B.A. 1st Year
Mathematics(Core Course)

Course Code: MATH102TH
New Syllabus : CBCS

# Differential Equations

## Units 1 to 20


Centre for Distance and Online Education (CDOE)
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# B.A.-1st Mathematics MATH102TH

## Differential Equations

**CONTENTS**

**HIMACHAL PRADESH UNIVERSITY**
**B.A. with Mathematics (Annual System)**
**Syllabus and Examination Scheme**

| | |
|---|---|
| Course Code | MATH102TH |
| Name of the Course | Differential Equations |
| Type of the Course | Core Course |
| Assignments | Max. Marks:30 |
| Yearly Based Examination | Max Marks: 70 |
| | Maximum Times: 3 hrs. |

## Instructions

**Instructions for paper setter:** The question paper will consist of two Sections A & B of 70 marks, Section A will be Compulsory and will contain 8 questions of 16 marks (each of 2 marks) of short answer type having two questions from each Unit of the syllabus. Section B of the question paper shall have four Units I, II, III, and IV. Two questions will be set from each unit of the syllabus and the candidates are required to attempt one question from each of these units. Each question in Units 1, II, III and IV shall be of 13.5 marks each.

**Instructions for Candidates:** Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

## Core 1.2: Differential Equations

### Unit-1

Basic theory of linear differential equations, Wronskian, and its properties. First order exact differential equations. Integrating factors, rules to find an integrating factor. First order higher degree equations solvable for x, y, p. Clairut's form

### Unit-II

Methods for solving higher-order differential equations. Solving a differential equation by reducing its order. Linear homogenous equations with constant coefficients, Linear non-homogenous equations.

### Unit-III

The method of variation of parameters with constant coefficients. The Cauchy-Euler equationand Legendre equation. Simultaneous differential equations, Total differential equations.

### Unit-IV

Order and degree of partial differential equations, Concept of linear and non-linear partial differential equations. Formation of first order partial differential equations (PDE). Linear partial differential equation of first order, Lagrange's method. Classification of second order partial differential equations into elliptic, parabolic and hyperbolic through illustrations only.

**Books Recommended**

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Sneddon, Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

**Unit - 1**

# Some Basic Concepts

**Structure**



## 1.1 Introduction

Many practical problems in Science are formulated by finding how one quantity is related to or depends upon one or more (other) quantities defined in the problem. Often, it is easier to model a relation between the rates of changes in the variables rather than between the vriables themselves. The study of this relationship gives rise to differential equations. Derivatives can always be interpreted as rates. For example, if x, is a function of t then $\frac{dx}{dt}$ is the rate of change of x with respect to t. If x denotes the displacement of a particle, then $\frac{dx}{dt}$ represents the velocity of a particle. If x represents the electric charge (q), then $\frac{dx}{dt}$ or $\frac{dq}{dt}$ represents the rate of flow of charge, that is the current. Derivatives of higher orders represent rate of rates. If x denotes the displacement of a particle, then $\frac{d^2x}{dt^2}$ represents the acceleration.

In physics, engineering, chemistry and, on occasion, in such subjects as biology, economics etc. It is necessary to build a mathematical model to represent certain problems. It is often the case that these mathematical models involve the search for an unknown function that satisfies an equation in which the derivatives of unknown functions are involved. Such equations are called differential equations.

## 1.2 Learning Objectives

After studying this unit, you should be able to:-

- Define ordinary and partial differential equations.
- Define and find order and degree of a differential equation.
- Define and find solution of differential equation.
- Formulate the differential equations.

## 1.3 Differential Equations

**Def:** An equation which involves derivatives of one or more dependent variables w.r.t. one or more independent variables is called a differential equation.

The following are some of the examples of differential equations:

(i) $\frac{dy}{dx} + 3xy = x^3$ (ii) $\sin^2 x \frac{d^2y}{dx^2} + \cos x \frac{dy}{dx} + y = 0$

(iii) $y = \frac{dy}{dx} + a\frac{dx}{dy}$ (iv) $\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^3 = a^2\left(\frac{d^2y}{dx^2}\right)^2$

(v) $\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 = 3$ (vi) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

There are two types of differential equations:

(I) Ordinary Differential Equations

(II) Partial Differential Equations

**(I) Ordinary Differential Equation:** An ordinary differential equation is one which involves one independent variable and differential coefficients w.r.t. it. In the above examples, equations (i), (ii), (iii) and (iv) are ordinary differential equations.

**(II) Partial Differential Equation:** A differential equation which involves two or more independent variables and partial derivatives w.r.t. these independent variables is called a partial differential equation.

In the above examples, equation (v) and (vi) are partial differential equations.

## 1.4 Order And Degree Of Differential Equation

The order of a differential equation is the order of the highest derivative term occurring in the differential equation.

In the above examples, the equations (i) and (iii) are differential equations of order 1 and equations (ii) and (iv) are differential equations of order 2.

An ordinary differential equation of order n is generally written as

$$F\left( x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ........, \frac{d^ny}{dx^n} \right) = 0$$

The degree of a differential equation the power of the highest derivative in the equation, when it has been made free from the radicals and fractions as far as derivatives are concerned. In the above examples, the equations (i) and (ii) are of degree 1 and the equations (iii) and (iv) are of degree 2.

To clarify what we have just said, consider the following examples:-

**Example 1:** Find the order and degree of the following differential equations:-

(i) $x\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} = a^2$

(ii) $y = x\frac{dy}{dx} + k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$

**Solution:**

(i) Here order is 1 as highest derivative occurring in it is $\frac{dy}{dx}$ And degree is 2 as the power of the highest derivative is 2.

(ii) The given equation is

$$y = x\frac{dy}{dx} + k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

Transposing, $y - x\frac{dy}{dx} = k\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$

Squaring, $\left(y - x\frac{dy}{dx}\right)^2 = k^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$ [Making it free from fractions]

This is of order 1 $\left[\because \frac{dy}{dx} \text{ is present}\right]$

and is of degree 2 $\left[\because \text{Power of } \frac{dy}{dx} \text{ is } 2\right]$

**Example 2:** Write the order and degree of the following differential equations:

(i) $y = x \frac{dy}{dx} + \frac{a}{dy/dx}$

(ii) $y_2 = \sqrt[3]{y + y_1^2}$

**Solution:**

(i) Given equation is

$$y = x \frac{dy}{dx} + \frac{a}{dy/dx}$$

It can be rewritten as

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + a = 0$$

Now order is 1 and degree is 2

(ii) Given equation is

$$y_2 = \sqrt[3]{y + y_1^2}$$

It can be rewritten as

$$y_2^3 = y + y_1^2$$

Now order is 2 and degree is 3

**Self-Check Exercise-1**

Q.1 Write one order and degree of the following differential equation:

$dy + (x - \sin x)\, dx = 0$

Q.2 Write the order and degree of the following differential equation:

$y_3 + xy_2 + 2y\, y_1^2 + xy = 0$

Q.3 Write the order and degree of the following differential equation:

$y_1 + x = (y - xy_1)^{-4}$

## 1.5 Solutions of A Differential Equations

The solution of a differential equation is a relation between the variables involved such that this relation and the differential coefficients obtained there from satisfy the given differential equation. This is also called primitive or integral of the differential equation.

