dx$$

$$= \frac{1}{3}\int_0^1 \left[\frac{e^{3-x}}{-1} - \frac{e^{2x}}{2}\right]_0^1$$

$$= \frac{1}{3}\left[\left(-e - \frac{1}{2}e^2\right) - \left(-e^3 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{3}\left[e^2 - e^2 - \frac{1}{2}e^2 + \frac{1}{2}\right]$$

$$= \frac{1}{3}\left[e^2(e-1) - \frac{1}{2}(e^2-1)\right]$$

$$= \frac{1}{3}\left[e^2 - \frac{1}{2}(e+1)\right]$$

$$= \frac{1}{6}(e - 1)\,[2e^2 - e - 1]$$

$$= \frac{1}{6}(e - 1)(e - 1)(2e + 1)$$

$$= \frac{1}{6}(e - 1)^2(2e + 1)$$

Example 9: Evaluate $\iint_A$ xy dx dy, where A is the region common to the circles

$x^2 + y^2 = x$, $x^2 + y^2 = y$

Sol: The equations of two circles are

$x^2 + y^2 = x$ ....(1)

and $x^2 + y^2 = y$ .....(2)

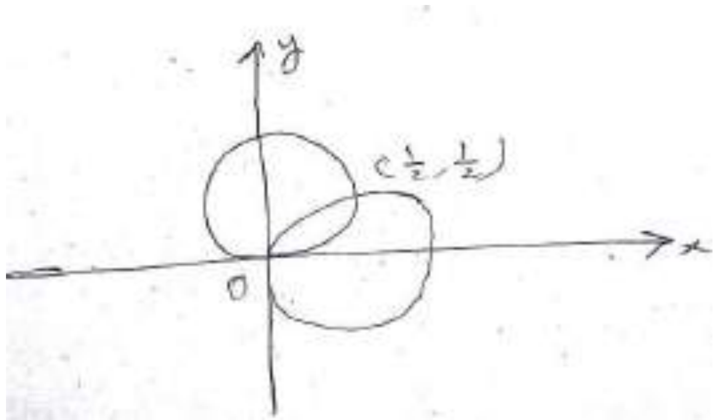
From (1) and (2), we get

$x = y$

Putting x = y in (1), we get

$y^2 + y^2 = y$ or $2y^2 - y = 0$

$\therefore$ $y(2y - 1) = 0 \Rightarrow y = 0, \frac{1}{2}$



$\therefore$ x = 0, $\frac{1}{2}$

$\therefore$ circles (1) and (2) intersect in points (0, 0) and $\left(\frac{1}{2}-\frac{1}{2}\right)$

Now from (1), y = $\pm\sqrt{x-x^2}$

and from (2), y = $\frac{1\pm\sqrt{1-4x^2}}{2}$

But $0 \le x \le \frac{1}{2}$, $0 \le y \le \frac{1}{2}$

$\therefore$ y varies from $\frac{1-\sqrt{1-4x^2}}{2}$ to $\sqrt{x-x^2}$

$\therefore$ A = $\left\{(x,y):\sqrt{x-x^2} \le y \le \frac{1-\sqrt{1-4x^2}}{2}, 0 \le x \le \frac{1}{2}\right\}$

Now f(x, y) = x y is continuous over A.

$\therefore$ $\iint\limits_A$ xy dx dy

$$= \int_0^{1/2} x\left[\int_{\frac{1-\sqrt{1-4x^2}}{2}}^{\sqrt{x-x^2}} y\,dy\right]dx$$

$$= \int_0^{1/2} x\left[\frac{y^2}{2}\right]_{\frac{1-\sqrt{1-4x^2}}{2}}^{\sqrt{x-x^2}} dx$$

$$= \frac{1}{2}\int_0^{1/2} x\left[\left(\sqrt{x-x^2}\right)^2 - \left(\frac{1-\sqrt{1-4x^2}}{2}\right)^2\right]dx$$

$$= \frac{1}{2}\int_0^{1/2} x\left[\frac{1+(1-4x^2)-2\sqrt{1-4x^2}}{4} + (x-x^2)\right]dx$$

$$= \frac{1}{8}\int_0^{1/2} x\left[-\left\{1+1-4x^2-2\sqrt{1-4x^2}\right\} + 4x - 4x^2\right]dx$$

$$= -\frac{1}{8}\int_0^{1/2} x\left[2-4x-2\sqrt{1-4x^2}\right]dx$$

$$= -\frac{1}{8}\int_0^{1/2} \left[2x-4x^2+\frac{1}{4}(1-4x^2)^{\frac{1}{2}}(-8x)\right]dx$$

$$= -\frac{1}{8}\left[\frac{2x^2}{2}-\frac{4x^3}{3}+\frac{1}{4}\frac{(1-4x^2)^{3/2}}{3/2}\right]_0^{1/2}$$

$$= -\frac{1}{8}\left[x^2-\frac{4}{3}x^3+\frac{1}{6}(1-4x^2)^{\frac{3}{2}}\right]_0^{1/2}$$

$$= -\frac{1}{8}\left[\left(\frac{1}{4}-\frac{1}{6}+0\right)-\left(0-0+\frac{1}{6}\right)\right]$$

$$= -\frac{1}{8}\left[\frac{1}{4}-\frac{1}{3}\right]$$

$$= -\frac{1}{8}\left[\frac{3-4}{12}\right] = \frac{1}{96}$$

Self-Check Exercise-1

Q.1 Evaluate $\iint_A xy\, dx\, dy$

where $A = \{(x, y): 1 \leq x \leq 2, 1 \leq y \leq 2\}$

Q.2 If a region A is defined as

$A = \{(x, y): 0 \leq x \leq 3, 2 \leq y \leq 5\}$, show that

$108 < \iint_A (2x^2 + 3y^2)\, dx\, dy < 837$

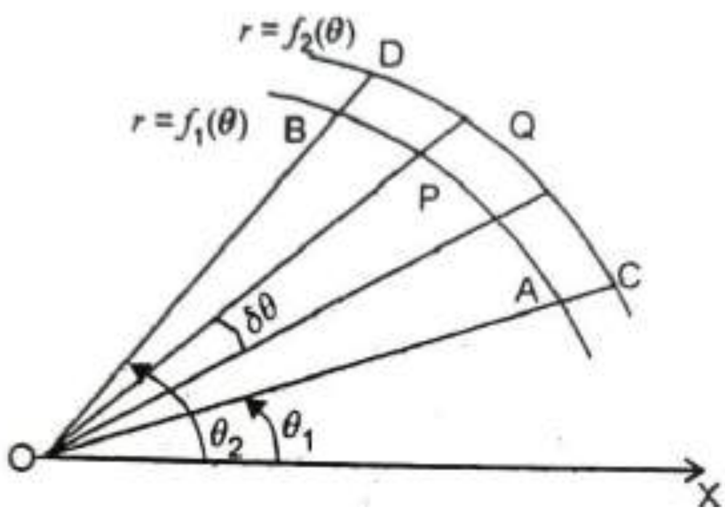
Q.3 Evaluate $\int_0^{\pi} \int_0^{\sin x} y\, dy\, dx$

Q.4 If $A = \{(x, y) : x \geq 0, x^2 + y^2 = 1\}$, then evaluate $\iint_A dx\, dy$

Q.5 Evaluate $\iint_A \sqrt{4x^2 - y^2}\, dx\, dy$, where A is the triangle bounded by the lines $y = 0, y = x, x = 1$

17.5 Double Integral in Polar Co-ordinates

The integral $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta$ is double integral in Polar co-ordinates bounded by the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$. We first integrate w.r.t. r and then w.r.t. θ between the limits $\theta = \theta_1$. $\theta = \theta_2$.



Note: If $\theta_1 = a$, $\theta = b$ and $r_1 = c$, $r_2 = d$ then integration can be evaluated separately w.r.t. θ and r.

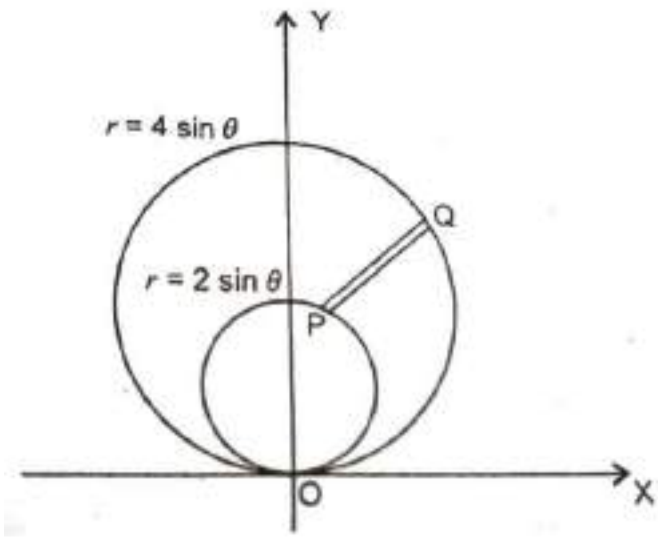
[More over $f(r, \theta) = f(r) f(\theta)$]

i.e. $\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta)\, dr\, d\theta = \int_{\theta_1}^{\theta_2} f(\theta)\, d\theta \int_{r_1}^{r_2} f(r)\, dr$

Let us consider the following examples to clear the idea:-

Example 10: Evaluate $\iint$ r^2 dr dθ over the Area included between the circles r = 2 sin θ and r = 4 sin θ

Sol:



$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\sin\theta}^{4\sin\theta} r^3 \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{4}\left[r^4\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (256 \sin^4\theta - 16 \sin^4\theta)\, d\theta$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (240 \sin^4\theta \, . \, d\theta$$

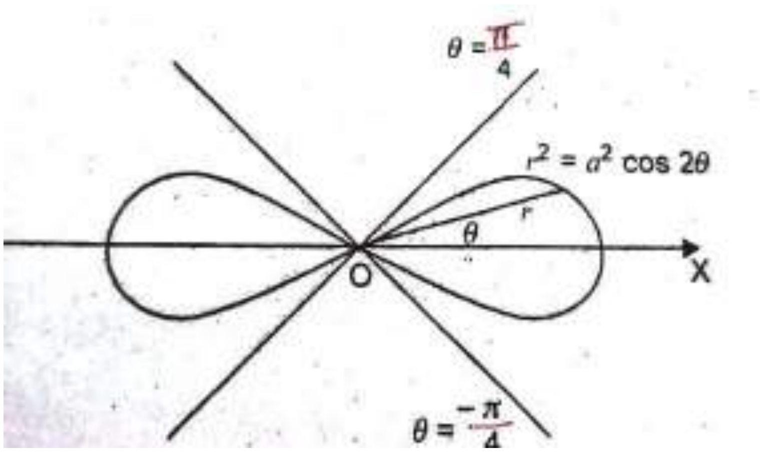
$$= \frac{240}{4} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 \theta \, . \, d\theta$$

$$= 2 \times 60 \int_{0}^{\frac{\pi}{2}} \sin^4 \theta d\theta$$

$$= 120 \times \frac{1 \times 3}{4 \times 2} \times \frac{\pi}{2} = \frac{45\pi}{2}$$

Example 11. Evaluate $\iint \frac{r\, dr\, d\theta}{\sqrt{a^2 + r^2}}$ over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$.

Sol. : Given lemniscate is $r^2 = a^2 \cos 2\theta$.



Now for the one loop of the Lemniscate r varies from 0 to $a\sqrt{\cos 2\theta}$ and θ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

$$\therefore \qquad \iint \frac{r\, dr\, d\theta}{R\sqrt{a^2 + r^2}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \int_{0}^{a\sqrt{\cos 2\theta}} \frac{1}{2}(a^2 + r^2)^{\frac{1}{2}} .2r \; d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\frac{1}{2} . \frac{(a^2 + r^2)}{\frac{1}{2}} \right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(a^2 + r^2 \cos 2\theta\right)^{\frac{1}{2}} - a\right] d\theta$$

$$= \text{a} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(1 + \cos 2\theta\right)^{\frac{1}{2}} - 1\right] d\theta$$

$$= \text{a} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\left(2\cos^2 \theta\right)^{\frac{1}{2}} - 1\right] d\theta$$

$$= \text{a} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta - 1\right) d\theta$$

$$= 2\text{a} \int_{\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sqrt{2}\cos\theta - 1\right) d\theta$$

$$\left[\because \quad \int_{-a}^{a} f(x)dx = 2\int_{-a}^{a} f(x)dx \quad if \quad f(-x) = f(x)\right]$$

$$= 2\text{a}\left[\sqrt{2}\sin\theta - \theta\right]_0^{\frac{\pi}{4}}$$

$$= 2\text{a}\left[\sqrt{2}.\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

$$= 2\text{a}\left(1 - \frac{\pi}{4}\right)$$

Self-check Exercise-2

Q. 1 Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\ dr\ d\theta$

Q. 2 Evaluate $\iint r\sin\theta\, dr\ d\theta$ over the area of the cardioid r = a (1+ cosθ) above the initial line.

17.6 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Defined double integral over a general region.
2. Discussed properties of double integral.
3. Solved questions related to double integral by using properties of double integral.
4. Discussed double integral in polar coordinates.
5. Solved questions of double integral in polar coordinates.

17.7 Glossary

1. If f : B → R be a bounded function where B is bounded subset of R^2. Let A be any rectangle containing B.

   We define a function F : A → R as

   $$F(x, y) = \begin{cases} f(x,y), & if\,(x,y) \in B \\ 0, & if\,(x,y) \in A - B \end{cases}$$

   Then the function f is said to be integrable over B if the function F is integral over A and in this case

   $$\iint_B F(x,y)dx\,dy = \iint_A F(x,y)dx\,dy$$

2. If e be a non-zero number and F : B→R be a bounded function where B is a bounded sub-set of R^2. Then $\iint_B cf$ exists iff $\iint_B cf$ = c $\iint_B f$

3. The integral $\int_{\theta_1}^{\theta_2}\int_{r_2}^{r_2} f(r,\theta)\ dr\ d\theta$ is double integral in polar co-ordinates bounded be the lines $\theta = \theta_1$, $\theta = \theta_2$ and the curves $r = r_1$, $r = r_2$.

17.8 Answer To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\frac{9}{4}$

Ans. 2 Hint : $12 \leq 2x^2 + 3y^2 \leq 93$, then after double integration, we get the result.

Ans. 3 $\frac{\pi}{4}$

Ans. 4 $\frac{\pi}{2}$

Ans. 5 $\frac{1}{3}\left(\frac{\sqrt{3}}{2}+\frac{2\pi}{3}\right)$

Self-Check Exercise-2

Ans. 1 $\frac{\pi a^4}{4}$

Ans. 2 $\frac{4}{3}$ a^2

17.9 References/Suggested Readings

1. H. Anton, I. Bivens and S. Davis, *Calculus*, John Wiley and Sons (Asia) P. Ltd, 2002
2. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, Delhi, 2005

17.10 Terminal Questions

1. Evaluate $\iint_A (x^3+y^3)$ dx dy, where A is the rectangle bounded by the lines x = 0, x = 1 and y = 0, y = 2.
2. Let A = {(x, y) : 2 $\le$ x $\le$ 3, 4 $\le$ y $\le$ 5}

   Show that 56 $\le \iint_A (2x^2+3y^2)$ dx dy $\le$ 93
3. Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx}{1+x^2+y^2}$
4. Evaluate $\iint x^2 y^2 \ dx \ dy$, where A is the region in the first quadrant enclosed by x = 0, y = 0 and $x^2 + y^2 = 1$
5. Evaluate $\iint (x+y) \ dx \ dy$ over the region bounded by x = 0, y = 0, x+y = 1
6. Evaluate $\int_0^{\pi/2} \left[\int_0^{a\cos\theta} r\sqrt{a^2+r^2)} \ dr \ \right] d\theta$
7. Show that $\iint_R r^2 \sin\theta \ dr \ d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi circle r = 2a cos θ, above the initial line.

