B.A. IInd Year
Mathematics

Course Code :MATH201TH
New Syllabus- DSC IC

# Real Analysis

## Unit 1 to 20

Centre for Distance And Online Education (CDOE)
Himachal Pradesh University,
Gyan Path, Summer Hill, Shimla - 171005

# B.A. Mathematics MATH201TH

## Real Analysis

### CONTENTS

# SYLLABUS

# BA IIndYear Mathematics

# MATH201TH DSC-IC New Syllabus

## Himachal Pradesh University

## B.A. with Mathematics

## Syllabus and Examination Scheme

| | |
|---|---|
| Course Code | MATH201TH |
| Credits | 6 |
| Name of the Course | Real Analysis |
| Type of the Course | Core Course |
| Assignments | Max. Marks:30 |
| Yearly Based Examination | Max Marks: 70Maximum Times: 3 hrs. |

### Instructions

**Instructions for Candidates:** Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

### Core 2.1 : Real Analysis

### Unit-I

Real line, bounded sets. suprema and infima, completeness property of R, Archimedean property of R, intervals. Concept of cluster points and statement of Bolzano-Weierstrass theorem.

### Unit-II

Real Sequence, Bounded sequence, Cauchy convergence criterion for sequences. Cauchy's theorem on limits, order preservation and squeeze theorem, monotone sequences and their convergence (monotone convergence theorem without proof).

### Unit-III

Infinite series. Cauchy convergence criterion for series, positive term series, geometric series, comparison test, convergence of p-series, Root test, Ratio test, alternating series, Leibnitz's test (Tests of convergence without proof). Definition and examples of absolute and conditional convergence.

## Unit-IV

Sequences and series of functions, pointwise and uniform convergence, $M_n$-test, M-test, Results about uniform convergence. Power series radius of convergence.

**Books Recommended**


1. T.M. Apostol, Calculus (Vol. I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. R.G. Bartle and D.R. Sherbert, Introduction to real Analysis, John Wiley and Sonce (Asia) P. Ltd., 2000.
3. E.Fischer, Intermediate Real Analysis, Springer Verlag, 1983.
4. K.A. Ross, Elementary Analysis- The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag, 2003.

**Unit - 1**

# Some Basic Concepts

**Structure**



## 1.1 Introduction

Dear students,you are already aware of the concept of a set. From your previous knowledge you must be aware of the fact that infinite set has infinite number of elements. Now, the question arises that is there any connection between the sets of natural numbers N, the set of integer z, the set of real numbers etc.? In this direction the concept of equivalent sets is being introduced and consequently the concept of countable set and uncountable set comes into picture.

## 1.2 Learning Objectives:

The main objective of this unit are

(i) to study the concept of equivalent sets.

(ii) to study finite and infinite sets

(iii) to study countable and uncountable sets

(iv) to learn the concept of algebraic and transcendental numbers.

## 1.3 Equivalent Sets.

Let A and B be two sets. Then a set A is said to be equivalent to a set B, written as A ~ B, if there exists a mapping (function)

$f : A \rightarrow B$

Which is one-one and onto (Bijective). Let us understand the above concept with the help of some examples.

**Example 1 :** If $A = \{2n : n \in z\}$

and $B = \{2n - 1 : n \in z\}$

then $A \sim B$

**Solution :** Given $A = \{2n : n \in z\}$, $B = \{2n - 1 : n \in z\}$

Let us define a function

$f : A \rightarrow B$ such that

$f(x) = x - 1 \ \forall \ x \in A$

Claim : $f$ is one-one and onto i.e. bijective

For one-one - Let $x_1, x_2 \in A$ s.t.

$f(x_1) = f(x_2)$

$\Rightarrow \quad x_1 - 1 \quad x_2 - 1$

$\Rightarrow \quad x_1 = x_2$

$\therefore \quad f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Thus $f$ is one-one.

$f$ is onto - set $y \in B$ be any element

$\therefore \quad \exists$ some $n \in z$ such that

$y = 2n - 1$

Thus $\exists \ x = 2n \in A$ s.t.

$f(x) = f(2n) = 2n - 1 = y$

$\Rightarrow \quad f$ is onto

Hence $f$ is one-one and onto.

Thus $A \sim B$.

**Example 2 :** Consider $A = \{ x : x \in R \text{ s.t. } 0 \leq x \leq 1\}$

$B = \{ x : x \in R \text{ s.t. } 3 \leq x \leq 7\}$

then show that $A \sim B$.

**Solution :** Let us define a function

$f : A \rightarrow B$ as

$f(x) = 4x + 3 \ \forall \ x \in A$

Claim : $f$ is bijective

$f$ is one-one - Let $x_1, x_2 \in$ A s.t.

$f(x_1) = f(x_2)$

$\Rightarrow$ $4x_1 + 3 = 4x_2 + 3$

$\Rightarrow$ $x_1 = x_2$

$\therefore$ $f$ is one- one

$f$ is onto-

Let $y \in$ B be any element

$3 \leq y \leq 7$

$\Rightarrow$ $0 \leq y - 3 \leq 4$

$\Rightarrow$ $0 \leq \frac{y-3}{4} \leq 1$

Let $x = \frac{y-3}{4} \in$ A s.t.

$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y$

Thus $\forall\, y \in B\ \exists\, x \in A$ s.t. $f(x) = y$

$\therefore$ $f$ is onto

$\Rightarrow$ $f$ is one-one and onto i.e. bijective

Hence A ~ B.

**Example 3 :** Show that any open interval is equivalent to (0, 1).

**Solution :** Let A = (0, 1) and B = (a, b), a < b are reals.

Consider the function $f$ A $\rightarrow$ B as

$f(x) = a + (b - a)x \ \forall x \in A$

Claim : $f$ is bijective

$f$ is one-one. - Let x1, x2 $\in$ A s.t.

$f$ (x1) = $f$ (x2)

$\Rightarrow$ a + (b - 1) x1 = a + (b - a) x2

$\Rightarrow$ x1 = x2

$\therefore$ $f$ is one-one.

$f$ is onto - Let $y \in$ B be any element

$\therefore$ a < y < b

$\Rightarrow$ 0 < y - 1 < b - a

$\Rightarrow$ $0 < \frac{y-a}{b-a} < 1$

Let $x = \frac{y-a}{b-a}$, then 0 < x < 1 i.e. x $\in$ A

Now $f(x) = f\left(\frac{y-a}{b-a}\right)$

$= a + \left(\frac{y-a}{b-a}\right)(b - a)$

= y

$\therefore$ $f$ (x) = y

$\Rightarrow$ $f$ is onto

$\Rightarrow$ $f$ is bijective

Hence A~B.

**Remark :** Example 3 can be restated as :

Show that (0, 1) ~ (a, b) for every real number a and b, a < b.

**Some More illustrated Examples**

**Example 4 :** Show that the equivalent relation ~ is an equivalent relation

**Solution :** (i) Consider the identity function

I : A $\rightarrow$ A defined by

I (x) = x $\forall x \in$ A

clearly A ~ A $\forall$ set A

$\therefore$ ~ is reflexive

(ii) A is symmetric -

Let A ~ B $\Rightarrow \exists$ a function

$f$ : A $\rightarrow$ B, such that

$f$ is one-one and onto

$\Rightarrow$ $\exists$ inverse function $f^{-1}$ : B $\rightarrow$ A which is also one-one and onto

$\therefore$ B ~ A

Hence ~ is symmetric

(iii) A is transitive -

Let A ~ B and B ~ C

Claim : A ~ C

Now A ~ B, B ~ C

$\Rightarrow$ $\exists$ functions $f$ : A $\to$ B and g : B $\to$ C

Which are both one-one and onto

$\therefore$ the composite function h = go $f$ : A $\to$ C is also one-one and onto

$\therefore$ A ~ C

Hence ~ is transitive

Thus the equivalent relation ~ is an equivalence relation.

**Example 5 :** Show that $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ ~ R

**Solution :** Let A = $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and B = R

Consider a function

$f$ : A $\to$ B defined by

$f(x) = \tan x \ \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

Claim : $f$ is bijective

$f$ is one-one - set $x_1, x_2 \in$ A be s.t.

$f(x_1) = f(x_2)$

$\Rightarrow$ $\tan x_1 = \tan x_2$

$\Rightarrow$ $x_1 = x_2 + n\pi$ , $n \in z$

$\Rightarrow$ $x_1 = x_2 = n\pi$, $n \in z$

But $(x_1, x_2) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$\therefore$ $|x_1 - x_2| < \frac{\pi}{2}$

$\therefore$ $x_1 - x_2 = n\pi \Rightarrow n = 0$ i.e. $x_1 = x_2$

$\therefore$ $f$ is one-one

$f$ is onto I Let $y \in$ R then

$\exists$ x = $\tan^{-1}$ y $\in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

s.t.

$f$ (x) = y

$\therefore f$ is onto

$\therefore f$ is bijective

Hence A ~ B i.e. $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ ~ R

**Example 6 :** A, B are two sets, then show that A $\times$ B is equivalent B $\times$ A.

**Solution :** Let us consider a function

$f$ : A $\times$ B $\rightarrow$ B $\times$ A defined by

$f$ ((a, b)) = (b, a) $\forall$ (a, b) $\in$ A $\times$ B

Claim :$f$ is bijective

$f$ is one-one -

Let ($a_1$ $b_1$), ($a_2$ $b_2$) $\in$ A $\times$ B s.t.

$f$ (($a_1$ $b_1$)) = $f$ (($a_2$ $b_2$))

$\Rightarrow$ ($b_1$, $a_1$) = ($b_2$ , $a_2$)

$\Rightarrow$ $b_1$ = $b_2$, $a_1$ = $a_2$

$\Rightarrow$ ($a_1$ $b_1$) = ($a_2$ $b_2$)

$\therefore$ $f$ is one-one

$f$ is onto -

Let y $\in$ B $\times$ A be any element

$\therefore$ $\exists$ b $\in$B and a $\in$As.t.

y = (b, a)

Now, we fine x = (a, b) $\in$ A $\times$ B s.t.

$f$ (x) = $f$ (a, b) = (b, a) = y

$\therefore$ $f$ is onto

$f$ is bijective. Hence A $\times$ B ~ B $\times$ A

## 1.4 Finite and Infinite Set

A set A is said to be finite set iff either A = $\phi$ or there exists a positive integer n such that A ~ {1, 2, 3, ...n}

Otherwise A is said to be infinite set

In other words, A is said to be infinite if ∃ a function

$f$ : A → A s.t.

$f$ (A) ≠ A.

**Remark:** (i) If a set A ~ {1, 2, 3, ...., n}, then we say that A is a finite set having n elements.

(ii) In case of finite sets, two sets are equivalent iff they have the same number of elements.

(iii) If B is infinite and B ⊂ A, then A is infinite i.e. superset of infinite set is infinite.

## 1.5 Countable and Uncountable sets

(i) **Countable infinite set (or Denumerable or Enumerable set) :** An infinite set A is said to be countably infinite set or denumerable or enumerable set iff A ~ N.

(ii) **Countable set :** A set A is said to be countable if A is either finite set or a countably infinite set.

(iii) **Almost Countable :** A set A is said to be almost countable if A is either a finite set or a countable set.

(iv) **Uncountable set :** A infinite set, which is not countably infinite (or denumerable) set is said to be uncountable set.

**Art-1.** A set A is countably infinite or denumerable iff its elements can be put in the form of an infinite sequence of distinct elements.

**Proof :** Let A be a countable infinite set. Then by definition A ~ ℕ.

Therefore, ∃ a function $f:\mathbb{N} \to A$ which is bijective

i.e. ∀ n ∈ ℕ, ∃' s. $a_n \in A$ such that $f(n) = a_n$ and

$f(n_1) = f(n_2) = n_2$ [or $n_1 \neq n_2 \Rightarrow f(n_1) \neq f(n_2)$]

Thus A = $\{a_1, a_2, ....., a_n, ....\}$ where $a_1, a_2, ........, a_n$ ..... are distinct of A.

Conversely, let every elements of A can be put in the form of an elements of A.

Let A = $\{a_1, a_2, ....., a_n, ....\}$ where $a_1, a_2, ........, a_n$ ..... are distinct elements of A.

Define a function $g:\mathbb{N} \to A$ as g (n) = an ∀ n ∈ ℕ

Then we claim g is a bijective function.

For one-one Let $n_1 \neq n_2 \in \mathbb{N} \Rightarrow a_{n_1} \neq a_{n_2} \Rightarrow f(n_1) \neq f(n_2)$

∴ $n_1 \neq n_2 \Rightarrow f(n_1) \neq f(n_2)$

So, $f$ is one-one

For onto Let y ∈ A be any element

∴ $\exists\, a_n \in A$ such that $y = a_n$

Thus $\exists\ n \in \mathbb{N}$ such that $g(n) = a_n = y$

$\therefore$ g is onto.

Thus g is bijective function

Hence $A \sim \mathbb{N}$ i.e., A is countably infinite set.

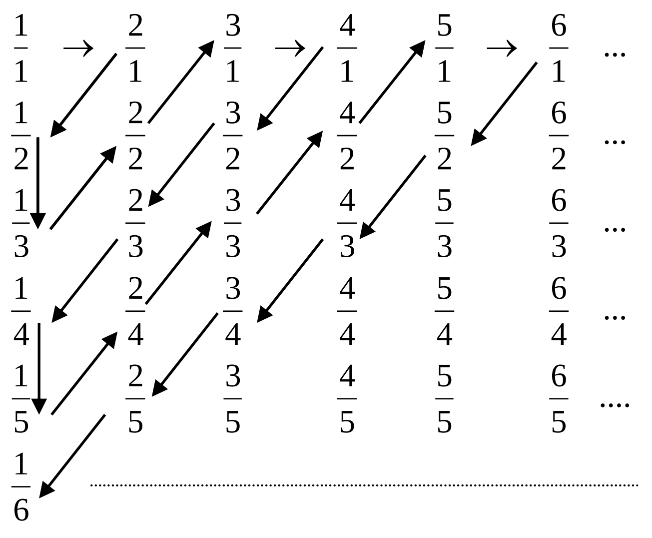
**Remark :** Every infinite set contains a denumerable set.

**Art 2.** Prove that the set of rational numbers is denumerable.

**Or**

Find a bijection between the set of rational numbers and the set of all positive integers.

**Proof :** Firstly, we will take positive rational numbers only and write them all in the order of magnitude i.e. all numbers whose denominator is 1, then all fraction with denominator is 2, then all fraction with denominator 3 and so on as shown below by the arrow.



[Here arrow follows the addition of numerator and denominator in each fraction as 2, 3, 4, ......]

If we write down numbers in the order of succession indicated by arrow after leaving out those numbers, which have already appeared, then every positive rational number can be written as the sequence $\left\{\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, ........\right\}$ and if we denote the sequence by $\{s_1, s_2, s_2, .....\}$ then

Clearly,

$$Q = \{0 - s_1, s_1, -s_2, s_2, -s_3, ........\}$$

Thus, the set of rational numbers can be written as an infinite sequence with distinct elements.

Then we can define a bijection map $f$ : N → Q as $f$ (1) = 0 and

$$f(n) = \begin{cases} s_{\frac{n}{2}} & ;if\ n\, is\ even \\ -s_{\frac{n-1}{2}} & ;if\ n\, is\ odd \end{cases}$$

N ~ Q.

Hence Q is denumerable.

**Example:** Prove that the set of all sequences whose elements are either zero or one is not countable.

**Solution:** Let A be the set of all sequences with elements 0 and 1 only.

For example the sequence < 0, 1, 0, 1, 0, 1, ....... >∈ A

Suppose that A is countable. Then A can be written P$f_1$, $f_2$, $f_3$,.......} where each $f_i$, i∈ N is a sequence such that $f_i$ (n) = 0 or 1; ∀ n ∈N, i∈ N.

Consider the sequence $f$ such that

$$f(n) = \begin{cases} 1\ if\ f_n(n) = 0 \\ 0\ if\ f_n(n) = 1 \end{cases} \qquad .... (1)$$

Since all the elements of the sequence $f$ are either 0 or 1, therefore $f$∈ A.

But $f \neq f_n$, ∀ n ∈N (∵ of (1))

So, we arrive at a contradiction.

Hence our supposition is wrong. Therefore A is uncountable.

**Some More Examples**

**Example 7:** If a function $f$ : A → B is one-one and B is countable, there prove that A is atmost countable.

**Solution:** Let A be any set

Case (i) When A is a finite set, then A is almost countable

Case (ii) When A is infinite set.

Since $f$ : A → B is one-one. Then A ~$f$ (A)

∴ $f$ (A) is also infinite set.

⇒ B is infinite set [∵ super set of infinite set is infinite $f$ (A) ⊆ B]

As B is countable set.

∴ B is countably infinite set.

Let B = {$b_1$, $b_2$, $b_3$, ......}

Let $n_1$ be the first positive integer such that $b_{n_1} \in f$ (A)

Let $n_2 > n_1$ be the next positive integer such that $b_{n_2} \in f$ (A)

Continuing in this way.

We choose $n_1, n_2, \ldots, n_{k-1}, n_k \in$ N such that $n_k > n_{k-1} > n_{k-2} > \ldots$

Such that $b_{n_k} \in f$ (A). We get

$f$ (A) = $\{b_{n_1}, b_{n_2}, \ldots\}$

In other word the element of $f$ (A) can be arranged in the form of a sequence

$\therefore$ $f$ (A) ~ N also A ~ $f$ (A)

$\Rightarrow$ A ~ N [$\therefore$ ~ is transitive relation]

i.e. A is countable set

Thereforecombinding the two cases, we find that either A is a finite set or a countable set.

Hence A is atmost countable.

**Example 8:** Prove that every infinite set is equivalent to a proper subset of itself.

**Solution:** Let A be infinite set

$\therefore$ by previous theorem A has a countable set (say) B.

Let C = A - B $\Rightarrow$ A = B $\cup$ C and B $\cap$ C = $\phi$

Since B is countable set $\therefore$ its elements can be arranged in the form of an infinite sequence with distinct elements.

Let B = $\{b_1, b_2, b_3, \ldots\}$ be countable set

We take another set $B_1 = \{b_2, b_3, b_4 \ldots\}$

$\therefore$ $B_1$ is a proper subset of B.

Define a function $\phi : B \rightarrow B_1$ by $\phi(b_1) = b_{i+1}$.

Clearly $\phi$ is one-one and onto $\therefore$ B ~ $B_1$

Let $A_1 = B_1 \cup C$

Since $B_1$ is a proper subset of B $\therefore$ $B_1 \cup C$ is a proper subset of B $\cup$ C.

$\Rightarrow$ $A_1$ is a proper subset of A.

But B ~ $B_1$ and B $\cap$ C = $\phi$

Also C ~C and $B_1 \cap C = \phi$

Implies that B $\cup$ C ~ $B_1 \cup C$ i.e. A ~ $A_1$

Where $A_1$ is a proper subset of A

Hence every infinite set is equivalent to a proper subset of itself.

**Example 9:** Prove that the set of rational numbers is denumerable.

**Solution:** We know that the set of integers is a denumerable set as ∃ a bijection $f$ : N → Z defined by

$$f(n) = \begin{cases} \frac{n}{2} & ; if\ n\ is\ even \\ -\frac{n-1}{2} & ; if\ n\ is\ odd \end{cases}$$

i.e. N ~ Z

Let for each n ∈N, Define the set

$$A_n \left\{ \frac{m}{n} : m \in Z \right\}$$

Then $A_n$~ Z for ∃ a bijection g : Z → $A_n$ by

$$g(m) = \frac{m}{n}; \forall m \in Z$$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left\{ \frac{m}{n} : m \in Z \right\} = Q$$

Since each An is denumerable set and countable union of denumerable sets is denumerable set implies that $\bigcup_{n=1}^{\infty} A_n$ is denumerable set.

**Example 10:** Prove that Cartesians product of two countable sets is a countable set.

**Solution:** Let A and B be two countable sets

∴ ∃'s bijection $f$ : N → A and g : N → B

Let us defined a map h : N ×N → A × B by

h (m, n) = ($f$ (m), g (n))

For one-one Let h ($m_1$, $n_1$) = h ($m_2$, $n_2$)

⇒ ($f$ ($m_1$), g ($n_1$) = ($f$ ($m_2$), g ($n_2$))

⇒ $f$ ($m_1$) = $f$ ($m_2$) and g ($n_1$) = g ($n_2$)

⇒ $m_1 = m_2$ and $n_1 = n_2$ (∵ both $f$ and g are one-one)

∴ ($m_1$, $n_1$) = ($m_2$, $n_2$)

So h is one-one

For onto. Let (a, b) $\in$ A $\times$ B be any element where a $\in$A and b $\in$B.

As $f$ and g are onto maps $\therefore \exists$ m $\in$ N and n $\in$N such that $f$ (m) = a and g (n) = b

Thus $\exists$ (m, n) $\in$ N $\times$N such that

h (m, n) = ($f$ (m), g(n)) = (a, b)

$\therefore$ h is onto

Hence h is a bijection between N $\times$N to A $\times$ B

$\therefore$ N $\times$N~ A $\times$ B

As N $\times$N is countable set $\therefore$ A $\times$ B is also countable set

Thus the product of two countable sets is a countable set.

## 1.6 Algebraic and Transcendental Numbers

**Algebraic Number -** The roots of a polynomial equation with integral coefficients are called algebraic numbers.

If þ (x) = $a_0 + a_1 x + a_2 x^2 + ..... + a_n x^n = 0$ a polynomial equation, where each of ai $\in$ z. Then the roots of þ(x) are algebraic numbers

**Transcendental Numbers -** A number which is not an algebraic number is said to be transcendental number. In other words a number which is not a root of a polynomial equation with integral coefficient is called a transcendental number

For Instance, e, $\pi$, log 2 etc. are transcendental numbers.

**Art. 3** Show that the set of algebraic numbers is countable set.

**Proof:** Consider a polynomial equation

p = (þ (x) = þ (x) is a polynomial with integral coefficient}

Where

þ (x) = $a_0 + a_1 x + ...... + a_m x^m$5 : $a_i$tz we first prove that p is countable.

For each ordered pair of natural number (m, n), let

$P_{mn}$ = {þ(x) : þ (x) $\in$ P s.t. $|a_0| + |a_1| + ..... + |a_m| = n$}

$\Rightarrow$ $P_{mn}$ is a finite set

$\Rightarrow$ $P_{mn}$ is countable ($\because$ degree m of a polynomial is fixed and sum of finite number if terms is finite)

Now

P = U {$P_{mn}$ : (m, n) $\in$ N $\times$ N}

= countable union of countable set

= countable set

Further, Let

$E = \{þ_i (x) = 0, i \in A\}$, A is contable set.

For each $i \in A$ Let

$Ai = \{x : x \text{ is a root of } pi (x) = 0, þ_i (x) = 0, þ_i \in E\}$

Then $A = UA_i$ = set of all algebraic number.

Since each $þ_i (x) = 0$ is of finite degree m has atmost m roots,

$\therefore$ Each $A_i$ is finite and so countable

$\therefore$ $A = U A_i$ is the countable union of countable $i \in Ai$

set is also countable set

Hence set of all algebraic numbers is a countable set

**Remark:** If a finite number of elements are added in a countable set, then remitting set is also countable.

**Art. 4:** Show that set of transcendental numbers is uncountable set

**Proof:** Let T = set of all transcendental numbers

A = set of all algebraic numbers

If possible, let T is countable, Then T U A is countable

Since $R \subseteq AUT$, R set of real numbers

$\Rightarrow$ R is countable ($\because$ subset of countable set is countable)

our supposition is wrong

Hence the set of all transcendental number is uncountable.

Let us understand the concept with the keep of some examples.

**Example 11:** Prove that the set of rational number is denumerable.

**Solution:** We know that the set of integers is a denumerable as $\exists$ a bijection $f : N \to Z$ defined as

$$f(n) = \begin{cases} \frac{n}{2}, & n \text{ even} \\ -\frac{n-1}{2}, & n \text{ odd} \end{cases}$$

i.e. $N \sim z$.

For each $n \in N$, define

$$A_n = \left\{\frac{m}{n} : m \in Z\right\}$$

Then    $A_n$~ z for ∃ a bijection g : z → $A_n$ by g (m) = $\frac{m}{n}$ ∀ m ∈ z

∴    $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left\{ \frac{m}{n} : m \in Z \right\}$ = Q

Since each An is denumerable set and countable union of there sets is denumerable let implies that $\bigcup_{n=1}^{\infty} A_n$ is denumerable set.

**Example 12:** Show that Cartesian product of two countable sets is a countable set

**Solution:** Let A, B be two countable sets

∴    ∃ a bijection

$f$ : N → A, g : N → B

Define a map

h : n ×N → A × B by

h (m, n) = ($f$(m), g(n))

Claim : h is one-one

Let h ($m_1$, $n_1$) = h/$m_2$, $n_2$)

⇒    ($f$ ($m_1$), g ($n_1$) = ($f$ ($m_2$), g ($n_2$))

⇒    $f$ ($m_1$) = $f$ ($m_2$) and g ($n_1$) = g ($n_2$)

⇒    $m_1 = m_2$ and $n_1 = n_2$    (∵ $f$, g are one-one)

∴    ($m_1$, $n_1$) = ($m_2$, $n_2$)

h is one-one

h is onto:

Let    (a, b) ∈ A × B, a ∈A, b ∈B

Since $f$, g are onto,

∴    ∃    m ∈ N, n ∈Ns.t.

$f$ (m) = a, g (n) = b

⇒    ∃ (m, n) ∈ N ×Ns.t.

h (m, n) = ($f$ (m), g (n)) = (a, b)

∴    h is onto

Hence h is a bijection between N × N to A × B further, N ×N is countable

⇒    A × B is also countable

Thus product of two countable set is a countable set.

## 1.7 Self Check Exercise

Q.1 Prove that set of all sequences whose elements are either zero or one is not countable

Q.2 Prove that the set [0, 1] defined as

$[0, 1] = \{x \in R : 0 \leq x \leq 1\}$ is uncountable

Q.3 Prove that every subset of countable set is atmost countable.

## 1.8 Summary

In this unit we have learnt

(i) equivalent set and equivalence relation

(ii) finite and infinite sets

(iii) countable and uncountable set

(iv) algebraic and transcendental numbers

## 1.9 Glossary:

**1. Atmost Countable -** A set A is called atmost countable if A is either finite set or a countable set

**2. Denumerable or Enumerable set -** An infinile set is called countable infinite or denumerable or enumerable set iff $A \sim N$.

## 1.10 Answer to Self Check Exercise

Ans.1 Consider the set <0, 1, 0, 1, 0, 1, .....> and then proceed.

Ans.2 Suppose A = [0, 1] be denumerable and prove it to be wrong.

Ans.3 Consider a set A be a subset of countable set B and then proceed.

## 1.11 References/Suggested Readings

1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) p. Ltd., 2002

2. E. Fischer, Intermediate Real thalysis, springer Verlag, 1983.

## 1.12 Terminal Questions

1. Prove that the set of complex number is uncountable

2. Show that union of two denumerable sets is denumerable

3. If A is a countable set and $A \sim B$, then prove that B is also countable

4. Prove that union of binite number of countable sets is a countable set.

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**Unit - 2**

# Real Line

**Structure**



## 2.1 Introduction

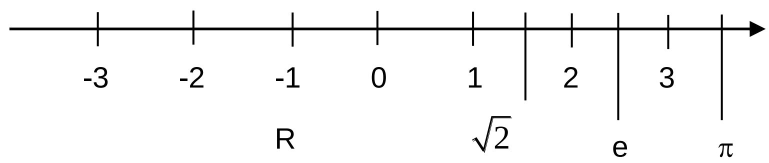
Dear students, you are already familiar with the concept of a set defined as the collection of well defined distinct objects of our perception of thought. Set are usually denoted by capital letters A, B, C etc and the elements of the set by small letters a, b, c etc. 1, 2, 3, ..... are called natural numbers and their set is denoted by N.

$\therefore$ N = {1, 2, 3, .....}

The set of natural numbers is closed under the operations of addition and multiplication but is not closed under the operation of subtraction. So the operation of subtraction intended the set of natural numbers by the introduction of numbers 0, -1, -2, ....0, 1, 2, 3, .... to it. This extended set is called the set of integers, denoted by Z or I.

$\therefore$ Z or I = {-3, -2, -1, 0, 1, 2, 3, .....}.

The set of inligers is not closed under the operation of division. So here comes the existence of rational numbers. The set of real numbers is denoted by R, which include the rational numbers such as. The integer-5 and the fraction $\frac{4}{3}$. The rest of real numbers are called irrational numbers. Some irrational numbers (as well as rational) are the root of a polynomial with integer coefficients, such as square root of $\sqrt{2}$ = 1.414 ......, there are called algebraic number. There are also real numbers which are not, such as $\pi$ = 3.1415.....; these are called transcendental numbers. Real number can be thought of as all points on a line called the number line or real line, where the points corresponding to integers (....2, -1, 0, 1, 2,.....) are equally spaced.




## 2.2 Learning Objectives

The main objectives of this unit are

(i) to study the concept of rational number

(ii) to study real numbers and real line

(iii) to learn about the concept of field of real numbers

(iv) to study order relation in R

(v) to learn the concept of 'between'.

## 2.3 Rational Number :

Any number of the form $\frac{p}{q}$ where p, q $\in$ I, q $\neq$ 0 and (p, q) = 1 is called a rational number.

The set of rational numbers is denoted by Q.

$$\therefore \quad Q = \left\{\frac{p}{q} : p, q \in I, q \neq 0, (p,q) = 1\right\}$$

The set of rational numbers consists of integers and fractions.

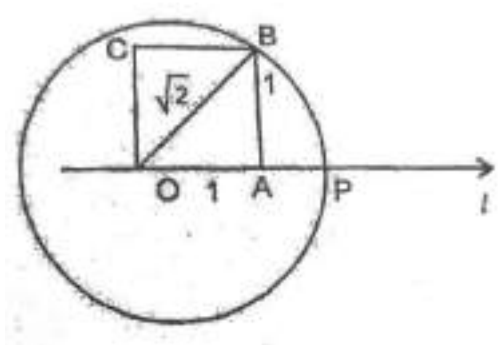
Any number which is not rational, is called an irrational number, $\sqrt{2}$, $\sqrt{3}$ etc. are irrational numbers.

Here is should be noted properly that every rational number can be expressed as a terminating or recurring decimal whereas every irrational number can be expressed as a non-terminating infinite decimal.

**Inadequacy of the Rational Number System**

Every rational number can be represented as a point on the number line but every point on the number line cannot always correspond to a rational number.

For a given rational number x, we can find a point on the number line I such that the difference of P from O is |x|. But for each point P on I we cannot always find a rational number x. For example, consider a unit length on the number line I such that |OA| = 1. Complete the square OABC.




Then $OB^2 = 2$ or $OB = \sqrt{2}$

Draw a circle of radius $\sqrt{2}$ with centre O which meets the number line at P so that $|OP| = \sqrt{2}$. This point P on I corresponds to the no. $\sqrt{2}$ which is not a rational number.

## 2.4 Real Number :

A number which is either rational or irrational is called a real number. The set of real numbers is denoted by R. The set R consists of all rational and irrational numbers.

**Prime Number :** A positive integer, greater than 1, which has only two factors 1 and itself is called a prime number.

It should be noted that if p is a prime number and p divides a b then either p divides a or p divides b where $a, b \in I$.

**Note :** 1 is not a prime number and 2 is the only even number which is prime.

## 2.5 Field of Real Numbers

Let R be the set of real numbers. Then R together with binary operations + and called respectively the addition and multiplication of real numbers, satisfies the following properties:

### I. Properties of addition operation (+)

(1) Closure Property

$a + b \in R \quad \forall \quad a, b \in R$

(2) Commutative Law

$a + b = b + a \quad \forall \quad a, b \in R$

(3) Associative Law

$(a + b) + c = a + (b + c) \quad \forall \quad a, b, c \in R$

(4) Additive Identity

There exists $0 \in R$, called zero-element of R, such that $a + 0 = \alpha \forall \alpha \in R$ '0' is called additive identity.

(5) Additive Inverse

For each $\alpha \in R$, there exists $-\alpha \in R$ such that $\alpha + (-\alpha) = 0 = (-\alpha) + \alpha$ '$-\alpha$' is called additive inverse (or negative) of $\alpha$.

**II. Properties of multiplication operation (.)**

(1) Closure Property

$a. b \in R \quad \forall \quad a, b \in R$

(2) Commutative Law

$a.b = b.a \quad \forall \quad a, b$ R

(3) Associative Law

$(a.b).c = a.(b.c) \quad \forall \quad a, b, c \in R$

(4) Multiplicative identity

There exists $1 \in R$, called unity of R, such that

$\alpha.1 = \alpha = 1.\alpha \quad \forall \quad \alpha \in R$

'1' is called multiplicative identity.

(5) Multiplicative Inverse

For each $\alpha\ (\neq 0) \in R$, there exists $\frac{1}{\alpha} \in R$ such that $\alpha.\frac{1}{\alpha} = 1 = \frac{1}{\alpha}.\alpha$ '$\frac{1}{\alpha}$' is called multiplicative inverse (or reciprocal) of $\alpha$.

**III. Distributive Property**

$a.(b + c) = a.b + a.c$ (Left distributive law)

$(b + c).a = b.a + c.a$ (Right distributive law)

$\forall\ a, b, c \in R$

R, the set of reals satisfying the above properties, is called a field.

**Important Properties of field of Real Numbers**

Some important properties of field R are

(i) The additive identity element 0 is unique.

(ii) The multiplicative identity element 1 is unique.

(iii) The additive inverse of a real number is unique.

(iv) The multiplicative inverse of a non-zero real number is unique.

(v) If $a, b, c \in R$ and $a + c = b + c$ then $a = b$

i.e. cancellation law holds for addition

(vi) If a, b, c ($\neq$ 0) $\in$ R and a.c = b.c then a = b

i.e. cancellation law holds for multiplication

(vii) If a, b $\in$ R and a.b = 0 then either a = 0 or b = 0

(viii) If a $\in$ R, then a.0 = 0.a

(ix) - (-a) = a$\forall$ a$\in$ R and $\frac{1}{\frac{1}{\alpha}}$ = a $\forall$a ($\neq$ 0) $\in$ R

(x) $(a^{-1})^{-1}$ = a, a ($\neq$ 0) $\in$ R where a-1 is multiplicative inverse of a.

(xi) (-a) + (-b) = - (a + b) $\forall$ a, b $\in$ R

(xii) (-a).(-b) = a.b$\forall$ a, b $\in$ R

(xiii) (-a).(b) = - a.b$\forall$ a, b $\in$ R

(xiv) (a).(-b) = -a.b$\forall$ a, b $\in$ R

(xv) (-1) a = a = (a) (-1) $\forall$ a $\in$ R

**Proof:** (v) Here a, b, c $\in$ R Also - c $\in$ R

Now a + b = b + c

$\Rightarrow$ (a + c) + (- c) = (b + c) + (- c)

$\Rightarrow$ a + {c + (-c)} = b + {c + (-c)}

$\Rightarrow$ a + 0 = b + 0

$\Rightarrow$ a = b

**Positive and Negative Real Numbers**

We know that a real number 'a' is either positive or negative or zero. If a is positive then we write a > 0 and if a is negative then we write a < 0.

Following results may be kept in mind :

(1) a > 0 $\Rightarrow$ a $\leq$ 0

(2) a < 0 $\Rightarrow$ a $\geq$ 0

(3) a > 0 $\Rightarrow$ -a < 0

(4) a < 0 $\leftrightarrow$ -a > 0

(5) a > 0, b > 0 $\Rightarrow$ a + b > 0

(6) a > 0, b > 0 $\Rightarrow$ a.b> 0

(7) a < 0, b < 0 $\Rightarrow$ a + b < 0

(8) $a < 0, b < 0 \Rightarrow a.b > 0$

(9) $a > 0, b < 0 \Rightarrow a.b < 0$

(10) $a < 0, b > 0 \Rightarrow a.b < 0$

$\forall a, b \in R$

## 2.6 Order Relation in R

Let a and b be any two real numbers. We say that a is greater then b if $a - b > 0$. Then we write $a > b$ if real point representing a is to the right of the point representing b.

**Properties of Order Relation in R**

**(1) Trichotomy Law**

If a and b are any two real numbers, then one and only one of the following possibilities holds :

(i) $a > b$ (ii) $a = b$ (iii) $a < b$

**(2) Order relation is transitive in R**

If $a, b, c \in R$, then $a < b$ and $b > c \Rightarrow a < c$

**(3) Order relation is preserved under addition**

If $a, b, c \in R$, then $a > b \Rightarrow a + c > b + c$

**(4) Order relation is preserved under multiplication**

If $a, b, c \in R$, then $a > b$ and $c > 0 \Rightarrow a.c > b.c$

**Note:**

(1) If $a, b, c \in R$, then $a < b \Rightarrow a + c < b + c$

(2) If $a, b, c \in R$, then $a < b$ and $c > 0 \Rightarrow a.c < b.c$

(3) If $a, b, c \in R$, then $a > b$ and $b > c \Rightarrow a > c$

(4) $a \geq b$ and $b \geq c \Rightarrow a \geq c \ \forall a, b, c \in R$

(5) $a \leq b$ and $b \leq c \Rightarrow a \leq c \ \forall a, b, c \in R$

**Art.7.** Prove that

(i) $a > b \Rightarrow -a < -b \ \forall a, b \in R$

(ii) $a > 0 \Rightarrow \frac{1}{\alpha} > 0 \ \forall a \in R$

(iii) $a < 0 \Rightarrow \frac{1}{\alpha} < 0 \ \forall a \in R$

(iv) $a > b, c < 0 \Rightarrow a.c < b.c \ \forall a, b, c \in R$

(v) $a < b$ and $c > 0 \Rightarrow a.c < b.c \ \forall a, b, c \in R$

**Proof:** Let a > b

∴ $a - b > 0$ ..... (1)

Now $(-b) - (-a) = -b + a = a - b > 0$ [∵ of (1)]

∴ $-b > -a$ ⇒ $-a < -b$

∴ either $a > 0$ or $a < 0$

In case $a > 0$, then $a, b < 0$

⇒ $\frac{1}{a}, (a, b) < \frac{1}{a}, 0$ $\left[\because \frac{1}{a} > 0\right]$

⇒ $\left(\frac{1}{a}, a\right), b < 0$

⇒ $1, b < 0$

⇒ $b < 0$

In case $a < 0$, then $a, b < 0$

⇒ $\frac{1}{a}, (a, b) > \frac{1}{a}, 0$ $\left[\because \frac{1}{a} < 0\right]$

⇒ $\left(\frac{1}{a}, a\right), b > 0$

⇒ $1, b > 0$

⇒ $b > 0$

∴ either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

**Art.** (i) If $a, b \in R$, then $a > b > 0 \Leftrightarrow \frac{1}{a} < \frac{1}{b}$

(ii) If a and b be positive real numbers, then $a < b$ iff $a^2 < b^2$

(iii) If a, b, c and d are all positive real numbers and $a \leq b$, $c \leq d$, then $\frac{a}{c} \geq \frac{b}{d}$.

**Proof :** (i) We have $a > 0$, $b > 0$

∴ $a . b > 0$ ⇒ $\frac{1}{a.b} > 0$

Now $a > b$

⇒ $\left(\frac{1}{a.b}\right). a > \left(\frac{1}{a.b}\right).b$

$\Rightarrow \quad \left(\frac{1}{b.a}\right)$. a > $\left(\frac{1}{a.b}\right)$. b

$\Rightarrow \quad \frac{1}{b} \cdot \left(\frac{1}{a}.a\right) > \frac{1}{a} \cdot \left(\frac{1}{b}.b\right)$

$\Rightarrow \quad \frac{1}{b}$ . 1 > $\frac{1}{a}$ . 1

$\Rightarrow \quad \frac{1}{b} > \frac{1}{a}$

$\Rightarrow \quad \frac{1}{a} < \frac{1}{b}$

(ii) a > 0, b > 0

$\Rightarrow$ b + a > 0

$\Rightarrow \quad \frac{1}{b+a} > 0$

Now a2 < b2

iff b2 - a2 > 0

i.e., iff (b + a) (b - a) > 0

i.e., iff $\frac{1}{b+a}$ . (b + a) (b - a) > $\frac{1}{b+a}$ . 0

i.e., iff b - a > 0

i.e., iff a < b

which is true.

Hence the result.

(iii) a $\geq$ b, c $\leq$ d

$\Rightarrow$ a $\geq$ b, $\frac{1}{c} \geq \frac{1}{d}$

$\Rightarrow$ a. $\frac{1}{c} \geq$ b. $\frac{1}{d}$

$\Rightarrow \quad \frac{a}{c} \geq \frac{b}{d}$

## 2.7 Between

Let a, b and c be any three real numbers. If a < c < b or b < c < a then c is said to lie between a and b.

**Art.** (b) Prove that the arithmetic average of two (distinct) real numbers is a real number and it lies between them.

**Proof.** Let a and b be any two distinct real numbers. Without any loss of generality, we take a < b.

The arithmetic mean of a and b is $\frac{a+b}{2}$

Now $a + b \in R \quad \forall \quad a, b \in R$

Also $\frac{1}{2} \in R$

$\therefore \quad \frac{a+b}{2} \in R$ i.e., A.M. of a and b is a real number.

Further $\quad a < b$

$\Rightarrow \quad a + a < b + a$

$\Rightarrow \quad 2a < a + b$

$\Rightarrow \quad a < \frac{a+b}{2}$ ....(1)

Again $\quad a < b$

$\Rightarrow \quad a + b < b + b$

$\Rightarrow \quad a + b < 2\,b$

$\Rightarrow \quad \frac{a+b}{2} < b$ ...(2)

From (1) and (2), we get

$a > \frac{a+b}{2} > b$

$\therefore$ the arithmetic average of two distinct real numbers a and b is a real number $\frac{a+b}{2}$ and lies between a and b.

Cor. Between two distinct real numbers, there lie infinitely many real numbers.

Proof. We know that if $a, b \in R$ then $\frac{a+b}{2} \in R$.

$\therefore$ there lies a real number $\frac{a+b}{2}$ between two given distinct real numbers a and b.

Again the real number $\frac{a+\frac{a+b}{2}}{2}$ i.e., $\frac{3a+b}{4}$ lies between a and $\frac{a+b}{2}$

$\therefore$ real number $\frac{3a+b}{4}$ lies between two real numbers a and b.

$$\left[\because \text{ any number which lies between a and } \frac{a+b}{2} \text{ lies between a and b as } \frac{a+b}{2} < b\right]$$

Proceeding is this way, we can show that there lies any infinity of real numbers between two distinct real numbers a and b.

**Note.** This property of real numbers is called denseness property of reals.

**Some Illustrated Examples**

**Example 1 :** If b > a > 0 and c > 0, then $\frac{a+c}{b+c} > \frac{a}{b}$

**Solution :** Here b > a > 0 and c > 0

Now $\frac{a+c}{b+c} - \frac{a}{b}$

$= \frac{b(a+c)-a(b+c)}{b(b+c)}$

$= \frac{ba+bc-ab+ac}{b(b+c)}$

$= \frac{bc-ac}{b(b+c)}$

$= \left\{\frac{c}{b(b+c)}\right\}$ (b - a) > 0

$$\left[\because \text{ b - a} > 0,\ \frac{c}{b(b+c)} > 0 \text{ as b} > \text{a} > 0,\ \text{c} > 0\right]$$

$\therefore$ $\frac{a+c}{b+c} - \frac{a}{b} > 0$

$\Rightarrow \qquad \frac{a+c}{b+c} > \frac{a}{b}$

**Example 2 :** If a > b > 0, then prove that $\sqrt{ab}$ lies between a and b.

**Solution :** Here a > b >

Now $\quad a > b$

$\Rightarrow \quad a.a > a.b$

$\Rightarrow \quad a^2 > ab$

$\Rightarrow \quad a > \sqrt{ab}$ ....(1)

Again a > b

$\Rightarrow \quad a.b > b.b$

$\Rightarrow \quad ab > b^2$

$\Rightarrow \quad \sqrt{ab} > b$ ...(2)

From (1) and (2).

$a > \sqrt{ab} > b$

$\Rightarrow \quad \sqrt{ab}$ lies between a and b.

**Example 3 :** If a $\geq$ 0, b $\geq$ 0, then prove that $\frac{a+b}{2} \geq \sqrt{ab}$ , with equality holding iff a = b.

**Solution :** Here a $\geq$ 0, b $\geq$ 0

Now $\frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{a+b-2\sqrt{a}\sqrt{b}}{2}$

$= \frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} \geq 0$

$= \frac{a+b}{2} - \sqrt{ab} \geq 0$

$= \frac{a+b}{2} \geq \sqrt{ab}$

Then equality $\frac{a+b}{2} = \sqrt{ab}$ holds

iff $a + b = 2\sqrt{a}\sqrt{b}$

i.e., iff $a + b - 2\sqrt{a}\sqrt{b} = 0$

i.e., iff $\left(\sqrt{a}-\sqrt{b}\right)^2 = 0$

i.e., iff $\sqrt{a} - \sqrt{b} = 0$

i.e., iff $\sqrt{a} = \sqrt{b}$

i.e., iff $a = b$

**Example 4 :** If x, y are rationals such that x < y, then prove that $x < x + \frac{y-x}{n} < y$, there n is a natural number > 1.

**Solution :** $x < y \Rightarrow y > x$

$\Rightarrow \quad y - x > 0$

$\Rightarrow \quad \frac{y-x}{n} > 0$

$\Rightarrow \quad \frac{y-x}{n} + x > 0 + x$

$\Rightarrow \quad x < x + \frac{y-x}{n}$ ....(1)

Again $y - x > 0$

$\Rightarrow \quad (n - 1)(y - x) > 0$

$\Rightarrow \quad n(y - x) - (y - x) > 0$

$\Rightarrow \quad n(y - x) > y - x$

$\Rightarrow \quad y - x > \frac{y-x}{n}$

$\Rightarrow \quad y > x + \frac{y-x}{n}$

$\Rightarrow \quad x + \frac{y-x}{n} < y$ .... (2)

From (1) and (2), we have

$x < x < \frac{y-x}{n} < y$, n $\in$N, n > 1.

**Example 5:** Ifþ is a prime number, then $\sqrt{p}$ is not a rational number.

Solution : Let us suppose that $\sqrt{p}$ is rational number.

$\therefore \quad \sqrt{p} = \frac{a}{b}$, a, b $\in$ I, b $\neq$ 0 (a, b) = 1 .... (1)

$\Rightarrow$ þ = $\frac{a^2}{b^2}$

$\Rightarrow$ $a^2$ = þ $b^2$ .... (2)

Now þ is a factor of þ $b^2$

$\therefore$ þ is factor of $a^2$

$\therefore \quad \frac{p}{a^2} \quad \Rightarrow \quad \frac{p}{a}$

Let þ = þk, k $\in$ I

From (2) þ$^2$ $k^2$ = þ $b^2$

$\Rightarrow$ $b^2$ = þ$k^2$ ...(3)

Now þ is factor of þ$k^2$

$\Rightarrow$ þ is factor of $b^2$

$\Rightarrow \quad \frac{p}{b^2} \quad \Rightarrow \quad \frac{p}{b}$

$\therefore$ þ is a common factor of a and b which contradicts the fact that (a, b) = 1 as p is prime number.

$\therefore$ our supposition is wrong

$\therefore$ $\sqrt{p}$ is not a rational number

**Example 6:** Prove that $\sqrt{3}$ - $\sqrt{2}$ is not a rational number.

**Solution:** We shall first show that $\sqrt{3}$ is not a rational number.

Let if possible $\sqrt{3}$ is a rational number.

$\therefore \quad \sqrt{3} = \frac{p}{q}$, q $\neq$ 0 (p, q) = 1, p, q $\in$ I .... (1)

From (1) we have

$$3 = \frac{p^2}{q^2}$$

$\Rightarrow$ $p^2 = 3q^2$ .... (2)

Clearly R.H.s. of (2) has a factor 3

$\therefore$ 3 is a factor of L.H.S. of (2)

$$\therefore \quad \frac{3}{p^2} \quad \Rightarrow \quad \frac{3}{p}$$

Let $p = 3k, k \in I$

$\therefore$ from (2) $9k^2 = 3q^2$

$\Rightarrow$ $q^2 = 3k^2$ .... (3)

Now 3 is a factor of R.H.S. of (3)

$\therefore$ 3 is factor of L.H.S. of (3)

$$\therefore \quad \frac{3}{p^2} \quad \Rightarrow \quad \frac{3}{p}$$

$\therefore$ 3 is a common factor of p and q, which contradicts the fact that (p, q) = 1

$\therefore$ our supposition is wrong

$\therefore$ $\sqrt{3}$ is not a rational number.

Now, if possible, Let $\sqrt{3} - \sqrt{2}$ is a rational number and let $\sqrt{3} - \sqrt{2} = r$ (say)

$\Rightarrow$ $\sqrt{2} = \sqrt{3} - r$

$\Rightarrow$ $2 = 3 + r^2 - 2\sqrt{3}\, r$

$$\Rightarrow \quad \sqrt{3} = \frac{r^2 + 1}{2r}$$

Now $\frac{r^2+1}{2r}$ is a rational number as r is rational

$\Rightarrow$ $\sqrt{3}$ is rational, which is contrary to the fact that $\sqrt{3}$ is not a rational number

$\therefore$ our supposition is wrong

Hence $\sqrt{3} - \sqrt{2}$ is not a rational numbers.

## 2.8 Self Check Exercise

Q.1 Show that $\sqrt{5}$ is not a rational number.

Q.2 Prove that $\sqrt{2}$ - $\sqrt{5}$ is an irrational number

Q.3 If a > b > 0 and 0 < c < d, then show that $\frac{a}{c} > \frac{b}{d}$, a, b, c, d $\in$ R

Q.4 If x, y are positive reals and n is a positive integer, then prove that $x^n<y^n$ iff x < y.

## 2.9 Summary

In this unit we have learnt the following:

(i) rational numbers

(ii) concept of real numbers and real line

(iii) field of real numbers

(iv) order relation in R

(v) the concept of 'between'.

## 2.10 Glossary:

1. The set of all real number is denoted by R or R. As it is naturally endwed with the structure of field, the expression field of real numbers is frequently used when its algebraic properties are under consideration.

2. The sets of positive real numbers and negative real numbers are often noted $R^+$ and R- respectively.

3. The notation $R^n$ refers to the set of the n-triples of elements of R.

## 2.11 Answer to Self Check Exercise

Ans.1 Use the concept of contradiction

Ans.2 Proceed similar to Example 6

Ans.3 Easy to prove

Ans.4 Assume x < y and then proceed. Also assume $x^n<y^n \forall$ n $\in$N and then proceed.

## 2.12 References/Suggested Readings

1. K.A. Ross, Elementary Analysis- The theory of calculus series - undergraduate Texts in Mathematics, Springer Verlag, 2003

2. R.G. Bartle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000.

## 2.13 Terminal Questions

1. Show that $\sqrt{2}$ is not a rational number

2. If a, b, c $\in$ R then

(i) $a > b, c > 0 \Rightarrow a.c>b.c$

(ii) $a < b, c < 0 \Rightarrow a.c>b.c$

3. If $x \in R$, show that

   $x^2 \geq 0$

4. If a, b, c d are distinct reals such that

   $(a^2 + b^2 + c^2)\ p^2 - 2\ (ab + bc + cd)\ p + (b^2 + c^2 + d^2) < 0$ then a, b, c, d are in G.P.

5. If (i) $a > 0, b > 0$ then $a < b$ iff $\frac{1}{a} < \frac{1}{b}$

   (ii) $a < 0, b > 0$ then $a < b$ iff $\frac{1}{a} > \frac{1}{b}$

   (iii) $a < 0, b > 0$ then $a < b$ iff $\frac{1}{a} < \frac{1}{b}$

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## Unit - 3
# Intervals

**Structure**



### 3.1 Introduction

Dear students, we shall study the concept of intervals in this unit. A interval (real) is the set of all numbers lying between two fixed end points with no gaps. Each end point is either a real number or positive or negative infinity, indicating the interval extends with out a bound. An interval contains neither end point, either end point, or both end points. For instance, the set of real numbers consisting of 0, 1, and all numbers in between is an interval, denoted by [0, 1] and is called unit interval ; the set of all positive real number is an interval, denoted by $(0, \infty)$; the set all real numbers is an interval denoted by $(-\infty, \infty)$; and any single real number a is an interval, denoted by [a, a].

### 3.2 Learning Objectives

The main objectives of this unit are to

(i) study interval

(ii) study open and closed intervals

(iii) study semi-closed (or semi open) intervals

(iv) learn about infinite interval

(v) study absolute value or Numerical value or modulus of a real number.

### 3.3 Intervals

An interval is a subset of the real number that contains all real numbers lying between any two numbers of the subset.

The end point of an interval are its supremum, and its infimum, if they exist ∞ real number. If the infimum does not exist, one say often that the corresponding end point is $-\infty$. Similarly if the supremum does not exist, we say that the corresponding end point is $+\infty$.

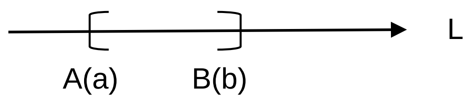
Intervals are completely determined by their end point belong to the interval. This is a consequence if the least upper bound property of real numbers. This characterization is used to specify intervals by mean of interval notation, which is described below.

Let a and b be two distinct real numbers with $a < b$ (say) then,

**(i) Open Interval :** The set of all real numbers between a and b is said to form an open interval from a to b denoted by (a, b). In symbols

$(a, b) = \{x : a < x < b \text{ and } x \in R\}$

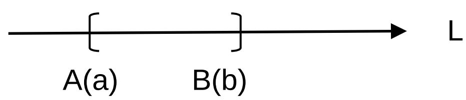
Geometrically the open interval (a, b) is represented on the real line as




(ii) **Closed Interval :** The set of all real numbers between a and b including the end points a and b is said to form a closed interval and is denoted by [a, b]

In symbols $[a, b] = \{x : a \leq x \leq b \text{ and } x \in R\}$

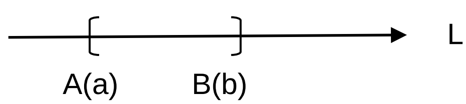
Geometrically, the closed interval [a, b] is represented on the real line as




(iii) **Semi-closed (or Semi-open) intervals:** An interval in which one end point is included and the other end point is excluded is called semi-closed interval.

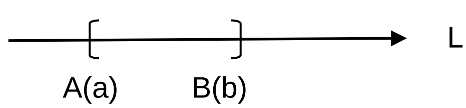
In symbols, $[a, b] = \{x : a \leq x < b \text{ and } x \in R\}$

Geometrically, [a, b] is represented on the real line as




Similarly $(a, b] = \{x : a < x \leq b \text{ and } x \in R\}$

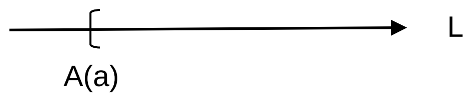
Geometrically, (a, b] is represented on the real line as




The intervals defined above are called finite intervals. Now we defined infinite intervals.

**Infinite Interval.** The set of all the real numbers x such that $x > a$ forms an infinite set and is denoted by $(a, \infty)$.

Symbolically (a, ∞) = {x : x > a and x ∈ R}. Geometrically




Some examples of infinite intervals are :

(i) (a, ∞) = {x : x > a and x ∈ R}

(ii) (-∞, a) = {x : x < a and x ∈ R}

(iii) (-∞, a) = {x : x ≤ a and x ∈ R}

(iv) (-∞, ∞) = R = {x : x ∈ R}

**Some Illustrated Examples**

**Example 1.** Find the solution set of the following :

(a) 3x - 2 > 0 and 5x - 1 < 0

(b) $-0.03 < \frac{2x+3}{5} \leq 0.03$

(c) 3 + x ≤ 5x - 2 < 7 + x

**Solution:** (a) 3x - 2 ≥ 0 and 5x - 1 < 0

∴ 3x ≥ 2 and 5x < 1

∴ $x \geq \frac{2}{3}$ and $x < \frac{1}{5}$

∴ $\frac{2}{3} \leq x$ and $x < \frac{1}{5}$

∴ $\frac{2}{3} \leq x < \frac{1}{5}$, which is impossible

∴ solution set is $\phi$

(b) $-0.03 \leq \frac{2x+3}{5} \leq 0.03$

Multiplying by 5,

-0.15 ≤ 2x + 3 ≤ 0.15

Subtracting 3 from each,

-3.15 ≤ 2x ≤ 2.85

Dividing by 2,

-1.575 ≤ x ≤ 1.425

$\therefore$ x $\in$ [-1.575, -1.425]

$\therefore$ solution set is [-1.575, -1.425]

(c) $\because$ 3 + x $\leq$ 5x - 2 < 7 + x

Subtracting x from each,

3 $\leq$ 4x - 2 < 7

Adding 5 to each,

5 $\leq$ 4x < 9

Dividing by 4,

$$\frac{5}{4} \leq x < \frac{9}{4}$$

$\therefore$ solution set is $\left[\frac{5}{4}, \frac{9}{4}\right)$

**Example 2:** Solve $\frac{x+3}{x-2} < 5$

**Solution:** Here $\frac{x+3}{x-2} < 5$

Clearly x - 2 $\neq$ 0 i.e., x $\neq$ 2

Two cases arise :

Case I x - 2 > 0 i.e. x > 2. Then

$$\frac{x+3}{x-2} < 5$$

$\Rightarrow$ x + 3 < 5x - 10

$\Rightarrow$ -4x < - 13

$\Rightarrow$ $x > \frac{13}{4}$

$\Rightarrow$ $x \in \left(\frac{13}{4}, \infty\right)$

Case II x - 2 < 0 i.e. x < 2. Then

$$\frac{x+3}{x-2} < 5$$

$\Rightarrow$ x + 3 > 5x - 10

$\Rightarrow$ -4x > - 13

$\Rightarrow$ $x < \frac{13}{4}$

But x < 2

$\therefore$ x $\in$ (-∞, 2)

$\therefore$ solution set is (-∞, 2)

Hence the solution set of the given inequality is (-∞, 2) $\cup \left(\frac{13}{4}, \infty\right)$

**Example 3:** Solve for x : $\frac{2}{x-2} < \frac{x+2}{x-2} < 2$

**Solution:** Here $\frac{2}{x-2} < \frac{x+2}{x-2} < 2$

Clearly x - 2 ≠ 0 i.e., x ≠ 2

Two cases arise :

Case I. x - 2 > 0 i.e., x > 2

Then $\frac{2}{x-2} < \frac{x+2}{x-2} < 2$

$\Rightarrow$ 2 < x + 2 < 2x - 4

$\Rightarrow$ 2 < x + 2 and x + 2 < 2x - 4

$\Rightarrow$ 0 < x and 6 < x

$\Rightarrow$ x > 0 and x > 6

$\Rightarrow$ x > 6

$\Rightarrow$ x $\in$ (6, ∞)

Case II. x - 2 < 0 i.e., x < 2

Then $\frac{2}{x-2} < \frac{x+2}{x-2} < 2$

$\Rightarrow$ 2 > x + 2 2x - 4

$\Rightarrow$ 2 > x + 2 and x + 2 > 2x - 4

$\Rightarrow$ 0 > x and 6 > x

$\Rightarrow$ x < 0 and x < 6

⇒ $x < 0$

⇒ $x \in (-\infty, 0)$

Combining the results of two cases, solution set is $(-\infty, 0) \cup (6, \infty)$

**Example 4:** Find real values of x, which satisfy the inequality $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

**Solution:** The given inequality is $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

Clearly $x + 2 \neq 0$, $4x - 1 \neq 0$ i.e. $x \neq -2$, $x \neq \frac{1}{4}$

Following cases arise:

Case I. When $x < -2$ and $x < \frac{1}{4}$

∴ $x + 2 < 0$ and $x - \frac{1}{4} < 0$

∴ $x + 2 < 0$ and $4x - 1 < 0$

∴ $(x + 2)(4x - 1) > 0$

∴ $\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$

⇒ $(x - 2)(4x - 1) > (2x - 3)(x + 2)$

⇒ $4x^2 - 9x + 2 < 2x^2 + x - 6$

⇒ $2x^2 - 10x + 8 > 0$

⇒ $x^2 - 5x + 4 > 0$

⇒ $(x - 1)(x - 4) > 0$

∴ $(x + 2)(x + 1)(3x + 2)(2x + 1) < 0$

∴ $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

∴ required set is $(-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

## 3.4 Art. Absolute Value or Numerical Value or Modulus of a Real Number

**Definition :** Let x be any real number. The absolute value of x denoted by |x| (read as modulus of x or numerical value of x is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Therefore, the absolute value of a +ve real number or zero is the number itself whereas the absolute value of a negative real number is the negative of the number itself e.g.

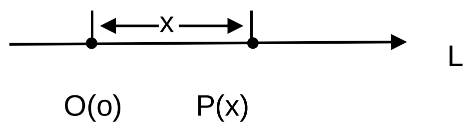
|3| = 3, |-3| = -(-3) = 3.|0| = 0

Alternatively, we may define,

|x| = max.(x, -x) or |x| = $+\sqrt{x^2}$

Geometrical meaning of |x|

If P be the point on the real line L corresponding to real number x, then the distance between the origin O and the point P is |x|




∴ |x| = |OP|

Important results of |x| where x ∈ R, are

(i) |x| ≥ 0 ∀ x ∈ R

(ii) |x| = 0 iff x = 0

(iii) |x| = |-x|

(iv) |x| = max.(x, -x) or - |x| = min.(x, -x)

(v) -|x| ≤ x ≤ |x|

(vi) |x| = x iff x > 0

(vii) |x| > x iff x < 0

(viii) |x| ≥ x iff x ∈ R

(ix) |x| = -x iff x ≤ 0

(x) |x|2 = x2 ↔ |x| = $\sqrt{x^2}$

**Art.** Prove that |ab| = |a||b| where a, b ∈ R

**Proof.** |ab| = $\sqrt{(ab)^2}$ $\left[\because |x| = \sqrt{x^2}\right]$

= $\sqrt{a^2 b^2}$

= $\sqrt{a^2}\ \sqrt{b^2}$

= |a||b|

**Art.** $\left|\frac{a}{b}\right| = \left|\frac{a}{b}\right|$ where a, b $\in$ R and b $\neq$ 0

**Proof.** $\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2}$

$= \sqrt{\frac{a^2}{b^2}}$

$= \frac{\sqrt{a^2}}{\sqrt{b^2}}$

$= \left|\frac{a}{b}\right|$, b $\neq$ 0

**Art.** Prove that $|a + b| \leq |a| + |b|$ where a, b $\in$ R

Further show that equality holds when a and b have same signs.

**Proof:** $|a + b|^2 = (a + b)^2$ $[\because\ |x|^2 = x^2]$

$= a^2 + b^2 + 2ab$

$= |a|^2 + |b|^2 + 2ab$

$\therefore$ $|a + b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$ $[\because\ ab < |ab|]$

$\therefore$ $|a + b|^2 \leq |a|^2 + |b|^2 + 2|a||b|$ $(\because\ |ab| = |a||b|)$

$\therefore$ $|a + b|^2 \leq (|a| + |b|)^2$

$\therefore$ $|a + b| \leq |a| + |b|$

Hence the result

Now equality holds when

$|a + b| = |a| + |b|$

$\therefore$ $|a + b|^2 = (|a| + |b|)^2$

$\therefore$ $(a + b)^2 = |a|^2 + |b|^2 + 2|a||b|$

$a^2 + b^2 + 2ab = a^2 + b^2 + 2|ab|$

$\therefore$ $ab = |ab|$

$\therefore$ $ab \geq 0$ $[\because\ |x| = x \text{ when } x \geq 0]$

$\therefore$ a and b have same signs.

Hence equality holds when a and b have same signs.