Also, let

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, ........, \frac{d^ny}{dx^n}\right) = 0 \qquad .....(1)$$

be a differential equation of order n.

Then, (i) a real valued function $y = f(x)$ with domain $Df$ is called an explicit solution or simply a solution of the differential equation (1), if on substituting $y = f(x)$ in (1), the equality sign holds for all $x \in Df$, i.e. $F(x, f(x), f'(x), f''(x), ........, f^n(x)) = 0$ for all $x \in Df$.

(ii) a relation $g(x, y) = 0$ is called an implicit solution of the differential equation (1), if from this relation, we can find a real valued function $y = \phi(x)$ which is an explicit solution of (1).

**Various Types of Solution**

(I) General Solution The solution of a differential equation, which contains as many arbitrary constants as the order of the differential equation, is said to be the general solution. This is also called complete primitive or complete solution of the differential equation.

(II) Particular Solution The particular solution of a differential equation is that which is obtained from the general solution by giving particular values to arbitrary constants.

(III) Singular Solution Any solution of the differential equation which cannot be obtained from its general solution by assigning values to arbitrary constants and which is independent of arbitrary constants, is called a singular solution of differential equation.

**Let us do some examples:-**

**Example 3:** Show that y = A sin Bx is a solution of the differential equation

$$\frac{d^2y}{dx^2} + B^2y = \sigma$$

**Solution:** Since y = A sin Bx

$$\therefore \quad \frac{dy}{dx} = A (\cos B x) B = AB \cos Bx$$

and $$\frac{d^2y}{dx^2} = AB (- \sin Bx) B = -AB^2 \sin Bx$$

$$= - B^2 (A \sin Bx)$$

Putting the values of y, $\frac{d^2y}{dx^2}$ in the given differential equation, we get

- $B^2$ (A sin B x) + $B^2$ (A sin B x) = 0

i.e. 0 = 0 which is true.

Hence y = A sin Bx is a solution of $\frac{d^2y}{dx^2}$ + $B^2$y = 0

**Example 4:** Verify that y = $e^{mx}$ + $e^{-mx}$ is an explicit solution of the differential equation $\frac{d^2y}{dx^2}$ - $m^2$y = 0. Also determine the interval in which the solution holds.

**Solution:** The given real valued function y = $e^{mx}$ + $e^{-mx}$ is defined for all x.

Also on differentiating w.r.t. x, we have

$$\frac{dy}{dx} = m(e^{mx} - e^{-mx})$$

and $$\frac{d^2y}{dx^2} = m^2(e^{mx} + e^{-mx})$$

On substituting the values of y and $\frac{d^2y}{dx^2}$ in the given differential equation $\frac{d^2y}{dx^2}$ - $m^2$y = 0, we find that the equality sign holds for all x $\in$ R. Hence y = $e^{mx}$ + $e^{-mx}$ is an explicit solution of $\frac{d^2y}{dx^2}$ - $m^2$y = 0 and it holds for all x $\in$ R.

**Example 5:** Show that y = mx + $\sqrt[a]{1+m^2}$ is solution of y = x $\frac{dy}{dx}$ + $\sqrt[a]{1+\left(\frac{dy}{dx}\right)^2}$

**Solution:** Given equation is

$$y = x\frac{dy}{dx} + \sqrt[a]{1+\left(\frac{dy}{dx}\right)^2}$$

Now y = mx + $\sqrt[a]{1+m^2}$ $\Rightarrow$ $\frac{dy}{dx}$ = m

Putting the values of y and $\frac{dy}{dx}$ in the given differential equation, we get

mx + $\sqrt[a]{1+m^2}$ = x m + $\sqrt[a]{1+m^2}$ , which is true.

Hence y = mx + $\sqrt[a]{1+m^2}$ is a solution of the given differential equation.

**Example 6:** Verify that $4x^2 + y^2 = 25$ is an implicit solution of the differential equation $y\frac{dy}{dx} + 4x = 0$. Also determine the interval in which the solution holds.

**Solution:** Given relation is $4x^2 + y^2 = 25$

$\Rightarrow$ $\quad y = \sqrt{25-4x^2}$

or $\quad y = -\sqrt{25-4x^2}$

Clearly the domain of $y = \sqrt{25-4x^2}$ is the interval $\left[-\frac{5}{2}, \frac{5}{2}\right]$

Differentiating, $y = \sqrt{25-4x^2}$ w.r.t., we get

$\frac{dy}{dx} = \frac{1}{2}\frac{-8x}{\sqrt{25-4x^2}} = \frac{-4x}{\sqrt{25-4x^2}}$ and it exists for $x \in \left[-\frac{5}{2}, \frac{5}{2}\right]$.

On substituting the values of y and $\frac{dy}{dx}$ in the given differential equation $y\frac{dy}{dx} + 4 = 0$, we find that the equality sign holds for all $x \in \left[-\frac{5}{2}, \frac{5}{2}\right]$.

Hence $4x^2 + y^2 = 25$ is an implicit solution of the given differential equation and it holds for all $x \in \left[-\frac{5}{2}, \frac{5}{2}\right]$.

Note: The given differential equation is satisfied even if we take $y = \sqrt[-]{25-4x^2}$ instead of y $\sqrt[=]{25-4x^2}$ .

**Example 7:** Show that $y = (a + bx) e^2x$ is a solution of the differential equation.

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Examine the nature of the solution.

**Solution:** Given differential equation is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Also we have $y = (a + bx) e^{2x}$

$\therefore \quad \frac{dy}{dx} = (a + bx) e^{2x} 2 + be^{2x}$

$= (2a + 2bx + b) e^{2x}$

and $\frac{d^2y}{dx^2}$ = (2a + 2bx + b) $e^{2x}$ 2 + (0 + 2b + 0) $e^{2x}$

= (4a + 2b + 4bx) $e^{2x}$ + 2b$e^{2x}$

= (4a + 4b + 4bx) $e^{2x}$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the U.H.S. of the given differential equation, we get (4a + 4b + 4bx) $e^{2x}$ - 4 (2a + 2bx + b) $e^{2x}$ + 4 (a + bx) $e^{2x}$

$\Rightarrow$ (4a + 4b + 4bx - 8a - 8bx - 4b + 4a + 4bx) $e^{2x}$

$\Rightarrow$ 0.$e^{2x}$ = 0 $\forall$ x $\in$ R

Hence y = (a + bx)$e^{2x}$ is a solution of the differential equation $\frac{d^2y}{dx^2}$ - 4 $\frac{dy}{dx}$ + 4y = 0.