Unit - 18

Change of Order of Integration and Change of Variables in Double Integral

Structure

18.1 Introduction

Dear students, when changing the order of integration in a double integral, we exchange the roles of the variables of integration. This technique is useful in simplifying the evaluation of integrals, especially when the original order of integration leads to complicated expressions on limits of integration. Suppose we have the following integral $\iint_R f(x,y)\ dA$, where R is a region in the xy-plane and f(x,y) is the integrand. To change the order of integration, we need to determine the new limits of integration based on the new order. Let us say we want to change the order from integrating with respect x first to integrating with respect to y first. To do this, we need to express the region R in terms of the new variables of integration. This involves finding the bounds of y as function of x. Let's denote the new region S. Once we have determined the new limits of integration, the integral becomes :

$\iint_S f(x,y)\ dA$, where the limits of integration are defined by the region S.

Also, the change of variables techniques is a powerful tool used to simplify the evaluation of integrals. Changing variables can help transform the original integral into a more manageable form. In this UNIT, we are going to study change of order of integration and change of variables in double integral.

18.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss change of order of integration and solve the questions related to change of integration in double integrals.
- Discuss the change of variables in double integral.
- Discuss the particular case of change to polar coordinates.
- Solve questions of double integral by change variables.

18.3 Change of Order of Integration

While evaluating double integrals if the limits of integration are variables, then the change or order of integration changes the limits of integration. In changing the order of integration sometimes it is required to split up the region and express the given double integral as a sum of the number of double integrals with changed limits. Sometimes it is advisable to draw rough sketch of the region of integration.

The following examples will illustrate the idea more clearly.

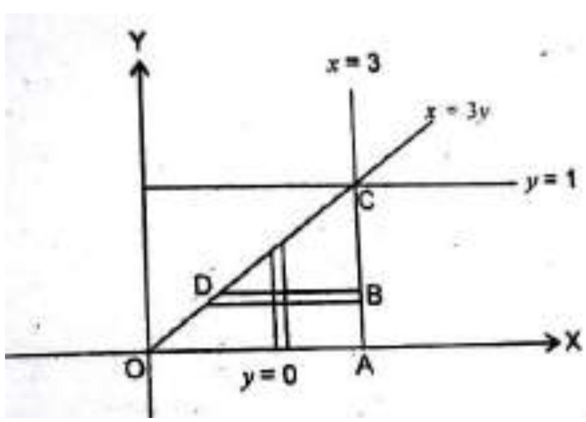
Example 1 : Evaluate $\int_0^1 \int_{3y}^3 e^{x^2} dx\ dy$ by reversing the order of integration.

Sol. : Given integral is $\int_0^1 \int_{3y}^3 e^{x^2} dx\ dy$.

Here the region of integration is R = {(x, y) : 0 $\le$ y $\le$ 1; 3y $\le$ x $\le$ 3}

i.e., it is bounded by the curves

x = 3y, x = 3; y = 0, y = 1



While changing the order of integration the horizontal strip is changed into vertical strip.

Region R can be written as

R = $\left\{(x, y); 0 \le x \le 3; 0 \le y \le \frac{x}{3}\right\}$

$\therefore$ given integral on change of order of integration becomes

I = $\int_0^3 \int_0^{\frac{x}{3}} e^{x^2} dy\ dx$

= $\int_0^3 e^{x^2} \left[y\right]_0^{\frac{x}{3}} dx$

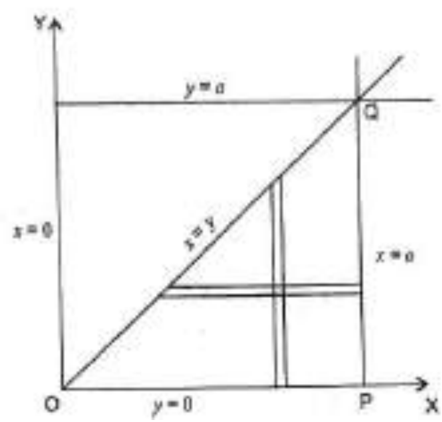
= $\int_0^3 e^{x^2} \frac{x}{3} dx$

= $\frac{1}{6} \int_0^3 e^{x^2} 2x\ dx$

= $\frac{1}{6}\left[e^{x^2}\right]_0^3$ = $\frac{1}{6}\left[e^9 - 1\right]$

Example 2 : Change the order of integration in $\int_0^a \int_y^a \frac{x\ dx\ dy}{x^2 + y^2}$ and hence evaluate the same.

Sol. Let I = $\int_0^a \int_y^a \frac{x\ dx\ dy}{x^2 + y^2}$



From the limits of integration, it is clear that the region of integration is bounded by x = y, x = a, y = 0 and y = a. Thus the region of integration is the ΔOPQ and is divided into horizontal strips. For changing the order of integration, we divide the region of integration into vertical strips. The new limits of integration become : y varies from 0 to x and varies from 0 to a.

$$\therefore \qquad I = \int_0^a \int_y^a \frac{x\ dx\ dy}{x^2 + y^2}$$

$$= \int_0^a \int_0^x \frac{x\ dx\ dy}{x^2 + y^2}$$

$$= \int_0^a x \left[\frac{1}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx$$

$$= \int_0^a \frac{\pi}{4} dx$$

$$= \frac{\pi}{4} [x]_0^a$$

$$= \frac{\pi a}{4}$$

Example 3 : Write an equivalent double integral with order of integration reversed for $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\ \ dx\ dy$. check your answer by evaluating both double integrals.

Sol. : Given is $\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\ \ dx\ dy$

The region Fr of integration is given by

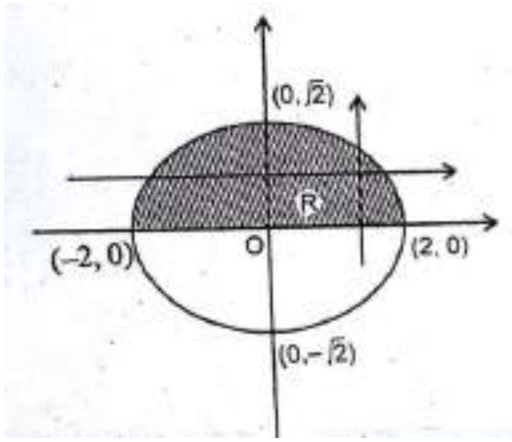
R = $\left\{(x, y) : -\sqrt{4-2y^2} \le x \le \sqrt{4-2y^2}, 0 \le y \le \sqrt{2}\right\}$

Now, x = ± $\sqrt{4-2y^2}$ $\Rightarrow$ x^2 = y - $2y^2$ $\Rightarrow$ $x^2 + 2y^2 = 4$

which is an ellipse

$\therefore$ given region of integration is bounded by ellipse $x^2 + 2y^2 = 4$ for $0 \le y \le \sqrt{2}$ and is shown shaded in the figure.

To change the order of integration, the vertical line enters the region where y = 0 and leaves it where y = $\sqrt{\frac{4-x^2}{2}}$. Also minimum value of x is -2 and maximum value is 2.



$$\therefore \qquad \int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \; dx \; dy = \int_{-2}^{2} \int_0^{\sqrt{\frac{4-x^2}{2}}} y \; dy \; dx$$

Now, $\displaystyle\int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \; dx \; dy = \int_0^{\sqrt{2}} y\left[x\right]_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} dy$

$$= \int_0^{\sqrt{2}} y\left(\sqrt{4-2y^2}+\sqrt{4-2y^2}\right) dy$$

$$= \int_0^{\sqrt{2}} y\sqrt{4-2y^2} \; dy$$

$$= -\frac{1}{2}\int_0^{\sqrt{2}} y\left(4-2y^2\right)^{\frac{1}{2}} (-4y \; dy)$$

$$= -\frac{1}{2}\left[\frac{\left(4-2y^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{\sqrt{2}}$$

$$= -\frac{1}{3}\left[0-4^{\frac{3}{2}}\right]$$

$$\therefore \qquad \int_0^{\sqrt{2}} \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y \; dx \; dy = \frac{8}{3} \qquad \text{....(1)}$$

Again, $\int_{-2}^{2}\int_{0}^{\sqrt{\frac{4-x^2}{2}}} y\ dy\ dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_0^{\sqrt{\frac{4-x^2}{2}}} dx$

$= \frac{1}{2}\int_{-2}^{2}\left(\frac{4-x^2}{2}\right) dx$

$= \frac{1}{2}\int_{0}^{2}\left(4-x^2\right) dx$

$= \frac{1}{2}\left[4x-\frac{x^3}{3}\right]_0^2$

$= \frac{1}{2}\left[8-\frac{8}{3}\right]$ - 0

$\therefore \qquad \int_{-2}^{2}\int_{0}^{\sqrt{\frac{4-x^2}{2}}} y\ dx\ dy = \frac{8}{3}$ ....(2)

From (1) and (2), we get

$$\int_{0}^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\ dx\ dy == \int_{-2}^{2}\int_{0}^{\sqrt{\frac{4-x^2}{2}}} y\ dx\ dy = \frac{8}{3}$$

Example 4 : Change the order of integration in the integral $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{dy\,dx}{(1+e^y)\sqrt{1-x^2-y^2}}$ and evaluate it.

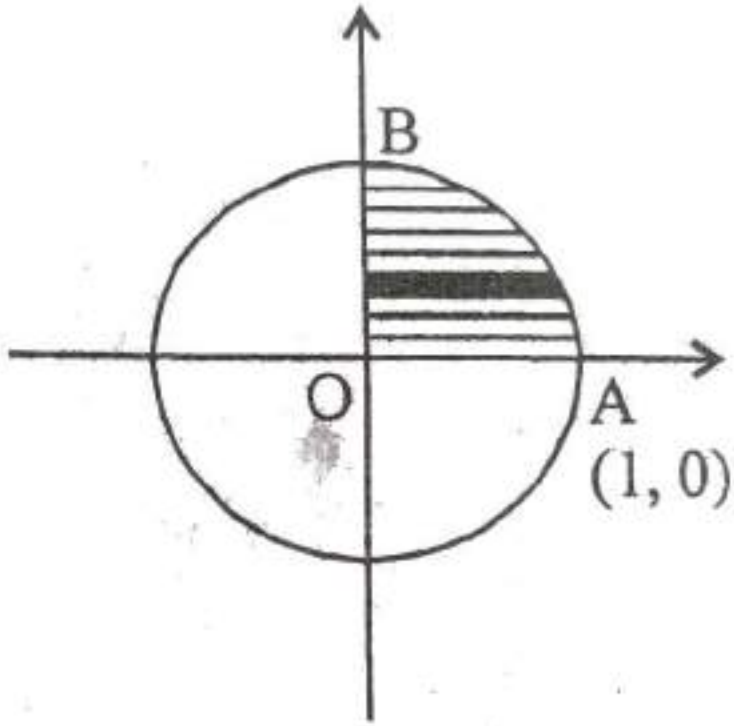
Sol: Let I = $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} \frac{dy\,dx}{(1+e^y)\sqrt{1-x^2-y^2}}$

Here integration is done first w.r.t. y and later w.r.t. x. So y varies from 0 to $\sqrt{1-x^2}$ while x varies from 0 to 1. Now y = $\sqrt{1-x^2}$ = 1, which is circle with centre at (0, 0) and radius 1.

So domain of integration OAB is shaded as shown in figure and is bounded by unit circle in the first quadrant.

When the order of integration is changed integrate first w.r.t. and later w.r.t. y.

For this, consider a horizontal strip whose length varies from 0 to $\sqrt{1-y^2}$ and width y varies from 0 to 1.



Thus the given integral can be written with change of order of integration as

$$I = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \frac{dx}{(1+e^y)\sqrt{1-x^2-y^2}} \right) dy$$

$$= \int_0^1 \frac{1}{1+e^y} \left(\int_0^{\sqrt{1-y^2}} \frac{dx}{(1+y^y)-x^2} \right) dy$$

$$= \int_0^1 \frac{1}{1+e^y} \left(\sin^{-1} \frac{x}{\sqrt{1-y^2}} \right)_0^{1-y^2} dy$$

$$= \int_0^1 \frac{1}{1+e^y} \left(\sin^{-1} \frac{\sqrt{1-y^2}}{\sqrt{1-y^2}} - \sin^{-1} \frac{0}{\sqrt{1-y^2}} \right) dy$$

$$= \int_0^1 \frac{1}{1+e^y} (\sin^{-1} 1 - \sin^{-1} 0)\, dy$$

$$=\frac{\pi}{2}\int_0^1 \frac{dy}{1+e^y}$$

$$=\frac{\pi}{2}\int_0^1 \frac{e^{-y}dy}{e^{-y}+1}$$

$$= -\frac{\pi}{2}\left\{\log(e^{-y}+1)\right\}_0^1 \qquad \left(U\sin g\int\frac{f'(y)}{f(y)}dy=\log f(y)\right)$$

$$= -\frac{\pi}{2}\left(\log(e^{-1}+1)-\log(e^0+1)\right)$$

$$= -\frac{\pi}{2}\left(\log\left(\frac{1+e}{e}\right)-\log 2\right)$$

$$= \frac{\pi}{2}\left(\log 2-\log\frac{1+e}{e}\right)$$

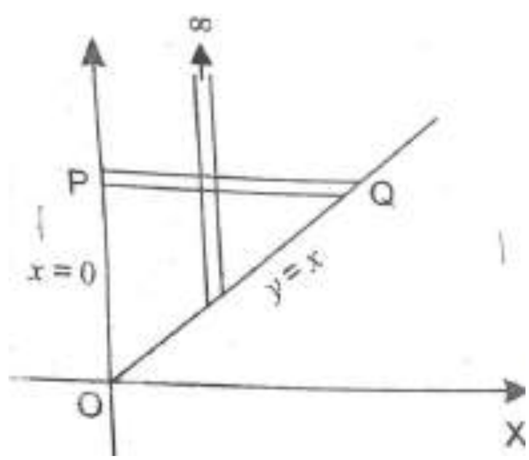
$$=\frac{\pi}{2}\log\left(\frac{2e}{1+e}\right)$$

Example 5: Evaluate after changing the order of integration.

$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,\text{dy dx}$$

Sol: Given integral is

$$I = \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,\text{dy dx}$$



Here y varies from x to ∞ along vertical strip and x varies from 0 to ∞.

After changing the order of integration the limit of x becomes x = 0 to x = y along horizontal strip PQ and y varies from y = 0 to y = ∞.

$$\therefore \qquad I = \int_0^{\infty}\int_{y=x}^{\infty} \frac{e^{-y}}{y} \text{ dy dx}$$

$$= \int_0^{\infty} \left[\int_0^{y} \frac{e^{-y}}{y} dx\right] \text{dy}$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} \left[x\right]_0^{y} \text{dy}$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} (y - 0) \text{ dy}$$

$$= \int_0^{\infty} e^{-y} \text{ dy}$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= (e^{-\infty} - e^{0})$$

$$= 1$$

Example 6: Change the order of integration and evaluate the integral

$$\int_0^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \text{ dy dx}$$

Sol: Given integral is $\int_0^{a}\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}}$ dy dx

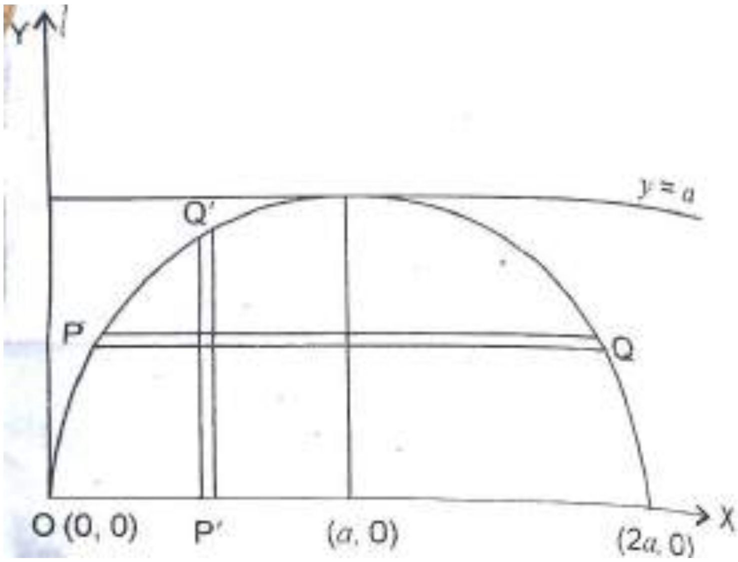
Here x varies from a-$\sqrt{a^2-y^2}$ to a+$\sqrt{a^2-y^2}$ and y varies from 0 to a.