**Cor. I.** If $x_1, x_2, \ldots\ldots, x_n \in R$ then $|x_1 + x_2 + \ldots + x_n| \leq |x_1| + |x_2| + \ldots + |x_n|$

**Proof:** We shall prove it by Induction

When n = 2

$|x_1 + x_2| \leq |x_1| + |x_2|$

$\therefore$ result is true for n = 2

Let us assume that the result is true for n = m (an integer greater than 2)

$\therefore$ $|x_1 + x_2 + \ldots + x_m| < |x_1| + |x_2| + \ldots + |x_m|$

Now when n = m + 1

$|x_1 + x_2 + \ldots + x_m + x_{m+1}| = |(x_1 + x_2 + \ldots + x_m) + x_{m+1}|$

$\leq |x_1 + x_2 + \ldots + x_m| + |x_{m+1}|$ [$\because$ $|a + b||a|+|b|$]

$\leq |x_1| + |x_2| + |x_m| + |x_{m+1}|$

$\therefore$ the result is proved for n = m + 1

Hence the result is proved by induction

**Cor. II.** $\because$ $|a + b| \leq |a| + |b|$

Changing b to -b

$|a - b| \leq |a| + |-b|$

$\therefore$ $|a - b| < |a| + |b|$ ($\because$ $|b| = |-b|$)

**Cor. III.** $|a - b| \geq |a| - |b|$ and $|a - b| \geq ||a| - |b||$

**Proof:** Let a - b = c

$\therefore$ a = b + c

$\therefore$ $|a| = |b + c| \leq |b| + |c|$

$\therefore$ $|a| \leq |b| + |a - b|$

$|a| - |b| \leq |a - b|$

$\therefore$ $|a - b| \geq |a| - |b|$ ..... (1)

Interchanging a and b in (1), we get,

$|b - a| \geq |b| - |a|$

$\therefore$ $|a - b| \geq -(|a| - |b|)$ ...... (2)

{$\because$ $|x| = |-x|$}

From (1) and (2),

$|a - b| \geq \max\{|a| - |b|, -(|a| - |b|)\}$

$\therefore$ $|a - b| \geq ||a| - |b||$

Hence the result

Another Method

$|a - b|^2 = (a - b)^2 = a^2 + b^2 - 2ab$

$\geq |a|^2 + |b|^2 - 2|ab|$ [$\because$ $-ab \geq |ab|$ and $|x|^2 = x^2$]

$= |a|_2 + |b|_2 - 2|a||b|$

$= ||a| - |b||_2$

$\therefore$ $|a - b| \geq ||a| - |b||$

**Art. Some Useful results**

(I) $|x| = \ell \Rightarrow x = \pm\ell$

**Proof:** $\because$ $|x| = \ell$

$\therefore$ $\sqrt{x^2} = \ell$

$\Rightarrow$ $x^2 = \ell^2$

$\therefore$ $x = \pm\ell$

(II) $|x| = |\ell|$ $\Rightarrow$ $x = \pm\ell$

**Proof:** $|x| = |\ell|$

$|x|^2 = |\ell|^2$

$x^2 = \ell^2$

$\therefore$ $x = \pm l$

(III) If $|x| < \ell$ (where $\ell > 0$), then $-\ell < x < \ell$ or $x \in (-\ell, \ell)$

**Proof:** $\because$ $|x| < \ell$ .... (1)

Two possibilities arise :

(i) $x \geq 0 \therefore |x| = x$

$\therefore$ (1) becomes $x < \ell$ .... (2)

(ii) $x < 0 \therefore |x| = -x$

$\therefore$ (1) becomes $-x < \ell$

or $x > -\ell$

or $-\ell < x$ ...... (3)

From (2) and (3), we have,

$-\ell < x < \ell$

**Note:** Similarly if $|x| \leq \ell$ then $-\ell \leq x \leq \ell$

(IV) if $|x| > \ell$ then $x > \ell$ or $x < -\ell$

**Proof:** Given $|x| > \ell$

Two possibilities arise:

(i) $x \geq 0 \therefore |x| = x$

$\therefore$ (1) becomes $x < \ell$ ..... (2)

(ii) $x < 0 \therefore |x| = -x$

$\therefore$ (1) becomes $-x < \ell$

$\therefore$ $-x > \ell$ ...... (3)

From (2) and (3), we conclude,

$|x| > \ell \Rightarrow x > \ell$ or $x < -\ell$

Hence $\{x : |x| > \ell\} = (-\infty, -\ell) \cup (\ell, \infty)$

**Note:** (1) Similarly $|x| \geq \ell \Rightarrow$ either $x \leq -\ell$ or $x > \ell$

(2) $|x - a| < \ell$ $(\ell > 0)$ $\Rightarrow$ $x \in (a - \ell, a + \ell)$

(3) $|x - a| > \ell$ $\Rightarrow$ $x \in (-\infty, a - \ell) \cup (a + \ell, \infty)$

(4) It is clear from above article that an interval can be represented by the use of modulus.

**Art.** If a, b are real numbers, then

(i) $a^2 \leq b^2$ iff $|a| \leq |b|$

(ii) $a^2 < b^2$ iff $|a| < |b|$

(iii) $a^2 \leq b^2$ iff $|a| \leq |b|$

(iv) $a^2 < b^2$ iff $|a| < |b|$

**Proof:** $\because$ $a, b \in R$

$\therefore$ $|a| \geq 0, |b| \geq 0$

$\Rightarrow$ $|a| + |b| \geq 0$

$\Rightarrow$ either $|a| + |b| = 0$ or $|a| + |b| > 0$

(i) To prove $a^2 \leq b^2$ iff $|a| \leq |b| = 0$

When $|a| + |b| = 0$, then $|a| = 0$ and $|b| = 0$

$\therefore$ $a = b = 0$

$\therefore$ the result holds in this case.

When $|a| + |b| > 0$

$a^2 \leq b^2$

iff $b^2 - a^2 \geq 0$

i.e. iff $|b|^2 - |a|^2 \geq 0$

i.e. iff $(|b| + |a|)(|b| - |a|) \geq 0$

i.e. iff $|b| - |a| \geq 0$ $[\because |b| + |a| > 0]$

i.e. iff $|b| \geq |a|$

i.e. iff $|a| \leq |b|$

(ii) To prove $a^2 < b^2$ iff $|a| < |b|$

When $|a| + |b| = 0$, then $|a| = 0$ and $|b| = 0$

$\therefore$ $a = b = 0$

$\Rightarrow$ $a^2 = b^2$

Which contradicts the given hypothesis that $a^2 < b^2$ and hence rejected.

$\therefore$ $|a| + |b| > 0$

Now $a^2 < b^2$

iff $b^2 - a^2 > 0$

i.e. iff $|b|^2 - |a|^2 > 0$

i.e. iff $(|b| + |a|)(|b| - |a|) > 0$

i.e. iff $|b| - |a| > 0$ $[\because |b| + |a| > 0]$

i.e. iff $|b| > |a|$

i.e. iff $|a| < |b|$

(iii) to prove $a^2 \geq b^2$ iff $|a| \geq |b|$

When $|a| + |b| = 0$, then $|a| = 0$ and $|b| = 0$

$\therefore$ $a = b = 0$

$\therefore$ the result holds in this case.

When $|a| + |b| > 0$

$a^2 \geq b^2$

iff $a^2 - b^2 \geq 0$

i.e. iff $|a|^2 - |b|^2 \geq 0$

i.e. iff $(|a| + |b|)(|a| - |b|) \geq 0$

i.e. iff $|a| - |b| \geq 0$ $[\because |a| + |b| > 0]$

i.e. iff $|a| \geq |b|$

(iv) To prove $a^2 > b^2$ iff $|a| > |b|$

When $|a| + |b| = 0$, then $|a| = 0$ and $|b| = 0$

$\therefore$ a = b = 0

$\Rightarrow$ $a^2 = b^2$

Which contradicts the given hypothesis that $a^2 > b^2$ and hence rejected.

$\therefore$ $|a| + |b| > 0$

Now $a^2 > b^2$

iff $a^2 - b^2 > 0$

i.e. iff $|a|^2 - |b|^2 > 0$

i.e. iff $(|a| + |b|)(|a| - |b|) > 0$

i.e. iff $|a| - |b| > 0$ [$\because$ $|a| + |b| > 0$]

i.e. iff $|a| > |b|$

**Example 5 :** Show that $\frac{1}{2}(a + b + |a - b|)$ is the larger of the two real numbers a and b, and $\frac{1}{2}(a + b - |a - b|)$ is the smaller.

**Solution :** Hear a and b are two real numbers

Now max. {a, b} is either a or b.

Let max. {a, b} = a ...(1)

$\Rightarrow$ $a > b$

$\Rightarrow$ $a - b > 0$

$\Rightarrow$ $|a - b| = a - b$

$\Rightarrow$ $\frac{1}{2}(a + b + |a - b|)$

$= \frac{1}{2}(a + b + a - b)$

$= \frac{1}{2}(2a)$

$\therefore$ $\frac{1}{2}(a + b + |a - b|) = a$ ...(2)

$\therefore$ $\frac{1}{2}(a + b + |a - b|) = \max. \{a, b\}$ ...(I)

[$\because$ of (1) and (2)]

Let min. {a, b} = b ...(3)

$\Rightarrow$ $a \geq b$

$\Rightarrow$ $a - b \geq 0$

$\Rightarrow$ $|a - b| = a - b$

$\Rightarrow$ $\frac{1}{2}(a + b - |a - b|)$

$= \frac{1}{2}(a + b - (a - b))$

$= \frac{1}{2}(a + b - a + b)$

$= \frac{1}{2}(2a)$

$\therefore$ $\frac{1}{2}(a + b - |a - b|) = b$ ...(4)

From (3) and (4), we get

$\therefore$ $\frac{1}{2}(a + b - |a - b|) = \min. \{a, b\}$ ...(II)

From (I) and (II), we see that

$\frac{1}{2}(a + b - |a - b|)$ is larger of the two real numbers a and b.

and $\frac{1}{2}(a + b - |a - b|)$ is the smaller.

**Example 6 :** For a, b $\in$ R, show that

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

**Solution :** We have

$\therefore$ $1 + |a + b| < 1 + |a| + |b|$

$\Rightarrow$ $\frac{1}{1+|a+b|} \geq \frac{1}{1+|a|+|b|}$

$\Rightarrow$ $\frac{1}{1+|a+b|} \leq \frac{1}{1+|a|+|b|}$

$\Rightarrow$ $1 - \frac{1}{1+|a+b|} \leq 1 - \frac{1}{1+|a|+|b|}$

$\Rightarrow$ $\frac{|a+b|}{1+|a+b|} < \frac{|a|+|b|}{1+|a|+|b|}$ ...(1)

Also 1 + |a| + |b| $\geq$ 1 + |a| [$\because$ |b| $\geq$ 0]

$\Rightarrow$ $\frac{1}{1+|a|+|b|} \leq \frac{1}{1+|a|}$

$\Rightarrow$ $\frac{|a|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|}$

[$\because$ |a| $\geq$ 0]

Similarly, $\frac{|b|}{1+|a|+|b|} \leq \frac{|b|}{1+|b|}$ ....(3)

Adding (2) and (3), we have

$$\frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

$\therefore$ $\frac{|a|+|b|}{1+|a|+|b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$ ...(4)

From (1) and (4), we get

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

**Example 7 :** Prove that a $\leq$ x $\leq$b $\Rightarrow$ |x| $\leq$ |a| + |b|

**Solution :**a $\leq$ x $\leq$ b $\Rightarrow$ x $\leq$ b

and x $\geq$ a $\Rightarrow$ x $\leq$ b $\leq$ |b| $\leq$|a| + |b|

and -x $\leq$ -a $\leq$ |a| $\leq$ |a| + |b| $\Rightarrow$ x $\leq$ |a| + |b|

and -x $\leq$ |a| + |b|

$\Rightarrow$ max {x, -x} $\leq$ |a| + |b|

$\Rightarrow$ |x| $\leq$ |a| + |b|

**Example 8 :** (a) Convert a < x < b into single inequality by making use of absolute values.

(b) Write the inequality -2 < x < 4 in the form |x - a| < l.

**Solution :** (a) We have a < x < b

Subtracting $\frac{a+b}{2}$ (i.e. A.M. of a and b) from each, we get

$$a - \frac{a+b}{2} < x - \frac{a+b}{2} < b - \frac{a+b}{2}$$

or $$\frac{2a-a-b}{2} < x - \frac{a+b}{2} < \frac{2a-a-b}{2}$$

or $$\frac{a-b}{2} < x - \frac{a+b}{2} < \frac{-a+b}{2}$$

or $$-\frac{b-a}{2} < x - \frac{a+b}{2} < \frac{b-a}{2}$$

$\therefore$ $$\left|x - \frac{a+b}{2}\right| < \frac{b-a}{2}$$

(b) We have -2 < x < 4

Subtracting 1 (i.e. A.M. of -2 and 4 i.e., $\frac{-2+4}{2} = \frac{2}{2} = 1$) from each, we get

-2 - 1 < x - 1 < 4 - 1

or -3 < x - 1 < 3

or |x - 1| < 3

or |x - a| < l, where a = 1, l = 3

**Example 9.** Solve $2x^2 + |5x| + 2 = 0$

**Solution :** The given equation is

$2x^2 + |5x| + 2 = 0$

or $2x^2 + 5|x| + 2 = 0$ [$\because$ |ab| = |a||b|]

or $2|x|^2 + 5|x| + 2 = 0$ [$\because$ $x^2 = |x|^2$]

$\therefore$ $$|x| = \frac{-5 \pm \sqrt{25-16}}{4}$$

$$= \frac{-5 \pm 3}{4}$$

$$= -2, -\frac{1}{2}$$

Both these values are impossible as |x| can not be negative.

∴ Solution set is $\phi$.

## 3.5 Self Check Exercise

Q.1 For what values of x is $x^3 + 1 > x^2 + x$ ?

Q.2 Solve $\dfrac{2x}{2x^2+5x+2} > \dfrac{1}{x+1}$

Q.3 Solve |2x - 1| = |4x + 3|

Q.4 Solve |x - 1| - |x + 3| < 6

## 3.6 Summary

In this unit we have learnt:

(i) intervals

(ii) open, closed and infinite intervals

(iii) Absolute value of a real number etc.

## 3.7 Glossary:

1. A set of real numbers is an interval, if and only if it is an open interval, closed interval or a half open interval.

2. Degenerate interval - Any set consisting of a single real number (interval of the form [a, a] is called degenerate interval.

   Note : A real interval that is neither empty not degenerate is said to be proper, and has infinite may elements.

## 3.8 Answer to Self Check Exercise

Ans.1 $(-1, 1) \cup (1, \infty)$

Ans.2 $(-2, -1) \cup \left(\dfrac{-2}{3}, \dfrac{-1}{2}\right)$

Ans.3 $\left(-2, \dfrac{-1}{3}\right)$

Ans.4 Set of all real numbers.

## 3.9 References/Suggested Readings

1. K.A. Ross, Elementary Analysis- The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag, 2003

2. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

3. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

## 3.10 Terminal Questions

1. Solve for x ; $\frac{2}{x-2} < \frac{x+2}{x-2} < 1$

2. Solve $\sqrt{x^2+1}$ = x

3. For what x is 4x2 + 9x < 9.

4. Find the solution set of the equation

   $|x^2 - 5x + 6| = |x - 3||x - 2|$

5. Prove that $\left|\frac{1}{x} - 5\right| < 1$ iff $\frac{1}{6} < x < \frac{1}{4}$.

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## Unit - 4

# Bounds

**Structure**



## 4.1 Introduction

Dear students, in this unit we shall study the concept of bounded set. In mathematical analysis and related areas of mathematics, a set is called bounded if all its points are within a certain distance of each other. Conversely, a set which is not bounded is called unbounded. The word bounded makes no sense in a general topological space without a corresponding metric. We not here that boundary is a distinct concept, for example, a circle in isolation is a boundaryless bounded set, while the half plane is unbounded yet has a boundary. Further it may also be noted that a bounded set is not necessarily a closed set and vice-versa. For example, a subset S of two-dimensional real space $R^2$ constrained by two parabolic curves $x^2+1$ and $x^2-1$ defined in a cartesion coordinate system is closed by the curves but not bounded (so unbounded)

## 4.2 Learning Objectives

The main objectives of this unit are

(i) to study the concept of bounded sets, least upper bound and greatest lower bound (l.u.b. and g.l.b.)

(ii) l.u.b. property of reals or order completeness property of reals.

(iii) to learn about Archemedian property of real numbers

(iv) to know the concept of nhd. of a point

(v) to study cluster point and limit point of a set

(vi) to study Bolzano-Weirstrass property.

## 4.3 Art. Bounds

Let S be a non-empty sub-set of real numbers.

(i) If there exists a real number K such that $x < K \ \forall \ x \in S$ then K is said to be an upper bound of S. The set S is said to be bounded above or bounded to the right if it has an upper bound.

(ii) If there exists a real number l such that $l < x \ \forall x \in S$ then l is said to be lower bound of S. The set S is said to be bounded below or bounded to the left if it has a lower bound.

(iii) The set S is said to be bounded if it is both bounded below and above.

Geometrically, K is an upper bound of S if the point K is to the right of each point x of S when plotted and l is a lower bound of S if the point l is to the left of each point x of S.

**Note :** Every finite set is bounded.

**Remark :** Let S be a non-empty sub-set of R which is bounded above. If K is an upper bound of S then clearly every real number greater than K is also an upper bound of S. There may exist real numbers less than K which may also be upper bounds of S. We are interested in a 'leftmost' real number which is an upper bound of S and that such a left most real number exists, is suggested by geometrical intuition and we shall accept it as an axiom. Similarly, if S is bounded below then there exists a 'right most' real number which is less than or equal to the elements of S.

**Least upper bound**

Let S be a non-empty subset of R which is bounded above. Then there exists a real number u which is the smallest of all the upper bounds of S i.e., for any other upper bound u' of S we have $u \leq u'$.

This number u is called the least upper bound of S or Supremum of S and is written as l.u.b. of S = Sup. S = u.

Least upper bound of a set S bounded above is unique. For if u and u' are two least upper bounds of S, then by definition $u \leq u'$ and $u' \leq u$ which implies that $u = u'$.

**Greatest lower bound**

Let S be a non-empty subset of R which is bounded below. Then there exists a real number l which is the greatest of all the lower bounds of S i.e., for any other lower bound l' of S we have $l' < l$.

The number l is called the greatest lower bound of S or infimum of S and is written as g.l.b. of S = Inf. S = l.

**Examples :** Let $A_1$ = (-3, 10), $A_2$ = (-3, 10], $A_3$ = [-3, 10), $A_4$ = [-3, 10]

We have for all $x \in A_i$, where i = 1, 2, 3, 4

$-3 \leq x \leq 10$

∴ $A_i$'s are bounded

∴ g.l.b. of $A_i = -3$, l.u.b. of $a_i = 10$

It is worth nothing that g.l.b. of a set may many not belong to the set.

(ii) Let S = {x : x ∈ Q and x ≤ 0}

The set S is unbounded because it is not bounded below. However, l.u.b. of S = 0.

(iii) The set N of natural numbers is bounded below but it is not bounded from above. Its g.l.b. = 1.

(iv) The set Z of integers is neither bounded below not above.

(v) $S = \left\{\frac{1}{x} : x \in N\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, .....\right\}$ has g.l.b. = 0 and l.u.b. = 1.

**Art.** Prove that a non-empty subset of a bounded set is bounded.

**Proof :** Let A be any non-empty subset of a bounded set S.

∵ S is a bounded set

∴ there exist real numbers l and u such that $l < x < u \ \forall \ x \in S$

∵ $A \subset S \Rightarrow \forall x \in A \Rightarrow x \in S$

∴ $\forall x \in A$ we have $l \leq x \leq u$

∴ A is a bounded set.

**4.4 Art. Least upper bound property of Reals or Order Completeness Property of Reals**

Every non-empty set S of real numbers which is bounded above has a least upper bound in R.

This property is called the order completeness property of reals. Due to this property, the set R of real numbers is said to be a complete order field. By an example, we will show that set Q of rationals is not a complete ordered field, though it is a field.

**Art.** Given an example to show that the set Q of rationals does not possess the least upper bound property.

Or

Give an example of a field which is not complete. Justify your answer.

**Proof :** Let S {x : x ∈ Q and $x^2 < 3$}

Clearly $1 \in S \Rightarrow \Sigma \neq \phi$

Also if u is any positive number such that $u^2 > 3$, then u is an upper bound of S. Thus S is a non-empty set of rational numbers which is bounded above. We want to show that S has no least upper bound in Q.

If possible, let $\alpha \in$ Q be the least upper bound of S. Clearly $\alpha$ is positive. then one and only one of

(i) $\alpha^2 < 3$ (ii) $\alpha^2 = 3$ (iii) $\alpha^2 > 3$ holds.

Let $\beta = \dfrac{2\alpha+3}{\alpha+2}$. Here $\beta$ is a positive rational number as $\alpha$ is positive rational number.

Case I. $\alpha^2 < 3$

Now $\beta 2 - 3 = \left(\dfrac{2\alpha+3}{\alpha+2}\right)^2 - 3$

$$= \frac{4\alpha^2+12\alpha+9-3\left(\alpha^2+4\alpha+4\right)}{(\alpha+2)^2}$$

$$= \frac{\alpha^2-3}{(\alpha+2)^2}$$

$$= \frac{3-\alpha^2}{(\alpha+2)^2} < 0$$

Also $\beta - \alpha = \dfrac{2\alpha+3}{\alpha+2} - \alpha$

$$= \frac{2\alpha+3-\alpha^2-2\alpha}{\alpha+2}$$

$$= \frac{3-\alpha^2}{\alpha+2} > 0$$

$\Rightarrow$ $\beta > \alpha$

Thus $\beta \in$ S and $\beta > \alpha$, which contradicts the result that $\alpha$ is an upper bound.

$\therefore$ our supposition is wrong.

Case II. When $\alpha^2 = 3$, then $\alpha = \sqrt{3}$ is not a rational number. Thus this case is not possible.

Case III. $\alpha^2 > 3$

Now $\beta^2 - 3 = \left(\frac{2\alpha+3}{\alpha+2}\right)^2 - 3$

$= \frac{\alpha^2-3}{(\alpha+2)^2} > 0$

$\therefore$ $\beta^2 > 3$ $\Rightarrow$ $\beta$ is an upper bound of S.

Also $\beta - \alpha = \frac{2\alpha+3}{\alpha+2} - \alpha$

$= -\frac{\alpha^2-3}{\alpha+2} < 0$

$\Rightarrow$ $\beta < \alpha$

Thus we get an upper bound $\beta$ of S such that $\beta < \alpha$, which is again a contradiction.

Hence in all the cases, it follows that S has no least upper bound in Q.

$\therefore$ the set of rational numbers is not a complete ordered field.

**Note:** Let S be a non-empty subset of real numbers bounded below. Then by using the least upper bound property of real numbers, it can be proved that S has a greatest lower bound in R. This is called the greatest lower bound property of reals. Again this property is not satisfied by the set of rational numbers.

**Art.** (a) Let S be a non-empty sub set of R, which is bounded above. Then a real number u is the least upper bound of S iff

(i) u is an upper bound of S

(ii) Given $\varepsilon > 0$, there exists a real number $x \in S$ such that $u - \varepsilon < x$.

**Proof:** The condition is necessary

Given u is l.u.b. of S and we want to show that conditions (i) and (ii) are satisfied.

$\because$ u is the l.u.b. of S.

$\therefore$ u is an upper bound of S.

Also $\forall\ \varepsilon > 0$, however small, $u - \varepsilon < u$.

$\Rightarrow$ $u - \varepsilon$ is not an upper bound of S

$\Rightarrow$ there exists a real number $x \in S$ such that $x > u - \varepsilon$

Thus both the conditions (i) and (ii) are satisfied.

The condition is sufficient

Here it is given that a real number u satisfies both the conditions (i) and (ii). We are to show that u is the l.u.b. of S.

If possible, let a real number u' <u be any other upper bound of S.

Take $\varepsilon = u - u' > 0$

∴ by conditions (ii), there exists a real number $x \in S$ such that

$u - \varepsilon < x$ i.e., $u - (u - u') < x$ i.e., $u' < x$

∴ u' is not upper bound of S, which is a contradiction.

Hence u is the l.u.b. of S.

**Art.** (b) Let S be a non-empty subset of R, which is bounded below. Then a real number is the g.l.b. of S iff.

(i) l is a lower bound of S.

(ii) Given $\varepsilon > 0$, there exists a real number $x \varepsilon S$ s.t. $x > l + \varepsilon$

**4.5 Proof: Please try yourself.**

**Art.** Archmedian Property of Real Numbers

For given a > 0 and $b \in R$, there exists a natural number n such that na> b.

**Proof:** Here $b \in R$

∴ either $b \leq 0$ or $b > 0$

If $b \leq 0$, then na> b holds for all $n \in N$ [∵ na is positive]

∴ we discuss the case when b > 0.

If b > 0, suppose the theorem is false

i.e., na< b $\forall n \in N$

∴ the set $A = \{n a : n \in N, na < b\}$ is bounded above by b.

∴ A has l.u.b., say u (By order Completeness Property).

$na \leq u \quad \forall n \in N$

∴ $(n + 1) a \leq u \quad \forall n \in N$

i.e., $na \leq u - a \quad \forall n \in N$

∴ u - a is an upper bound of A which is less than l.u.b u

∴ we arrive at a contradiction

∴ $\exists n \in N$ such that na> b.

**Art.** (i) Given any $x \in R$, there exists a unique integer m such that $m < x < m + 1$.

(ii) Between any two distinct real numbers, there is always a rational number and therefore, infinitely many rational numbers.

(iii) Between any two distinct real numbers, there is always an irrational number and, therefore, infinitely many irrationals.

**Proof:** (i)Consider the set S = (n : n $\in$ Z, n < x}. Here S is a non-empty set of reals which is bounded above. Therefore, by the least upper bound property, S has a unique least upper bound, say, m. Clearly m is an integer and is the largest element of S.

$\therefore$ m $\in$ S

$\Rightarrow$ m + 1 $\notin$ S

$\Rightarrow$ m $\leq$ x

and m + 1 > x

$\Rightarrow$ m $\leq$ x < m + 1, where m $\in$ Z

Since l.u.b. of a set is unique.

$\therefore$ m is unique

Here for any real number x, there exists a unique integer m such that

m $\leq$ x < m + 1

(ii) Let x and y be two distinct real numbers such that x < y i.e., y - x > 0

$\therefore$ by Archimedian Property of R, $\exists$ a natrual number n such that

n (y - x) > 1 $\left\{\begin{matrix} Take\ a = y = x \\ b = 1 \\ then\ \ na > b \end{matrix}\right\}$

i.e., ny>nx + 1

i.e., nx + 1 <ny .... (1)

As nx$\in$ R, there exists a unique integer m such that

m $\leq$nx< m + 1 [$\because$ of (1)]

$\Rightarrow$ m $\leq$nx and nx< m + 1 .... (2)

Now m $\leq$nx

$\Rightarrow$ m + 1 <nx + 1

Also nx + 1 <ny [$\because$ of (1)]

$\therefore$ we have m + 1 <ny ..... (3)

From (2) and (3), we get,

nx< m + 1 <ny

$\Rightarrow$ x < $\frac{m+1}{n}$ < y [$\because$ n > 0]

Now $\frac{m+1}{n}$ is a rational number and this shows that a rational number lies between two given distinct real numbers.

Continuing in this way, an infinite number of rational numbers can be found between two given distinct real numbers.

(iii) Let x and y be two distinct real numbers with y - x > 0

Then by Archimedian property of R there exists a natural number n such that

$$n(y - x) > \sqrt{2} \qquad \left\{\begin{array}{l} Take\ a = y - x \\ b = \sqrt{2} \\ then\ \ na > b \end{array}\right\}$$

$\therefore$ $y - x > \frac{\sqrt{2}}{n}$

$\Rightarrow$ $y > x + \frac{\sqrt{2}}{n} > x + \frac{\sqrt{2}}{2n} > x$

Now, out of the number $x + \frac{\sqrt{2}}{n}$ and $x + \frac{\sqrt{2}}{2n}$, at least one must be irrational because their difference = $\frac{\sqrt{2}}{2n}$ is an irrational number.

Therefore, there lies an irrational number and hence infinite irrational numbers between x and y.

**Example 1:** Find g.l.b. and l.u.b. (if they exist) of the set

$$S = \left\{\frac{2x-1}{x+4} : |x-5| < 2\right\}$$

**Solution:** $S = \left\{\frac{2x-1}{x+4} : |x-5| < 2\right\}$

Now $\frac{2x-1}{x+4} = 2 - \frac{9}{x+4}$

Again $|x - 5| < 2$

$\Rightarrow$ $-2 < x - 5 < 2$

$\Rightarrow$ $3 < x < 7$

$\Rightarrow$ $3 + 4 < x + 4 < 7 + 4$

$\Rightarrow$ 7 < x + 4 < 11

$\Rightarrow$ $\frac{1}{7} > \frac{1}{x+4} > \frac{1}{11}$

$\Rightarrow$ $-\frac{9}{7} < -\frac{9}{x+4} < -\frac{9}{11}$

$\Rightarrow$ 2 - <u>97</u>< 2 - $\frac{9}{x+4}$ < 2 - $\frac{9}{11}$

$\Rightarrow$ $\frac{5}{7} < \frac{2x-1}{x+4} < \frac{13}{11}$

$\therefore$ S is bounded and l.u.b. = $\frac{13}{11}$, g.l.b. = $\frac{5}{7}$

**Example 2:** Find the l.u.b. and g.l.b. in each case :

(i) $\left\{\frac{n+1}{n} : n \in \mathrm{N}\right\}$

(ii) $\left\{\frac{2x+1}{x+5} : |x-4| < 2\right\}$

(iii) $\left\{\frac{1}{2+x^2} ; -6 \le x \le 4\right\}$

(iv) $\left\{-\sqrt{1-x^2} : |x| \le \frac{\sqrt{3}}{2}\right\}$

**Solution:** (i) Let S = $\left\{\frac{n+1}{n} : n \in \mathrm{N}\right\}$

Now n $\in$N

$\Rightarrow$ n > 0 and n $\geq$ 1

$\Rightarrow$ $\frac{1}{n}$ > 0 and $\frac{1}{n} \leq 1$

$\Rightarrow$ 1 + $\frac{1}{n}$ > 0 and 1 + $\frac{1}{n} \leq 2$

$\Rightarrow$ 1 < 1 + $\frac{1}{n} \leq 2$

$\Rightarrow$ $1 < \frac{n+1}{n} \leq 2$

$\therefore$ S is a bounded set, where l.u.b. = 2 and g.l.b. = 1

OR Let S = $\left\{\frac{n+1}{n} : n \in \mathrm{N}\right\} = \left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, ......\right\}$

Clearly, no member of S is less then 1 and no member of S is greater than 2

$\therefore$ l.u.b. = 2, g.l.b. = 1

(ii) Let S = $\left\{\frac{2x+1}{x+5} : |x-4| < 2\right\}$

Now $\frac{2x+1}{x+5} = 2 - \frac{9}{x+5}$

Again |x - 4| < 2

$\Rightarrow$ $-2 < x - 4 < 2$

$\Rightarrow$ $2 < x < 6$

$\Rightarrow$ $2 + 5 < x + 5 < 6 + 5$

$\Rightarrow$ $7 < x + 5 < 11$

$\Rightarrow$ $\frac{1}{7} > \frac{1}{x+5} > \frac{1}{11}$

$\Rightarrow$ $-\frac{9}{7} < -\frac{9}{x+5} < -\frac{9}{11}$

$\Rightarrow$ $2 - \frac{9}{7} < 2 - \frac{9}{x+5} < 2 - \frac{9}{11}$

$\Rightarrow$ $\frac{5}{7} < \frac{2x+1}{x+5} < \frac{13}{11}$

$\therefore$ S is bounded and l.u.b = $\frac{13}{11}$, g.l.b. = $\frac{5}{7}$

(iii) Let S = $\left\{\frac{1}{2+x^2} ; -6 \leq x \leq 4\right\}$

Now $-6 \leq x \leq 4$

$\Rightarrow$ $-6 \leq x \leq 4 \leq 6$

$\Rightarrow$ $-6 \leq x \leq 6$

$\Rightarrow$ $|x| \le 6$

$\Rightarrow$ $0 \le x^2 \le 36$

$\Rightarrow$ $2 \le x^2 + 2 \le 38$

$\Rightarrow$ $\frac{1}{2} \ge \frac{1}{x^2+2} \ge \frac{1}{38}$

$\Rightarrow$ $\frac{1}{38} \le \frac{1}{x^2+2} \le \frac{1}{2}$

$\therefore$ S is bounded where l.u.b. = $\frac{1}{2}$, g.l.b. = $\frac{1}{38}$

(iv) Let S = $\left\{-\sqrt{1-x^2} : |x| \le \frac{\sqrt{3}}{2}\right\}$

Let $y = -\sqrt{1-x^2}$

$\because$ $|x| \le \frac{\sqrt{3}}{2}$, $0 \le |x| \le \frac{\sqrt{3}}{2}$

$\Rightarrow$ $0 \le x^2 \le \frac{3}{4}$

$\Rightarrow$ $0 \ge -x^2 \ge -\frac{3}{4}$

$\Rightarrow$ $1 \ge 1 - x^2 \ge 1 - \frac{3}{4}$

$\Rightarrow$ $1 \ge 1 - x^2 \ge \frac{1}{4}$

$\Rightarrow$ $\frac{1}{4} \le 1 - x^2 \le 1$

$\Rightarrow$ $\frac{1}{2} \le \sqrt{1-x^2} \le 1$

$\Rightarrow$ $-\frac{1}{2} \ge -\sqrt{1-x^2} \ge -1$

$\Rightarrow$ $-1 \le -\sqrt{1-x^2} \le -\frac{1}{2}$

$\Rightarrow \quad -1 \leq y \leq -\frac{1}{2}$

$\therefore$ S is bounded where l.u.b. = $-\frac{1}{2}$, g.l.b. = -1

**Example 3:** Find l.u.b. and g.l.b. (if they exist) of the set S = {3 sin x + 4 cos x} where $x \in R$.

**Solution:** S = {3 sin x + 4 cos x : $x \in R$}

Let y = 3 sin x + 4 cos x

Put $3 = r \cos \alpha$ ..... (1)

And $4 = r \sin \alpha$ ...... (2)

Squaring and (1) and (2), we get

$9 + 16 = r^2 (\cos^2\alpha + \sin^2\alpha)$

or $25 = r^2$

$\Rightarrow$ r = 5

$\therefore$ $y = r \cos \alpha \sin x + r \sin \alpha \cos x$

$= r \sin (x + \alpha)$

$= 5 \sin (x + \alpha)$

Now $-1 \leq \sin (x + \alpha) \leq 1 \quad \forall x \in R$

$\Rightarrow$ $-5 \leq 5 \sin (x + \alpha) \leq 5 \quad \forall x \in R$

$\Rightarrow$ $-5 \leq y \leq 5 \quad \forall x \in R$

$\therefore$ S = {-5, 5} is bounded and g.l.b. = -5, l.u.b. = 5

**Example 4:** Show that the following sets are bounded. Also find their l.u.b. and g.l.b. :

(i) $\{(\sin x + \cos x)^2 : 0 \leq x \leq \pi\}$

(ii) {2 sin x - 3 cos x}

**Solution:** (i) Let $S = \{(\sin x + \cos x)^2 : 0 \leq x \leq \pi\}$

Now $(\sin x + \cos x)^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$

$= 1 + \sin 2x$

Also $0 \leq x \leq \pi$

$\Rightarrow$ $0 \leq 2x \leq 2\pi$

$\Rightarrow$ $-1 \leq \sin 2x \leq 1$

$\Rightarrow$ $0 \leq 1 + \sin 2x \leq 2$

$\Rightarrow$ $0 \leq (\sin x + \cos x)^2 \leq 2$

$\therefore$ S is bounded and its l.u.b. = 2, g.l.b. = 0

(ii) Let S = (2 sin x - 3 cos x}

Let y = 2 sin x - 3 cos x

Put $2 = r \cos \alpha$ ..... (1)

and $3 = r \sin \alpha$ ...... (2)

Squaring and adding (1) and (2), we get

$4 + 9 = r^2 (\cos^2\alpha + \sin^2\alpha)$

$\Rightarrow \quad r^2 = 13$

$\Rightarrow \quad r = \sqrt{13}$

$\therefore$ $y = r \cos \alpha \sin x - r \sin \alpha \cos x$

$= (\sin x \cos \alpha - \cos x \sin \alpha)$

$= \sqrt{13} \sin (x - \alpha)$

Now $-1 < \sin (x - \alpha) < 1$

$\Rightarrow \quad -\sqrt{13} \leq \sqrt{13} \sin (x - \alpha) \leq \sqrt{13}$

$\Rightarrow \quad -\sqrt{13} \leq y \leq \sqrt{13}$

$\therefore$ S is bounded and its l.u.b. = $\sqrt{13}$, g.l.b. = $-\sqrt{13}$

**Example 5:** Find g.l.b. and l.u.b. of the following sets

(i) $\{\sin^2 x + \cos^4 x : x \in R\}$

(ii) $\{a \sin x + b \cos x + c : x \in R\}$

**Solution:** (i) Let S = {sin2 x + cos4 x : x $\in$ R}

Now $\sin^2 x + \cos^4 x = \sin^2 x + \cos^2 x (1 - \sin^2 x)$

$= \sin^2 x + \cos^2 x - \sin^2 x \cos^2 x$

$= 1 - \sin^2 x \cos^2 x$

$= 1 - \frac{1}{4} \sin^2 2x$

Also $0 \leq \sin^2 2x \leq 1$

$\Rightarrow \quad 0 \geq \frac{1}{4} \sin^2 2x \geq -\frac{1}{4}$

$\Rightarrow \quad 1 \geq 1 - \frac{1}{4} \sin^2 2x \geq 1 - \frac{1}{4}$

$\Rightarrow$ $1 - \frac{1}{4}\sin^2 2x \geq \frac{3}{4}$

$\Rightarrow$ $\frac{3}{4} \leq 1 - \sin^2 2x \leq 1 \ \forall\ x \in R$

$\therefore$ $S = \left[\frac{3}{4}, 1\right]$ is bounded and l.u.b = 1, g.l.b. = $\frac{3}{4}$

(ii) Let S = {a sin x + b cos x + c : x $\in$ R}

Let y = a sin x + b cos x + c

Put a = r cos a ..... (1)

b = r sin a ..... (2)

Squaring (1) and (2), we get

$a^2 + b^2 = r^2 (\cos^2 a + \sin^2 a)$

$\Rightarrow$ $a^2 + b^2 = r^2$

$\Rightarrow$ $r = \sqrt{a^2 + b^2}$

$\therefore$ y = r cos a sin x + r sin a cos x = r (sin x cos a + cos x sin a)

= r sin (x + a)

$= \sqrt{a^2 + b^2}\ \sin (x + a)$

Now $-1 \leq \sin (x + a) \leq 1 \ \forall\ x \in R$

$\Rightarrow$ $-\sqrt{a^2 + b^2} \leq \sqrt{a^2 + b^2}\sin (x + a) \leq \sqrt{a^2 + b^2} \ \forall\ x \in R$

$\Rightarrow$ $c - \sqrt{a^2 + b^2} \leq c + \sqrt{a^2 + b^2}\sin (x + a) \leq c + \sqrt{a^2 + b^2} \ \forall\ x \in R$

$\Rightarrow$ $c - \sqrt{a^2 + b^2} \leq y \leq c + \sqrt{a^2 + b^2} \ \forall\ x \in R$

$\Rightarrow$ $S = \left[c - \sqrt{a^2 + b^2}, c + \sqrt{a^2 + b^2}\right]$ is bounded

and g.l.b = $c - \sqrt{a^2 + b^2}$, l.u.b. = $c + \sqrt{a^2 + b^2}$

**4.6 Art. Neighbourhood of a point**

Let c $\in$ R and $\delta > 0$. Then the $\delta$- neighbourhood ($\delta$- nhd.) of c written as $N_\delta(c)$ or N (c, $\delta$) is the set

$N_\delta(c) = \{x \in R : |x - c| < \delta\}$

$= \{ x \in R : c - \delta < x < c + \delta \}$

$= ( c - \delta , c + \delta )$

**Remark :**

(i) The deleted $\delta$ - neighbourhood of c is the set

$N_\delta (c) - \{c\} = (c - \delta , c + \delta ) - \{c\} = (c - \delta , c) \cup (c, c + \delta )$

(ii) $(c - \delta , c)$ is the left heighbourhood of c denoted by $N \delta$ (c) and $(c, c + \delta )$ is the right neighbourhood of c and is denoted by $N \delta = (c)$.

(iii) Neighbourhood of a point c are open intervals around c or one sided open intervals from c.

**Another definition of neghbourhood of a point**

A set $G \subseteq R$ is said to be neighbourhood of a point c in R if $\exists \delta > 0$ such that $c \in N_\delta (c) \subseteq G$ i.e., if $(c - \delta , c + g) \subseteq G$.

A deleted neighbourhoodG of c is the set $G - \{c\}$

**Example.** (i) Since for each $c \in R$, $\exists \delta > 0$ such that $(c - \delta , c + \delta ) \subseteq R$, therefore the set R of real numbers is the neighbourhood of each of its points.

(ii) The set of rational numbers is not a neighbourhood of each of its points for if $c \in Q$, then $(c - \delta , c + \delta )$, $( \delta > 0)$ contains infinite many irrational numbers and thus $(c - \delta , c + \delta ) \subseteq Q$.

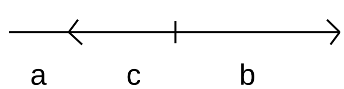
**Art.** Every open interval is a neighbourhood of each of its points.

Proof : Let I be any open interval in R, then we have the following cases

(i) $I = (a, b)$ (ii) $I = (a, \infty)$

(iii) $I = (-\infty, b)$ (iv) $I = (-\infty, \infty)$

**Case (i)** When $I = (a, b)$ where $a, b \in R$ with $a < b$. Let $c \in I$ be any point i.e., $a < c < b$. Choose $\delta = \min \{c - a, b - c\}$ then $\delta > 0$. We show that $N_\delta (c) = \{c - \delta , c + \delta \} \subseteq I$.




Let $x \in N_\delta (c)$ be arbitrary point. Then

$x \in (c - \delta , c + \delta )$

or $c - \delta < x < c + \delta$

or $- \delta < x - c < \delta$

Since $\delta \leq c - a$

$\therefore$ $-(c - a) \leq - \delta \leq x - c < \delta \leq b - c$

$\Rightarrow$ $-(c - a) < x - c < b - c$

$\Rightarrow$ $\alpha < x < b$ or $x \in (a, b) = I$

$\therefore$ $\forall\, x \in N_\delta (c)$

$\Rightarrow$ $x \in I$

$\therefore$ $N\delta\ (c) \subseteq I$ is neighbourhood of $c \in I$

Since c is any arbitrary point of I, therefore I is neighbourhood of each of its point.

Case (ii) When $I = (a, \infty)$.

Let $c \in I$ any point, then $c < a$

Let $\delta = c - a > 0$.

Now $N_\delta (c) = (c - \delta, c + \delta)$

$= (a, 2c - a) \subseteq (a, \infty) = I$

$\Rightarrow$ I is neighbourbood of c and hence I is the heighbourhood of each of its point.

Case (iii) When $I = (-\infty, b)$

Let $c \in I$ be any point, then $c < b$

Choose $\delta = b - c > 0$.

Now $N_\delta (c) = (c - \delta, c + \delta)$

$= (2c - b, b) \subseteq (-\infty, b) = I$

$\Rightarrow$ I is neighbourbood of c and hence I is the heighbourhood of each of its point.

Case (iv) When $I = (-\infty, \infty)$

Let $c \in I$ and let $\delta > 0$, then

$N_\delta (c) = (c - \delta, c + \delta) \subseteq (-\infty, \infty) = I$ and hence I is the heighbourhood of each of its point.

**Remark :**

(i) As shown above, the open interval (a, b) is a heighbourhood of each of its points.

(ii) The intervals [a, b) is a neighbourhood of each point of (a, b), but is not a neighbourhood of a, as the interval $(a - \delta, a + \delta) \not\subseteq [a, b)$ for any $\delta > 0$. Similarly, the interval (a, b] is a heighbourhood of each point (a, b) but not a heighbourhood of b.

(iii) The set N, Z and Q are not neighbourhood of any real number, because for any $x \in R$, $N_\delta (x) = (x - \delta, x + \delta)$ is a neighbourhood of x, $\delta > 0$ but $N_\delta (x) \not\subseteq N$ or $N_\delta(x) \not\subseteq Z$ and $N_\delta (x) \not\subseteq Q$. Therefore N, Z and Q are not neighbourhood of any point of R.

**Art.** If G and H are two neighbourhood of a point c, then $G \cap H$ is also a neighbourhood of c.

**Proof :** Since G and H are neighbourhood of a point c, therefore these exists $\delta_1$, $\delta_2$> 0 such that $N_{\delta_1}$ (c) $\subseteq$ G and $N_{\delta_2}$ (c) $\subseteq$ H.

Choose $\delta = \min\{\delta_1, \delta_2\}$, then

$N_\delta$ (c) = (c - δ , c + δ ) $\subseteq$ (c - $\delta_1$, c + $\delta_1$) $\subseteq N_{\delta_1}$ (c) $\subseteq$ G and

$N_\delta$ (c) = (c - δ , c + δ ) $\subseteq$ (c - $\delta_2$, c + $\delta_2$) $\subseteq N_{\delta_2}$ (c) $\subseteq$ G

i.e., $N_\delta$ (c) $\subseteq$ G and $N_\delta$ (c) $\subseteq$ H

$\Rightarrow$ $N_\delta$ (c) $\subseteq$ G $\cap$ H.

Hence G$\cap$H is a neighbourhood of c.

**Art.** If G is a neighbourhood c and G $\subseteq$ H, then his also a neighbourhood of c.

**Proof :** Since G is a neighbourhood of c $\Rightarrow\exists$ δ > 0 such that $N_\delta$ (c) $\subseteq$ G

As G $\subseteq$ H $\therefore N_\delta$ (c) $\subseteq$ H and there H is heighbourhood of c.

**Remark :** If G and H are two neighbourhood of a point c, then G$\cup$H is also a neighbourhood of c.

## 4.7 Art. Cluster Point and Limit Point of a Set

**Cluster Point or adherent point of a set**

A real number c is called a cluster point (or adherent point) of the set A. in R if and only if every δ -neighbourhood of c contains atleast one point of A. In other words the real number c is a cluster point of the set A if and only if for every δ > 0, N δ (c) $\cap$ A $\neq\phi$.

The set of all cluster points of A is called the closure of the set A and it is denoted Ā or CI (A).

**Remark :**

(i) A cluster point of a set A need not belong to the set A.

(ii) Every point of A is a cluster point of the set A.

**Limit point of a set**

A real number c is called a limit point of the set A in R if and only if every δ - Neighbourhood of c contains atleast one point of A other then c. In other words, the real number c is a limit point of the set A if and only if for every δ > 0,

($N_\delta$ (c) - {c}) $\cap$ A $\neq\phi$ (or ($N_\delta$ (c) $\cap$ A $\neq$ δ , {c})

The set of all limit points of A is called derived set of A and it is denoted by A' or D (A).

**Isolated point of a set**

A cluster point of a set which is not a limit point of A is called an isolated point of A.

**Note :**

(i) Every limit point of a set A is a cluster point of the set A.

(ii) c is not limit point of A if there exsits a $\delta$-neighbourhood of c such that $N_\delta(c) \cap A \neq \phi, \{c\}$.

**Art.** If c is a limit point of $A \subseteq R$, then each neighbourhood or c contains infinite many distinct points of A.

**Proof :** Since c is a limit point of A, therefore, each neighbourhood of c contains atleast on point of A other than c.

Let N be a neighbourhood of c, than N contains atleast one print (say) $x_0$ of A such that $x_0 \neq c$.

Choose $\delta_1 > 0$ such that $(c - \delta_1, c + \delta_1) = I_1 \subseteq N$ such that $x_0 \notin I_1$.

Now by definition of limit point, the neighbourhood $I_1$ of c contain atleastone point (say) $x_1$ of A such that $x_1 \neq c$. Also $x_1 \neq x_0$ ($\because x_0 \notin I_1$).

Let $\delta_2 = \min\{|c - x_1|, \delta_1\}$, then $(c - \delta_2, c + \delta_2) = I_2 \subseteq I_1$, is a neighbourhood $I_2$ of c such that $x_1 \notin I_2$. Again by definition of limit point, the neighbourhood $I_2$ of c contains atleast one point (say) $x_2$ of A where $x_2 \neq c$. Also $x_2 \neq x_1$ [$\because x_1 \notin I_2$].

Clearly $N \supseteq I_1, I_2 \Rightarrow x_0, x_1, x_2$ are distinct point of A and they belong to N, the neighbourhood of c.

Again let $\delta_3 = \min\{|c - x_2|, \delta_2\}$, then $(c - \delta_3, c + \delta_3) = I_3 \subseteq I_2$, is a neighbourhood of c and as discussed above, the neighbourhood $I_3$ of c contains atleast one point (say) $x_3$ of A where $x_3 \neq c$ and $x_3$ is distinct from each of $x_0, x_1, x_2$.

**Corollary.** A finite subset of R has no limit point.

**Proof :** Let A be a finite subset of R. if $c \in R$ is a limit point of A, then by the above theorem, every neighbourhood of c contains infinitely many distinct points of A, which is not possible, A has finite many distinct points. Hence a finite subset of R has not limit point.

**Remark :** Since $\phi$ is an empty set and hence it has not limit point i.e. $\phi' = \phi$.

### 4.8 Art Bolzano-Weierstrass property

Every infinite and bounded set of real numbers has atleast one limit point.

**Remark :** The above theorem gurantees the existence of a limit point of set which is both bounded and infinite. The converse of above theorem need not be true i.e., an infinite set having a limit point need not be bounded.

e.g. the set Q of a rational numbers has limit points but it is not bounded.

Example 5 : Show that every non-empty finite set is not a neighbourhood of any point of R.

Solution : Since neighbourhood of a point is an interval containing the point and every open interval contains infinite many real numbers, therefore it is not contained in a finite set.

Hence every non-empty finite set is not a neighbourhood of any print of R.

Example 6 : Show that intersection of all arbitrary neighbourhood of a point $x \in R$ is $\{x\}$.

Solution : Let y ($\neq$ x) be any real number.

Let $\delta_1 = |x - y| > 0$. Then $N_{\delta_1}(x) = (x - \delta_1, x + \delta_1)$ is aneighbourhood of x and it does not contain y

$\therefore$ $y \notin (x - \delta_1, x + \delta_1) \Rightarrow y \notin \bigcap_{\delta>0} (x - \delta, x + \delta)$ for $\delta_1 > \delta > 0$

But $\{x\} \subseteq \bigcap_{\delta>0} (x - \delta, x + \delta), \forall \delta > 0$

Thus the intersection of arbitrary neighbourhood of x does not contain any real number other than x.

Hence the intersection of all arbitrary neighbourhood of x is {x}.

**Example 7:** Show that the set of natural numbers N has no limit point.

**Solution :** Let $x \in R$ be any real number. Then either $x \in N$ or $x \in R - N$.

**Case (i)** When $x \in N$, then $\exists$ a neighbourhood $N_1(x)$ of x such that

$N_1(x) \cap N = (x - 1, x + 1) \cap N = \phi, \{x\}$

$\Rightarrow$ the neighbourhood $N_1(x)$ contains no point of N other than x.

So, x is not a limit point of N.

**Case (ii)** When $x \in R - N$, then

Subcase (i) When $x > 0$, then $\exists$ a positive integer n such that $n < x < n + 1$.

Choose $\delta = \min \{x - n, n + 1 - x\}$. Then $\exists$neighbourhood

$N_\delta(x) = (x - \delta, x + \delta) \subseteq (n, n + 1)$ and $N_\delta(x) \cap N = \phi$.

So, x is not a limit point of N.

Subcase (ii) When $x < 0$. Take $\delta = -x > 0$, then $\exists$neighbourhood$N_\delta(x)$ of x such that $N_\delta(x) \cap N = (2x, 0) \cap N = \phi$.

So, x is not a limit point of N.

Hence the set of natural number has no limit points.

**Example 8 :** Show that the set of integers Z has no limit point.

**Solution :** Let $n \in Z$ be any integer. Choose $0 < \delta < 1$. Then $\exists$ a $\delta$-neighbourhood$N_\delta(n) = (n - \delta, n + \delta)$ of n such that $N_\delta(n) \cap Z = \{n\}$

$\therefore$ $n \in Z$ is not a limit point of Z.

Let $x \in R$ be any real number such that $x \notin Z$. Then $\exists$ a unique integer m such that $m < x < m + 1$.

So that neighbourhood (m, m +1) of x contains no integer i.e., $(m, m + 1) \cap Z = \phi$.

Thus x is not a limit point of Z.

Hence no real number is a limit point of Z.

**Example 9 :** Show that 0 is the only limit point of set $\left\{\frac{1}{n} : n \in N\right\}$.

**Solution :** Let A = $\left\{\frac{1}{n} : n \in N\right\}$ = $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, .....\right\}$ be given set.

Clearly A is an infinite bounded subset of R. Because $0 < x < 1 \ \forall \ x \in A$.

Let $\delta > 0$. By the Archimedian property of real numbers, $\exists$ a positive integerm such that $m\delta > 1$ i.e., $\frac{1}{m} < \delta$

$\therefore$ the $\delta$-neighbourhood$N_\delta$ (0) = ($-\delta$, $\delta$) of 0 contains a point $\frac{1}{m}$ of A other than 0.

Since $\delta > 0$ is any arbitrary positive real number.

$\therefore$ the $\delta$-neighbourhood$N_\delta$ (0) of 0 contains atleast one point of A other than 0.

Hence 0 is a limit point of A.

Further, we show that A has no other limit point.

Let $x \neq 0 \in R$ be any real number

**Case (i)** When x < 0 than take $\delta = -x > 0$ so that $\delta$-neighbourhood of x i.e.$N_\delta$ (x) = (2x, 0) contains no point of A.

$\therefore$ x < 0 cannot be a limit point of A.

**Case (ii)** When x > 1, then take $\delta$ = x - 1 so that $\delta$-neighbouthood of x

i.e., $N_\delta$ (x) = (1, 2x - 1) contains no point of A.

$\therefore$ x > 1 cannot be a limit point of A.

**Case (iii)** When x = 1, then take $\delta = \frac{1}{10}$ so that the $\delta$-neighbourhood of x = 1

i.e. $N_{\frac{1}{10}} = \left(1 - \frac{1}{10}, 1 + \frac{1}{10}\right) = \left(\frac{9}{10}, \frac{11}{10}\right)$contains no point of A other than 1.

$\therefore$ x = 1 cannot be a limit point of A

**Case (iv)** When 0 < x < 1, then $\exists$ a positive integer m such that $m \leq \frac{1}{x} < m + 1$

Subcase (i) if $x \notin A$, then $x \neq \frac{1}{m}$ ($m \in N$) $\Rightarrow m \neq \frac{1}{x}$, then we have m < $\frac{1}{x}$ < m + 1 ⇒ $\frac{1}{m+1}$ < x < $\frac{1}{m}$ and x ∉ A.

Hence x is not a limit point of A.

Subcase (ii) if x ∈A then $\frac{1}{m}$ (m ∈N) and m ≠ 1. (∵ x ≠ 1)

Now m ≤ $\frac{1}{x}$ < m + 1

⇒ m - 1 < $\frac{1}{x}$ < m + 1

⇒ $\frac{1}{m+1}$ < x < $\frac{1}{m-1}$

∴ ∃ a neighbourhood $\left(\frac{1}{m+1}, \frac{1}{m-1}\right)$ of x $\left(=\frac{1}{m}\right)$ contains no point of A other than x $\left(=\frac{1}{m}\right)$ hence x is not a limit point of A.

Thus for 0 < x < 1. x is not a limit point of A.

From (i) to (iv) we notice that x ≠ 0 is not a limit point of A.

Hence 0 is the only limit point of A.

**Example 10:** Show that (A ∪ B)' = A' ∪ B'.

i.e., the set of limit points of A ∪ B is equal to the union of limit points of A and B.

**Solution:** Let x be a limit point of A ∪ B

⇔ ∃ a positive real number δ such that

$N_\delta(x) \cap (A \cup B) \neq \phi, \{x\}$

⇔ $(N_\delta(x) \cap A) \cup (N_\delta(x) \cap B) \neq \phi, \{x\}$

⇔ $N_\delta(x) \cap A \neq \phi, \{x\}$ or N δ (x) ∩ B ≠φ, {x}

⇔ x ∈ A' or x ∈ B'

⇔ x ∈ A' ∪ B'

Hence (A ∪ B)' = A' ∪ B'

**Example 11:** (i) If u is the supermum of an infinite set A in R and u ∉ A, then u is a limit point of A.

(ii) If l is the infimum of an infinite set A in R and l ∉ A, then l is a limit point of A.

**Solution:** (i) Since sup A = u, where A $\subseteq$ R and is infinite set

$\therefore$ by completeness property of R, u $\in$ R

Also sup A = u $\notin$ A (given)

$\therefore$ for any $\delta > 0$ however small $\exists$'s atleast one point x $\in$ A such that $u - \delta < x < u$

$\Rightarrow$ each neighbourhood of u contains atleast one point x of A different from u.

Hence u is a limit point of A.

(ii) Since Inf A = l, where A $\subseteq$ R and is an infinite set

$\therefore$ by completeness property of R, l $\in$ R

Also inf A = l $\notin$ A (given)

$\therefore$ for any $\delta > 0$ however small $\exists$'s atleast one point x $\in$ A such that $l < x < l + 1$

$\Rightarrow$ each neighbourhood of l contains atleastoen point x of A different from l.

Hence l is a limit point of A.

## 4.9 Self Check Exercise

Q.1 Find l.u.b. and g.l.b. (if exist) of the set

S = {3 sin x + 4 cos x}, x $\in$ R

Q.2 If $0 < a < \frac{1}{n}$ $\forall$ n $\in$N, then show that a = 0.

Q.3 Prove that the set

$\left\{\frac{n-1}{n+1}, n \in N\right\}$ is bounded.

Find l.u.b. and g.l.b.

## 4.10 Summary

In this unit we have learnt the following

(i) definition of a bounded set, l.u.b and g.l.b

(ii) how to find l.u.b and g.l.b of a set ?

(iii) l.u.b. property of real numbers

(iv) Archemedian Property of reals

(v) nhd. of a point, duster point and limit point of a set

(vi) Bolzano-Weirstrass property

## 4.11 Glossary:

1. Complete order field - Due to the property of order completeness of reals, the set R of real-numbers is called complete order field.

2. g.l.b. property of real numbers-

   Let S be a set bounded below (S is non-empty subset of reals). Then by using l.u.b. property of reals it can be shown that S has g.l.b. on R. This is called g.l.b. property of reals.

## 4.12 Answer to Self Check Exercise

Ans.1 S = [-5, 5], g.l.b. = -5 l.u.b. = 5

Ans.2 Prove it

Ans.3 l.u.b. = 1, g.l.b. = 0

## 4.13 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983
3. K.A. Ross, Elementary Analysis- The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag, 2003


## 4.14 Terminal Questions

1. Prove that the union of two bounded set is bounded subset of R. Is the converse true? Justify.

2. Find l.u.b and g.l.b. (if exist) of the set

   $$\left\{\frac{1}{1+x^2} : -5 \le x \le 1\right\}$$

3. Prove that the set

   $$\left\{\frac{(-1)^n . n}{n+1}, n \in N\right\}$$

   is bounded Find l.u.b. and g.l.b.

4. State whether the set

   $$\left\{x \sin x : \pi \le x \le \frac{3\pi}{2}\right\}$$

   is bounded or not. Find l.u.b. and g.l.b. (if they exist)

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## Unit - 5

# Real, Bounded And Convergent Sequences

**Structure**



## 5.1 Introduction

Dear students,you are already familiar with the notion of a set as the collection of a well defined distinct objects of our perception or thought. IN mathematics, a sequence is an enumerated collection of objects in which repetitions are allowed and order matters. Like a set, it contains members (also called elements of terms). The number of elements (possibly infinite) is called the length of the sequence. Unlike a set, the same elements can appear multiple times at different positions in a sequence and unlike a set, the order matters. Sequences are useful in a number of mathematical disciplines for studying functions, spaces and other mathematical structures using the convergent property of a sequence. Sequences are also of interest in their own right, and can be studied as patterns or puzzles, such as in the study of prime numbers.

## 5.2 Learning Objectives

The main objectives of this unit are to

(i) define a sequence

(ii) to study bounded and unbounded sequence

(iii) study convergent, divergent and oscillatory sequences

(iv) to study null sequence, equal sequence etc.

## 5.3 Sequences

**Definition:** If N is a set of natural numbers and X any set, then function $f : N \to X$ is called a sequence.

## 5.4 Real Sequence

If X = R or a subset of R, then $f$ is called a real sequence and if X = C or a subset of C, then $f$ is called a complex sequence.

In the remaining part of this unit, we will be concerned with real sequences only. Hence by a sequence, we will mean real sequence.

Thus, a sequence is a function whose domain is the set N of natural numbers and range is any set X.

Let $a : N \to R$ be a sequence. The image of $n \in N$, instead of being denoted by a (n) is generally denoted by $a_n$. Thus $a_1, a_2, a_3$, ..... are the real numbers associated to 1, 2, 3, ..... by this mapping. $a_n$ is called the general term or the nth term of the sequence.

If the nth term $a_n$ of a sequence is give, we can find the first second, third,...... terms of the sequence by putting n = 1, 2, 3, .....

Thus a sequence whose nth term is $a_n$ is written as $\{a_n\}$ or $\{a_n\}$ or $(a_n)$ or $< a_n>$ where $n \in N$

Sometimes, it is denoted by writing all its terms within the brackets

i.e. $\{a_1, a_2, a_3, ......., a_n.........\}$

Another definition

A set of numbers $a_1, a_2, a_3, ......., a_n.........$ such that to very positive integer n, there corresponds a number an of the set, is called a sequence.

$a_1, a_2, a_3$, ....... are called the elements of the sequence.

### Range of a sequence

The range of a sequence $\{a_n\}$ is the set of values $\{a_n, n \in N\}$ consisting of distinct terms, without repetition and irrespective of their position.

In other words, the set of all distinct terms of sequence $\{a_n\}$ is called its range.

### Some Examples of sequences

1. The sequence $\{a_n\}$, where $a_n = n$ is $\{1, 2, 3,......., n, ....\}$
2. The sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ is $\left\{1, \frac{1}{2}, \frac{1}{3}, ......, \frac{1}{n}, .....\right\}$
3. The sequence $\{a_n\}$, where $a_n = (-1)^n$ is $\{-1, 1, -1, 1, -1, 1,.......\}$. Its range is $\{-1, 1\}$
4. The sequence $\{a_n\}$, where $a_n = c \;\forall\; n \in N$ is $\{c, c, c,.......\}$. Its range of $\{c\}$. This sequence is a constant sequence.

5. The sequence $\{a_n\}$, where $a_n = (-1)^n \frac{n}{n+1}$, is

$$\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, ......(-1)^n \frac{n}{n+1}, ....\right\}$$

**Note 1 :** The mth and nth terms $a_m$ and $a_n$ for $m \neq n$ are treated as distinct even if $a_m = a_n$ i.e., the terms occurring at different positions are treated as distinct terms even if they have the same value.

So, in a sequence $\{a_n\}$, the order of elements cannot be changed.

**Note 2 :** The number of terms in a sequence is always infinite where as the range of a sequence may have a finite number of elements.

**Note 3 :** Sometimes a sequence has the zero-th term. In this case, its domain is $N \cup \{0\}$ so that the sequence is $\{a_0, a_1, a_2, .........., a_n, .........\}$

or $\{a_n\}_{n \geq 0}$

Similarly, we can start a sequence from any positive integer m and in this case, the sequence is written as $\{a_n\}_{n>m}$.

**Note 4 :** Equal Sequences.

Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equal if $a_n = b_n$ for every n.

If ranges of two sequences are equal, even then the sequences may not be equal.

Let $a_n = (-1)^n$, $b_n = (-1)^{n+1}$

$\therefore$ ranges of $\{a_n\}$ and $\{b_n\}$ are equal.

But $a_1 \neq b_1$ as $a_1 = -1$, $b_1 = 1$

$\therefore$ two sequences are not equal

## 5.5 Bounded and unbounded Sequences

(i) A sequence {an} is said to be bounded above if there exists a real number k such that $a_n < k \ \forall \ n \in N$.

k is called an upper bound of the sequence $\{a_n\}$.

(ii) A sequence $\{a_n\}$ is said to be bounded below if there exists a real number h such that $h \leq a_n \forall \ n \in N$. h is called a lower bound of the sequence $\{a_n\}$.

(iii) A sequence $\{a_n\}$ is said to be bounded if it is bounded above as well as bounded below i.e., if there exist two real numbers h and k such that

$h \leq a_n \leq k \quad \forall \ n \in N$

(iv) S sequence $\{a_n\}$ is said to be unbounded if it is not bounded.

A sequence $\{a_n\}$ is said to be unbounded if given $\Delta > 0$, however large, $\exists$ m $\in$ N such that $|a_n| > \Delta \forall \ n \geq m$.

**Note 1 :** A sequence is bounded above, bounded below or bounded according as its range is bounded above, bounded below or bounded.

**Note 2 :** If a sequence $\{a_n\}$ is bounded, then its range is bounded. The l.u.b. and g.l.b. of range of sequence $\{a_n\}$ are called the l.u.b. and g.l.b. of the sequence. They are also called the supremum and infimum of the sequence.

**Properties of l.u.b. u of $\{a_n\}$**

(i) $a_n \le u \quad \forall \quad n \in N$

(ii) Given $\varepsilon > 0$, however small, $\exists$ at least one positive integer m such that $a_m > u - \varepsilon$.

**Properties of g.l.b. l of $\{a_n\}$**

(i) $a_n \ge l \quad \forall \quad n \in N$

(ii) Given $\varepsilon > 0$, however small, $\exists$ at least one positive integer m such that $a_m < l + \varepsilon$.

Note 3 : Assume that sequence {an} is bounded

$\therefore$ $\exists$ two real numbers h and k such that

$$h \le a_n \le k \quad \forall \quad n \in N \qquad ....(1)$$

Let $M = \text{Max.} (|h|, |k|)$

$\therefore$ $|h| \le M, |k| \le M$

$\Rightarrow$ $-M \le h \le M, -M \le k \le M \qquad ....(2)$

From (1) and (2), $-M \le a_n \le M \quad \forall \quad n \in N$

$\therefore$ $|a_n| \le M \quad \forall \quad n \in N$

Now assume that $|a_n| < M \quad \forall \quad n \in N$

$\therefore$ $-M \le a_n \le M \quad \forall \quad n \in N$

$\Rightarrow$ Sequence $\{a_n\}$ is bounded

$\therefore$ Sequence $\{a_n\}$ is bounded iff $\exists$ M > 0 such that

$|a_n| < M \quad \forall \quad n \in N$

$\therefore$ Sequence $\{a_n\}$ is bounded iff $\exists$ real number M such that $|a_n| < M \ \forall \ n \in N$.