The solution is a general one because it involves two arbitrary constants and the given q equation is also of order 2.

**Self-Check Exercise - 2**

Q.1 Is y = 3 cos x + 4 sin x is a solution of the differential equation

$\frac{d^2y}{dx^2}$ + y = 0 ?

Q.2 Show that y = 4 sin 3x is a solution of the differential equation

$\frac{d^2y}{dx^2}$ + 9y = 0

Q.3 Show that

y = 2x + 3 cos x is a solution of

$\frac{d^4y}{dx^4}$ - cot x $\frac{d^3y}{dx^3}$ = 0

## 1.6 Formation of Differential Equations

The differential equation whose solution is $f$(x, y, c) = 0, c being an arbitrary constant, can be obtained by eliminating c between this relation and the relation obtained by differentiating it w.r.t. the independent variable 'x'. This will give us a differential equation of order one.

In general, if we have to form a differential equation whose general solution is $f(x, y, c_1, c_2, ......, c_n) = 0$ where $c_1$, $c_2$, ......., $c_n$ are 'n' arbitrary constants, then we have to proceed as follow:-

The given general solution is

$f(x, y, c_1, c_2, \ldots\ldots., c_n) = 0$

Where $c_1$, $c_2$,......, $c_n$ are 'n' arbitrary constants cor. parameters). Since (1) contains 'n' arbitrary constants, therefore, the differential equation, of which (1) is the general solution, must contain derivatives of $n^{th}$ order.

To eliminate these n constants, differentiating (1) w.r.t. x successively n times, we get

$f(x, y, y_1, c_1, c_2, \ldots\ldots, c_n) = 0$ .......2

$f(x, y, y_1, y_2, c_1, c_2, \ldots\ldots., c_n) = 0$ ......3

.............................................

..............................................

$f(x, y, y_1, y_2, \ldots\ldots.., y_n, c_1, c_2, \ldots\ldots.., c_n) = 0$ (n+1)

Eliminating the n arbitrary constants from (1), (2),.... (n+1), we get

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots., \frac{d^ny}{dx^n}\right) = 0,$$

which is the required differential equation.

Method form the differential equation whose solution in x, y is given:

(i) Write down the equation of the general solution.

(ii) Differentiate it w.r.t. x successively as many times as the number of arbitrary constants it contains.

(iii) Eliminate the arbitrary constants from the equations obtained in steps (i) and (ii)

Ultimately we get an ordinary differential equation:

(i) Whose order is equal to the number of arbitrary constants in the given equation,

(ii) Which is consistent with the given equation, and

(iii) Which is free from arbitrary constants.

To clarity what we have just said, consider the following examples:-

**Example 8:** Find the differential equations whose solutions are given by

(i) $y = ce^x \cos x$ (ii) $y = A x + B x^2$

**Solution:** (i) Differentiating the given relation

$y = ce^x \cos x$

We get

$$\frac{dy}{dx} = c\,[e^x(-\sin x) + e^x \cos x]$$

$$= c\, e^x [\cos x - \sin x] \quad .....(2)$$

Divide (2) by (1), we get

$$\frac{dy/dx}{y} = \frac{\cos x - \sin x}{\cos x}$$

or $\cos x \dfrac{dy}{dx} = y(\cos x - \sin x)$,

is the required differential equation,

(ii) Given relation is

$y = Ax + Bx2$ ....(1)

$\Rightarrow \dfrac{dy}{dx} = A + 2Bx$ .....(2)

$\Rightarrow \dfrac{d^2y}{dx^2} = 2B$ ......(3)

To get the required differential equation, we eliminate A and B from (1), (2) and (3).

From (3), $B = \dfrac{1}{2}\dfrac{d^2y}{dx^2}$ .....(4)

Putting in (2), we get $\dfrac{dy}{dx} = A + 2.\dfrac{1}{2}\left(\dfrac{d^2y}{dx^2}\right)x$

$\Rightarrow A = \dfrac{dy}{dx} - x\dfrac{d^2y}{dx^2}$ ......(5)

Using (4) and (5) in (1), we get

$$y = \left[\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right]x + \frac{1}{2}\left(\frac{d^2y}{dx^2}\right)x^2$$

$\Rightarrow 2y = 2x\dfrac{dy}{dx} - 2x^2\dfrac{d^2y}{dx^2} + x^2\dfrac{d^2y}{dx^2}$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} + 2y = 2x\dfrac{dy}{dx}$, which is the required differential equation.

**Example 9:** Form the differential equation of simple Harmonic Motion given by $x = A\cos(nt + \propto)$, where n is fixed and A and $\propto$ are arbitrary constants.

**Solution:** Given

$x = A\cos(nt + \propto)$ .....(1)

$\Rightarrow$ $\frac{dx}{dt}$ = - n A sin (nt + $\propto$)

$\Rightarrow$ $\frac{d^2x}{dt^2}$ = $n^2$ A cos (nt + $\propto$)

= - $n^2x$ [By (1)]

$\Rightarrow$ $\frac{d^2x}{dt^2}$ - $n^2x$ = 0, which is the required equation of S.H.M.

**Example 10:** Form the differential equation of which $c(y + c)^2 = x^3$ is the complete integral

**Solution:** The given integral is

$c\,(y + c)^2 = x^3$ ......(1)

which contains one arbitrary constant

Differentiating (1), w.r.t. x, we get

$2c\,(y + c)\,\frac{dy}{dx} = 3x^2$ .......(2)

Squaring (2), we get

$4c^2\,(y + c)^2\left(\frac{dy}{dx}\right)^2 = 9x^4$ .....(3)

Divide (3) by (1), we get

$4c\,\left(\frac{dy}{dx}\right)^2 = 9x \Rightarrow c = \frac{9x}{4y_1^2}$ $\left[\text{where } y_1 = \frac{dy}{dx}\right]$

Putting the values of c in (1), we get

$$\frac{9x}{4y_1^2}\left[y + \frac{9x}{4y_1^2}\right]^2 = x^3$$

or $\frac{9x}{4y_1^2}\left[\frac{4y\,y_1^2 + 9x}{4y_1^2}\right]^2 = 64\,y_1^6\,x^3$

or $9\,(4y\,y_1^2 + 9x)^2 = 64\,y_1^6\,x^2$

which is the required differential equation.