As x varies from a-$\sqrt{a^2-y^2}$ to a+$\sqrt{a^2-y^2}$

$\Rightarrow \qquad x = a + \sqrt{a^2-y^2} \Rightarrow (x - a)^2 = a^2 - y^2$

$\Rightarrow \qquad x^2 + y^2 - 2ax = 0$

which is the circle with centre at (a, 0) and radius = a



This integration is performed along the horizontal strip PQ. Now change the strip horizontal to vertical strip P'Q', then y varies from 0 to $\sqrt{2ax-x^2}$ and x varies from 0 to 2a.

$\therefore$ $$I = \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dy\, dx$$

$$= \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} dy\, dx$$

$$= \int_0^{2a} \left[\int_0^{\sqrt{2ax-x^2}} dy \right] dx$$

$$= \int_0^{2a} \sqrt{2ax-x^2}\, dx$$

$$= \int_0^{2a} \sqrt{a^2-(x-a)^2}\, dx$$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} \right]_0^{2a}$$

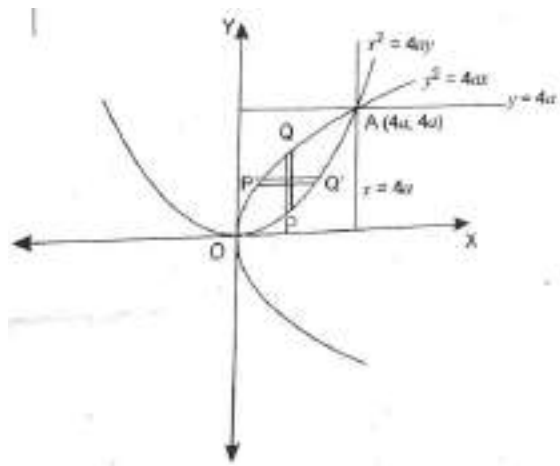
$$= \frac{a^2}{2}\,\frac{\pi}{2} - \frac{a^2}{2}\left(-\frac{\pi}{2}\right)$$

$= \dfrac{a^2\pi}{2}$

Example 7: Change the order of integration and evaluate the integral:

$$\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$$

Sol: Given integral is $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx$



Here y varies from $\dfrac{x^2}{4a}$ to $2\sqrt{ax}$ and k varies from 0 to 4a

Here integration is carried along the vertical strip PQ. When we change the order of integration the vertical strip must be changed to horizontal strip P'Q' where x varies from x = $\dfrac{y^2}{4a}$ to x = $2\sqrt{ay}$ and y varies from 0 to 4a. The point of intersection of the curves is A(4a, 4a).

$\therefore \quad I = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\, dx = \int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dx\, dy$

$= \int_0^{4a} [x]_{y^2/4a}^{2\sqrt{ay}}\, dy$

$= \int_0^{4a} \left[2\sqrt{ay} - \dfrac{y^2}{4a}\right] dy$

$$= \left[2\sqrt{a}\, \frac{y^{3/2}}{3/2} - \frac{y^3}{3.4a} \right]_0^{4a}$$

$$= \frac{4}{3}\, \sqrt{a}\, (4a)^{3/2} - \frac{64a^3}{12a}$$

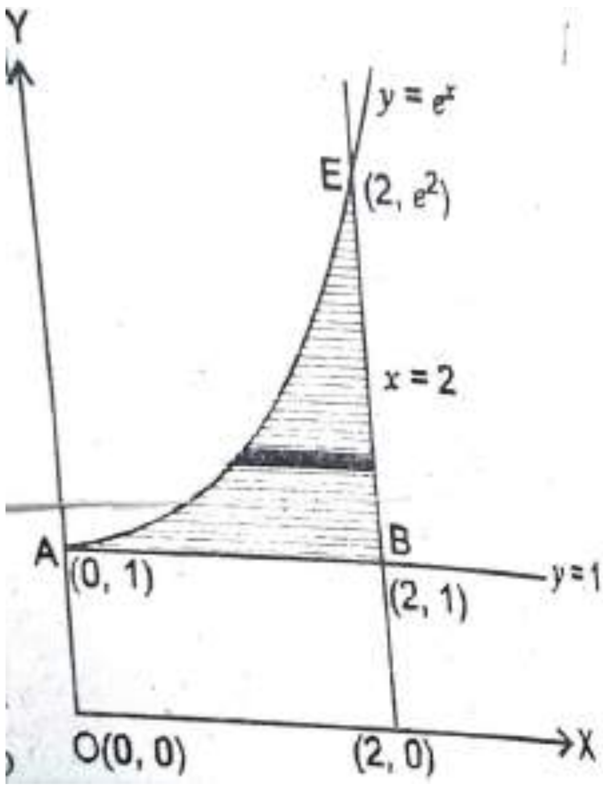
$$= \frac{32}{3} a^2 - \frac{16a^2}{3}$$

$$= \frac{16a^2}{3}$$

Example 8: Change the order of integration and then evaluate the double integral

$$\int_0^2 \int_1^{e^x} dy\, dx$$

Sol: Here integration is done first w.r.t. y and later w.r.t x. So y varies from 1 to ex while x varies from 0 to 2



Thus the domain of integration ABE is shaded as shown in figure, and is bounded by the curve $y = e^x$ and the straight lines y = 1, x = 0 and x = 2.

Here vertices of region are A(0), B(2,1), E(2, e^2). When the order of integration is changed, integrate first w.r.t. x and later w.r.t. y. For this, consider a horizontal strip whose length x varies from x = log y to x = 2 and width y varies from y = 1 to y = e^2

Thus the given integral can be written with change of order of integration as

$$\int_0^2 \int_1^{e^x} dy\,dx = \int_1^{e^2} \left[\int_{x=\log y}^{2} dx \right] dy$$

$$= \int_1^{e^2} [x]_{\log y}^{2}\, dy$$

$$= \int_1^{e^2} [2 - \log y]\, dy$$

$$= \left[2y - y\log y + y\right]_1^{e^2}$$

$$= \left[3y - y\log y\right]_1^{e^2}$$

$= (3e^2 - e^2 \log e^2) - (3 - \log 1)$

$= 3e^2 - 2e^2 - 3$

$= e^2 - 3$

Example 9: Write an equivalent double integral with order of integration reversed for

$$\int_{-2a}^{a} \int_{x^2/a}^{2a-x} f(x, y)\, dy\, dx$$

Hence evaluate for $f(x, y) = 1$

Sol: Given integral is $\displaystyle\int_{-2a}^{a} \int_{x^2/a}^{2a-x} f(x, y)\, dy\, dx$

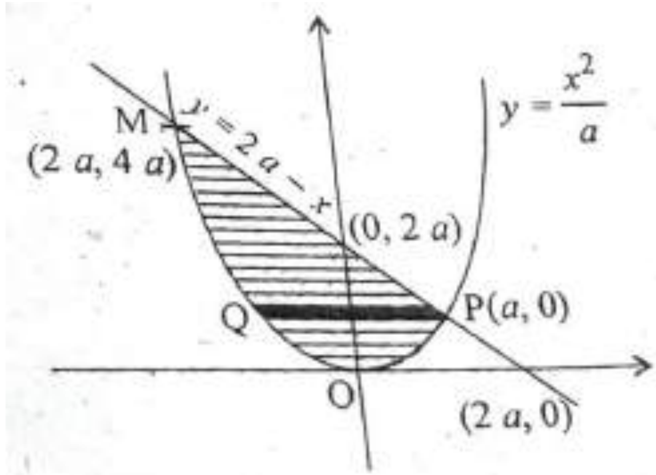
The region of integration is

$$\left\{(x, y) : -2a \le x \le a, \frac{x^2}{a} \le y \le 2a - x\right\}$$

which is bounded by vertical strips between

y = $\dfrac{x^2}{a}$ and

y = 2a - x and



Shown by shaded region in the figure.

To change order of integration, we change vertical to horizontal strips. The region is divided into two regions R_1, R_2 represented by region POQ and PQM respectively in figure.

where $R_1 = \left\{(x, y) \mid 0 \le y \le a, -\sqrt{ay} \le x \le \sqrt{ay}\right\}$

and $R_2 = \left\{(x, y) \mid a \le y \le 4a, -\sqrt{ay} \le x \le 2a - y\right\}$ $\left(\begin{array}{cccc} \because & y & = & \dfrac{x^2}{a} \\ \Rightarrow & x^2 & = & ay \\ \Rightarrow & x & = & \pm\sqrt{ay} \end{array}\right)$

$\therefore$ $\quad I = \int_0^a \int_{-\sqrt{ay}}^{\sqrt{ay}} f(x, y)\, dx\, dy + \int_a^{4a} \int_{-\sqrt{ay}}^{2a-y} f(x, y)\, dx\, dy$

IInd Part : When $f(x, y) = 1$

Then $I = \int_0^a \int_{-\sqrt{ay}}^{\sqrt{ay}} 1\, dx\, dy + \int_a^{4a} \int_{-\sqrt{ay}}^{2a-y} f(x, y)\, dx\, dy = \int_0^a (x)_{-\sqrt{ay}}^{\sqrt{ay}}\, dy + \int_0^{4a} (x)_{-\sqrt{ay}}^{2a-y}\, dy$

$= \int_0^a \left(\sqrt{ay} + \sqrt{ay}\right) dy + \int_0^{4a} \left(2a - y + \sqrt{ay}\right) dy$

$= \left(2\sqrt{a}\, \frac{y^{3/2}}{3/2}\right)_0^a + \left(2ay - \frac{y^2}{2} + \sqrt{a}\, \frac{y^{3/2}}{3/2}\right)_a^{4a}$

$$= \left(\frac{4}{3}\sqrt{a}(y^{3/2})-0\right)+\left(48a^2-8a^2+\frac{2}{3}\sqrt{a}(8a^{3/2})\right)-\left(2a^2-\frac{a^2}{2}+\frac{2}{3}a^2\right)$$

$$= \frac{4}{3}a^2 + \frac{16}{3}a^2 - 2a^2 + \frac{a^2}{2} - \frac{2}{3}a^2 = \frac{a}{2}a^2$$

Self-Check Exercise-1

Q.1 Evaluate $\int_0^1 \int_{4y}^4 ex^2\, dx\, dy$ by changing the order of integration.

Q.2 Change the order of integration of $\int_{-a}^{a} \int_{\frac{1}{2}\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy\, dx$.

Hence evaluate it when $f(x, y) = 1$

Q.3 Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} f(x, y)\, dy\, dx$

when $f(x, y) = xy$

Q.4 Evaluate after changing the order of integration.

$$\int_0^a \int_{y^2/a}^{y} \frac{y\, dx\, dy}{(a-x)\sqrt{ax-y^2}}$$

18.4 Change of Variables In Double Integral

If A is a simple closed subregion of the xy-plane which is mapped into the region B of the uv-plane by the transformations

$x = \phi(u, v)$, $y = \psi(u, v)$

and

(i) ϕ, ψ have continuous partial derivatives on B.

(ii) $$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

$\forall$ (u, v) EB

then $\iint_A f(x, y)\, dx\, dy = \iint_B f\left(\phi(u,v),\psi(u,v)\right) |J|\, du\, dv$

We accept this result without proof.

Particular Case : Change to Polar Coordinates

Here x = r cos θ, y = r sin θ

$$\therefore \quad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore \quad \iint_A f(x, y)\,dx\,dy = \iint_A f(r\cos\theta, r\sin\theta)\,r\,d\theta\,dr$$

The following examples will illustrate the idea more clearly:-

Example 10: Evaluate $\iint \sin\pi(x^2 + y^2)\,dx\,dy$ over the circle $x^2 + y^2 \leq 1$

Sol: Here the region is A = $\left\{(x, y)x^2 + y^2 \leq 1\right\}$

Changing to polar co-ordinates by x = r cos θ, y = r sin θ, it becomes

A = $\left\{(r, \theta); 0 \leq r \leq 1; 0 \leq \theta \leq 2\pi\right\}$ where $x^2 + y^2 = r^2$

$$\therefore \quad \iint_A \sin\pi(x^2 + y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 \sin(\pi r^2) r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta . \frac{1}{2\pi}\int_0^1 \sin\pi r^2 . 2\pi r\,dr$$

$$= 2\pi . \frac{1}{2\pi}\left[-\cos\pi r^2\right]_0^1$$

$$= -[\cos\pi - \cos 0]$$

$$= -[-1 - 1] = 2$$

Example 11: Evaluate $\iint_A \frac{1}{xy}dx\,dy$ where A is the region bounded by the four circles $x^2 + y^2$ = ax, a_1x, by, b1y such that $a_1 > a > 0$ and $b_1 > b > 0$.

Sol: Let I = $\iint_A \frac{1}{xy}dx\,dy$ where A is the region bounded by the four circles $x^2 + y^2$ = ax, a_1x, by, b_1y such that $a_1 > a > 0$ and $b_1 > b > 0$.