**Note 4 :** (i) If each $a_n > 0$ and sequence $\{a_n\}$ is unbounded, then it is unbounded above.

(ii) If each $a_n < 0$ and sequence $\{a_n\}$ is unbounded, then it is unbounded below.

**Examples :** (i) Consider the sequence $\{a_n\}$ defined by

$$a_n = \frac{1}{n} \ \forall \quad n \in N$$

Now $n \ge 1$

$\Rightarrow \quad \frac{1}{n} \leq 1$ ....(1)

Also $\frac{1}{n} > 0 \quad \forall\, n \in N$ ....(2)

From (1) and (2),

$$0 < \frac{1}{n} \leq 1 \qquad \forall \qquad n \in N$$

$\Rightarrow$ $\{a_n\}$ is bounded.

(ii) Consider the sequence {an} defined by

$a_n = n^2$

$\therefore$ Sequence is $\{1^2, 2^2, 3^2, \ldots\ldots, n^2, \ldots..\}$

Here $a_n \geq 1 \quad \forall \quad n \in N$, but $\exists$ no real number k such that $a_n \leq k$

$\therefore$ the sequence $\{a_n\}$ is not bounded above.

(iii) consider the sequence $\{a_n\}$ defined by $a_n = -n$

It is bounded above and 0 is an upper bound. But it is not bounded below.

(iv) the sequence $\{a_n\}$ where $a_n = (-2)^n$ is neither bounded above not bounded below.

## 5.6 Convergent, Divergent and Oscillatory Sequences.

A sequence $\{a_n\}$ is said tbe converge to a limit l, if given $\varepsilon > 0$. however small, there exists a positive integer m (depending upon $\varepsilon$) such that

$$|a_n - l| < \varepsilon \qquad \forall \qquad n \geq m$$

l is called a limit of the sequence $\{a_n\}$ and we write it as

$\underset{n \to \infty}{Lt}\, a_n = l$ or Lt $a_n = l$

or $a_n \to l$ as $n \to \infty$

or simply $a_n \to l$

**Examples**

(i) Consider the sequence $\left\{\frac{1}{n}\right\}$.

Given $\varepsilon > 0$, however small, we choose a natural number m such that m $> \frac{1}{\varepsilon} \Rightarrow \frac{1}{m} < \varepsilon$

∴ $\forall\ n \geq m,\ \frac{1}{n} \leq \frac{1}{m} < \varepsilon$

∴ $\left|\frac{1}{n} - 0\right| = \frac{1}{n < \varepsilon}\ \forall \quad n \geq m$

∴ Sequence $\{a_n\}$ converges to 0.

(ii) Consider the sequence $\left\{\frac{n}{n+1}\right\}$.

Given $\varepsilon > 0$, however small, we choose a natural number m such that m $> \frac{1}{\varepsilon} \Rightarrow \frac{1}{m} < \varepsilon$

∴ $\forall\ n \geq m$, we have

$$\left|\frac{n}{n+1} - 1\right|$$

$$= \left|\frac{n-n-1}{n+1}\right|$$

$$= \left|\frac{-1}{n+1}\right|$$

$$= \frac{1}{n+1} < \frac{1}{n} \leq \frac{1}{m} < \varepsilon$$

$$= \left|\frac{n}{n+1} - 1\right| < \varepsilon \qquad \forall \quad n \geq m$$

∴ $\underset{n\to\infty}{Lt} \frac{n}{n+1} = 1$

⇒ Sequence $\left\{\frac{n}{n+1}\right\}$ converges to 1.

**Note 1:** We know that $|a_n - l| < \varepsilon \Rightarrow a_n \in (l - \varepsilon, l + \varepsilon)$. Now a neighbourhood V of l always contains $(l - \varepsilon, l + \varepsilon)$. Thus we give another definition of convergent sequence.

A sequence $\{a_n\}$ is said to converge to a limit l, if given a neighbourhood V of i, there exists an integer m such that $a_n \in V\ \forall\ n \geq m$.

**Note 2 :** A sequence $\{a_n\}$ is said to be convergent if an converges to some limit, otherwise $\{a_n\}$ is said to be non-convergent.

Divergent Sequence

(i) A sequence $\{a_n\}$ is said to be diverge to $\infty$ if given k > 0, however large, there exists a positive integer m (depending upon k) such that

$$a_n > k \quad \forall \quad n \geq m$$

We write it as $a_n \to \infty$ or $\underset{n\to\infty}{Lt}\ a_n = \infty$

(ii) A sequence $\{a_n\}$ is said to be diverge to $-\infty$ if given k > 0, however large, there exists a positive integer m (depending upon k) such that

$$a_n < -k \quad \forall \quad n \geq m$$

Note. A sequence {an} is said to be divergent if $\underset{n\to\infty}{Lt}\ a_n$ is not finite

i.e. if $\underset{n\to\infty}{Lt}\ a_n = +\infty$ or $-\infty$

(i) Consider the sequence $\{a_n\}$ where $a_n = 2 - n^2$.

Let k be any positive real number, however large

Now $a_n = 2 - n^2 < -k$ iff $n > \sqrt{k+2}$

$\Rightarrow$ $a_n < -k \quad \forall \quad n \geq m$ where $m > \sqrt{k+1}$, $m \in N$

$\therefore$ sequence $\{a_n\}$ diverges to $-\infty$.

(ii) Consider the sequence $\{a_n\}$ where $a_n = n^2 + 3n$.

Let k be any positive real number, however large

Now $a_n > k$

if $n^2 + 3n > k$

i.e. if $n^2 > k$

i.e. if $n > \sqrt{k}$

$\therefore$ If m is a positive integer such that $m > \sqrt{k}$, then $a_n > k\ \forall\ n \geq m$

$\therefore$ sequence $\{a_n\}$ diverges to $+\infty$

**Note :** A divergent sequence is always unbounded.

**Oscillatory Sequence**

A sequence which is neither convergent not divergent is said to be oscillatory.

**Note.** (i) A bounded sequence $\{a_n\}$ which is not convergent is said to be oscillate finitely.

(ii) An unbounded sequence $\{a_n\}$ which diverges neither to $+\infty$ not to $-\infty$ is said to oscillate infinitely.

**Examples :**

(i) The sequence $\{(-1)^n\}$ oscillates finitely.

(ii) The sequence $\{(-1)^n n\}$ oscillates infinitely.

**Uniqueness of Limit of a Convergent Sequence**

If a sequence is convergent, then it converges to a unique limit.

**Proof :** If possible, suppose that the sequence $\{a_n\}$ converges to two different limits l and l'.

Then given $\varepsilon > 0$, however small, there exist natural numbers $m_1$ and $m_2$ such that

$$|a_n - l| < \frac{\varepsilon}{2} \quad \text{for } n \geq m_1$$

and $$|a_n - l'| < \frac{\varepsilon}{2} \quad \text{for } n \geq m_2$$

Let m = max. $(m_1, m_2)$

$$|a_n - l| < \frac{\varepsilon}{2} \quad \text{for } n \geq m \qquad \text{....(1)}$$

and $$|a_n - l'| < \frac{\varepsilon}{2} \quad \text{for } n \geq m \qquad \text{....(2)}$$

Now $|l - l'| = |(l - a_n) + (a_n - l')|$

$\leq |l - a_n| + |a_n - l'|$

$= |a_n - l| + (a_n - l')$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore$ $|l - l'| < \varepsilon$ for every $\varepsilon > 0$ and $\forall\, n \geq m$

$\Rightarrow$ $l - l' = 0$ [$\therefore$ the only non-negative real number which is less than all Positive numbers is zero.

$\Rightarrow$ $l = l'$

$\therefore$ our supposition is wrong.

$\therefore$ If a sequence is convergent, then it converges to a unique limit.

**Art.** Prove that a convergent sequence is bounded. Is its converse true?

**Proof :** Let the sequence $\{a_n\}$ converge to l.

$\therefore$ given $\varepsilon > 0$, however small, there exists a natural number m such that

$$|a_n - l| < \varepsilon \quad \forall \quad n \geq m$$

$\therefore$ $l - \varepsilon < a_n < l + \varepsilon \quad \forall \quad n \geq m$

Let $h = \min(l - \varepsilon, a_1, a_2, ......, a_{m-1})$

and $k = \max.(l + \varepsilon, a_1, a_2, ......, a_{m-1})$

$\therefore$ $h \leq a_n \leq k \quad \forall \quad n \in N$

$\therefore$ sequence $\{a_n\}$ is bounded.

The converse is not true i.e. a bounded sequence may not be convergent.

Consider the sequence $\{a_n\}$ where $a_n = (-1)^n$

$\therefore$ Sequence is $\{-1, 1, -1, 1, -1, 1,......\}$ which is bounded as 1 and -1 are its l.u.b. and g.l.b. respectively.

Now we will prove that $\{a_n\}$ is not convergent.

If possible, suppose that $\{a_n\}$ converges to l.

$\therefore$ given $\varepsilon > 0$, however small, there exists a natural number m such that

$|a_n - l| < \varepsilon \quad \forall \quad n \geq m$

$\therefore$ $|a_m - l| < \varepsilon$ ....(1)

and $|a_{m+1} - l| < \varepsilon$ ....(2)

Now $|a_m - a_{m+1}| = |a_m - l| - (a_{m+1} - l)$

$\leq |a_m - l| + |a_{m+1} - l|$

$< \varepsilon + \varepsilon$ [$\because$ of (1), (2)]

$= 2\varepsilon$

or $2 < 2\varepsilon$ [$\because |a_m - a_{m+1}| = |(-1)^m - (-1)^{m+1}| = |(-1)^m (1 + 1)| = 2$]

which is not true of $\varepsilon \leq 1$

$\therefore$ our supposition is wrong.

$\therefore$ $\{a_n\}$ is not convergent.

**Note.** If a sequence is unbounded, then it is non-convergent.

For example, $\{a_n\}$ where $an = n^2$ is not convergent as it is not bounded.

(i) Prove that a sequence which diverges to $+\infty$ is bounded below but unbounded above.

(ii) Prove that a sequence which diverges to $-\infty$ is bounded above but unbounded below.

**Proof:** (i) Assume that sequence $\{a_n\}$ diverges to $+\infty$

∴ given k > 0, however, large, there exists a positive integer m such that

$a_n > k \quad \forall \quad n \geq m$

∴ there are infinitely many terms of $\{a_n\}$ which are greater than k

∴ sequence $\{a_n\}$ is not bounded above.

Let a = min. $(a_1, a_2, \ldots, a_{m-1}, k)$

Then an $\geq \alpha \quad \forall \quad n \in N$

∴ $\{a_n\}$ is bounded below.

(ii) Assume that sequence $\{a_n\}$ diverges to $-\infty$

∴ given k > 0, however, large, there exists a positive integer much that

$a_n > -k \ \forall \quad n \geq m$

∴ there are infinitely many terms of $\{a_n\}$ which are less than -k

Let $\beta$ = max. $(-k, a_1, a_2, \ldots, a_{m-1})$

Then $a_n \geq \beta \quad \forall \quad n \in N$

∴ $\{a_n\}$ is bounded above.

**Note.1:** The converse of (i) is not true.

Consider the sequence $\{a_n\}$ where

$$a_n = \begin{cases} n, \ n \ is \ odd \\ \frac{1}{n}, n \ is \ even \end{cases}$$

This sequence is bounded below as $a_n > 0 \ \forall \ n \in N$ and is not bounded above. but it does not diverge to $+\infty$

**Note 2:** The converse of (ii) is not true.

Consider the sequence $\{a_n\}$ where

$$a_n = \begin{cases} -2n, \ n \ is \ odd \\ -\frac{1}{2n}, n \ is \ even \end{cases}$$

This sequence is bounded below as $a_n < 0 \ \forall \ n \in N$ and is not bounded above, but it does not diverge to $-\infty$.

**Note 3:** The three behaviours of a sequence namely convergence, divergence to $+\infty$ and divergence to $-\infty$ are mutually exclusive i.e., only one of them is true at a time.

**Null Sequence**

A sequence $\{a_n\}$ is said to be null sequence if $a_n \to 0$ as $n \to \infty$.

e.g., the sequence $\left\{\frac{1}{n}\right\}$ is a null sequence as $\frac{1}{n}$ is a null sequence as $\frac{1}{n} \to 0$ as $n \to \infty$.

**Another Def.** A sequence $\{a_n\}$ is said to be null sequence if for every $\varepsilon > 0$, however small, there exists a positive integer m such that

$|a_n| < \varepsilon \quad \forall \quad n \geq m$

**Result:** If $b_n \to 0$ and $|a_n| < |b_n|$ $\forall$ n, then $a_n \to 0$.

**Proof:** Since $b_n \to 0$

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$|b_n - 0\} < \varepsilon \quad \forall\ n \geq m$

or $|b_n| < \varepsilon \quad \forall\ n \geq m$ .... (1)

Now $|a_n| \leq |b_n|$

$\Rightarrow$ $|a_n| < \varepsilon \quad \forall\ n \geq m$ [$\because$ of (1)]

$\Rightarrow$ $a_n \to 0$

**Note.** The above result is not true when $b_n \to l\ (\neq 0)$

**Example:** Take $b_n = 1$ and $a_n = (-1)^n \forall n$. Now $\{b_n\}$ converges but $\{a_n\}$ does not converge.

**Proof:** (i) Assume that $\{a_n\}$ is a null sequence

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$a_n| < \varepsilon \quad \forall\ n \geq m$ ...... (1)

Now $|a_n| - 0| = |a_n| < \varepsilon \quad \forall\ n > m$ ($\because$ of (1)}

$\therefore$ $\{|a_n|\}$ converges to 0.

$\Rightarrow$ $\{|a_n|\}$ is a null sequence.

(ii) Assume that $\{|a_n|\}$ is a null sequence

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$||a_n|-0| < \varepsilon \quad \forall\ n \geq m$

or $||a_n|| < \varepsilon \forall\ n \geq m$

or $|a_n| < \varepsilon \quad \forall\ n \geq m$

$\therefore$ $\{a_n\}$ converges to 0.

$\therefore$ $\{a_n\}$ is a null sequence.

**Another Statement.** for any sequence $\{a_n\}$, show that

$$\underset{n\to\infty}{Lt}\, a_n = 0 \text{ iff } \underset{n\to\infty}{Lt}\, |a_n| = 0$$

**Example:** The sequence $\{a_n\}$ where $a_n = (-1)^{n-1}\frac{1}{n}$ is a null sequence as the sequence $\{|a_n|\}$ = $\left\{\frac{1}{n}\right\}$ is a null sequence.

Prove that a sequence $\{a_n\}$ converges to the limit l iff the sequence $\{a_n - l\}$ is a null sequence.

**Proof:** (i) Assume that sequence $\{a_n\}$ converges to the limit l

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$|a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\therefore$ $|(a_n - l) - 0| < \varepsilon \ \forall\ n \geq m$

$\Rightarrow$ $(a_n - l) \to 0$ as $n \to \infty$

$\Rightarrow$ $(a_n - l)$ is a null sequence

(ii) Assume that

$(a_n - l)$ is a null sequence

$\therefore$ $(a_n - l) \to 0$ as $n \to \infty$

$\therefore$ $|a_n - l) - 0| < \in n \geq m$

$\Rightarrow$ $|a_n - l| < \in \forall\ n \geq m,\ m \in N$

$\Rightarrow$ sequence $\{a_n\}$ converges to the limit l.

If $\{a_n\}$ and $\{b_n\}$ be two sequences such that

$|a_n| \leq |a_n|$, then $\{a_n\}$ is a null sequence of $\{b_n\}$ is a null sequence.

**Proof:** Let us assume that $\{b_n\}$ is a null sequence

$\therefore$ given $\in > 0$, however small, $\exists$ a positive integer m s.t.

$|b_n| < \in \forall\ n \geq m$

Now $|a_n - 0| = |a_n| \leq |b_n| \leq \in \forall\ n \geq m$

$\therefore$ $a_n \to 0$ as $n \to \infty$

$\Rightarrow$ $\{a_n\}$ is a null sequence.

## 5.7 Sandwich Theorem or Squeeze Principle

**Theorem:** If $b_n \leq a_n \leq c_n \forall\ n \in N$ and $b_n \to l$, $c_n \to l$, then $a_n \to l$

**Proof:** Here $b_n \leq a_n \leq c_n$ ..... (1)

$\because$ $b_n \to l$, $c_n \to l$

$\therefore$ given $\varepsilon < 0$, $\exists$ $m_1, m_2 \varepsilon$ N s.t.

$|b_n - l| < \varepsilon \quad \forall\ n \geq m_1$

and $|c_n - l| < \varepsilon \quad \forall\ n \geq m_2$

Let m = max. $(m_1, m_2)$

$\therefore$ $|b_n - l) < \varepsilon \quad \forall\ n \geq m$

i.e., $l - \varepsilon < b_n < l + \varepsilon \quad \forall\ n \geq m$ ...... (2)

and $|c_n - l| < \varepsilon \quad \forall\ n \geq m$

i.e., $l - \varepsilon < c_n < l + \varepsilon \quad \forall\ n \geq m$ ...... (3)

From (1), (2) and (3), we get

$l - \varepsilon < b_n \leq a_n \leq c_n < l + \varepsilon \quad \forall\ n \geq m$

or $l - \varepsilon < a_n < l + \varepsilon \quad \forall\ n \geq m$

or $|a_n - l| < \varepsilon \quad \forall\ n \geq m$

$\therefore$ $a_n \to l$

**Art.** If $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = l$, where $|l| < 1$, then $\underset{n\to\infty}{Lt}\ a_n = 0$.

**Proof:** Since $|l| < 1$, therefore we can choose $\varepsilon > 0$ such that $|l| + \varepsilon < 1$

$\because$ $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = l$

$\therefore$ there exists a positive integer m such that

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \quad \forall\ n \geq m$$

$$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| - |l| \leq \left|\frac{a_{n+1}}{a_n} - l\right| \qquad [\because |a| - |b| \leq |a - b|]$$

$$< \varepsilon \quad \forall\ n \geq m$$

$$\therefore \quad \left|\frac{a_{n+1}}{a_n}\right| < |l| + \varepsilon \quad \forall\ n \geq m$$

$\Rightarrow$ $|a_{n+1}| < |a_n| (|l| + \varepsilon) \quad \forall\ n \geq m$

$\Rightarrow$ $|a_{n+1}| < k\,|a_n| \quad \forall\ n \geq m$

where $k = |l| + \varepsilon < 1$

Putting n = m, m + 1, m + 2, ....., n - 1, we get

$|a_{m+1}| < k\,|a_m|$

$|a_{m+2}| < k\,|a_{m+1}|$

$|a_{m+3}| < k\,|a_{m+2}|$

... ... ...

$|a_n| < k\,|a_{n-1}|$

Multiplying these, we get

$|a_n| < k^{n-m}\,|a_m|$

or $|a_n| < \frac{k^n}{k^m}\,|a_m|$

$\Rightarrow$ $|a_n| \to 0$ as $n \to \infty$ [Since $0 < k < 1$, $\therefore k^n \to 0$ as $n \to \infty$]

$\therefore$ $a_n \to 0$ as $n \to \infty$

$\therefore$ $\underset{n\to\infty}{Lt}\, a_n = 0$

**Art.** If $a_n > 0$ and $\underset{n\to\infty}{Lt}\, \frac{a_{n+1}}{a_n} = l > 1$, then $\underset{n\to\infty}{Lt}\, a_n = +\infty$

**Proof:** Since $l > 1$, therefore we can choose $\varepsilon > 0$ such that $l - \varepsilon > 1$

$\therefore$ $\underset{n\to\infty}{Lt}\, \frac{a_{n+1}}{a_n} = l$

$\therefore$ there exists a positive integer m such that

$\left|\frac{a_{n+1}}{a_n} - l\right| < \varepsilon \;\forall\; n \geq m$

Or $l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon \;\forall\; n \geq m$

$\Rightarrow$ $\frac{a_{n+1}}{a_n} > l - \varepsilon = k$ (say) where $k > 1$, $\forall\; n \geq m$

Putting n = m, m + 1, m + 2, ......, n - 1 we get,

$\frac{a_{m+1}}{a_m} > k$

$\frac{a_{m+2}}{a_{m+1}} > k$

$$\frac{a_{m+3}}{a_{m+2}} > k$$

... ... ...

$$\frac{a_n}{a_{n-1}} > k$$

Multiplying these inequations, we get,

$$\frac{a^n}{a^m} > k^{n-m} \qquad \text{or} \qquad a_n > \frac{k^n}{k^m}\, a_m$$

$\therefore$ $a_n \to +\infty$ as $n \to \infty$ $\qquad$ [$\because$ $k > 1 \Rightarrow k^n \Rightarrow +\infty$ as $n \to \infty$]

$\therefore$ $\underset{n\to\infty}{Lt}\, a_n = +\infty$

**Art.** Prove that

(i) $\underset{n\to\infty}{Lt}\, \sqrt[n]{n} = 1$

(ii) $\underset{n\to\infty}{Lt}\, \sqrt[n]{a} = 1 \quad \forall\, a > 0$

**Proof:** (i) Let $a_n = \sqrt[n]{n}$

$\therefore$ $a_n > 1$ for $n \geq 2$

Put $a_n = 1 + b_n$ where $b_n > 0$ for $n \geq 2$

Now $n = (a_n)^n$

$= (1 + b_n)^n$

$$= 1 + n.b_n + \frac{n(n-1)}{2}\, b_n^2 + \ldots\ldots + b_n^n$$

$$\therefore \quad n > \frac{n(n-1)}{2}\, b_n^2 \text{ for } n \geq 2$$

$$\Rightarrow \quad b_n^2 < \frac{2}{n-1} \text{ for } n \geq 2$$

$$\Rightarrow \quad 0 < b_n \sqrt{\frac{2}{n-1}} \text{ for } n \geq 2$$

$\Rightarrow$ by Sandwich Theorem

$$\underset{n\to\infty}{Lt}\, b_n = 0 \qquad \left[\because \underset{n\to\infty}{Lt} \sqrt{\frac{2}{n-1}} = 0\right]$$

$\therefore \quad \underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt}\,(1 + b_n) = 1 + 0 = 1$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\sqrt[n]{n} = 1$

(ii) Three cases arise :

**Case I.** a > 1, $\therefore \sqrt[n]{a} > 1$

Put $\sqrt[n]{a} = 1 + h, h > 0$

$\therefore \quad a = (1 + h)^n$

$$= 1 + nh + \frac{n(n-1)}{2} h^2 + \ldots\ldots. + h^n$$

$\Rightarrow \quad a \geq 1 + nh > nh$

$\Rightarrow \quad nh < a$

$\Rightarrow \quad 0 < h < \frac{a}{n}$

$\therefore$ by Sandwich Theorem

$\underset{n\to\infty}{Lt}\, h = 0 \qquad \left[\underset{n\to\infty}{Lt}\,\frac{a}{n} = 0\right]$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\, 0 = 1 + 0 = 1$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\sqrt[n]{a} = 1$

**Case II.** a = 1,

$\therefore \quad \sqrt[n]{a} = 1 \ \forall\ n$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\sqrt[n]{a} = 1$

**Case III.** 0 < a < 1,

$\therefore \quad b = \frac{1}{a} > 1$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\sqrt[n]{b} = 1$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\frac{1}{\sqrt[n]{a}} = 1$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\,\sqrt[n]{a} = 1$

Combing the results of three cases, we get,

$$\underset{n\to\infty}{Lt} \sqrt[n]{a} = 1 \quad \forall\ a > 0$$

**Art.** Prove that sequence $\{a^n\}$

(i) diverges to $+\infty$ if $a > 1$

(ii) converges if $-1 < a < 1$

(iii) oscillates finitely if $a = -1$

(iv) oscillates infinitely if $a < -1$

**Proof:** (i) Since $a > 1$, $\therefore$ $a = 1 + h$, $h > 0$

$\therefore$ an = (1 + h)n

$$= 1 + nh + \frac{n(n-1)}{2} h^2 + ......+ h^n > 1 + nh \text{ for } n > 1$$

Let k be any positive number, however large and m be any positive integer such that

$$m > \frac{k-1}{h}$$

or mh< k - 1

or 1 + mn> k

$\therefore$ $\forall\ n \geq m,\ 1 + nh \geq 1 + mh > k$

$\therefore$ $a^n > 1 + nh \geq 1 + mh > k \qquad \forall\ n \geq m$

$\Rightarrow$ $a^n > k \qquad \forall\ n \geq m$

$\therefore$ $\underset{n\to\infty}{Lt}\ a^n = +\infty$

$\therefore$ sequence $\{a^n\}$ diverges to $+\infty$

(ii) Four cases arise:

**Case I.** If a = 1, then an = 1 $\forall$ n

$\therefore$ sequence {an} converges to 1.

**Case II.** If $0 < a < 1$, then put $b = \frac{1}{a}$ so that

$b > 1 \quad \Rightarrow \quad b^n \to +\infty$

Let $\varepsilon > 0$, however small. Then $\frac{1}{\in}$ is a positive real number sufficiently large.

Since $b^n \to +\infty$

∴ there exists m ∈ N such that

$b^n > \frac{1}{\in}$ ∀ n ≥ m

⇒ $\frac{1}{a^n} > \frac{1}{\in}$ ∀ n ≥ m

⇒ $a^n < \varepsilon$ ∀ n ≥ m

⇒ $|a^n| < \varepsilon$ ∀ n ≥ m [∵ 0 < a < 1]

⇒ $a^n \to 0$ as $n \to \infty$

∴ sequence $\{a^n\}$ converges to 0.

**Case III.** If a = 0, then an = 0 ∀ n

⇒ $a^n \to 0$ as $n \to \infty$

⇒ sequence $\{a^n\}$ converges to 0.

**Case IV.** If - 1 < a < 0, then put a = - $\frac{1}{b}$ so that b > 1

⇒ $b^n \to +\infty$

⇒ $\frac{1}{b^n} \to 0$

⇒ $(-a)^n \to 0$

⇒ $(-1)^n a^n \to 0$

⇒ $a^n \to 0$

∴ sequence $\{a^n\}$ converges to 0.

(iii) Let a = -1

∴ an = $(-1)^n = \begin{cases} -1, & n \text{ is odd} \\ 1, & n \text{ is even} \end{cases}$

Now sequence $\{a^n\}$ does not converge

Also $|a^n| = 1$ ∀ n

∴ sequence $\{a^n\}$ is bounded

⇒ sequence {an} osculates binately

(iv) If a < -1 then put a = -b so that

b > 1

⇒ $b^n \to +\infty$

∴ $a^n = (-1)^n b^n$ and $b^n \to +\infty$

∴ $a^n \to +\infty$ if n is even

and $a^n \to -\infty$ if n is odd

∴ sequence $\{a^n\}$ oscillates finitely.

**Some More Illustrated Examples**

**Example 1:** Prove that the sequence

$$\left\{\frac{1}{n^2}+\frac{1}{(n+1)^2}+......+\frac{1}{(2n)^2}\right\}$$

is bounded

**Solution:** Let $\{a_n\}$ be given sequence, where

$$a_n = \frac{1}{n^2}+\frac{1}{(n+1)^2}+......+\frac{1}{(2n)^2}$$

$$\left\langle \frac{1}{n^2}+\frac{1}{n^2}+\frac{1}{n^2} + \text{........... to (n+1) terms}\right.$$

$$\left\{\begin{array}{l} \because (m+p)^2 > n^2 \;\; for \;\; p=1,2,.....n \\ \because \dfrac{1}{(n+p)^2} < \dfrac{1}{n^2} \;\; for \;\; p=1,2,.....n \end{array}\right\}$$

$$= \frac{n+1}{n^2}$$

$$= \frac{1}{n} + \frac{1}{n^2}$$

$$\left\langle \frac{1}{n}+\frac{1}{n}=\frac{2}{n} < 2 \right. \qquad \forall\ n \in N$$

$$(\because\ n \geq 1 \ \therefore\ n^2 \geq n \Rightarrow \frac{1}{n^2} \leq \frac{1}{n})$$

∴ $a_n < 2 \quad \forall\ n \in N$

Also $a_n > 0 \quad \forall\ n \in N$

∴ $0 < a_n < 2 \qquad \forall\ n \in N$

⇒ $\{a_n\}$ is a bounded sequence.

**Example 2:** Prove that the sequence

$\{2-n^2\}$ is unbounded

**Solution:** Let $\{a_n\} = \{2-n^2\}$ where

$a_n = 2 - n^2$

Let M > 0, however large

For $n \geq 2$ $|a_n| = |2 - n^2| = n^2 - 2 > M$

If $n^2 - 2 > M$

if $n^2 > 2 + M$

i.e. if $n > \sqrt{2+M}$

$\therefore$ $|a_n| > M$ $\forall$ n

$> m = \max(2, \sqrt{2+M}), m \in N$

$\Rightarrow$ $\{a_n\}$ is unbounded}

**Example 3:** Show that the sequence

$\left\{\frac{(-1)^n}{n}\right\}$ is convergent sequence

**Solution:** Let $\{a_n\} = \left\{\frac{(-1)^n}{n}\right\}$, $a_n = \frac{(-1)^n}{n}$

Now $|a_n - 0| = \left\{\frac{(-1)^n}{n} - 0\right\} = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$

$|a_n - 0| < \in$

if $\frac{1}{n} < \in$ i.e. $n > \frac{1}{\in}$

Let $m \in N$ just greater than $\frac{1}{\in}$

$\therefore$ $|a_n - 0| < \in$ $\forall$ n > m

$\Rightarrow$ $\{a_n\}$ converges to 0

$\Rightarrow$ $\left\{\frac{(-1)^n}{n}\right\}$ is a convergent sequence.

**Example 4:** Show that the sequence

$\{1 + (-1)^n\}$ oscillates finitely

**Solution:** Let $\{a_n\} = \{1 + (-1)^n\}$ be given sequence,

where $a_n = 1 + (-1)^n$

$\therefore$ $|a_n| = |1 + (-1)^n| \leq |1| + )(-1)^n|$

$= 1 + 1 = 2 \ \forall \ n \in N.$

$\Rightarrow$ $a_n$ is a bounded sequence

Let if possible, $\{a_n\}$ converges to l, therefore given $\in > 0$, however small $\exists$ a +ve integer m s.t.

$|a_{n-l}| < \in \qquad \forall \ n \geq m$

$\Rightarrow$ $|a_{m-1}| < \in$ and $|a_{m+1} - 1| < \in$

Now $|a_{m+1} - a_m| = |(a_{m+1} - l) - (a_{m-l})|$

$< \quad |a_{m+1} - l| + |a_m - l|$

$< \quad \in + \in$

$\therefore$ $|1 + (-1)^{m+1} - 1 - (-1)^m| < 2 \in$

or $2 < 2\in$ or $1 < \in$ which is not true if $0 < \in < 1$.

$\therefore$ our supposition is wrong.

$\therefore$ $\{a_n\}$ is not convergent but bounded

$\therefore$ $\{a_n\}$ oscillates finitely.

**Example 5:** Show that $\left\{\dfrac{n!}{n^n}\right\}$ is a null sequence.

**Solution:** Here $a_n = \dfrac{n!}{n^n} = \dfrac{n(n-1)(n-2)......3.2.1}{n.n.n.........n.n}$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} ..... \frac{2}{n} \cdot \frac{1}{n}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)........\left(1-\frac{n-2}{n}\right) \cdot \frac{1}{n}$$

$$\leq \quad 1.1.1.........\frac{1}{n} = \frac{1}{n}$$

$\therefore$ $|a_n| \leq \dfrac{1}{n}$ and $\dfrac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow$ $a_n \to 0$ as $n \to \infty$

$\Rightarrow$ $|a_n|$ is a null sequence

or $\left\{\frac{n!}{n^n}\right\}$ is a null sequence

**Example 6:** Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

**Solution:** Let $a_n = \sqrt[n]{n}$

$\therefore$ $a_n > 1$ for $n \geq 2$

Put $a_n = 1 + b_n$, $b_n > 0$ for $n \geq 2$

Now $n = (a_n)^n = (1 + b_n)^n = 1 + n \cdot b_n + \frac{n(n-1)}{2!} b_n^2 + ......+ ....+b_n^n$

$\Rightarrow$ $n > \frac{n(n-1)}{2!} b_n^2$ for $n \geq 2$

$\Rightarrow$ $b_n^2 < \frac{2}{n-1}$ for $n \geq 2$

$\Rightarrow$ $0 < b_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$

$\therefore$ by Sandwich theorem

Lim bn = 0

$n \to \infty$

$\therefore$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1 + b_n) = 1 + 0 = 1$

$\Rightarrow$ $\lim_{n\to\infty} \sqrt[n]{n} = 1$

## 5.8 Self Check Exercise

Q.1 Prove that the sequence

$\left\{\frac{2n-2}{3n+7}\right\}$ is bounded

Q.2 Prove that the sequence

$\{-3^n\}$ is unbounded

Q.3 Show that the sequence

$\left\{\frac{n^2+2n+5}{2n^2+5n+7}\right\}$ converges to $\frac{1}{2}$

Q.4 Prove that the sequence

$a_n$ = $\frac{1}{\sqrt{n^2+1}}$ + $\frac{1}{\sqrt{n^2+2}}$ + ........+ $\frac{1}{\sqrt{n^n+n}}$

## 5.9 Summary

In this unit we have learnt the following

(i) sequence

(ii) real sequence

(iii) bounded and unbounded sequences

(iv) convergent, divergent and oscillatory sequence.

## 5.10 Glossary:

1. Equal Sequences - Two sequences {an} and {bn} are said to be equal if $a_n = b_n$ for every n.

2. Divergent sequence - A sequence $\{a_n\}$ is said to diverge to ∞if given k > 0, however large, ∃ a +ve integer m (∈) s.t.

   $a_n > k \ \forall \ n \geq m$

   and we write it as $\lim_{n\to\infty}$ an = ∞ or $a_n \to \infty$.

## 5.11 Answer to Self Check Exercise

Ans.1 Hint an = $\frac{2}{3}$ - $\frac{23}{3(3n-T)}$

Ans.2 Hint $|a_n|$ $|-3^n| = 3^n = M$

Ans.3 Hint $\left|a_n - \frac{1}{2}\right|$ = $\frac{1}{2}\left|\frac{n+3}{2n^2+5n+7}\right|$

Ans.4 Hint Let $x_r$ = $\frac{1}{\sqrt{n^2+r}}$ , r = 1, 2,.......n.

## 5.12 References/Suggested Readings


1. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

2. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

3. R.G. Bartle and D.R. Sherbert, Introductions to Real Analysis, John Wiley and Sons (Asia) P. Ltd, 2000.

## 5.13 Terminal Questions

1. Prove that the sequence

   $\left\{ \frac{n^2+1}{2^n+3} \right\}$ is bounded.

2. Find a positive integer m such that

   $\sqrt[1/]{n+1} < 0.03 \ \forall \ n \geq m$

3. By definition, show that

   $\underset{n\to\infty}{Lim}. \ \frac{3n}{n+\sqrt[5]{n}} = 3$

4. Prove that the sequence $\{n^2 + 3n\}$ diverges to $+\infty$

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## Unit - 6

# Monotonic Sequences

**Structure**



### 6.1 Introduction

Dear students,in this unit we shall learn the concept of monotonic sequences. We are already familiar with the concept of monotonic function. In our previous classes we have studied that a function $f$ defined on a subset of real numbers with real values is called monotonic iff it is either entirely non-decreasing or entirely non-increasing. In mathematics, a monotonic function is a function between ordered sets that preserve or reverse the given order. The concept first arose in calculus, and was later generalized to the more abstract setting of order theory.

A monotonic sequence is a sequence in which its elements follow a consistent trend-either increasing or decreasing. A sequence is monotonic if every term is greater than or equal to one before (monotonically increasing) or every term is less than or equal to one before (monotonically decreasing).

### 6.2 Learning Objectives

The main objectives of this unit are

(i) to study the concept of monotonic sequences

(ii) to learn the behaviour of monotonic sequences

### 6.3 Art-Monotonic Sequences

(i) A sequence $\{a_n\}$ is said to be monotonically increasing

if $a_n \leq a_{n+1} \quad \forall \; n \in N$

(ii) A sequence $\{a_n\}$ is said to be monotonically decreasing

if $a_n \geq a_{n+1} \quad \forall \; n \in N$

(iii) A sequence which is monotonically increasing or decreasing is called a monotonic (or a monotonic) sequence.

(iv) A sequence $\{a_n\}$ is said to be strictly monotonically increasing

if $a_n < a_{n+1} \quad \forall \; n \in N$

(v) A sequence $\{a_n\}$ is said to be strictly monotonically decreasing

if $a_n > a_{n+1} \quad \forall \; n \in N$

**Examples.**

(i) Suppose is $\left\{\frac{1}{n}\right\}$ strictly monotonically decreasing.

(ii) Suppose is $\{n^4\}$ strictly monotonically increasing.

(iii) Suppose is $\{(-1)^n\}$ is neither monotonically increasing nor monotonically decreasing.

**6.4 Behaviour of Monotonic Sequences**

(i) Prove that a monotonically increasing sequence $\{a_n\}$ converges iff it is bounded above. The limit of $\{a_n\}$, when it converges, is l.u.b. of $\{a_n\}$

(ii) Prove that a monotonically decreasing sequence $\{a_n\}$ converges iff it is bounded below. The limit of $\{a_n\}$, when it converges, is g.l.b. of $\{a_n\}$

(iii) Prove that a monotonically increasing sequence $\{a_n\}$ diverges to $+\infty$iff it is unbounded above.

(iv) Prove that a monotonically decreasing sequence $\{a_n\}$ diverges to $-\infty$iff it is unbounded below.

**Proof :** (i) Let the monotonic sequence $\{a_n\}$ converge to l

$\therefore$ given $\varepsilon > 0$, however small, there exists a natural number m such that

$|a_n - l| < \varepsilon \quad \forall \; n \geq M$

$\therefore$ $1 - \varepsilon < a_n < 1 + \varepsilon \quad \forall \; n \geq M$

Let $k = \max (l + \varepsilon, a_1, a_2 .......a_{m-1})$

$\therefore$ $a_n < k \quad \forall \quad n \geq N$

$\therefore$ sequence $\{a_n\}$ is bounded above

**Converse :** Set us assume that the monotonic sequence $\{a_n\}$ is bounded above and u is the l.u.b. of $\{a_n\}$

$\therefore$ given $\varepsilon > 0, \quad \exists \quad m \in N$ s.t.

$u - \varepsilon < a_n$ ...(1)

$\Rightarrow$ $a_n < a_n$ for every $n \geq m$

($\because$ the sequence $\{a_n\}$ is monotonically increasing)

....(2)

From (1) and (2), we have

$u - \varepsilon < a_n \quad \forall \quad n \geq m$

Also $a_n < u + \varepsilon \quad \forall \quad n \geq m$

$\therefore$ $a_n < u + \varepsilon$ , in particular $\forall \; n \geq m$

$\therefore$ $u - \varepsilon < a_n < u + \varepsilon \, \forall \; n \geq m$

$\Rightarrow$ $|a_n - u| < \varepsilon \quad \forall \quad n \geq m$

$\therefore$ the sequence $\{a_n\}$ converges, its limit being l.u.b. of $\{a_n\}$

(ii) Let monotonic sequence {an} converges to l

$\therefore$ given $\varepsilon < 0$, however small, $\exists$ a m $\varepsilon$ N s.t.

$|a_n - l| < \varepsilon \quad \forall \; n \geq m$

$\therefore$ $l - \varepsilon < a_n < l + \varepsilon \forall \; n \geq m$

Let k = min ($l - \varepsilon$, $a_1$, $a_2$.......$a_{m-1}$)

$\therefore$ $h \leq a_n \quad \forall \quad n \geq N$

$\therefore$ sequence $\{a_n\}$ is bounded below

**Converse :** Let us assume that $\{a_n\}$ is bounded below and l is the g.l.b. of $\{a_n\}$

$\therefore$ given $\varepsilon > 0$, $\exists$ $m \in N$ s.t.

$a_m \leq l + \varepsilon$ ...(1)

Since {an} is monotonically decreasing

$\therefore$ $a_n \leq a_m$ for every $n \geq m$ ....(2)

Using (1) and (2), we have

$a_n < l + \varepsilon \quad \forall \quad n \geq m$

Also $a_n < l - \varepsilon \quad \forall \quad n \in m$

$\therefore$ in particular $a_n < l - \varepsilon \forall \; n \geq m$

$\therefore$ $l - \varepsilon < a_n < l + \varepsilon \quad \forall \; n \geq m$

$\therefore$ $|a_n - l| < \varepsilon \quad \forall \quad n \geq m$

$\Rightarrow$ the sequence $\{a_n\}$ converges and its limit being g.l.b. of $\{a_n\}$

(iii) Let us assume that monotonic sequence {an} converges to $+\infty$

$\therefore$ given $k < 0$, however small, $\exists$ a positive integer m s.t.

$a_n > k \quad \forall\ n \geq m$

$\Rightarrow$ There are infinitely many terms of {an} which are greater than k.

$\therefore$ the sequence $\{a_n\}$ is not bounded above.

**Converse :** Let us assume that monotonic sequence $\{a_n\}$ is unbounded above.

$\therefore$ given $k > 0$, however large, $\exists$ a term $a_m$ s.t.

$a_m < k$ ...(1)

Now, since {an} is monotonic decreasing

$\therefore$ $a_n \leq a_m$ for every $n \geq m$ ....(2)

From (1) and (2), we have

$a_n > k \quad \forall \quad n \geq m$

$\therefore$ the sequence $\{a_n\}$ diverges to $+\infty$.

(iv) Assume that the sequence {an} diverges to $-\infty$.

$\therefore$ given $k > 0$, however small, $\exists$ a positive integer m s.t.

$a_n < -k \quad \forall\ n \geq m$

$\therefore$ There are infinitely many terms of $\{a_n\}$ which are less than -k.

$\therefore$ $\{a_n\}$ is not bounded below.

**Converse :** Let us assume that monotonic sequence $\{a_n\}$ is unbounded below.

$\therefore$ given $k > 0$, however large, $\exists$ a term $a_m$ s.t.

$a_m < -k$ ...(1)

Since $\{a_n\}$ is monotonic decreasing

$\therefore$ $a_n \leq a_m$ ....(2)

From (1) and (2), we have

$a_n < -k \quad \forall \quad n \geq m$

$\therefore$ the sequence $\{a_n\}$ diverges to $-\infty$.

**Note :** 1. A monotonic increasing sequence is either convergent or diverges to $+\infty$.

2. A monotonic decreasing sequence is either convergent or diverges to $-\infty$.

3. A necessary and sufficient condition for convergence of a monotonic sequence is that it is bounded.

4. A necessary and sufficient condition for the divergence of the monotonic sequence is that it is unbounded.

5. A monotonic sequence can never be oscillatory.

**Some Illustrated Examples**

**Example 1.** Prove that the sequence

$$\left\{\frac{2n-7}{3n+2}\right\} \text{ is}$$

(i) monotonically increasing

(ii) bounded

(iii) convergent

**Solution :** (i) Let $\{a_n\}$ be the given sequence

$\therefore \quad a_n = \frac{2n-7}{3n+2}$

$\therefore \quad a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2} = \frac{2n-5}{3n+5}$

Now $\quad a_n + 1 - an = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$

$$= \frac{(2n-5)(3n+2)-(2n-7)(3n+5)}{(3n+5)(3n+2)}$$

$$= \frac{6n^2+4n-15n-10-6n^2-10n+21n+35}{(3n+5)(3n+2)}$$

$$= \frac{25}{(3n+5)(3n+2)} > 0 \; \forall \; n \in N$$

$\therefore \quad a_n + 1 > a_n \qquad \forall \; n$

$\Rightarrow \quad \{a_n\}$ is monotonically increasing.

(ii) $\quad a_n = \frac{2n-7}{3n+2} > 0 \qquad \forall \; n \geq 4$

Also $\quad a_1 = \frac{-5}{5} = -1$

$\quad a_2 = \frac{-3}{8}$

$a_3 = \frac{-1}{11}$

$\therefore$ $a_n \geq -1 \quad \forall\, n$

$\therefore$ $\{a_n\}$ is bounded below

Again $a_n = \frac{2n-7}{3n+2}$

$$= \frac{2}{3} - \frac{25/3}{3n+2} \qquad \left\{ \begin{array}{c} 2/3 \\ {}^{3n+2}\sqrt[]{2n-7} \\ \dfrac{2n+4/3}{-25/3} \end{array} \right.$$

$$= \left\langle \; \frac{2}{3} \qquad \forall\, n \right.$$

$\therefore$ $\{a_n\}$ is bounded above

$\Rightarrow$ $\{a_n\}$ is bounded

(iii) Since $\{a_n\}$ is monotonically increasing and bounded above

$\therefore$ $\{a_n\}$ is convergent

**Example 2:** Prove that the sequence $\left\{\frac{n}{n^2+1}\right\}$ is convergent

**Solution:** Let $\{a_n\} = \left\{\frac{n}{n^2+1}\right\}$ be given sequence

$\therefore$ $a_n = \frac{n}{n^2+1}$

$\therefore$ $a_{n+1} = \frac{n+1}{(n+1)^2+1} = \frac{n+1}{n^2+2n+2}$

Now $a_{n+1} - a_n = \frac{n+1}{n^2+2n+2} - \frac{n}{n^2+1}$

$$= \frac{n^3+n+n^2+1-n^3-2n^2-2n}{(n^2+1)(n^2+2n+2)}$$

$= -\dfrac{n^2+n-1}{(n^2+2n+2)(n^2+1)} < 0 \ \forall\ n \in N$

$\therefore$ $a_{n+1} < a_n \quad \forall\ n \in N$

$\Rightarrow$ $\{a_n\}$ is a monotonically decreasing sequence

Also $a_n = \dfrac{n}{n^2+1} > 0 \ \forall\ n \in N$

$\therefore$ $\{a_n\}$ is bounded below

$\Rightarrow$ $\{a_n\}$ is convergent.

**Example 3:** Discuss the convergence of the sequence $\{n - n^2\}$

**Solution:** Let $\{a_n\} = \{n - n^2\}$ be given sequence.

$\therefore$ $a_n = n - n^2$

$a_{n+1} = (n+1) - (n+1)^2$

$= n + 1 - n^2 - 2n - 1$

$= -n^2 - n$

Now $a_{n+1} - a_n = (-n^2 - n) - (n - n^2)$

$= -2n < 0 \quad \forall \quad n \in N$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall \quad n \in N$

$\Rightarrow$ $\{a_n\}$ is monotonically decreasing sequence but $\{a_n\}$ is not bounded below

$(\because\ a_n = n-n^2 = n(1-n) < 0\ \forall n \in N)$

$\Rightarrow$ $\{a_n\}$ diverges to $-\infty$

**Example 4:** Examine the convergence of the sequence $\{a_n\}$, $a_n = \propto^n$, $\propto > 1$

**Solution:** Here $a_n = \propto^n$, $\propto > 1$

$\therefore$ $a_{n+1} = \propto^{n+1}$

$\Rightarrow$ $a_{n+1} - a_n = \propto^{n+1} - \propto^n > 0\ [\because \propto^{n+1} > \propto^n$ as $\propto > 1]$

$\Rightarrow$ $a_{n+1} > a_n \quad \forall \quad n \in N$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing

But $\{a_n\}$ is not bounded above, since

$an = \propto^n$, $\propto > 1$

$> 0 \quad \forall n \in N$.

$\therefore$ $\{a_n\}$ diverges to $+\infty$

**Example 5:** Prove that the sequence $\{a_n\}$ defined by

$x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2+x_n}$ is bounded, monotonic and converges to 2.

**Solution:** Step I. We shall use the method of induction to prove that $\{x_n\}$ is monotonically increasing sequence i.e. we shall show that

$x_{n+1} > x_n \ \forall \ n \in N$

Now $x_2 = \sqrt{2+x_1} = \sqrt{2+\sqrt{2}} > \sqrt{2+0} = \sqrt{2} = x_1$

$\Rightarrow$ $x_2 > x_1$

$\therefore$ result is true for n = 1

Let us assume that the result is true for n = k i.e.

$x_{n+1} > x_k$

$\therefore$ $2 + x_{k+1} > 2 + x_k$

$\Rightarrow$ $\sqrt{2+x_{k+1}} > \sqrt{2+x_k}$

$\Rightarrow$ $x_{k+2} > x_{k+1}$

$\therefore$ result is true for n = k + 1.

Also it is true for n = 1

Hence by induction $\{x_n\}$ is monotonically increasing

**Step II.** **Claim:** $\{x_n\}$ is bounded above by 2 i.e. $x_n < 2 \ \forall \ n \in N$

Now $x_1 = \sqrt{2} < 2$

$\Rightarrow$ result is true for n = 1

Assume it to be true for n = m

$\Rightarrow$ $x_m < 2$

$\therefore$ $2 + x_m < 2 + 2 = 4$

$\Rightarrow$ $\sqrt{2+x_m} < \sqrt{4} = 2$

$\Rightarrow$ $x_{m+1} < 2$

$\therefore$ result is true for n = m + 1

Also it is true for n = 1

$\therefore$ by mathematical induction $\{x_n\}$ is bounded above by 2.

From Step I and Step II it is clear that $\{x_n\}$ is monotonically increasing and bounded above.

$\therefore$ $\{x_n\}$ is convergent

Let $\lim_{n\to\infty} x_n = x$

$\therefore \quad \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2+x_n}$

$\therefore \quad x = \sqrt{2+x} \qquad (\because \lim_{n\to\infty} x_{n+1} = x)$

$\Rightarrow \quad x^2 = 2 + x$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\therefore \quad x = \dfrac{1 \pm \sqrt{1+8}}{2} = \dfrac{1 \pm 3}{2} = 2, -1$

But $x_n > 0 \quad \forall\, n \in N \Rightarrow \quad x \nless 0$

$\therefore \quad x = 2$

Hence $\{x_n\}$ converges to 2

## 6.5 Self Check Exercise

Q.1 Examine whether the sequence $\dfrac{4n+3}{3n+2}$ is monotonically increasing or decreasing.

Q.2 Prove that the sequence $\dfrac{3n-1}{4n+5}$ is monotonically increasing, bounded and convergent.

Q.3 Show that the sequence

$1 + \dfrac{1}{3} + \dfrac{1}{3^2} + \dots\dots + \dfrac{1}{3^n}$ is convergent.

Q.4 Prove that the sequence $\{a_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$ is convergent

## 6.6 Summary

Dear students, we have learnt the following concepts in this unit.

(i) Monotonic sequences (monotonically increasing or monotonically decreasing sequences)

(ii) The behaviour of monotonic sequences

(iii) For better understanding of these concepts sufficient examples have been given.

## 6.7 Glossary:

1. Monotonic Sequence - A sequence which is monotonically increasing or decreasing is called monotonic (or a monotonic) sequence

2. Strictly Monotonically increasing sequence - A seq. $\{a_n\}$ is said to be strictly monotonically increasing if $a_{n+1} > a_n$ or $a_n < a_{n+1} \forall n \in N$

3. Strictly Monotonically decreasing sequence - A seq $\{a_n\}$ is said to be strictly monotonically decreasing of $a_n > a_{n+1} \forall n \in N$.

## 6.8 Answer to Self Check Exercise

Ans.1 Monotonically decreasing

Ans.2 Prove it

Ans.3 $a_n < 3/2 \forall n \in N \therefore a_n$ is convergent

Ans.4 Prove it.

## 6.9 References/Suggested Readings


1. K.A. Ross, Elementary thalysis - The Theory of calculus series - Undergraduate Texts in Mathematics, Springer Verlag, 2003.

2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

3. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.


## 6.10 Terminal Questions

1. show that the sequence {an},

   an = $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...... + \frac{1}{n-1!}$ is convergent.

2. Prove that the sequence $\left\{\left(1+\frac{3}{n}\right)^n\right\}$ is monotonically increasing and bounded.

   Show that it converges to limit $e^3$.

3. Prove that the sequence $\{a_n$ 3,$\}$

   $a_n = 1 + \frac{1}{2} + \frac{1}{3} + ...... + \frac{1}{n}$ diverges.

   Deduce that {an} is unbounded

4. Prove that the sequence

   $\sqrt{2} + \sqrt{2+\sqrt{2}} + \sqrt{2+\sqrt{2}+\sqrt{2}}$ ........ converges to 2.

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## Unit - 7

# Cauchy's Theorem On Limits

**Structure**



### 7.1 Introduction

Dear students, in this unit we shall study Cauchy's limit theorem, named after the french mathematician Augustin-Louis Cauchy. It describes a property of converging sequences. It states that for a converging sequence the sequence of arithmetic means of first n numbers or members converges against the same limit as the original sequence. This theorem was founded by Canchy in 1821. Subsequently a number of related and generalized results were published, in part cularby otto Stolz (1885) and Eonesto Cesaro (1888).

### 7.2 Learning Objectives

The main objectives of this unit are to

(i) learn about Cauchy's first theorem on limits

(ii) study Cauchy's second theorem on limits

(iii) learn Cauchy's Stolze Theorem

(iv) to prove Cesar's theorem etc.

### 7.3 Theorem: Cauchy's First Theorem on Limits

If $a_n \to l$, then $x_n = \dfrac{a_1 + a + a_3 + ...... + a_n}{n} \to l$

**Proof:** Let $t_n = a_n - l$

i.e. $a_n = t_n + l$

Now $x_n = \dfrac{a_1 + a + a_3 + ...... + a_n}{n}$

$$= \frac{(t_1 + l) + (t_2 + l) + (t_3 + l) + ..... + (t_n + l)}{n}$$

$$= \frac{nl + (t_1 + t_2 + t_3 + ....... + t_n)}{n}$$

$\therefore$ $x_n = l + \dfrac{(t_1 + t_2 + t_3 + ....... + t_n)}{n}$ .... (1)

Now $a_n \to 1$

$\Rightarrow$ $t_n \to 0$

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$|t_n| < \dfrac{\varepsilon}{2}\ \forall\ n \geq m$ ..... (2)

Again as $t_n \to 0$

$\therefore$ $\{t_n\}$ is convergent.

$\Rightarrow$ $\{t_n\}$ is bounded

$\therefore$ there exists a real number k such that

$|t| < k\ \forall\ n$ ..... (3)

Now $\left|\dfrac{t_1 + t_2 + t_3 + ....... + t_n}{n}\right| = \left|\dfrac{t_1 + t_2 + t_3 + t_m + t_{m+1} + ....... + t_n}{n}\right|$

$$= \frac{|t_1 + t_2 + t_m|}{n} + \frac{|t_{m+1} + ....... + t_n|}{n}$$

$$\leq \frac{|t_1 + t_2 + t_m|}{n} + \frac{|t_{m+1} + ....... + t_n|}{n}$$

$$\leq \frac{|t_1| + |t_2| + ....... + |t_m|}{n} + \frac{|t_{m+1}| + ....... + |t_n|}{n}$$

$< \dfrac{mk}{n} + \dfrac{n-m}{n} \cdot \dfrac{\varepsilon}{2}$ [$\because$ of (2) and (3)]

$< \dfrac{mk}{2} + \dfrac{\varepsilon}{2}$ $\qquad\qquad [\because \dfrac{n-m}{n} < 1]$

$\therefore \qquad \left|\dfrac{t_1+t_2+t_3+.......+t_n}{n}\right| < \dfrac{mk}{n} + \dfrac{\varepsilon}{2}$ ..... (4)

Now $\dfrac{mk}{n} \to 0$ as $n \to \infty$

$\therefore$ there exists a positive integer p such that $\dfrac{mk}{n} < \dfrac{\varepsilon}{2}$ for $n \geq p$ ..... (5)

Let $q = \max.(m, p)$

$\therefore \qquad \left|\dfrac{t_1+t_2+t_3+.......+t_n}{n} - 0\right| < \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$ for $n \geq q$ $\qquad [\because$ of (4), (5)]

$\therefore \qquad \left|\dfrac{t_1+t_2+t_3+.......+t_n}{n} - 0\right| < \varepsilon$ for $n \geq q$

$\therefore$ from (1), $\underset{n\to\infty}{Lt}\, x_n = l$

$$\Rightarrow \qquad \underset{n\to\infty}{Lt} \frac{a_1+a_2+a_3+......+a_n}{n} = l$$

**Note 1.** Sequence $\{x_n\}$ is called the sequence of means of the sequence $\{a_n\}$

**Note 2.** The converse of the above result is not true.

**Example:** Let an = $(-1)^n$. Here an does not tend to a limit as $\{a_n\}$ oscillates finitely

$$\therefore \qquad x_n = \frac{a_1+a_2+......+a_n}{n} = \begin{cases} 0, & n \text{ is even} \\ -\dfrac{1}{n}, & n \text{ is odd} \end{cases}$$

$$\therefore \qquad \underset{n\to\infty}{Lt}\, x_n = 0$$

### 7.4 Theorem: Cauchy's Second Theorem on Limits

If $\{a_n\}$ is a sequence of positive terms and $\underset{n\to\infty}{Lt} \dfrac{a_{n+1}}{a_n}$ exists whether finite or infinite, then

$$\underset{n\to\infty}{Lt} (a_n)^{\frac{1}{n}} = \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n}$$

**Proof:** Two cases arise:

Case I. $\underset{n\to\infty}{Lt}\ \frac{a_{n+1}}{a_n}$ = l, where l is finite

$\therefore$ given $\varepsilon$> 0, however small, $\exists$ m $\in$ N s.t.

$$\left|\frac{a_{n+1}}{a_n} - l\right| < \frac{\varepsilon}{2} \qquad \forall \qquad n \geq m$$

$$\Rightarrow \qquad l = \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < l + \frac{\varepsilon}{2} \qquad \forall \qquad n \geq m$$

Putting n = m, m + 1, m + 2,......, n-1, we get

$$l - \frac{\varepsilon}{2} < \frac{a_{m+1}}{a_m} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{a_{m+2}}{a_{m+1}} < l + \frac{\varepsilon}{2}$$

$$l - \frac{\varepsilon}{2} < \frac{a_{m+3}}{a_{m+2}} < l + \frac{\varepsilon}{2}$$

... ... ... ...

$$l - \frac{\varepsilon}{2} < \frac{a_n}{a_{n-1}} < l + \frac{\varepsilon}{2}$$

Multiplying these (n - 1) - (m - 1) i.e. (n - m) inequations, we get

$$\left(l - \frac{\varepsilon}{2}\right)^{n-m} < \frac{a_n}{a_m} < \left(l + \frac{\varepsilon}{2}\right)^{n-m}$$

$$\Rightarrow \qquad am\left(l - \frac{\varepsilon}{2}\right)^{n-m} < an < am\left(l + \frac{\varepsilon}{2}\right)^{n-m}$$

$$\Rightarrow \qquad (a_m)^{\frac{1}{n}}\left(l - \frac{\varepsilon}{2}\right)^{\frac{n-m}{n}} < (a_n)^{\frac{1}{n}} < (a_m)^{\frac{1}{n}}\left(l + \frac{\varepsilon}{2}\right)^{\frac{n-m}{n}}$$

$$\Rightarrow \qquad (a_m)^{\frac{1}{n}}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < (a_n)^{\frac{1}{n}} < (a_m)^{\frac{1}{n}}\left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \qquad ..... (1)$$

Let $\quad n \to \infty$

Now $\quad (a_m)^{\frac{1}{n}} \left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \to l - \frac{\varepsilon}{2}$

[$\because (a_m)^{\frac{1}{n}} \to 1$ as $n \to \infty$ for a > 0 and $1 - \frac{m}{n} \to 1$, $1+\frac{m}{n} \to 1$ as $n \to \infty$]

and $\quad (a_m)^{\frac{1}{n}} \left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} \to l + \frac{\varepsilon}{2}$

$\therefore$ given $\varepsilon > 0$, $\exists\, m_1, m_2 \in N$ s.t.

$$\left|(a_m)^{\frac{1}{n}} \left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} - \left(l - \frac{\varepsilon}{2}\right)\right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m_1$$

and
$$\left|(a_m)^{\frac{1}{n}} \left(l + \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} - \left(l + \frac{\varepsilon}{2}\right)\right| < \frac{\varepsilon}{2} \qquad \forall\, n \geq m_2$$

$\therefore$
$$\left(l - \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} < (a_m)^{\frac{1}{n}} \left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < \left(l - \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$$

i.e. $\quad l - \varepsilon < (a_m)^{\frac{1}{n}} \left(l - \frac{\varepsilon}{2}\right)^{1-\frac{m}{n}} < l \;\; \forall\, n \geq m_1 \qquad \text{..... (2)}$

Let $\quad$ p = max (m, m1, m2)

$\therefore$ from (1), (2), (3), we get

$$l - \varepsilon < (a_n)^{\frac{1}{n}} < l + \varepsilon \qquad \forall\, n \geq p$$

Or $\quad \left|(a_n)^{\frac{1}{n}} - l\right| < \varepsilon \;\forall\, n \geq p$

$\therefore \quad \underset{n\to\infty}{Lt}\, (a_n)^{\frac{1}{n}} = l$

Or $\quad \underset{n\to\infty}{Lt}\, (a_n)^{\frac{1}{n}} = \underset{n\to\infty}{Lt}\, \frac{a_{n+1}}{a_n}$

**Another Form:** If $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to l$, then prove that $\sqrt[n]{a_n} \to l$.

Case II. $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = +\infty$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = 0$$

$$\Rightarrow \quad \underset{n\to\infty}{Lt} \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} = 0 \text{ (finite)}$$

$$\therefore \quad \underset{n\to\infty}{Lt} \left(\frac{1}{a_n}\right)^{\frac{1}{n}} = 0$$

$$\Rightarrow \quad \underset{n\to\infty}{Lt} (a_n)^{\frac{1}{n}} = +\infty \qquad \text{(By applying case I for } \frac{1}{a_b}\text{)}$$

**Note :** Converse of the above theorem is not true.

**Example :** Let an = $5^{-n+(-1)n}$

$$\therefore \quad \underset{n\to\infty}{Lt} (a_n)^{\frac{1}{n}}$$

$$= \underset{n\to\infty}{Lt} 5^{\frac{-n+(-1)^n}{n}}$$

$$= \underset{n\to\infty}{Lt} 5^{-1+\frac{(-1)^n}{n}}$$

$$= 5^{-1} = \frac{1}{5} \text{, which exists.}$$

But $\frac{a_{n+1}}{a_n} = \frac{5^{-(n+1)+(-1)^{n+1}}}{5^{-n+(-1)^n}}$

$$= 5^{-1+(-1)^{n+1}-(-1)^n}$$

$$= \begin{cases} 5, & \text{n is odd} \\ 5^{-3}, & \text{n is even} \end{cases}$$

$\therefore$ $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n}$ does not exist.

**Some Illustrative Examples**

**Example 1 :** Prove that the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ is bounded.

**Solution :** The given sequence is $\{a_n\}$

Where $a_n = \left(1+\frac{1}{n}\right)^n \; \forall \; n \in N$

$\therefore$ $a_n = \left(1+\frac{1}{n}\right)^n$

$$= 1 + n.\frac{1}{n} + \frac{n(n-1)}{\lfloor 2} \cdot \left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{\lfloor 3} \cdot \left(\frac{1}{n}\right)^3 + \text{..... upto (n+1) terms}$$

$$= 1 + 1 \frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right)+\frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \text{... upto (n+1) terms}$$

$$\leq 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{.... upto (n+1) terms}$$

$$< 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{....} \infty$$

$$= e$$

$\therefore$ $a_n < e \qquad \forall \; n \in N$

Also $a_n > 1 \qquad \forall \; n \in N$

$\therefore$ $1 < an < e \qquad \forall \; n \in N$

$\therefore$ sequence {an} is bounded

**Example 2 :** Prove that the sequence $\left\{\frac{n^2+1}{2n+3}\right\}$ is unbounded.

**Solution :** Let $\{a_n\} = \left\{\dfrac{n^2+1}{2n+3}\right\}$ where an + $\dfrac{n^2+1}{2n+3} > 0 \quad \forall\ n \in N$

Now $a_n = \dfrac{n^2+1}{2n+3} = \dfrac{n}{2} - \dfrac{3}{4} + \dfrac{\frac{13}{4}}{2n+3}$

Let $\Delta > 0$, however large

Now $|a_n| > \Delta$

$$\begin{array}{r} \frac{n}{2} - \frac{3}{4} \\ 2n+3\overline{)\,n^2+1} \\ n^2 \quad \frac{3n}{2} \\ \underline{\phantom{n^2}} \\ - \quad - \\ -\frac{3n}{2}+1 \\ -\frac{3n}{2}-\frac{9}{4} \\ \underline{\phantom{n^2}} \\ + \quad + \\ \frac{13}{4} \end{array}$$

if $\quad a_n > \Delta$

i.e. if $\quad \dfrac{n}{2} - \dfrac{3}{4} + \dfrac{13}{4(2n+3)} > \Delta \qquad [\because |a_n = a_n|]$

i.e. if $\quad \dfrac{n}{2} - \dfrac{3}{4} > \Delta \qquad \left[\because \dfrac{13}{4(2n+3)} > 0\right]$

i.e. if $\quad n > 2\Delta + \dfrac{3}{2}$

$\therefore \quad |a_n| > \Delta \qquad \forall\ n > m > 2\Delta + \dfrac{3}{2},\ m \in N$

$\Rightarrow \quad \{a_n\}$ is unbounded.

**Example 3 :** Prove that $\left\{\dfrac{\lfloor n}{n^2}\right\}$ is a null sequence.

**Solution :** The given sequence is $\{a_n\}$ where

$a_n = \dfrac{\lfloor n}{n^n}$

$$= \frac{n(n-1)(n-2)...2.1}{n.n.n...n.n}$$

$$= \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} ... \frac{2}{n} \cdot \frac{1}{n}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)...\left(1-\frac{n-2}{n}\right) . \frac{1}{n} \leq 1.1.1... \frac{1}{n} = \frac{1}{n}$$

$\therefore$ $|a_n| \leq \frac{1}{n}$ and $\frac{1}{n} \to 0$ $n \to \infty$

$\therefore$ $\{a_n\}$ is a null sequence.