Putting the value of c in (2), we shall get the required differential equation as

$$12y\,y_1^2 + 27x = 8x\,y_1^3$$

**Example 11:** Form the differential equation corresponding to

$y^2 = 2ay - x^2 + a^2$, a being an arbitrary constant.

**Solution:** We have

$$y^2 = 2ay - x^2 + a^2 \qquad ......(1)$$

Differentiating w.r.t. x, we get

$2y\, y_1 = 2ay_1 - 2x$

$\Rightarrow \quad yy_1 = ay_1 - x \quad \Rightarrow \quad a = \dfrac{yy_1 + x}{y_1}$

Putting in (1), we get

$$y^2 = 2\left[\frac{y\,y_1 + x}{y_1}\right]y - x^2 + \left[\frac{y\,y_1 + x}{y_1}\right]^2$$

$\Rightarrow \quad y_1^2\, y^2 = 2y\, y_1\, (y\, y_1 + x) - x^2\, y_1^2 + (y\, y_1 + x)^2$

$\Rightarrow \quad y^2\, y_1^2 = 2y^2\, y_1^2 + 2xyy_1 - x^2\, y_1^2 + y^2\, y_1^2 + x^2 + 2xyy_1$

$\Rightarrow \quad 2y^2\, y_1^2 + 4xyy_1 - x^2\, y_1^2 + x^2 = 0$

$\Rightarrow \quad (2y^2 - x^2)\, y_1^2 + 4xyy_1 + x^2 = 0$

which is the required differential equation.

**Example 12:** Form the differential equation corresponding to $y = a \sin^{-1} x$, a being a parameter.

**Solution:** We have

$$y = a \sin^{-1} x \qquad .....(1)$$

Differentiating w.r.t. x, we get

$y_1 = a\, \dfrac{1}{\sqrt{1-x^2}} \Rightarrow \quad a = \sqrt{1-x^2}\;\; y_1$

Putting in (1),

$y = \sqrt{1-x^2}\, y_1 \sin^{-1} x$

$\Rightarrow \quad y_1 = \dfrac{y}{\sqrt{1-x^2}\;\, sub^{-1}x}$ ,

which is the required differential equation.

**Example 13:** Find the differential equation of the family of circles $(x - h)^2 + (y - k)^2 = r^2$, where h, k are arbitrary constants.

**Solution:** Given equation of the family of circles is $(x - h)^2 + (y - k)^2 = r^2 \qquad .....(1)$

where h, k are arbitrary constants.

Since (1) contains two arbitrary constants, we shall differentiate (1) w.r.t. x twice only.

Differentiating (1) w.r.t x, we get

$$2(x - h) + 2(y - k) \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x - h) + (y - k) \frac{dy}{dx} = 0 \qquad .....(2)$$

Differentiating (2) w.r.t x, we have

$$1 + (y - k) \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad .....(3)$$

$$\Rightarrow \quad y - k = -\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} \qquad .....(4)$$

Substituting it in (2), we have

$$x - h = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}} \qquad ......(5)$$

Substituting the values of (x - h) and (y - k) from (4) and (5) in (1), we get

$$\left[\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}\right]^2 + \left[\frac{-\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}\right]^2 = r^2$$

$$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 \left[\left(\frac{dy}{dx}\right)^2+1\right] = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

$$\Rightarrow \quad \left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

**Example 14:** Find the differential equation of all parabolas whose axes are parallel to y - axis.

**Solution:** The general equation of any parabola whose axes is parallel to y - axis is

$(x + h)^2 = 4a (y + k)$ .....(1)

where h, k, a are arbitrary constants.

Since (1) contains three arbitrary constants, we shall differentiate (1), w.r.t. x three times only.

Differentiating (1) w.r.t. x, we get

$$2(x + h) = 4a \frac{dy}{dx}$$

$\Rightarrow$ $2a \frac{dy}{dx} = x + h$

Again differentiate w.r.t. x, we have

$2a \frac{d^2y}{dx^2} = 1$

or $\frac{d^2y}{dx^2} = \frac{1}{2a}$

Again differentiate w.r.t. x, we get

$\frac{d^3y}{dx^3} = 0$

which is the required differential equation.

**Example 15:** Find the differential equation of all circles in a plane.

**Solution:** The general equation of all circles in a plane is

$x^2 + y^2 + 2gx + 2fy + c = 0$ ......(1)

where g, $f$ and c are arbitrary constants.

Since (1) contains three arbitrary constants, we shall differentiate (1), w.r.t. x three times only.

Differentiating (1) w.r.t. x, we get

$2x + 2yy_1 + 2g + 2fy_1 = 0,$ $\left[\text{where } y_1 = \frac{dy}{dx}\right]$

$\Rightarrow$ $x + yy_1 + g + fy_1 = 0$ ......(2)

Again differentiate w.r.t. x, we have

$1 + y_1^2 + yy_2 + fy_2 = 0$ .....(3)

Again differentiate w.r.t. x, we get

$0 + 2y_1y_2 + yy_3 + y_1y_2 + fy_3 = 0$

$\Rightarrow \quad 3y_1y_2 + yy_3 + fy_3 = 0 \qquad \text{......(4)}$

$\dfrac{\text{To eliminate f from (3) and (4)}}{\text{Multiplying (3) by } y_3\text{, we have}}$

$y_3 + y_1^2 y_3 + yy_2y_3 + fy_2y_3 = 0 \quad \text{.....(5)}$

Multiplying (4) by y2, we have

$3y_1 y_2^2 + yy_2y_3 + fy_2y_3 = 0 \qquad \text{.....(6)}$

Subtracting (6) from (5), we get

$y_3 + y_1^2 y_3 - 3y_1 y_2^2 = 0$

$\Rightarrow \quad y_3 (1 + y_1^2) - 3y_1 y_2^2 = 0,$

which is the required differential equation.

**Example 16:** Find the differential equation of all circles touching a given straight line at a given point.

**Solution:** Let us take the given line as y - axis and given point as origin.

Then its centre is (a, 0) and radius is a.

The general equation of all such circles is

$(x - a)^2 + y^2 = a^2 \qquad \text{.....(1)}$

Equation (1) can be written as

$x^2 + y^2 - 2ax = 0 \qquad \text{.......(2)}$

Differentiating (2) w.r.t. x, we have

$2x + 2yy_1 - 2a = 0$

$\Rightarrow \quad x + yy_1 - a = 0$

$\Rightarrow \quad a = x + yy_1$

Substituting in (2), we get

$x^2 + y^2 - 2(x + yy_1)\, x = 0$

$\Rightarrow \quad x^2 + y^2 - 2x^2 - 2xyy_1 = 0$

$\Rightarrow \quad x^2 - y^2 + 2xyy_1 = 0,$

which is the required differential equation.