Put $\quad \frac{x^2 + y^2}{x} = u$ and $\frac{x^2 + y^2}{y} = v$

Then the transformed region A' in UV-plane is

A' = $\{(u,v): a \le u \le a_1, b \le v \le b_1\}$

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1-\frac{y^2}{x^2} & \frac{2y}{x} \\ \frac{2x}{y} & 1-\frac{x^2}{y^2} \end{vmatrix}$

$= \left(1-\frac{y^2}{x^2}\right)\left(1-\frac{x^2}{y^2}\right) - \frac{2x}{y} \cdot \frac{2y}{x}$

$= \frac{(x^2+y^2)^2}{x^2y^2} - 4$

$= -\left[\frac{(x^2+y^2)+4x^2y^2}{x^2y^2}\right]$

$= -\frac{(x^2+y^2)^2}{x^2y^2}$

$= -\frac{x^2+y^2}{x} \cdot \frac{x^2+y^2}{y} \cdot \frac{1}{xy} = \frac{uv}{xy}$

$\therefore$ $|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{xy}{uv}$

$\therefore$ $I = \iint_{A} \frac{1}{xy} \cdot \frac{xy}{uv}$ du dv

$= \int_b^{b_1} \left(\int_a^{a_1} \frac{1}{u} du\right) \frac{1}{v}$ dv

$= \int_b^{b_1} [\log u]_a^{a_1} \frac{1}{v}$ dv

= (log a_1 - log a) $\int_b^{b_1} \frac{1}{v}$ dv

= log $\frac{a_1}{a}$ $[\log v]_b^{b_1}$

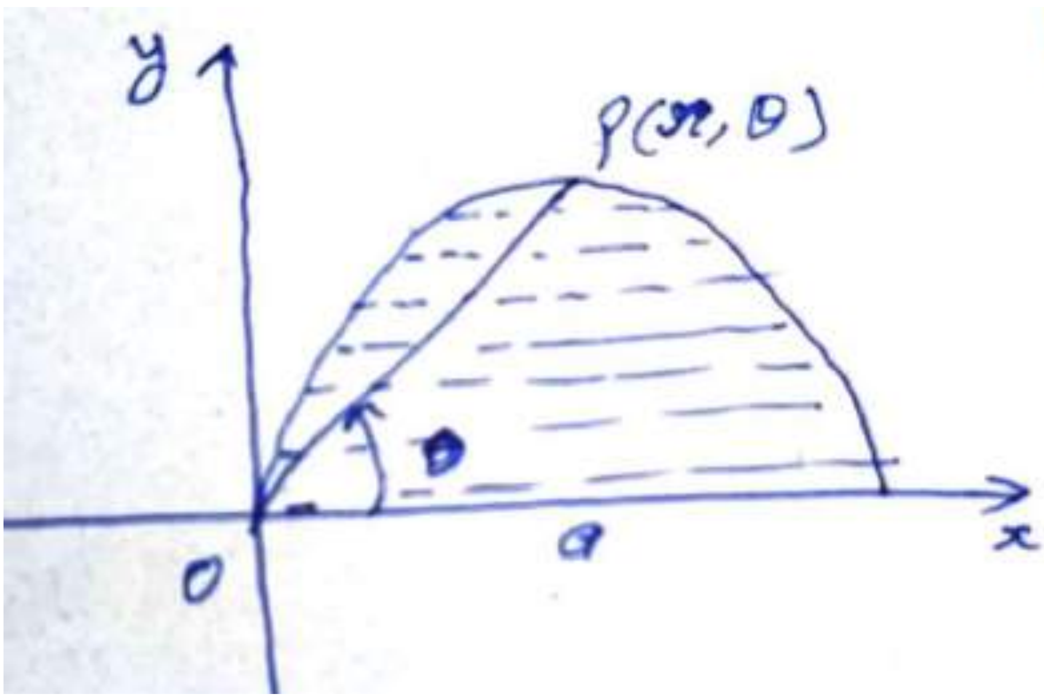
$= \log \dfrac{a_1}{a}$ (log b_1 - log b)

$= \log \dfrac{a_1}{a} \log \dfrac{b_1}{b}$

Example 12: Evaluate $\iint \sqrt{a^2 - x^2 - y^2}$ dx dy over the circle $x^2 + y^2 \leq ax$ in the positive quadrant where a > 0

Sol: Consider the circle $x^2 + y^2 = ax$ or $x^2 + y^2 - ax = 0$

Its centre is $\left(\dfrac{a}{2}, 0\right)$ and radius = $\dfrac{a}{2}$



Put $x = r \cos \theta$, $y = r \sin \theta$

$\therefore$ given circle becomes re $\cos^2 \theta + r^2 \sin^2 \theta = ar \cos \theta$

i.e. $r = a \cos \theta$

$\therefore$ region of integration A is shown in the figure and for this region θ varies from 0 to $\pi/2$ and r varies from 0 to a cos θ

$\therefore \quad \iint\limits_A \sqrt{a^2 - x^2 - y^2}$ dx dy = $\int_0^{\pi} \int_0^{a\cos\theta} \sqrt{a^2 - r^2 \cos^2 \theta - r^2 \sin^2 \theta}$. rdr dθ

$$= \int_0^{\pi/2} \left[\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\, r dr \right] d\theta$$

$$= \int_0^{\pi/2} \left[-\frac{1}{2} \int_0^{a\cos\theta} (a^2 - r^2)^{\frac{1}{2}} (-2r) dr \right] d\theta$$

$$= -\frac{1}{2} \left[\frac{(a^2 - r^2)^{\frac{3}{2}}}{3/2} \right]_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left[(a^2 - a^2\cos^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}} \right] d\theta$$

$$= -\frac{1}{3} a^3 \int_0^{\pi/2} (\sin^3\theta - 1)\, d\theta$$

$$= -\frac{a^3}{3} \int_0^{\pi/2} \sin 3\theta\, d\theta + \frac{a^3}{3} \int_0^{\pi/2} 1\, d\theta$$

$$= -\frac{a^3}{3} \left[\frac{2}{3.1} \right] + \frac{a^3}{3} [\theta]_0^{\pi/2}$$

$$= -\frac{2a^3}{9} + \frac{a^3\pi}{6}$$

$$= \frac{a^3}{6} \left(\pi - \frac{4}{3} \right)$$

Example 13: Evaluate $\iint \sqrt{\dfrac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\, dx\, dy$

over the positive quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Sol: Put $\frac{x}{a} = u$, $\frac{y}{b} = v$

i.e. x = au, y = bv

$\therefore$ dx = adu, dy = bdv

$\therefore$ ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ = 1 in the positive quadrant of xy-plane transforms into the circle $u^2 + v^2$ = 1 in the positive quadrant of the uv-plane.

$$\therefore \quad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

$$\therefore \quad \iint \sqrt{\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}}\, dx\, dy = \iint_A \sqrt{\frac{1-u^2-v^2}{1+u^2+v^2}}\, ab\, du\, dv$$

where A is the region $u \geq 0$, $v \geq 0$, $u^2 + v^2 \leq 1$

Put $u = r\cos\theta$, $v = r\sin\theta$

$$\therefore \quad J = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r$$

Also $A = \left\{(r,\theta) : 0 \leq r \leq 1, 0 \leq \theta \leq \frac{\pi}{2}\right\}$

$$\therefore \quad \iint \sqrt{\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}{1+\frac{x^2}{a^2}+\frac{y^2}{b^2}}}\, dx\, dy = \int_0^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2\cos^2\theta-r^2\sin^2\theta}{1+r^2\cos^2\theta+r^2\sin^2\theta}}\, ab.r\, dr\, d\theta$$

$$= ab\int_0^{\pi/2} d\theta \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r dr$$

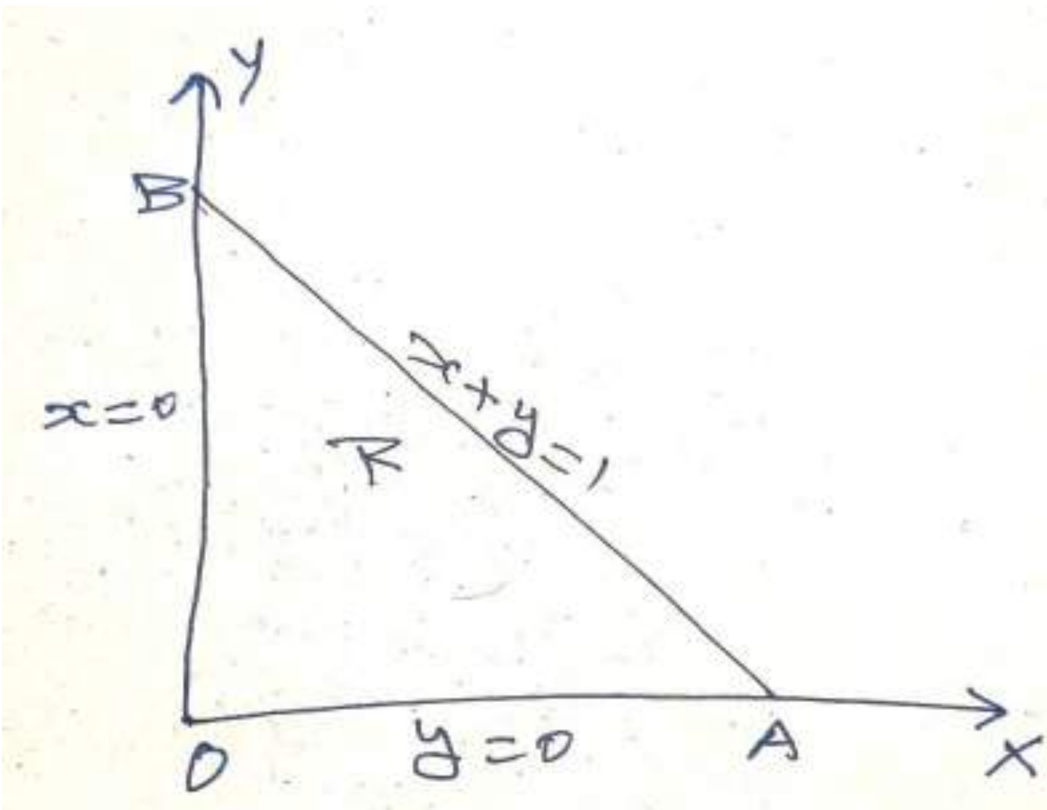
$= ab\frac{\pi}{2}.\frac{1}{2}\left(\frac{\pi}{2}-1\right)$

$= \frac{\pi ab}{8}(\pi - 2)$

Example 14: Using the transformation x + y = u, y = uv, show that

$\iint \sqrt{xy(1-x-y)}\, dx\, dy = \frac{2\pi}{105}$, where the integration being taken over the area of the triangle bounded by the lines x = 0, y = 0, x + y = 1

Sol: Given triangle is x = 0, y = 0, x + y = 1



Given transformation is x + y = u, and y = uv

$\Rightarrow$ x + uv = u

$\therefore$ x = u (1 - v) ....(1)

and y = uv ....(2)

when x = 0, from (1), we get

u = 0, v = 1

when y = 0, from (2), we get

u = 0, v = 0

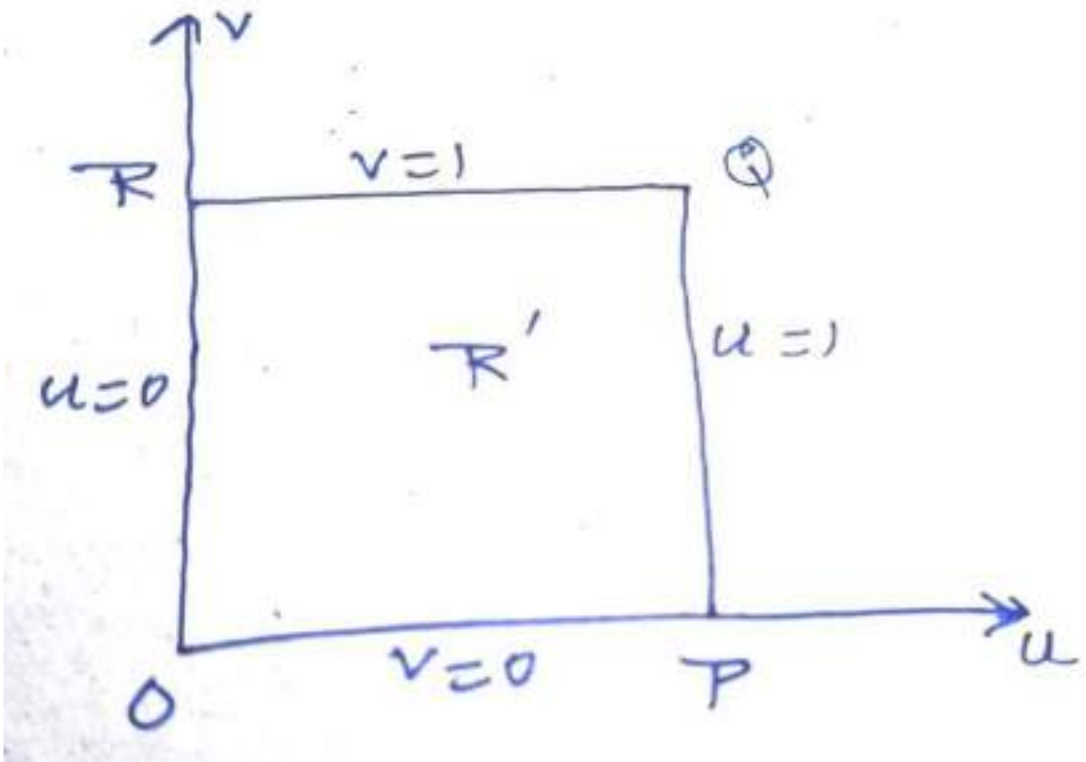
when x + y = 1 ⇒ u - uv + uv = 1 ⇒ u = 1

which is a square in uv-plane as shown.

Now

$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

= u - uv + uv = u



∴ $I = \iint_R \sqrt{xy(1-x-y)}\, dx\, dy$

$= \iint_R \sqrt{u(1-v)uv(1-u)}\, u\, du\, dv$

$= \int_0^1 \int_0^1 \sqrt{u^2(1-u)v(1-v)}\, u\, du\, dv$

$= \int_0^1 u^2\sqrt{1-u}\,du \int_0^1 \sqrt{v(1-v)}\,dv$

$= I_1 \times I_2$; where $I_1 = \int_0^1 u^2\sqrt{1-u}\,du$

and $I_2 = \int_0^1 \sqrt{v(1-v)}\,dv$

Now

$I_1 = (1-u)^{\frac{1}{2}}\,du$

[Put $u = \sin^2\theta \therefore du = 2\sin\theta\cos\theta\,d\theta$]

$\therefore \quad I_1 = \int_0^{\pi/2} \sin^4\theta\cos\theta\, 2\sin\theta\cos\theta\,d\theta$

$= \int_0^{\pi/2} 2\sin^5\theta\cos^2\theta\,d\theta$

$= 2.\dfrac{4.2.1}{7.5.3.1}$

$= \dfrac{16}{105}$

and $\quad I_2 = \int_0^1 v\sqrt{1-v}\,dv$

$= \int_0^1 \sqrt{\left(\frac{1}{2}\right)^2 - \left(v-\frac{1}{2}\right)^2}\,dv$

$$= \left[\frac{\left(v-\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2-\left(v-\frac{1}{2}\right)^2}}{2} + \frac{\left(\frac{1}{2}\right)^2}{2}\sin^{-1}\left\{\frac{\left(v-\frac{1}{2}\right)}{\frac{1}{2}}\right\}\right]_0^1$$

$= \left[0+\frac{1}{8}.\frac{\pi}{2}-\frac{1}{8}\left(-\frac{\pi}{2}\right)\right] = \dfrac{\pi}{8}$

$\therefore \quad I = \frac{16}{105} \times \frac{\pi}{8}$

$= \frac{2\pi}{105}$

Self-check Exercise-2

Q. 1 Change into polar coordinates and evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\, dx$

Q. 2 Evaluate $\iint_A (x^2 + y^2)\, dx\, dy$, where A is the region bounded by the four hyperbolas $x^2 - y^2 = 2, 9$ and $xy = 2,4$.

Q. 3 Evaluate $\iint_E \sin\left(\frac{x-y}{x+y}\right) dx\, dy$, where E is the region bounded by the coordinate axes and $x + y = 1$ in the first quadrant.

18.5 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Discussed change of order of integration.
2. Solved questions related to change of order of integration in double integral.
3. Discussed the change of variables in double integral. Also discussed the particular case of change to polar coordinates.
4. Solved questions of double integrals by change of variables.

18.6 Glossary

1. While evaluating double integrals if the limits of integration are variables, then change of order of integration changes the limits of integration.
2. If A is a simple closed subregion of the xy-plane which is mapped into the region B of the uv-plane by the transformations $x = \phi(u, v)$, $y = \psi(u, v)$ and (i) ϕ, ψ have continuous partial derivatives on B, (ii) $J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \neq 0, \forall (u, v) \in B$

   Then $\iint_A f(x, y)\, dx\, dy = \iint_B f(\phi(u, v), \psi(u, v))\, |J|\, du\, dv$.