$\therefore$ $\left\{\frac{\underline{|n}}{n^n}\right\}$ is a null sequence.

**Example 4 :** Show that $\underset{n\to\infty}{Lt}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+...+\frac{1}{(2n)^2}\right] = 0$

**Solution :** Let $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(2n)^2}$

$> \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + \frac{1}{(2n)^2} + ...$ to (n + 1) terms

$$\left[\begin{array}{l} \because n^2 < (n+1)^2 < (n+2)^2 < ... < (n+n)^2 \\ \because \frac{1}{n^2} > \frac{1}{(n+1)^2} > \frac{1}{(n+2)^2} > ... > \frac{1}{(2n)^2} \end{array}\right]$$

$$= \frac{n+1}{(2n)^2}$$

$$= \frac{n+1}{4n^2}$$

$$= \frac{1}{4n}\left(1+\frac{1}{n}\right)$$

$\therefore$ $\frac{1}{4n}\left(1+\frac{1}{n}\right) < a_n$ ....(1)

Again $a_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + ... + \frac{1}{(2n)^2}$

$< \frac{1}{n^2} + \frac{1}{n^2} + \frac{1}{n^2} +$.... to (n +1) terms

$= \frac{n+1}{n^2}$

$= \frac{1}{n} + \frac{1}{n^2}$

$\therefore \quad a_n < \frac{1}{n} + \frac{1}{n^2}$ ....(2)

From (1) and (2) , $\frac{1}{4n}\left(1+\frac{1}{n}\right) <$ an $< a_n < \frac{1}{n} + \frac{1}{n^2} \quad \forall$ n

Now $\frac{1}{4n}\left(1+\frac{1}{n}\right)$ and $\frac{1}{n} + \frac{1}{n^2} \to 0$ as $n \to \infty$

$\therefore$ by squeeze principle, an $\to 0$ as $n \to \infty$

$\therefore \quad \underset{n\to\infty}{Lt}\, a_n = 0$

$\Rightarrow \quad \underset{n\to\infty}{Lt}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+...+\frac{1}{(2n)^2}\right] = 0$

**Example 5 :** Prove that the sequence $\left\{\frac{2n-7}{3n+2}\right\}$ is

(i) monotonically increasing

(ii) bounded

(iii) convergent

**Solution :** (i) the given sequence is $\{a_n\}$ where $a_n = \frac{2n-7}{3n+2}$

$\therefore \quad a_{n+1} = \frac{2(n+1)-7}{3(n+1)+2}$

$= \frac{2n-5}{3n+5}$

$\therefore \quad a_{n+1} - a_n = \frac{2n-5}{3n+5} - \frac{2n-7}{3n+2}$

$$= \frac{6n^2+4n-15n-10-6n^2+21n-10n+35}{(3n+5)(3n+2)}$$

$$= \frac{25}{(3n+5)(3n+2)} > 0 \qquad \forall\ n \in N$$

$\Rightarrow$ $a_{n+1} - a_n =$ $\forall\ n \in N$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing.

(ii) $a_n = \frac{2n-7}{3n+2} > 0 \qquad \forall\ n \geq 4$

Also $a_1 = \frac{-5}{5} = -1$, $a_2 = \frac{-3}{8}$, $a_3 = \frac{1}{11}$

$\therefore$ $a_n > -1 \ \forall\ n$

$\therefore$ {an} is bounded below

Again $a_n = \frac{2n-7}{3n+2}$

$$= \frac{2}{3} - \frac{\frac{23}{3}}{3n+2} < \frac{2}{3} \ \forall\ n$$

$\therefore$ $\{a_n\}$ is bounded above

$\therefore$ $\{a_n\}$ is bounded.

$$\frac{2}{3}$$

$$3n+2 \overline{)\, 2n-7}$$

$$2n + \frac{4}{3}$$

$$- \quad -$$

$$-\frac{25}{3}$$

(iii) Since $\{a_n\}$ is monotonically increasing and bounded above.

$\therefore$ $\{a_n\}$ is convergent.

**Example 6 :** Prove that the sequence $\{a_n\}$ where $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$ is convergent.

**Solution :** Here $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n}$

$\therefore$ $a_{n+1} = \frac{1}{n+1} + \frac{1}{n+3} + \frac{1}{n+4} + \dots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$

$\therefore$ $a_{n+1} - a_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$

$$= \frac{1}{2n+1}\left[\frac{1}{2(n+1)} - \frac{1}{n+1}\right]$$

$$= \frac{1}{2n+1} - \frac{1}{2(n+1)}$$

$$= \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= \frac{2n+2-2n-1}{(2n+1)(2n+2)}$$

$$= \frac{1}{(2n+1)(2n+2)} > 0 \ \forall\ n \in N$$

$\therefore$ $a_{n+1} - a_n > 0 \quad \forall\ n \in N$

$\therefore$ $a_{n+1} > a_n \quad \forall\ n \in N$

$\Rightarrow$ $\{a_n\}$ is monotonically increasing.

Also $a^n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n} < \frac{1}{n} + \frac{1}{n} + \ldots\ldots + \frac{1}{n}$

$$= \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \ldots\ldots \text{ to n terms}$$

$$= \frac{n}{n} = 1 \quad \forall\ n \in N$$

$\therefore$ $a_n < 1 \quad \forall\ n \in N$

$\Rightarrow$ $\{a_n\}$ is bounded above.

$\therefore$ $\{a_n\}$ is convergent.

**Example 7:** If $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n$, then prove that $\{a_n\}$ is a monotonically decreasing sequence. Prove that it is convergent.

**Solution:** Here $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} - \log n$

$\therefore$ $a_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n} + \frac{1}{n+1} - \log(n+1)$

$\therefore$ $a_{n+1} - a_n = \frac{1}{n+1} - \log(n+1) + \log n$

$$= \frac{1}{n+1} - \log\left(\frac{n+1}{n}\right)$$

$$= \frac{1}{n+1} - \log\left(1+\frac{1}{n}\right)$$

$$= \frac{1}{n+1}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + .......\right)$$

$$\left[\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ......\right]$$

$$= \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2} - \left(\frac{1}{3n^3} - \frac{1}{4n^4}\right) - ......$$

$$< \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2} \qquad \left[\because \frac{1}{3n^3} - \frac{1}{4n^4} > 0, etc.\right]$$

$$= \frac{2n^2 - 2n^2 - 2n + n + 1}{2n^2(n+1)}$$

$$= \frac{2n^2 - 2n^2 - 2n + n + 1}{2n^2(n+1)}$$

$$= \frac{n+1}{2n^2(n+1)}$$

$\leq 0 \quad \forall\, n \in N$

$\therefore$ $a_{n+1} - a_n < 0 \quad \forall\, n \in N$

$\Rightarrow$ $a_{n+1} < a_n \quad \forall\, n \in N$

$\therefore$ $\{a_n\}$ is monotonically decreasing.

Also $a_n \geq 0 \quad \forall\, n \in N$

$a_n > 0$

$\therefore$ $\{a_n\}$ is bounded above.

$\therefore$ $\{a_n\}$ is convergent.

**7.5 Cesaro's Theorem**

If the sequences $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, then

$$\underset{n\to\infty}{Lt}\ \frac{a_1b_n + a_2b_{n-1} + a_3b_{n-2} + ......+ a_nb_1}{n} = AB$$

**Proof:** Let an = A + $t_n$ $\forall$ n $\in$N

$\because$ $a_n \to A$ as $n \to \infty$

$\therefore$ $t_n \to 0$ as $n \to \infty$

$\Rightarrow$ $|t_n| \to 0$ as $n \to \infty$

$\therefore$ by Cauchy's First Theorem on limits, $\frac{|t_1| + |t_2| + ..... + |t_n|}{n} \to 0$

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$\frac{|t_1| + |t_2| + ..... + |t_n|}{n} < \varepsilon \qquad \forall\ n \geq m \qquad ...... (1)$$

Now $d_n = \frac{a_1b_n + a_2b_{n-1} + a_3b_{n-2} + ......+ a_nb_1}{n}$

$$= \frac{(A+t_1)b_n + (A+t_2)b_{n-1} + (A+t_3)b_{n-2} + ......+ (A+t_n)b_1}{n}$$

$$= \frac{A(b_1 + b_2 + .....b_n) + (t_1b_n + t_2b_{n-1} + t_3b_{n-2} + ......+ t_nb_1)}{n}$$

$$= A\left(\frac{b_1 + b_2 + ......+ b_n}{n}\right)\left(\frac{(t_1b_n + t_2b_{n-1} + t_3b_{n-2} + ......+ t_nb_1)}{n}\right) \quad .... (2)$$

Now $b_n \to B$

$\therefore$ by Cauchy's First Theorem on limits,

$$\frac{b_1 + b_2 + ......+ b_n}{n} \to B \qquad .... (3)$$

Since $\{b_n\}$ is convergent.

$\therefore$ it is bounded.

$\therefore$ there exists a positive real number k such that

$$|b_n| \leq k \qquad \forall\ n \qquad ...... (4)$$

$\therefore$ $\left|\frac{(t_1b_n + t_2b_{n-1} + t_3b_{n-2} + ......+ t_nb_1)}{n}\right| < \frac{1}{n}\left[|t_1b_n| + |t_2b_{n-1}| + ..... + |t_nb_1|\right]$

$= \frac{1}{n}\left[|t_1||b_n| + |t_2||b_{n-1}| + ..... + |t_n||b_1|\right]$

$$\le \frac{1}{n}\left[|t_1|.k + |t_2|.k + ..... + |t_n|.k\right] \qquad [\because \text{ of } (4)]$$

$$= k\frac{|t_1| + |t_2| + ..... + |t_n|}{n} < k\,\varepsilon \;\forall\; n > m \qquad [\because \text{ of } (1)]$$

$$\therefore \qquad \frac{t_1b_n + t_2b_{n-1} + t_3b_{n-2} + ...... + t_nb_1}{n} \to \text{ as } n \to \infty.$$

From (2),

$$\underset{n\to\infty}{Lt}\, d_n = \underset{n\to\infty}{Lt}\left[A\left(\frac{b_1 + b_2 + ...... + b_n}{n}\right) + \left(\frac{t_1b_n + t_2b_{n-1} + ...... + t_nb_1}{n}\right)\right]$$

$$= A\,\underset{n\to\infty}{Lt}\,\frac{b_1 + b_2 + ...... + b_n}{n} + \underset{n\to\infty}{Lt}\,\frac{t_1b_n + t_2b_{n-1} + ...... + t_nb_1}{n}$$

$$= AB + 0 \qquad [\because \text{ of (3) and (5)}]$$

$$= AB$$

$$\therefore \qquad \underset{n\to\infty}{Lt}\,\frac{a_1b_n + a_2b_{n-1} + a_3b_{n-2} + ...... + a_nb_1}{n}$$

### 7.6 Cauchy-stolze Theorem

If $\{b_n\}$ is a strictly monotonic increasing sequence so that $b_{n+1} > b_n \;\forall\; n$ and if $b_n \to \infty$ as $n \to \infty$, and $\{a_n\}$ is any sequence, then

$$\underset{n\to\infty}{Lt}\,\frac{a_n}{b_n} = \underset{n\to\infty}{Lt}\,\frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

provided that the limit on the right hand side exists whether finite or infinite.

**Proof:** Two cases arise :

**Case I.** Let $\underset{n\to\infty}{Lt}\,\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l$, where $l$ is finite

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer $m_1$ such that

$$\left|\frac{a_{n+1} - a_n}{b_{n+1} - b_n} - l\right| < \frac{\varepsilon}{2} \;\forall\; n \ge m_1$$

$$\therefore \qquad l - \frac{\varepsilon}{2} < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < l + \frac{\varepsilon}{2} \;\forall\; n > m_1$$

Multiplying by $b_{n+1} - b_n > 0$, we have,

$$\left(l-\frac{\varepsilon}{2}\right)(b_{n+1} - b_n) < a_{n+1} - a_n < \left(l-\frac{\varepsilon}{2}\right)(b_{n+1} - b_n) \ \forall\ n \geq m_1$$

Replacing n by n, n + 1, n + 2, ....., n + p - 1 and adding, we get,

$$\left(l-\frac{\varepsilon}{2}\right)(b_{n+p} - b_n) < a_{n+p} - a_n < \left(l-\frac{\varepsilon}{2}\right)(b_{n+p} - b_n) \ \forall\ n \geq m_1$$

Dividing by $b_{n+p} > 0$ and adding $\frac{a_n}{b_{n+p}}$, we get

$$\left(l-\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} < \frac{a_{n+p}}{b_{n+p}}$$

$$< \left(l+\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} \ \forall\ n \geq m_1 \text{ and } p \in N \qquad \text{.... (1)}$$

Keep n fixed and let $p \to \infty$

$\therefore$ $$\left(l+\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} \to l - \frac{\varepsilon}{2}$$

and $$\left(l+\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} \to l + \frac{\varepsilon}{2}$$

$$\left[\begin{array}{l} \because a_n, b_n \text{ are finite and } b_{n+p} \to \infty \text{ as } b_n \to \infty \text{ for } n \to \infty \\ \therefore \frac{a_n}{b_{n+p}} \to 0 \text{ and } \frac{b_n}{b_{n+p}} \to 0 \end{array}\right]$$

$\therefore$ there exists a positive integer $m_2$ such that $\forall\ n \geq m_2$, we have

$$\left(l-\frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} < \left(l-\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} < \left(l-\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$$

i.e. $$l - \varepsilon < \left(l-\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} < l \qquad \text{....(2)}$$

and $$\left(l+\frac{\varepsilon}{2}\right) - \frac{\varepsilon}{2} < \left(l+\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} < \left(l+\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$$

i.e. $$l < \left(l+\frac{\varepsilon}{2}\right)\left(1-\frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} + l + \varepsilon \qquad \text{....(3)}$$

From (1), (2), (3), we get

$$l - \varepsilon < \frac{a_{n+p}}{b_{n+p}} < l + \varepsilon \ \forall \ n \geq m_1, \ p \geq m_2$$

or $$l - \varepsilon < \frac{a_n}{b_n} < l + \varepsilon \qquad \forall \ n > m_1 + m_2$$

$\Rightarrow$ $$\underset{n\to\infty}{Lt} \frac{a_n}{b_n} = l$$

**Case II.** Let $\frac{a_{n+1}}{b_{n+1}} \to \infty$ as $n \to \infty$

$\therefore$ given $\Delta > 0$, however large, there exists a positive integer $m_1$ such that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} > \Delta \ \forall \ n \geq m_1$$

Multiplying by $b_{n+1} - b_n > 0$, we get,

$$a_{n+1} - a_n > \Delta (b_{n+1} - b_n) \quad \forall \ n \geq m_1$$

Replacing n with n, n + 1, n + 2,......n + p - 1 and adding, we get

$$a_{n+p} - a_n \Delta (b_{n+p} - b_n)$$

Dividing by $b_{n+p}$ and adding $\frac{a_n}{b_{n+p}}$, we get,

$$\frac{a_{n+p}}{b_{n+p}} > \Delta\left(1 - \frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} \qquad ....(1)$$

Keep n fixed and let $p \to \infty$

Now $\frac{a_n}{b_{n+p}}$ and $\frac{b_n}{b_{n+p}} \to 0$ as $n \to \infty$

$$\therefore \quad \Delta\left(1 - \frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} \to \Delta \text{ as } p \to \infty$$

$\therefore$ there exists a positive integer $m_2$ such that

$$\Delta - \varepsilon < \Delta\left(1 - \frac{b_n}{b_{n+p}}\right) + \frac{a_n}{b_{n+p}} < \Delta + \varepsilon \ \forall \ p > m_2 \qquad ....(2)$$

From (1) and (2), we get,

$$\frac{a_{n+1}}{b_{n+1}} > \Delta - \varepsilon \; \forall \; n \geq m_1 + m_2$$

$\Rightarrow$ $$\frac{a_n}{b_n} \to \infty \text{ as } n \to \infty$$

Combining the results of two cases,

$$\underset{n\to\infty}{Lt} \frac{a_n}{b_n} = \underset{n\to\infty}{Lt} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

**Cor.** Deduce Cauchy's First Theorem on limits from Cauchy's Stolze Theorem.

**Proof:** Let $A_n = a_1 + a_2 + a_3 + .....+ a_n$, $B_n = n$

Now $\{A_n\}$ is any sequence and $\{B_n\}$ is a strictly monotonic increasing sequence and $B_n \to \infty$ as $n \to \infty$.

$\because$ by Cauchy Stolze Theorem,

$$\underset{n\to\infty}{Lt} \frac{A_n}{B_n} = \underset{n\to\infty}{Lt} \frac{A_n - A_{n-1}}{B_n - B_{n-1}}$$

or $$\underset{n\to\infty}{Lt} \frac{a_1 + a_2 + ......+ a_n}{n}$$

$$= \underset{n\to\infty}{Lt} \frac{a_n}{n-(n-1)}$$

$$= \underset{n\to\infty}{Lt} a_n$$

$\therefore$ $$\underset{n\to\infty}{Lt} \frac{a_1 + a_2 + ......+ a_n}{n} = l \text{ where } l = \underset{n\to\infty}{Lt} a_n$$

Which is Cauchy's First Theorem on limits

**Some More Illustrated Examples**

**Example 8:** Prove that $\underset{n\to\infty}{Lim} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + .... + \frac{1}{n}\right) = 0$

**Solution:** Here $a_n = \frac{1}{n}$

$\therefore$ $$\underset{n\to\infty}{Lim} a_n = \underset{n\to\infty}{Lim} \frac{1}{n} = 0$$

$\therefore$ By Cauchy's First theorem on limits, we have

$$\lim_{n\to\infty} \frac{a_1 + a_2 + ...... + a_n}{n} = 0$$

or $$\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{1}{2} + \frac{1}{3} + .... + \frac{1}{n}\right) = 0$$

**Example 9:** Prove that $\lim_{n\to\infty} \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3} ....... \frac{n}{n-1}\right)^{\frac{1}{n}} = 1$

**Solution:** Here, $a_n = \frac{2}{1}, \frac{3}{2}, \frac{4}{3} ....... \frac{n}{n-1}$

$\therefore$ $$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$\therefore$ $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$$

Now $a_n > 0 \ \forall\ n \in N$

$\therefore$ by Cauchy's Second theorem on limits, we have

$$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = 1$$

or $$\lim_{n\to\infty} \left(\frac{2}{1}, \frac{3}{2}, \frac{4}{3} ....... \frac{n}{n-1}\right)^{\frac{1}{n}} = 1$$

**Example 10:** Calculate $\lim_{n\to\infty} \frac{u^n}{n}$

**Solution:** Let an = $u^n$, $b_n = n$

Here $\{a_n\}$ is any sequence and $\{b_n\}$ is a strictly monotonically increasing sequence, and $b_n \to \infty$ as $n \to \infty$

$\therefore$ by Cauchy Stolze Theorem

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$$

or $$\lim_{n\to\infty} \frac{u_n}{n} = \lim_{n\to\infty} \frac{u^{n+1} - u^n}{n + 1 - n}$$

$$= \lim_{n\to\infty} u^n(u - 1)$$

$$= \begin{cases} 0, & u \le 1 \\ \to \infty & u > 1 \end{cases}$$

**Example 11:** Prove that $\underset{n\to\infty}{Lim} \frac{\log n}{n} = 0$

**Solution:** Let an = log n, $b_n$ = n

Now sequence $\{a_n\}$ is any sequence and $\{b_n\}$ is a strictly monotonically increasing and $b_n \to \infty$ as n - ∞.

∴ by Cauchy Stole theorem

$$\underset{n\to\infty}{Lim} \frac{a_n}{b_n} = \underset{n\to\infty}{Lim} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

or $$\underset{n\to\infty}{Lim} \frac{\log n}{n} = \underset{n\to\infty}{Lim} \frac{\log(n+1)-\log n}{n+1-n} = \underset{n\to\infty}{Lim} \log\left(\frac{n+1}{n}\right)$$

$$= \underset{n\to\infty}{Lim} \log\left(1+\frac{1}{n}\right)$$

∴ $$\underset{n\to\infty}{Lim} \frac{\log n}{n} = 0$$

## 7.7 Self Check Exercise

Q.1 If $\underset{n\to\infty}{Lim}$ $(a_{n+1} - a_n) = l$ then

find $\underset{n\to\infty}{Lim} \frac{a_n}{n}$

Q.2 Show that $\underset{n\to\infty}{Lim} \sqrt[n]{n} = 1$

Q.3 Prove that $\underset{n\to\infty}{Lim} \left(\frac{1}{n!}\right)^{\frac{1}{n}} = 0$

## 7.8 Summary

In this unit we have learnt the following

(i) Cauchy's first theorem on limits

(ii) Cauchy's Second theorem on limits

(iii) Ceasaro's theorem

(iv) Cauchy-Stolze theorem

## 7.9 Glossary:

1. Sequence of means - The sequence $\{x_n\}$, $x_n = \frac{a_1 + a_2 + ...... + a_n}{n}$ is called sequence of means of sequence $\{x_n\}$

2. Deduction of Cauchy's first theorem on limits from Cauchy's Stolze Theorem -

   Let $A_n = a_1 + a_2 + ..... + a_n$, $B_n = n$

   then $\{A_n\}$ is any sequence and $\{B_n\}$ is strictly monotonically increasing sequence and $B_n \to \infty$ as $n \to \infty$]

$\because$ by Cauchy - Stolze theorem

$$\lim_{n\to\infty} \frac{A_n}{B_n} = \lim_{n\to\infty} \frac{A_n - A_{n-1}}{B_n - B_{n-1}}$$

or $$\lim_{n\to\infty} \frac{a_1 + a_2 + ...... + a_n}{n} = \lim_{n\to\infty} \frac{a_n}{n-(n-1)} = \lim_{n\to\infty} a_n$$

$\therefore$ $$\lim_{n\to\infty} \frac{a_1 + a_2 + ...... + a_n}{n} = l, \; l = \lim_{n\to\infty} a_n$$

which is Cauchy's first theorem on limits

## 7.10 Answer to Self Check Exercise

Ans.1 $l$

Ans.2 Prove it

Ans.3 Prove it

## 7.11 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983
3. K.A. Ross, Elementary thalysis - The Theory of calculus series - Undergraduate Texts in Mathematics, Springer Verlag, 2003.


## 7.12 Terminal Questions

1. If $\{x_n\}$ is convergent and $\{y_n\}$ is divergent, then show that $\{x_n + y_n\}$ is divergent
2. If $x_n \to 0$ and $\{y_n\}$ oscillates finitely, then show that the sequence $\{x_n y_n\}$ converges to 0.
3. Give examples to show that
   (i) $\{x_n + y_n\}$ can be convergent without
   $\{x_n\}$, $\{y_n\}$ being convergent
   (ii) $\{x_n y_n\}$ can be convergent without $\{x_n\}$ and $\{y_n\}$ being convergent.

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## Unit - 8

# Sub Sequences

**Structure**



### 8.1 Introduction

Dear students,after having the knowledge of a sequence, we shall, in this limit study the concept of a Subsequence. In mathematics, a Subsequence of a given sequence is a sequence that can be derived from the given sequence by deleting some or no elements without changing the order of the remaining elements. For example, the sequence {P, Q, S} is a Subsequence of {P,Q,R,S,T,U} obtained after removal of elements R.T, U. the relation of one sequence being the Subsequence of another is preorder.

### 8.2 Learning Objectives

The main objectives of this unit are

(i) to know about subsequence

(ii) how to construct a subsequence from a given sequence

(iii) to study peak point of a sequence

(iv) to study Bolzano - Weirstrars theorem

(v) to learn about duster point of a sequence or Subsequentrat limit.

## 8.3 Subsequence

A sequence $\{y_n\}$ is called a subsequence of the sequence $\{x_n\}$ if there exists a sequence $\{n_k\}$ of positive integers such that $n_1 < n_{2<...} < n_3$ and $y_k = x_{n_k}$ .

In other words, if we are given a sequence $\{x_n\}$ and a sequence $n_1 < n_2 < n_3 <.....$ of positive integers, we select the terms of $\{x_n\}$ corresponding to the sequence $\{n_k\}$ and place them in the same order. This new obtained sequence is called a subsequence of $\{x_n\}$.

## 8.4 Method to construct a subsequence

Step I. Find a strictly monotonic increasing sequence of positive integers $n_1, n_2, n_3.....$ i.e. $n_1 < n_2 < n_3 < ....$

Step II. Images $x_{n_1}$, $x_{n_2}$, $x_{n_3}$,..... of $n_1, n_2, n_3,.....$ under sequence $\{x_n\}$ are the elements of the subsequence $\{x_{n_k}\} = \{y_n\}$.

**Examples**

(i) Let $n_k = 2k$, $k = 1, 2, 3, .......$

Now $\{n_k\} = \{2, 4, 6, .....\}$ is a strictly monotonic increasing sequence of positive integers.

$\therefore$ $\{x_{n_k}\} = \{x_{2k}\} = \{x_2, x_4, x_6,......\}$ is a subsequence of $\{x_n\}$.

(ii) Let $n_k = 2k - 1$, $k = 1, 2, 3,.....$

Now $\{n_k\} = \{1, 3, 5,......\}$ is a strictly monotonic increasing sequence of positive integers.

$\therefore$ $\{x_{n_k}\} = \{x_{2k-1}\} = \{x_1, x_3, x_5,......\}$ is a subsequence of $\{x_n\}$.

(iii) Let $n_k = K^2$, $k = 1, 2, 3,......$

Now $\{n_k\} = \{1, 4, 9,.....\}$ is a strictly monotonic increasing sequence of positive integers.

$\therefore$ $\{x_{n_k}\} = \left\{x_{k^2}\right\} = \{x_1, x_4, x_9,.....\}$ is a subsequence of $\{x_n\}$.

(iv) Let $n_k = k^3$, $k = 1, 2, 3,......$

Now $\{n_k\} = \{1, 8, 27,.....\}$ is a strictly monotonic increasing sequence of positive integers.

$\therefore$ $\left\{x_{n_k}\right\} = \left\{x_{k^3}\right\} = \{x_1, x_8, x_{27},.....\}$ is a subsequence of $\{x_n\}$.

**Note.**

(1) Every sequence is a subsequence of itself.

(2) As $\{n_k\}$ is a strictly increasing sequence of positive integers, therefore the order in which the various terms of subsequence occur is the same as that in which they

occur in the given sequence. Thus {8, 2, 4, 6,....} is not a subsequence of {1,2, 3, 4, 5, 6, 7, 8,......}

(3) The interval between two consecutive terms of a subsequence is not always the same.

(4) If $x_m \in \{x_n\}$, then there exists an $n_i > m$ such that $x_{n_i}$ belongs to the subsequence.

(5) Any subsequence of sequence is itself a sequence.

(6) A sequence has an infinite number of subsequence.

**Art.** (i) If a sequence $\{x_n\}$ converges to l, then prove that every subsequence of $\{x_n\}$ also converges to l.

(ii) If a sequence $\{x_n\}$ converges to $+\infty$, then prove that every subsequence of $\{x_n\}$ also converges to $+\infty$.

(iii) If a sequence $\{x_n\}$ converges to $-\infty$, then prove that every subsequence of $\{x_n\}$ also converges to $-\infty$.

**Proof:** (i) Since $\{x_n\}$ converges to l

$\therefore$ given $\varepsilon > 0$, however small, there exists a positive integer m such that

$$|x_n - l| < \varepsilon \qquad \forall\ n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p$, $n_k \geq n_p \geq m$

$\therefore$ $|x_n - l| < \varepsilon \qquad \forall n_k \geq m$

$\therefore$ subsequence $\{x_{n_k}\}$ also converges to l.

(ii) Since $\{x_n\}$ diverges to $+\infty$

$\therefore$ given $\Delta > 0$, however large, there exists a positive integer m such that

$$x_n > \Delta \quad \forall\ n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p$, $n_k \geq n_p \geq m$

$\therefore$ $x_{n_k} > \Delta \qquad \forall n_k \geq m$

$\Rightarrow$ subsequence $\{x_{n_k}\}$ diverges to $+\infty$.

(iii) Since $\{x_n\}$ diverges to $-\infty$

$\therefore$ given $\Delta > 0$, however, large, there exists a positive integer m such that

$$x_n > -\Delta \ \forall\ n \geq m$$

If $n_p \geq m$ is a natural number, then for $k \geq p$, $n_k \geq n_p \geq m$

$\therefore$ $x_{n_k} > -\Delta \qquad \forall n_k \geq m$

$\Rightarrow$ subsequence $\{x_{n_k}\}$ diverges to $-\infty$.

**Note.** The converse of the above theorem is not true.

**Examples:**

(i) Let $x_n = (-1)^n = \{-1, 1, -1, 1, -1, 1, \ldots\}$

Two subsequences $\{-1, -1, -1, \ldots\}$ and $\{1, 1, 1, \ldots\}$ converges to -1 and 1 respectively. But $\{x_n\}$ does not converge.

(ii) Let $x_n = \begin{cases} n^2, & n \text{ is evn} \\ 0, & n \text{ is odd} \end{cases}$

Now $\{x_{2n}\}$ diverges to $+\infty$ but $\{x_n\}$ does not diverge to $+\infty$.

(iii) Let $x_n = \begin{cases} -n^2, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

Now $\{x_{2n-1}\}$ diverges to $-\infty$ but $\{x_n\}$ does not diverge to $-\infty$.

**8.5 Art. Peak Point of Sequence**

A natural number m is called peak point of the sequence $\{x_n\}$ if $x_n < x_m \quad \forall\ n > m$.

**Examples:**

(i) Every natural number is a peak point of the sequence $\left\{\frac{1}{n}\right\}$. In fact every natural number is a peak point of strictly monotonic decreasing sequence.

(ii) Let $x_n = \begin{cases} \frac{1}{n}, & n = 1, 2, 3, \ldots, m \\ -1, & n > m \end{cases}$

$\therefore$ $\{x_n\}$ has exactly m peak points 1, 2, 3,......, m.

**Note:** A sequence may have no peak point, finite number of peak points or infinite number of peak points.

**Art.** Prove that every sequence contains a monotonic subsequence.

**Proof:** Three cases arise :

**Case I.** The sequence $\{x_n\}$ has an infinite number of peak points.

Let the peak points be $n_1, n_2, n_3, \ldots$ such that

$n_1 < n_2 < n_3 < \ldots$

∴ $\{x_{n_1}, x_{n_2}, x_{n_3}, ......\}$ i.e., $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$

∵ $n_1$ is a peak point and $n_2 > n_1$

∴ $x_{n_2} < x_{n_1}$

⇒ $x_{n_1} > x_{n_2}$

Again $n_2$ is a peak point and $n_3 > n_2$

∴ $x_{n_3} < x_{n_2}$

⇒ $x_{n_2} > x_{n_3}$

∴ $x_{n_1} > x_{n_2} > x_{n_3}$

Proceeding in this way, we get,

$x_{n_1} > x_{n_2} > x_{n_3} > ......$

⇒ $\{x_{n_k}\}$ is a monotonic decreasing subsequence of $\{x_n\}$.

**Case II.** The sequence $\{x_n\}$ has a finite number of peak points.

Let $m_1, m_2, m_3, ...., m_p$ be the peak points of $\{x_n\}$.

Let $n_1$ be a natural number strictly greater than each of $m_1, m_2, m_3, ...., m_p$

∵ $n_1$ is not a peak point

∴ there exists a natural number $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$

Again $n_2$ is not a peak point of $\{x_n\}$

∴ there exists a natural number $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$

Therefore we have $n_1 < n_2 < n_3$ such that $x_{n_1} \leq x_{n_2} \leq x_{n_3}$

Proceeding in this way, we get a monotonic increasing subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

**Case III.** The sequence $\{x_n\}$ has no peak point.

∴ 1 is not a peak point of $\{x_n\}$

∴ there exists a natural number $n_2 > 1 = n_1$ such that $x_{n_2} \geq x_{n_1}$

Again $n_2$ is not a peak point of $\{x_n\}$

∴ there exists a natural number $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$

Therefore we have $n_1 < n_2 < n_3$ such that $x_{n_1} \le x_{n_2} \le x_{n_3}$

Proceeding in this way, we get a monotonic increasing subsequence $\{x_{n_k}\}$ of $\{x_n\}$.

Hence every sequence contains a monotonic subsequence.

**8.6 Cor. Bolzano-Weierstrass Theorem**

Prove that every bounded sequence has a convergent subsequence.

Proof: Let $\{x_n\}$ be a bounded sequence.

$\because$ $\{x_n\}$ is a sequence, therefore $\{x_n\}$ has a monotonic subsequence $\{x_{n_k}\}$

$\because$ $\{x_n\}$ is bounded, therefore $\{x_{n_k}\}$ is also bounded

[$\because$ every subsequence of a bounded sequence is bounded.]

$\therefore$ $\{x_{n_k}\}$ is a bounded monotonic sequence

$\Rightarrow$ $\{x_{n_k}\}$ is convergent.

$\therefore$ $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$

**8.7 Art. Sub sequential Limit or Cluster Point of a Sequence**

A real number l is called a sub sequential limit or cluster point of the sequence $\{x_n\}$ if there exits of sub-sequence of $\{x_n\}$ which converges to l.

Note. (1) If a sequence $\{x_n\}$ converges to l, then l is the only cluster point of $\{x_n\}$. This is so as every subsequence of $\{x_n\}$ converges to l.

(2) If a sequence has more than one cluster point, then it cannot be convergent.

(3) If a sequence $\{x_n\}$ diverges to $+\infty$, then $+\infty$ is the only cluster point of $\{x_n\}$.

(4) If a sequence $\{x_n\}$ diverges to $-\infty$, then $-\infty$ is the only cluster point of $\{x_n\}$.

**Example:**

(i) Consider the sequence $\{x_n\}$ where $x_n = \frac{1}{n}$. Then the sequence $\{x_n\}$ converges to 0. Hence 0 is the only cluster point of $\{x_n\}$.

(ii) Consider the sequence $\{x_n\}$ where $x_n = (-1)^n$. The subsequence {-1,-1,-1,....} converges to 1 and subsequence {1,1,1,....} converges to 1. Therefore -1 and 1 are two cluster points of $\{x_n\}$

(iii) Let $x_n = \begin{cases} n, \ n \text{ is odd} \\ -n, \ n \text{ is even} \end{cases}$

Subsequence $\{x_1,x_3,x_5,....\}$ diverges to $+\infty$ and subsequence $\{x_2,x_4,x_6,.....\}$ diverges to $-\infty$. Thus the sequence $\{x_n\}$ has cluster points as $+\infty$ and $-\infty$.

**Art.** Prove that a real number l is a limit point of a set A iff there exists a sequence of distinct points of A converging to l.

**Proof:** (i) Assume that $\{x_n\}$ is a sequence of distinct points of A convergent to l.

$\therefore$ every neighbourhood of l contains infinitely many points of $\{x_n\}$ which are also points of A. Thus every nbd. Of l contains infinitely many points of A, which in turn shows that l is the limit point of A.

(ii) Assume that l is a limit point of A.

$\therefore$ every nbd. Of l contains infinitely many points of A.

$\therefore$ $\forall$ n $\in$N, $I_n = \left(1-\frac{1}{n}, l+\frac{1}{n}\right)$ contains infinitely many points of A.

Choose $x_1 \in \cap$ A and the choose $x_2 \in I_2 \cap$ A such that $x_2 \neq$ x1.

Proceeding in this way, we choose $x_k \in I_k \cap$ A such that $x_k$ is different from $x_1,x_2,...,x_{k-1}$. [This is possible as $I_k$ contains infinitely many points of A]

$\therefore$ we get a sequence $\{x_n\}$ of distinct points of A such that $x_n \in I_n$

Let m be any fixed positive integer

$\therefore$ $\forall$ $n \geq m$, $\frac{1}{n} \leq \frac{1}{m}$ and $\frac{1}{n} \geq - \frac{1}{m}$

$\therefore$ $x + \frac{1}{n} \leq x + \frac{1}{m}$ and $x - \frac{1}{n} \geq x - \frac{1}{m}$

$\therefore$ $\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset \left(x-\frac{1}{m}, x+\frac{1}{m}\right)$

or $I_n \subset I_m$ $\forall$ $n \geq m$

$\therefore$ $\forall$ $n \geq m$, $x_n \in I_n$

$\Rightarrow$ $x_n \in I_m$

$\Rightarrow$ $x_n \in \left(x-\frac{1}{m}, x+\frac{1}{m}\right)$ $\forall$ $n \geq M$

$\Rightarrow$ $|x_n - l| < \frac{1}{m} = \in$ $\forall$ $n \geq m$

$\Rightarrow$ $\underset{n\to\infty}{Lt}\, x_n = i$

$\therefore$ sequence $\{x_n\}$ converges to l.

**Art :** Prove that a real number l is a cluster point of real sequence $\{x_n\}$ iff given $\in$> 0, the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many points of $\{x_n\}$.

**Proof :** (i) Assume that l is a cluster point of $\{x_n\}$

$\therefore$ there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to l.

$\therefore$ given $\varepsilon$> 0, there exists a natural number m such that

$$x_{n_k} \in (l - \varepsilon, l + \varepsilon) \qquad \forall n_k \geq m$$

In particular, $x_n \in (l - \varepsilon, l + \varepsilon)$ for infinitely many n i.e. the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms $\{x_n\}$.

(ii) Assume that the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many terms of $\{x_n\}$, where $\varepsilon > 0$.

$\therefore$ $x_n \in \left(l - \frac{1}{n}, l + \frac{1}{n}\right)$ for infinitely many n, where $\frac{1}{n} = \varepsilon$

In particular we can find $x_{n_1} \in (l - l, l + 1)$

Again $\left(l - \frac{1}{2}, l + \frac{1}{2}\right)$ contains $x_n$ for infinitely many n, we can find $n_2 > n_1$ such that $x_{n_2} \in \left(l - \frac{1}{2}, l + \frac{1}{2}\right)$.

Proceeding in this way, we can find natural numbers $n_1 < n_2 < n_3 < \ldots\ldots < n_k < \ldots\ldots$ such that

$$x_{n_k} \in \left(l - \frac{1}{k}, l + \frac{1}{k}\right)$$

i.e. $|x_{n_k} - l| < \frac{1}{k}$

$\Rightarrow$ $x_{n_k} \to l$

$\therefore$ l is a cluster point of $\{x_n\}$

**Art. :** Prove that a sequence $\{x_n\}$ converges to a real number l iff $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$.

**Proof :** (i) Assume that $\{x_n\}$ converges to l

$\therefore$ $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$.

(ii) Assume that $\{x_n\}$ is bounded and l is the only cluster point of $\{x_n\}$.

If possible, suppose that $\{x_n\}$ does not converges to l.

$\therefore$ there exists a $_n\varepsilon > 0$, such that for $m \in N$, there exists and $n \geq m$ such that

$x_n \notin (l - \varepsilon, l + \varepsilon)$

In particular, there exists an integer $n_1$ such that $x_{n_1} \notin (l - \varepsilon, l + \varepsilon)$ on the basis of same argument, there exists an integer $n_2 > n_1$ such that $x_{n_2} \notin (l - \varepsilon, l + \varepsilon)$.

Proceeding in this way, we get a subsequence $\{x_{n_k}\}$ such that

$x_{n_k} \notin (l - \varepsilon, l + \varepsilon) \ \forall K$

**Some Illustrated Examples**

**Example 1 :** If a sequence $\{x_n\}$ converges to l, then its subsequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ also converges to l.

**Solution :** Since $\{x_n\}$ converges to l

$\therefore$ given $\varepsilon > 0$, however small, $\exists$ a positure integer m such that

$|x_n - l| < \in \quad \forall\ n \geq m$

Now $2n > m$ and $2n + 1 > m$

$\therefore$ $|x_{2n} - l| < \in \quad \forall\ n \geq m$

and $|x_{2n+1} - l| < \in \quad \forall\ n \geq m$

$\therefore$ $x_{2n} \rightarrow l$ and $x_{2n+1} \rightarrow l$

$\therefore$ subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to l.

**Example 2 :** If two subsequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ of a sequence $\{x_n\}$ converge to the same limit l, then $\{x_n\}$ also converges to l.

**Solution :** Since $\{x_{2n}\}$, $\{x_{2n+1}\}$ converge to l

$\therefore$ $x_{2n} \rightarrow l$ and $x_{2n+1} \rightarrow l$.

$\therefore$ given $\varepsilon > 0$, $\exists$ natural number $m_1$, $m_2$ s.t.

$|x_{2n} - l| < \in \quad \forall\ n \geq m_1$ ...(1)

and $|x_{2n+1} - l| < \in \quad \forall\ n \geq m_2$ ...(2)

Two cases arise.

**Case 1.** n is even

Let $n = 2k$

$\therefore$ $|x_{2n} - l| = |x_{2n+1} - l| < \in \forall \quad n \geq m_1$

Now $n = 2k \Rightarrow n > 2m_1$

∴ $|x_n - l| < \in \quad \forall\ n \geq 2m_1$ ...(3)

**Case 2.** n is odd

Let n = 2k+1

∴ $|x_n - l| = |x_{2n+1} - l| < \in \forall \quad n \geq m_2$

Now $n = 2k+1 \Rightarrow n > 2m_2 + 1$

∴ $|x_n - l| < \in \forall\ n \geq 2m_2 + 1$ ...(4)

Let m = maso $(2m_1, 2m_2 + 1)$

∴ From (3) and (4), we have

$$|x_n - l| < \in \forall\ n \geq m$$

⇒ $x_n \to l$ as $n \to \infty$

⇒ $\{x_n\}$ converges to l.

## 8.8 Self Check Exercise

Q.1 Prove that every sequence contains a monotone subsequence.

Q. 2 Prove that a real number l is a cluster point of real sequence $\{x_n\}$ if given $\in > 0$, the interval $(l - \varepsilon, l + \varepsilon)$ contains infinitely many points of $\{x_n\}$.

## 8.9 Summary

In this unit we have learnt the following

(i) Subsequences

(ii) Method to construct a subsequence

(iii) Peak point of a sequence

(iv) Bolzano-Weirstrass theorem

(v) Subsequential limit or cluster point if a sequence

## 8.10 Glossary:

1. **Substring :** A subsequence which consists of a consecutive run of elements from the sequence, such as {Q, R, S}, from {P, Q, R, S, T, V} is called a substring.

2. **Preorder :** The relation of one sequence being the subsequence of another is called preorder.

## 8.11 Answer to Self Check Exercise

Ans.1 Prove it

Ans.2

## 8.12 References/Suggested Readings

1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

## 8.13 Terminal Questions

1. Prove that every sequence is a subsequence of itself.
2. Prove that every sequence contains or monotone subsequence.
3. Prove that every bounded sequence was a convergent subsequence.

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## Unit - 9

# Cauchy Sequences

**Structure**



### 9.1 Introduction

Dear students, in this unit we shall study the concept of Cauchy sequence. A Cauchy sequence is a sequence whose elements become arbitrarily close to each other as the sequence progresses. Cauchy sequences are named after Augustin-Lovis Cauchy; they may occasionally be known as fundamental sequences.

### 9.2 Learning Objectives

The main objectives of this unit are

(i) to study Cauchy sequences

(ii) to learn about Cauchy's general principle of convergence.

(iii) to prove contor-intersection theorem

(iv) to study limit superior and limit inferior of a sequence etc.

### 9.3 Cauchy Sequence

**Definition :** A sequence $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, $\exists$ a positive integer k $(\in)$s.t.

$|x_n - x_m| < \in \quad \forall\ n, m \geq K$

Or

A sequence $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$, however small, $\exists$ a positive integer m $(\in)$s.t.

$|x_{n+p} - x_n| < \in \quad \forall\ n, m \geq m$ and $\rho \in N$.

**Art 1 :** Prove that a Cauchy sequence is bounded.

**Proof :** Set $\{x_n\}$ be a Cauchy sequence. Therefore by definition, given $\in > 0$, $\exists$ a positive integer $\rho$s.t.

$|x_n - x_m| < \in \quad \forall\ n, m \geq \rho$ ...(1)

In particular, we have

$|x_n - x_p| < \in \quad \forall\ n \geq \rho$ ...(2)

Now

$|x_n| = |x_n - x_p + x_p|$

$\leq |(x_n - x_p)| + |x_p|$

$< \in + |x_p|$ [using (2)]

$\therefore \quad |x_n| < \in + |x_p| \quad \forall\ n \geq \rho$

Let $M = \max \{|x_1|, |x_2|, \ldots\ldots. |x_{p-1}||, \in + |x_p|\}$

$\therefore \quad |x_n| \leq M \quad \forall\ n$

$\Rightarrow \quad |x_n|$ is bounded

**Art. 2:** Show that a convergent sequence is always a Cauchy sequence.

**Proof:** Let the sequence {xn} converges to l.

$\therefore$ given $\in > 0$, however small, $\exists\ k \in N$ s.t.

$|x_{n-1}| < \varepsilon/2 \quad \forall\ n \geq k$ ..... (1)

Let m> k be a natrual number

$\therefore \quad |x_{m-l}| < \varepsilon/2 \quad \forall\ m \geq k$ ..... (2)

Now $|x_n - x_m| = |x_{n-l} + l - x_m|$

$\langle |(x_{n-l}) + |x_{m-l}|$

$< \varepsilon/2 + \varepsilon/2 = \in$

$\Rightarrow \quad |x_n - x_m| < \in \quad \forall\ n, m > k$

$\therefore \quad \{x_n\}$ is a Cauchy sequence

**Art 3:** Prove that a Cauchy sequence is always convergent.

**Proof:** Let $\{x_n\}$ be a Cauchy sequence

given $\in > 0$, $\exists$a the integer p s.t.

$|x_n - x_m| < \in \quad \forall\ n, m \geq p$ ...... (1)

or in particular

$|x_n - x_p| < \in \quad \forall\ n \geq p$ ...... (2)

Now

$|x_n| = |x_n - x_p + x_p|$

$< \quad |(x_n - x_p) + |x_p|$

$< \quad \in + |x_p| \forall\ n \geq p$

$\therefore \quad |x_n| < \in + |x_p| \quad \forall\ n \geq p$

Let $\quad M = \max \{|x_1|, |x_2|, ...... |x_{p-1}|, \in + |x_p|\}$

$|x_n| \in M \quad \forall\ n$

$\Rightarrow \quad \{x_n\}$ is bounded

$\therefore$ by Bolzano - Weirrtrass theorem $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$.

Let $\{x_{n_k}\}$ be convergent to l.

Our claim is $\{x_n\}$ also converges to l

Since $\{x_{n_k}\} \to l$,

$\therefore$ given $\in > 0$, $\exists$a the integer p s.t.

$|x_{n_k} - l| < \in \quad \forall\ k \geq p$ ..... (3)

$\therefore$ For $n \geq p$, $n_k \geq np \geq p$, from (1), we get

$|x_n - x_{n_k}| < \varepsilon/2$ ..... (4)

$\therefore \quad |x_n - l| = |x_n - x_{n_k} + x_{n_k} - l|$

$\langle \left(x_n - x_{n_k}\right) + |x_{n_k} - l|$

$\langle \varepsilon/2 + \varepsilon/2 = \in$ ($\because$ of (3) & (4))

$\therefore \quad |x_n - l| < \in \quad \forall\ n \geq p$

$\Rightarrow$ $\{x_n\}$ is convergent

## 9.4 Cauchy's General Principle of convergence

OR

**Art 4:** Prove that a necessary and sufficient condition for the convergence of a sequence $\{x_n\}$ of real numbers is that it is a Cauchy sequence.

**Proof:** See Art 2. and Art 3.

**Note:** In the system of rational numbers, every Cauchy sequence does not converge to a rational number.

For example, consider the sequence

1.4, 1.41, 1.414, 1.4142.

It is a Cauchy sequence but does not converge to a rational number. It can be seen to converge to $\sqrt{2}$ (not a rational number)

## 9.5 Cantor Intersection Theorem

**Theorem:** Let $\{I_n\}$, $I_n = \{a_n, b_n\}$ be a sequence of closed intervals such that

(i) $I_{n+1} \subset I_n \ \forall\ n$

(ii) $l(I_n) = b_n - a_n \to 0$ as $n \to \infty$, then $\exists$ a unique point c such that $c \in In\ \forall\ n$, $l(I_n)$ being length of the interval In:

**Proof:** As $I_{n+1} \subset I_n \quad \forall\ n$

$\therefore$ $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \quad \forall\ n$

$\therefore$ $a_1 \leq a_2 \leq a_3 \leq ....... \leq a_n \leq a_{n+1} \leq$ ...... (1)

and $b_1 \geq b_2 \geq b_3 \geq ....... \geq b_n \geq b_{n+1} \geq$ ....... (2)

$\therefore$ the sequence $\{a_n\}$ is monotonically increasing and is bounded as $a_n < b_n \forall\ n$.

$\Rightarrow$ $\{a_n\}$ is monotonically decreasing and is bounded below as $b_n > a_n \quad \forall\ n$.

Now Let $a_n \to \propto$, $b_n \to \beta$

$\therefore$ $b_n = (b_n - a_n) + a_n$

$\Rightarrow$ $\lim\limits_{n\to\infty} bn = \lim\limits_{n\to\infty} (bn - an) + \lim\limits_{n\to\infty} an$

$\therefore$ $\beta = 0 + \propto$

$\Rightarrow$ $\beta = \propto = c$ (say)

Now c is l.u.b. of $\{a_n\}$ and g.l.b of $\{b_n\}$

$\therefore$ $a_n \leq c \leq b_n \quad \forall\ n$

$\Rightarrow$ $c \in [a_n, b_n]\ \forall\ n$

$\Rightarrow$ $c \in I_n \quad \forall\ n$.

**c is unique**

Let of possible that there exists two real numbers $c_1$ and $c_2 \in I_n \quad \forall$ n

$\therefore$ $\{b_n - a_n\}$ cannot converge to 0 (zero) which is a contradiction to the fact that $b_n - a_n \to 0$

$\therefore$ our supposition is wrong

$\therefore$ c is unique

**Note:** This theorem is also known as 'Cantor's Theorem on Nested Intervals' or 'Nested Interval Property'.

**9.6 Art. Limit Superior and Limit Inferior of a Sequence**

We know that a sequence of a real numbers always contains a monotone subsequence and a monotone sequence converges or diverges to $+\infty$ or $-\infty$. Thus if E denotes the set of all the cluster points (i.e. subseequential limits), including +) and -), of a sequence $\{x_n\}$ of real numbers, then E has at lest one element i.e., E is non-empty.

(i) The l.u.b of E in the extended real number system is called the limit superior or upper limit of $\{x_n\}$ and is denoted by $\overline{\underset{n\to\infty}{Lt}}\, x_n$ or $\underset{n\to\infty}{Lt}$ sup $x_n$ or $\overline{Lt}\, x_n$ or Lt sup $x_n$.

(ii) The g.l.b. of E in the extended real number system is called the limit inferior or lower limit of $\{x_n\}$ and is denoted by $\underline{\underset{n\to\infty}{Lt}}\, x_n$ or $\underset{n\to\infty}{Lt}$ inf $x_n$ or $\underline{Lt}\, x_n$ or Lt inf $x_n$.

**Examples:**

(i) Let $x_n = (-1)^n$

$\therefore$ $\{x_n\}$ has only two cluster points -1 and 1.

$\therefore$ E = {-1, 1}

$\therefore$ $\overline{Lt}\, x_n = 1$, $\underline{Lt}\, x_n = -1$

(ii) Let $\{x_n\}$ converge to l. Then every subsequence of $\{x_n\}$ converges to l.

$\therefore$ E = {l}

$\therefore$ $\overline{Lt}\, x_n = l$, $\underline{Lt}\, x_n = l$

(iii) Let $x_n = \begin{cases} n, n \text{ is odd} \\ -n, n \text{ is even} \end{cases}$

$\therefore$ $E = \{-\infty, \infty\}$

$\therefore$ $\overline{Lt}\, x_n = \infty$, $\underline{Lt}\, x_n = -\infty$

**Properties of Limit Superior**

For a bounded sequence $\{x_n\}$,

$\overline{Lim}\, x_n$ = u iff for every $\in > 0$

(i) $\exists$ a m $\in$ N s.t. $x_n < u + \in \forall n > m$

(ii) $x_n > u - \in$ for infinitely many values of n.

**Properties of Limit Inferior**

For a bounded sequence $\{x_n\}$,

$\underline{Lim}\, x_n$ = l iff for every $\in > 0$

(i) $\exists$ m $\in$ N s.t $x_n > l - \in \quad \forall n \geq m$

(ii) $x_n < l + \in$ for infinitely many values of n.

**Some Illustrated Examples.**

**Example 1:** Prove that $\left\{\frac{1}{n}\right\}$ is a Cauchy sequence.

**Solution:** Let $a_n = \frac{1}{n} \therefore a_m = \frac{1}{m}$

Without loss of generality, we take n > m

$\therefore$ given $\in > 0$, however small, we have

$|a_n - a_m| < \in$

if $\left|\frac{1}{n} - \frac{1}{m}\right| < \in$

i.e. if $\frac{1}{m} - \frac{1}{n} < \in$ $\left(\because n > \frac{1}{m} \Rightarrow \frac{1}{m} > \frac{1}{n}\right)$

$\Rightarrow \frac{1}{m} < \frac{1}{n} + \in$

i.e. if $\frac{1}{m} < \in$

i.e. if $m > \frac{1}{\in}$

Let p be any positive integer just greater than $\frac{1}{\in}$

$\therefore |a_n - a_m| < \in \quad \forall n, m \geq p$

$\Rightarrow \{a_n\}$ is a Cauchy sequence

**Example 2:** Prove that the sequence $\{a_n\}$ where an = 8 + $\frac{1}{n^3}$ is a Cauchy sequence and find its limit.

**Solution:** Here $a_n = 8 + \frac{1}{n^3}$

$\therefore$ am = $8 + \frac{1}{n^3}$

Without loss of generality, we take n > m.

Let $\varepsilon > 0$, however small. Then $|a_n - a_m| < \varepsilon$

if $\left|\left(8+\frac{1}{n^3}\right)-\left(8+\frac{1}{n^3}\right)\right| < \varepsilon$

i.e. if $\left|\frac{1}{n^3}-\frac{1}{m^3}\right| < \varepsilon$

i.e. if $\frac{1}{n^3}-\frac{1}{m^3} < \varepsilon$ $\left[\because n > m \Rightarrow \frac{1}{n^3} < \frac{1}{m^3}\right]$

i.e. if $\frac{1}{n^3} < \frac{1}{m^3} + \varepsilon$

i.e. if $\frac{1}{m^3} < \varepsilon$

i.e. if $m^3 > \frac{1}{\varepsilon}$

i.e. if $m > \left(\frac{1}{\varepsilon}\right)^{\frac{1}{3}}$

Let p be any positive integer just greater than $\left(\frac{1}{\varepsilon}\right)^{\frac{1}{3}}$

$\therefore$ $|a_n - a_m| < \varepsilon \; \forall$ n, m $\geq$ p

$\Rightarrow$ $\{a_n\}$ is a Cauchy Sequence

$\Rightarrow$ $\{a_n\}$ is convergent as every Cauchy sequence is convergent.

Now $\underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt}\left(8+\frac{1}{n^3}\right)$

$$= \underset{n\to\infty}{Lt}\, 8 + \underset{n\to\infty}{Lt} \left(\frac{1}{n}\right)^3$$

$$= 8 + \left(\underset{n\to\infty}{Lt} \frac{1}{n}\right)^3$$

$$= 8 + 0 = 8 \qquad \left[\because \left\{\frac{1}{n}\right\} \textit{ is a null sequence}\right]$$

∴ sequence $\{a_n\}$ converges to 8.

**Example 3:** Prove that $\left\{\frac{n^3}{n^3+1}\right\}$ is a Cauchy sequence.

**Solution:** Let $a_n = \frac{n^3}{n^3+1}$. So $a_m = \frac{m^3}{m^3+1}$

Without loss of generality, we take n > m. Let $\varepsilon > 0$, however small. Then

$|a_n - a_m| < \varepsilon$

if $\left|\frac{n^3}{n^3+1} - \frac{m^3}{m^3+1}\right| < \varepsilon$

i.e. if $\left|\left(1 - \frac{1}{n^3+1}\right) - \left(1 - \frac{1}{m^3+1}\right)\right| < \varepsilon$

i.e. if $\left|\frac{1}{m^3+1} - \frac{1}{n^3+1}\right| < \varepsilon$

i.e. if $\frac{1}{m^3+1} - \frac{1}{n^3+1} < \varepsilon$

$$\left[\begin{array}{lll} \because n > m & \Rightarrow & n^3 > m^3 \\ \Rightarrow n^3+1 > m^3+1 & \Rightarrow & \frac{1}{m^3+1} > \frac{1}{n^3+1} \end{array}\right]$$

i.e. if $\frac{1}{m^3+1} < \frac{1}{n^3+1} + \in$

i.e. if $\frac{1}{m^3+1} < \in$

i.e. if $m^3 + 1 > \frac{1}{\in}$

i.e. if $m > \left(\frac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}$

Let p be any positive integer just greater than $\left(\frac{1}{\varepsilon} - 1\right)^{\frac{1}{3}}$

$\therefore$ $|a_n - a_m| < \varepsilon \quad \forall\ n, m \geq p$

$\Rightarrow$ $\{a_n\}$ is a Cauchy sequence

$\Rightarrow$ $\{a_n\}$ is a convergent sequence as every Cauchy sequence is convergent.

**Example 4:** Prove that $\frac{n}{n+1}$ is a Cauchy sequence

**Solution:** Here $a_n = \frac{n}{n+1}$, $a_m = \frac{m}{m+1}$

Take n > m (w.l.o. generality)

$\therefore$ $|a_n - a_m| = \left|\frac{1}{m+1} - \frac{1}{n+1}\right|$

$= \frac{1}{m+1} - \frac{1}{n+1}$ $\left[\because n > m \Rightarrow \frac{1}{m+1} > \frac{1}{n+1}\right]$

Let $\varepsilon > 0$. then

$|a_n - a_m| < \varepsilon$

if $\frac{1}{m+1} - \frac{1}{n+1} < \varepsilon$

i.e. if $\frac{1}{m+1} < \frac{1}{n+1} + \varepsilon$

i.e. if $\frac{1}{m+1} < \varepsilon$

i.e. if $m + 1 > \frac{1}{\varepsilon}$

i.e. if $m > \frac{1}{\varepsilon} - 1$

Let p be a positive integer just greater than $\frac{1}{\varepsilon} - 1$

$\therefore$ $|a_n - a_m| < \varepsilon \quad \forall\ n, m \geq p$

$\Rightarrow$ $\{a_n\}$ is a Cauchy sequence

## 9.7 Self Check Exercise

Q.1 Show that $\left\{\frac{(-1)^n}{n}\right\}$ is a Cauchy sequence

Q. 2 Prove that $\left\{(-1)^n\ n\right\}$ is not a Cauchy sequence

Q.3 Apply Cauchy's General principle of convergence to prove that $\{a_n\}$, $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2}$, converges.

## 9.8 Summary

In this unit we have learnt the following

(i) Cauchy sequence

(ii) Cauchy's General Principle of Convergence

(iii) Cantor-Intersection Theorem or Cantor Nested interval

(iv) Limit superior and Limit-Inferior of a sequence.

## 9.9 Glossary:

1. Periodic Sequence :A periodic sequence is a sequence for which the same terms are repeated over and over.

2. The l.u.b. of E (set of all cluser point) of a seq. $x_n$ in the extended real number system is called Limit-Superior, denoted by $\overline{\lim_{n\to\infty}}\, x_n$.

3. The g.l.b of E in the extended real number system is called Limit Inferior, denoted by $\underline{\lim}_{n\to\infty}\, x_n$.

## 9.10 Answer to Self Check Exercise

Ans.1 Prove it with the help of illustrated examples.

Ans.2 Prove it by taking help of examples.

Ans.3 Prove it by using idea of examples.

## 9.11 References/Suggested Readings

1. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

**2**. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

## 9.12 Terminal Questions

1. Prove that $\left\{\frac{1}{n^2}\right\}$ is a Cauchy sequence

2. Show that the sequence $\{a_n\}$,

   $a_n = 1 + \frac{1}{3} + \frac{1}{5} + ..... + \frac{1}{2n-1}$ is not convergent.

3. If $\{x_n\}$, $\{y_n\}$ are convergent sequences then show that

   (i) $\{x_n + y_n\}$

   (ii) $\{x_n y_n\}$, are also convergent.

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## Unit - 10

# Infinite Series

**Structure**



## 10.1 Introduction

Dear students, in this unit we shall learn the concept of infinite series. A series is roughly speaking, the operation of adding infinitely many terms, one after the other, to a given starting term. Series are used in most areas of mathematics, even for studying finite structure (such as combinatorics) through generating functions. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, Computer Science and Finance. In modern terminology, any (ordered) infinite sequence ($a_1$, $a_2$,.....) of terms defines a series which is operation of adding the ai's one after the other. To emphasize that there are infinite number of terms, a series may be called an infinite series. Such a series is represented by an expression like $a_1 + a_2 + ..... + ....$ or using summation sign $\sum_{i=1}^{\infty} a_i$ .

## 10.2 Learning Objectives

In this unit we shall study the following concepts.

(i) Definition of an infinite series

(ii) The partial sums of an infinite series

(iii) Behaviour of an infinite series

(a) Convergent Series

(b) Divergent Series

(c) Oscillatory Series

(d) Absolutely convergent Series

(e) Conditionally convergent Series

(f) Non-convergent Series

(iv) Cauchy's Criterion of convergence of a series

Or

Cauchy's General Principle of convergence of a series

(v) Infinite geometric Series.

## 10.3 Infinite Series

Let $\{a_n\}$ be a sequence of real numbers.

The expression $a_1 + a_2 + a_3 + ...... + a_n + ....$ is called an infinite series and is denoted by $\sum_{i=1}^{\infty}$ $a_n$ or by $\sum$ $a_n$ and $a_n$ is called the nth term of the series.

Partial Sums

We define

$s_1 = a_1$

$s_2 = a_1 + a_2$

$s_3 = a_1 + a_2 + a_3$

... ... ...

$s_n = a_1 + a_2 + a_3 + ..... + a_n = \sum_{1}^{n} a_n$

These s's are called the Partial Sums of the series $\sum$ $a_n$. The nth partial sum is denoted by $S_n$ or $s_n$ or $\sigma_n$.

**Behaviour of an Infinite Series**

The behaviour of the infinite series $\sum$ $a_n$ is the same as that of the sequence $\{s_n\}$. In other words, the series $\sum$ $a_n$ is said to be convergent, divergent or oscillating as the sequence $\{s_n\}$ converges, diverges or oscillates.

**(i) Convergent Series :** The series $\sum a_n$ is said to converge if the sequence $\{s_n\}$ of its partial sums converges to s and s is called the sum of the convergent infinite series $\sum_{i=1}^{\infty} a_n$ . We write $\sum_{i=1}^{\infty} a_n = s$

Note: Here $s_n \to s$ as $n \to \infty$.

**(ii)** **Divergent Series :** The series $\sum a_n$ is said to diverge to $\infty$ if the sequence $\{s_n\}$ diverges to $\infty$. We write $\sum_{i=1}^{\infty} a_n = \infty$.

The series $\sum a_n$ is said to diverge to $-\infty$ if the sequence $\{s_n\}$ diverges to $-\infty$. We write $\sum_{i=1}^{\infty} a_n = -\infty$.

**(iii)** **Oscillatory Series :** The series $\sum a_n$ is said to oscillate finitely or infinitely if the sequence $\{s_n\}$ oscillates finitely or infinitely.

In other words, if the series $\sum a_n$ neither converges nor diverges to $+\infty$ (or $-\infty$), it is said to oscillate finitely (or infinitely) if the sequence $\{s_n\}$ is bounded (or unbounded).

**(iv)** **Absolutely Convergent Series :** The series $\sum a_n$ is said to converge absolutely if the series $\sum_{i=1}^{\infty} |a_n|$ i.e. the series $|a_1| + |a_2| + |a_3| + \ldots + |a_n| + \ldots$ is convergent.

**(v)** **Conditionally Convergent Series :** The series $\sum a_n$ is said to be conditionally convergent if $\sum a_n$ is convergent but $\sum |a_n|$ is not convergent.

Note: In this case, series $\sum a_n$ converges but not absolutely. For that reason, it is also called non-absolutely convergent series or semi-convergent series.

**(vi)** **Non-Convergent Series :** Series which diverge or oscillate are said to be non-convergent.

### 10.4 Cauchy's General Principle (Cauchy's Criterion) of Convergence of a Series

Prove that a necessary and sufficient condition for the convergence of the series $\sum a_n$ is that for every $\varepsilon > 0$, however small, $\exists\ p \in \mathbb{N}$ such that $\forall\ n > m \geq p$,

$$|s_n - s_m| < \varepsilon$$

**or** $$|a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon \ \forall\ n > m \geq p$$

**Note 1:** Above result can also be stated as :

A necessary and sufficient condition for the convergence of the series $\sum a_n$ is that for every s > 0, however small, $\exists\ m \in N$ such that

$$|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon \qquad \forall\ n \geq m,\ p \in N$$

2. If $\sum a_n$ is a series whose each term is positive, then the above condition becomes

$$a_{n+1} + a_{n+2} + ..... + a_{n+p} < \varepsilon \qquad \forall\ n > m,\ p \in N$$

**Art.** Show that the geometric series $\sum_{1}^{\infty} r^{n-1} = 1 + r + r^2 + ..... + r^{n-1} + .....$

(i) converges to $\frac{1}{1-r}$ for $|r| < 1$

(ii) diverges to $\infty$ if $r \geq 1$

(iii) oscillates finitely between 0 and 1 if $r = -1$

(iv) oscillates infinitely if $r < -1$

(v) converges absolutely for $|r| < 1$.