**Example 17:** Find the differential equation of all conics whose axes coincide with the axes of co-ordinates.

OR

Find the differential equation of the family of curves $ax^2 + by^2 = 1$, where a, b are arbitrary constants.

**Solution:** Let the general equation of the conic, whose axes are axes of co-ordinates, be

$$ax^2 + by^2 = 1 \qquad .......(1)$$

Differentiating (1) w.r.t. x, we have

$$2ax + 2by \frac{dy}{dx} = 0$$

or $$ax + by \frac{dy}{dx} = 0 \qquad ......(2)$$

or $$\frac{a}{b} = -\frac{y}{x}\frac{dy}{dx} \qquad .......(3)$$

Again differentiating (2) w.r.t. x, we get

$$a + b\left(\frac{dy}{dx}\right)^2 + by\frac{d^2y}{dx^2} = 0$$

or $$\frac{a}{b} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0$$

or $$-\frac{y}{x}\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = 0 \qquad \text{[Using (3)]}$$

or $$xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0,$$

which is the required differential equation.

**Example 18:** Find the differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$, where A, B are arbitrary constants.

**Solution:** The given equation of the family of curves is

$$y = e^x (A \cos x + B \sin x) \qquad .......(1)$$

where A, B are arbitrary constants.

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x (-A \sin x + B \cos x) + (A \cos x + B \sin x) e^x$$

$$\Rightarrow \quad \frac{dy}{dx} = e^x [(B - A) \sin x + (B + A) \cos x] \qquad .......(2)$$

Differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = e^x [(B - A) \cos x - (B + A) \sin x] + [(B - A) \sin x + (B + A) \cos x]e^x$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} = 2e^x [B \cos x - A \sin x] \qquad ......(3)$$

Multiplying (2) by 2 and subtracting it from (3), we get

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x [-2 A \cos x - 2 B \sin x]$$

$$= - 2 e^x [A \cos x + B \sin x]$$

$$= - 2y \qquad \text{[Using (1)]}$$

$\Rightarrow$ $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0,$$

which is the required differential equation.

**Example 19:** Find the differential equation of all ellipses centred at origin.

**Solution:** The general equation of ellipse centred at origin is

$ax^2 + 2h\,xy + by^2 = 1$ ......(1)

where a, b, h are arbitrary constants.

Differencing (1) three times as it contains three arbitrary constants.

Differentiate (1) w.r.t. x, we have

$2ax + 2h (xy_1 + y) + 2byy_1 = 0$

$\Rightarrow$ $ax + h (xy_1 + y) + byy_1 = 0$ .......(2)

Again differentiate (2) w.r.t. x, we get

$a + h [xy_2 + y_1 + y_1] + b [yy_2 + y_1 y_1] = 0$

$\Rightarrow$ $a + h [xy_2 + 2y_1] + b [yy_2 + y_1^2] = 0$ .......(3)

Again differentiate (3) w.r.t. x, we get

$0 + h [xy_3 + y_2 + 2y_2] + b [yy_3 + y_2y_1 + 2y_1y_2] = 0$

$\Rightarrow$ $h [xy_3 + 3y_2] + b [yy_3 + 3y_1y_2] = 0$ ......(4)

Multiplying (3) with x, we get

$ax + h (x_2\, y_2 + 2xy_1) + b (xyy_2 + x\, y_1^2) = 0$ ......(5)

Subtracting (5) from (2), we get

$h [y - xy_1 - x^2 y_2] + b [yy_1 - xyy_2 - x\, y_1^2] = 0$

$\Rightarrow$ h $[y - xy_1 - x^2y_2] = -b\,[yy_1 - xyy_2 - x\,y_1^2]$ ......(6)

From (4), $h[xy_3 + 3y_2] = -b\,[yy_2 - 3y_1y_2]$ .....(7)

Divide (6) by (7), we get

$$\frac{y - xy_1 - x^2y_2}{xy_3 + 3y_2} = \frac{y\,y_1 - xy\,y_2 - x\,y_1^2}{y\,y_3 + 3y_1y_2}$$

$\Rightarrow$ $(y - xy_1 - x^2y_2)\,(yy_3 + 3y_1y_2) = (xy_3 + 3y_2)\,(yy_1 - xyy_2 - x\,y_1^2)$

$\Rightarrow$ $y^2\,y_3 + 3yy_1\,y_2 - xy\,y_1\,y_3 - 3x\,y_1^2\,y_2 - x^2\,y\,y_2\,y_3$

$- 3x^2\,y_1\,y_2^2 = x\,y\,y_1\,y_3 - x^2\,y\,y_2\,y_3 - x^2\,y_1^2\,y_3 + 3y\,y_1\,y_2 - 3x\,y\,y_2^2 - 3x\,y_1\,y_2$

$\Rightarrow$ $y_2\,y_3 - 2x\,y\,y_1\,y_3 + x^2\,y_1^2\,y_3 + 3x\,y\,y_2^2 - 3x^2\,y_1\,y_2^2 = 0$

$\Rightarrow$ $y_3\,[y^2 - 2xy\,y_1 + x^2\,y_1^2] + 3\,x\,y_2^2\,[y - xy_1] = 0$

$\Rightarrow$ $y_3\,[y - xy_1]^2 + 3x\,y_2^2\,[y - xy_1] = 0$

$\Rightarrow$ $(y - xy_1)\,[y_3\,(y - xy_1) + 3x\,y_2^2] = 0$

$\Rightarrow$ $y - xy_1 = 0$ or $y_3\,(y - xy_1) + 3x\,y_2^2 = 0$

Because (1) contains three arbitrary constants, therefore the differential equation of which (1) is a solution must contain third order derivative.

Hence $yy_3 - xy_1\,y_3 + 3x\,y_2^2 = 0$, is the required differential equation.

**Example 20:** Obtain the differential equation associated with the primitive $y = a + be^{5x} + ce^{-7x}$, where a, b, c are arbitrary constants.