18.7 Answer To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\dfrac{e^{16}-1}{18}$

Ans. 2 $\displaystyle\int_{-a}^{a}\int_{\frac{1}{2}\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y)\, dy\, dx = \int_{0}^{a/2}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-4y^2}} f(x, y)\, dx\, dy +$

$\displaystyle + \int_{0}^{a/2}\int_{-\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy + \int_{a/2}^{a/2}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x, y)\, dx\, dy$

and

$\dfrac{\pi}{2}$

Ans. 3 $\displaystyle\int_{0}^{1}\int_{x^2}^{2-x} f(x, y)\, dy\, dx = \int_{0}^{1}\int_{0}^{\sqrt{y}} f(x, y)\, dx\, dy + \int_{1}^{2}\int_{0}^{2-y} f(x, y)\, dx\, dy$

and $\dfrac{3}{8}$

Ans. 4 $\dfrac{\pi a}{2}$

Self-Check Exercise-2

Ans. 1 $\dfrac{\pi}{4}$

Ans. 2 7

Ans. 3 0

18.8 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, Delhi, 2005
2. H. Anton, I. Bivens and S. Davis, *Calculus*, John Wiley and Sons (Asia) P. Ltd, 2002

18.9 Terminal Questions

1. Evaluate $\displaystyle\int_{0}^{\infty}\int_{0}^{x}$ x $e^{-x^2/y} dy$ dx by change of order of integration.

2. Evaluate after changing the order of integration:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \text{ - } dy\, dx$$

3. Evaluate after changing the order of integration:

$$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$$

4. Change the order of integration and then evaluate the double integral

$$\int_{-2}^{1} \int_{x^2+4x}^{3x+2} dy\, dx$$

5. By changing into Cartesian coordinates evaluate $\int_0^{\pi} \int_0^{a}$ $r^3 \sin\theta \cos\theta$ dr dr

6. Evaluate $\iint_A$ x^2 dx dy, where A is the region enclosed be the four parabolas y^2 = 4ax, y^2 = bx, x^2 = cy, x^2 = dy where a, b, c, d are positive reals. (a < b, c < d)

7. Show that $\iint_A \dfrac{dxdy}{4-x^2-y^2} = \pi \log 3$ over the region A between the concentric circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 3$

Unit - 19

Triple Integral

Structure

19.1 Introduction

Triple integration is a mathematical technique used to calculate the volume of three-dimensional regions and evaluate various quantities within those regions. It extends the concept of integration from one dimension (single integration) and two dimensions (double integration) to three dimensions. In triple integration, we integrate a function over a three-dimensional region in space. This region can be described using Cartesian, Cylindrical or Spherical coordinates, depending on the nature of the problem and the symmetry of the region. The general form of a triple integral is given by:-

$$\iiint f(x, y, z)\, dv$$

Here $f(x, y, z)$ represents the integrand, which is the function being integrated and dv represents an infinitesimal volume element. The triple integral is performed over the region of interest. To evaluate a triple integral, we divide the three-dimensional region into small volume elements, calculate the contribution of each element, and sum them up over the entire region. This process involves setting up the limits of integration for each variable and applying appropriate coordinate transformations if necessary.

19.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss triple integration and solve questions related to it.
- Discuss change of variables in triple integral.
- Discuss change to cylindrical co-ordinates.
- Discuss change to spherical co-ordinates.
- Solve questions related to change of variables in triple integral.

19.3 Triple Integration

Let $f(x, y, z)$ be a continuous function of three independent variables x, y and z, defined over a closed and bounded region enclosing a volume V in R^3. Divide the Region into a number of parallelepipeds by drawing planes parallel to the co-ordinate planes inside the Region enclosing volumes δV_1, δV_2,......, δV_n, then the sum;

$$\underset{n\to\infty}{Lt}\ \underset{\substack{n\to\infty\\ \delta V_r\to 0}}{Lt} f(x_r,y_r,z_r)\,\delta V_r \to \iiint_V f(x, y, z)\, dV \text{ or } \iiint_V f(x, y, z)\, dx\, dy\, dz$$

For evaluation, it can be expressed as the repeated integral

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y, z)\, dx\, dy\, dz \qquad .....(1)$$

and the order of integration depends upon the limits.

Let $z_1 = f_1(x, y)$ and $z_2 = f_2(x, y)$; $y_1 = \phi_1(x)$ and $y_2 = \phi_2(x)$ and x_1, x_2 be constants say $x_1 = a$, $x_2 = b$.

Then (1) can be written as

$$\int_{x=a}^{x=b}\left[\int_{y=\phi_1(x)}^{y=\phi_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)dz\right)dy\right]dx$$

i.e. we integrate $f(x, y, z)$ first w.r.t. z (treating x, y as constants), then resulting expression w.r.t. y (keeping x constant) and finally w.r.t. x.

When x_1, x_2, y_1, y_2 and z_1, z_2 are constant then the order of the integration is immaterial i.e.

$$\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x, y,z)\, dx\, dy\, dz = \int_{x_1}^{x_2}\int_{z_1}^{z_2}\int_{y_1}^{y_2} f(x, y, z)\, dy\, dz\, dx$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x, y, z)\, dz\, dy\, dx$$

Let us consider the following examples to clear the idea:

Example 1: Evaluate $\int_1^e \int_0^{\log y} \int_1^{e^x}$ log z dz dx dy

Sol: Let I = $\int_1^e \int_0^{\log y} \int_1^{e^x}$ log z dz dx dy

$\therefore$ I = $\int_1^e \left\{ \int_0^{\log y} \left(\int_0^{e^x} \log z dz \right) dx \right\}$ dy .....(1)

Now $\int_0^{e^x}$ log z dz = $\int_0^{e^x}$ log z, 1 dz

= $\left[\log z.z \right]_1^{e^x}$ - $\int_0^{e^x} \frac{1}{z}$ .z dz

= (e^x log e^x - log 1) - $\int_0^{e^x}$ 1 dz

x e^x - $\left[z \right]_1^{e^x}$

= x e^x - (e^x - 1)

= x e^x - e^x + 1

$\therefore$ from (1), we get

I = $\int_1^e \left\{ \int_0^{\log y} \left(xe^x - e^x + 1 \right) dx \right\} dy$

= $\int_1^e \left[x\ e^x - e^x - e^x + x \right]_0^{\log y} dy$

= $\int_1^e \left[\left(x - 2 \right)\ e^x + x \right]_0^{\log y} dy$

$$\left[\because \quad \int x\ e^x dx = xe^x - \int 1.e^x dx = x\ e^x - e^x \right]$$

= $\int_1^e \left\{ \left(\log y - 2 \right)\ e^{\log y} + \log y - (-2) \right\}\ dy$

$$= \int_1^e \left\{ y\left(\log y - 2\right) + \log y + 2 \right\} dy$$

$$= \int_1^e \left[(y+1) - 2y + 2 \right] dy$$

$$= \int_1^e (y+1).\log y \; dy - 2\int_1^e y \; dy + 2\int_1^e 1.dy$$

$$= \left[\log y.\left(\frac{y^2}{2} + y \right) \right]_1^e - \int_1^e \frac{1}{y}.\left(\frac{y^2}{2} + y \right) dy - 2\left[\frac{y^2}{2} \right]_1^e + 2\left[y \right]_1^e$$

$$= \left(\frac{e^2}{2} + e \right). \text{ log e} - 0 - \int_1^e \left(\frac{1}{2}y + 1 \right) dy - (e^2 - 1) + 2(e-1)$$

$$= \left(\frac{e^2}{2} + e \right).1 - \left[\frac{y^2}{4} + y \right]_1^e - (e^2 - 1) + 2(e-1)$$

$$= \frac{e^2}{2} + e - \left[\left(\frac{e^2}{2} + e \right) - \left(\frac{1}{4} + 1 \right) \right] - (e^2 - 1) + 2(e-1)$$

$$= \frac{e^2}{2} + \frac{e^2}{4} - e + \frac{5}{4} - e^2 + 1 + 2e - 2$$

$$= -\frac{3e^2}{4} + 2e + \frac{1}{4}$$

$$= \frac{1}{4}(1 + 8e - 3e^2)$$

Example 2 : Evaluate $\iiint (x + y + z)$ dx dy dz over the tetrahedron bounded by the planes x = 0, y = 0, z = 0, x + y + z = 1

Sol. The tetrahedron is bounded by the planes

x = 0, y = 0, z = 0, x + y + z = 1

$\because$ x + y + z $\leq$ 1

$\therefore$ x $\leq$ 1, x + y $\leq$ 1, x + y + z $\leq$ 1

$\therefore$ x $\leq$ 1, y $\leq$ 1 – x, z $\leq$ 1 – x – y

$\therefore$ V = {(x, y, z) : 0 $\leq$ x $\leq$ 1, 0 $\leq$ y $\leq$ 1 – x, 0 $\leq$ z $\leq$ 1 – x – y}

$\therefore \qquad \iiint\limits_V (x+y+z)\ dx\ dy\ dz = \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} (x+y+z)\ dz \right) dy \right\} dx$

$$= \int_0^1 \left\{ \int_0^{1-x} \left[\frac{(x+y+z)^2}{2} \right]_0^{1-x-y} dy \right\} dx$$

$$= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-x} \left[(x+y+1-x-y)^2 - (x+y)^2 \right] dy \right\} dx$$

$$= \frac{1}{2} \int_0^1 \left\{ \int_0^{1-x} \left[1-(x+y)^2 \right] dy \right\} dx$$

$$= \frac{1}{2} \int_0^1 \left[y - \frac{(x+y)^3}{3} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 \left[\left(1-x-\frac{1}{3} \right) - \left(0 - \frac{x^3}{3} \right) \right] dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{x^3}{3} - x + \frac{2}{3} \right) dx$$

$$= \frac{1}{2} \left[\frac{x^4}{12} - \frac{x^2}{2} + \frac{2}{3} x \right]_0^1$$

$$= \frac{1}{2} \left[\left(\frac{1}{12} - \frac{1}{2} + \frac{2}{3} \right) - 0 \right]$$

$$= \frac{1}{2} \left[\frac{1-6+8}{12} \right] = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

Example 3 : Evaluate $\iiint\limits_{x^2+y^2+z^2 \le 1} (a\ x + b\ y + c\ z)^2$ dx dy dz

Sol. Since $x^2 + y^2 + z^2 = 1$

$\therefore \qquad x^2 \le 1,\ x^2 + y^2 \le 1,\ x^2 + y^2 + z^2 \le 1$

$\Rightarrow \qquad - \le x \le 1,\ -\sqrt{1-x^2}\ ,\ -\ \sqrt{1-x^2}\ ,\ -\ \sqrt{1-x^2-y^2}\ \le z \le \sqrt{1-x^2-y^2}$

$$\therefore \qquad \iiint\limits_{x^2+y^2+z^2\le 1} x^2 dx\ dy\ dz = \int_{-1}^{1} x^2 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left\{ \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz \right\} dy \right) dx$$

$$= \int_{-1}^{1} x^2 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \{z\}_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\ dy \right) dx$$

$$= \int_{-1}^{1} x^2 \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1-x^2-y^2}\, dy \right) dx$$

$$= \int_{-1}^{1} x^2\, 2(2) \left(\int_{0}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy \right) dx$$

$$= 4\int_{-1}^{1} x^2 \left(\frac{y\sqrt{1-x^2-y^2}}{2} + \frac{1-x^2}{2} \sin^{-1} \frac{y}{\sqrt{1-x^2}} \right)_0^{\sqrt{1-x^2}} dx$$

$$= 4\int_{-1}^{1} x^2 \left(\frac{\sqrt{1-x^2}\sqrt{1-x^2(1-x^2)}}{2} + \frac{1-x^2}{2} \sin^{-1} \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} - 0 \right) dx$$

$$= 4\int_{-1}^{1} x^2 \left(0 + \frac{1-x^2}{2} \frac{\pi}{2} \right) dx$$

$$= \pi \int_{-1}^{1} x^2 \left(1-x^2\right) dx$$

$$= 2\pi \int_{-1}^{1} (x^2 - x^4)\ dx$$

$$= 2\pi \left(\frac{x^3}{3} - \frac{x^5}{5} \right)_0^1$$

$$= 2\pi = \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{4\pi}{15}$$

Similarly $\iiint\limits_{x^2+y^2+z^2\le 1} (y)^2 dx\ dy\ dz = \iiint\limits_{x^2+y^2+z^2\le 1} (z)^2 dx\ dy\ dz = \dfrac{4\pi}{15}$

and $\iiint\limits_{x^2+y^2+z^2\le 1} xy\ dx\ dy\ dz\ \int\limits_{-1}^{1} x\left(\int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\left\{\int\limits_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\right\}dy\right)dx$

$$= \int\limits_{-1}^{1} x\left(\int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} y\left(\sqrt{1-x^2-y^2}\right)dy\right)dx$$

$$= \int\limits_{-1}^{1} x(0)\,dx$$

$= 0$ $\left(\because\ 2y\sqrt{1-x^2-y^2}\ \text{is an odd function w.r.t. y}\right)$

Similarly $\iiint\limits_{x^2+y^2+z^2\le 1} yz\ dx\ dy\ dz\ = \iiint\limits_{x^2+y^2+z^2\le 1} zx\ dx\ dy\ dz = 0$

$\therefore \quad \iiint\limits_{x^2+y^2+z^2\le 1} (a\ x + b\ y + c\ z)^2 dx\ dy\ dz$

$$= \iiint\limits_{x^2+y^2+z^2\le 1} (a^2\ x^2 + b^2\ y^2 + c^2\ z^2 + 2ab\,x\,y + 2bc\,y\,z + 2ca\,z\ x)\ dx\ dy\ dz$$

$$= a^2 \iiint x^2 dx\ dy\ dz + b^2 \iiint y^2 dx\ dy\ dz + c^2 \iiint z^2 dx\ dy\ dz$$

$$+ 2ab \iiint xy\ dx\ dy\ dz + 2bc \iiint yz\ dx\ dy\ dz + 2ca \iiint zx\ dx\ dy\ dz$$

$$= a^2\left(\frac{4\pi}{15}\right) + b^2\left(\frac{4\pi}{15}\right) + c^2\left(\frac{4\pi}{15}\right) + 2ab(0) + 2bc(0) + 2ca(0)$$

$= (a2 + b2 + c2)\ \dfrac{4\pi}{15}$

Example 4 : Evaluate $\int\limits_{1}^{3}\int\limits_{1/x}^{1}\int\limits_{0}^{\sqrt{xy}} x\ y\ z\ \ dz\ \ dy\ dx$

Sol. : $\int\limits_{1}^{3}\int\limits_{1/x}^{1}\int\limits_{0}^{\sqrt{xy}} x\ y\ z\ \ dz\ \ dy\ dx = \int\limits_{1}^{3}\left\{\int\limits_{1/x}^{1} xy\left(\int\limits_{0}^{\sqrt{xy}} x\ y\ z\right)\ dy\right\} dx$

$$= \int\limits_{1}^{3}\left\{\int\limits_{1/x}^{1} xy\left(\left[\frac{z^2}{2}\right]_0^{\sqrt{xy}}\right)\ dy\right\} dx$$

$$= \int_1^3 \left\{ \int_{1/x}^1 xy \frac{xy}{2} \ dy \right\} dx$$

$$= \int_1^3 \left\{ \int_{1/x}^1 \frac{x^2 y^2}{2} \ dy \right\} dx$$

$$= \int_1^3 \left[\frac{x^2 y^3}{6} \ \right]_{1/x}^1 dx$$

$$= \int_1^3 \left[\frac{x^2}{6} - \frac{1}{6x} \right] dx$$

$$= \int_1^3 \left[\frac{x^2}{18} - \frac{1}{6} \log x \right]_1^3$$

$$= \left[\frac{27}{18} - \frac{1}{6} \log 3 \right] - \left[\frac{1}{18} - \frac{1}{6} \log 1 \right]$$

$$= \frac{26}{18} - \frac{1}{6} \log 1$$

$$= \frac{13}{9} - \frac{1}{6} \log 3$$

Example 5 : Evaluate the following triple integral over the region given :