**Example 1:** Show that the sequence $\{a_n\}$ where

$a_n = 1+\frac{1}{2} + \frac{1}{3}+......+\frac{1}{n}$ does not converge, by showing that it is not a Cauchy Sequence.

**Solution:** $a_n = 1+\frac{1}{2} + \frac{1}{3}+......+\frac{1}{n}$

$\therefore \quad a_m = 1+\frac{1}{2} + \frac{1}{3}+......+\frac{1}{m}$

Without any loss of generality, we take m > n,

$$\therefore \quad |a_m - a_n| = \left|\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{m}\right|$$

$$\text{or} \quad |a_m - a_n| = \frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{m}$$

Take m = 2n

$$\therefore \quad |a_{2n} - a_n| = \frac{1}{n+1}+\frac{1}{n+2}+.....+\frac{1}{2n}$$

$$\geq \frac{1}{2n}+\frac{1}{2n}+......+\frac{1}{2n}$$

$$\left[\begin{array}{l}\therefore\ n+1 \leq n+n, n+2 \leq n+n, ....., n+n \leq n+n \\ \because\ \frac{1}{n+1} \geq \frac{1}{n+n}, \frac{1}{n+2} \geq \frac{1}{n+n}, ......, \frac{1}{n+n} \geq \frac{1}{n+n}\end{array}\right]$$

$= \frac{n}{2n} = \frac{1}{2} \forall$ n

$\therefore$ $|a_{2n} - a_n| > \frac{1}{2} \forall$ n

$\Rightarrow$ $\{a_n\}$ is not Cauchy sequence

$\therefore$ $\{a_n\}$ does not converge.

**Example 2:** Apply Cauchy's General Principle of convergence to show that $\{a_n\}$ where $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ..... + \frac{1}{n^2}$ converges.

Solution : Here $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ..... + \frac{1}{n^2}$

$\therefore$ $a_m = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ..... + \frac{1}{m^2}$

Let n > m

$\therefore$ $|a_n - a_m| = \left| \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + .... + \frac{1}{n^2} \right|$

$= \frac{1}{(m+1)^2} + \frac{1}{(m+2)^2} + ...... + \frac{1}{n^2}$

$< \frac{1}{m(m+1)} + \frac{1}{(m+1)(m+2)^2} + ...... + \frac{1}{(n-1)n}$

$= \frac{(m+1)-m}{m(m+1)} + \frac{(m+2)-(m+1)}{(m+1)(m+2)} + ...... + \frac{n-(n-1)}{(n-1)n}$

$\left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m+1} - \frac{1}{m+2}\right) + \left(\frac{1}{m+2} - \frac{1}{m+3}\right) + ..... + \left(\frac{1}{n-1} - \frac{1}{n}\right)$

$= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}$

$\therefore$ $|a_n - a_m| = \frac{1}{m} < \varepsilon$

if $\frac{1}{m} < \varepsilon$

i.e. if $m > \frac{1}{\varepsilon}$

Let p be a positive integer just greater than $\frac{1}{\varepsilon}$

$\therefore$ $|a_n - a_m| < \varepsilon \; \forall$ n, m > p

$\therefore$ $\{a_n\}$ is a Cauchy sequence

$\therefore$ by Cauchy General Principle of convergence, $\{a_n\}$ is convergent.

**10.5 Art. Infinite Geometric Series**

Discuss the convergence of series $\sum_{i=1}^{\infty} a\,r^{n-1}$, where $a \neq 0$, $r \neq 0$ and a, r are fixed real numbers.

**Proof:** The given series is

$$\sum_{i=1}^{\infty} a\,r^{n-1} = a + ar + ar^2 + ..... + ar^{n-1} + .....$$

(i) Let $|r| < 1$

$\therefore$ $s_n = \frac{a\left(1-r^n\right)}{1-r}$

$= \frac{a}{1-r} - \frac{ar^n}{1-r}$

$\therefore$ $\underset{n\to\infty}{Lt}\, s_n = \frac{a}{1-r} - 0$

$= \frac{a}{1-r}$

$[\because \underset{n\to\infty}{Lt}\, r^n = 0$ as $|r| < 1]$

$\therefore$ $\{s_n\}$ converges and given series converges to the sum $\frac{1}{1-r}$.

(ii) Let r = 1

$\therefore$ $s_n = a + a + a + ....$ to n terms = na

$\therefore$ $s_n \to \infty$ as $n \to \infty$.

$\therefore$ $\{s_n\}$ diverges to $\infty$ and hence given series diverges to $\infty$.

Let r > 1

∴ $s_n = \dfrac{a\left(r^n - 1\right)}{r - 1}$

∴ $s_n \to \infty$ as $n \to \infty$ [∵ $r^n \to \infty$ as $n \to \infty$ for $r > 1$]

∴ $\{s_n\}$ diverges to $\infty$ and hence given series diverges to $\infty$.

(iii) Let r = -1

∴ $s_n = a - a + a + \ldots$ to n terms

$$\begin{cases} 0, \text{ if } n \text{ is even} \\ a, \text{ if } n \text{ is odd} \end{cases}$$

∴ $\{s_n\}$ oscillates finitely between 0 and a.

∴ given series oscillates finitely between 0 and a.

**Art.** If $\sum_{1}^{\infty} a_n$ is convergent, then prove that $\underset{n\to\infty}{Lt}\, a_n = 0$

**Proof:** Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$

∴ $\sum a_n$ is convergent to s (say)

∴ $\{s_n\}$ is convergent to s

⇒ $\{s_{n-1})$ is convergent to s

[∵ every subsequence of a convergent sequence is convergent to the same limit]

Now $a_n = s_n - s_{n-1}$

∴ $\underset{n\to\infty}{Lt}$ an $= \underset{n\to\infty}{Lt}\,(s_n - s_{n-1})$

$= \underset{n\to\infty}{Lt}\, s_n - \underset{n\to\infty}{Lt}\, s_{n-1}$

$= s - s$

∴ $\underset{n\to\infty}{Lt}\, a_n = 0.$

**Cor.** If the sequence $\{a_n\}$ does not converge to zero, then $\sum a_n$ does not converge.

This is so as if $\sum a_n$ converges, then $a_n \to 0$.

**Note 1.** The converse of the theorem is not true.

Consider the series $\sum a_n = \sum \dfrac{1}{n}$

Let $s_n = 1 + \frac{1}{2} + \frac{1}{3} + ..... + \frac{1}{n}$

$\therefore$ $|s_{2n} = s_n| = \left|\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}\right|$

$$= \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}$$

$$> \frac{1}{2n} + \frac{1}{2n} + ... + \frac{1}{2n}$$

$$= \frac{n}{2n} = \frac{1}{2} \forall n$$

$\therefore$ $\{s_n\}$ is not a Cauchy sequence and is not convergent.

Also $s_{n+1} - s_n = \frac{1}{n+1} > 0 \ \forall n$

$\therefore$ $\{s_n\}$ is monotonic increasing sequence.

$\Rightarrow$ $\{s_n\}$ is divergent.

$\therefore$ $\sum_{1}^{\infty} \frac{1}{n}$ is divergent.

But $\underset{n\to\infty}{Lt}\, a_n = \underset{n\to\infty}{Lt} = \frac{1}{n}\ 0$

**Note 2.** A necessary condition for the convergence of the series $\sum a_n$ is that $a_n \to 0$. The condition is not sufficient.

**Art.** Prove that a positive terms series (i.e., a series in which all the terms ae positive) either converges or diverges to $+\infty$.

**Proof :** Let $\sum a_n$ be a positive terms series i.e. $a_n > \forall n$

Let $s_n = a_1 + a_2 + .... + a_n$

Since all the a's are positive, therefore, $s_n$ is positive.

Now $s_{n+1} = s_n + a_{n+1}$

$\therefore$ $s_{n+1} - s_n = a_{n+1} > 0$

$\Rightarrow$ $s_{n+1} > s_n \forall n$

$\therefore$ $\{s_n\}$ is a monotonically increasing sequence.

Now there are two possibilities:

(i) If the increasing sequence $\{s_n\}$ is bounded above, then $\{s_n\}$ converges and so $\sum a_n$ converges.

(ii) If the increasing sequence $\{s_n\}$ is not bounded above, then $\{s_n\}$ diverges to $+\infty$ and so $\sum a_n$ diverges to $+\infty$.

$\therefore$ a positive terms series either converges or diverges to $+\infty$.

**Cor.** If $a_{n>0} \forall$ n and $a_n$ does not tend to zero, then the positive $\sum a_n$ diverges to $+\infty$.

**Proof :** Since $\sum a_n$ is a positive terms series.

$\therefore$ $\sum a_n$ either converges or diverges to $+\infty$

$\because$ $a_n$ does not tend to zero

$\therefore$ $\sum a_n$ does not converges

$\Rightarrow$ $\sum a_n$ diverges to $+\infty$.

**Art :** Let $a_n > 0$ and $s_n = a_1 + a_2 + .....+a_n$. Show that a necessary and sufficient condition for the convergence (or divergence) of the positive terms series $\sum a_n$ is that sequence $\{s_n\}$ is bounded above (or not bounded above).

**Proof :** Here $\{s_n\}$ is a strictly increasing sequence.

$\therefore$ $\sum a_n$ converges (or diverges) iff the increasing sequence $\{s_n\}$ converges (or diverges) or iff the sequence $\{s_n\}$ is bounded above (or not bounded above).

**Note.** (i) The positive terms series $\sum a_n$ converges iff there exists a positive constant k such that $s_n < k \ \forall$ n.

(ii) The positive terms series $\sum a_n$ diverges iff given a positive constant $\Delta$, however large, $\exists$ m $\in$ N such that $s_n > \Delta \forall$ n $\geq$ m.

**Art. :** If $\sum a_n$ and $\sum b_n$ converges respectively to $\sigma$ and $\tau$and$\lambda$, $\mu$ are real numbers, then $\sum_{n=1}^{\infty} (\lambda a_n + \mu b_n)$ converges to $\lambda\sigma + \mu\tau$.

**Proof :** Let $\sigma_n = a_1 + a_2 +..... + a_n$

and $\tau_n = b_1 + b_2 +..... + b_n$

Again let $s_n = (\lambda a_1 + \mu b_1) + (\lambda a_2 + \mu b_2) +.... + (\lambda a_n + \mu b_n)$

$= \lambda\sigma_n + \mu\tau_n$

Since $\sum a_n$ converges to $\sigma$, $\therefore \sigma_n \to \sigma$

Since $\sum b_n$ converges to $\tau$, $\therefore \tau_n \to \tau$

$\therefore$ $s_n = \lambda\sigma_n + \mu\tau_n \to \lambda\sigma + \mu\tau$

$\therefore$ $\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n)$ converges to $\lambda\sigma + \mu\tau$.

**Cor 1.** Putting $\lambda = \mu = 1$, we see that if $\sum a_n$ and $\sum b_n$ converges respectively to $\sigma$ and $\tau$, then $a_1 + b_1 + a_2 + b_2 + ....$ converges to $\sigma + \tau$.

**Note 1.** From above, it is clear that sum of two convergent series is convergent.

**Cor 2.** Putting $\lambda = 1$, $\mu = -1$, we see that difference of two convergent series is convergent.

**Art.** Let $\sum a_n$ and $\sum b_n$ be two series. Suppose that $\exists$ m $\in$ N and an integer $p \geq 0$ such that $b_n = a_{n+p} \forall$ $n \geq m$. Then the two series $\sum a_n$ and $\sum b_n$ behave alike.

Or

Prove that the behaviour of a series is not changed by the removal, addition or alternation of a finite number of its terms.

**Proof :** Let $\sigma_n = a_1 + a_2 + ..... + a_n$

and $\tau_n = b_1 + b_2 + ..... + b_n$

$\therefore$ for $n \geq m$,

$\tau_n = (b_1 + b_2 + ..... + b_m) + (b_{m+1} + ..... + b_n)$

$= \tau_m + (a_{m+p+1} + ..... + a_{n+p})$, as $b_n = a_{n+p}$

$= \tau_m + (a_1 + ....+a_{m+p} + a_{m+p+1} + ..... + a_{n+p})-$ , as $b_n = (a_1 + ....+ a_{m+p})$

$= \tau_m + \sigma_{m+p} - \sigma_{m+p}$

$= \sigma_{n+p} + (\tau_m - \sigma_{m+p})$

$= \sigma_{n+p} + K$

where $k = \tau_m - \sigma_{m+p}$ is independent of n and hence a constant.

$\therefore$ the sequence $\{\sigma_n\}$ and $\{\tau_n\}$ of the partial sums of the series $\sum a_n$ and $\sum b_n$ behave alike.

$\therefore$ the two series $\sum a_n$ and $\sum b_n$ behave alike.

**Art.** If $\sum_{n=1}^{\infty} a_n$ converges to the series σ and $\{n_k\}$ is a strictly increasing sequence of natural numbers, then the series $(a_1 + ...... + a_n) + (a_{n_1} + .... + a_{n_2}) + ...$ also converges to a.

**Proof :** Let $\sigma_k$ and $\tau_k$ denote the kth partial sums of the series $\sum a_n$ and the new series $(a_1 + ...... + a_n) + (a_{n_1} + .... + a_{n_2}) + ...$

∴ $\{\tau_k\}$ being a subsequence of $\{\sigma_n\}$ also converges to σ.

∴ new series also converges to σ.

**Note.** the introduction of parentheses (brackets) in a convergent series does not affect its convergence.

the removal of parentheses may affect the convergence.

**For example :** The series (1 - 1) + (1 - 1) +(1 - 1) + ... is convergent, but the series obtained after removing the parentheses i.e. 1 - 1 + 1 - 1 + ... oscillates finitely.

**Art.** Prove that convergent series $\sum a_n$ of a positive terms remains convergent when each term $a_n$ is multiplied by a factor $\nu_n$ whose numerical value does not exceed a positive constant k.

**Proof :** Since $\sum a_n$ is a positive terms convergent series.

∴ by Cauchy's Criterion of convergence of a series, given ε > 0, however small, there exists a positive integer m such that

$$a_{m+1} + a_{m+2} + .... + a_n < \frac{\varepsilon}{2} \ \forall\ n \geq m \qquad ...(1)$$

∴ $|a_{m+1}\nu_{m+1} + a_{m+2}\nu_{m+2} + .... + a_n\nu_n|$

$\leq |a_{m+1}\nu_{m+1}| + |a_{m+2}\nu_{m+2}| + .... + |a_n\nu_n|$

$= |a_{m+1}||\nu_{m+1}| + |a_{m+2}||\nu_{m+2}| + .... + |a_n||\nu_n|$

$\leq k|a_{m+1}| + k\,|a_{m+2}| + .... + k\,|a_n|$ [$\because |\nu_n|\ k\ \forall\ n$]

$= k\,(|a_{m+1}| + |a_{m+2}| + .... + |a_n|)$

$= k\,(a_{m+1} + a_{m+2} + .... + a_n)$ [$\because |a_n| = a_n$ as $a_n > 0\ \forall\ n$]

$= k \cdot \frac{\varepsilon}{k}$

$= \varepsilon \forall\ n \geq m$

∴ $\sum a_n\, \nu n$ is convergent.

**Art.** Prove that a positive terms series $\sum a_n$ is divergent if $a_n > \delta\ \forall\ n$ where δ is a finite positive constant.

**Proof :** Let $s_n = a_1 + a_2 + \ldots + a_n > \delta + \delta + \ldots + \delta = n.\delta$

$\therefore$ $s_n > n\delta$

$\Rightarrow$ $s_n \to \infty$ as $n \to \infty$

$\therefore$ $\sum a_n$ is divergent

**Art.** If k is a fixed number and $\sum a_n$ converges to the sums s, then $\sum ka_n$ converges to the sum ks. Also if $\sum a_n$ diverges or oscillates, so does $\sum ka_n$ unless k = 0.

**Proof :** Let $s_n = a_1 + a_2 + \ldots + a_n$

If $\sum a_n$ is convergent, then $s_n \to$ as $n \to \infty$

$\therefore$ $Ks_n \to ks$

$\Rightarrow$ $\sum ka_n$ converges to ks.

If $k \neq 0$ and $s_n \to \pm\infty$, so does $ks_n$

$\therefore$ If $\sum a_n$ diverges, then so does $\sum ka_n$

If $s_n$ tends to no fixed limit, finite or infinite, so does $ks_n$

$\therefore$ if $\sum ka_n$ oscillates, so does $\sum ka_n$

**Example 3:** Show that the series $1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{n-1}}+\ldots$ converges to the sum 2.

**Solution:** The given series is $1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots+\frac{1}{2^{n-1}}+\ldots$

$\therefore$ $s_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots\ldots$ to n terms

$$= \frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}}$$

$$= 2\left(1-\frac{1}{2^n}\right)$$

$$2 - \frac{1}{2^{n-1}}$$

$\therefore$ $\underset{n\to\infty}{Lt}$ $s_n$ = $\underset{n\to\infty}{Lt}\left(2-\frac{1}{2^{n-1}}\right)$

= 2 - 0 = 2 $\left[\because \underset{n\to\infty}{Lt}\frac{1}{2^{n-1}}=0\right]$

$\therefore$ $\{s_n\}$ converges to 2

$\therefore$ given series $1+\frac{1}{2}+\frac{1}{2^2}+......+\frac{1}{2^{n-1}}+....$ converges to the sum 2.

**Example 4:** Show that $\frac{1}{n(n+1)}$ = 1

**Solution:** The given series is $\sum_{n=1}^{\infty} a_n$

where $a_n = \frac{1}{n(n+1)}$

$= \frac{1}{n(0+1)} + \frac{1}{(-1)(n+1)}$

$\therefore$ $\frac{1}{n} - \frac{1}{n+1}$

Putting n = 1, 2, 3,....., n

$a_1 = \frac{1}{1} - \frac{1}{2}$

$a_2 = \frac{1}{2} - \frac{1}{3}$

$a_3 = \frac{1}{3} - \frac{1}{4}$

... ... ...

... ... ...

... ... ...

... ... ...

$a_n = \frac{1}{n} - \frac{1}{n+1}$

Adding, we have

$s_n = 1 - \frac{1}{n+1}$

$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\frac{1}{1-n+1}\right)$

= 1 - 0

= 0

$\Rightarrow \quad \{s_n\}$ converges to 1

$\therefore$ the series {an converges to the sum 1.

$\therefore \quad \frac{1}{n(n+1)} = 1$

**Some More Illustrated Examples**

**Example 5:** Show that the series

$1^2 + 2^2 + 3^2 + ..... + n^2 + ......$

diverges to + ∞

**Solution:** The given series is

$1^2 + 2^2 + 3^2 + ..... + n^2 + ......$

$\therefore \quad s_n = 1^2 + 2^2 + 3^2 + ..... + n^2 + ......$

= sum of squares of first n natural numbers

$= \frac{n(n+1)(2n+1)}{6}$

$\therefore \quad \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6}$

$\to +\infty$

$\therefore \quad \{s_n\}$ diverges to + ∞

Hence the given series $1^2+2^2+.....+n^2+......$ diverges to + ∞.

**Example 6:** show that the series

$- 1^2 - 2^2 - 3^2 - ..... - n^2 - ......$

diverges to - ∞.

**Solution:** We have the given series as

$- 1^2 - 2^2 - 3^2 - ..... - n^2 - ......$

$\therefore \quad s_n = - 1^2 - 2^2 - 3^2 - ..... - n^2 - ......$

$= -(1^2 + 2^2 + ..... + n^2)$

$= -\left\{\frac{n(n+1)(2n+1)}{6}\right\}$

$\to -\infty$ as $n \to \infty$

$\therefore$ the sequence $\{s_n\}$ diverges to $-\infty$

Hence the given series

$-1^2 - 2^2 - 3^2 - ..... - n^2 - ......$

diverges to $-\infty$

**Example 7:** Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, \ p < 1 \text{ is divergent}$$

**Solution:** The given series is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ........+ \frac{1}{n^p}$$

Now

$$|s_{2n} - s_n| = \frac{1}{(n+1)^p} + \frac{1}{(n+2)^p} + ....... \frac{1}{(2n)^p}$$

$$\geq \frac{1}{n+1} + \frac{1}{n+2} + ......+ \frac{1}{2n} \qquad (\because \ p \leq 1)$$

$$> \frac{1}{2n} + \frac{1}{2n} + .......+ \frac{1}{2n}$$

$$= \frac{n}{2n} = \frac{1}{2} \qquad \forall \ n$$

$\therefore$ $|s_{2n} - s_n| > \frac{1}{2} \quad \forall \ n$

$\therefore$ $\{s_n\}$ is not a Cauchy sequence and hence not convergent

Also $s_{n+1} - s_n = \frac{1}{(n+1)^p} > 0 \ \forall \ n$

$\therefore$ $s_{n+1} > s_n \ \forall \ n$

$\Rightarrow$ $\{s_n\}$ is monotonically increasing sequence

∴ $\{s_n\}$ is divergent $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$, p ≤ 1 is divergent.

Now you can try the following exercises

## 10.6 Self Check Exercise

Q.1 Show that the series

$\sum (-1)^{n-1}$ oscillates finitely

Q. 2 Use Cauchy's Criterion to show that the series $\sum \frac{1}{2n-1}$ diverges

Q.3 Show that the series

$\frac{2}{1} + \frac{3}{2} + \frac{4}{3} +$....... is not convergent.

Q.4 Show that the series $\sum \frac{n(n+1)}{(n+2)^2}$ cannot converge.

## 10.7 Summary

In this unit we have learnt the following

(i) Infinite Series and its Partial Sums

(ii) Behaviour of an Infinite Series

(iii) Cauchy's Criterion of Convergence of a Series

(iv) Infinite Geometric Series

## 10.8 Glossary:

1. Absolutely Convergent Series -

The series $\sum a_n$ $|a_n| = |a_1| = |a_2| +.....+ |a_n| +......$ is convergent.

2. Non-Convergent Series -

The Series which diverge or oscillate are called non-convergent Series.

## 10.9 Answer to Self Check Exercise

Ans.1 Here $s_n = \begin{cases} 0, & n \text{ even} \\ 1, & n \text{ odd} \end{cases}$

Ans.2 Here $s_n = 1 + \frac{1}{3} + ....... + \frac{1}{2n-1}$

Find $|s_{2n+1} - s_n|$. and then prove.

Ans.3 Here $a_n = \frac{n+1}{n} = 1 + \frac{1}{n}$. Now prove.

Ans.4 Here $a_n = \frac{n(n+1)}{(n+2)^2} = \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$ Find Limit and proceed to prove.

## 10.10 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983
3. R.G. Bartle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000
4. K.A. Ross, Elementary Analysis - The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag, 2003.


## 10.11 Terminal Questions

1. Prove that the series

$$\sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \ldots\ldots + \sqrt{\frac{n}{n(n+1)}} + \ldots\ldots$$ is not convergent.

2. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$ converges.

3. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$. Justify your answer.

4. Show that the series

$$\sum \left(\frac{n}{n+1}\right)^n$$ is divergent.

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## Unit - 11

# Alternating Series

**Structure**



### 11.1 Introduction

Dear students, in this unit we shall study the concept of alternating Series. In mathematics, the alternating test is the method used to show that an alternating series is convergent when (1) etc term decrease in absolute value and (2) approach to zero in the limit. The test was used by Gottfried Leibnitz and some times known as Leibnitz Test. This test is only sufficient, not necessary, so some convergent series may fail the first part of test.

### 11.2 Learning Objectives

The main objectives of this unit are

(i) to define an alternating series

(ii) to test the convergence of a series by Leibnitz Test or by an alternating Series test.

### 11.3 Alternating Series

A series whose terms are alternatively positive and negative, is called an alternating series.

For example, $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ......$ is an alternating series.

### 11.4 Leibnitz (Or Alternating Series) Test

If $\{a_n\}$ is monotone decreasing sequence of positive terms and converges to zero, then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent.

**Proof:** Here $\{a_n\}$ is a monotone decreasing sequence of positive terms converging to zero.

$\therefore$ we have

(i) $a_n > 0 \ \forall \ n$

(ii) $a_n \geq a_{n+1} \quad \forall \ n$ and

(iii) $a_n \to 0$ as $n \to \infty$

Let $\{s_n\}$ be the sequence of partial sums of $\sum (-1)^{n-1} a_n$.

$\therefore$ $s_{2n} = (a_1 - a_2) + (a_3 - a_4) + \ldots\ldots + (a_{2n-1} - a_{2n})$

$\Rightarrow$ $s_{2n} \geq 0 \ \ \forall \ n$ .....(1)

[$\because$ content of each bracket is $\geq 0$]

Again $s_{2n} = a_1 - [(a_2 - a_3) + (a_4 - a_5) + \ldots\ldots + (a_{2n-2} - a_{2n-1}) + a_{2n}]$

$\therefore$ $s_{2n} < a_1 \ \forall \ n$ ..... (2)

From (1) and (2), we get

$0 \leq s_{2n} < a_1 \quad \forall \ n$

Also $s_{2n+2} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0$

$\therefore$ $s_{2n+2} \geq s_{2n} \quad \forall \ n$

$\Rightarrow$ $\{s_{2n}\}$ is an increasing sequence

Also $\{s_{2n}\}$ is bounded above

$\therefore$ $\{s_{2n}\}$ is convergent

Let $s_{2n} \to s$ as $n \to \infty$

Now $s_{2n+1} = s_{2n} + a_{2n+1}$

$\Rightarrow$ $Lt \ s_{2n+1} = Lt \ s_{2n} + Lt \ a_{2n+1}$

$\Rightarrow$ $Lt \ s_{2n+1} = s + 0$ [$\because$ $a_n \to 0$ as $n \to \infty$]

$\Rightarrow$ $Lt \ s_{2n+1} = s$

Now $s_{2n} \to s$ as $n \to \infty$ and also $s_{2n+1} \to s$ as $n \to \infty$

$\therefore$ $\underset{n \to \infty}{Lt} s_n = s$

$\therefore$ given series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges to s.

**Note: Another Form :** If the alternating series

$u_1 - u_2 + u_3 - u_4 + \ldots\ldots \ (u_n > 0 \ \forall n \in N)$

is such that

(i) $u_{n+1} \leq u_n \quad \forall$ n, and (ii) $\underset{n\to\infty}{Lt}\ u_n = 0$

then the series converges.

**Cor. 1.** We have

$0 \leq s_{2n} \leq a_1$

$\Rightarrow \quad 0 \leq \underset{n\to\infty}{Lt}\ s_{2n} \leq a_1 \qquad [\because$ of squeeze principle]

$\Rightarrow \quad 0 \leq s \leq a_1$

$\therefore$ sum of the convergent series $\sum_{n=1}^{\infty} (-1)^n a_n$ lies between 0 and $a_1$ (both inclusive)

**Cor. 2 :** If $a_n \to a \neq 0\ \forall$ n, then $\sum (-1)^n a_n$ oscillates finitely.

**Proof:** We have

$s_{2n+1} = s_{2n} + a_{2n+1}$

$\Rightarrow \quad Lt\ s_{2n+1} = Lt\ s_{2n} + Lt\ a_{2n+1}$

$\Rightarrow \quad Lt\ s_{2n+1} = s + a \neq s$

$\Rightarrow \quad Lt\ s_{2n+1} = Lt\ s_{2n}$

$\therefore$ $\{s_n\}$ has two subsequential limits s and s + a which are finite. Hence $\{s_n\}$ oscillates finitely.

$\therefore$ $\sum (-1)^n a_n$ oscillates finitely between s and s + a which differ by a.

**Art.** If the series $\sum a_n$ is absolutely convergent, then prove that $\sum a_n$ is convergent. Is its converse true?

**Proof :** Since $\sum a_n$ is absolutely convergent.

$\therefore$ $\sum |a_n|$ is convergent

$\therefore$ by Cauchy's Criterion for convergence of a series, given $\varepsilon > 0$, however small, $\exists$ $m \in N$ such that

$|a_{n+1}| + |a_{n+2}| + .... + |a_{n+p}| < \varepsilon\ \forall\ n \geq m$ and $\rho \in N$

Now $|a_{n+1} + a_{n+2} + .... + a_{n+p}| \leq |a_{n+1}| + |a_{n+2}| + .... + |a_{n+p}|$

$< \varepsilon\ \forall\ n \geq m$ and $\rho \in N$

$\therefore$ by Cauchy's Criterion for convergence of series, $\sum a_n$ is convergent.

$\therefore$ if $\sum a_n$ is absolutely convergent, then $\sum a_n$ is convergent.

The converse is not true.

Let $a_n = (-1)^{n-1}\frac{1}{n}$

$\therefore$ $\sum |a_n| = \sum \frac{1}{n}$ is not convergent (Prove it)

Now we will prove that $\sum (-1)^{n-1}\frac{1}{n}$ is convergent

Comparing $\sum (-1)^{n-1}\frac{1}{n}$ with $\sum (-1)^{n-1} a_n$, we get

$a_n = \frac{1}{n}$

$\therefore$ we have

(i) $a_n > a_{n+1} > 0 \ \forall \ n$ $\left[\because \frac{1}{n} > \frac{1}{n+1} > 0\right]$

$\therefore$ $\{a_n\}$ is a monotonic decreasing sequence.

(ii) $a_n = \frac{1}{n} \to 0$ as $n \to \infty$

$\therefore$ by Leibnitz Test, $\sum (-1)^n \frac{1}{n}$ converges.

$\therefore$ $\sum a_n$ converges but $\sum |a_n|$ does not converge.

**Art.** If $\sum a_n$ is absolutely convergent and $\{\lambda_n\}$ is bounded sequence, then prove that $\sum \lambda_n a_n$ is absolutely convergent.

**Proof :** Since $\{\lambda_n\}$ is bounded

$\therefore$ there exists a real constant $\lambda > 0$ such that $|\lambda_n| < \lambda \ \forall \ n$ ....(1)

Now $\sum a_n$ converges absolutely

$\Rightarrow$ $|\sum a_n|$ is convergent

$\therefore$ by Cauchy's Criterion of convergence of series, given $\varepsilon > 0$, however small, $\exists \ m \in N$ such that

$|a_{n+1}| + |a_{n+2}| + .... + |a_{n+p}| < \frac{\varepsilon}{\lambda} \ \forall \ n \geq m$ and $\rho \in N$ ...(2)

$\therefore$ $|\lambda_{n+1}\, a_{n+1}| + |\lambda_{n+2}\, a_{n+2}| + .... + |\lambda_{n+p} a_{n+p}|$

$= |\lambda_{n+1}||a_{n+1}| + |\lambda_{n+2}||a_{n+2}| + .... + |\lambda_{n+p}||a_{n+p}|$

$< \lambda|a_{n+1}| + \lambda|a_{n+2}| + .... + \lambda|a_{n+p}|$ [$\because$ of (1)]

$= \lambda(|a_{n+1}| + |a_{n+2}| + .... + |a_{n+p}|) < \lambda . \frac{\varepsilon}{\lambda}$ [$\because$ of (2)]

$= \varepsilon \forall\ n > m$ and $p \in N$

$\therefore$ by Cauchy's Criterion for convergence of a series $\sum |\lambda_n a_n|$ converges i.e. $\sum \lambda_n a_n$ converges absolutely.

**Note.** the absolute converges of $\sum a_n$ is a must for the validity of the above theorem.

**Example.** $\sum (-1)^{n-1} \frac{1}{n}$ is convergent but not absolutely.

Also $\left\{(-1)^{n-1}\right\}$ is bounded.

But $\sum (-1)^{n-1} . (-1)^{n-1} \frac{1}{n} = \sum \frac{1}{n}$ is divergent and not convergent.

**Some Illustrated Examples**

**Example 1** : Show that the alternative series $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ is convergent.

**Solution :** The given alternating series is $\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...$ ....(1)

Comparing $\sum \frac{(-1)^{n-1}}{n}$ with $\sum (-1)^{n-1}$, we get,

$a_n = \frac{1}{n}$

$\therefore$ we have

(i) $a_n > 0\ \forall\ n$

(ii) $a_n > a_{n+1} \forall\ n$

(iii) $a_n = \frac{1}{n} \to$ as $n \to \infty$

$\therefore$ by Leibnitz's Test, series (1) is convergent.

**Example 2 :** Use Leibnitz's Test to show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n (n+5)}{n(n+1)}$ converges.

**Solution :** The given series is $\sum_{n=1}^{\infty}$ $(-1)^n$ $a_n$ = $\sum_{n=1}^{\infty}$ $(-1)^{n-1}$ $a_n$

Where $a_n = \frac{(n+5)}{n(n+1)}$

$\therefore$ we have

(i) $a_n > 0 \ \forall\ n$

(ii) $a_{n+1} - a_n = \frac{n+5}{(n+1)(n+2)} - \frac{n+5}{n(n+1)}$

$$= \frac{n(n+6)-(n+2)(n+5)}{n(n+1)(n+2)}$$

$$= \frac{n^2+6-n^2+7n-10}{n(n+1)(n+2)}$$

$$= \frac{n-10}{n(n+1)(n+2)} < 0 \ \forall\ n$$

$\therefore$ $a_{n+1} < a_n \forall\ n$

(iii) $\underset{n\to\infty}{Lt}\ a_n = \underset{n\to\infty}{Lt}\ \frac{n+5}{n(n+1)}$

$$= \underset{n\to\infty}{Lt}\ \frac{\frac{1}{n}+\frac{5}{n^2}}{1+\frac{1}{n}}$$

$$= \frac{0+0}{1+0} = \frac{0}{1} = 0$$

$\therefore$ by Leibnitz test, series $\sum$ $(-1)^{n-1}a_n$ converges.

$\therefore$ $\sum_{n=1}^{\infty}$ $(-1)^{n-1}$ $a_n$ i.e. $\sum_{n=1}^{\infty}$ $(-1)^n$ $a_n$ converges.

**Example 3 :** Show that the series

$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} +$..... is convergent.

**Solution :** The given series is

$\sum (-1)^{n-1}a_n = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + .....$

where

$a_n = \frac{1}{\sqrt{n}}$

We therefore have

(i) $a_n > 0 \ \forall \ n$

(ii) Since $\sqrt{n+1} > \sqrt{n}$

$\Rightarrow \quad \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \ \forall \ n$

$\therefore \quad a_{n+1} < a_n \forall \ n.$

(iii) an - $\frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$

$\therefore$ by Leibnitz test, the given series is convergent.

**Example 4 :** Prove that the series.

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + .....$ is convergent.

**Solution** : The given series is

$\sum (-1)^{n-1}a_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + .....$

Where

$a_n = \frac{1}{\text{nth term of A.P. } 1, 3, 5, ...}$

$= \frac{1}{1+(n-1)2} = \frac{1}{2n-1}$

$\therefore$ We have

(i) $a_n > 0 \ \forall \ n$

(ii) Since $2n + 1 > 2n - 1$

$\Rightarrow \quad \frac{1}{2n+1} < \frac{1}{2n-1} \ \forall \ n$

$\therefore \quad a_{n+1} < a_n \forall \ n.$

(iii) $a_n - \frac{1}{2n-1} \to 0$ as $n \to \infty$

∴ by Leibnitz test, the given series is convergent.

**Example 5 :** Prove by Leibnitz Test, the given series

$$\frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots \text{ is convergent.}$$

**Solution** : The given series is

$$\sum (-1)^{n-1} a_n = \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \ldots$$

Where $a_n = \frac{1}{\log(n+1)}$

∴ We have

(i) $a_n > 0 \ \forall \ n$

(ii) Since $\log(n+2) > \log(n+1)$

$$\Rightarrow \quad \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)} \ \forall \ n$$

∴ $a_{n+1} < a_n \forall \ n$.

(iii) $a_n = \frac{1}{\log(n+1)} \to 0$ as $n \to \infty$

Hence by Leibnitz test, the given series is convergent.

**Example 6 :** Prove that the given series $= 1 - \frac{1}{2p} + \frac{1}{3p} + \frac{1}{4p} + \ldots,\ p > 0$ is convergent.

**Solution :** We have

$$\sum (-1)^{n-1} a_n = 1 - \frac{1}{2p} + \frac{1}{3p} + \frac{1}{4p} + \ldots,\ p > 0$$

Where $a_n = \frac{1}{np}$

∴ we have

(i) $a_n = \frac{1}{n^{\rho}} > 0 \forall \ n$

(ii) Since $(n+1)^{\rho} > n^{\rho}$

$\Rightarrow \qquad \frac{1}{(n+1)^\rho} < \frac{1}{n^\rho}\ \forall$ n

$\Rightarrow \qquad a_{n+1} < a_n \forall$ n

(iii) $\quad a_n = \frac{1}{n^\rho} \rightarrow 0$ as $n \rightarrow \infty$

Hence by Leibnitz's Test, the given series is convergent.

**Example 7 :** Show that the series

$$\frac{\sum(-1)^{n-1}}{Log^n}$$ is convergent

**Solution:** The given series is

$$\frac{\sum(-1)^{n-1}}{Log^n}$$

Now, We have

$$\sum(-1)^{n-1} a_n = \frac{\sum(-1)^{n-1}}{Log^n}$$

Here $\quad a_n = \frac{1}{\log n}$, n > 2

(i) $\quad a_n = \frac{1}{\log n} > 0\ \forall$ n > 2

(ii) $\quad$ Since log (n + 1) > log n

$\Rightarrow \qquad \frac{1}{\log(n+1)} < \frac{1}{\log n}$

$\Rightarrow \qquad a_{n+1} < a_n \forall$ n > 2

(iii) $\quad a_n = \frac{1}{\log n} \rightarrow 0$ as $n \rightarrow \infty$

$\therefore$ by Leibnilz Test $\sum_{n=2}^{\infty}(-1)^n$ an is convergent

**Example 8:** Prove that the series $\sum(-1)^{n-1} . \frac{n}{5^n}$ is convergent

**Solution:** The given series is $\sum(-1)^{n-1} a_n$, where $a_n = \frac{n}{5^n} > 0 \quad \forall$ n

(i) $a_n = \frac{n}{5^n} > 0 \quad \forall$ n

(ii) $\frac{d}{d^n} a_n = \frac{d}{d^n}\left(\frac{n}{5^n}\right) = \frac{5^n.1 - n}{5^n} 5^n \log n = \frac{1 - n\log 5}{5^n} < 0 \;\forall$ n

⇒ $a_n$ is a decreasing function of n

⇒ $a_n > a_{n+1} \quad \forall$ n

(iii) $\underset{n\to\infty}{Lim}$ an = $\underset{n\to\infty}{Lim} \frac{n}{5^n}$

$= \underset{n\to\infty}{Lim} \frac{1}{5^n \log n}$ (use of L'stospital Rule)

$\to 0$ ($\because$ $5^n \to \infty$ n $\to \infty$)

∴ by Leibnitz let the series $\sum(-1)^n \frac{n}{5^n}$ is convergent.

**Example 9:** Show that the series

$$\sum(-1)^{n-1} \frac{n+1}{n} = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + .........$$

oscillates finitely.

**Solution:** The given series is $\sum(-1)^{n-1} a_n$,

where $a_n = \frac{n+1}{n} = 1 + \frac{1}{n} \quad \forall$ n

∴ we have

(i) $a_n > 0 \quad \forall$ n

(ii) ∵ n + 1 > n

⇒ $\frac{1}{n+1} < \frac{1}{n}$

⇒ $1 + \frac{1}{n+1} < 1 + \frac{1}{n} \quad \forall$ n

⇒ $a_{n+1} < a_n \quad \forall$ n

(iii) $a_n = 1 + \frac{1}{n} \to 1 \neq 0$ as $n \to \infty$

∴ the given series oscillates finitely.

**Example 10:** Show that the series

$$\frac{1}{Log\,2} - \frac{1}{Log\,3} + \frac{1}{Log\,4} - \frac{1}{Log\,5} + \text{..... converges conditionally.}$$

**Solution:** The given series is

$$\frac{1}{Log\,2} - \frac{1}{Log\,3} + \frac{1}{Log\,4} - \frac{1}{Log\,5} + \text{.....} \qquad \text{....(1)}$$

The Series (1) can be written as $\frac{\sum(-1)^n}{Log^n}$

Comparing it with $\sum_{n=2}^{\infty} (-1)^n$, $u_n$, $u_n = \frac{1}{Log\,n}$

Now $\frac{d}{d^n} u_n = n = \frac{d}{d^n}\left(\frac{1}{1og\,n}\right)$

$$= \frac{\log n.0 - \frac{1}{n}.1}{(Log\,n)^2}$$

$$= - \frac{1}{n(Log\,n)^2} < 0 \qquad \forall\ n > 2$$

⇒ $u_n$ is a monotonically decreasing sequence

Also $\lim_{n\to\infty} un = \lim_{n\to\infty} \frac{1}{\log n} = 0$

∴ by Leibnitz Test, the given series is convergent

Now $|(-1)^n un| = un = \frac{1}{\log n} > \frac{1}{n}$ $\qquad (\because \log n < n\ \forall n \geq 2)$

Now, the series $\sum \frac{1}{n}$ is divergent and so $\sum\left|(-1)^n u_n\right|$ $u_n$ is divergent i.e. the given series is not absolutely convergent.

Hence the given series is conditionally convergent.

## 11.5 Self Check Exercise

Q.1 Show that the series

$$\frac{1}{2^3} - \frac{1+2}{3^3} + \frac{1+2+3}{4^3} - \frac{1+2+3+4}{5^3} + \ldots \text{ is convergent.}$$

Q. 2 Show that the series, is convergent 1 - $\frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ +......, is convergent

Q.3 Show that the series

$\sum(-1)^{n-1}\left(\frac{\log n}{n^2}\right)$ is not convergent.

Q.4 Show that if $\sum a_n$ is convergent and $\sum b_n$ is divergent, then $\sum(a_n + b_n)$ is divergent.

## 11.6 Summary

In this unit we have learnt the following concepts.

(i) Alternating Series

(ii) To Test the conconvergence of a series by Leibnitz Test or by Alternating Series Test.

## 11.7 Glossary:

1. Alternating Series - A series whose terms are alternatively positive and negative, is called an alternating series.

   For example 1 - $\frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ + ........

2. Anther form of convergent series -

   If $u_1 - u_2 + u_3$ .......... ($u_n > 0 \ \forall \ n \in N$) is an alternating series s.t.

   (i) $u_{n+1} < u_n \forall \ n$

   (ii) $\lim_{n\to\infty} u_n = 0$ then the given series is convergent.

## 11.8 Answer to Self Check Exercise

Ans.1 Hint. an = $\frac{1+2+3+\ldots\ldots+n}{(n+1)^3} = \frac{n(n+1)}{2(n+1)^3} = \frac{n}{2(n+1)^2}$

Now proceed to prove the convergence.

Ans.2 Prove it (easy to prove)

Ans.3 Here an = $\frac{\log n}{n^2}$, an+1 an = $\frac{\log n}{n^2}$

$$\left(\because \frac{d}{dx}\frac{\log x}{x^2}\right) = \frac{1-2\log x}{x^3} = < 0 \qquad \forall \ x > \sqrt{e}$$

Ans.4 Easy to prove

## 11.9 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. R.G. Bartle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000
3. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983
4. K.A. Ross, Elementary Analysis - The Theory of Calculus Series - Undergraduate Texts in Mathematics, Springer Verlag, 2003.


## 11.10 Terminal Questions

1. Prove that the series

$$\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+.......+\sqrt{\frac{n}{2(n+1)}}$$ is not convergent.

2. Show that the series

$$\frac{1}{xy} - \frac{1}{(x+1)(y+1)} + \frac{1}{(x+2)(y+2)} ........$$, x > 0, y > 0 is convergent.

3. Prove that the series

$$\sum \frac{(-1)^{n-1}}{\sqrt{n+\sqrt{a}}}$$, $a \geq 0$

4. Show that the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{n}{2n-1}$$, osculates finitely.

5. Show that if u1, u2........un..... is a decreasing sequence of positive terms tending to zero, then the series

$$\sum (-1)^{n-1} \left\{\frac{u_1+u_2+......+u_n}{n}\right\}$$ is convergent.

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## Unit - 12

# Comparison Tests For Series of Positive Terms

**Structure**



### 12.1 Introduction

Dear students, in this unit we shall extend our knowledge by studying comparison test for series of positive terms. The comparison test, sometimes called the direct comparison test to distinguish it from similar related test. (especially the limit comparison test), provides a way of deducing the convergence or divergence of an infinite series or an improper integral. In both the cases the test works by comparing the given series or integral to one whose convergence properties are known.

### 12.2 Learning Objectives

The main objectives of this unit are

(i) To study comparison tests for series of positive terms

(ii) Test 1 to Test 4 are studied under comparison tests.

(iii) To study another test of convergence namely, p-test.

### 12.3 Comparison Tests for series of positive terms

**Test I.** Let $\sum a_n$ and $\sum b_n$ be positive terms series. Let k be a positive constant independent of n and m be a fixed positive integer.

(i) If $a_n < k\, b_n \forall$ n > m and $\sum b_n$ is convergent then $\sum a_n$ is divergent.

(ii) If $a_n > k\, b_n \forall$ n > m and $\sum b_n$ is divergent then $\sum a_n$ is divergent.

**Proof:** Let $s_n = a_1 + a_2 + ....... + a_n$

and $s_n' = b_1 + b_2 + ........ + b_n$

(i) Since $\sum b_n$ converges $\Rightarrow s_n' \to s'$ (finite) as $n \to \infty$

$\because$ $b_n > 0 \ \forall\ n \ \therefore s_n' < s \ \forall\ n$ $\quad (\because s_n' < \sum_{n=1}^{\infty} b_n = s')$

$\because$ $a_n < k\, b_n \quad \forall\ n > m$

$\therefore$ $a_{m+1} < k\, b_{m+1},\ a_{m+2} < k\, b_{m+2},\ .......\ a_n < kb_n$

$\therefore$ $a_{m+1} + a_{m+2} + ....... + a_n < k\,(b_{m+1} + b_{m+2} + ....... + b_n)$

$\Rightarrow$ $s_n - s_m < k\,(s_n' - s_m')$

$\Rightarrow$ $s_n < s_m + k\,(s' - s_m')\ \forall\ n\ (\because s_n' < s'\ \forall\ n)$

$\Rightarrow$ $s_n <$ a finite quantity $\quad (\because s_m$ of $s_m'$ are finite)

$\Rightarrow$ the sequence $\{s_n\}$ is bounded

Also $\{s_n\}$ is increasing

$\therefore$ sequence $\{s_n\}$ converges

$\Rightarrow$ $\sum a_n$ converges.

(ii) Since an > k bn $\quad \forall$ n > m

$\therefore$ proceeding as in (i), we have

$s_n > s_m + k\,(s_n' - s_m')$.

Now since $\sum b_n$ diverges

$\therefore$ $s_n' \to \infty$

$\therefore$ sn $\to \infty \Rightarrow \{s_n\}$ diverges to $+\infty$.

Hence $\sum a_n$ is divergent

**Test II.** Let $\sum a_n$ and $\sum b_n$ be two positive terms series

(i) If $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, l being a finite non-zero constant, then $\sum a_n$ and $\sum b_n$ both converges or diverges together.

(ii) If $\lim_{n\to\infty} \frac{a_n}{b_n}$ = 0 and $\sum b_n$ converges, then $\sum a_n$ also converges.

(iii) If $\lim_{n\to\infty} \frac{a_n}{b_n}$ = ∞ and $\sum b_n$ diverges, then $\sum a_n$ also diverges.

**Proof:** (i) Since $\lim_{n\to\infty} \frac{a_n}{b_n} = l$

∴ given $\in > 0$, however small, $\exists$ m $\in$ N, s.t

$$\left|\frac{a_n}{b_n} - l\right| < \in \quad \forall\ n > m$$

⇒ $l - \in < \frac{a_n}{b_n} < l + \in \quad \forall\ n > m$

⇒ $(l - \in)\, b_n < a_n < (l + \in)\, b_n$ $\quad (\because b_n > 0\ \forall\ n)$

⇒ $l > 0$ $\quad (\because$ both $a_n$, $b_n$ are positive for all n)

Now choose $\in$ in a such a way that $l - \in > 0$.

Let $l - \in = k_1$, $l + \in = k_2$, $k_1\ k_2$ are positive constants independent of n.

∴ $k_1 b_n < a_n < k_2 b_n \quad \forall\ n > m$

Now following cases arise:

Case 1. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent $(\because a_n > k_2 b_n \forall\ n \geq m)$

Case 2. If $\sum b_n$ is divergent, then $\sum a_n$ is also divergent $(\because a_n > k_1 b_n \forall\ n \geq m)$

Case 3. If $\sum a_n$ is divergent, then $\sum b_n$ is also convergent $(\because b_n < \frac{1}{k_1} a_n \forall\ n \geq m)$

Case 4. If $\sum a_n$ is divergent, then $\sum b_n$ is also divergent $(\because b_n < \frac{1}{k_2} a_n \forall\ n \geq m)$

Hence $\sum a_n$ and $\sum b_n$ both converge or diverge together.

(ii) Since $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

∴ given $\in > 0$, however smell, $\exists$ m $\in$ N s.t.

$$\frac{a_n}{b_n} < \in \quad \forall\ n \geq m$$

or $a_n < \in b_n \quad \forall\ n \geq m$

Since $\sum b_n$ converges $\quad (\because\ b_n > 0\ \forall\ n)$

$\therefore$ by comparison test $\sum a_n$ converges

(iii) Since $\underset{n\to\infty}{Lim} \frac{a_n}{b_n} = \infty$

$\therefore$ given $\delta > 0$, however small, $\exists\ m \in N$ s.t

$\therefore$ $\frac{a_n}{b_n} > \delta\ \forall\ n > m$

$\therefore$ $a_n > \delta\ bn \quad \forall\ n > m \quad (\because\ b_n > 0\ \forall\ n)$

But $\sum b_n$ diverges

$\therefore$ by comparison test, $\sum a_n$ diverges

**Test III.** Let $a_n > \sum b_n$ be convergent and $\sum c_n$ be divergent. Then we show that

(i) $\sum a_n$ converges if $\overline{Lim}\left(\frac{a_n}{b_n}\right) = \lambda \neq \infty$

(ii) $\sum a_n$ diverges if $\overline{Lim}\left(\frac{a_n}{c_n}\right) = \mu \neq 0$

**Proof:** $\because \overline{Lim}\left(\frac{a_n}{b_n}\right) = \lambda$

$\therefore$ by limit superior property, for every $\in > 0$, $\exists\ m \in N$ s.t.

$\frac{a_n}{b_n} < \lambda + \in \quad \forall\ n > m$

or $\frac{a_n}{b_n} < \lambda'\ \forall\ n > m,\ \lambda' = \lambda + \in$

$\Rightarrow$ $a_n < \lambda'\ b_n \quad \forall\ n > m$

$\therefore$ $a_n < \lambda'\ b_n \quad \forall\ n > m$

Now as $\sum b_n$ converges

$\Rightarrow$ by comparison test, $\sum a_n$ also converges

(ii) Since $\mu \neq 0 \therefore \mu > 0$

Now $\underline{Lim} \frac{a_n}{c_n} = \mu$

$\therefore$ by Limit inferior property, for every $\in > 0$, $\exists$ m $\in$ N s.t.

$\frac{a_n}{c_n} > \mu - \in \quad \forall n > m$

or $\frac{a_n}{c_n} > u' \ \forall n > m \qquad (\because \mu' = r - t)$

Choose $\in$ in such a manner that r' = r - $\in$> 0

$\therefore$ an $>\mu'$ cn $\quad \forall n > m$

Since $\sum c_n$ diverges

$\therefore$ by comparison test, $\sum a_n$ diverges.

**Test IV.** Let $\sum a_n$ and $\sum b_n$ be two positure terms series.

(i) If $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$ $\forall$ n > m, m $\in$ N and $\sum b_n$ is convergent, then $\sum a_n$ is also convergent.

(ii) If $\frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n}$ $\forall$ n > m, m $\in$ N and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

**Proof:** Let $s_n = a_1 + a_2 + ...... + a_n$

and $s_n = b_1 + b_2 + ....... + b_n$

$\therefore$ $a_{m+1} + a_{m+2} + ...... + a_n = (a_1 + a_2 + ...... + a_{m+1} + ....... + a_n)$

$- (a_1 + a_2 + ....... + a_n)$

Likewise $= s_n - s_m$

$b_{m+1} + b_{m+2} + ....... + b_n = s_n' - s_m'$

(i) $s_n - s_m = a_{m+1} + a_{m+2} + ....... + a_n$

$= a_{m+1}\left\{1 + \frac{a_{m+2}}{a_{m+1}} + \frac{a_{m+3}}{a_{m+1}} + ........\right\}$

$= am+1 \left\{1 + \frac{a_{m+2}}{a_{m+1}} + \frac{a_{m+3}}{a_{m+2}} . \frac{a_{m+2}}{a_{m+1}} + ........\right\}$

$< a_{m+1}\left\{1+\frac{a_{m+2}}{a_{m+1}}+\frac{a_{m+3}}{a_{m+2}}.\frac{a_{m+2}}{a_{m+1}}+........\right\}$ $\left(\because \frac{a_{n+1}}{a_n}<\frac{b_{n+1}}{b_n}\right)$

$= \frac{a_{m+1}}{b_{m+1}}\{b_{m+1} + b_{m+2} + .......+b_n\}$

$= \frac{a_{m+1}}{b_{m+1}}(s_n' - b_m')$

$\therefore$ $s_n - s_m < \frac{a_{m+1}}{b_{m+1}}(s_n' - s_m')$ ..... (1)

As $\sum b_n$ is convergent. Let $\sum b_n$ converges to s'.

$\therefore$ $s_n' < s' \ \forall$ n

$\therefore$ from (1) $s_n < s_m + \frac{a_{m+1}}{b_{m+1}}(s_n' - s_m')$

$\Rightarrow$ $s_n<$ (a positive quantity)

$\Rightarrow$ $\{s_n\}$ is an increasing sequence

$\therefore$ $\{s_n\}$ converges $\Rightarrow \sum a_n$ converges.

(ii) Since $\frac{a_{n+1}}{a_n}>\frac{b_{n+1}}{b_n}$ $\quad \forall$ n > m

$\therefore$ As in part (i), we have

$s_n > s_m + \frac{a_{m+1}}{b_{m+1}}(s_n' - s_m')$ .... (2)

Now $\frac{a_{m+1}}{b_{m+1}}$ is positive

Also $s_m$ and $s_n'$ are finite as these are the sum of finite number of terms of $\sum a_n$ and $\sum b_n$ respectively.

$\because$ $\sum b_n$ diverges $\Rightarrow s_n' \to \infty$ as $n \to \infty$.

$\therefore$ From (2)

$s_n \to \infty$ as $n \to \infty$

$\Rightarrow$ $\{s_n\}$ is divergent

$\Rightarrow \qquad \sum a_n$ is divergent.

**12.4 Art. p-Test**

Prove that the series $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ....... + \frac{1}{n^p} + ......$, p > 0

converges if p > 1 and diverges if $p \leq 1$.

**Proof:** The given series is $\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + ....... + \frac{1}{n^p} + ......$, p > 0

Let $\qquad \sum a_n = \sum \frac{1}{n^p}$ so that $a_n = \frac{1}{n^p} \; \forall$ n

Now $\quad$ an = $\frac{1}{n^p} > 0 \; \forall$ n. Therefore $\sum a_n$ is a positive terms series. Hence converges or divergence of $\sum a_n$ is not affected by grouping the terms in any manner, we like.

**Case I.** Let p > 1.

Group the terms in such a way that first group contains first term of the series, second group contains next two terms, third group contains next four terms and so on. In other words, $\forall$ r $\in$ N, the rth group contains $2^{r-1}$ terms.

$$\therefore \qquad \sum a_n = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \left(\frac{1}{8^p} + \frac{1}{9^p} + ........ + \frac{1}{15^p}\right) + ......$$

$$= \sum b_n$$

where $b_1 = \frac{1}{1^p} = 1 = a_1$

$$b_2 = \frac{1}{2^p} + \frac{1}{3^p} < \frac{1}{2^p} + \frac{1}{2^p} = \frac{2}{2^p} = \frac{1}{2^{p-1}}$$

$$b_3 = \frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} < \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}$$

$$= \frac{4}{4^p} = \frac{1}{4^{p-1}} = \frac{1}{2^{2p-2}} = \frac{1}{\left(2^{p-1}\right)^2}$$

.......................................................................

$$b_n < \frac{1}{\left(2^{p-1}\right)^{n-1}} \; \forall \; n \geq 2$$

Now $\sum \frac{1}{\left(2^{p-1}\right)^{n-1}}$ is an infinite G.P. with common ratio $\frac{1}{2^{p-1}}$.

∴ $\sum \frac{1}{\left(2^{p-1}\right)^{n-1}}$ converges. $\left[\because 0 < \frac{1}{2^{p-1}} < 1 \; for \;\; p > 1\right]$

∴ by Comparison Test, $\sum b_n$ is convergent.

⇒ $\sum a_n$ is convergent.

**Case II.** Let p = 1

∴ $\sum a_n = \sum \frac{1}{n}$, which is divergent.

**Case III.** Let 0 < p < 1

∴ $\frac{1}{n^p} > \frac{1}{n}$ $\forall$ n $\geq$ 2

But $\sum \frac{1}{n}$ is divergent.

∴ by Comparison Test, $\sum \frac{1}{n^p}$ is also divergent.

∴ we see that $\sum \frac{1}{n^p}$, p > 0 converges.

If p > 1 and diverges ir p $\leq$ 1

**Note.** (i) let p = 0

∴ $\frac{1}{n^p} = 1 \; \forall$ n

∴ $\underset{n\to\infty}{Lt} \frac{1}{n^p} \neq 0$

⇒ $\sum \frac{1}{n^p}$ does not converge.

But $\sum \frac{1}{n^p}$ is a positive terms series.

∴ $\sum \frac{1}{n^p}$ diverges to $+\infty$.

(ii) Let $p < 0$

Put $p = -q$ so that $p > 0$

$\therefore$ $\frac{1}{n^p} = \frac{1}{n^{-q}} = nq \to \infty$ as $n \to \infty$

$\therefore$ $\sum \frac{1}{n^p}$ diverges to $+\infty$ $\left[\because \sum \frac{1}{n^p}, p < 0 \text{ is a positive terms series}\right]$

$\therefore$ $\sum \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$

**Note.** The series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots\ldots + \frac{1}{n^p} + \ldots$ and geometric series

$1 + r + r^2 + r^3 + \ldots\ldots r^n + \ldots$ with $|r| < 1$ are called auxiliary series.

**Note. Method to apply Comparison Test**

We know that if $\underset{n\to\infty}{Lt} \frac{a_n}{b_n} = l$, where l is neither zero no infinite, then the two series $\sum a_n$ and $\sum b_n$ are both convergent or both divergent. So to test the convergence or divergence of $\sum a_n$, we find $\sum b_n$ known as auxiliary series. Two auxiliary series are given above. If $a_n$ can be expanded in ascending powers of $\frac{1}{n}$, then we retain the least power of $\frac{1}{n}$ and take it as $b_n$. Also if an = sin $\frac{1}{n}$, $\sin^{-1}\frac{1}{n}$, tan $\frac{1}{n}$, $\tan^{-1}\frac{1}{n}$, we take $b_n = \frac{1}{n}$.

Also $b_n = \frac{1}{n^p}$, where p = Deg. of denominator - Deg. of numerator.

**Illustrative Examples**

**Example 1:** Discuss the convergence or divergence of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$$

**Solution :** The given series is $\sum a_n = \frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots$

Where $a_n = \frac{1+(n-1).2}{[1+(n-1).1][2+(n-1).1][3+(n-1).1]}$

$\therefore$ $a_n = \frac{2n-1}{n(n+1)(n+2)}$

Let $b_n = \frac{1}{n^2}$ [$\because \rho = 3 - 1 = 2$]

$\therefore$ $\frac{a_n}{b_n} = \frac{2n-1}{n(n+1)(n+2)} \times \frac{n^2}{1}$

$= \frac{n(2n-1)}{(n+1)(n+2)}$

$= \frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$\therefore$ $\underset{n\to\infty}{Lt} \frac{2-0}{(1+0)(1+0)}$ = 0, which is non-zero and finite

$\therefore$ by Comparison Test, $\sum a_n$ and $b_n$ converge or diverge together.

But $\sum b_n = \sum \frac{1}{n^2}$ is convergent. $\left[\because \sum \frac{1}{n^p} \text{ converges if } p > 1\right]$

$\therefore$ given series $\sum a_n$ is convergent.

**Example 2 :** Test the following series for convergence

$\sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + .....$

**Solution :** The given series is $\sum a_n$

Where $a_n = \sqrt{\frac{1}{2^3}} + \sqrt{\frac{2}{3^3}} + \sqrt{\frac{3}{4^3}} + .....$

$\therefore$ $a_n = \sqrt{\frac{n}{(n+1)^3}}$

$= \sqrt{\frac{n}{n^3\left(1+\frac{1}{n}\right)^3}}$

$$= \sqrt{\frac{1}{n\left(1+\frac{1}{n}\right)^{\frac{3}{2}}}}$$

$$= \frac{1}{n}\left(1+\frac{1}{n}\right)^{\frac{3}{2}}$$

$$= \frac{1}{n}\left[1-\frac{3}{2n}+...\right]$$

$$= \frac{1}{n} - \frac{3}{2n^2} + ...$$

Take $b_n = \frac{1}{n}$

$\therefore \quad \frac{a_n}{b_n} = 1 - \frac{3}{2n} + ...$

$\therefore \quad \underset{n\to\infty}{Lt} \frac{a_n}{b_n} = 1$, which is finite and non-zero

$\therefore \quad \sum a_n$ and $\sum b_n$ converge or diverge together.

But $\sum b_n = \sum \frac{1}{n}$ diverges the p-Test

$\therefore$ by Comparison test, $\sum a_n$ diverges.

**Some More Illustrated Examples**

**Example 3 :** Discuss the convergence of the series

$$\frac{1}{1.3} + \frac{3}{5.7} + \frac{5}{9.11} + ....$$

**Solution :** (i) We have

$$\sum a_n = \frac{1}{1.3} + \frac{3}{5.7} + ....$$

$$\therefore \quad a_n = \frac{1+(n-1)2}{\{1+(n-1).4\}\{3+(n-1).4\}}$$

$$= \frac{2n-1}{(4n-3)(4n-1)}$$

Let $b_n = \frac{1}{n}$ $\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ ($\because \rho = 2 - 1 = 1$)

$$\therefore \quad \frac{a_n}{b_n} = \frac{2n-1}{(4n-3)(4n-1)} \cdot \frac{n}{1}$$

$$= \frac{2n^2 - n}{(4n-3)(4n-1)}$$

$$\therefore \quad \underset{n\to\infty}{Lim} \frac{a_n}{b_n} = \underset{n\to\infty}{Lim} \frac{2 - \frac{1}{n}}{\left(4 - \frac{3}{n}\right)\left(4 - \frac{1}{n}\right)}$$

$= \frac{2}{4.4} = \frac{2}{16} = \frac{1}{8}$ (non zero and finite)

$\Rightarrow \quad \sum a_n$ and $\sum b_n$ converge or diverge together.

But $\sum b_n = \sum \frac{1}{n}$ diverges

$\therefore \quad$ series $\sum a_n$ diverges.

**Example 4 :** Examine the convergence of the series

$$\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + ...$$

**Solution :** The given series is

$$\sum a_n = \frac{2}{1^p} + \frac{3}{2^p} + ...$$

$$\Rightarrow \quad a_n = \frac{2+(n-1).1}{\{1+(n-1).1\}^p} = \frac{n+1}{n^p}$$

Let $b_n = \frac{1}{n^{p-1}}$

$$\therefore \quad \frac{a_n}{b_n} = \frac{n+1}{n^p} \times \frac{n^{p-1}}{1} = \frac{n+1}{n} = 1 + \frac{1}{n}$$

$\Rightarrow \quad \underset{n\to\infty}{Lim} \frac{a_n}{b_n} = \underset{n\to\infty}{Lim}\left(1+\frac{1}{n}\right)$ = 1 (finite and non zero)

$\therefore \quad \sum a_n$ and $\sum b_n$ converge or diverge together.

But $\sum b_n = \sum \frac{1}{n^{p-1}}$ is convergent if ρ - 1 > 1 i.e. if p > 2.

and divergent if p - 1 ≤ 1 i.e. p ≤ 2

$\therefore$ given series $\sum a_n$ converges for p > 2 and diverges for p ≤ 2.

**Example 5 :** Examine the convergence of the series

$$\sum \frac{n^2+n+1}{n^4+1}$$

**Solution:** We have the given series

$\sum a_n$ where $a_n = \frac{n^2+n+1}{n^4+1}$

Let $\quad b_n = \frac{1}{n^2}$

$$\therefore \quad \frac{a_n}{b_n} = \frac{n^2+n+1}{n^4+1} \cdot \frac{n^2}{1}$$

$$= \frac{n^2+n+1}{n^4+1} = \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^4}}$$

$$\therefore \quad \underset{n\to\infty}{Lim} \frac{a_n}{b_n} = \underset{n\to\infty}{Lim} \frac{1+\frac{1}{n}+\frac{1}{n^2}}{1+\frac{1}{n^4}}$$

$= \frac{1+0+0}{1+0}$ = 0 ≠ 0 and finite.