**Solution:** Given equation is

$y = a + be^{5x} + ce^{-7x}$ ......(1)

Differentiating (1) w.r.t. x, we get

$y_1 = 5be^{5x} - 7c\,e^{-7x}$ .......(2)

Multiply (1) by 7, we get

$7y = 7a + 7b\,e^{5x} + 7c\,e^{-7x}$ .......(3)

Equations (2) + (3) give

$y_1 + 7y = 7a + 12be^{5x}$

Differentiating it w.r.t. x, we have

$y_2 + 7y_1 = 60\,b\,e^{5x}$ ......(4)

Again differencing w.r.t. x, we get

$y_3 + 7y_2 = 300\, b\, e^{5x}$ ......(5)

Multiply equation (4) by (5), we get

$5y_2 + 35y_1 = 300\, b\, e^{5x}$ ......(6)

Now Equations (5) - (6) gives

$y_3 + 2y_2 - 35y_1 = 0,$

which is the required differential equation

**Example 21:** Obtain the differential equation of the family of circles through the fixed points (-a, 0) family of circles through the fixed points (-a, 0) and (a, 0). Also sketch different members of the family.

**Solution:** Let A (-a, 0) and B $\leftrightarrow$ (a, 0) be two points.

Then the equation of the circle with segment [AB] as diameter is

$(x + a)(x - a) + (y - 0) = 0$ [Diameter Form]

i.e. $x^2 - a^2 + y^2 = 0$

$\Rightarrow$ $x^2 + y^2 - a^2 = 0$ ......(1)

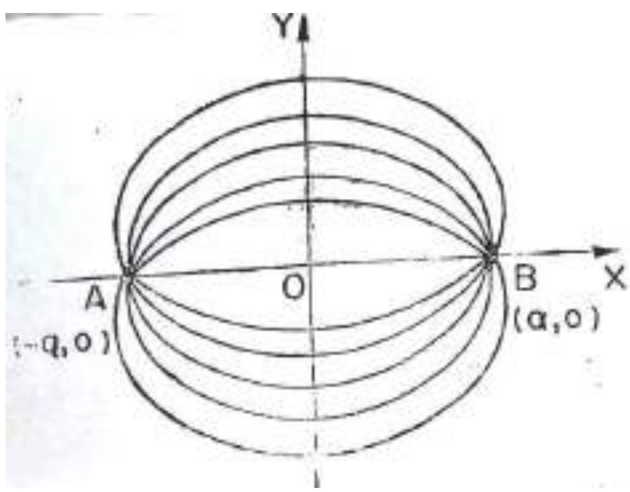
And the equation of $\overleftrightarrow{AB}$ is $y = 0$ .......(2)

Now any circle through the points A and B can be considered as the circle through the points of intersection of (1) and (2).

Thus its equation is

$(x^2 + y^2 - a^2) + k(y) = 0$ .......(3)

Equation (3) represents the family of circles passing through the fixed points A $\leftrightarrow$ (-a, 0) and B $\leftrightarrow$ (a, 0) for different values of k, which is as shown below:




**To find the differential Equation of the family**

Equation (3) can be written as

$$\frac{x^2 + y^2 - a^2}{y} = -k \qquad .......(4)$$

Differentiating w.r.t. x, we get

$$\frac{y(2x + 2y\, y_1)-(x^2 + y^2 - a^2)y_1}{y^2} = 0$$

$\Rightarrow$ $y\,(2x + 2y\, y_1) - (x^2 + y^2 - a^2)\, y_1 = 0$

$\Rightarrow$ $2xy + 2y^2\, y_1 - x^2\, y_1 - y^2\, y_1 + a^2\, y_1 = 0$

$\Rightarrow$ $(y^2 - x^2 + a^2)\, y_1 + 2xy = 0$

which is the required differential equation.

**Self-Check Exercise-3**

Q.1 Form the differential equation whose solution is

$y = A \cos 3x + B \sin 3x$

where A and B are arbitrary constants.

Q.2 Form the differential equation whose solution is

$y = ax + bx^2$,

where a and b are arbitrary constants.

Q.3 Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.

Q.4 Find the differential equation of all circles centred at the origin.

Q.5 Find the differential equation of all hyperbolas centred at the origin.

## 1.7 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Elementary concepts of differential equations.
2. Discussed types of differential equations.

   i.e. Ordinary differential equation and Tartial differential equation.
3. Defined Order and Degree of differential equations.
4. Discussed different types of solutions of differential equations. Thereafter, we find solutions of differential equations.
5. Explained the concept of formation of differential equations.

## 1.8 Glossary:

1. An equation which involves derivatives of one or more dependent variables w.r.t. one or more independent variables is called a differential equation.

2. An ordinary differential equation is one which involves one independent variable and differential coefficients w.r.t. it.

3. The order of a differential equation is the order of the highest derivative term occurring in the differential equation.

4. The degree of a differential equation is the power of the highest derivative in the equation, when it has been made free from the radicals and fractions as for as derivatives are concerned.

## 1.9 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 order : 1 and degree : 1

Ans.2 Order : 3 and degree : 1

Ans.3 Order : 1 and degree : 5

**Self-Check Exercise-2**

Ans.4 Yes

Ans.2 Verified

Ans.3 Verified

**Self-Check Exercise-3**

Ans.1 $\frac{d^2y}{dx^2} + 9y = 0$

Ans.2 $x^2\frac{d^2y}{dx^2} + 2y = 2x\frac{dy}{dx}$

Ans.3 $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

Ans. 4 $x + y y_1 = 0$

Ans. 5 $y y_3 - x y_1^3 + 3x y_2^2 = 0$

## 1.10 References/Suggested Readings

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.

2. Boyce, w. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.

## 1.11 Terminal Questions

1. Find the order and degree of the following differential equation:

$$\frac{d^2y}{dx^2} = \left[ y + \left( \frac{dy}{dx} \right)^2 \right]^{1/3}$$

2. Find the order and degree of the following differential equations:

(i) $\frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + x \left( \frac{dy}{dx} \right)^2 + y = 0$

(ii) $L \frac{d^2\theta}{dt^2} + R \frac{d\theta}{dt} + \frac{a}{c} = 0$

3. Show that $y = mx + \frac{a}{m}$ and $y^2 = 4ax$ are solutions of $y = x \frac{dy}{dx} + \frac{a}{dy/dx}$.

4. Show that $4y + x^2 = 0$ is a solution of $\left( \frac{dy}{dx} \right)^2 + x \frac{dy}{dx} - y = 0$

5. Show that $y = c\,x + \frac{a}{c}$ is a solution of $y = x \frac{dy}{dx} + \frac{a}{dy/dx}$

Is it the general solution?

6. Form the differential equation whose solution is

$y = Ae^{2x} + Be^{-2x}$,

where A and B are arbitrary constants.

7. Eliminate c from $y = (x + c - c^3$.

8. Find the differential equation of all parabolas with latus rectum 49 and whose axes are parallel to the x-axes.

9. Find the differential equation of all circles touching y-axis at the origin.

10. Find the differential equation which has y = a cos (mx + b) as its complete integral, where a and b are the arbitrary constants.