$\iiint_V x \ dv$ where V = $\therefore$ {(x, y, z) : 2 $\leq$ x $\leq$ 4; 1 $\leq$ y $\leq$ x; 0 $\leq$ z $\leq$ x;

Sol. : $\iiint_V x \ dv = \int_2^4 \int_1^x \int_0^x x \ \ dz \ \ dy \ \ dx$

$$= \int_2^4 \int_1^x x \, x \, dy \, dx$$

$$= \int_2^4 x^2 (x-1) dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_2^4$$

$$= \left[64 - \frac{64}{3}\right] - \left[4 - \frac{8}{3}\right]$$

$$= 60 - \frac{56}{3}$$

$$= \frac{124}{3}$$

Example 6: Evaluate $\iiint\limits_V \frac{dxdydz}{(x+y+z+1)^3}$ over the region x $\geq$ 0, y $\geq$ 0, x + y + z $\leq$ 1

Sol: Since x + y + z $\leq$ 1, x $\geq$ 0, y $\geq$ 0, z $\geq$ 0

$\therefore$ x $\leq$ 1, x + y $\leq$ 1, x + y + z $\leq$ 1

$\Rightarrow$ x $\leq$ 1, y $\leq$ 1 - x, z $\leq$ 1 - x - y

$\therefore$ V = $\{(x,y,z): 0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y\}$

$$\therefore \quad \iiint\limits_V \frac{dxdydz}{(x+y+z+1)^3} = \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^3} \right) dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \left(\int_0^{1-x-y} (x+y+z+1)^3 dz \right) dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} \left[\frac{(x+y+z+1)^{-2}}{-2} \right]_0^{1-x-y} dy \right\} dx$$

$$= -\frac{1}{2} \int_0^1 \left\{ \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x+y+1)^2} \right] dy \right\} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\frac{1}{4} y - \frac{(x+y+1)^{-1}}{-1} \right]_0^{1-x} dx$$

$$= -\frac{1}{2} \int_0^1 \left[\left(\frac{1-x}{4} + \frac{1}{2} \right) - \left(0 + \frac{1}{x+1} \right) \right] dx$$

$$= -\frac{1}{8} \int_0^1 \left(1 - x + 2 - \frac{4}{x+1} \right) dx$$

$$= -\frac{1}{8}\int_0^1 \left(1-x-\frac{4}{x+1}+3\right)dx$$

$$= -\frac{1}{8}\left[-\frac{x^2}{2}-4\log(1+x)+3x\right]_0^1$$

$$= -\frac{1}{8}\left[\left(-\frac{1}{2}-4\log 2+3\right)-(0-4\log 1+0)\right]$$

$$= -\frac{1}{8}\left[\frac{5}{6}-4\log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

Example 7: Evaluate $\iiint xyz\,dx\,dy\,dz$ over the ellipsoid $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

Sol: Since $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

$\therefore$ $\frac{x^2}{a^2} \le 1$, $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$

Or $x^2 < a^2$, $y^2 < \frac{b^2}{a^2}(a^2 - x^2)$, $z^2 < c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$

$\therefore$ $-a \le x \le a$, $\frac{b}{a}\sqrt{a^2-x^2}$, $z^2 \le c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$,

$-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

$$\therefore \quad \int\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1}\int\int xyz\;\;dx\;\;dy\;\;dz = \int_{-a}^{a}\left\{\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}}\left(xy\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} z\;\;dz\right)dy\right\}dx$$

$$= 0 \qquad \left[\because \text{ z is an odd function of x and } \therefore \int_{-a}^{a} z\,dz = 0\right]$$

Example 8: Evaluate $\iiint\limits_{x^2+y^2+z^2\le a^2} (lx^2+my^2+nz^2)$ dx dy dz.

Sol: Since $x^2 + y^2 + z^2 < a^2$

$\therefore$ $x^2 \le a^2$, $x^2 + y^2 \le a^2$, $x^2 + y^2 + z^2 \le a^2$

$\Rightarrow$ $-a \le x \le a$, $-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$, $-\sqrt{a^2-x^2-y^2} \le z \le \sqrt{a^2-x^2-y^2}$

Now $\iiint\limits_{x^2+y^2+z^2\le a^2} x^2 dx\,dy\,dz = \int\limits_{-a}^{a} x^2 \left\{ \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(\int\limits_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} 1\,dz \right) dy \right\} dx$

$$= \int\limits_{-a}^{a} x^2 \left\{ \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} [z]_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}}\, dy \right\} dx$$

$$= \int\limits_{-a}^{a} x^2 \left\{ \int\limits_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2-y^2}\,dy \right\} dx$$

$$= \int\limits_{-a}^{a} 2x^2 \left\{ 2\int\limits_{0}^{\sqrt{a^2-x^2}} \sqrt{(a^2-x^2)-y^2}\,dy \right\} dx$$

$$= 4\int\limits_{-a}^{a} x^2 \left[\frac{y\sqrt{a^2-x^2-y^2}}{2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{a^2-x^2} dx$$

$$= 4\int\limits_{-a}^{a} x^2 \left[\frac{\sqrt{a^2-x^2}\,\sqrt{a^2-x^2-\left(a^2-x^2\right)}}{2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} - 0 \right] dx$$

$$= 4\int\limits_{-a}^{a} x^2 0 + \left[\frac{a^2-x^2}{2}\left(\frac{\pi}{2}\right) \right] dx$$

$$= \pi\ (2) \int\limits_{0}^{a} x^2 \left[a^2\,\frac{x^3}{3} - \frac{x^5}{5} \right]_0^a$$

$$= 2\pi \left[\frac{a^5}{3} - \frac{a^5}{5} \right] = \frac{4\pi}{15}\ a^5$$

Similarly $\iiint\limits_{x^2+y^2+z^2\le a^2} y^2 dx\,dy\,dz = \iiint\limits_{x^2+y^2+z^2\le a^2} z^2 dx\,dy\,dz = \frac{4\pi}{15}\ a^5$

$\therefore \quad \iiint\limits_{x^2+y^2+z^2 \le a^2} (lx^2 + my^2 + nz^2)$ dx dy dz

$= \iiint\limits_{x^2+y^2+z^2 \le a^2} x^2\, dx\, dy\, dz + m \iiint\limits_{x^2+y^2+z^2 \le a^2} y^2\, dx\, dy\, dz$ + n $\iiint\limits_{x^2+y^2+z^2 \le a^2} z^2\, dx\, dy\, dz$

$= l\left(\dfrac{4\pi}{15} a^5\right) + m\left(\dfrac{4\pi}{15} a^5\right) + n\left(\dfrac{4\pi}{15} a^5\right)$

$= (l + m + n)\, \dfrac{4\pi a^5}{15}$

Self-Check Exercise-1

Q.1 Evaluate $\int_0^{\pi}\int_0^{3}\int_0^{2} z^2 r^3 \sin\theta$ dr dz dz

Q.2 Evaluate the following triple integral over the region given

$\iiint\limits_V xy dv,$ where V = $\{(x, y, z) : 1 \le x \le 2, 1 \le z \le x, 1 \le y \le z\}$

Q.3 Evaluate $\iiint (x + y + z)^9$ dx dy dz over the region defined by

$x \ge 0, y \ge 0, z \ge 0, x + y + z \le 1$

Q.4 Evaluate $\int\int\int\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1} x\, dx\, dy\, dz$

14.4 Change of Variables in Triple Integral

Let V be a simple closed sub-region of the xyz space and f : V → R be integrable over V. If the region V of the xyz space is mapped on the region V' of the uvw space by transformations x = f_1(u,v,w), y = f_2(u,v,w), z = f_2(u,v,w) and

(i) f_1, f_2, f_3 have continuous partial derivatives on V'

(ii) $J = \dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} \ne \forall$ (u, v, w) $\in$ V', then

$$\iiint_V f(x,y,z)\ \text{dx dy dz} = \iiint_V f(f_1, f_2, f_3)\ |J|\ \text{du dv dw}.$$

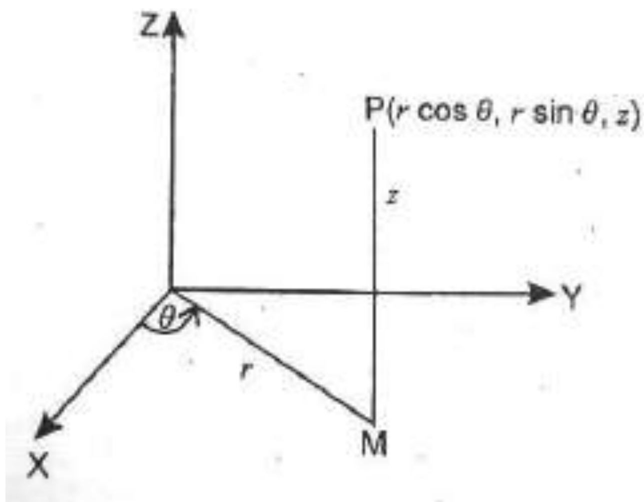
19.5 Change to Cylindrical Coordinates

Let P(x, y, z) be any point in the region V. From P draw PM $\perp$ xy-plane, Join OM.

Let OM = r, $\underline{|XOM}$ = θ. Then

x = r cos θ, y = r sin θ, z = MP

$\therefore$ (r cos θ, r sin θ, z) are called cylindrical coordinates of P.



$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

= r cos^2 θ + r sin^2 θ = r (cos$^2\theta$ + sin$^2\theta$) = r(1) = r

Here V = V'

$\therefore$ $$\iiint_V f(x,y,z)dx\,dy\,dz = \iiint_V f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$$

19.6 Change to Spherical Coordinates

Let P(x, y, z) be any point in the region V. From P, draw PM $\perp$ xy-plane. Join OM.

Let OP = r, $\underline{|XOM}$ = θ, $\underline{|ZOP}$ = ϕ

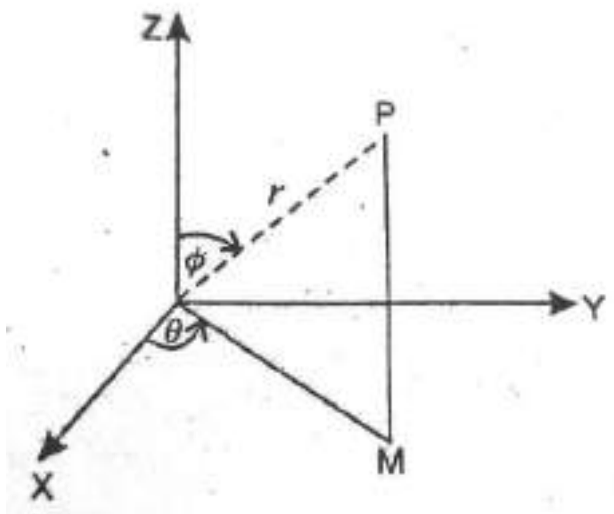
$\therefore$ x = r sin ϕ cos θ, y = r sin ϕ sin θ, z = r cos ϕ

$$\therefore \qquad J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -\sin\phi & 0 \end{vmatrix} = r^2 \sin\phi$$

Here V = V'

$$\therefore \qquad \iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\; r^2 \sin\phi \; dr\, d\theta\, d\phi$$

Note 1: The polar spherical coordinates are useful when the region of integration is a part of a sphere.



Note 2: Under these transformations V = $\{(x,y,z): x^2+y^2+z^2 \le a^2\}$ is mapped onto

V' = $\{(r,\theta,\phi): 0 \le r \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$

Let us consider the following examples to clear the idea:-

Example 9: Evaluate $\iiint_V z(x^2+y^2)\,dx\,dy\,dz$

where V = $\{(x,y,z): x^2+y^2 \le 1, 2 \le z \le 3\}$

Sol: Since the region of integration is a part of a right circular cylinder, so we change to cylindrical coordinates.

$\therefore$ V = $\{(r,\theta,z): 0 \le r \le 1, 0 \le \theta \le 2\pi, 2 \le z \le 3\}$

Now $\iiint\limits_V z(x^2+y^2)dx\,dy\,dz = \int_0^1\int_0^{2\pi}\int_2^3 z(r^2\cos^2\theta + r^2\sin^2\theta)r\,dz\,d\theta\,dr$

$= \int_0^1\int_0^{2\pi}\int_2^3 zr^3\,dz\,d\theta\,dr$

$= \int_0^1 r^3\,dr \int_0^{2\pi} d\theta \int_2^3 z\,dz$

$= \left[\frac{r^4}{4}\right]_0^1 [\theta]_0^{2\pi} \left[\frac{z^2}{2}\right]_2^3$

$= \left(\frac{1}{4}-0\right)(2\pi - 0)\left(\frac{9}{2}-\frac{4}{2}\right)$

$= \frac{1}{4}\times 2\pi \times \frac{5}{2}$

$= \frac{5\pi}{4}$

Example 10: Evaluate $\iiint\limits_R (ax^2+by^2+cz^2)dx\,dy\,dz$

where R is the region $x^2 + y^2 + z^2 \leq 1$

Sol: Since the region of integration is a ball bounded by the sphere $x^2 + y^2 + z^2 \leq 1$, so we change to spherical coordinates by substituting

$r = r \sin\phi \cos\theta$, $y = r \sin\phi \sin\theta$, $z = r \cos\phi$

$\therefore$ $V = \{(r,\theta,z) : 0 \le r \le 1, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$

$\therefore$ $\iiint\limits_R (ax^2+by^2+cz^2)dx\,dy\,dz = a\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\sin\phi\cos\theta)^2\, r^2\sin\phi\, dr\, d\theta\, d\phi$

$+b\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\sin\phi\cos\theta)^2\, r^2\sin\phi\, dr\, d\theta\, d\phi + c\int_0^1\int_0^{2\pi}\int_0^{\pi} (r\cos\phi)^2\, r^2\sin\phi\, dr\, d\theta\, d\phi$

$= a\int_0^1 r^4\,dr \int_0^{2\pi}\cos^2\theta\,d\theta \int_0^{\pi}\sin^3\phi\theta\,d\phi + b\int_0^1 r^4 dr \int_0^{2\pi}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi$

$+ c\int_0^1 r^4\,dr \int_0^{2\pi} d\theta \int_0^{\pi}\cos^2\phi\sin\phi\,d\phi$

$$= \left[\frac{r^5}{5}\right]_0^1 4\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta . 2\int_0^{\frac{\pi}{2}} \sin^2\phi\, d\phi + b\left[\frac{r^5}{5}\right]_0^1 .4\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta . 2\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi$$

$$+ \text{c}\left[\frac{r^5}{5}\right]_0^1 \left[\theta\right]_0^{2\pi} .2\left[-\frac{\cos^3\phi}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \text{a}\left(\frac{1}{5}-0\right).4\frac{1}{2}\times\frac{\pi}{2}.2\times\frac{2}{3} + \text{b}\left(\frac{1}{5}-0\right).4\frac{1}{2}\times\frac{\pi}{2}.2\times\frac{2}{3} + \text{c}\left(\frac{1}{5}-0\right).(2\pi - 0).\left(\frac{2}{3}\right)(0 + 1)$$

$$= \frac{a}{5}\times\pi\times\frac{4}{3} + \frac{b}{5}\times\pi\times\frac{4}{3} + \frac{c}{5}\times 2\pi\times\frac{2}{3}$$

$$= \frac{4\pi}{15}(\text{a} + \text{b} + \text{c})$$

Example 11: Evaluate $\iiint\limits_V \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}$ dx dy dz, where

$$\text{V} = \left\{(x,y,z): x\le 0, y\le 0, z\le 0, x^2+y^2+z^2\le 1\right\}$$

Sol: Let I = $\iiint\limits_V \frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}$ dx dy dz

where V = $\left\{(x,y,z): x\le 0, y\le 0, z\le 0, x^2+y^2+z^2\le 1\right\}$

Put x = r sin θ cos ϕ, y = r sin ϕ sin θ, z = r cos θ

∴ |J| = r^2 sin θ and V is mapped into V, where

$$\text{V}' = \left\{(r,\theta,\phi): 0\le r\le 1, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\right\}$$

∴ I = $\iiint\limits_V \frac{1-r^2}{1+r^2}$ r^2 sin θ dr dθ dϕ

$$= \int_0^1 \left(\int_0^{\frac{\pi}{2}}\left(\int_0^{\frac{\pi}{2}} d\phi\right)\sin\theta\, d\theta\right)\frac{1-r^2}{1+r^2}\, r^2\, dr$$

$$= \int_0^1 \left[\int_0^{\pi/2} \frac{\pi}{2}\sin\theta\, d\theta\right]\frac{1-r^2}{1+r^2}\, r^2\, dr$$

$$= \frac{\pi}{2}\int_0^1 \frac{r^2(1-r^2)}{1+r^2}\,dr$$

$$= \frac{\pi}{2}\int_0^1 \left[2-r^2-\frac{2}{1+r^2}\right]dr$$

$$= \frac{\pi}{2}\left[2r-\frac{r^2}{3}-2\tan^{-1}r\right]_0^1$$

$$= \frac{\pi}{2}\left[\left(2-\frac{1}{3}-2\tan^{-1}1\right)-(0-0-2\tan^{-1}0\right]$$

$$= \frac{\pi}{2}\left[2-\frac{1}{3}-2\frac{\pi}{2}-0+0+0\right]$$

$$= \frac{\pi}{2}\left(\frac{5}{3}-\frac{\pi}{2}\right)$$

Example 12: Evaluate $\iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$ over the positive octant of the sphere $x^2 + y^2 + z^2 = 1$

Sol: Since the region of integration is that part of the ball of radius 1 and centred at origin which lies is the positive octant.