$\therefore$ by comparison Test, $\sum a_n$ and $\sum b_n$ converge or diverge together.

But $\quad \sum b_n = \sum \frac{1}{n^2}$ converges by p test

$\therefore$ the given series $\sum a_n$ converges.

**Example 6:** Examine the convergence or divergence of the series $\sum \frac{1}{3n-1}$

**Solution:** The given series is $\sum a_n = \sum \frac{1}{3n-1}$

$\therefore$ $a_n = \frac{1}{3n-1}$

Let $b_n = \frac{1}{n}$

$\therefore$ $\frac{a_n}{b_n} = \frac{1}{3n-1} \cdot \frac{n}{1} = \frac{n}{3n-1} = \frac{1}{3-\frac{1}{n}}$

$\therefore$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{3-\frac{1}{n}} = \frac{1}{3}$ (non zero and finite)

$\therefore$ $\sum a_n$ and $\sum b_n$ converge or diverge together.

But $\sum b_n = \sum \frac{1}{n}$ diverges by p - test

$\therefore$ the given series $\sum a_n$ diverges.

**Example 7:** Discuss the convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 \log n}$$

**Solution:** The given series is $\sum_{n=2}^{\infty} a_n$ ,

where $a_n = \frac{1}{n^2 \log n}$

Take $b_n = \frac{1}{n^2}$

$\therefore$ $\frac{a_n}{b_n} = \frac{1}{n^2 \log n} \frac{n^2}{1} = \frac{1}{\log n}$

$\therefore$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$

But $\sum b_n = \sum \frac{1}{n^2}$ is convergent by p-test.

$\therefore$ by comparison test, the given series is convergent.

**Example 8:** For the series $\sum \frac{1}{\sqrt{n}}$, discuss the convergence or divergence.

**Solution:** The given series is $\sum \frac{1}{\sqrt{n}} = \sum a_n$

where $a_n = \frac{1}{\sqrt{n}}$

Take $b_n = \frac{1}{n}$

$\therefore \quad \frac{a_n}{b_n} = \frac{1}{\sqrt{n}} \times \frac{n}{1} = \sqrt{n}$

$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \infty$

But $\sum b_n = \sum \frac{1}{n}$ is divergent by p-test

$\therefore$ By comparison test, $\sum a_n$ is divergent.

## 12.5 Self Check Exercise

Q.1 Examine the convergence of the series

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^3} + \dots\dots\dots$$

Q. 2 Discuss the convergence or divergence of the series $\sum \frac{2+n}{3+2n^2}$

Q.3 Discuss the convergence or divergence of the series $\sum \frac{1}{n^1 + \frac{1}{n}}$

Q.4 Discuss the convergence of the series

$$\sum \left( \sqrt[3]{n+1} - \sqrt[3]{n} \right)$$

## 12.5 Summary

We have learnt the following concepts in this unit.

(i) Comparison Test for series of positive terms

(ii) Test 1 to Test 4, Method to apply comparison test

(iii) p-test etc.

## 12.7 Glossary:

Method to apply comparison test -

We know that if $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ = l, l is nether zero nor infinite. then the two series $\sum a_n$ , $\sum b_n$ are both convergent or both divergent. So the test of convergence or divergence of $\sum a_n$ . We find $\sum b_n$ known as Auxiliary series. Two auxiliary are given above. If $a_n$ can be expanded in ascending power of $\frac{1}{n}$ and take it as $b_n$. If $a_n$ = sin $\frac{1}{n}$ , sin-1$\frac{1}{n}$ , tan $\frac{1}{n}$ , tan-1$\frac{1}{n}$ we take bn = $\frac{1}{n}$ .

Also bn = $\frac{1}{n^p}$ , p = degree of denominator-degree of numerator

## 12.8 Answer to Self Check Exercise

Ans.1 $a_n = \frac{n^n}{(n+1)^{n+1}}$ , $b_n = \frac{1}{n}$ and now proceed

Ans.2 $a_n = \frac{2+n}{3+2n^2}$ 1. , $b_n = \frac{1}{n}$ and now proceed

Ans.3 $a_n = \frac{1}{n^1 + \frac{1}{n^p}}$ , $b_n = \frac{1}{n}$ and then proceed.

Ans.4 Take $a_n = (n+1)^{\frac{1}{3}} - n^{\frac{1}{3}} = n^{\frac{1}{3}} \left\{ \left(1+\frac{1}{n}\right)^{\frac{1}{3}} \right\} - n^{\frac{1}{3}}$

$= \frac{1}{3n^{2/3}} - \frac{1}{9n^{5/3}} + ......$, $b_n = \frac{1}{n^{2/3}}$ and now proceed.

## 12.9 References/Suggested Readings

**1**. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. R.G. Bartle and D.R. Sherbert, Introduction to Real trellises, John Wiley and Sons (Asia) P. Ltd., 2000.

3. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

4. K.A. Ross, Elementary Analysis - The Theory of Calculus Series-undergraduate Tests in Mathematics, Springer Verlag, 2003.

## 12.10 Terminal Questions

1. Discuss the convergence or divergence of the series $\sum \frac{a+bn}{a+bn^2}$

2. Discuss the convergence or divergence of the series

$$\sum \sqrt{\frac{n}{n^4+1}}$$

3. Examine the convergence of divergence of the series

$$\sum \frac{1}{n^p + \frac{q}{n}}$$

4. Discuss the convergence or divergence of the series

$$\sum \left( \sqrt{n^2+1} - \sqrt{n^2-1} \right)$$

5. Discuss the convergence of the series

$$\sum \left\{ \frac{1}{n} - \log\left( \frac{n+1}{n} \right) \right\}$$

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## Unit - 13

# Cauchy's Condensation Test
# And
# Cauchy's Integral Test
# For Infinite Series

**Structure**



### 13.1 Introduction

Dear students, in this unit we shall study the concept of Cauchy condensation Test and Cauchy integral Test. The Cauchy condensation Test named after Augustion-Locis Cauchy, is a standard convergence test for infinite series. For a non increasing sequence $f$(n) of non-negative real numbers the series $\sum_{n=1}^{\infty} f(n)$ converges iff the conduced series $\sum_{n=0}^{\infty} 2^n f(2^n)^{n=1}$ converges. Moreover if there converge, the sum of the conduced series is no more then twice as large as sum of the original. On the other hand the Cauchy's integral test compares a series with an integral. The test compares the area of a series of unit width rectangles with the area under the curve. The Cauchy integral test is also known as Euler-Maclaurin summation formula. The integral test of convergence was developed by colin Maclurin and Augustin Cauchy and is sometimes known as Maclaurin-Cauchy Test. This test is used to test the convergence of an infinite series of non negative terms.

### 13.2 Learning Objectives

The main objectives of this unit are

(i) to study Cauchy's condensation test of convergence for infinite series.

(ii) to prove Pringsheim's Theorem

(iii) to learn about Cauchy-integral test for finding the convergence of an infinite series.

## 13.3 Cauchy's Condensation Test

If $f$(n) $\geq f$(n+1) $\geq$ 0 $\forall$ n, then prove that the two series

$$\sum_{n=1}^{\infty} f(n) \text{ and } \sum_{n=0}^{\infty} 2^n f(2^n)$$

Converge or diverge together

**Proof:** We accept the result without proof.

**Another form of Cauchy condensation Test.**

If $a_n \geq a_{n+1} \geq 0 \ \forall$ n, then prove that the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converge or diverge together.

**Example 1:** Show that the series

$\sum \frac{1}{n^p}$, p > 0 converges if p > 1 and diverges if p $\leq$ 1.

**Solution:** Let $\sum f(n) = \sum \frac{1}{n^p}$ so that $f(n) = \frac{1}{n^p}$, p > 0 clearly, $f$(n) is a positive decreasing function of n $\forall$n.

$\therefore$ by Cauchy's condensation Test,

$\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together.

Now $\sum 2^n f(2^n) = \sum \frac{2^n}{\left(2^n\right)^p} = \sum \frac{2^n}{2^{np}} = \sum \frac{1}{2^{np-n}}$

$= \sum \frac{1}{2^{(p-1)n}}$, which is a G.P. with common ratio = $\frac{1}{2^{p-1}}$

Following two Cases arise

**Case 1:** If p > 1, then $2^{p-1} > 2^0 = 1$

$\therefore \frac{1}{2^{p-1}} < 1$

$\therefore \sum 2^n f(2^n) = \sum \frac{1}{2^{(p-1)n}}$ is a G.P. with common ratio = $\frac{1}{2^{p-1}}$ where $0 < \frac{1}{2^{p-1}} < 1$ and hence $\sum 2^n f(2^n)$ converges

∴ $\sum f(n) = \sum \frac{1}{n^p}$ converges for p > 1.

**Case 2:** If 0 < p $\leq$ 1 then 1 - p > 0

⇒ $2^{1-p} \geq 2^0 - 1$

∴ $\frac{1}{2^{p-1}} \geq 1$

∴ $\sum 2^n f(2^n) = \sum \frac{1}{2^{(p-1)n}}$ is a G.P. with common rates = $\frac{1}{2^{p-1}}$ where $\frac{1}{2^{p-1}} \geq 1$.

Hence $\sum 2^n f(2^n)$ diverges

∴ $\sum f(n) = \sum \frac{1}{n^p}$ diverges for 0 < p $\leq$ 1

Combining the results of two cases we see that $\sum \frac{1}{n^p}$, p > 0 converges for p > 1 and diverges for p $\leq$ 1.

**Note:** If p = 0 then $f$(n) = 1 which does not tend to zero as n → ∞.

∴ positive terms series $\sum f(n)$ diverges

Again if p < 0, Let p = -q (q > 0) then $f$(n) = $\sum \frac{1}{n^p}$ = $n^q$ → ∞ as n → ∞.

∴ $f$(n) → 0 as n → ∞.

but $\sum f(n)$ diverges for p < 0

∴ we can say that

$\sum \frac{1}{n^p}$ converges for p > 1 and diverges for p $\leq$ 1.

## 13.4 Art. Pringsheim's Theorem

If the terms of the positive terms series $\sum a_n$ is that $\underset{n\to\infty}{Lt}$ $na_n$ = 0.

The condition is not sufficient.

**Proof: Necessary condition**

Assume that positive terms series $\sum a_n$ converges

∴ by Cauchy's General Principle of convergence for a series, given $\varepsilon > 0$, however small, ∃ m ∈ N such that

$a_{m+1} + a_{m+2} + ......+ a_n < \varepsilon \quad \forall\ n > m$ ..... (1)

$\because$ terns of $\sum a_n$ steadily decrease

$\therefore$ each of the (n - m) terms $a_{m+1}, a_{m+2}, ....., a_n$ is $\geq a_n$

$\therefore$ $(n - m)\ a_n \leq a_{m+1} + a_{m+}2 + ..... + a_n$

$\Rightarrow$ $(n - m)\ a_n < \varepsilon \quad \forall\ n \geq m$ [$\because$ of (1)]

$\Rightarrow$ $n\ a_n < m\ a_n + \varepsilon\ \forall\ n \geq m$ ..... (2)

Since $\sum a_n$ converges,

$\therefore$ $a_n \to 0$ as $n \to \infty$

$\Rightarrow$ $m\ a_n \to 0$ as $n \to \infty$

$\therefore$ we can find a positive integer $m_1 > m$ such that

$|m\ a_n| < \quad \forall\ n \geq m_1$

i.e. $m\ a_n < \varepsilon\ \forall\ n \geq m_1$ ...... (3)

From (2) and (3), we have

$n\ a_n < \varepsilon + \varepsilon \quad \forall\ n \geq m_1$

or $n\ a_n < 2\varepsilon \quad \forall\ n \geq m_1$

$\Rightarrow$ $Lt\ n\ a_n = 0$

(ii) Condition is not sufficient

Consider the series $\sum_{n=2}^{\infty} a_n$ where $a_n = \dfrac{1}{n \log n}$

Now by Cauchy's Condensation Test, the series $\sum_{2}^{\infty} \dfrac{1}{n \log n}$ is divergent.

But $\displaystyle \underset{n\to\infty}{Lt} (n\ a_n) = \underset{n\to\infty}{Lt} \frac{1}{\log n} = 0$

**13.5 Art. Cauchy's Integral Test**

If $f$ be defined, non-negative and decreasing for $x \geq 1$, then the series $\sum_{n=1}^{\infty} f(n)$ and the integral $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

We accept this result without proof.

**Example:** Using Cauchy's Integral Test, discuss the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, p > 0.

**Solution:** The given series is $\sum_{n=1}^{\infty}\frac{1}{n^p}$, p > 0.

Here $f$ (x) = $\frac{1}{x^p}$, p > 0

$\therefore$ $f$ (x) is positive and decreasing for x $\geq$ 1

$\therefore$ Cauchy Integral Test is applicable.

**Case I.** p = 1

$$\therefore \int_1^{\infty} f(x)\,dx = \underset{n\to\infty}{Lt}\int_1^{\infty}\frac{1}{x}\,dx$$

$$= \underset{n\to\infty}{Lt}\left[\log x\right]_1^n$$

$$= \underset{n\to\infty}{Lt}\,(\log n - \log l)$$

$$= \infty - 0 = \infty$$

$\therefore$ $\int_1^{\infty} f(x)\,dx$ diverges

$\Rightarrow$ $\sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges

**Case II.** p $\neq$ 1

$$\therefore \int_1^{\infty} f(x)\,dx \;\; \underset{n\to\infty}{Lt}\int_1^{\infty}\frac{1}{x}\,dx$$

$$= \underset{n\to\infty}{Lt}\int_1^{n} x^{-p}\,dx$$

$$= \underset{n\to\infty}{Lt}\left[\frac{x^{1-p}}{1-p}\right]_1^n$$

$$= \frac{1}{1-p}\,\underset{n\to\infty}{Lt}\left[n^{1-p} - 1\right]$$

Two sub-cases arise :

(i) If p < 1, then $\underset{n\to\infty}{Lt}$ $n^{1-p} = \infty$

$\therefore \quad \int_1^\infty f(x)\,dx = \infty$

$\Rightarrow \quad \int_1^\infty f(x)\,dx$ diverges

$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

(ii) If p > 1, then $\underset{n\to\infty}{Lt}$ $n^{1-p} = \underset{n\to\infty}{Lt} \frac{1}{n^{p-1}} = 0$

$\therefore \quad \int_1^\infty f(x)\,dx = \frac{1}{1-p}\ (0-1)$

$= \frac{1}{p-1}$, which is finite

$\therefore \quad \int_1^\infty f(x)\,dx$ converges

$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges

Combining the results of the two cases, we see that $\sum_{n=1}^{\infty} \frac{1}{n^p}$, p > 0 converges for p > 1 and diverges for $p \leq 1$.

**Some Illustrated Examples**

**Example 2:** Show that the series

$\frac{1}{n(\log n)^p}$, p > 0 converges for p > 1 and diverges for $p \leq 1$.

**Solution:** The given series is $\sum_{n=2}^{\infty} f(n)$

where $f(n) = \frac{1}{n(\log n)^p}$, p > 0

clearly, $f(n)$ is a decreasing function of n.

$\therefore$ by Cauchy's condensation test

$\sum f(n)$ and $\sum 2^n f(2^n)$ converges or diverge together

Now $\sum 2^n f(2^n) = \sum 2^n \dfrac{1}{2^n \left(\log 2^n\right)^p}$

$= \dfrac{1}{(n \log 2)^p}$

$= \dfrac{1}{\left(\log 2\right)^p} \sum \dfrac{1}{n^p}$

which converges for p > 1 and diverges for p $\leq$ 1. (see Example 1)

$\therefore$ $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \dfrac{1}{n(\log n)^p}$ converges for p > 1 and diverges for p $\leq$ 1.

**Example 3:** Show that the series

$\sum_{n=2}^{\infty} \dfrac{1}{(\log n)^p}$ is convergent for p > 1 and diverges for p $\leq$ 1

**Solution:** The given series is $\sum_{n=2}^{\infty} f(n)$,

where $f$(n) = $\dfrac{1}{(\log n)^p}$

clearly $f$(n) is a positive decreasing function of n.

$\therefore$ by Cauchy's condensation test,

$\sum f(n)$ and $\sum 2^n f(2^n)$ converge or diverge together

Now $\sum 2^n f(2^n) = \sum 2^n \dfrac{1}{\left(\log 2^n\right)^p}$

$= \dfrac{2^n}{\left(\log 2\right)^p} \sum \dfrac{1}{n^p}$

which is convergent for p > 1 and divergent for p $\leq$ 1. (see Ex.1)

$\therefore$ $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \dfrac{1}{(\log n)^p}$ converges for p > 1 and diverges for p $\leq$ 1

**Example 4:** Discuss the convergence or divergence of the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$, p > 0, by using Cauchy's integral Test.

**Solution:** The given series is $\sum_{n=1}^{\infty}\frac{1}{n^p}$, p > 0

Here $f(x) = \frac{1}{n^p}$, p > 0

$\Rightarrow$ $f(x)$ is positive and decreasing function for $x \geq 1$.

$\therefore$ Cauchy Integral Test is applicable.

Two cases arise

**Case 1:** For p = 1

$$\int_1^{\infty} f(x)\,dx = \underset{n\to\infty}{Lim}\int_1^n \frac{1}{x}dx$$

$$= \underset{n\to\infty}{Lim}\left[\log x\right]_1^n$$

$$= \underset{n\to\infty}{Lim}\,[\log n - \log 1]$$

$$= \infty - 0 = \infty$$

$\therefore$ $\int_1^{\infty} f(x)\,dx$ diverges $\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^p}$ diverges.

**Case 2:** For p ≠ 1

$$\int_1^{\infty} f(x)\,dx = \underset{n\to\infty}{Lim}\sum_{n=1}^{\infty}\frac{1}{n^p}\,dx$$

$$= \underset{n\to\infty}{Lim}\left[\frac{x^{1-p}}{1-p}\right]_1^n$$

$$= \frac{1}{1-p}\underset{n\to\infty}{Lim}[n^{1-p} - 1]$$

Two subcases arise:

(i) It p < 1, then $\underset{n\to\infty}{Lim}\, n^{1-p} = \infty$

$\therefore \quad \int_1^\infty f(x)\,dx = \infty \quad \Rightarrow \quad \int_1^\infty f(x)\,dx$ διϖεργεσ

$\Rightarrow \quad \sum_{n=1}^\infty \frac{1}{n^p}$ diverges

(ii) It p > 1, then $\lim_{n\to\infty} n^{1-p} = \lim_{n\to\infty} \frac{1}{n^{p-1}} = 0$

$\therefore \quad \int_1^\infty f(x)\,dx = \frac{1}{1-p}(0-1) = \frac{1}{1-p}$, a finite quantity.

$\therefore \quad \int_1^\infty f(x)\,dx$ converges $\Rightarrow \quad \sum_{n=1}^\infty \frac{1}{n^p}$ converges combining the above two cases we see that $\sum_{n=1}^\infty \frac{1}{n^p}$, p > 0 converges for p > 1 and diverges for $p \leq 1$.

**Example 5:** Discuss the convergence or divergence of the series

$$\sum_{n=1}^\infty n e^{n^2}$$

**Solution:** The given series is $\sum_{n=1}^\infty n e^{-n^2}$

$\therefore \quad f(x) = x e^{-x^2}$

$\Rightarrow \quad f'(x) = x e^{-x^2}(-2x) + e^{-x^2}.1$

$= (1 - 2x^2)\, e^{-x^2} < 0$ for $x \leq 1$

$\therefore \quad f(x)$ is positive and decreasing function for $x \geq 1$.

$\therefore$ Cauchy Integral Test is applicable

Now $\int_1^\infty f(x)\,dx = \lim_{n\to\infty} \int_1^n x e^{-x^2}\,dx$

$= \frac{-1}{2} \lim_{n\to\infty} \int_1^n e^{-x^2}(-2x)\,dx$

$= \frac{-1}{2} \lim_{n\to\infty} \left[ e^{-x^2} \right]_1^n = \frac{-1}{2} \left[ e^{-x^2} - e^{-1} \right]$

$= \frac{-1}{2}\left[0 - e^{-1}\right]$ $\left(\because \lim_{n\to\infty} e^{-n^2} = 0\right)$

$= \frac{-1}{2}\left(0 - e^{-1}\right)$

$= \frac{1}{2e}\left(0 - e^{-1}\right)$

$= \frac{1}{2e}$, a finite quantity

$\therefore$ $\int_1^{\infty} f(x)\,dx$ converges, consequently $\sum_{n=1}^{\infty} n e^{-n^2}$ .

## 13.6 Self Check Exercise

Q.1 Prove that the series $\sum_{n=2}^{\infty} a_n$ , where an = $\frac{1}{n^p (\log n)^q}$ converges if either p > 1 or p = 1 and diverges for p < 1 and q > 1.

Q. 2 Discuss the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

Q.3 Discuss the convergence or divergence of the series ]

$$\frac{1}{n \log n (\log\log n)^p}, \ p > 0$$

## 13.7 Summary

In this unit we have learnt the following concepts.

(i) Cauchy's condensation Test

(ii) Pringshiem's Theorem

(iii) Cauchy's Integral Test etc.

## 13.8 Glossary:

1. Cauchy - Machaurin Integral Test -

   The Cauchy Integral Test is also known as

   Cauchy - Maclaurin Integral Test

2. Euler's Constant - The limit to which the sequence $\{t_n\}$ converges is called Euler's constant, denoted by Y or C, 0 < Y < 1.

## 13.9 Answer to Self Check Exercise

Ans.1 Hint. $a_n = \frac{1}{n^p(\log n)^q}$, Take $b_n = \frac{1}{n^{1+\delta}}$ for first case and $b_n = \frac{1}{n^{1+\delta}}$ for second case.

Ans.2 Hint. Take $f(x) = \frac{x}{(x^2+1)^2}$, find $f(x) < 0$

$\forall$ n $\geq$ 1 and then proceed.

Ans.3 Hint. Take $f(x) = \frac{1}{x\log x}$ (log logx)$^p$, p > 0, find $f'(x)$.

$f(x)$ is positive and decreasing for x $\geq$ 3. Now proceed.

## 13.10 References/Suggested Readings


**1**. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

3. R.G. Bartle and D.R. Sherbert, Introduction to Real trellises, John Wiley and Sons (Asia) P. Ltd., 2000.

4. K.A. Ross, Elementary Analysis - The Theory of Calculus Series-undergraduate Tests in Mathematics, Springer Verlag, 2003.


## 13.11 Terminal Questions

1. Discuss the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{n^4}{2n^5+2}$

2. Show that the series

   $\sum_{n=2}^{\infty}\left(\frac{n+5}{n^2+1}\right)(\log n)^{5/4}$ is convergent.

3. Prove that the sequence $\{t_n\}$, where

   $t_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots. + \frac{1}{n} - \log n$

   Converges to a limit which lies between 0 and 1.

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## Unit - 14

# Cauchy's Toot Test And D' Alembert's Ratio Test For Infinite Series

**Structure**



### 14.1 Introduction

Dear students, we have already discussed various test of convergence for an infinite series in our previous units. In this unit we shall discuss about Cauchy's root test, D' Alembert's ratio test and Kummer's test for the convergence of an infinite series. The root test is a criterion for the convergence of an infinite series. It depends upon the quantity $\underset{n\to\infty}{Lim}\sup\left(|a_n|\right)^{\frac{1}{n}}$ where an are the terms of the series. It is particularly useful in connection with power series.

### 14.2 Learning Objectives

The main objectives of this unit are

(i) to study first form of Cauchy's Root Test

(ii) to learn about Cauchy's Root Test

(iii) to study Kummer's Test

(iv) to study D' Alembert's Ratio Test of convergence etc.

### 14.3 First Form of Cauchy's Root Test

If $a_n > 0$ and $\overline{Lim}\left(a_n\right)^{\frac{1}{n}} = l$, then $\sum a_n$ is convergent if l < 1 and divergent if l > 1.

**Proof:** (a) Let $\overline{Lim}\,(a_n)^{\frac{1}{n}} = l < 1$.

Choose r such that 0 < l < r < 1

Now, by limit superior property, ∃ m ∈ N s.t

$(a_n)^{\frac{1}{n}} < r \quad \forall\ n \geq m$

⇒ $a_n < r_n \quad \forall\ n \geq m$

clearly $\sum r^n$ is a G.P. with common ratio i 0 < r < 1.

∴ $\sum r^n$ converges

∴ by comparison Test $\sum a_n$ converges.

(b) Let $\overline{Lim}\,(a_n)^{\frac{1}{n}} = l > 1$

∴ by limit superior property,

$(a_n)^{\frac{1}{n}} > 1$ for infinitely many n ∈N.

∴ $a_n > 1$ for infinitely many n ∈N

⇒ $a_n \to 0$

⇒ $\sum a_n$ cannot converge

But $\sum a_n$ is a positive term series

∴ $\sum a_n$ diverges to + ∞.

Note: Cauchy's Root Test fails when l = d.

**14.4 Cauchy's Root Test**

If Lim $(a_n)^{\frac{1}{n}} = l$, $a_n > 0$, the series $\sum_{n=1}^{\infty} a_n$ is convergent if l < 1 and divergent if l > 1.

**Proof:** (i) Let Lim $(a_n)^{\frac{1}{n}} = l < 1$

Choose ∈> 0 s.t. l < l + ∈< 1.

Now as Lim $(a_n)^{\frac{1}{n}} = l$

⇒ ∃ m ∈ N s.t

$l - \in < (a_n)^{\frac{1}{n}} < l + \in \quad \forall\ n \geq m$

Let us put $l + \in = r$ so that $0 < r < 1$

$\therefore \quad (a_n)^{\frac{1}{n}} < r \quad \forall\ n \geq m$

$\Rightarrow \quad a_n < r^n \quad \forall\ n \geq m$

clearly $\sum r^n$ is a G.P. with common ratio r $0 < r < 1$.

$\therefore \quad \sum r^n$ converges

$\therefore$ by comparison test, $\sum a_n$ converges

(ii) Let $\text{Lim } (a_n)^{\frac{1}{n}} = l > 1$

Choose $\in > 0$ s.t. $l > l - \in > 1$

Now as $\text{Lim } (a_n)^{\frac{1}{n}} = l$,

$\exists\ m \in N$ s.t.

$l - \in < (a_n)^{\frac{1}{n}} < l + \in \quad \forall\ n \geq m$

Take $l - \in = r$ so that $r > 1$

$\therefore \quad (a_n)^{\frac{1}{n}} > r \quad \forall\ n \geq m$

$\Rightarrow \quad a_n > r^n \quad \forall\ n \geq m$

Now, $\sum r^n$ is a G.P. with common ratio $r > 1$

$\therefore \quad \sum r^n$ diverges

$\therefore$ by comparison test, $\sum a_n$ diverges

Note: This test is applied when power of every factor of $a_n$ is a multiple of n.

**Some Illustrated Examples**

**Example 1:** Test the convergence or divergence of the series $\sum \left(\frac{n+1}{3n}\right)^n$

**Solution:** The given series is $\sum a_n$, where

$a_n = \left(\frac{n+1}{3n}\right)^n$

$\therefore \quad (a_n)^{\frac{1}{n}} = \left\{\left(\frac{n+1}{3n}\right)^n\right\}^{\frac{1}{n}} = \frac{n+1}{3n} = \frac{1+\frac{1}{n}}{3}$

$\Rightarrow \quad \underset{n\to\infty}{Lim}\,(a_n)^{\frac{1}{n}} = \underset{n\to\infty}{Lim}\left(\frac{1+\frac{1}{n}}{3}\right) = \frac{1}{3} < 1.$

Thus given series $\sum a_n$ is convergent by Cauchy's Root Test.

**Example 2:** Examine the convergence or divergence of the series

$$\sum\left(\frac{n}{n+1}\right)^{n^2}$$

**Solution:** The given series is $\sum a_n$ , where

$a_n = \left(\frac{n}{n+1}\right)^{n^2}$

$\therefore \quad (a_n)^{\frac{1}{n}} = \left\{\left(\frac{n}{n+1}\right)^{n^2}\right\}^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n$

$= \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \quad \underset{n\to\infty}{Lim}\,(a_n)^{\frac{1}{n}} = \underset{n\to\infty}{Lim}\left(\frac{1}{\left(1+\frac{1}{n}\right)^n}\right)$

$= \frac{1}{e} < 1$

$\therefore$ given series $\sum a_n$ converges by Cauchy's Root Test.

**Example 3:** Discuss the convergence or divergence of the series $\sum\left(1+\frac{1}{n}\right)^{n}\left(\frac{1}{2}\right)^{n}$

**Solution:** We have

$\sum a_n$ , where $a_n = \left(1+\frac{1}{n}\right)^{n}\left(\frac{1}{2}\right)^{n}$

$\therefore \quad (a_n)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)\left(\frac{1}{2}\right)$

$\therefore \quad \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = (1 + 0)\left(\frac{1}{2}\right) = \frac{1}{2} < 1.$

$\therefore$ The given series $\sum a_n$ converges by Cauchy's Root Test.

**Example 4:** Examine the convergence or divergence of the series $\sum_{n=2}^{\infty}\frac{n^{\log n}}{(\log n)^{n}}$

**Solution:** We have the given series as $\sum_{n=2}^{\infty} a_n$,

where $a_n = \frac{n^{\log n}}{(\log n)^{n}}$

$= \frac{e^{\log n.\log n}}{(\log n)^{n}} \qquad \left(\because a^{x} = e^{\log a^{x}} = e^{x\log a}\right)$

$\therefore \quad (a_n)^{\frac{1}{n}} = \frac{e^{\frac{\log n}{\sqrt{n}}.\frac{\log n}{\sqrt{n}}}}{\log n}$

$\Rightarrow \quad \lim_{n\to\infty} (a_n)^{\frac{1}{n}} = 0 < 1 \qquad \left(\because \lim_{n\to\infty}\frac{\log n}{\sqrt{n}} = 0 \, as \lim_{n\to\infty}\frac{\log n}{\sqrt{n}} = 0\right)$

$\therefore$ by Cauchy root test, $\sum a_n$ converges.

## 14.5 Kummer's Test

If $\sum\frac{1}{d_n}$ is a divergent series. Let

$T_n = D_n \dfrac{a_n}{a_{n+1}} - D_{n+1}$. Then the series $\sum a_n$ converges if $\underline{\text{Lim}}\ T_n > 0$ and diverges if $\overline{\text{Lim}}\ T_n < 0$; both the series being positive terms series.

**Proof :** (i) Set Lim $T_n = g > 0$

Let h be a number s.t. $0 < h < a$

$\therefore$ we can find a positive integer m(h) s.t.

$T_n > h$ for $n \geq m$.

i.e. $D_n \dfrac{a_n}{a_{n+1}} - D_{n+1} > h$ for $n \geq m$

$\therefore$ $h\, a_{n+1} < a_n D_n - a_{n+1} D_{n+1}$ for $n \geq m$

Changing n to m, m+1, m+2, ......, n-1 and adding we get

$h\,(a_{m+1} + a_{m+2} + ..... + a_n) < a_m d_m - a_n D_n$

$< am\ dm$

$\therefore$ $a_{m+1} + a_{m+2} + ..... + a_n < \dfrac{a_m d_m}{h} = k$ for $n \geq m$

$\therefore$ the series converges by Cauchy's General Principle of the positive terms series.

(ii) Let $\overline{\text{Lim}}\ T_n < 0$

$\therefore$ $T_n <$ for $n \geq m$

$\therefore$ $D_n \dfrac{a_n}{a_{n+1}} - D_{n+1} < 0$ for $n \geq m$

or $a_n D_n < a_{n+1} D_{n+1}$ for $n \geq m$

$\therefore$ Sequence $\{a_n D_n\}$ is an increasing sequence.

$\therefore$ $a_m D_m < a_{m+1} D_{m+1} < a_{m+2} D_{m+2} .... < a_n D_n$ for $n \geq m$

$\therefore$ $a_n D_n > a_m D_m = k$ for $n \geq m$

$\therefore$ $a_n > \dfrac{k}{D_n}$ for $n \geq m$

$\therefore$ $\sum \dfrac{1}{D_n}$ diverges.

Hence by comparison Test, $\sum a_n$ diverges.

**Note : Particular form of Kummer's Test.**

If $\sum \frac{1}{D_n}$ is a divergent series and $T_n = D_n \frac{a_n}{a_{n+1}} - D_{n+1}$ , then the series $\sum a_n$ converges if $\underline{\text{Lim}} \frac{a_n}{a_{n+1}} > 1$ and diverges if $\overline{\text{Lim}} \frac{a_n}{a_{n+1}} < 1$.

**Proof :** Take Dn = 1 in Kummer's Test.

$\therefore$ Dn+1 = 1

$\therefore$ $\frac{1}{D_n} = 1 \therefore \sum \frac{1}{D_n}$ is divergent.

$\therefore$ $T_n = D_n \frac{a_n}{a_{n+1}} - D_{n+1} = \frac{a_n}{a_{n+1}} - 1$

$\therefore$ $\sum a_n$ is convergent if $\underline{\text{Lim}}\ T_n > 0$. i.e. if Lim $\left(\frac{a_n}{a_{n+1}} - 1\right) > 0$ or if $\underline{\text{Lim}} \left(\frac{a_n}{a_{n+1}}\right) > 1$.

Likewise $\sum a_n$ is divergent if $\overline{\text{Lim}} \frac{a_n}{a_{n+1}} < 1$.

14.6 D' Alembert's Ratio Test

If $\underset{n \to \infty}{Lim} \frac{a_{n+1}}{a_n} = l$, then the positive terms series $\sum a_n$ converges if $l < 1$ and diverges if $l > 1$.

**Proof :** (i) Set $l < 1$.

Choose $\in > 0$ s.t. $l < l + \in < 1$

$\because$ $\underset{n \to \infty}{Lim} \frac{a_{n+1}}{a_n} = l$

$\therefore$ $\exists$ a positive integer m s.t.

$l - \in < \frac{a_{n+1}}{a_n} < l + \in \forall\ n \geq m$

Put $l + \in = r$ so that $0 < r < 1$.

$\therefore$ $\frac{a_{n+1}}{a_n} < r \ \forall\ n \geq m$

Charging n to n-1, n-2, .... m, we have

$\frac{a_n}{a_{n+1}} <$ r

$\frac{a_{n-1}}{a_{n-2}} <$ r

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$\frac{a_{m+2}}{a_{m+1}} <$ r

$\frac{a_{m+1}}{a_m} <$ r

Multiplying these, we get

$\frac{a_n}{a_m} < r^{n-m}$

or $\quad a_n < a_m' \frac{r^n}{r^m}$

$\therefore \quad a_n < k\, r^n \forall\ n \geq m$ where $k = \frac{a_m}{r^m} > 0$.

Now $\sum k\, r^n$ is a G.P. with common ratio r such that 0 < r < 1.

$\therefore \quad \sum k\, r^n$ converges

$\therefore \quad$ by comparison test $\sum a_n$ converges.

(ii) Let l < 1

Choose $\in > 0$ s.t. $l > l - \in > 1$

Since $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l$

$\therefore \quad \frac{a_{n+1}}{a_n} > l - \in \forall\ n \geq m$

$\therefore \quad \frac{a_{n+1}}{a_n} < 1\ \forall\ n \geq m$

$\Rightarrow \quad a_{n+1} > a_n \forall\ n \geq m$

$\therefore$ $\{a_n\}$ is an increasing so that an does not tend and zero as n → ∞.

$\therefore$ $\sum a_n$ cannot converge.

Since $\sum a_n$ is a positive terms series

$\therefore$ $\sum a_n$ diverges.

Note : Cauchy's Root Test is stronger than D' Alembert's Ratio Test. This is so because $\underset{n\to\infty}{Lim}(a_n)^{\frac{1}{n}}$ exists whenever $\underset{n\to\infty}{Lim}\frac{a_{n+1}}{a_n}$ exists, but the converse is not true.

**Some Illustrated Examples**

**Example 5 :** Show that the series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots.$$ converges for all finite values of x > 0.

Solution : The given series is $\sum a_n$ , where

$$\sum a_n = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots.$$

$\therefore$ $a_n = \frac{x^{n-1}}{n-1!}$, $a_{n+1} = \frac{x^n}{n!}$

Now $\frac{a_{n+1}}{a_n} = \frac{x^n}{n!}\,\frac{n-1!}{x^{n-1}}$

$= \frac{x}{n}$

$\therefore$ $\underset{n\to\infty}{Lim}\frac{a_{n+1}}{a_n} = x\ \underset{n\to\infty}{Lim}\frac{1}{n} = x.0 = 0 < 1$

$\therefore$ by ratio test, $\sum a_n$ converges for all finite values of x > 0.

**Example 6 :** Discuss by a suitable example that Cauchy's Root Test is better than D' Alembert's ratio test.

**Solution :** Set us consider a series $\sum a_n$ as

$$\frac{1}{2^1} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots$$

∴ $a_{2n-1} = \frac{1}{2^{n-1}}$ , $a_{2n} = \frac{1}{3^{2n}}$ , $a_{2n+1} = \frac{1}{2^{2n+1}}$

$(a_{2n-1})^{\frac{1}{2n-1}} = \frac{1}{2}$ , $(a_{2n})^{\frac{1}{2n}} = \frac{1}{3}$

∴ $\lim_{n\to\infty} (a_{2n-1}) = \frac{1}{2}$ , $\lim_{n\to\infty} (a_{2n})^{\frac{1}{2n}} = \frac{1}{3}$

∴ $\overline{\text{Lim}}\, (a_n)^{\frac{1}{n}} = \frac{1}{2} < 1$

∴ $\sum a_n$ is convergent by Cauchy's root Test

Again $\frac{a_{2n-1}}{a_{2n}} = \frac{3^{2n}}{2^{2n-1}} = 3\left(\frac{3}{2}\right)^{2n-1}$

∴ $\lim_{n\to\infty} \frac{a_{2n-1}}{a_{2n}} = \infty$

Also $\frac{a_{2n}}{a_{2n+1}} = \frac{2^{2n}}{3^{2n}} = 2\left(\frac{2}{3}\right)^{2n}$

∴ $\lim_{n\to\infty} \frac{a_{2n}}{a_{2n+1}} = 2\,(0) = 0$

∴ $\underline{Lim}\, \frac{a_n}{a_{n+1}} = 0 < 1$

and $\overline{Lim}\, \frac{a_n}{a_{n+1}} = \infty > 1$

∴ Ratio test does not provide us the result.

∴ Root Test is better than the Ratio Test

**Example 7:** Discuss the convergence of the series

$$\frac{x}{x+1} + \frac{x^2}{x+2} + \frac{x^3}{x+3} + \ldots\ldots. \text{ for } x > 0$$

**Solution:** We have

$$\sum a_n = \frac{x}{x+1} + \frac{x^2}{x+2} + \frac{x^3}{x+3} + \ldots\ldots.$$

$\therefore$ $a_n = \dfrac{x^n}{x+n}$ , $a_{n+1} = \dfrac{x^{n+1}}{x+n+1}$

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ = x. $\lim\limits_{n\to\infty} \dfrac{x+n}{x+n+1}$

= x. $\lim\limits_{n\to\infty} \dfrac{\frac{x}{n}+1}{\frac{x}{n}+1+\frac{1}{n}}$

= x. $\dfrac{0+1}{0+1+0}$ = x . 1 = x

$\therefore$ by Ratio test $\sum a_n$ converges for x < 1 and diverges for x > 1.

For x = 1, the ratio test fails.

For x = 1, $a_n = \dfrac{1}{1+n}$

Take $b_n = \dfrac{1}{n}$

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{\frac{1}{n}+1} = \dfrac{1}{0+1} = 1$, finite and non zero

$\therefore$ $\sum a_n$ and $\sum b_n$ converges or diverges together.

But $\sum b_n = \sum \dfrac{1}{n}$ diverges by p-test

$\therefore$ $\sum a_n$ diverges

**Example 8:** Discuss the convergence of the series $\sum \dfrac{x^n}{n}$, x > 0.

**Solution:** Here the given series is $\sum a_n$

$a_n = \dfrac{x^n}{n}$

$a_{n+1} = \dfrac{x^{n+1}}{n+1}$

∴ $\frac{a_{n+1}}{a_n} = x\left(\frac{n}{n+1}\right) = x\left(\frac{1}{1+\frac{1}{n}}\right)$

∴ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = x\left(\frac{1}{1+0}\right) = x$

∴ by ratio test $\sum a_n$ converges for x < 1 and diverges for n > 1

For x = 1, the ratio test fails

When x = 1 $\sum a_n = \sum \frac{1}{n}$ which diverges by p-test.

**Example 9:** Discuss the convergence of the series $\frac{n}{n^2+1} x^n$, x > 0.

**Solution:** Let the given series is $\sum a_n$ ,

$a_n = \frac{n}{n^2+1} x^n.$

∴ $a_{n+1} = \frac{n+1}{(n+1)^2+1} x^{n+1}$

Now $\frac{a_{n+1}}{a_n} = \frac{n+1}{n^2+1+2n+1} x^{n+1} \times \frac{x^2+1}{nx^n}$

$= \frac{n+1}{n} \cdot \frac{n^2+1}{n^2+2n+2} \cdot x$

$= \left(1+\frac{1}{n}\right) \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \cdot x$

∴ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = (1+0) \cdot \frac{1+0}{1+0+0} = x$

∴ by ratio test, $\sum a_n$ converges for x < 1 and diverges for x > 1.

For x = 1, the ratio test fails

For x = 1, $a_n = \frac{n}{n^2+1}$

Take $b_n = \frac{1}{n}$

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1$$

which is finite and non-zero

$\therefore$ $\sum a_n$ and $\sum b_n$ converges or diverges together.

But $\sum b_n = \sum \frac{1}{n}$ diverges by p-test.

$\therefore$ $\sum a_n$ diverges for x = 1.

## 14.7 Self Check Exercise

Q.1 Examine the convergence or divergence of the series $\sum \frac{n^2}{n!}$

Q. 2 Examine the convergence or divergence of the series

$$\sum \frac{x^{n-1}}{1+x^n}, \ x > 0$$

Q.3 Show that the series $\sum \frac{x^n}{n!}$ converges absolutely for all x.

Q.4 Prove that the series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\dots\dots$ is convergent for $-1 \leq x \leq 1$.

## 14.8 Summary

We have learnt the following concepts in this unit

(i) First form of Cauchy's Root Test

(ii) Cauchy's Root Test

(iii) D' Alembert Ratio Test

We have noticed here that Cauchy's Root test is more powerful than the D' Alembert's Ratio Test.

## 14.9 Glossary:

1. Cauchy's Radical Test - The Cauchy's Root Test is also known as Cauchy's radical test.

2. D' Alembert's Ratio Test - The positive terms series converges if

$$\underline{Lim}\ \frac{a_n}{a_{n+1}} > 1$$

## 14.10 Answer to Self Check Exercise

Ans.1 $\sum a_n$ converges

Ans.2 Take $U_n = \frac{x^{n-1}}{1+x^n}$, $u_{n+1} = \frac{x^{n-1}}{1+x^{n+1}}$, then proceed for 0 < x < 1, x > 1, x = 1.

Ans.3 Take $a_n = \frac{x^n}{n!}$, ∴ $a_{n+1} = \frac{x^{n+1}}{n+1}$ and then proceed.

Ans.4 Take $a_n = (-1)^{n-1}\frac{x^{2n-1}}{2n-1}$, $a_{n+1} = (-1)^n\frac{x^{2n+1}}{2n+1}$ then proceed.

## 14.11 References/Suggested Readings


**1**. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

3. R.G. Bartle and D.R. Sherbert, Introduction to Real trellises, John Wiley and Sons (Asia) P. Ltd., 2000.


## 14.12 Terminal Questions

1. Show that the series $\sum u_n = \frac{(-1)^n (n+2)}{2^n + 5}$ is absolutely convergent.

2. Prove that $\sum \frac{(n-2)^n}{3^n n^2}$ is convergent for $-1 \leq x \leq 5$.

3. Discuss the convergence of the series $\sum \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$

4. Examine the convergence or divergence of the series

$$\sum \frac{x^n}{x^n + a^n},\ x > 0,\ a > 0$$

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## Unit - 15

# Raabe's Test, Guass Test And Logarithmic Test Of Convergence For Infinite Series

**Structure**



### 15.1 Introduction

Dear students, in this unit we shall study few more tests of convergence for an infinite series. The ratio test may be inconclusive when the limit of the ratio is 1. Extension to the ratio test, however, sometimes allow one to deal with this case. Raabe - Duhamel's test. On the this hand, Gauss Test is another root test the convergence of an infinite series.

### 15.2 Learning Objectives

The main objectives of this unit are

(i) to study Raabe Test of convergence

(ii) to learn about D' Morgan and Bertrand Test

(iii) to study the Gauss Test of convergence

(iv) to learn about logarithmic test of convergence for an infinite series.

### 15.3 Art. Raabe's Test

If $\sum a_n$ be a positive terms series, then $\sum a_n$ converges or diverges according as $\underset{x\to a}{Lt}$ n $\left(\frac{a_n}{a_{n+1}}-1\right)$> or < 1.

We accept this result without proof.

**Note. Two Important Notations**

1. If $\underset{x\to a}{Lt}\ \frac{f(x)}{\phi(x)}$ is finite and non-zero, we say that $f(x) = O\{\phi(x)\}$ as $x \to \alpha$.

In other words $f(x) = O\{\phi(x)$ means $|f(x)| < A\,|\phi(x)|$ where A is positive constant.

It must be kept in mind that

$O\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$ and $O\left(\frac{\log n}{n}\right) \to 0$ as $n \to \infty$

Again if $f(x) = O(1)$ means that $f(x)$ is bounded.

2. If $\underset{x\to a}{Lt}\ \frac{f(x)}{\phi(x)} = 0$, then we say that $f(x) = 0\{\phi(x)\}$ as $x \to a$

## 15.4 Art. D'Morgan and Bertrand's Test

If $\sum a_n$ is a positive terms series and

$\underset{n\to\infty}{Lt}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right]\log n = l$, then $\sum a_n$ converges if $l > 1$ and diverges if $l > 1$.

We accept this result without proof.

## 15.5 Art. Gauss's Test

If an > 0 and $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$, then $\sum a_n$ converges for $\mu > 1$ and diverges for $\mu \leq 1$.

**Proof:** (i) Let $\mu \neq 1$

Now $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$

$\Rightarrow \frac{a_n}{a_{n+1}} - 1 = \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$

$\Rightarrow n\left(\frac{a_n}{a_{n+1}}-1\right) = \mu + O\left(\frac{1}{n}\right)$

$\Rightarrow \underset{n\to\infty}{Lt}\left[n\left(\frac{a_n}{a_{n+1}}-1\right)\right] = \mu$

$\therefore$ by Raabe's Test, $\sum a_n$ converges for $\mu > 1$ and diverges for $\mu < 1$.

(ii) Let $\mu = 1$

$\therefore$ $n\left(\frac{a_n}{a_{n+1}} - 1\right) = 1 + O\left(\frac{1}{n}\right)$

$\Rightarrow$ $n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 = O\left(\frac{1}{n}\right)$

$\Rightarrow$ $\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \log n = O\left(\frac{\log n}{n}\right)$

$\Rightarrow$ $\underset{n\to\infty}{Lt}\left[\left\{n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right\}\log n\right] = 0 < 1$ $\left[\because \underset{n\to\infty}{Lt}\frac{\log n}{n} = 0\right]$

$\therefore$ by D' Morgan and Bertrand's Test, $\sum a_n$ diverges.

**Note. Another Form of Gauss's Test.**

If an>0 and $\frac{a_n}{a_{n+1}} = 1 + \frac{\mu}{n} + \frac{\alpha_n}{n^p}$, where p > 1 and $\{\alpha_n\}$ is a bounded sequence, then the series $\sum a_n$ converges for $\mu > 1$ and diverges for $\mu \leq 1$.

**15.6 Art. Logarithmic Test**

If $a_n > 0$, then the series $\sum a_n$

(i) converges if $\underset{n\to\infty}{Lt}\, n \log \frac{a_n}{a_{n+1}} > 1$ and

(ii) diverges if $\underset{n\to\infty}{Lt}\, n \log \frac{a_n}{a_{n+1}} < 1$

We accept this result without proof.

**Illustrative Examples**

**Example 1:** Discuss the convergence of the series $\sum \frac{(n)^2}{2n} x^n$, x > 0

**Solution:** The given series is $\sum a_n$ where

$$a_n = \sum \frac{(n)^2}{2n} x^n$$

∴ an+1 = $\sum \frac{(n+1)^2}{2n+2}$ xn+1

∴ $\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 x^{n+1}}{2n+2} \times \frac{2n}{(n)^2 x^n}$ .x

= $\frac{[(n+1)n]^2}{(2n+2)(2n+1)2n} \times \frac{2n}{(n)^2}$ .x

= $\frac{n+1}{2(2n+1)}$ .x

= $\frac{1+\frac{1}{n}}{2\left(2+\frac{1}{n}\right)}$ x

∴ $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \frac{1+0}{2(2+0)}$ .x = $\frac{x}{4}$

∴ by Ratio Test, $\sum a_n$ converges for $\frac{x}{4}$ < 1 i.e. x < 4 and diverges for $\frac{x}{4}$ < 1 i.e. x < 4 and diverges for $\frac{x}{4}$ > 1 i.e. x > 4.

When x = 4, Ratio Test fails.

For x = 4, $\frac{a_n}{a_{n+1}} = \frac{2(2n+1)}{n+1} . \frac{1}{4} = \frac{2n+1}{2(n+1)}$

∴ $\frac{a_n}{a_{n+1}}$ - 1 = $\frac{2n+1}{2(n+1)}$ - 1

= $\frac{2n+1-2n-2}{2(n+1)}$

= $\frac{1}{2(n+1)}$

∴ n$\left(\frac{a_n}{a_{n+1}} - 1\right)$ = - $\frac{n}{2(n+1)}$

$$= \frac{1}{2\left(1+\frac{1}{n}\right)}$$

$\therefore$ $\underset{n\to\infty}{Lt}$ n$\left(\frac{a_n}{a_{n+1}}-1\right)$ - $\frac{1}{2(1+0)}$ = - $\frac{1}{2}$ < 1

$\therefore$ by Raabe's Test, $\sum a_n$ converges fro x < 4 and diverges for x $\geq$ 4.

**Note. Another form of above question.**

Discuss the convergence or divergence of the series

$$1 + \frac{(1)^2}{2} x + \frac{(2)^2}{4} x^2 + \frac{(3)^2}{6} x^3 + ....., x > 0.$$

**Example 2:** Discuss the convergence or divergence of the series

$$1 + \frac{4}{5} x + \frac{4.6}{5.7} x^2 + \frac{4.6.8}{5.7.9} x^3 + ........$$

**Solution:** The given series is $\sum a_n$ where

$$a_n = \frac{4.6.8........(2n+2)}{5.7.9.........(2n+3)} x^n$$

$\therefore$ $a_{n+1} = \frac{4.6.8........(2n+2)(2n+4)}{5.7.9.........(2n+3)(2n+5)} x^{n+1}$

$\therefore$ $\frac{a_{n+1}}{a_n} = \frac{2n+4}{2n+5}$ x

$$= \frac{1+\frac{4}{2n}}{1+\frac{5}{2n}} x$$

$\therefore$ $\underset{n\to\infty}{Lt} \frac{a_{n+1}}{a_n} = \frac{1+0}{1+0}$ x = x

$\therefore$ by Ratio Test, $\sum a_n$ converges for x < 1 and diverges for x > 1

When x = 1, Ratio Test fails.

For x = 1, $\frac{a_n}{a_{n+1}} = \frac{2n+5}{2n+4}$

$\therefore \quad \frac{a_n}{a_{n+1}} - 1 = \frac{2n+5}{2n+4} - 1$

$= \frac{2n+5-2n-4}{2n+4}$

$= \frac{1}{2n+4}$

$\therefore \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+4}$

$= \frac{1}{2\left(1+\frac{2}{n}\right)}$

$\therefore \quad \underset{n\to\infty}{Lt}\ n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2(1+0)} = \frac{1}{2} < 1$

$\therefore$ by Raabe's Test, $\sum a_n$ converges for x < 1 and diverges for x $\geq$ 1.

**Example 3:** Show that the series

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \ldots\ldots + \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)} + \ldots\ldots$$

Converges if $\beta > \alpha$ and diverges if $\beta \leq \alpha$.

**Solution:** The given series is $\sum a_n$ where

$a_n = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)}$ (Neglecting first term)

$a_{n+1} = \frac{(\alpha+1)(2\alpha+1)\ldots\ldots(n\alpha+1)(\overline{n+1}\alpha+1)}{(\beta+1)(2\beta+1)\ldots\ldots(n\beta+1)(\overline{n+1}\beta+1)}$

$\therefore \quad \frac{a_n}{a_{n+1}} = \frac{\overline{n+1}\alpha+1}{\overline{n+1}\beta+1}$

$= \frac{n\alpha+\alpha+1}{n\beta+\beta+1}$

$$= \frac{\alpha + \frac{\alpha+1}{n}}{\beta + \frac{\beta+1}{n}}$$

$\therefore \quad \underset{n\to\infty}{Lt} \frac{a_n}{a_{n+1}} = \frac{\alpha+0}{\beta+0} = \frac{\alpha}{\beta}$

$\therefore$ by Ratio Test, $\sum a_n$ converges for $\frac{\alpha}{\beta} < 1$ i.e. $\alpha<\beta$

and diverges for $\frac{\alpha}{\beta} > 1$ i.e. $\alpha>\beta$

When $\frac{\alpha}{\beta} = 1$ i.e. $\alpha = \beta$, Ration Test fails.

When $\alpha = \beta$,

$$\frac{a_n}{a_{n+1}} = \frac{n\alpha+\alpha+1}{n\alpha+\alpha+1} = 1$$

$\therefore \quad \frac{a_n}{a_{n+1}} - 1 = 1 - 1 = 0$

$\therefore \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) = 0$

$\therefore \quad \underset{n\to\infty}{Lt}\, n\left(\frac{a_n}{a_{n+1}} - 1\right) = 0 < 1$

$\therefore$ by Raabe's Test, $\sum a_n$ diverges

$\therefore$ given series converges if $\beta>\alpha$ and diverges if $\beta\le\alpha$

**Example 4:** Find whether the series

$x + x^{1+\frac{1}{2}} + x^{1+\frac{1}{2}+\frac{1}{3}} + x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} + ......$ is convergent or divergent.

**Solution:** The given series is $\sum u_n$ where

$u_n = x^{1+\frac{1}{2}+\frac{1}{3}+......\frac{1}{n}}$

$u_{n+1} = x^{1+\frac{1}{2}+\frac{1}{3}+......\frac{1}{n}+\frac{1}{n+1}}$

$$\underset{n\to\infty}{Lt}\frac{u_n}{u_{n+1}} = \underset{n\to\infty}{Lt}\frac{1}{x^{\frac{1}{n+1}}} = \frac{1}{x^0} = \frac{1}{1} = 1$$

$\therefore$ Ratio Test fails.

Now $\log \frac{u_n}{u_{n+1}} = \log\left(\frac{1}{x}\right)^{\frac{1}{n+1}}$

$= \frac{1}{n+1}\log\left(\frac{1}{x}\right)$

$\therefore$ $n \log \frac{u_n}{u_{n+1}} = \frac{n}{n+1}\log \frac{1}{x}$

$= \frac{1}{1+\frac{1}{n}}\log \frac{1}{x}$

$\therefore$ $\underset{n\to\infty}{Lt}\ n \log \frac{u_n}{u_{n+1}} = \frac{1}{1+0}\log \frac{1}{x} = \log \frac{1}{x}$

$\therefore$ By Logarithmic Test, given series is convergent if $\log \frac{1}{x} > 1$

i.e. $\frac{1}{x} > e$

i.e. $x < \frac{1}{e}$ and the series is divergent if $x > \frac{1}{e}$.

When $x = \frac{1}{e}$, Test fails.

Now $n \log \frac{u_n}{u_{n+1}} = \frac{n}{n+1}\log e = \frac{n}{n+1}$

$\therefore$ $n \log \frac{u_n}{u_{n+1}} - 1 = \frac{n}{n+1} - 1$

$= \frac{1}{n+1}$

$\therefore$ $\log n\left[n\log\frac{u_n}{u_{n+1}} - 1\right] = -\frac{\log n}{n+1}$

$\therefore \quad \underset{n\to\infty}{Lt}$ log n $\left[n\log\frac{u_n}{u_{n+1}}-1\right]$ = 0 < 1 $\quad \left[\because \underset{n\to\infty}{Lt}\frac{\log n}{n+1}=0\right]$

$\therefore$ given series is divergent for x = $\frac{1}{e}$

Hence given series $\sum u_n$ is convergent for x < $\frac{1}{e}$ and divergent for x > $\frac{1}{e}$

**Example 5:** Discuss the convergence of the series

$$\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\text{.......} \text{ to infinity.}$$

**Solution:** The given series is $\sum a_n = \frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+$.......

$\therefore \quad a_n = \frac{1^2.5^2.9^2.....(4n-3)^2}{4^2.8^2.12^2.......(4n)^2}$

$\therefore \quad a_{n+1} = \frac{1^2.5^2.9^2.....(4n-3)^2(4n+1)^2}{4^2.8^2.12^2.......(4n)^2(4n+4)^2}$

$\therefore \quad \frac{a_n}{a_{n+1}} = \frac{(4n+4)^2}{(4n+1)^2}$

$$= \frac{\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{1}{4n}\right)^2}$$

$$= \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{4n}\right)^2$$

$$= \left[1+\frac{2}{n}+O\left(\frac{1}{n^2}\right)\right]\left[1-\frac{2}{4n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= 1 + \frac{2}{n} - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{3}{2n} + O\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$$

Where $\mu = \frac{3}{2} > 1$

$\therefore$ by Gauss's Test, given series $\sum a_n$ converges.

**Some More Illustrated Examples**

**Example 6:** Discuss the convergence of the series

$$\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + ......$$

**Solution:** Here

$$\sum a_n \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + ......$$

Where $a_n = \frac{1.3.5.....2n-1}{2.4.6......2n}$

$$\therefore \quad a_{n+1} = \frac{1.3.5.....(2n-1)(2n+1)}{2.4.6......(2n)(2n+2)}$$

Now $$\frac{a_n}{a_{n+1}} = \frac{2n+1}{2n+2} = \frac{2+\frac{1}{2n}}{2+\frac{2}{n}}$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{2+0}{2+0} = 1$$

$\therefore$ Ratio Test fails

Now $$\frac{a_n}{a_{n+1}} - 1 = \frac{2n+2}{2n+1} - 1 = \frac{1}{2n+1}$$

$$\therefore \quad n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\therefore \quad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2+0} = \frac{1}{2} < 1$$

$\therefore$ by Raabe's Test $\sum a_n$ diverges

**Example 7:** Show that the series

$$\sum \frac{1.2.3.......n}{3.5.7.....(2n+1)} x^n, \; x > 0$$

Converges of x < 2 and diverges of x > 2

**Solution:** We have the given series as $\sum a_n$

Where $a_n = \frac{1.2.3........n}{3.5.7.....(2n+1)} x^n$

$\therefore$ $a_{n+1} = \frac{1.2.3........n.(n+1)}{3.5.7.....(2n+1)(2n+3)} x^{n+1}$

Now $\frac{a_{n+1}}{a_n} = \frac{n+1}{2n+3} = \frac{1+\frac{1}{n}}{2+\frac{3}{n}} x$

$\therefore$ $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+0}{2+0} . x = \frac{x}{2}$

$\therefore$ by Ratio test, $\sum a_n$ converges for $\frac{x}{2} < 1$ i.e. x < 2 and diverges

for $\frac{x}{2} > 1$ i.e. x > 2.

when $\frac{x}{2} = 1$ i.e. x = 2, Ratio test fails.

For x = 2

$$\frac{a_n}{a_{n+1}} = \frac{2n+3}{2(n+1)}$$

$\therefore$ $\frac{a_n}{a_{n+1}} - 1 = \frac{2n+3}{2(n+1)} - 1 = \frac{1}{2n+2}$

$\therefore$ $n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{n}{2n+3} = \frac{1}{2+\frac{2}{n}}$

$\therefore \quad \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1}{2+0} = \frac{1}{2} < 1$

$\therefore$ by Raabe's test, $\sum a_n$ converges for x < 2 and diverges for x $\geq$ 2.

**Example 8:** Discuss the convergence of divergence of the series

$$\sum_{n=1}^{\infty} \frac{3.6.9...3n}{4.7.10...(3n+1)} x^n$$

**Solution :** The given series is $\sum a_n$ where

$$a_n = \frac{3.6.9...(3n)}{4.7.10...(3n+1)} x^n$$

$$\therefore \quad a_{n+1} = \frac{3.6.9...(3n)(3n+3)}{4.7.10...(3n+1)(3n+4)} x^{n+1}$$

Now

$$\frac{a_{n+1}}{a_n} = \frac{3n+3}{3n+4} x = \frac{1+\frac{1}{n}}{1+\frac{4}{3n}} x$$

$$\therefore \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+0}{1+0} x = x$$

$\therefore$ by ratio test $\sum a_n$ converges for x < 1 and diverges for x > 1

For x = 1

$$\frac{a_n}{a_{n+1}} = \frac{3n+4}{3n+3} = \frac{1+\frac{4}{3n}}{1+\frac{1}{n}}$$

$$\therefore \quad \frac{a_n}{a_{n+1}} = \left(1+\frac{4}{3n}\right)\left(1+\frac{1}{n}\right)^{-1}$$

$$= \left(1+\frac{4}{3n}\right)\left[1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= 1+\frac{4}{3n}-\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

$$= \quad 1+\frac{1}{3n}+O\left(\frac{1}{n^2}\right)$$

$$= \quad 1+\frac{\mu}{n}+O\left(\frac{1}{n^2}\right)$$

Where $\mu \frac{1}{3} < 1$

$\therefore$ by Gauss's Test, the given series $\sum a_n$ diverges

$\therefore$ The given series $\sum a_n$ converges x < 1 and diverges for x $\geq$ 1.

**Example 9 :** Test the convergence or divergence of the series $\sum \frac{1}{n^\rho}$, ρ> O.

**Solution :** The given series is $\sum a_n$, where $a_n = \frac{1}{n^\rho}$

$\therefore$ $a_{n+1} = \frac{1}{(n+1)^\rho}$

Now $\frac{a_{n+1}}{a_n} = \frac{(n+1)^\rho}{n^\rho} = \left(\frac{(n+1)}{n}\right)^\rho = \left(\frac{1+\frac{1}{n}}{1}\right)^\rho$

$$= 1 + \frac{p}{n} + O\left(\frac{1}{n^2}\right) = 1 + \frac{\mu}{n} + O\left(\frac{1}{n^2}\right)$$

where μ = ρ

$\therefore$ by Gauss's Test, $\sum a_n$ converges for ρ> 1 and diverges for ρ$\leq$ 1

**Example 10 :** Discuss the convergence or divergence of the series

$$1 + \frac{a}{b} + \frac{a}{b}\frac{(a+1)}{(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + ....$$

**Solution :** The given series is $\sum u_n$ where

$$u_n = \frac{a\ (a+1)(a+2)...(a+n-1)}{b\ (b+1)(b+2)...(b+n-1)}$$

$$\therefore \quad u_{n+1} = \frac{a\ (a+1)(a+2)...(a+n-1)(a+n)}{b\ (b+1)(b+2)...(b+n-1)(b+n)}$$

$\therefore \qquad \dfrac{u_n}{u_{n+1}} = \dfrac{b+n}{a+n} = \dfrac{1+\frac{b}{n}}{1+\frac{a}{n}}$

$$= \left(1+\frac{b}{n}\right)\left(1+\frac{a}{n}\right)^{-1}$$

$$= \left(1+\frac{b}{n}\right)\left[1-\frac{a}{n}+O\left(\frac{1}{n^2}\right)\right]$$

$$= 1+\frac{b-a}{n}+O\left(\frac{1}{n^2}\right)$$

$$= 1+\frac{\mu}{n}+O\left(\frac{1}{n^2}\right), \ \mu = b - a.$$

$\therefore \qquad$ by Gauss's Test $\sum u_n$ converges for b - a > 1 and diverges for b - a < 1

**Example 10 :** Discuss the convergence or divergence of the series

$$1+ \frac{2x}{2!} + \frac{3^2 x^2}{3!} + \frac{4^3 x^3}{4!} + ......, \ x> 0.$$

**Solution :** We have the given series as $\sum a_n$ .

Where

$$a_n = \frac{n^{n-1}x^{n-1}}{n!}$$

$\therefore \qquad a_{n+1} = \dfrac{(n+1)^n x^n}{(n+1)!}$

Now

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^n x^n}{(n+1)!} \times \frac{n!}{n^{n-1}x^{n-1}}$$

$$= \frac{(n+1)^{n-1}}{n^{n-1}} \ . \ x = \left(\frac{n+1}{n}\right)^{n-1} . \ x$$

$$= \left(1+\frac{1}{n}\right)^{-1}\left(1+\frac{1}{n}\right)^{n} . \ x$$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ = (1 + 0)$^{-1}$ . e. x = ex

∴ by Ratio test $\sum a_n$ converges for ex < 1

i.e. x .< $\frac{1}{e}$ and diverges for ex > 1 i.e. x > $\frac{1}{e}$

But at x = $\frac{1}{e}$, ratio test fails.