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## Unit - 2

# Linear Equations, Linear Independence And Wronskian

**Structure**



## 2.1 Introduction

A linear differential equation is said to be linear if the unknown function and all of its derivatives occurring in the equation occur only in the first degree and are not multiplied together.

e.g. the differential equations.

$\frac{d^2y}{dx^2} + 4y = 0$, $\frac{dy}{dx} = \sin x$ are linear whereas

$\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^2 = 0$ is non-linear.

It should be noted that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear e.g. the differential equation.

$\frac{d^2y}{dx^2}$ + 2 $\left(\frac{dy}{dx}\right)^2$ + $y^2 = 0$ is not linear though its degree is 1.

The most general linear differential equation of order n in the dependent variable y and the independent variable x is an equation which can be expressed in the form.

$$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \text{.........} + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x) \qquad \text{......(1)}$$

where $a_0(x)$ is not identically zero. This is called non-homogeneous differential equation.

Here we assume that $a_0$, $a_1$, $a_2$,......$a_n$ and F are continuous real function on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The right hond member F(x) is called the non-homogeneous term.

If F is identically zero, equation (1) becomes $a_0(X)\frac{d^ny}{dx^n}$ + $a_1(x)\frac{d^{n-1}y}{dx^{n-1}}$ + .......+ $a_{n-1}(x)$ $\frac{dy}{dx}$ + $a_n(x)\,y = 0$ .......(2)

This is called homogeneous differential equation.

For n = 2, equation (1) reduces to the second-order non-homogeneous linear differential equation $a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = F(x)$ .....(3)

and equation (2) reduces to the corresponding second order homogeneous linear differential equation.

$$a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad \text{......(4)}$$

In this case, we assume that $a_0$, $a_1$, $a_2$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$.

## 2.2 Learning Objectives

After studying this unit, you should be able to:

- Define and discuss linear combination, linear dependence and linear independence.
- Discuss Wronskian and its properties.
- Define differential operator and discuss important properties of differential operators.

## 2.3 Existence And Uniqueness Theorem

In this section we shall state the basic existence theorem for initial value problems associated with $n^{th}$ order linear differential equation. (Without Proof)

**Theorem 1**

(i) Consider the $n^{th}$ order linear differential equation

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = F(x) \quad \dots\dots(1)$$

Where $a_0, a_1, a_2, \dots, a_n$ and F are continuous real functions on a real interval $a \le x \le b$ and $a_0(x) \neq 0$ for any x on $a \le x \le b$.

(ii) Let $x_0$ be any point of the interval $a \le x \le b$ and let $c_0, c_1, \dots, c_{n-1}$ be n arbitrary real constants.

Then there exists a unique solution $f$ of (1) such that

$f(x_0) = c_0$, $f'(x_0) = c_1$, ......., $f^{(n-1)}(x_0) = c_{n-1}$ and this solution is defined over the whole interval $a \le x \le b$.

Cor. Let $f$ be a solution of the $n^{th}$ order homogeneous linear differential equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0 \text{ such that}$$

$f(x_0) = 0$, $f'(x_0) = 0$, ........, $f^{(n-1)}(x_0) = 0$

Where $x_0$ is a point of the interval $a \le x \le b$ in which the coefficients $a_0, a_1, \dots, a_n$ are all continuous and $a_0(x) \neq 0$ then $f(x) = 0\ \forall$ x/on $a \le x \le b$.

## 2.4 Linear Combination

Let $f_1, f_2, \dots, f_n$ be any functions of x defined over on interval I, then the function $c_1f_1 + c_2f_2 + \dots + c_nf_n$ is called a linear combination of the functions $f_1, f_2, \dots, f_n$ over I, where $c_1, c_2, \dots, c_n$ are arbitrary constants.

**Illustration**

The expression

$2 \sin x - 3e^x + 5 \log|x|$ is a linear combination of the functions $\sin x$, $e^x$ and $\log|x|$.

The expression

$2e^x + 4e^{2x} + 3e^{5x}$

is a linear combination of the functions $e^x$, $e^{2x}$ and $e^{5x}$.

We now take up the fundamental results w.r.t. the homogeneous equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0 \qquad \dots\dots(1)$$

**Theorem 2: Basic Theorem on Linear Homogeneous Differential Equation**

Let $f_1, f_2, \dots, f_m$ be any m solutions of the homogeneous differential equation (1). Then $c_1f_1 + c_2f_2 + \dots + c_mf_m$ is also a solution of (1) where $c_1, c_2, \dots, c_m$ are m arbitrary constants.

**Another form of Theorem 2**

Any linear combination of solutions of a linear homogeneous differential equation is also a solution of the equation.

**Proof:** Let $f_1, f_2, \ldots\ldots, f_m$ be m solutions of the linear homogeneous equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0$$

or $\quad a_0(x)\,y^{(n)} + a_1(x)\,y^{(n-1)} + \ldots\ldots + a_{n-1}(x)\,y^{(1)} + a_n(x)\,y = 0$

Then we have

$$a_0(x)\,f_1^{(n)} + a_1(x)\,f_1^{(n-1)} + \ldots\ldots + a_{n-1}(x)\,f_1^{(1)} + a_n(x)\,f_1 = 0 \quad \ldots\ldots(1)$$

$$a_0(x)\,f_2^{(n)} + a_1(x)\,f_2^{(n-1)} + \ldots\ldots + a_{n-1}(x)\,f_2^{(1)} + a_n(x)\,f_2 = 0 \quad \ldots\ldots(2)$$

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$$a_0(x)\,f_m^{(n)} + a_1(x)\,f_m^{(n-1)} + \ldots\ldots + a_{n-1}(x)\,f_m^{(1)} + a_n(x)\,f_m = 0$$

Multiplying (1), (2), ......, (m) by $c_1, c_2, \ldots\ldots, c_m$ (arbitrary constants) respectively and adding, we get

$$a_0(x)\,[c_1 f_1^{(n)} + c_2 f_2^{(n)} + \ldots\ldots + c_n f_m^{(n)}] + a_1(x)\,[c_1 f_1^{(n-1)} +$$

$$c_2 f_2^{(n-1)} + \ldots\ldots + c_n f_m^{(n-1)}] + \ldots\ldots + a_n(x)\,[c_1 f_1 + c_2 f_2 + \ldots\ldots + c_m f_m] = 0$$

or

$$a_0(x)\,[c_1 f_1 + c_2 f_2 + \ldots\ldots + c_n f_m]^{(n)} + a_1(x)\,[c_1 f_1 +$$

$$c_2 f_2 + \ldots\ldots + c_n f_m]^{(n-1)} + \ldots\ldots + a_n(x)\,[c_1 f_1 + c_2 f_2 + \ldots\ldots + c_m f_m] = 0$$

This shows that $c_1 f_1 + c_2 f_2 + \ldots\ldots + c_m f_m$ is a solution of the given homogeneous linear equation. But $c_1 f_1 + c_2 f_2 + \ldots\ldots + c_m f_m$ is any linear combination of $f1, f_2, \ldots\ldots, f_m$.