$\therefore$ V = $\left\{(x,y,z): x\geq 0, y\geq 0, \geq\geq 0, x^2+y^2+z^2\leq 1\right\}$

Changing to spherical coordinates by substituting

x = r sin ϕ cos θ, y = r sin ϕ sin θ, z = r cos ϕ, we get

V = $\left\{(r,\phi,\theta): 0\leq r, 0\leq\theta\leq\frac{\pi}{2}, 0\leq\phi\leq\frac{\pi}{2}\right\}$

$$\therefore \quad \iiint \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}} = \int_0^1\int_0^{\pi/2}\int_0^{\pi/2} \frac{r^2\sin\phi\, dr\, d\theta\, d\phi}{\sqrt{1-r^2}}$$

$$= \int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{2}} \sin\phi\ d\theta$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2}\times 1$$

$= \dfrac{\pi^2}{8}$

$$\left[\begin{array}{l} By\, letting\ I = \int_0^1 \frac{r^2}{\sqrt{1-r^2}} dr \\ Put\ \ r = \sin t,\ \therefore dr = \cos dt \\ when\ \ r = 0,\ t = 0 \\ when\ \ r = 1,\ t = \frac{\pi}{2} \\ \therefore\ I = \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\ dt = \int_0^{\pi/2} \sin^2 t \\ \qquad = \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4} \\ Also\ \int_0^{\pi/2} \sin\phi\ d\phi = [-\cos\phi]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 1 \end{array}\right]$$

Self-check Exercise-2

Q. 1 Evaluate $\iiint z\ (x^2+y^2+z^2)$ dv where

$v = \{(x,y,z) : x^2+y^2 \le a^2, 0 \le z \le h\}$

Q. 2 Evaluate $\iiint\limits_{x^2+y^2+z^2\le 1} z^2\ dx\ dy\ \ dz$

Q. 3 Evaluate $\iiint (x^2+y^2) dx\, dy\ \ dz$ over the region bounded by $x^2+y^2+z^2$ =1.

19.7 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Defined and discussed triple integration.
2. Solved some questions related to triple integration.
3. Discussed change of variables in triple integral. Also discussed change to cylindrical coordinates and change to spherical co-ordinates.
4. Solved some questions related to change of variables in triple integral.

19.8 Glossary

1. Let f(x, y, z) be a continuous function of three independent variables x, y and z, defined over a closed and bounded region enclosing a volume V in R^3. Divide the Region into a number of parallelepipeds by drawing planes parallel to the co-ordinate planes inside the Region enclosing volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$, then the sum $\underset{\substack{n\to\infty \\ \delta V_n \to 0}}{Lt} \sum_{r=1}^{n} (x_r, y_r, z_r)\, \delta V_r \to \iiint_V f(x, y, z)\, \delta V$ or $\iiint_V f(x, y, z)$ dx, dy dz is called the triple integration and for evaluation, it can be expressed as the repeated integral $\int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f\ (x, y, z)\, dx\, dy\, dz$

19.9 Answer To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 72

Ans. 2 $\frac{11}{30}$

Ans. 3 $\frac{1}{24}$

Ans. 4 0

Self-Check Exercise-2

Ans. 1 $\frac{\pi a^2 h^2}{4}(a^2 + h^2)$

Ans. 2 $\frac{4\pi}{15}$

Ans. 3 $\frac{8\pi}{15} a^5$

19.10 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, Delhi, 2005

2. H. Anton, I. Bivens and S. Davis, *Calculus*, John Wiley and Sons (Asia) P. Ltd, 2002

19.11 Terminal Questions

1. Evaluate $\int_0^1\int_0^{1-x}\int_0^{2-x} dyz\ dz\ dy\ dx$

2. Evaluate $\int_0^{\pi/2}\int_0^4\int_0^{\sqrt{16-z^2}} \sqrt{16-r^2}\, rz\ dr\ dz\ d\theta$

3. Compute the integral $\iiint_V xyz\ dx\ dy\ dz$ over a domain bounded by x = 0, y = 0, z = 0, x + y + z = 1

4. Evaluate $\iiint_{x^2+y^2+z^2\le 1} (z^5+z)\ dx\ dy\ dz$

5. Show that $\iiint_{x^2+y^2+z^2\le 1} (ax+by+cz)\ dx\ dy\ dz = 0$

6. Show that $\iiint (x^2+y^2+z^2) dx\, dy\ dz = \frac{4\pi}{5}$ over the region $x^2+y^2+z^2 \le 1$

7. Evaluate $\iiint \frac{dx\, dy\, dz}{a^2+x^2+y^2+z^2}$ over the region $a^2+x^2+y^2 < a^2$

8. Prove that $\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}$ dx dy dz = $\frac{\pi^2 abc}{4}$

where $v = \left\{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1\right\}$

Unit - 20

Applications of Double And Triple Integral

Structure

20.1 Introduction

Double and triple integration have various applications in mathematics, physics, engineering and other fields. Double integration is commonly used to calculate areas and volumes of irregular shapes. By integrating a function over a region in the plane, you can determine the area enclosed by the curve or the volume under a surface. Double integration is also employed to find the center of mass of an object with non-uniform density. By using the concept of moments, you can calculate the coordinates of the object's center of mass. Triple integration extends the concept of double integration by calculating volumes of three-dimensional objects or regions. It is especially useful for irregular or complex shapes. Triple integration is also used to calculate the mass and density distribution of three-dimensional objects with varying densities. Triple integration is also used to calculate the moment of inertia of three-dimensional objects.

20.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss the formulae for calculation of area by use of double integration.
- Solve questions related to finding the area by double integration.
- Discuss the formulae for calculation of volume by use of triple integration.

- Solve questions related to finding the volume by triple integration.

20.3 Area by Use of Double Integration

(1) The area A of the region $\{(x, y): a \le x \le b, f_1(x) \le y \le f_2(x)\}$ is given by

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\ dx$$

(2) The area A of the region $\{(x, y): c \le y \le d, g_1(y) \le x \le g_2(y)\}$ is given by

$$A = \int_c^d \int_{g_1(y)}^{g_2(y)} dx\ dy$$

(3) The area A of the region $\{(r, \theta): \alpha \le \theta \le \beta, f_1(\theta) \le r \le f_2(\theta)\}$ is given by

$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\ dr\ d\theta$$

(4) The area A of the region $\{(r, \theta): r_1 \le r \le r_2, g_1(r) \le \theta \le g_2(r)\}$ is given by

$$A = \int_{r_1}^{r_2} \int_{g_1(r)}^{g_2(r)} r\ d\theta\ dr$$

The following examples will illustrate the idea more clearly:

Example 1: Find the area of the circle using the double integration

Sol: Let the equation of circle be $x^2 + y^2 = a^2$

We know that circle $x^2 + y^2 = a^2$ is symmetrical about both the axes.

Also in first quadrant, $y = \sqrt{a^2 - x^2}$, $0 \le x \le a$

$\therefore$ required area $= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} dy\ dx$

$$= 4\int_0^a [y]_0^{\sqrt{a^2-x^2}}\ dx$$

$$= 4\int_0^a \sqrt{a^2 - x^2}\ dx$$

$$= 4\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\left[\left(0+\frac{a^2}{2}\sin^{-1}1\right)-\left(0+\frac{a^2}{2}\sin^{-1}0\right)\right]$$

$$= 4\left[\frac{a^2}{2}\times\frac{\pi}{2}\times\frac{a^2}{2}\times 0\right]$$

$$= 4 \times \frac{\pi a^2}{4} = \pi a^2$$

Example 2: Find the are enclosed using double integration by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

Sol: The equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$

We know that ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is symmetrical about both the axes.

Also in first quadrant, y = $\frac{b}{a}\sqrt{a^2-x^2}$,0 $\le$ x $\le$ a

$\therefore$ required area = $4\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} dy\ dx$

$$= 4\int_0^a [y]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 4\int_0^a \sqrt{a^2-x^2}\ dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\ dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$= \dfrac{4b}{a}\left[\left(0+\dfrac{a^2}{2}\sin^{-1}1\right)-\left(0+\dfrac{\pi a^2}{4}\sin^{-1}0\right)\right]$

$= \dfrac{4b}{a}\left[\dfrac{a^2}{2}\times\dfrac{\pi}{2}\right]$

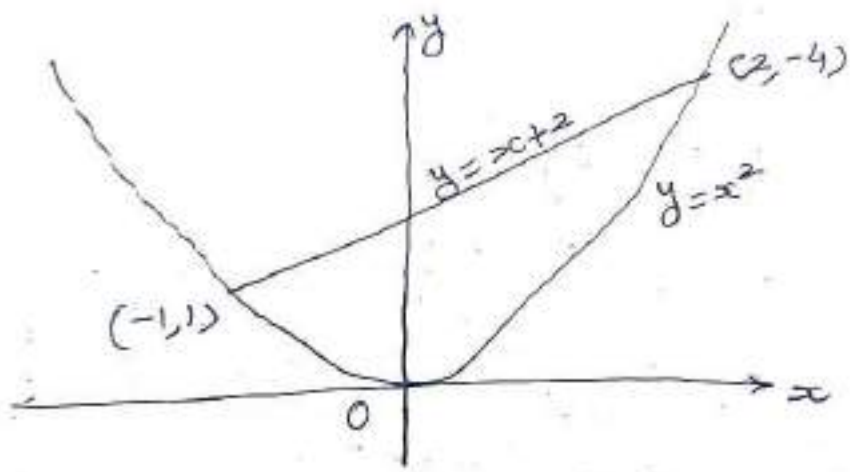
$= \pi ab$

Example 3: Find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$

Sol: The equation of the parabola is $y = x^2$ ....(1)

The equation of the line is $y = x + 2$ ....(2)

From (1) and (2), $x + 2 = x^2$



$\therefore$ $x^2 - x - 2 = 0$

$\Rightarrow$ $(x - 2)(x + 1) = 0$

$\Rightarrow$ $x = -1, 2$

$\therefore$ $y = 1, 4$

$\therefore$ region of integration A is given by

A = $\left\{(x, y) : -1 \le x \le 2, x^2 \le y \le x+2\right\}$

Required are = $\displaystyle\int_{-1}^{2}\int_{x^2}^{x+2} dy\ dx$

$= \displaystyle\int_{-1}^{2}[y]_{x^2}^{x+2}$

$$= \int_{-1}^{2} (x + 2 - x^2)\, dx$$

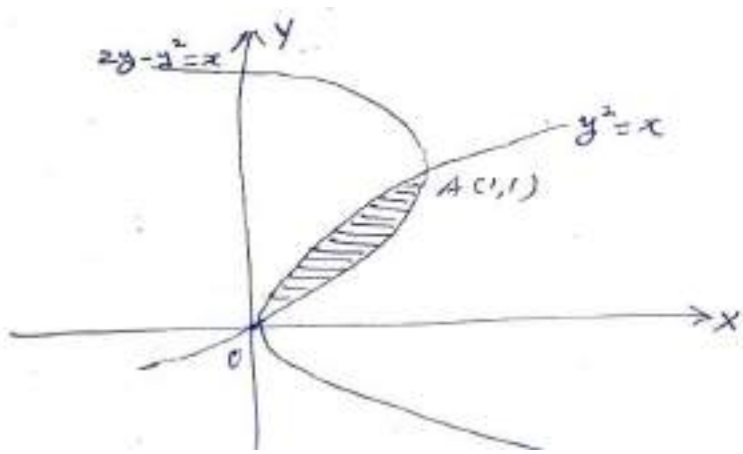
$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2}$$

$$= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right)$$

$$= \frac{10}{3} + \frac{7}{6} = \frac{20+7}{6} = \frac{27}{6} = \frac{9}{2}$$

Example 4: Using double integration find the area bounded by the curves $x = 2y - y^2$ and $x = y^2$

Sol: The equation $y^2 = x$ is a parabola with vertex at (0, 0) and $x = 2y - y^2$ is also a parabola with vertex (1, 1) and can be written as $(y - 1)^2 = -(x - 1)$



They will intersect when $y^2 = 2y - y^2$

Or $\quad 2y^2 - 2y = 0 \quad \Rightarrow \quad 2y(y - 1) = 0$

Or $\quad y = 0, 1$

Put $\quad y = 0, 1$ in $y^2 = x$, we get $x = 0, 1$

i.e. $\quad$ Point of intersection are (0, 0) and (1, 1)

The required area = $\int_0^1 \int_{y^2}^{2y-y^2} dx\, dy = \int_0^1 [x]_{y^2}^{2y-y^2} dy$

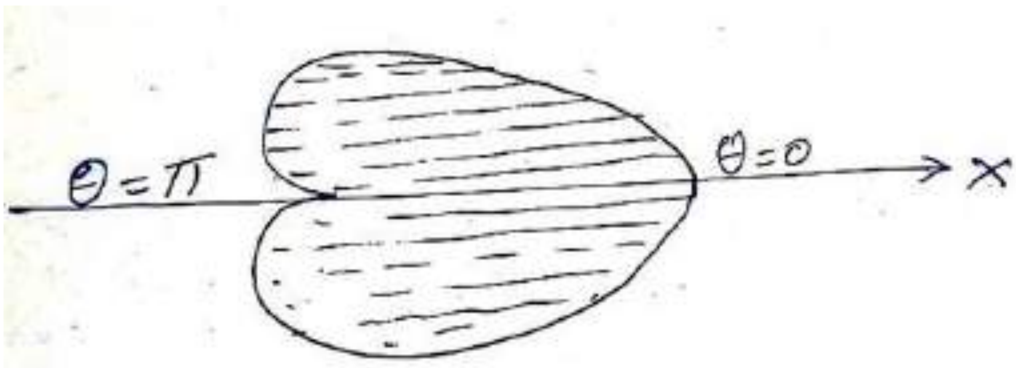
$$= \int_0^1 (2y - y^2 - y^2)\, dy = \int_0^1 (2y - 2y^2)\, dy$$

$\therefore$ Area = 2 $\int_0^1 (y - 2y^2)\, dy$ = 2 $\left[\frac{y^2}{2} - \frac{y^2}{3}\right]_0^1$

= $2\left[\frac{1}{2} - \frac{1}{3}\right]$ = $\frac{1}{3}$

Example 5: Find the area enclosed by the cardioids r = a (1 + cos θ)

Sol: The equation of the cardioids is r = a (1 + cos θ)



From the figure, it is clear that cardioids is symmetrical about the initial line.