For x = $\frac{1}{e}$, $\frac{a_n}{a_{n+1}}$ = log e + log $\left(1+\frac{1}{n}\right)^{-(n-1)}$

= log e - (n - 1) log $\left(1+\frac{1}{n}\right)$

= 1 - (n -1) $\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}.....\right]$

∴ log $\frac{a_n}{a_{n+1}}$ = $\frac{3}{2n}$ - $\frac{5}{6n^2}$ + O$\left(\frac{1}{n^3}\right)$

⇒ n log $\frac{a_n}{a_{n+1}}$ = $\frac{3}{2}$ - $\frac{5}{6n}$ + O $\left(\frac{1}{n^2}\right)$

∴ $\lim_{n\to\infty}$ n log $\frac{a_{n+1}}{a_n}$ = $\frac{3}{2}$ > 1

∴ by log arithmic test, $\sum a_n$ converges

Hence the given series $\sum a_n$ converges for x ≤ $\frac{1}{e}$ and diverges for x > $\frac{1}{e}$.

## 15.7 Self Check Exercise

Q.1 Discuss the convergence or divergence of the series

$$\frac{n!}{(n+1)^n} x^n, x > 0$$

Q. 2 Discuss the convergence or divergence of the series

$$\sum \sqrt{\frac{n+1}{n^2+1}}, x^n$$

Q.3 Discuss the convergence or divergence of the series.

$$\sum \frac{n!}{a(a+1).......(a+n-1)}, \text{ a > 0}$$

## 15.8 Summary

We have learnt the following concepts in this unit

(i) Raabe's test of convergence and its application

(ii) D' Morgan and Bertrand Test of convergence

(iii) Gaun's Test and its application to test the convergence or divergence of an infinite series.

(iv) Logarithmic test and its application of an infinite series.

## 15.9 Glossary:

1. Raabe's Test (another form) - If $\sum a_n$ be a positive terms series when $\sum a_n$ converges or diverges according as $\lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1\right)$ > or < 1.

2. D' Morgan and Bertrand Test (another form) and $\lim_{n\to\infty} \left[ n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right]$ log n = l, then $\sum a_n$ converges if l > 1 and diverges if l < 1.

## 15.10 Answer to Self Check Exercise

Ans.1 $\sum a_n$ converges for x < e and diverges for x $\geq$ e

Ans.2 By Gauss's test $\sum a_n$ converges for x < 1 and diverges for x $\geq$ 1

Ans.3 By Gauss's test, $\sum u_n$ converges for a - 1 > 1 i.e. a > 2 and diverges for a - 1 < 1 i.e. a < 2.

## 15.11 References/Suggested Readings


1. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

2. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

3. K.A. Ross, Elementary Analysis - The Theory of Calculus Series - Undergrounuate Texts in Mathematics, Springer Verlog, 2003.


## 15.12 Terminal Questions

1. Discuss the convergence of divergence of the following series.

   (i) $\frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} +$.........

(ii) $\sum \frac{1.3.5......(2n-1)}{2.4.6......(2n)} \cdot \frac{1}{n^{\propto}}$

(iii) $\frac{x}{1} + \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5.x^7}{2.4.6.7}$ +.......

2. Discuss the convergence of divergence of the series

$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + ......., a > 0.$

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## Unit - 16

# Sequences of Functions
# (Pontwise and Uniform Convergence)

**Structure**



## 16.1 Introduction

Dear students, you are already familiar with the concept of sequence and series from your previous knowledge's. There we have discussed the case when the terms of the sequence were numbers. In the present unit we shall discuss. Sequences whose terms are real valued functions defined on an interval as domain. We shall denote the term by $f_n$ (x), sequence by $\{f_n\}$ or $<f_n>$ etc.

## 16.2 Learning Objectives

The main objective of this unit are

(i) to study the concept of sequence in a set

(ii) to learn about sequences of functions

(iii) to study pointwise and uniform convergence of sequence of functions

(iv) to study Mn-test for uniform convergence etc.

## 16.3 Sequence in a Set

A sequence in a set E is a mapping of the set N of positive integers into E.

**Example:** A sequence in the set of reals $f : N \to R$ is defined by

$$f(n) = n^2 \forall\ n \in N$$

The image $f(n)$ is denoted by $f_n$ and we call $f_n$ the nth term of sequence. The sequence is often written as $\{f_1, f_2, f_3, \ldots, f_n\}$ or simply $\{f_n\}$.

## 16.4 Sequence of Functions

A sequence whose terms are real valued functions defined on a set E is called sequence of functions.

**Example.** The sequence $\{f_n(x)\}$ where $f_n(x) = \frac{1}{nx} \; \forall \; x \in R, n \in N$ is a sequence of functions.

### Uniformly Bounded Sequence

A sequence $\{f_n\}$ of real valued functions defined on a set E is said to be uniformly bounded on set E if $\exists$ a real number M s.t. $|f_n(x)| < M \; \forall \; x \in E$ and for every positive integer n.

**Example 1.** Consider $\{f_n(x)\}$ defined by $f_n(x) = \sin nx \; \forall \; x \in r$. Show that the sequence $\{f_n\}$ is uniformly bounded.

**Solution:** $f_n(x) = \sin nx \; \forall \; x \in R$

We know that $|\sin nx| < 1 \; \forall \; x \in R, \forall \; n \in N$

$\therefore \quad \{f_n(x)\}$ is uniformly bounded.

**Example 2.** Show that the sequence $\{f_n(x)\}$ defined by $fn(x) = \frac{1}{nx}$ is not uniformly bounded.

**Solution:** $f_n(x) = \frac{1}{nx}$

Since $\frac{1}{nx} \to \infty$ as $x \to 0$

$\therefore \quad f_n(x) = \frac{1}{nx}$ is not uniformly bounded.

## 16.5 Pointwise Convergence

A sequence of functions $\{f_n\}$ defined on set E is said to be pointwise convergent if for each $x \in E$ sequence $\{f_n(x)\}$ or real numbers converges.

Let $\{f_n(a)\}$ converges to $f(a)$ say for $a \in E$. Similarly let $\{f_n(b)\}, \{f_n(c)\}, \ldots$ at points b, c, ...... or E converge to $f(b), f(c), \ldots$ i.e. let the sequences of numbers $\{f_n(x)\}$ converge for $x \in E$.

We can define a real value function $f$ with domain E and range

$\{f(a), f(b), f(c), \ldots\}$ s.t. $f(x) = \lim_{x \to \infty} f_n(x) \; \forall \; x \in E$.

The limit of the function $f_n$ (x) to which it converges for each x ∈ E will itself be a function of x, say $f$(x), then $f$(x) is called limit function of $f_n$(x) on E. We simply express as $f_n(x) \to f$ (x) for each x ∈ E as $n \to \infty$ and say $\{f_n (x)\}$ converges pointwise to $f$ (x) on E.

Thus for x ∈ E and given ε> 0 ∃ m ∈ N such that

$$|f_n (x) - f (x)| < \varepsilon \forall\ n \geq m$$

**Note.** Thus number m in general depends upon x and ε both.

## 16.6 Art. Uniform Convergence on an Interval

A sequence of functions $\{f_n\}$, n = 1,2,3,..... is said to converge uniformly on an interval E to a function $f$ if for any ε> 0 and for all x ∈ E, there exists a + ve integer N (independent of x but dependent on ∈) such that for all x ∈ E.

$$|f_n (x) - f (x)| < \epsilon \forall\ n \geq N$$

**Point of Non-uniform Convergence**

A point such that the sequence does not converge uniformly in any neighbourhood of it however small, is said to be a point of non-uniform convergence of the sequence.

**Art. Difference between Pointwise Convergence and Uniform Convergence.**

In the case of pointwise convergence, for each ε> 0 and for each x ∈ E, there exist a positive integer N (depending upon ε and x both) such that $|f_n (x) - f (x)| < \varepsilon, \forall\ n \geq N$. But in uniform convergence, for each ε> 0, it is possible to find a positive integer N (depending upon ε but not upon x ) such that

$$|f_n (x) - f (x)| < \varepsilon, \forall\ n \geq N \text{ and } \forall\ x \in E$$

**Note 1.** Every point-wise convergent sequence need not be uniformly convergent.

**Note 2.** A sequence which is not pointwise convergent cannot be uniformly convergent.

**Art. Cauchy Criterion for Uniform Convergence**

A sequence of functions $\{f_n\}$ defined on E converges uniformly on E iff for every ε> 0 and for all x ∈ E there exists a positive integer N such that

$$|f_{n+p}(x) - f_n (x)| < \varepsilon, \forall\ n > N,\ p \geq 1$$

**Proof: The condition is necessary**

Assume that the sequence $\{f_n\}$ converges uniformly on E to the limit function $f$.

∴ to a given ε> 0, and for all x ∈ [a, b] there exists positive integers $m_1$, $m_2$ such that

$$|f_n (x) - f (x)| < \frac{\varepsilon}{2} \qquad \forall\ n \geq m_1$$

and $$|f_{n+p} (x) - f (x)| < \frac{\varepsilon}{2} \qquad \forall\ n \geq m_2$$

Let N = max $\{m_1, m_2\}$

$\therefore$ $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall$ n $\geq$ N

and $|f_{n+p}(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall$ n $\geq$ N, p $\geq$ 1

Now $|f_{n+p}(x) - f_n(x)| = |f_{n+p}(x) - f(x) + f(x) - f_n(x)|$

$$= \left|\left[f_{n+p}(x) - f(x)\right] - \left[f_n(x) - f(x)\right]\right|$$

$$< \left|f_{n+p}(x) - f(x)\right| + \left|f_n(x) - f(x)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\ n \geq N,\ p \geq 1$$

$\therefore$ $\left|f_{n+p}(x) - f(x)\right| < \varepsilon\ \forall$ n $\geq$ N, p $\geq$ 1 ..... (1)

**The condition is sufficient.**

Assume that the given condition holds. By Cauchy's general principle of convergence $\{f_n\}$ converges for each x $\in$ E to a limit say $f$. Thus the sequence converges pointwise to $f$. We shall prove that this convergence is uniform.

For a given $\varepsilon > 0$, choose a positive integer N such that (1) holds. Fix n, and let $p \to \infty$ in (1).

$\therefore$ $f_{n+p} \to f$ as $p \to \infty$

$\therefore$ $|f(x) - f_n(x)| < \varepsilon\ \forall$ n $\geq$ N and for all x $\in$ E [$\because$ of (1)]

i.e. $\left|f_n(x) - f(x)\right| < \varepsilon\ \forall$ n $\geq$ N and $\forall$ x $\in$ E

$\therefore$ $f_n(n) \to f(x)$ uniformly on E

i.e., sequence $\{f_n(x)\}$ converges uniformly to $f(x)$ on E.

**Note.** Above result can be written as

$|f_n(x) - f_m(x)| < \varepsilon$ $\forall$ n, m $\geq$ N

**Art.** Let $\{f_n(x)\}$ and $\{g_n(x)\}$ be two sequence defined on set E. If $\{f_n(x)\}$ and $\{g_n(x)\}$ converge uniformly to $f(x)$ and g (x) respectively on set E then $\{f_n(x) + g_n(x)\}$ and $\{f_n(x) - g_n(x)\}$ converge uniformly to $f(x) + g(x)$ and $f(x) - g(x)$ respectively.

**Proof:** Since $\{f_n(x)\}$ and $\{g_n(x)\}$ converge uniformly to $f(x)$ and g (x) respectively on E.

Therefore for every $\varepsilon > 0$ $\exists$ two positive integers $m_1$ and $m_2$ such that

$|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ $\forall$ n > $m_1$ and $\forall$ x $\in$ E ..... (1)

and $|g_n(x) - g(x)| < \frac{\varepsilon}{2}\ \forall\ n \geq m_2$ and $\forall\ x \in E$ ..... (2)

Take m = max $\{m_1, m_2\}$

$\therefore$ $|f_n(x) - f(x)| < \frac{\varepsilon}{2}\ \forall\ n \geq m$ and $\forall\ x \in E$ ..... (3)

and $|g_n(x) - g(x)| < \frac{\varepsilon}{2}\ \forall\ n \geq m$ and $\forall\ x \in E$ ..... (4)

Now for $n \geq m$, $x \in E$, we have

$|f_n(x) + g_n(x) - (f(x) + g(x)| = |\{f_n(x) - f(x)\} + \{g_n(x) - g(x)\}|$

$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ [$\because$ of (3) and (4)]

$\therefore$ $\{f_n(x) + g_n(x)\}$ converges uniformly on E to $f(x) + g(x)$

Again $|(f_n(x) - g_n(x)) - (f(x) - g(x))| = |(f_n(x) - f(x)) - (g_n(x) - g(x))|$

$\leq |(f_n(x) - f(x) + |g_n(x) - g(x)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ [$\because$ of (3) and (4)]

$\therefore$ $\{f_n(x) - g_n(x)\}$ converges uniformly to $f(x) - g(x)$ on E.

**Art.** Verify that product of two uniformly convergent sequences need not be uniformly convergent.

**Proof:** Let $f_n(x) = \frac{n+x+1}{(n+1)x}\ \forall\ n \in N,\ x \in (0, 1)$

$g_n(x) = \frac{(n+1)x^2}{1+n^2x^2}\ \forall\ n \in N,\ x \in (0, 1)$

$\underset{n\to\infty}{Lt} f_n(x) = \frac{1}{x}$ and $\underset{n\to\infty}{Lt} g_n(x) = 0$ where $x \in (0, 1)$

$\therefore$ the sequences $\{f_n(x)\}$ and $\{g_n(x)\}$ converge pointwise to $\frac{1}{x}$ and 0 respectively on (0, 1).

$$\left|f_n(x) - \frac{1}{x}\right| = \left|\frac{n+x+1}{(n+1)x} - \frac{1}{x}\right|$$

$= \dfrac{n+x+1-n-1}{(n+1)x}$

$= \dfrac{1}{n+1} \to 0$ as $n \to \infty$

∴ $\{f_n(x)\}$ converges uniformly on (0, 1)

$|g_n(x) - 0| = \left|\dfrac{(n+1)x^2}{1+n^2x^2} - 0\right|$

$= \dfrac{(n+1)x^2}{1+n^2x^2} < \dfrac{n+1}{n^2} \to 0$ as $n \to \infty$

∴ $\left\{\dfrac{(n+1)x^2}{1+n^2x^2}\right\}$ converges uniformly to 0 on (0, 1)

Product = $f_n(x)\, g_n(x) = \dfrac{nx+x^2+x}{1+n^2x^2}$ for each $x \in (0, 1)$

$\underset{n\to\infty}{Lt} f_n(x)\, g_n(x) = \underset{n\to\infty}{Lt} \dfrac{nx+x^2+x}{1+n^2x^2}$

$= \underset{n\to\infty}{Lt} \dfrac{\dfrac{x}{n} + \dfrac{x^2+x}{n^2}}{\dfrac{1}{n^2+x^2}}$

$= \dfrac{0+0}{0+x} = 0$

Assume that $\{f_n(x)\, g_n(x)\}$ converges uniformly to 0 so that for $\varepsilon = \dfrac{1}{4}$ $\exists$ $m \in N$

$|f_n(x)\, g_n(x) - 0| = \left|\dfrac{nx+x^2+x}{1+n^2x^2} - 0\right|$

$= \left|\dfrac{nx+x^2+x}{1+n^2x^2}\right| < \dfrac{1}{4}$ $\forall$ $n > m$ and $x \in (0, 1)$

Taking $x = \dfrac{1}{n} \in (0, 1)$, we get

$$\left|\frac{1+\frac{1}{n^2}+\frac{1}{n}}{1+1}\right| = \left|\frac{1+\frac{1}{n^2}+\frac{1}{n}}{2}\right| > \frac{1}{2} > \frac{1}{4}$$ which contradicts the supposition.

∴ $\{f_n(x)\, g_n(x)\}$ is not uniformly convergent.

**Illustrative Examples**

**Example 3:** Does pointwise convergence imply uniform convergence ? Justify your answer.

**Solution:** Let $f_n(x) = \dfrac{nx}{1+n^2x^2} \quad \forall\, x \in R$

Then $f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$$= \underset{n\to\infty}{Lt} \frac{nx}{1+n^2x^2}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{x}{n}}{\frac{1}{n^2}+x^2}$$

$$= \frac{0}{0+x^2} = 0 \ \forall\, x \in R \text{ with } x \neq 0$$

Also, when x = 0, $f_n(x) = 0$

$\Rightarrow$ $f(x) = 0$

∴ $f(x) = \underset{n\to\infty}{Lt} f_n(x) \ \forall\, x \in R$

∴ the sequence $\{f_n)$ is pointwise convergent.

We shall show that convergence is not uniform in any interval [a, b] including '0'.

Assume that $\{f_n\}$ converges uniformly in [a, b]

∴ to a given $\varepsilon > 0$, ∃s a positive integer N such that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \in N$$

i.e. $$\left|\frac{nx}{1+n^2x^2} - 0\right| < \varepsilon \qquad \forall\, n \in N$$

Take $\varepsilon = \frac{1}{4}$ and an integer K with $K \geq N$

Such that $\frac{1}{K} \in$ [a, b]

Taking n = K, x = $\frac{1}{K}$, we have

$$\frac{nx}{1+n^2x^2} = \frac{K.\frac{1}{K}}{1+K^2.\frac{1}{K^2}}$$

$$= \frac{1}{1+1} = \frac{1}{2} < \frac{1}{4}$$

which contradict the supposition.

$\therefore$ the sequence is not uniformly convergent in the interval [a, b] which contains the point $\frac{1}{K}$.

Since $\frac{1}{K} \to 0$ as $K \to \infty$,

$\therefore$ the interval contains the point '0'.

Hence, the sequence is not uniformly convergent on any interval [a, b] containing '0'.

**Another Statement :** Show that the sequence $\{f_n (x)\}$, When $f_n (x) = \frac{nx}{1+n^2x^2}$ is not uniformly convergent on any interval containing 0.

**Example 4.** Show that the sequence $\{f_n\}$, where $f_n (x) = \frac{1}{x+n}$ is uniformly convergent in any interval [0, b], b > 0.

**Solution:** Here $f_n (x) = \frac{1}{x+n}$

$$f (x) = \underset{n\to\infty}{Lt} f_n (x)$$

$$= \underset{n\to\infty}{Lt} \frac{1}{x+n}$$

$$= 0 \ \forall \ x \in [0, b]$$

$\therefore$ the sequence converges pointwise to '0'.

Again for any $\varepsilon > 0$,

$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$

if $x + n > \frac{1}{\varepsilon}$

i.e., if $n > \frac{1}{\varepsilon} - x$,

which decreases with x and its maximum value = $\frac{1}{\varepsilon}$

Let N be an integer $\geq \frac{1}{\varepsilon}$, so that for $\epsilon > 0$, there exists a positive integer N such that

$|f_n(x) - f(x)| < \varepsilon \quad \forall\, n \in N$

$\therefore$ the sequence is uniformly convergent in any interval [0, b], b > 0

**Example 5:** Show that the sequence $\{f_n(x)\}$ defined by

$f_n(x) = nxe^{-nx^2} \ \forall\, n \in N$

Convergent pointwise but not uniformly in $[0, \infty]$

**Solution:** Here $f_n(x) = nxe^{-nx^2} \ \forall n \in N$

We have

$$f(x) = \lim_{x\to\infty} nxe^{-nx^2}$$

$$= \lim_{x\to\infty} \frac{nx}{e^{nx^2}}$$

$$= \lim_{x\to\infty} \frac{x}{x^2 e^{nx^2}}$$

$$= \lim_{x\to\infty} \frac{1}{xe^{nx^2}} \qquad [\because \text{ of L' Hospital Rule}]$$

$= 0 \ \forall\, x \in [0, \infty]$

$\therefore$ sequence converges pointwise to 0.

Assume that the sequence is uniformly convergent in $[0, \infty]$

$\therefore$ given $\varepsilon > 0 \ \exists$ t such that $\forall\, n \geq n$ and $\forall\, x \geq 0$

$$|f_n(x) - f(x)| = nxe^{-nx^2} < \varepsilon \qquad \text{...... (1)}$$

Let $t_0$ be an integer greater than t and $e^2\varepsilon^2$

Putting t = $t_0$, x = $\frac{1}{\sqrt{t_0}}$ in (1) we get $t_0$ = $\frac{1}{\sqrt{t_0}}$ $e^{-1} < \varepsilon$

or $\sqrt{t_0} < e\varepsilon$

or $t_0 < e^2 e^2$ which is contradiction to fact that $t_0 > e^2\varepsilon^2$

∴ given sequence is not uniformly convergent in [0, ∞]

**16.7 $M_n$-Test for Uniform Convergence**

If $\{f_n\}$ is a sequence of functions such that $\underset{n\to\infty}{Lt} f_n(x) = f(x)$, x ∈ [a, b] and

$M_n = \underset{x\in[a,b]}{Sup} \{|f_n(x) - f(x) : x \in [a,b]|\}$

then $f_n \to f$ uniformly on [a, b] iff $M_n \to 0$ as $n \to \infty$

**Proof:** The condition is necessary

Assume that $f_n \to f$ uniformly on [a, b]

∴ to given $\varepsilon > 0$, there exists a +ve integer N such that

$|f_n(x) - f(x)| < \varepsilon \; \forall \; n \in N$ and $\forall \; x \in [a, b]$

∴ $M_n = \underset{n\to\infty}{\sup} |f_n(x) - f(x)| < \varepsilon \forall n \in N$

i.e. $M_n < \varepsilon \; \forall \; n \in N$

∴ $M_n \to 0$ as $n \to \infty$

**The condition is sufficient.**

Assume that $M_n \to 0$ as $n \to \infty$

∴ to a given $\varepsilon > 0$, there exists a +ve integer N such that

$M_n < \varepsilon \forall n \in N$

∴ $\underset{x\in[a,b]}{Sup} |f_n(x) - f(x)| < \varepsilon \; \forall \; n \in N$

or $|f_n(x) - f(x)| < \varepsilon \; \forall \; n \in N$ and $\forall \; x \in [a, b]$

∴ $f_n \to f$ uniformly on [a, b]

**Illustrative Examples**

**Example 6.** Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}$, x ∈ R converges uniformly on any closed interval.

**Solution:** Here $f_n(x) = \frac{x}{1+nx^2}$

$\therefore \quad f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$$= \underset{n\to\infty}{Lt} \frac{x}{1+nx^2}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{x}{n}}{\frac{1}{n}+x^2}$$

$$= \frac{0}{0+x^2} = 0 \text{ if } x \neq 0$$

Also for x = 0, each $f_n(x) = 0$,

$\therefore \quad f(x) = 0$

Thus $f(x) = 0 \quad \forall\, x \in R$

Now $\quad |f_n(x) - f(x)| = \left|\frac{x}{1+nx^2} - 0\right|$

$$= \left|\frac{x}{1+nx^2}\right|$$

Let $\quad y = \frac{x}{1+nx^2}$

$\therefore \quad \frac{dy}{dx} = \frac{(1+nx^2).1 - x(2nx)}{\left(1+nx^2\right)^2}$

$$= \frac{1+nx^2-2nx^2}{\left(1+nx^2\right)^2}$$

$$= \frac{1-nx^2}{\left(1+nx^2\right)^2}$$

and $\frac{d^2y}{dx^2} = \frac{\left(1+nx^2\right)^2(0-2nx)-(1-nx^2).2(1+nx^2).2nx}{\left(1+nx^2\right)^4}$

$$= \frac{-2(1+nx^2)nx - 4nx(1-nx^2)}{(1+nx^2)^3}$$

$$= \frac{-2nx[1+nx^2+2-2nx^2]}{(1+nx^2)^3}$$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{-2nx(3-nx^2)}{(1+nx^2)^3}$

For y to be maximum or minimum,

$$\frac{dy}{dx} = 0$$

$\Rightarrow \quad \dfrac{1-nx^2}{\left(1+nx^2\right)^2} = 0$

$\Rightarrow \quad 1 - nx^2 = 0$

$\Rightarrow \quad x^2 = \dfrac{1}{n}$

$\Rightarrow \quad x = \dfrac{1}{\sqrt{n}}$

When $x = \dfrac{1}{\sqrt{n}}$, $\dfrac{d^2y}{dx^2} = \dfrac{-2\sqrt{n}(3-1)}{(1+1)^3}$

$$= \frac{-4\sqrt{n}}{8}$$

$$= \frac{-\sqrt{n}}{2} < 0$$

$\therefore \quad$ y is maximum when $x = \dfrac{1}{\sqrt{n}}$

and maximum value $\dfrac{\frac{1}{\sqrt{n}}}{1+1} = \dfrac{1}{2\sqrt{n}}$

$\therefore \quad M_n = \sup_{x\in[a,b]} \left\{\left| f_n(x) - f(x)\right|\right\}$

$$= \underset{x \in [a,b]}{Sup} \left\{ \left| \frac{x}{1+nx^2} \right| \right\}$$

$$= \frac{1}{2\sqrt{n}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Therefore by $M_n$ Test, given sequence converges uniformly to $f$ on [a, b]

**Example 7:** Test the sequence $\{f_n(x)\}$, where $f_n(x) = \frac{n^2x}{1+n^3x^2}$ for uniform convergence on [0, 1]

**Solution:** We have $f_n(x) = \frac{n^2x}{1+n^3x^2}$

$\therefore$ $f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$$= \underset{n\to\infty}{Lt} \frac{n^2x}{1+n^3x^2}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{x}{n}}{\frac{1}{n^3}+x^2}$$

$= 0 \ \forall \ x \in [0, 1]$

Now, $|f_n(x) - f(x)| = \left| \frac{n^2x}{1+n^3x^2} - 0 \right|$

$$= \frac{n^2x}{1+n^3x^2}$$

Let $y = \frac{n^2x}{1+n^3x^2}$

Then $\frac{dy}{dx} = \frac{\left(1+n^3x^2\right)n^2 - n^2x(2n^3x)}{\left(1+n^3x^2\right)^2}$

$$= \frac{n^2 - n^5x^2}{(1+n^3x^2)^2}$$

For maxima or minima, $\frac{dy}{dx} = 0$

$\Rightarrow \quad n^2 - n^5 x^2 = 0$

$\Rightarrow \quad x = \dfrac{1}{n^{\frac{3}{2}}}$

Now $\quad \dfrac{d^2y}{dx^2} = \dfrac{\left(1+n^3x^2\right)^2(-2n^5x)-(n^2-n^5x^2)2(1+n^3x^2)2n^3x}{\left(1+n^3x^2\right)^4}$

$= \dfrac{-2n^2x(3n^2-n^5x^2)}{\left(1+n^3x^2\right)^2}$

$\therefore \quad = \left.\dfrac{d^2y}{dx^2}\right]_{x=\frac{1}{n^{\frac{3}{2}}}} = \dfrac{-2n^3\left(\dfrac{1}{n^{\frac{3}{2}}}\right)\left(3n^2-n^5.\dfrac{1}{n^3}\right)}{\left(1+n^3.\dfrac{1}{n^3}\right)}$

$= \dfrac{-n^{\frac{7}{2}}}{2} < 0$

$\therefore$ y is maximum at $x = \dfrac{1}{n^{\frac{3}{2}}}$ and

$y_{max} = \dfrac{n^2.\dfrac{1}{n^{\frac{3}{2}}}}{1+n^3.\dfrac{1}{n^3}}$

$= \dfrac{\sqrt{n}}{2}$

$\therefore \quad M_n = \sup \{|f_n(x) - f(x)| : x \in [0, 1]\}$

$= \sup\left\{\dfrac{n^2x}{1+n^3x^2} : x \in [0,1]\right\}$

$= \dfrac{\sqrt{n}}{2} \to \infty$ as $n \to \infty$

$\therefore$ by $M_n$ - test, $\{f_n(x)\}$ does not converge uniformly on [0, 1]

**Example 8 :** Show that seq. $\{f_n\}$ where $f_n(x) = nx(1-x)^n$ does not converge uniformly on [0,1].

Solution : $f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$$= \underset{n\to\infty}{Lt}\, nx(1-x)^n$$

$$= \underset{n\to\infty}{Lt} \frac{nx}{(1-x)^{-n}} \qquad \left[\frac{\infty}{\infty}\right]$$

(By L-Hospital's Rule)

$$= \underset{n\to\infty}{Lt} \frac{x}{-(1-x)^{-n}\log(1-x)}$$

$$= \underset{n\to\infty}{Lt} \frac{x(1-x)^n}{\log(1-x)}$$

= 0 since $(1-x)n \to 0$ as $n \to \infty$

$f(x) = 0\ \forall\ x \in [0, 1]$

$M_n = \sup\{|f_n(x) - f(x)| : x \in [0, 1]\}$

$= \sup[nx(1-x)^n : x \in [0, 1]]$

$$\geq n\,\frac{1}{n}\left(1-\frac{1}{n}\right)^n \qquad \left[\text{Take } x\ \frac{1}{n}\right]$$

$$= \left(1-\frac{1}{n}\right)^n \to \frac{1}{\varepsilon} \text{ as } n \to \infty$$

Hence by $M_n$ test, $\{f_n\}$ does not converge uniformly on [0, 1]

Hence 0 is a point of non-uniform convergence since as $n \to \infty$, $x \to 0$.

**Example 9 :** Show that 0 is a point of non-uniform convergence of the seq. $\{f_n(x)\}$ when $f_n(x) = nxe^{-nx^2}$

**Solution :** $f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$$= \underset{n\to\infty}{Lt}\, nxe^{-nx^2}$$

$$\underset{n\to\infty}{Lt} \frac{nx}{e^{nx^2}} = 0\ \forall\ n \in R$$

$\therefore$ $M_n = \sup\{|f_n(x) - f(x)| : x \in R\}$

$= \sup \{n (x)\ e^{-nx^2} : x \in R\}$

$\geq n \cdot \frac{1}{\sqrt{n}} \cdot e^{-n.\frac{1}{n}}$ [Taking $x = \frac{1}{\sqrt{n}} \in R$]

$= \frac{\sqrt{n}}{e} \to \infty$ as $n \to \infty$

$\therefore$ $M_n \not\to 0$ as $n \to \infty$

$\therefore$ the sequence does not cgs.

Since $x \to 0$ as $n \to \infty$

$\therefore$ 0 is a point of non-uniform convergence.

**Example 10 :** Show that the sequence $\{f_n (x)\}$ defined by $f_n (x) = \frac{\sin nx}{\sqrt{n}}$ is uniformly convergent on $[0, 2\pi]$

**Solution :** Here $f_n (x) = \frac{\sin nx}{\sqrt{n}}$

Now $f (x) = \underset{n\to\infty}{Lt} \frac{\sin nx}{\sqrt{n}} = 0$ $\left[\begin{array}{l} \because \sin nx \text{ is bounded} \\ \text{and } \frac{1}{\sqrt{n}} \to 0 \text{ as } n \to \infty \end{array}\right]$

$M_n = \sup \{|f_n (x) - f (x)| : x \in [0,2\pi]\}$

$= \sup \left\{\left|\frac{\sin nx}{\sqrt{n}}\right| : x \in [0, 2\pi]\right\}$

Maximum value of sin nx is 1 where $x = \frac{\pi}{2n}$

$\therefore$ $M_n = \frac{1}{\sqrt{n}}$

$\therefore$ $M_n \to 0$ as $n \to \infty$

$\therefore$ by $M_n$ test, given uniformly on $[0, 2\pi]$

## 16.8 Self Check Exercise

Q.1 Show that the sequence $\{f_n\}$ where $f_n (x) = \tan^{-1} (n\ x)$, $x \geq 0$ is uniformly convergent in [a, b], a > 0 but is only pointwise convergent in [a, b]

Q. 2 Show by $M_n$ - Test 0 is the point of non-uniform convergence of the sequence $\{fn (x)\}$, $fn(x) = 1 - (1-x^2)^n$.

Q. 3 Show that the sequence $\{f_n(x)\}$, defined by

$$f_n(x) = \frac{\sin nx}{\sqrt{n}}$$ is uniformly convergent on $[0, 2\pi]$.

## 16.9 Summary

We have learnt the following concepts in this unit:

(i) definition of sequence in a set

(ii) definition of sequence of functions

(iii) learnt pointwise convergence

(iv) the uniform convergence

(v) $M_n$-test for uniform convergence.

## 16.10 Glossary:

1. Limit Function : The limit of a function $f_n(x)$ to which it converges for each $x \in E$ will itself be a function of x, ray $f(x)$, then $f(x)$ is called limit function of $f_n(x)$. on E.

2. Uniformly Bounded Sequence : A sequence $\{f_n\}$ of real valued function defined on a set E is said to be uniformly bounded on set E if $\exists$ a real number M s.t. $|f_n(x)| < M \; \forall \; x \in E$, and for every positive integer n.

## 16.11 Answer to Self Check Exercise

Ans.1 Take $f(x) = \underset{n\to\infty}{Lim} f_n(x) = \begin{cases} \frac{\pi}{2}, x > 0 \\ 0, x = 0 \end{cases}$

and then proceed.

Ans.2 Here $M_n$ Sup $\{(1 - x^2)^n : x \in (0, \sqrt{2})\}$ and then proceed for $M_n$ - test.

Ans.3 Here $f(x) = \underset{n\to\infty}{Lim} \frac{\sin nx}{\sqrt{n}} = 0$, as sin nx is bounded.

Now proceed for $M_n$-test.

## 16.12 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

3. R.G. Bartle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000.

## 16.13 Terminal Questions

1. Show that 0 is a point of non-uniform convergence of the sequence $\{f_n(x)\}$, $f_n(x) = n\,x\,e^{-nx^2}$, $x \in R$.

2. Show that the sequence $\{f_n(x)\}$, $f_n(x) = \frac{x^n}{1+x^n}$, $n \in N$ ; converges uniformly on $\left[0, \frac{1}{2}\right]$

3. Use $M_n$ - test to show that $\{f_n(x)\}$, when $f_n(x) = \frac{nx}{1+n^2x^2}$, does not converge uniformly on [0, 1].

4. Show that the sequence $\{f_n(x)\}$, $f_n(x) = \frac{x}{n(1+nx^2)}$ is uniformly convergent on R.

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## Unit - 17

# Series of Functions and The Uniform Convergence

**Structure**



### 17.1 Introduction

Dear students, we have already discussed about the sequence of function and their convergence in our previous unit. In this unit we shall study the series of functions and their convergence. A series is the sum of the terms of an infinite sequence of functions. A function series is a series where the summands are not first-real or complex numbers but functions. Examples of function series include power series, Lamrent Series, Fourier series etc. We shall denote the series of functions by $\Sigma f$n.

### 17.2 Learning Objectives

The main objective of this unit are

(i) to define what do we mean by series of functions

(ii) to study sequence of partrol sums of given series

(iii) to learn about uniform convergence of series of functions

(iv) to test the convergence of a series we shall study Weierstrass M-Test.

### 17.3 Series of Function

A series of the form

$$u_1(x) + u_2(x) + u_3(x) + ....+ ... = \sum_{n=1}^{\infty} u_n(x)$$

Where $u_1$ (x), $u_2$ (x) ...... are real valued functions defined on a set E is called series of Functions.

**17.4 Sequence of Partial Sums of Given Series**

Let $\sum_{n=1}^{\infty}$ $u_n$ (x) = $u_1$ (x), $u_2$ (x) ...... be a series of function defined on a set E, then the sequence {$f_n$ (x)}, where

$$f_n (x) = u_1 (x) + u_2 (x) + ....+ u_n (x) = \sum_{r=1}^{n} u_r (x)$$

$\forall$ n $\in$N is called sequence of partial sums of the series $\sum_{n=1}^{\infty}$ $u_n$ (x) and $f_n$ (x) is called nth partial sum of the series.

**17.5 Uniform Convergence of the Series of Functions.**

The series $\sum_{n=1}^{\infty} u_n$ (x) is said to converge uniformly on set E if the sequence {$f$n (x)} where

$f_n$ (x) = $u_1$ (x) + $u_2$ (x) +.......+ $u_n$ (x) $\forall$ n $\in$N converges uniformly on E and limiting functions of {$f_n$ (x)} is called sum function of given series $\sum_{n=1}^{\infty} u_n$ (x).

**Note.** A series of functions $\sum f_n$ converges uniformly to $f$ on E if for every $\varepsilon$> 0 and for all x $\in$ E, there exists a positive integer N ($\varepsilon$, x) such that for all x $\in$ E,

$|\{f_1 (x) + f_2 (x) + f_3 (x) +........+ f_n (x)\} - f (x)| < \varepsilon \; \forall n \in N.$

Note. Cauchy Criterion for Uniform Convergence of Series

A series of functions $\sum f_n$ defined on [a, b] converges uniformly on [a, b] iff for every $\varepsilon$> 0 and for all x $\in$ [a, b], there exists a positive integer N such that

$|f_{n+1} (x) - f_{n+2} (x) +........+ f_{n+p} (x)| < \varepsilon \; \forall n > N, p \geq 1$

$|f_{m+1} (x) - f_{m+2} (x) +........+ f_n (x)| < \varepsilon \quad \forall n, m \geq N$

**Example 1:** The sum to n terms of a series is $f$n (x) $\frac{n^2x}{1+n^4x^2}$. Show that it converges non-uniformly in the interval [0, 1]

**Solution:** Here $f_n (x) = \frac{n^2x}{1+n^4x^2}$

$\therefore \quad f (x) = \underset{n\to\infty}{Lt} f_n (x)$

$$= \underset{n\to\infty}{Lt} \frac{n^2 x}{1+n^4x^2}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2} = 0$$

Now $|f_n(x) = f(x)| = \left|\frac{n^2x}{1+n^4x^2} - 0\right|$

$$= \left|\frac{n^2x}{1+n^4x^2}\right|$$

Let $y = \frac{n^2x}{1+n^4x^2}$

Then $\frac{dy}{dx} = \frac{(1+n^4x^2)n^2 - n^2x(2n^4x)}{(1+n^4x^2)^2}$

$$= \frac{n^2\left[1+n^4x^2-2n^4x^2\right]}{(1+n^4x^2)^2}$$

$$= \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$$

For y to be maximum or minimum, we have

$$\frac{dy}{dx} = 0$$

$\Rightarrow$ $\frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2} = 0$

$\Rightarrow$ $1 - n^4x^2 = 0$

$\Rightarrow$ $n^4x^2 = 1$

$\Rightarrow$ $x^2 = \frac{1}{n^4}$

$\Rightarrow$ x = $\frac{1}{n^2}$

Again $\frac{d^2y}{dx^2}$ = $\frac{n^2\left(1+n^4x^2\right)^2\left(-2n^4x\right)n^2\left(1-n^4x^2\right).2\left(1+n^4x^2\right).2n^4x}{\left(1+n^4x^2\right)^2}$

= $\frac{-2n^6x\left(1+n^4x^2\right)\left[1+n^4x^2+2\left(1-n^4x^2\right)\right]}{\left(1+n^4x^2\right)^4}$

= $\frac{-2n^6x\left(1+n^4x^2+2-2n^4x^2\right)}{\left(1+n^4x^2\right)^3}$

= $\frac{-2n^6x\left(3-n^4x^2\right)}{\left(1+n^4x^2\right)^3}$

When x = $\frac{1}{n^2}$ , $\frac{d^2y}{dx^2}$ = $\frac{-2n^6.\frac{1}{n^2}\left(3-n^4.\frac{1}{n^4}\right)}{\left(1+n^4.\frac{1}{n^4}\right)^3}$

= $\frac{2n^4\left(3-1\right)}{\left(1+1\right)^3}$

= - $\frac{4n^4}{8}$

= - $\frac{n^4}{2}$ < 0

$\therefore$ y is maximum when x = $\frac{1}{n^2}$

and maximum value = $\frac{n^2.\frac{1}{n^2}}{1+n^4.\frac{1}{n^4}}$ = $\frac{1}{2}$

Also x = $\frac{1}{n^2}$ $\rightarrow$ 0 as n $\rightarrow \infty$

∴ $M_n = \sup_{x\in[0,1]} |f_n(x) = f(x)|$

$$= \sup_{x\in[0,1]} \left|\frac{n^2x}{1+n^4x^2} - 0\right|$$

$$= \sup_{x\in[0,1]} \left|\frac{n^2x}{1+n^4x^2}\right| = \frac{1}{2}$$

Which does not tend to zero as $n \to \infty$.

Hence $\{f_n(x)\}$ does not converge uniformly to $f$ on [0, 1].

**Example 2:** Show that the series

$$\frac{x}{1+x^2} + \left(\frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}\right) + \ldots\ldots$$

does not converge uniformly on [0, 1]

**Solution:** The given series is $\frac{x}{1+x^2} + \left(\frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}\right) + \ldots\ldots$

Let $u_n(x)$ be nth term of the series and $\{f_n(x)\}$ be a sequence of partial sums of the given series.

∴ $u_1(x) = \frac{x}{1+x^2}$

$u_2(x) = \frac{2^2x}{1+2^3x^2} - \frac{x}{1+x^2}$

$u_3(x) = \frac{3^2x}{1+3^3x^2} - \frac{2^2x}{1+2^3x^2}$

..........................................

$u_n(x) = \frac{n^2x}{1+n^3x^2}$

Adding $f_n(x) = \frac{n^2x}{1+n^3x^2}$

Hence $f(x) = \lim_{x\to\infty} f_n(x) = 0 \ \forall \ x \in [0, 1]$

Now $M_n = \sup \left[\left|f_n(x) - f(x)\right| : x[0,1]\right]$

$$= \sup \left[ \frac{n^2 x}{1+n^3 x^2} : x \in [0,1] \right]$$

$$\geq \frac{n^2 n^{\frac{\frac{1}{3}}{2}}}{1+n^3 \frac{1}{n^3}} = \frac{\sqrt{n}}{2} \rightarrow \text{ as } n \rightarrow \infty.$$

Since $M_n$ does not tend to zero as $n \rightarrow \infty$, the series is non-uniformly convergent on [0, 1] by M-Test. Here 0 is a point of non-uniform convergence.

**Example 3:** Show that 0 is a point of non-uniform convergence of the series $\sum_{n=1}^{\infty} \frac{-2x(1+x)^{n-1}}{\left[1+(1+x)^{n-1}\right]\left[1+(1+x)^{n}\right]}$

**Solution:** The given series is $\sum_{n=1}^{\infty} \frac{-2x(1+x)^{n-1}}{\left[1+(1+x)^{n-1}\right]\left[1+(1+x)^{n}\right]}$

Let $u_n$ (x) be nth term of the series and $|f_n$ (x)| be a sequence of partial sums of the given series. Now proceed as above.

**Example 4:** Discuss for uniform convergence of the series

$$\sum_{n=1}^{\infty} \left[ \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2 x^2} \right] \text{ in } [0, 1]$$

**Solution:** The given series is $\sum_{n=1}^{\infty} \left[ \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2 x^2} \right]$

Let $u_n$ (x) be the nth term of the series and $|f_n$ (x)} be a sequence of partial sums of the given series.

$$\therefore \quad u_n(x) = \frac{-2x(1+x)^{n-1}}{\left[1+(1+x)^{n-1}\right]\left[1+(1+x)^{n}\right]} \qquad \text{..... (1)}$$

$$\text{or} \quad u_n(x) = \frac{2}{1+(1+x)^n} - \frac{2}{1+(1+x)^{n-1}}$$

Putting n = 1, 2, 3,......, n in (1), we get

$$u_1(x) = \frac{2}{1+(1+x)} - 1$$

$$u_2(x) = \frac{2}{1+(1+x)^2} - \frac{2}{1+(1+x)}$$

$u_3(x) = \frac{2}{1+(1+x)^3} - \frac{2}{1+(1+x)^2}$

....................................................

....................................................

$u_n(x) = \frac{2}{1+(1+x)^n} - \frac{2}{1+(1+x)^{n-1}}$

Adding vertically, we get

$f_n(x) = u_1(x) + u_2(x) + ......... + u_n(x)$

$= \frac{2}{1+(1+x)^n} = 1$

$\therefore \quad f(x) = \underset{n\to\infty}{Lt} f_n(x) = \begin{cases} -1 \text{ when } x > 0 \\ 0 \text{ when } x = 0 \\ 1 \text{ when } x < 0 \end{cases}$

Take x > 0

$\therefore \quad M_n = \sup \{ |f_n(x) - f(x)| : x > 0\}$

$= \sup \left\{ \frac{2}{1+(1+x)^n}; x > 0 \right\}$

$\geq \frac{2}{1+\left(1+\frac{1}{n}\right)^n} \qquad \left[ \text{Taking } x = \frac{1}{n} \right]$

$\therefore \quad M_n \geq \frac{2}{1+e}$ as $n \to \infty$

$\therefore$ $M_n$ does not tend to zero as $n \to \infty$ and so the sequence $\{f_n\}$ is non-uniformly convergent in any neighbourhood of zero. Thus 0 is a point of non-uniform convergence of the given series.

### 17.6 Art. Welerstrass's M-Test (or W.M-Test)

A series $\sum_{n=1}^{\infty} u_n(x)$ of functions will converge uniformly on E if there exists a convergent series $\sum_{n=1}^{\infty} M_n$ of positive constants such that $|u_n(x)| \leq M_n$ for all n and all $x \in E$.

**Proof:** Given series is $\sum_{n=1}^{\infty} u_n$ (x)

Let there exist a convergent series $\sum M_n$ of positive constants such that

$|u_n (x)| \leq M_n$ ...... (1)

Since $\sum M_n$ converges

$\therefore$ for given $\varepsilon > 0$ $\exists$ a +ve integer t such that

$M_{n+1} + M_{n+2} + ...... + M_{n+p} < \varepsilon \; \forall \; n > t, p > 0$ ...... (2)

Now for $n \geq t$, $p \geq 0$

$|u_{n+1} (x) + u_{n+2} (x) + ....... + u_{n+p} (x)|$

$< |u_{n+1} (x)| + |u_{n+2} (x)| + ....... + |u_{n+p} (x)|$

$< M_{n+1} + M_{n+2} + ........ + M_{n+p}$ [$\because$ of (1)]

$< \varepsilon$ [$\because$ of (2)]

$\therefore$ $\sum u_n$ (x) converges uniformly.

**Illustrative Examples**

**Example 5:** Show that the series $\sum_{1}^{\infty} \frac{1}{1+n^2x}$ converges uniformly in [1, ∞]

**Solution:** Let $\sum u_n$ (x) = $\sum_{1}^{\infty} \frac{1}{1+n^2x}$

$\therefore$ $u_n (x) = \frac{1}{1+n^2x}$

$\therefore$ $u_n (x) = \left|\frac{1}{1+n^2x}\right| \leq \frac{1}{1+n^2x} < \frac{1}{n^2}$

$= M_n \; \forall \; x \in [1, \infty]$

i.e. $M_n = \frac{1}{n^2}$

and $\sum M_n = \sum \frac{1}{n^2}$ is convergent.

$\therefore$ by W-M Test, the given series is uniformly convergent on [1, ∞]

**Example 6:** Apply W.M. Test to show that the series

(i) $\sum \frac{a_n x^n}{1+x^{2n}}$

(ii) $\sum \frac{a_n x^{2n}}{1+x^{2n}}$

converges uniformly $\forall$ x $\in$ R is $\sum a_n$ is absolutely convergent.

**Solution:** Consider $\frac{x^n}{1+x^{2n}}$

$$\frac{d}{dx}\left(\frac{x^n}{1+x^{2n}}\right) = \frac{\left(1+x^{2n}\right)nx^{n-1} - x^n 2nx^{2n-1}}{\left(1+x^{2n}\right)^2}$$

$$= \frac{nx^{n-1}\left[1+x^{2n}-2x^{2n}\right]}{\left(1+x^{2n}\right)^2}$$

$$= \frac{nx^{n-1}\left(1-x^{2n}\right)}{\left(1+x^{2n}\right)^2}$$

$\therefore$ $\frac{d}{dx}\left(\frac{x^4}{1+x^{2n}}\right) = 0$

$\Rightarrow$ $nx^{n-1}(1 - x^{2n}) = 0$

$\Rightarrow$ x = 0, 1, - 1

When x < 1 slightly, then $\frac{dy}{dx}$ is +ve.

When x > 1 slightly, then $\frac{dy}{dx}$ is -ve

$\therefore$ at x = 1, $\frac{d}{dx}\left(\frac{x^4}{1+x^{2n}}\right)$, changes sign from +ve to -ve.

$\therefore$ $\frac{x^n}{1+x^{2n}}$ is maximum at x = 1

Its maximum value = $\frac{1}{2} < 1$

Hence $|u_n(x)| = \left|\frac{x^4}{1+x^{2n}}\right| < |a_n|$

$\because$ $\sum\|a_n\|$ is convergent [$\because \sum a_n$ is convergent absolutely]

$\therefore$ by W.M. Test, the given series is convergent $\forall$ x $\in$ R

(ii) $\because$ $\frac{x^{2n}}{1+x^{2n}} < 1$ $\forall$ x $\in$ R

$\therefore$ $\left|a_n \frac{x^{2n}}{1+x^{2n}}\right| < |a_n|$

and $\sum|a_n|$ is convergent

$\therefore$ By W.M. Test, $\sum u_n$ is convergent $\forall$ x $\in$ R

**Example 7:** Test for uniform convergence the series $\sum \frac{x}{n(1+nx^2)}$

**Solution:** Let $\sum u_n$ (x) = $\frac{x}{n(1+nx^2)}$

$\therefore$ $u_n$ (x) = $\frac{x}{n(1+nx^2)}$

$\therefore$ $|u_n(x)| = \left|\frac{x}{n(1+nx^2)}\right|$

Now $\frac{d}{dx}$ ($u_n$ (x)) = $\frac{\left(1+nx^2\right).1-x^n 2nx}{n(1+nx^2)^2}$

$= \frac{1-nx^2}{n\left(1+nx^2\right)^2}$

For $u_n$ (x) to be maximum or minimum,

$\frac{d}{dx}$ ($u_n$ (x)) = 0

$\Rightarrow$ $1 - nx^2 = 0$

$\Rightarrow$ $nx^2 = 1$

$\Rightarrow \qquad x = \dfrac{1}{\sqrt{n}}$

When $x < \dfrac{1}{\sqrt{n}}$, $\dfrac{d}{dx}$ $(u_n(x)) > 1$

and When $x > \dfrac{1}{\sqrt{n}}$, $\dfrac{d}{dx}$ $(u_n(x)) < 0$

$\therefore \qquad \dfrac{d}{dx}$ $(u_n(x))$ change sign from +ve to -ve

$\therefore \qquad u_n(x)$ is maximum at $x = \dfrac{1}{\sqrt{n}}$

and maximum $|u_n(x)| = \left| \dfrac{\frac{1}{\sqrt{n}}}{n\left(1+\frac{n}{n}\right)} \right|$

$$= \left| \frac{1}{2n^{\frac{3}{2}}} \right|$$

$\Rightarrow \qquad |u_n(x)| < \dfrac{1}{2} \cdot \dfrac{1}{n^{\frac{3}{2}}}$ and $\sum \dfrac{1}{n^{\frac{3}{2}}}$ is convergent.

$\therefore \qquad$ by W.M. Test, the given series is uniformly convergent $\forall$ x.

**Example 8:** Prove that the series $\cos x + \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} + \ldots\ldots$ converges uniformly on R.

**Solution:** Let $\sum u_n(x) = \cos x + \dfrac{\cos 2x}{2^2} + \dfrac{\cos 3x}{3^2} + \ldots\ldots$

$u_n(x) = \dfrac{\cos nx}{n^2}$

$|u_n(x)| = \left|\dfrac{\cos nx}{n^2}\right| < \dfrac{1}{n^2} \; \forall \; x \in R$

But $\sum \dfrac{1}{n^2}$ is convergent.

$\therefore \qquad$ by W.M. Test the given series is uniformly converges in R.

**Example 9:** Show that the series $\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+$.......... converges in -1 < x 1.

**Solution:** The given series is

$$\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+..........$$

$\therefore \quad u_n(x) = \frac{2^n x^{2^n-1}}{1+x^{2^n}}$

$\therefore \quad |u_n(x)| = \left|\frac{2^n x^{2^n-1}}{1+x^{2^n}}\right| \leq 2^n K^{2^n-1}$ for $|x| \leq K < 1$

$= M_n$ (say)

**Some More Illustrated Example**

**Example 10 :** Show that the series

$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} +$........ is not uniformly convergent on [0, 1]

**Solution :** We have the given series as

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} + ........$$

$\therefore \quad f_n(x)$ = sum of n terms of the series

$$= x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{\left(1+x^4\right)^2} + ........ + \frac{x^4}{\left(1+x^4\right)^{n-1}}$$

$\therefore \quad f_n(x) = x^4\left\{\frac{1-\frac{1}{\left(1+x^4\right)^n}}{1-\frac{1}{1+x^4}}\right\} \qquad \left(\because S_n = \frac{a(1-r^n)}{1-r}\right)$

$$= (1+x^4)\left\{1-\frac{1}{\left(1+x^4\right)^n}\right\}$$

Set $f(x) = \lim_{n\to\infty} f_n(x)$

$$= \lim_{n\to\infty} (1 + x^4) \left\{1 - \frac{1}{(1+x^4)^n}\right\}$$

$$= \begin{cases} (1+x^4) & , \quad 0 < x < 1 \\ 0 & , \quad x = 0 \end{cases}$$

Since f(x) exists for all values of x in [0, 1]

$\Rightarrow$ the series is pointwise convergent in [0, 1]

For $0 \le x \le 1$ and for given $\in > 0$, we have

$$|f_n(x) - f(x)| = \left| (1+x^4), \left\{1 - \frac{1}{(1+x^4)^n}\right\} - (1+x^4) \right|$$

$$= \frac{1}{(1+x^4)^{n-1}} < \in$$

Now if $(1 + x^4)^{n-1} > \frac{1}{\in}$ or if (n - 1) log (1 + n4) > loge $\frac{1}{e}$

or if $n - 1 > \dfrac{\log\left(\frac{1}{\in}\right)}{\log(1+n^4)}$

if $n > 1 + \dfrac{\log\left(\frac{1}{\in}\right)}{\log(1+n^4)}$

which clearly implies that

if $x \to 0$ then $n \to \infty$ and so the given series is not uniformly convergent in [0, 1]

Moreover x = 0 is a point of non-uniform convergence of the series.

**Example 11 :** Show that the series

$(1 - x)^2 + (1 - x)^2 x + (1 - x)^2 x^2 + ........$ is

Uniformly convergent on [0, 1]

**Solution :** Set $\sum_{n=1}^{\infty} U_n(x)$ be the given series

Set $U_n(x)$ = nth term of the series

$\therefore$ $U_n(x) = (1 - x)^2 x^{n-1}$

Set $f_n$ (x) = sum of n terms of the given series

$= (1 - x)^2 + (1 - x)^2 x + (1 - x)^2 x^2 + ........+ (1 - x)^2 x^{n-1}$

$= (1 - x)^2 \left(\frac{1-x^n}{1-x}\right) = (1 - x)(1 - x^n)$

Now $f(x) = \underset{n\to\infty}{Lim} f_n(x) \underset{n\to\infty}{Lt} = (1 - x)(1 - x^n)$

$= (1 - x)$ for $0 \le x \le 1$

$\therefore$ $|f_n(x) - f(x)| = |(1 - x^n)(1 - x) - (1 - x)|$

$= x^n (1 - x) \; \forall \; x \in [0, 1]$ ....(1)

Set $y = x^n (1 - x) = x^n - x^{n+1}$

$\therefore$ $\frac{dy}{dx} = n x^{n-1} - (n + 1) x^n$

For y to be maximum or minimum

$\frac{dy}{dx} = 0 \Rightarrow x = \frac{n}{n+1}$

and $\frac{d^2y}{dx^2} = n x^{n-2} \{(n - 1) - (n + 1) x\}$

when $x = \frac{n}{n+1}$, $\frac{d^2y}{dx^2} = n \left(\frac{n}{n+1}\right)^{n-2} [n - 1 - n]$

$= -n \left(\frac{n}{n+1}\right)^{n-2} < 0$

$\therefore$ y is max when $x = \frac{n}{n+1}$

and $y_{max} = \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right)$

$= \left(\frac{n}{n+1}\right)^n \left(\frac{n}{n+1}\right)$

$= \left(\frac{n}{n+1}\right)^n . \frac{n}{n+1} < \frac{1}{n}$

∴ $|f_n(x) - f(x)| < \frac{1}{n} < \in \forall\ n \geq t > \frac{1}{\in}$, $n \in [0, 1]$

Hence the given series converges uniformly

to (1 - x) on [0, 1]. Hence the result.

## 17.7 Self Check Exercise

Q.1 Test the convergence of the series $\sum_{n=1}^{\infty} \left\{ \frac{2n^2x^2}{e^{n2}x^2} - \frac{2(n-1)^2x^2}{e^{(n-1)2}x^2} \right.$

Q. 2 Test the series $\sum f_n(x) = \sum_{n=1}^{\infty} x\, e^{-nx}$ in [0, 1] for uniform convergence

Q. 3 By Mn-Test examine the convergence of the series

$$\frac{1}{(1+x)^4} + \frac{2}{(2+x)^4} + \frac{3}{(3+x)^4} + \ldots, \ x \geq 0$$

## 17.8 Summary

We have learnt the following concepts to find the convergence of an infinite series:

(i) series of functions

(ii) sequence of partial sums of the given series

(iii) uniform convergence of series of functions

(iv) Test of convergence, namely weierstrass Mn-test. etc.

## 17.9 Glossary:

1. Cauchy Criterion for uniform convergence of series. A series of function $\sum f_n$ defined on [a, b] converges uniformly on [a, b] iff for every $\in > 0$ and for all $x \in [a, b]$, $\exists$ a positive integer N s.t.

   $|f_{n+1}(x) = f_{n+2}(x) \ldots\ldots + f_{n+p}(x)| < \in \forall\ n \geq N,\ p \geq 1$

   $|f_{m+1}(x) = f_{m+2}(x) \ldots\ldots + f_m(x)| < \in \forall\ n, m \geq N$

## 17.10 Answer to Self Check Exercise

Ans.1 Hint. Add $u_1, u_2, \ldots u_n$ vertically to find $f_n(x)$. Then proceed.

Ans.2 Use $M_n$ - test for uniform convergence

Ans.3 Take $U_n = \frac{n}{(n+x)^4}$, find $|u_n(n)|$, then proceed to apply $M_n$-test.

## 17.11 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983
3. R.G. Bartle and D.R. Sherbert, Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000.


## 17.12 Terminal Questions

1. Show that the series $\sum_{n=1}^{\infty} 3^n \sin\left(\frac{1}{4^n x}\right)$ converges absolutely and uniformly on (a, ∞), a > 0.

2. Prove that the series

   $\cos x + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots$ converges uniformly on R.

3. Test the series

   $1 + n + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$ in [-1, 1] for uniform convergence

4. Show that the series

   $x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots$ has x = 0 as a point of non-uniform convergence.

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## Unit - 18

# Test for Convergence

# (Abel's and Dirichlet's Tests)

**Structure**



### 18.1 Introduction

Dear students, we shall continue our discussion to best the convergence of series of functions in this unit also. Two important test, namely Abel's Test and Dirichlet's Test will be discussed in detail here. We shall see how these tests help us to test the uniform convergence of a given series.

### 18.2 Learning Objectives

The main objectives of this unit are

(i) to study Abel's test of convergence

(ii) to study Dirichlet's test of convergence for a infinite series of functions.

### 18.3 Abel's Test

Let the series $\sum u_n (x)$ converges uniformly in [a, b] and let the sequence $\{\nu_n (x)\}$ be monotonic for each x in [a, b] and be uniformly bounded in [a, b]. Then the series

$\sum u_n (x) \nu_n (x)$ is uniformly convergent in [a, b]

**Proof :** Let $R_{n,p} (x)$ be partial remainder for the series $\sum u_n (x) \nu_n (x)$ and let $S_n$ be sum of n terms and $r_{n,p} (x)$ be the partial remainder for the series $\sum u_n (x)$.

$$\therefore \quad R_{n,p} = u_{n+1} (x) \nu_{n+1} (x) + u_{n+2} (x) \nu_{n+2} (x) + \ldots + u_{n+p} (x) \nu_{n+p} (x)$$

$= r_{n,1}(x)\, \nu_{n+1}(x) + \{r_{n,2}(x) - r_{n,1}(x)\}\, \nu_{n+2}(x)$

$+ \ldots + \{r_{n,p}(x) - r_{n,p-1}(x)\}\, \nu_{n+p}(x)$

$= r_{n,1}(x)\, \{\nu_{n+1}(x) - \nu_{n+1}(x)\} + r_{n,2}(x)\, \{\{\nu_{n+2}(x) - \nu_{n+3}(x)\} +$

$+ \ldots + r_{n,p-1}(x)\, \{\nu_{n+p-1}(x) - \nu_{n+p}(x)\} + r_{n,p}\nu_{n+p}(x)$ --(1)

Since $\{\nu_n(x)\}$ is monotonic

$\therefore$ $\{\nu_{n+1}(x) - \nu_{n+2}(x)\}, \{\nu_{n+2}(x) - \nu_{n+3}(x)\} \ldots\ldots,$

$\{\nu_{n+p-1}(x) - \nu_{n+p}(x)\}$

have all of them the same sign for any fixed value of x in [a, b].

Also, since $\{\nu_n(x)\}$ is uniformly bounded on [a, b], we have

$|\nu_n(x)| < K \; \forall$ x in [a, b] and for all $n \in N$

And also since the $\sum u_n(x)$ cgs uniformly in [a, b]

$\therefore$ $|r_{n,1}(x)|, |r_{n,2}(x)|, \ldots, |r_{n,p}(x)|$ are each $< \dfrac{\varepsilon}{3K}$ when $n \geq m$.

the same m serving for all values of x in [a, b]

$\therefore$ $(1) \Rightarrow |R_{n,p}(x)| < \dfrac{\varepsilon}{3K} |\nu_{n+1}(x) - \nu_{n+p}(x)| + \dfrac{\varepsilon}{3K} |\nu_{n+p}(x)|$

$< \dfrac{\varepsilon}{3K}, 2K + \dfrac{\varepsilon}{3K}, K = \varepsilon$

$[\because |\nu_{n+1}(x) - \nu_{n+p}(x)| \leq |\nu_{n+1}(x)| + |\nu_{n+p}(x)| < K + K = 2K]$

$\Rightarrow$ $|R_{n+p}(x)| < \varepsilon$ when $n > m$ then m is fixed integer depending on $\varepsilon$ but independent of x.

$\Rightarrow$ $\sum u_n(x)\, \nu_n(x)$ cgs uniformly in [a, b]

## 18.4 Art Dirichlet's Test

The series $\sum u_n(x)\, \nu_n(x)$ is uniformly convergence [a, b] if

(i) $\{\nu_n(x)\}$ is a +ve monotonic decreasing sequence converging uniformly to zero for $a \leq x \leq b$.

(ii) $|f_n(x)| = \left|\sum_{r=1}^{n} u_r(x)\right| < K$ for every value of x in [a, b] and for all integral values of n, where K is a fixed number independent of x.

**Proof :** With usual notation

$R_{n,p} = u_{n+1}(x)\, \nu_{n+1}(x) + u_{n+2}(x)\, \nu_{n+2}(x) + \ldots + u_{n+p}(x)\, \nu_{n+p}(x)$

$= [f_{n+1}(x) - f_n(x)] + \nu_{n+1}(x)\, [f_{n+2}(x) - f_{n+1}(x)]\, \nu_{n+2}(x)$

$+ \ldots\ldots\ldots + [f_{n+p}(x) - f_{n+p-1}(x)]\, \nu_{n+p}(x)$

$= f_{n+1}(x)\, \{\nu_{n+1}(x) - \nu_{n+2}(x)\} + f_{n+2}(x)\, \{\nu_{n+2}(x) - \nu_{n+3}(x)\} + \ldots\ldots$

$+ f_{n+p-1}(x)\, \{\nu_{n+p-1}(x) - \nu_{n+p}(x)\} + f_{n+p}(x)\, \nu_{n+p}(x) - f_n(x)\, \nu_{n+1}(x)$ ....(1)

Now all $\nu_1(x), \nu_2(x), \ldots\ldots$ are +vie and $\nu_1(x) < \nu_2(x) > \nu_3(x) > \ldots\ldots.$

and $|f_n(x)| < K$ for any x on [a, b] and for all n.

(1) $\Rightarrow |R_{n,p}(x)| \le |f_{n+1}(x)|\, [\nu_{n+1}(x) - \nu_{n+2}(x)]$

$+ |f_{n+2}(x)|\, [\nu_{n+2}(x) - \nu_{n+3}(x)] + \ldots.. + |f_{n+p-1}(x)| [\nu_{n+p-1}(x) - \nu_{n+p}(x)]$

$+ |f_{n+p}(x)|\, [\nu_{n+p}(x) + |f_n(x)||\nu_{n+1}(x)|$

$< K\, [\nu_{n+1}(x) - \nu_{n+p}(x) + \nu_{n+p}(x) - \nu_{n+1}(x)]$

$= 2K\nu_{n+1}(x)$ ....(2)

Also, since $\{\nu_n(x)\}$ cgs to zero

$\therefore \quad |\nu_n(x)| < \dfrac{\varepsilon}{3K} \ \forall\ n \ge m$

Thus (2) $\Rightarrow \quad |R_{n,p}(x)| < 2K\, \dfrac{\varepsilon}{3K} = \varepsilon$ for $n \ge m$

$\Rightarrow \quad |R_{n,p}(x)| < \varepsilon\ \forall\ n \ge m$ and for every x in [a, b]

**Art.** Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued function defined as a set E and let $\underset{x\to a}{Lt}\, u_n(x)$ exist (n = 1, 2, 3, ....) where a is a limit point of E. If the series $\sum u_n(x)$ converges uniformly on E, then

$$\underset{x\to a}{Lt} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left( \underset{x\to a}{Lt}\, u_n(x) \right)$$

**Proof :** Let $f_n(x) = u_1(x) + u_2(x) + \ldots. + u_n(x)$

$$\therefore \quad \underset{x\to a}{Lt} \sum_{n=1}^{\infty} u_n(x) = \underset{x\to a}{Lt}\ \underset{n\to\infty}{Lt} \sum_{m=1}^{n} u_m(x)$$

or $$\underset{x\to a}{Lt} \sum_{n=1}^{\infty} u_n(x) = \underset{x\to a}{Lt}\ \underset{n\to\infty}{Lt}\, f_n(x) \qquad \ldots.(1)$$

and $$\sum_{n=1}^{\infty} \left( \underset{x\to a}{Lt}\, u_n(x) \right) = \underset{n\to\infty}{Lt} \sum_{m=1}^{n} \left[ \underset{x\to a}{Lt}\, u_m(x) \right]$$

or $\sum_{n=1}^{\infty} \left( \underset{x\to a}{Lt}\, u_n(x) \right) = \underset{n\to\infty}{Lt}\, \underset{x\to a}{Lt} \sum_{m=1}^{n} u_m(x)$

[∵ the limit of the sum of a finite number of function is the sum of their limits]

or $\sum_{n=1}^{\infty} \left( \underset{x\to a}{Lt}\, u_n(x) \right) = \underset{n\to\infty}{Lt}\, \underset{x\to a}{Lt}\, f_n (x)$ ....(2)

From (1) and (2) , we get

$\underset{n\to\infty}{Lim} \sum_{n=1}^{\infty} u_n |x| = \sum_{n=1}^{\infty} \left( \underset{n\to\infty}{Lim}\, u_n(x) \right)$

(∵ R.H.S. of (1) and (2) are equal)

Hence the proof.