$\therefore$ Area = $\int_0^{\pi} \int_0^{a(1+\cos\theta)} r\ dr\ d\theta$

= $\int_0^{\pi}\left[\int_0^{a(1+\cos\theta)} r\ dr\right]$dθ

= 2 $\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$

= a$^2\int_0^{\pi}(1+\cos\theta)^2\, d\theta$

= a$^2 \int_0^{\pi}\left[2\cos^2\frac{\theta}{2}\right]^2 d\theta$

= 4a$^2\int_0^{\pi}\cos^4\frac{\theta}{2} d\theta$

Put $\frac{\theta}{2}$ = t i.e. θ = 2t,

$\therefore$ $d\theta = 2dt$

when $\theta = 0, t = 0$

when $\theta = \pi, t = \frac{\pi}{2}$

$\therefore$ Area = $4a^2.2 \int_0^{\pi/2} \cos^4 t \, dt$

$= 8a^2 \frac{3.1}{4.2} . \frac{\pi}{2} = \frac{3\pi a^2}{2}$

Self-Check Exercise-1

Q.1 Find the area enclosed by the parabolas

$y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$

Q.2 Find the area of the region bounded by

$x = 0, y = 0, x^2 + y^2 = 1, y = \frac{1}{2}$

Q.3 Find the area enclosed by the cardioids

$r = a(1 - \cos\theta)$, $a > 0$

20.4 Volume by use of Triple Integration

(i) In case of Cartesian coordinates, V = $\iiint dx\, dy\, dz$

$\therefore$ element of volume is $\delta V = \delta x \; \delta y \; \delta z$

(ii) In case of cylindrical coordinates, V = $\iiint r \; d\theta \; dr \; dz$

$\therefore$ element of volume is $\delta V = r \, \delta\theta \; \delta r \; \delta z$

(iii) In case of spherical coordinates, V = $\iiint r^2 \sin\phi \; dr \; d\phi \; d\theta$

$\therefore$ element of volume is $\delta V = r^2 \delta \sin\phi \, \delta r \; \delta \phi \; d\theta$

where V stands for volume of the region.

Let us consider the following examples to clear the idea :-

Example 6 : Prove that the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi}{3}$ abc.

Sol. : Required volume $\iiint dx\, dy\, dz$

over the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

Put $\frac{x}{a} = u$, $\frac{y}{b} = v$, $\frac{z}{c} = w$

$\Rightarrow$ dx = a du, dy = b dv, dz = c dw

$\therefore$ volume = $\iiint abc\ du\ dv\ dw$

over the sphere $u^2 + v^2 + w^2 = 1$

Changing to spherical coordinates by the relations

$u = r \sin\phi \cos\theta$, $v = r \sin\phi \sin\theta$, $w = r \cos\phi$

$\therefore$ volume = abc $\iiint r^2 \sin\phi\ dr\ d\phi\ d\theta$

over the region $\{(r, \phi, \theta), 0 \le r \le 1, 0 \le \phi \le \pi, 0 \le \theta \le 2\pi\}$

$$= abc\int_0^1 r^2 dr\int_0^{\pi} \sin\phi\ d\phi\int_0^{2\pi} d\theta$$

$$= abc\left[\frac{r^3}{3}\right]_0^1 \left[-\cos\phi\right]_0^{\pi} \left[\theta\right]_0^{2\pi}$$

$$= abc\left(\frac{1}{3}-0\right)(-\cos\pi+\cos 0).(2\pi-0)$$

$$= abc\times\frac{1}{3}\times 2 \times 2\pi$$

$$= \frac{4\pi}{3}\ abc$$

Example 7 : (i) Find the volume of the tetrahedron bounded by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a, b, c are positive.

(ii) Find the volume of the tetrahedron bounded by the coordinate planes and the plane passing through points $(\alpha, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$.

Sol. (i) Required volume $\iiint_V dx\ dy\ dz$

where V = $\left\{(x, y, z) : x \ge 0, y \ge 0, z \ge 0, \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \le 1\right\}$

Put $\frac{x}{a}$ = X, $\frac{y}{b}$ = Y, $\frac{z}{c}$ = Z

$\therefore$ dx = a dX, dY = b dY, dz = c dZ

$\therefore$ V = {(X, Y, Z) : X $\geq$ 0, Y $\geq$ 0, Z $\geq$ 0, X + Y + Z $\leq$ 1}

= {(X, Y, Z) : 0 $\leq$ Z $\leq$ 1, 0 $\leq$ Y $\leq$ 1 – Z, 0 $\leq$ X $\leq$ 1 – Y – Z}

Required volume $\iiint_V dx\ dy\ dz$

$$= \iiint_V abc\ dX\ dY\ dZ$$

$$= abc\int_0^1\int_0^{1-Z}\int_0^{1-Y-Z} dX\ dY\ dZ$$

$$= abc\int_0^1\int_0^{1-Z}\left[X\right]_0^{1-Y-Z} dY\ dZ$$

$$= abc\int_0^1\left\{\int_0^{1-Z}\left(1-Y-Z\right)\right\} dZ$$

$$= abc\int_0^1\left[Y-\frac{Y^2}{2}-YZ\right]_0^{1-Z} dZ$$

$$= abc\int_0^1\left[(1-Z)-\frac{(1-Z)^2}{2}-Z(1-Z)\right]dZ$$

$$= \frac{abc}{2}\left[\frac{(1-Z)^3}{(-1)(3)}\right]_0^1$$

$$= \frac{abc}{6}[(1-1)^3 - (1-0)^3]$$

$$= \frac{abc}{6}.$$

(ii) the equation of plane passing through (α, 0, 0), (0, b, 0), (0, 0, c) is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

Now solution is same as that of part (i).

Example 8 : Prove that the volume of α tetrahedron bounded by the coordinate planes and the plane x + y + z = 1 is equal to $\frac{1}{6}$

Sol. : Required volume = $\iiint\limits_V dx\ dy\ dz$

where V = {(x, y, z) : x $\geq$ 0, y $\geq$ 0, z $\geq$ 0, x + y + z $\leq$ 1}

= {(x, y, z) : 0 $\leq$ z $\leq$ 1, 0 $\leq$ y $\leq$ 1 – z, 0 $\leq$ x $\leq$ 1 – y – z}

$\therefore$ Required volume = $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} dx\ dy\ dz$

$$= \int_0^1 \int_0^{1-z} [x]_0^{1-y-z}\ dy\ dz$$

$$= \int_0^1 \left\{ \int_0^{1-z} (1-y-z) \right\} dz$$

$$= \int_0^1 \left[y - \frac{y^2}{2} - yz \right]_0^{1-z} dz$$

$$= \int_0^1 \left[(1-z) - \frac{(1-z)^2}{2} - z(1-z) \right] dz$$

$$= \int_0^1 \left[(1-z)(1-z) - \frac{(1-z)^2}{2} - z(1-z) \right] dz$$

$$= \frac{1}{2}\int_0^1 (1-z)^2 dz$$

$$= \frac{1}{2} \left[\frac{(1-z)^3}{(-1)(3)} \right]_0^1$$

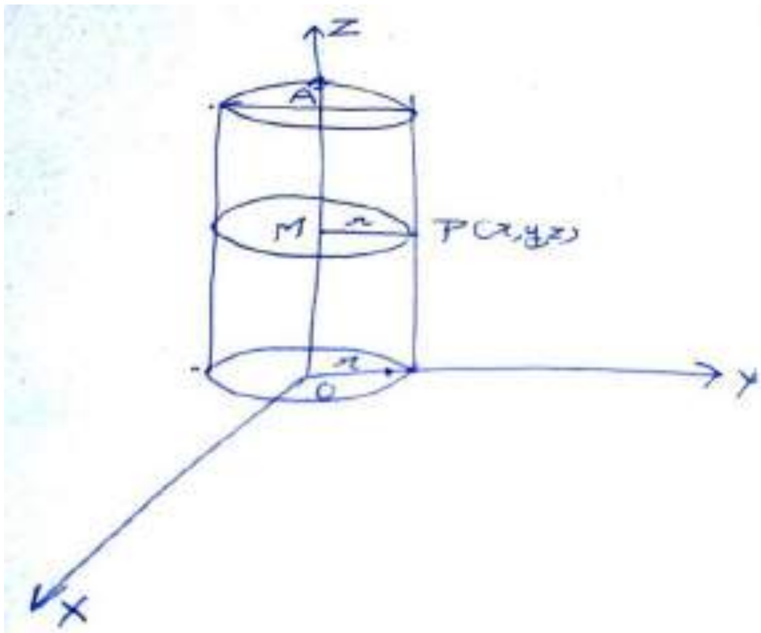
$$= -\frac{1}{6} [(1-1)^3 - (1-0)^3]$$

$$= -\frac{1}{6}(-1)$$

$$= \frac{1}{6}$$

Example 9 : Find the volume of a cylinder with base radius r and height h.

Sol. : Let O be the centre of the base and A that of the top. Take O as origin, OA as z-axis and two perpendicular lines through O in the plane of the base as x-axis and y-axis.



Let P(x, y, z) be any point on the cylinder Form P, draw PM $\perp$ OA such that MP = r

$\therefore \qquad \sqrt{x^2+y^2}$ = r or $x^2 + y^2 = r^2$,

which is one equation of cylinder.

Let V be the region bounded by the cylinder.

$\therefore \qquad V = \{(x, y, z) : 0 \leq z \leq h, -r \leq x \leq r, -y, \leq y \leq y_1\}$

where $y_1 = \sqrt{r^2+x^2}$

Required volume $= \iiint_V dx\ dy\ dz = \int_{-r}^{r}\int_{-y_1}^{y_1}\int_{0}^{h} dz\ dy\ dx$

$$= \int_{-r}^{r}\left\{\int_{-y_1}^{y_1}\left(\int_{0}^{h} dz\right)dy\right\}dx$$

$$= h\int_{-r}^{r}\left\{\int_{-y_1}^{y_1} dy\right\}dx = h\int_{-r}^{r} zy,\ dx$$

$$= 2h \int_{-r}^{r} \sqrt{r^2 - x^2}\ dx$$

$$= 2h \left[\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x}{r}\right) \right]_{-r}^{r}$$

$$= 2h \left[\frac{r^2}{2}\sin^{-1}(1) - \frac{r^2}{2}\sin^{-1}(-1) \right]$$

$$= 2h\ \frac{r^2}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right]$$

$= hr^2.\pi$

$= \pi r^2 h$

Example 10 : Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

Sol. : Given equation of the sphere is

$x^2 + y^2 + z^2 = a^2$

$\Rightarrow$ $z = \pm\sqrt{a^2 - x^2 - y^2} = \pm z_1$ (say)

Also, given equation of cylinder is

$x^2 + y^2 = ax.$

$\Rightarrow$ $y = \pm\sqrt{ax - x^2} = \pm$ y, (say)

$x^2 + y^2 = ax,\ y = 0 \Rightarrow x^2 = ax \Rightarrow x = 0, a$

$\therefore$ Required volume $= \int_0^a \int_{-y_1}^{y_1} \int_{-z_1}^{z_1} dz\ dy\ dx$

$$= \int_0^a \int_{-y_1}^{y_1} 2z_1\ dy\ dx$$

$$= 4\int_0^a \left\{ \int_0^{y_1} \sqrt{a^2 - x^2 - y^2}\ \right\} dx$$

Put $x = x \cos\theta,\ y = r\sin\theta$

$\therefore$ Required volume $= 4 \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\ r\ dr\, d\theta$

$$= -2 \int_0^{\pi/2} \left[\int_0^{a\cos\theta} (a^2 - r^2)^{\frac{1}{2}} (-2r) dr \right] d\theta$$

$$= -2 \int_0^{\pi/2} \left[\frac{(a^2 - r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= -\frac{4}{3} \int_0^{\pi/2} \left[(a^2 - a^2\cos^2\theta)^{\frac{3}{2}} - (a^2)^{\frac{3}{2}} \right] d\theta$$

$$= -\frac{4}{3} a^3 \int_0^{\pi/2} (\sin^3\theta - 1) d\theta$$

$$= -\frac{4}{3} a^3 \left(\frac{2}{3.1} - \frac{\pi}{2} \right)$$

$$= -\frac{8}{9} a^3 + \frac{2\pi}{3} a^3$$

$$= \frac{2a^3}{9} (3\pi - 4)$$

Self-check Exercise-2

Q. 1 Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$

Q. 2 Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$.

Q. 3 Find the volume of a right circular cone with base radius r and height h by triple integration.

Q. 4 Find the volume of the solid bounded by the coordinate planes and the planes.

$zx + y + z = 2, \quad 2x + y + z = 4.$

20.5 Summary

We conclude this unit by summarizing what we have covered in it:-

1. Discussed the formulae for calculation of area by use of double integration.
2. Solved some questions related to finding the area by double integration.

3. Discussed the formulae for calculation of volume by use of triple integration.

4. Solved some questions related to finding the volume by use of triple integration.

20.6 Glossary

1. The area A of the region

   {(x, y) : a $\leq$ x $\leq$ b, f_1 (x) $\leq$ y $\leq$ f_2(x) } is given by

   $$A = \int_{a}^{b} \int_{f_1(x)}^{f_2(x)} dy\ dx$$

2. The area A of the region

   {(r, θ) : x $\leq$ θ $\leq$ β, f_1 (θ) $\leq$ r $\leq$ f_2(θ) } is given by

   $$A = \int_{x}^{\beta} \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\ d\theta$$

3. In case of Cartesian coordinates,

   Volume V = $\iiint dx\ dy\ dz$

4. In case of Cylindrical coordinates,

   Volume V = $\iiint r\ d\theta\ dr\ dz$

5. In case of Spherical coordinates,

   Volume V = $\iiint r^2 \sin\phi\ \ dr\ d\phi\ d\theta$

20.7 Answer To Self-Check Exercise

Self-Check Exercise-1

Ans. 1 $\dfrac{16a^2}{3}$

Ans. 2 $\dfrac{3\sqrt{3}+2\pi}{24}$

Ans. 3 $\dfrac{3}{2}\pi a^2$

Self-Check Exercise-2

Ans. 1 $\dfrac{4}{3}\pi a^3$

Ans. 2 6

Ans. 3 $\frac{1}{3}\pi r^2 h$

Ans. 4 $\frac{14}{3}$

20.8 References/Suggested Readings

1. G.B. Thomas and R.L. Finney, *Calculus*, 9th Ed., Pearson Education, Delhi, 2005
2. H. Anton, I. Bivens and S. Davis, *Calculus*, John Wiley and Sons (Asia) P. Ltd, 2002

20.9 Terminal Questions

1. Find the area of the region in the first quadrant which is bounded by the parabola y2 = 4ax and the line x = 2a.
2. Using double integration find the area of the region bounded by the lines.
3. Find the area enclosed by the leminscate $r^2 = a^2 \cos = \theta$.
4. Show that the entire volume of the solid

   $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$ is $\frac{4\pi abc}{35}$
5. Find the volume of a truncated cone with end radii a and b and height h.
6. Show that the volume bounder by the cylinder $x^2 + y^2 = 4$ and planes y + z = 4, z = 0 is 16 π.
7. Find the volume common to the cylinder

   $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$.