**Some Illustrated Examples**

**Example 1 :** Test for uniform convergence of the series $\sum \frac{(-1)^{n-1}}{n}$ . xn in [0, 1].

**Solution :** Let $u_n$ (x) = $\frac{(-1)^{n-1}}{n}$ and $u_n$ (x) = $x^n$.

The sequence {$\nu n$ (x)} is uniformly bounded and monotonically increasing in [0, 1] and $\sum \frac{(-1)^{n-1}}{n}$ is convergent by Leibnitz test.

Hence by Abel's test, the given series is uniformly converges in [0, 1].

**Example 2 :** Show that $\sum \frac{a_n}{n^x}$ converges uniformly in [0, 1] if $\sum a_n$ converges.

**Solution :** Let $u_n$(x) = an , $\nu_n$ (x) = $\frac{1}{n^x}$

Let us assume that $\sum a_n$ is convergent.

∴ $\sum a_n$ is uniformly convergent in [0, 1] $u_n$(x) = $\frac{1}{n^x}$ is a +ve, monotonically decreasing and bounded in [0, 1]

∴ by Abel's test $\sum \frac{a_n}{n^x}$ is uniformly convergent in [0, 1]

**Example 3:** Show by Abel's test the series $\sum \frac{(-1)^n}{n} |x|^n$ is uniformly convergent in [-1, 1]

**Solution:** Let $u_n(x) = \frac{(-1)^n}{n}$, $u_n(x) = |x|^n$

Since $|x|^n$ is positive, monotonically decreasing and bounded for [-1, 1]

Further, $\sum \frac{(-1)^n}{n}$ is convergent by alternating series test and $\frac{(-1)^n}{n}$ is independent of x

$\therefore$ by Abel's test the given series is uniformly convergent in [-1, 1].

**Example 4:** Prove by Dirichlet's test the series $\sum \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent for all real x.

**Solution:** Let $u_n (=x) = (-1)^{n-1}$

and $u_n(x) = \frac{1}{n+x^2}$

$\therefore$ $f_n(x) = \sum_1^n u_n$ = 0 or 1 according as n even or odd.

$\therefore$ $f_n$ (x) is bounded.

Also $u_n(x)$ is a monotonically decreasing sequence converges to zero $\forall$ real x.

$\therefore$ by Dirichlet's test the given series is uniformly convergent for all real x.

**Example 5:** Test for uniform convergence the series $\sum \frac{(-1)^{n-1}}{n+n^2}$ $\forall$ n.

**Solution:** Take $u_n = (-1)^{n-1}$

and $u_n = \frac{1}{n+n^2}$

$\therefore$ $f_n$ (x) = $\sum u_n$ (x) = $\sum (-1)^{n-1}$ = 0 or 1

$\Rightarrow$ $f_n$ (x) is bounded $\forall$ n

Also $\{u_n (x)\}$ is a decreasing sequence and $\to 0$ as $n \to \infty \forall$ real x

$\therefore$ by Dirichlet's test the given series is uniformly convergent $\forall$ n.

**Example 6:** Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} x^n$ is uniformly convergent in $0 \leq x \leq m \leq 1$.

**Solution:** Let $u_n = (-1)^{n-1}$

and $\quad u_n = x^n$

$\therefore \quad \sum f_n(x) = \sum_{r=1}^{n} u_r$ = 0 or 1 according as n even or odd.

$\therefore \quad f_n(x)$ is bounded for all n.

Also $\{v_n(x)\}$ is a +ve monotonically decreasing sequence $\to 0 \ \forall$ x, $0 \leq x \leq m < 1$

Hence by Dirichlet test the given series is uniformly convergent in $0 \leq x \leq m < 1$.

## 18.6 Self Check Exercise

Q.1 Prove that $\underset{x \to 1}{Lim}\left[\sum_{n=1}^{\infty} \frac{nx^2}{n^3 + x^3}\right] = \sum_{n=1}^{\infty} \frac{n}{n^3 + 1}$

Q. 2 Prove that the series $\sum (-1)^n \frac{x^3 + n^2}{n^3}$ converges uniformly on every bounded subset of R.

Q. 3 Show that the series sin x + $\frac{1}{2}$ sin 2x + ....... converges uniformly for $0 < a \leq x \leq b < 2\pi$.

## 18.6 Summary

We have learnt the following in this unit.

(i) Abel's test

(ii) Dirichlet's test of convergence for infinite series.

## 18.7 Glossary:

Cluster point of a sequence -

A real number p is said to be a cluster point of the sequence $\{a_n\}$ if to each $\in > 0$ and each $m \in N$, $\exists$ a +ve integer $m_0 > m$ s.t.

$$p - \in < a_{m_0} < p + \in$$

e.g. $\{(-1)^n\}$ has two cluster point 1 and -1. respectively.

## 18.8 Answer to Self Check Exercise

Ans.1 Use W.M. Test and then Abel's test

Ans.2 Take $u_n = (-1)^n$, $v_n(x) = \frac{x^3 + n^2}{n^3}$ and proceed.

Ans.3 Take $u_n(x) = \sin nx$, $u_n(x) = \frac{1}{n}$ and then proceed.

## 18.9 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.
2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983


## 18.10 Terminal Questions

1. Prove that $\forall\,\theta \in$ R, the following series converges uniformly provided |x| < 1.

   (i) x cos $\theta$ + x2 cos 2$\theta$ + x3 cos 3$\theta$ + .......

   (ii) x coc$\theta$ + $\frac{x^2}{2}$cos 2$\theta$ + $\frac{x^3}{3}$cos 3$\theta$ + ......

2. Show that the series $\sum_{n=1}^{\infty}\frac{\cos nx}{n}$ converges uniformly on [0, 2π].

3. Prove that the series $\sum \frac{x^n}{n+1}$ is comiformly convergent in [-8, 8], if 8 is any fixed positive number less than 1.

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**Unit - 19**

# Some Important Results About Uniform Convergence

**Structure**



## 19.1 Introduction

Dear students, in this unit we shall study some important results about uniform convergence. These results are very useful in understanding the concepts of uniform convergence and continuity, uniform convergence and Riemann integration, term by term integration etc.

## 19.2 Learning Objectives

In this unit our main objectives are to study the following

(i) Uniform convergence and continuity.

(ii) Uniform convergence and Riemann integration

(iii) Term by term integration

(iv) Uniform convergence and Differentiation

(v) Term by term differentiation

(vi) Weierstars Approximation Theorem etc.

## 19.3 Art. Some Important Results

**Result I:** Let $f_n$ be real valued function defined on E and let the sequence $\{f_n\}$ converge uniformly to $f$ on E. Let c be a cluster point of E and suppose that

$$\underset{x\to c}{Lt}\, f_n(x) = L_n \quad (n = 1, 2, \ldots\ldots)$$

Then the seq. $\{L_n\}$ of real constants converges and

$$\underset{x\to c}{Lt}\, f_n(x) = \underset{x\to c}{Lt}\, L_n$$

i.e. $$\underset{x\to c}{Lt}\;\underset{n\to\infty}{Lt}\, f_n(x) = \underset{n\to\infty}{Lt}\;\underset{x\to c}{Lt}\, f_n(x)$$

**Result II: Uniform Convergence and Continuity**

Let $\{f_n\}$ be a sequence of real valued function on E which converges uniformly to $f$ on E. If each $f_n$ is continuous on E then $f$ is also continuous on E.

**Result III:** The sum function of a uniformly convergent series of continuous function is itself continuous.

**Illustrative Examples**

**Example 1:** Test for uniform convergence and continuity of the sum function of the series for which

(i) $f_n(x) = \frac{1}{1+nx}$ $0 < x < 1$

(ii) $f_n(x) = nx(1 - x)^n$ $0 < x < 1$

**Solution:** (i) Here $f_n(x) = \frac{1}{1+nx}$

$$f(x) = \underset{n\to\infty}{Lt}\, f_n(x) = \begin{cases} 0 \text{ when } 0 < x \le 1 \\ 1 \text{ when } x = 0 \end{cases}$$

Thus 0 is a point of discontinuity of the sum function.

Hence the series will be non-uniform convergent in the closed interval [0, 1]

On $0 < x \le 1$,

$$M_n = \sup \left[ |f_n(x) - f(x)| : x \in (0,1] \right]$$

$$= \sup \left[ \left|\frac{1}{1+nx}\right| : x \in (0,1] \right]$$

$$> \frac{1}{1+n.\frac{1}{n}} \qquad \left[ Take\ x = \frac{1}{n} \in (0,1] \right]$$

$$= \frac{1}{2}$$

$M_n \to 0$ as $n \to \infty$

Also x → 0 as n → ∞

∴ 0 is point of discontinuity of sum function.

(ii) $f_n(x) = nx(1 - x)^n \quad 0 < x < 1$

When $0 < x < 1$, $\underset{n\to\infty}{Lt} f_n(x) = \underset{n\to\infty}{Lt} \frac{nx}{(1-x)^{-n}}$

$$= \underset{n\to\infty}{Lt} \frac{-x}{(1-x)^{-n}\log(1-x)}$$

$$= \underset{n\to\infty}{Lt} - \frac{x(1-x)^n}{\log(1-x)} = 0$$

$f(x) = \underset{n\to\infty}{Lt} f_n(x) = 0$ where $0 < x \leq 1$

Also $f_n(x) = 0$ where x = 0 or 1

The sum function $f(x)$ is the convergent for all x in $0 \leq x \leq 1$

But 0 is a point of non-uniform convergence as proved earlier.

**Example 2:** Show that zero is a point of non-uniform convergence of the series

$\sum u_n(x)$... where $u_n(x) = \frac{1-(1+x)^n}{1+(1+x)^n} - \frac{1-(1+x)^{n-1}}{1+(1+x)^{n-1}}$

**Solution:** Here $u_n(x) = \frac{1-(1+x)^n}{1+(1+x)^n} - \frac{1-(1+x)^{n-1}}{1+(1+x)^{n-1}}$

∴ $u_1(x) = \frac{1-(1+x)^n}{1+(1+x)^n} - 0$

$u_2(x) = \frac{1-(1+x)^2}{1+(1+x)^2} - \frac{1-(1+x)}{1+(1+x)}$

$u_3(x) = \frac{1-(1+x)^3}{1+(1+x)^3} - \frac{1-(1+x)^2}{1+(1+x)^2}$

$u_n(x) = \frac{1-(1+x)^n}{1+(1+x)^n} - \frac{1-(1+x)^{n-1}}{1+(1+x)^{n-1}}$

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Adding vertically, we get

$$f_n(x) = \frac{1-(1+x)^n}{1+(1+x)^n}$$

$\therefore$ $$f(x) = \underset{n\to\infty}{Lt} f_n(x) = \begin{cases} -1 \ when \ x \neq 0 \\ 0 \ when \ x = 0 \end{cases}$$

$$\underset{n\to\infty}{Lt} f(x) = \underset{n\to\infty}{Lt} (-1) = -1$$

Hence $f$ (x) is discontinuous at x = 0. It follows that 0 is a point of non-uniform convergence of the series.

**Example 3:** Examine the series $\sum xe^{-nx}$ for uniform convergence and continuity of its sum function near x = 0.

**Solution:** $f_n(x) = \sum_{r=1}^{n} xe^{-rx} = x.\frac{1-e^{-nx}}{1-e^{-x}}$

and $$f_n(x) = \begin{cases} \underset{n\to\infty}{Lt} f_n(x) = \frac{x}{1-e^{-x}} \ when \ x \neq 0 \\ 0 \qquad when \ x = 0 \end{cases}$$

$$\underset{n\to 0}{Lt} f(x) = \underset{n\to 0}{Lt} \frac{x}{1-e^{-x}}$$

$$= \underset{n\to 0}{Lt} \frac{x}{\left(1-x+\frac{x^2}{2}-\frac{x^3}{3}\right)}$$

$$= \underset{n\to 0}{Lt} \frac{1}{1-x+\frac{x}{2}+\frac{x^2}{3}+.....} = 1$$

Also $f(0) = 0$

$\therefore$ $$\underset{n\to 0}{Lt} f(x) \neq f(0)$$

$\Rightarrow$ $f$ (x) is discontinuous at x = 0

$$= \underset{n\to\infty}{Lt} \int_0^1 \frac{n^2(1-x)^{n+1}}{n+1} dx$$

$$= \underset{n\to\infty}{Lt}\left[\frac{-n^2(1-x)^{n+1}}{(n+1)(n+2)}\right]_0^1$$

$$= \underset{n\to\infty}{Lt}\ \frac{n^2}{(n+1)(n+2)} = 1$$

So that the series is not integratable term by term.

**Example 4:** Show that the series for which sequence of partial sums is given by $f_n(x) = \frac{1}{1+nx}$, can be integrated term by term in $0 \le x \le 1$, although it is not uniformly convergent in this interval.

**Solution :** Here $f_n(x) = \frac{1}{1+nx}$

$$f(x) = \underset{n\to\infty}{Lt}\ f_n(x) = \underset{n\to\infty}{Lt}\ \frac{1}{1+nx} = 0$$

$$\therefore \qquad \int_0^1 f(x)\,dx = 0$$

and $$\underset{n\to\infty}{Lt}\int_0^1 f(x)\,dx = \underset{n\to\infty}{Lt}\int_0^1 \frac{1}{1+nx}\,dx$$

$$= \underset{n\to\infty}{Lt}\ \frac{1}{n}\log(1+n) \qquad \left[\frac{\infty}{\infty}\ form\right]$$

$$= \underset{n\to\infty}{Lt}\ \frac{\frac{1}{1+n}}{1} = 0$$

Hence the series can be integrated term by term.

But zero is a point of non-uniform convergence of the series.

**Example 5:** Examine for the continuity of the sum function and term by term integration the series whose nth term is

$$n^2 xe^{-n^2x^2} - (n-1)^2 xe^{-(n-1)^2x^2}$$

x having all values in the interval [0, 1]

**Solution:** Let $\{f_n(x)\}$ be a sequence of partial sums

$\therefore$ the series is not uniform convergent in any interval which includes x = 0.

**Result IV: Uniform Convergence and Riemann Integration**

Let $\{f_n\}$ be a sequence of real valued function defined on the closed interval [a, b] and bounded on [a, b] and let $f$n ∈ R [a, b] for n = 1,2,3,......

If $\{f_n\ (x)\}$ cgs uniformly the function $f$ on [a, b], then $f \in$ R [a, b]

and $\int_a^b f(x)dx = \underset{n\to\infty}{Lt} \int_a^b f_n(x)dx$

**Result V :** The $\{f$n (x)$\}$ be a sequence of real valued continuous function defined on [a, b] s.t. $f_n \to f$ uniformly on [a, b]. Then $f \in$ [a, b] and

$$\underset{n\to\infty}{Lt} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

**Result VI : Term by Term Integration**

Let $\sum_{n=1}^{\infty} u_n(x)$ by a series of real valued function defined on [a, b] s.t. $u_n$ (x) ∈ R [a, b]. If the series converges uniformly to $f$ on R [a, b] then $f \in$ R [a, b] and

$$\int_a^b \left[\sum_{n=1}^{\infty} u_n(x)\right] \text{dx} = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$$

**Illustrative Examples**

**Example 6:** Examine for term integrate the series the sum of whose n terms is $n^2$ x $(1 - x)^n$

**Solution:** $\int_0^1 f(x)dx = \int_0^1 \underset{n\to\infty}{Lt} \frac{n^2 x}{(1-x)^{n-n}}$ dx

$$= \int_0^1 0\, dx = 0 \qquad [0 \leq x < 1]$$

$$\left[\because \underset{n\to\infty}{Lt}\ n^2 x(1-x)^n = 0\right]$$

But $\underset{n\to\infty}{Lt} \int_0^1 f_n(x)dx = \underset{n\to\infty}{Lt} \int_0^1 n^2 x(1-x)^n$ dx

$$= \underset{n\to\infty}{Lt} \left[\left\{\frac{n^2 x(1-x)^{n+1}}{n+1}\right\}_0^1 + \int_0^1 \frac{n^2 x(1-x)^{n+1}}{n+1} dx\right]$$

$\therefore$ $u_1(x) = \frac{x}{1+x^2} = 0$

$u_2(x) = \frac{2^2 x^2}{1+2^4 x^2} - \frac{x}{1+x^2}$

$u_3(x) = \frac{3^2.x}{1+3^4.x^2} - \frac{2^2.x}{1+2^4.x^2}$

..........................................

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$u_n(x) = \frac{n^2 x}{1+n^4 x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}$

Adding vertically, we get

$f_n(x) = \frac{n^2 x}{1+n^4 x^2}$

$f(x) = \underset{n\to\infty}{Lt} f_n(x)$

$= \underset{n\to\infty}{Lt} \frac{\frac{x}{n^2}}{\frac{1}{n^4}+x^2} = 0 \ \forall\ x \in [0, 1]$

**Result VII. Uniform Convergence and Differentiation**

Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(i) $f_n$ is differentiable on [a, b] for n = 1,2,3

(ii) The sequence $\{f_n(d)\}$ converges for some point d of [a, b]

(iii) The sequence $\{f'_n\}$ converges uniformly on [a, b]

Then the sequence $\{f_n\}$ converges uniformly to a differentiable function $f$ and $\underset{n\to\infty}{Lt} f'_n(x) = f'(x)$ $a \le x \le b$.

**Result VIII: Term by Term differentiation**

Let $\sum_{n=1}^{\infty} u_n(x)$ be a series of real valued differentiable function on [a, b] such that $\sum_{n=1}^{\infty} u_n(d)$ converges for some point d of [a, b] and $\sum_{n=1}^{\infty} u'_n(x)$ converges uniformly on

$u_n(x) = n^2 xe^{-n^2x^2} - (n-1)^2 xe^{-(n-1)^2x^2}$

$u_1(x) = xe^{-x^2} - 0$

$u_2(x) = 2^2 xe^{-2^2x^2} - xe^{-2^2}$

$u_3(x) = 3^2 xe^{-3^2x^2} - 2^2 xe^{-2^2x^2}$

..............................................

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∴ $f_n(x) = n^2 xe^{-n^2x^2}$

Now $f(x) = \underset{n\to\infty}{Lt} f_n(x) = 0$ for $0 < x < 1$

∴ the sum function $f(x)$ is continuous for all values of x in [0, 1]

Now $\int_0^1 f(x)dx = 0$

and $\int_0^1 f_n(x)dx = \int_0^1 xe^{-n^2x^2}\,dx$

$$= -\frac{1}{2}\int_0^1 e^{-n^2x^2}(-2n^2x)\,dx$$

$$= \left[-\frac{1}{2}e^{-n^2x^2}\right]_0^1$$

$$= \frac{1}{2}\left[1 - e^{-n^2}\right]$$

$$\to \frac{1}{2} \text{ as } n \to \infty$$

Hence the series is not term by term integrable in $0 \leq x \leq 1$

**Example 7:** Test the series $\sum\left(\frac{n^2x}{1+n^4x^2} - \frac{(n-1)^2x}{1+(n-1)^4x^2}\right)$ for uniform convergence in the interval [0, 1]. Can the series be integrated term by term?

**Solution:** The given series is $\sum\left(\frac{n^2x}{1+n^4x^2} - \frac{(n-1)^2x}{1+(n-1)^4x^2}\right)$

∴ $u_n(x) = \frac{n^2 x}{1+n^4x^2} - \frac{(n-1)^2 x}{1+(n-1)^4 x^2}$

[a, b]. The series $\sum_{n=1}^{\infty} u_n$ (x) converges uniformly on [a, b] to a differentiable sum function $f$ and

$$f'(x) = \underset{n\to\infty}{Lt} \sum_{m=1}^{n} u'_m(x) \qquad (a \leq x \leq b)$$

In other words if $a \leq x \leq b$, then

$$\frac{d}{dx}\left(\sum_{n=1}^{\infty} u_n(x)\right) = \sum_{n=1}^{\infty}\left[\frac{d}{dx} u_n(x)\right]$$

**Result IX:** Let $\{f_n\}$ be a sequence of real valued functions defined on [a, b] such that

(i) $f_n$ is differentiable on [a, b] for n = 1,2,3,......

(ii) The sequence $\{f_n\}$ converges to $f$ on [a, b]

(iii) The sequence $\{f'_n\}$ converges uniformly on [a, b] to g

(iv) Each $f'_n$ is continuous on [a, b]

Then $g(x) = f'(x)$ $(a < x < b)$

That is

$$\underset{n\to\infty}{Lt} f'_n(x) = f'(x) \quad (a \leq x \leq b)$$

**Result X:** There exists a real continuous functions on the real line which is no where differentiable.

**Result XI :Weirstrass Approximation Theorem**

Let $f$ (x) be a continuous function defined on [a, b] there exists a sequence of polynomials which converges uniformly to $f$ on [a, b]

**Illustrative Examples**

**Example 8:** Show that the series $\sum \frac{1}{n^2 + n^4x^2}$ is uniformly convergent for all real values of x and that it can be differentiated term by term.

**Solution:** Let $u_n(x) = \frac{1}{n^2 + n^4x^2}$

Now $1 + n^2x^2 \geq 1 \ \forall \ x \in R$

$\therefore \quad \frac{1}{n^2 + n^4x^2} \leq 1$

$\Rightarrow \quad \frac{1}{n^2 + n^4x^2} < \frac{1}{n^2} = M_n$

But $\sum \frac{1}{n^2}$ is convergent.

By Weierstrans M-test, the given series is uniformly convergent for all x $\in$ R.

$\therefore$ it can be differentiated term by term.

**Example 9:** Given the series $\sum u_n$ (x) for which

$$f_n(x) = \frac{1}{2n^2} \log(1 + n^4x^2)$$

Show that the series $\sum u'_n$ (x) does not converge uniformly, but that the given series can be differentiated term by term.

**Solution:** Here $f_n(x) = \frac{1}{2n^2} \log(1 + n^4x^2)$, $0 \leq x \leq 1$

$$f(x) = \underset{n\to\infty}{Lt} f_n(x)$$

$$= \underset{n\to\infty}{Lt} \frac{\log(1+n^4x^2)}{2n^2} \qquad \frac{\infty}{\infty}$$

$$= \underset{n\to\infty}{Lt} \frac{\frac{4n^3x^2}{1+n^4x^2}}{4n}$$

$$= \underset{n\to\infty}{Lt} \frac{n^2x^2}{1+n^4x^2} = 0, \; 0 \leq x \leq 1$$

$\therefore \quad f'(x) = 0$

Now $f'_n(x) = \frac{2n^4(x)}{2n^2(1+n^4x^2)}$

$$= \frac{n^2x}{1+n^4x^2}$$

Also $\underset{n\to\infty}{Lt} f'_n(x) = \underset{n\to\infty}{Lt} \frac{n^2x}{1+n^4x^2} = 0, 0 \le x \le 1$

$\therefore \quad f'(x) = \underset{n\to\infty}{Lt} f'_n(x)$

$\therefore$ term by term differentiation can be done.

But the series $\sum u'_n(x)$ is not uniformly convergent for $0 \le x \le 1$ as the sequence $\{f'n(x)\}$ i.e. $\left\{\frac{n^2x}{1+n^4x^2}\right\}$, has 0 as a point of non-uniform convergence.

**Example 10:** Show that the function represented by $\sum_1^\infty \frac{\cos nx}{n^2}$

**Solution:** Let $f(x) = \sum_1^\infty \frac{\sin nx}{n^3}$

and $u_n(x) = \frac{\sin nx}{n^3}$

$u'_n(x) = \frac{\cos nx}{n^2}$

$\therefore \quad \sum_1^\infty u'_n(x) = \sum_1^\infty \frac{\cos nx}{n^2}$

But $\left|\frac{\cos nx}{n^2}\right| \le \frac{1}{n^2}$ for all x and $\sum \frac{1}{n^2}$ is cg.

By W.M. Test $\sum u'_n(x)$ is U.C. for all x and therefore $\sum u'_n(x)$ can be differentiable term by term

$$f'(x) = \sum_1^\infty u'_n(x) = \sum_1^\infty \frac{\cos nx}{n^2}$$

**Example 11:** Show that if $f(x) = \sum_1^\infty \frac{1}{n^3+n^4x^2}$ then it has a differential coefficient equal to $-2x\sum_1^\infty \frac{1}{n^2\left(1+nx^2\right)^2}$ for all values of x.

**Solution:** Here $\quad u_n(x) = \frac{1}{n^3+n^4x^2}$

$u'_{n(x)} = -\dfrac{2x}{n^2(1+nx^2)^2}$

Now $u'_n(x)$ is maximum value $\dfrac{d}{dx}(u'_n(x)) = 0$

∴ $(1 + nx^2)^2 - 4nx^2(1 + nx^2) = 0$

or $1 - 3nx^2 = 0$

or $n = \dfrac{1}{\sqrt{3n}}$

∴ $\max.\ |u'_n(x)| = \dfrac{1}{\sqrt{3n}^{\frac{5}{2}}\left(1+\frac{1}{3}\right)^2}$

$= \dfrac{3\sqrt{3}}{8n^{\frac{5}{2}}}$

Then $|u'_n(x)| < \dfrac{1}{n^{\frac{5}{2}}}$ for all value of x.

But $\dfrac{1}{n^{\frac{5}{2}}}$ is convergent.

Hence by W.M. Test, the series $\sum u'_n$ is U.C. for all real values of x. The term by term differentiable is therefore justified.

$$f'(x) = \sum_{n=1}^{\infty} u'_n(x)$$

$$= -2x\sum_{1}^{\infty}\frac{1}{n^2(1+nx^2)^2}$$

**Example 12:** Show that if $f(x) = \sum_{1}^{\infty}\dfrac{1}{n^3+n^4x^2}$ then it has a differential coefficient equal to $-2x\sum_{1}^{\infty}\dfrac{1}{n^2(1+nx^2)^2}$ for all values of x.

**Solution:** Here $u_n(x) = \dfrac{1}{n^3+n^4x^2}$

$$u'_{n(x)} = \frac{2x}{n^2(1+nx^2)^2}$$

Now $u'_n$ (x) is maximum value $\frac{d}{dx}$ ($u'_n$ (x)) = 0

∴ $(1 + nx^2)^2 - 4nx^2 (1 + nx^2) = 0$

or $1 - 3nx^2 = 0$

or $n = \frac{1}{\sqrt{3n}}$

∴ $\max. |u'_n (x)| = \frac{1}{\sqrt{3}n^{\frac{5}{2}}\left(1+\frac{1}{3}\right)^2}$

$$= \frac{3\sqrt{3}}{8n^{\frac{5}{2}}}$$

Then $|u'_n (x)| < \frac{1}{n^{\frac{5}{2}}}$ for all value of x.

But $\sum \frac{1}{n^{\frac{5}{2}}}$ is convergent.

Hence by W.M. Test, the series $\sum u'_n$ is U.C. for all real values of x. The term by term by term differentiable is therefore justified.

$$f'(x) = \sum_{n=1}^{\infty} u'_n (x)$$

$$= -2x \sum_{1}^{\infty} \frac{2x}{n^2(1+nx^2)^2}$$

**Example 13:** Let $f$ be continuous on [0, 1] and $\int_0^1 f(x)x^n dx = 0$ for all n = 0,1,2,....... Show that $f$(x) = 0 on [0, 1]

**Solution:** Since $f$ is continuous on [0, 1]

∴ by Weierstrass's Approximation Theorem, there exists a sequence $\{p_n\}$ of polynomials $P_n$ such that $P_n (x) \to f$ uniformly on [0, 1]

$\therefore f P_n(x) \to f^2$ uniformly on [0, 1], as $f$, being continuous is bounded on [0, 1]

$\therefore$ by theorem of uniform convergence and integration, we have

$$\int_0^1 f^2 dx = \underset{n\to\infty}{Lt} \int_0^1 f \, p_n(x)\, dx \qquad .....(1)$$

Let $P_n(x) = a_0 + a_1 x + a_2 x^2 + ....... + a_n x^n$

$$\therefore \int_0^1 f \, P_n(x)\, dx = \int_0^1 \left[a_0 x^0 f + a_1 x f + a_2 x^2 f + ..... + a_n x^n f\right] \qquad .....(2)$$

$$= \int_0^1 a_0 f . x^0 dx + ........ + \int_0^1 a_n f . x^n dx$$

$$= a_0 \int_0^1 f x^0 dx + a_1 \int_0^1 f \, x \, dx + a_2 \int_0^1 f . x^2 dx + ..... + a_n \int_0^1 f . x^n dx$$

$$= 0 + 0 + ...... + 0 = 0 \left[\because \int_0^1 x^n f(x) dx = 0 \;\; for \;\; n = 0, 1, 2, .......\right]$$

$\therefore$ from (1), we have

$$\int_0^1 f^2 dx = \underset{n\to\infty}{Lt} (0) = 0$$

$$\Rightarrow \int_0^1 f^2(x)\, dx = 0 \; \forall \; X \text{ in } \{0, 1\}$$

$\Rightarrow$ $f^2(x) = 0$ on [0, 1]

$\Rightarrow$ $f(x) = 0$ on [0, 1]

## 19.4 Self Check Exercise

Q.1 Examine for uniform convergence and continuity of the sum function of the series for which $f_n(x) = nx(1 - x)^n$, $x \in [0, 1]$

Q. 2 Examine for uniform convergence and continuity of the series $\sum xe^{-nx}$, near x = 0

Q. 3 Test for uniform convergence and term by term integration of the series $\sum \frac{x}{(n+x^2)^2}$, $0 \le x \le 1$

Q.4 Given by $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is a differentiable for every x and its derivative is $\sum_{1}^{\infty} \frac{\cos nx}{n^2}$.

## 18.5 Summary

In this unit we have learnt the following

(i) Some important results about uniform convergence

(ii) Uniform convergence and continuity,

(iii) Uniform convergence and Riemann integration

(iv) Term by term integration

(v) Uniform convergence and differentiation, term by term differentiation

(vi) Weierstrars approximation theorem etc.

## 19.6 Glossary:

(i) Uniform convergence and continuity -

Let $\{f_n\}$ be a sequence of real valued function on E which converges to $f$ on E. If each $f_n$ is continuous on E then $f$ is also continuous on E.

## 19.7 Answer to Self Check Exercise

Ans.1 Prove it.

Ans.2 Take $f_n(x) = \sum_{r=1}^{n} xe^{-rn} = x \frac{1-e^{-nx}}{1-e^{-n}}$

and $f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) = \frac{x}{1-e^{-n}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

and then proceed.

Ans.3 Prove it.

Ans. 4 Take $u_n(x) = \frac{\sin nx}{n^3}$, find $u_n'(x)$, ux$\left|\frac{\cos nx}{n^2}\right| < \frac{1}{n^2}$ $\forall$ n and $\sum \frac{1}{n^2}$ is convergent.

Then proceed.

## 19.8 References/Suggested Readings

1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. R.G. Bartle and D.R. Sharbert; Introduction to Real Analysis, John Wiley and Sons (Asia) P. Ltd., 2000

3. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

## 19.9 Terminal Questions

1. Show that if $f$(x) = $\sum_{1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then it has a differential coefficient equal to -2x $\sum \frac{1}{n^2\left(n^3 + n^4 x^2\right)}$ $\forall$ x.

2. Examine for term by term integration the series $\sum_{0}^{\infty} (-1)^n x^n$ , x $\in$ [0, 1]

3. Show that 0 is a point of non uniform convergence of the series $\sum u_n$ , $u_n$ (x) = $\frac{1-(1+x)^n}{1+(1+x)^n}$ - $\frac{1-(1+x)^{n-1}}{1+(1+x)^{n-1}}$

4. Test for uniform convergence and continuity of the sum function of the series for which $f_n$ (x) = $\frac{1}{1+nx}$ , 0 $\leq$ n $\leq$ 1.

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## Unit - 20

# Power Series And

# Radius of Convergence

**Structure**



## 20.1 Introduction

Dear students, we shall discuss about the concept of power series in this unit. In mathematics, a power series (in one variable) is an infinite series of the form

$\sum_{n=0}^{\infty} a_n (x-c)^n$ = $a_0$ +$a_1$ (x - c) + $a_2$ (x - c)$^2$ +...., where $a_n$ is the coefficient of the nth term and c is a constant. The power series are useful in mathematical analysis, where they arise as Taylor series of infinitely differentiable function. In many situations, c (the centre of the series) is equal to zero, for example when considering a Macoupin series. In such cases the above power series takes the form

$\sum_{n=0}^{\infty} a_n x^n$ = $a_0$ + $a_1$x +......+$a_n$x$^n$ +......, Beyond their role in Mathematical analysis, power series also occur in combinatory as generating functions and in electronic engineering.

## 20.2 Learning Objectives

The main objectives of this unit are

(i) to define a power series

(ii) to study radius of convergence and interval of convergence

(iii) to learn about uniform convergence of power series

(iv) to study some important results etc.

## 20.3 Definition of Power Series

A series of the type

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + ....a_n (z - z_0)^n + ....$$

is called a power series about $z = z_0$. In particular, z and constants an $z_0$ are often real, but if complex, the series can still be discussed. If $z - z_0 = x$ (change of variable) the above series can be reduced to $\sum_{n=0}^{\infty} a_n x^n$ . Hence it suffices to consider the series of the form $\sum_{n=0}^{\infty} a_n x^n$ about x = 0, x is real. If $z_0$ is replaced by a and z by x then power series reduced to $\sum_{n=0}^{\infty} a_n (x - a)^n$ about x = a.

The power series $\sum a_n x^n$ either

(i) converges for all values of x

(ii) converges only for x = 0

(iii) converges for x in same region on the real line. we give example in each of the about three cases.

(i) The series $\sum \frac{x^n}{\lfloor n}$ converges absolutely for all value of x

Let $u_n = \frac{x^n}{\lfloor n}$, $u_{n+1} = \frac{x^{n+1}}{\lfloor n+1}$, then

$$\underset{n\to\infty}{Lt} \frac{|u_n|}{|u_{n+1}|} = \underset{n\to\infty}{Lt} \frac{n+1}{|x|} = \infty > 1$$

Hence by D'Alembert's ratio test, the series converges absolutely for all value of x.

(ii) The series $\sum \lfloor n x^n$ converges only for x = 0

Let $u_n = \lfloor n x^n$,

$\therefore$ $u_{n+1} = \lfloor n+1 \, x^{n+1}$

$$\underset{n\to\infty}{Lt} \frac{|u_n|}{|u_{n+1}|} = \underset{n\to\infty}{Lt} \frac{1}{(n+1)|x|}$$

$= 0 < 1$ if $x \neq 0$

$\therefore$ $\sum \lfloor n x^n$ is divergent.

But if x = 0, then the given series has its sum = 0 [$\because$ each term = 0]

$\therefore$ $\sum \lfloor n x^n$ is convergent.

(iii) The series $\sum_{n=1}^{\infty} x^n$ converges for |x| < l and divergent for |x| > l

Take $u_n = x^n$

$\therefore$ $u_{n+1} = x^{n+1}$

$$\underset{n\to\infty}{Lt} \frac{|u_n|}{|u_{n+1}|} = \underset{n\to\infty}{Lt} \frac{1}{|x|} = \frac{1}{|x|}$$

By D'Alembert ratio test $\sum u_n$ is convergent for $\frac{1}{|x|} > 1$ i.e. |x| < 1 and divergent for $\frac{1}{|x|} < 1$ i.e. |x| > 1.

For |x| = 1, $\underset{n\to\infty}{Lt} \frac{|u_n|}{|u_{n+1}|} = 1$

$\therefore$ the test fails here.

Thus the given series is convergent for |x| < 1 i.e. -1 < x > 1 only.

**Art.** If a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for particular value $x_0$ of x, then it converges absolutely for all value of x for which $|x| < |x_0|$.

**Proof :** Let $\sum_{n=0}^{\infty} a_n x^n$ converges

$\therefore$ $un = a_n x_0^n \to 0$ as $n \to \infty$

$\therefore$ we can find a number M > 0 such that

$\left|a_n x_0^n\right| < M \quad \forall$ all n

$$|a_n x^n| \leq M \left|\frac{x^n}{x_0}\right| \qquad \text{...(1)}$$

Therefore for $|x| < |x_0|$, the G.P. series $\sum \left|\frac{x}{x_0}\right|^n$ converges.

Thus we conclude that the series $\sum |a_n x^n|$ converges for all value of x given by $|x| < |x_0|$

**Art.** If $\underset{n\to\infty}{Lt} |a_n|^{\frac{1}{n}} = \frac{1}{R}$, then the series $\sum a_n x^n$ converges absolutely for $|x| < R$ and diverges for $|x| > R$

**Proof :** Consider the series $\sum a_n x^n$

By Cauchy's Root test, $\sum_{n=0}^{\infty} a_n x^n$ converges if $\underset{n\to\infty}{Lt} |a_n x^n|^{\frac{1}{n}} < 1$

i.e. $\underset{n\to\infty}{Lt} |x| \; |a_n|^{\frac{1}{n}} < 1$

i.e. $|x| < \dfrac{1}{\underset{n\to\infty}{Lt} |a_n|^{\frac{1}{n}}}$

and $\sum_{0}^{\infty} a_n x^n$ diverges if $\underset{n\to\infty}{Lt} |a_n x^n|^{\frac{1}{n}} > 1$

i.e. $\underset{n\to\infty}{Lt} |x| \; (a_n)^{\frac{1}{n}} > 1$

i.e. $|x| > \dfrac{1}{\underset{n\to\infty}{Lt} (a_n)^{\frac{1}{n}}}$

From (1) and (2), we get

$$\underset{n\to\infty}{Lt} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left( \underset{n\to\infty}{Lt} u_n(x) \right)$$

[ ∵ R.H.S. of (1) and (2) are equal]

**Illustrative Examples**

**Example 1 :** Consider the series $\sum \frac{(-1)^{n+1}}{n} . x^n$ for uniform convergence in [0, 1]

**Solution :** Take $\nu_n(x) = x^n$ and $u_n(x) = \frac{(-1)^{n+1}}{n}$

The sequence $\{\nu n (x)\}$ is uniformly bounded and monotonically increasing [0, 1]

**Example 2 :** Prove that $\underset{x\to 1}{Lt}\left[\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}\right] = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$

**Solution :** We first prove that $\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}$ converges uniformly on (0, K) for any K > 0

Let $u_n(x) = \frac{1}{n^3+x^3}$ $v_n(x) = nx^2$

Now $|u_n(x) \le \frac{1}{n^3}$ $\forall x \in [0, K]$

But $\sum \frac{1}{n^3}$ is convergent by W.M. Test

$\therefore$ $\sum u_n(x)$ is uniformly convergent.

Also sequence $\{v_n(x)\} = \{nx^2\}$ is monotonically increasing in (0, K)

Hence by Abel's Test, $\sum u_n(x)\, v_n(x) = \sum \frac{nx^2}{n^3+x^3}$ converges uniformly in (0, K)

Now $\underset{x\to 1}{Lt}\left(\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}\right) = \sum_{n=1}^{\infty}\left(\underset{x\to 1}{Lt}\frac{nx^2}{n^3+x^3}\right) = \sum_{n=1}^{\infty}\frac{n}{n^3+1}$

**Example 3:** Consider $\sum\frac{(-1)^{n-1}}{n+x^2}$ for uniform convergence.

**Solution:** Let $u_n = (-1)^{n-1}$, $v_n = \frac{1}{n+x^2}$

$\therefore$ $f_n(x) = \sum_{1}^{n} u_n = 0$ or 1 according as n is even or odd

$\therefore$ $f_n(x)$ is bounded.

Also $\{vn(x)\}$ is a the monotonically decreasing sequence converging to zero for all real values x.

$\therefore$ by Dirichlet's Test the given series is uniformly convergent for all real values of x.

**Example 4:** Prove that $\sum(-1)^n\frac{n^3+x^2}{n^3}$ converges uniformly on every bounded subset of R.

**Solution:** The given series is $\sum(-1)^n\frac{n^3+x^2}{n^3}$

Let S be a bounded subset of R.

Since S is bounded

$\therefore$ $\exists$ a +ve real number M such that

$|x| < M$ .......(1)

Let $u_n = (-1)^n$

$\nu_n(x) = \dfrac{n^3 + x^2}{n^3}$

$|\nu_n(x)| = \left|\dfrac{n^3 + x^2}{n^3}\right| \le \dfrac{M + n^2}{n^3}$

Clearly $\nu_n$ is a +ve, monotonically decreasing function of x for $x \in S$ and $\nu_n(x) \to 0$ as $n \to \infty \forall x \in S$

$f_n(x) = \sum_1^n u_n(x) = \begin{cases} -1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$

$\therefore$ by Dirichlet, test the series $\sum(-1)^n \dfrac{n^3 + x^2}{n^3}$ converges uniformly on S.

**Example 5:** Show that the series sin x + $\frac{1}{2}$ sin 2x + $\frac{1}{3}$ sin 3x + ..... converges uniformly for $0 < a \le x \le b \le 2\pi$

Now $R = \dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}}$

$\therefore$ $\sum_0^\infty a_n x^n$ converges if : R and diverges if $|x| > R$.

## 20.4 Art. Radius of Convergence and Interval of Convergence.

A power series $a_0 + a_1x + a$ +...... i.e. $\sum a_n x^n$ is said to have R as radius of convergence if $R = \dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}}$.

$|x| = R$ is called the circle of convergence of the power series.

The set of value of x for when the power series converges is called interval of convergence of its region of convergence. So (-R, R) is called the interval of convergence.

**Note. 1.** R can be zero, finite or intervale.

**Note. 2.** By Cauchy's Second Theoreon limit for $a_n > 0$, we have

$$\underset{n\to\infty}{Lt}(a_n)^{\frac{1}{n}} = \underset{n\to\infty}{Lt}\left(\frac{a_{n+1}}{a_n}\right)$$

$\therefore$ R = $\dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}} = \underset{n\to\infty}{Lt}\left|\dfrac{a_{n+1}}{a_n}\right|$

**Illustrative Examples**

**Example 6:** Given the domain of convergence of the power series

$$\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+.......+\frac{x^n}{n}+........$$

**Solution:** The given power series is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ but x = 0

Here $a_n = \frac{1}{n}$ and $a_{n+1} = \frac{1}{n+1}$

Radius of convergence of the poor series is

$$R = \underset{n\to\infty}{Lt}\left|\frac{a_n}{a_{n+1}}\right|$$

$$= \underset{n\to\infty}{Lt}\left|\frac{n+1}{n}\right|$$

$$= \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)$$

$$= 1 + 0 = 1$$

Thus the series converges in the interval given by $|x| < 1$

$\therefore$ $-1 < x < 1$

$\Rightarrow$ $x \in (-1, 1)$

To discuss the convergence at the end points, first take x = 1 for which the series becomes

$$1 + \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.........+\frac{1}{n}+.......$$

which being a harmonic series diverges

Further for x = -1 the power series becomes

$-1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+.........+(-1)^n.\frac{1}{n}+.......\sum_{n=1}^{\infty}(-1)^n.\frac{1}{n}$

which converges conditionally by virtue of Leibnitz's test.

Thus domain of convergence of the given power series is [-1, 1].

**Example 7:** Find radius of convergence of power series $\sum(5+12i)z^n$.

**Solution:** The given power series is $\sum(5+12i)z^n$.

Here $a_n = 5 + 12i$

$\therefore$ $|a_n| = |5 + 12i|$

$$= \sqrt{(5)^2+(12)^2}$$

$$= \sqrt{25+144}$$

$$= \sqrt{169}$$

$$= 13$$

$$\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}} = \underset{n\to\infty}{Lt}(13)^{\frac{1}{n}}$$

$$= (13)^0 = 1$$

$\therefore$ radius of convergence $= \dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}} = \dfrac{1}{1} = 1$

**Example 8:** Find the radius of convergence of $\sum_{n=1}^{\infty}\left(\frac{nz}{n+1}\right)^n$

**Solution:** The given power series is $\sum_{n=1}^{\infty}\left(\frac{nz}{n+1}\right)^n$

Here $a_n = \left(\frac{n}{n+1}\right)^n$

$$\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}} = \underset{n\to\infty}{Lt}\left(\frac{n}{n+1}\right)$$

$$= \underset{n\to\infty}{Lt}\left(1+\frac{1}{\frac{1}{n}}\right) = \frac{1}{1+0} = \frac{1}{1} = 1$$

$\therefore$ radius of convergence = $\dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}} = \dfrac{1}{1} = 1$

**Example 9:** Find the radius of convergence of the power series $\sum\left(1+\frac{1}{n}\right)^{n^2} x^n$.

**Solution:** The given power series is $\sum\left(1+\frac{1}{n}\right)^{n^2} x^n$

$\therefore$ $a_n = \left(1+\frac{1}{n}\right)^{n^2}$

$\therefore$ $|a_n|^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{n}$

$\therefore$ $\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}} = \underset{n\to\infty}{Lt}\left(1+\frac{1}{n}\right)^{n} = e$

$\therefore$ $R = \dfrac{1}{\underset{n\to\infty}{Lt}|a_n|^{\frac{1}{n}}} = \dfrac{1}{e}$

**20.5 Art. Uniform Convergence of Power Series**

The power series $\sum a_n x^n$ is uniformly convergent for $|x| \le p < R$, where R is the radius of convergence.

**Proof.** Let p' be a number such that p < p' R

Since the series is convergent for |x| = p'

Therefore ∃ a number K, independent of n so that

$$\left|a_n p'^{\frac{1}{n}}\right| \le K \qquad \forall\ n$$

Hence for $|x| \le p$

$$|a_n x^n| = \left|a_n p'^n \left(\frac{x}{p'}\right)^n\right| < k \left(\frac{p}{p'}\right)^n$$

which is independent of x.

But the series K $\sum\left(\frac{p}{p'}\right)^n$ is convergent, being a G.P. series with common ratio $\frac{p}{p'} < 1$.

Hence by W.M. test the power series is uniformly convergent for $|x| \leq p < R$

Thus, every power series is U.C. within the circle of converges.

**20.6 Art. Some Important Results**

**Result I:** Properties of Power Series

(1) A power series $\sum a_n x^n$ is a continuous function of x within its interval of convergence.

(2) A power series can be integrated term by term so long as the limit of integration lies strictly within the range (-R, R)

(3) A power series can be differential term by term so long as the limit of differentiating lies strictly with in the range (-R, R)

**Result II:** The series obtained by differentiating a power term by term has the same radius of convergence as the original series.

**Result III:** The radius of convergence of the series obtained by integrating term by term has the radius of convergence as the original series.

**Result IV:Abel's Summability**

The series $\sum_{n=0}^{\infty} a_n$ is Abel's summable to a value S, if the associated power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $0 \leq x < 1$ to function $f$ and $\lim_{x \to 1-} f$ (x) = S.

**Result V: Abel's Theorem**

If the series $\sum_{0}^{\infty} a_n$ is convergent and has the sum S, the series $\sum_{0}^{\infty} a_n x^n$ is uniformly convergent for $0 \leq x \leq 1$

and $\lim_{x \to 1} \sum_{0}^{\infty} a_n x^n = S$

**Result VI: Taubers Theorem**

If the series $\sum_{n=0}^{\infty} a_n$ is Abel summable to s

and if $\lim_{n\to\infty}$ $na_n = 0$, the $\sum a_n$ converges to s.

**Result VII: Taylor Series**

Let $f$ be a function defined on some interval containing 0. If $f$ possesses derivatives of all order at 0, then the series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n} x^n$$

is called the Taylor Series for $f$ about 0. The remainder $R_n$ (x) is defined by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k} x^k$$

Here the Remainder $R_n$ depends on $f$.

Now $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n} x^n$ iff $\underset{n\to\infty}{Lt} R_n = 0$

**Result VIII: Taylor's Theorem**

If $\sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence R and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ for } |x| < R$$

then for any $a \in (-R, R)$, $f$ can be expanded in a power series about 'a' which converges for $|x - a| < R - |a|$ and $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n} (x - a)^n$, $|x - a| < R - |a|$

**Note :** Before taking up the solution of power series we must keep in mind the following explanations:

(1) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4$ .......

(2) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 +$ .......

(3) $(1 - x)^{-1} = 1 + x + x^2 + x^3 +$ .......

(4) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 +$ .......

(5) $\sin^{-1} x = x + \frac{1}{2}.\frac{x^3}{3} + \frac{1.3}{2.4}.\frac{x^5}{5} + \frac{1.3.5}{2.4.6}.\frac{x^7}{7} + ......$

(6) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} + ........$

(7) $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + .......$

(8) $\log(1 - x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + ........\right)$

**Example 8:** Show that

(i) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + .......$ for $-1 \leq x \leq 1$

(ii) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + .......$

(iii) $\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1 + \frac{1}{3}\right) + \frac{x^6}{6}\left(1 + \frac{1}{3} + \frac{1}{5}\right) + .........-1 < x \leq 1$

**Solution:** We have $(1 + x^2)^{-1} = 1 - x^2 + + x^4 - x^6 + .......$

$$= \sum_{n=\infty}^{\infty}(-1)^n . x^{2n}$$

$$= \sum_{n=\infty}^{\infty}(-1)^n . t^n \qquad \{\text{Taking x2 = t}\}$$

Here $a_n = (-1)^n$, $a_{n+1} = (-1)^{n+1}$

$$\underset{n\to\infty}{Lt}\left|\frac{a_{n+1}}{a_n}\right| = 1$$

$\therefore$ for interval of convergence

$|t| < 1$

$\Rightarrow$ $|x^2| < 1$

$\Rightarrow$ $|x| < 1$

$\Rightarrow$ $-1 < x < 1$

Therefore the series on the R.H.S. is a power series with radius of convergence unity and converges absolutely for $-1 < x < 1$

As such it converges uniformly in $(-\lambda, \lambda)$ where $|\lambda| < 1$

The series on R.H.S. does not converges for $x = \pm 1$, we get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{.........} + c$$

When

$$\therefore \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{.........}, \text{ for } -1 < x < 1$$

Because for $x = \pm 1$ the power series on R.H.S. of (1) becomes

$$\pm\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{........}\right) = \pm(-1)^{n-1} . \frac{1}{2n-1}$$

which is an alternating series.

$\therefore$ by Leibnitz's test the power series on the R.H.S. of (1) is convergent for $x = \pm 1$ also. Therefore it converges in $[-1, 1]$ and hence converges uniformly for $x \in [-1, 1]$.

Thus $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^7}{7} + \text{.........}$ for $-1 \le x \le 1$ ....(2)

(ii) At x = 1, we get

$$\tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{............}$$

$$\therefore \quad \frac{\pi}{4} = - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \text{........}$$

which is Gregory series.

(iii) From (2) and (1), we have

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{.......}, -1 \le x \le 1$$

and $(1 + x^2)^{-1} = 1 - x^2 + x^4 - x^6 +,,,,,,, -1 < x < 1$

Both the series are absolutely convergent in (-1, 1)

Their Cauchy product will also be absolutely convergent in (-1, 1) therefore by Abel's theorem.

$$\therefore \quad (\tan^{-1} x)(1 + x^2)^{-1} = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \text{......}\right)(1 - x^2 + x^4 - x^6 + \text{.........})$$

$$= x - x^3\left(1+\frac{1}{3}\right) + x^5\left(1+\frac{1}{3}+\frac{1}{5}\right)+\text{........ for } -1 < x < 1$$

Integrating both sides w.r.t. x, we get

$$\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1+\frac{1}{3}\right) + \frac{x^6}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)+\text{........}+c_1$$

When x = 0, $c_1 = 0$

$$\therefore \qquad \frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \left(1+\frac{1}{3}\right)\frac{x^4}{4} + \left(1+\frac{1}{3}+\frac{1}{5}\right)\frac{x^6}{6} + \text{....... for } -1 < x < 1$$

For x = 1, the power series on the R.H.S. becomes an alternating series and as such is convergent by Leibnitz's test.

Therefore by Abel's Theorem,

$$\frac{1}{2}(\tan^{-1} x)^2 = \frac{x^2}{2} - \frac{x^4}{4}\left(1+\frac{1}{3}\right) + \frac{x^6}{6}\left(1+\frac{1}{3}+\frac{1}{5}\right)+\text{........for } -1 < x \leq 1$$

**Example 9:** Find the sum of the series

$$x + \sum_{n=1}^{\infty}\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} \cdot \frac{x^{2n+1}}{2n+1} \text{ when } |x| \leq 1$$

Deduce that

(i) $\qquad \frac{\pi}{2} = 1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \text{.........}$

(ii) $\qquad \frac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2} + \frac{2}{3}.\frac{x^4}{4} + \frac{2.4}{3.5}.\frac{x^6}{6} + \text{........ for } -1 \leq x \leq 1$

**Solution:** We have

$$\left(1-x^2\right)^{\frac{-1}{2}} = 1 + \frac{1}{2}(x^2) + \frac{1.3}{2.4}(x^2)^2 + \frac{1.3.5}{2.4.6}(x^2)^3 + \text{.......} + \frac{1.3.5.......(2n-1)}{2.4.6.......(2n)}(x^2)^n + \text{.......}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)}.x^{2n}$$

$$\therefore \qquad \frac{1}{\sqrt{1-x^2}} = 1 + \sum_{n=1}^{\infty}\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} t^n \qquad \text{(where } x^2 = t) \qquad .....(1)$$

Here $\quad a_n = \frac{1.3.5.......(2n-1)}{2.4.6.......(2n)}$, $a_{n+1} = \frac{1.3.5.......(2n-1)(2n+1)}{2.4.6.......(2n)(2n+2)}$

$\therefore \quad R = \underset{n\to\infty}{Lt} \dfrac{2n+2}{2n+1}$

$= \underset{n\to\infty}{Lt} \dfrac{2+\frac{2}{n}}{2+\frac{1}{n}}$

$= \dfrac{2+0}{2+0}$

$= \dfrac{2}{2}$

$= 1$

Therefore the series on the R.H.S. is a power series whose radius of convergence is unity.

The series on the R.H.S. is convergent for $|t| < 1$

i.e. for $|x^2| < 1$

i.e. for $|x| < 1$

For $x^2 = 1$, the series on the R.H.S. becomes

$$1 + \frac{1}{2} + \frac{1.3}{2.4} + \frac{1.3.5}{2.4.6} + ........ + \frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} + ........$$

Here $u_n = \dfrac{1.3.5.......(2n-1)}{2.4.6.......(2n)}$,

$u_{n+1} = \dfrac{1.3.5.......(2n-1)(2n+1)}{2.4.6.......(2n)(2n+2)}$

$\therefore \quad \dfrac{u_n}{u_{n+1}} = \dfrac{2n+2}{2n+1}$

$\therefore \quad \underset{n\to\infty}{Lt}\, n\left\{\dfrac{u_n}{u_{n+1}}-1\right\} = \underset{n\to\infty}{Lt}\, n\left\{\dfrac{2n+2}{2n+1}-1\right\}$

$= \underset{n\to\infty}{Lt} \dfrac{n(2n+2-2n-1)}{2n+1}$

$= \underset{n\to\infty}{Lt} \dfrac{n}{2n+1}$

$$= \underset{n\to\infty}{Lt} \frac{1}{2+\frac{1}{2}}$$

$$= \frac{1}{2+0}$$

$$= \frac{1}{2} < 1$$

$\therefore$ by Raabe's test the power series on the R.H.S. is divergent for $x^2 = 1$ i.e. for $x = \pm 1$

Therefore series on the R.H.S. is absolutely convergent for $x \in (-1, 1)$ and uniformly convergent in $(-\lambda, \lambda)$ where $|\lambda| < 1$. The integrated series will have the same characteristics.

Thus integrating (1) on both sides of (1) w.r.t. x, we get

$$\sin^{-1} x = x + \sum_{n=0}^{\infty} \frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} \cdot \frac{x^{2n+1}}{2n+1} + c$$

For $x = 0$, $c = 0$

$$\therefore \quad \sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7} + .......$$

or $\sin^{-1} x = x + \dfrac{1.3.5.......(2n-1)}{2.4.6.......(2n)} \cdot \dfrac{x^{2n+1}}{2n+1}$ for $-1 < x < 1$ ..... (2)

Clearly radius of convergence on the R.H.S. is unity and as such the interval of convergence is $-1 < x < 1$

By Radbe's test for x = 1,

$$\frac{v_n}{v_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{2n+3}{2n+1}$$

$$\therefore \quad \underset{n\to\infty}{Lt}\ n\left[\frac{v_n}{v_{n+1}} - 1\right] = \underset{n\to\infty}{Lt}\ n\left[\frac{(2n+2)(2n+3)}{(2n+1)(2n+1)} - 1\right]$$

$$= \underset{n\to\infty}{Lt}\ n\left[\frac{4x^2 + 10x + 6 - 4x^2 - 4x - 1}{4x^2 + 4x + 1}\right]$$

$$= \underset{n\to\infty}{Lt} \frac{n(6n+5)}{4n^2 + 4n + 1}$$

$$= \underset{n\to\infty}{Lt} \frac{6+\frac{5}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}$$

$$= \frac{6+0}{4+0+0}$$

$$= \frac{6}{4} > 1$$

∴ the series on the R.H.S. is convergent for x = 1 by Rabbe's test Similarly the power weries on the R.H.S. is convergent for x = -1

Therefore the series is uniformly convergent for x ∈ [-1, 1]

∴ sin-1 x = x + $\sum_{n=1}^{\infty} \frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} \cdot \frac{x^{2n+1}}{2n+1}$ for -1 ≤ x ≤ 1 ..... (3)

(i) Put x = 1 on both sides of (2)

∴ sin-1 (1) = 1 + $\frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7}$ +.........

or $\frac{\pi}{2}$ = 1 + $\frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7}$ +.........

(ii) From above we have

$\left(1-x^2\right)^{\frac{-1}{2}}$ = 1 + $\frac{1}{2}$ $(x^2)$ + $\frac{1.3}{2.4}$ $(x^2)^2$ + $\frac{1.3.5}{2.4.6}$ $(x^2)^3$ +........

+ $\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)}$ $(x^2)^2$, for -1 < x < 1

and sin-1 x = x + $\frac{1}{2} \cdot \frac{x^3}{3} + \frac{1.3}{2.4} \cdot \frac{x^5}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^7}{7}$ +.......

+ $\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} \cdot \frac{x^{2n+1}}{2n+1}$ for -1 ≤ x ≤ 1.

Both the series are absolutely convergent in (-1, 1). Therefore their Cauchy product will have the same characteristics.

∴ $(\sin^{-1} x)$ $\left(1-x^2\right)^{\frac{-1}{2}} = \left[1+\frac{1}{2}(x^2)+\frac{1.3}{2.4}(x^2)^2+\frac{1.3.5}{2.4.6}(x^2)^3+........\right]$

$$\times\left[1+\frac{1}{2}.\frac{x^3}{3}+\frac{1.3}{2.4}.\frac{x^5}{5}+\frac{1.3.5}{2.4.6}.\frac{x^7}{7}+........\right]$$

$$= x + x^3\left(\frac{1}{2}+\frac{1}{6}\right) + x^5\left(\frac{3}{40}+\frac{1}{12}+\frac{3}{8}\right) +........$$

$$= x + x^3\left(\frac{4}{6}\right) + x^5\left(\frac{9+10+45}{120}\right) +.......$$

$$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}.x^3 + \frac{8}{15}.x^5 + ........ \text{ for } -1 < x < 1$$

$\therefore$ $$\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}.x^3 + \frac{2.4}{3.5}x^5 + ......... \text{ for } -1 < x < 1$$

Integrating both sides term by term w.r.t. x, we have

$$\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2}+\frac{2}{3}.\frac{x^4}{4}+\frac{2.4}{3.5}.\frac{x^6}{6}+......\ c_1 \text{ for } -1 < x < 1$$

When x = 0, $c_1 = 0$

$\therefore$ $$\frac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2}+\frac{2}{3}.\frac{x^4}{4}+\frac{2.4}{3.5}.\frac{x^6}{6}+......$$

$$+\frac{2.4...........2n}{3.5............(2n+1)}.\frac{x^{2n}}{2n}+........\text{for } -1 < x < 1$$

Here $$b_n = \frac{2.4...........2n}{3.5............(2n+1)}.\frac{1}{2n}$$

$$b_{n+1} = \frac{2.4...........2n(2n+2)}{3.5............(2n+1)(2n+3)}.\frac{1}{2n+2}$$

$\therefore$ $$\frac{b_n}{b_{n+1}} = \frac{2n+3}{2n+2}.\frac{2n+2}{2n}$$

$$= \frac{2n+3}{2n}$$

$\therefore$ $$n\left\{\frac{b_n}{b_{n+1}}-1\right\} = n.\left\{\frac{2n+3}{2n}-1\right\}$$

$= n.\frac{3}{2n}$

$= \frac{3}{2}$

$\therefore \quad \underset{n\to\infty}{Lt}\ n\left\{\frac{b_n}{b_{n+1}} - 1\right\} = \frac{3}{2} > 1$

$\therefore$ by Raabe's test, the power series on the R.H.S. is convergent for $x^2 = 1$ i.e. for x = $\pm 1$

$\therefore \quad \frac{1}{2}(\sin^{-1} x)^2 = \frac{x^2}{2} + \frac{2}{3}.\frac{x^4}{4} + \frac{2.4}{3.5}.\frac{x^6}{6} + ......$ for $-1 \le x \le 1$.

**Example 10:** Prove that $\sin^{-1} x = \sum_{n=0}^{\infty} \frac{^{2n}C_n}{4^n} \frac{x^{2n+1}}{2n+1}$, $x \in [-1, 1]$ (Prove it as in above example)

Now, $\frac{1.3.5.......(2n-1)}{2.4.6.......(2n)} = \frac{1.3.5.......(2n-1)}{2.4.6.......2n} \frac{2.4.6...........2n}{2.4.6............2n}$

$= \frac{1.2.3...........2n}{(2.4.6............2n)^2}$

$= \frac{2n}{2^{2n}(n)^n}$

$= \frac{2n}{2^n |n| n}$

$= \frac{^{2n}C_n}{4^n}$

$\therefore \quad \sin^{-1} x = x + \sum_{n=1}^{\infty} \frac{^{2n}C_n}{4^n} \frac{x^{2n+1}}{2n+1}$, $x \in [-1, 1]$

## 20.7 Self Check Exercise

Q.1 Find the interval of convergence of the power series $\sum\left(\frac{2^n}{n^2}\right)x^n$

Q. 2 Find the radius of convergence of $\sum_{n=2}^{\infty} \frac{(x-2)^{n-2}}{n \log n}$.

Q. 3 Find the radius of convergence and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{3n!}{(n!)^3}$ $x^n$.

## 20.8 Summary

We have learnt the following concepts in this unit:

(i) Definition of power series

(ii) Radius of convergence and interval of convergence

(iii) Uniform convergence of power series

(iv) Some important results like properties of power series, Abel's summabilily, Abel's theorem, Tanfer's Theorem, Taylor series and Taylor's Theorem etc.

## 20.9 Glossary:

1. Tauber's Theorem: If the series $\sum_{n=0}^{\infty} a_n$ is Abel's summable to s and if $\underset{n\to\infty}{Lim}$ n an = 0 then $\sum a_n$ converges to s.

2. Every power series is uniform convergent with in the circle of convergence.

## 20.10 Answer to Self Check Exercise

Ans.1 $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is the interval of convergence

Ans.2 Radius of convergence is 1.

Ans.3 Radius of convergence = $\frac{1}{27}$ and interval of convergence = $\left(-\frac{1}{27}, \frac{1}{27}\right)$

## 20.11 References/Suggested Readings


1. T.M. Apostal, Calculus (Vol I), John Wiley and Sons (Asia) P. Ltd., 2002.

2. E. Fischer, Intermediate Real Analysis Springer Verlag, 1983

3. K.A. Ross, Elementary analysis - The Theory of Calculus - undergraduate Tests in Mathematics, Springer Verlag, 2003.


## 20.12 Terminal Questions

1. Find radius of convergence and interval of convergence for the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{25^n}$

2. Find the radius of convergence of the power series x + $\frac{x^2}{2^2} + \frac{2x^3}{3^3}$ ! + $\frac{3x^4}{4^4}$ ! + .....

3. Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{n4^n}{3^n}$ $x^n$

4. Find the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n-1!}{n^n}$ $x^n$.

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