Hence the result follows.

## 2.5 Linear Dependence and Linear Independence

The n functions $f_1, f_2, \ldots\ldots, f_n$ are called linearly dependent (or L.D.) on $a \le x \le b$ if there exist constants $c_1, c_2, \ldots\ldots, c_n$ not all zero, such that

$$c_1 f_1(x)\, c_2 f_2(x) + \ldots\ldots + c_n f_n(x) = 0 \text{ for all x such that } a \le x \le b.$$

In particular, two functions $f_1$ and $f_2$ are linearly dependent on $a \le x \le b$ if there exist constants $c_1, c_2$, not both zero, such that

$$c_1 f_1(x) + c_2 f_2(x) = 0$$

for all x such that $a \le x \le b$.

For example, the functions x and 2x are linearly dependent on the interval $0 \leq x \leq 1$ because there exist constants ($c_1 = 2$, $c_2 = 1$)

s.t. $c_1 f_1 + c_2 f_2$

i.e. (2) (x) + (-1) (2x) = 0

The n functions $f_1, f_2, \ldots\ldots, f_n$ are called linearly independent (or L.I.) on the interval $a \leq x \leq b$ if they are not linearly dependent there. That is, the functions $f_1, f_2, \ldots\ldots, f_n$ are linearly independent on $a \leq x \leq b$ if the relation.

$c_1 f_1 (x) + c_2 f_2 (x) + \ldots\ldots + c_n f_n (x) = 0$ for all x such that $a < x < b$ implies that $c_1 = c_2 = \ldots\ldots = c_n = 0$

In other words, the only linear combination of $f_1, f_2, \ldots\ldots f_n$ that is identically zero on $a \leq x \leq b$ is the trivial linear combination $0.f_1 + 0.f_2 + \ldots\ldots + 0.f_n$.

In particular, two functions $f_1$ and $f_2$ are linearly independent on $a \leq x \leq b$ if the relation.

for all x on $a \leq x \leq b$ implies that $c_1 = c_2 = 0$.

For example, the functions x and $x_2$ are linearly independent on $0 \leq x \leq 1$, since $c_1 x + c_2 x_2 = 0$ for all x on $0 \leq x \leq 1$ implies that both $c_1 = 0$ and $c_2 = 0$.

We may verify this in the following way:-

We differentiate both sides of $c_1 x + c_2 x_2 = 0$. to get.

$c_1 + 2 c_2 x = 0$,

which must also hold for all x on $0 \leq x \leq 1$.

Then from this equation, we also have after multiplying both sides by x as

$c_1 x + 2 c_2 x^2 = 0$ for all such x.

Thus we have both

$c_1 x + c_2 x^2 = 0$

and $c_1 x + 2 c_2 x^2 = 0$

for all x on $0 \leq x \leq 1$.

Subtracting the first term from the second gives $c_2 x^2 = 0$ for all x on $0 \leq x \leq 1$, which at once implies $c_2 = 0$

Then either 1 of above two equations show similarly that $c_1 = 0$

From the above definitions of linearly dependent and linearly independent we conclude that:

(i) When $f_1, f_2, \ldots\ldots, f_n$ are linearly independent, then none of them is a linear combination of others.

The next theorem is concerned with the existence of sets of linearly independent solutions of an $n^{th}$-order homogeneous linear differential equation and with the significance of such linearly independent sets.

**Theorem 3:** The $n^{th}$-order homogeneous linear differential equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0 \qquad \dots\dots(1)$$

always possesses n solutions that are linearly independent. Further, i$f$ $f_1$, $f_2$,........,$f_n$ are n linearly independent solutions of (1), then every solution of (1) can be expressed as a linear combination $c_1f_1 + c_2f_2 + \dots\dots + c_nf_n$ of these n linearly independent solutions by proper choice of the constants $c_1, c_2, \dots\dots, c_n$.

Thus, given any nth, order homogeneous linear differential equation, theorem 2 assures that a set of n linearly independent solutions actually exists. Therefore, any solution of (1) can be written as a linear combination of such a linearly independent set of n solutions by suitable choice of the constants $c_1, c_2, \dots\dots, c_n$.

**Let us do some examples:-**

**Example 1:** For the second-order homogeneous linear differential equation.

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0, \quad \dots\dots(2)$$

Theorem 3, states that a set of two linearly independent solutions exist. For that, let $f_1$ and $f_2$ be a set of two linearly independent solutions. Then if $f$ is any solution of (2), then the theorem suggests that $f$ can be expressed as a linear combination $c_1f_1 + c_2f_2$ of the two linearly independent solutions $f_1$ and $f_2$ by proper choice of the constants $c_1$ and $c_2$.

**Example 2:** We know that sin x and cos x are solution of $\frac{d^2y}{dx^2} + y = 0 \qquad \dots\dots(3)$

for all x in $-\infty < x < \infty$.

We can show that two solutions are linearly independent. Next, let $f$ be any solution of (3). Then by Theorem 3, $f$ can be expressed as a linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions sin n and cos x by proper choice of $c_1$ and $c_2$.

i.e. there exist two particular constants $c_1$ and $c_2$ such that

$$f(x) = c_1 \sin x + c_2 \cos x \qquad \dots\dots(4)$$

for all x in $-\infty < x < \infty$.

Now, let $f_1, f_2, \dots\dots, f_n$ be a set of n linearly independent solutions of (1), then by Theorem 2, any linear combination of solutions of the homogeneous linear differential equation.

$$c_1f_1 + c_2f_2 + \dots\dots + c_nf_n, \qquad \dots\dots(5)$$

Where $c_1$, $c_2$, ....., $c_n$ are n arbitrary constants, is also a solution of (1). On the other hand, by Theorem 3, if $f$ is any solution of (1), then it can be expressed as a linear combination (5) of the n linearly independent solutions $f_1$, $f_2$,....., $f_n$ by a suitable choice of the constants $c_1$, $c_2$,...., $c_n$. Thus a linear combination (5) of the n linearly independent solutions $f_1$, $f_2$,....., $f_n$ in which $c_1$, $c_2$,....., $c_n$ are arbitrary constants must include all solutions of (1). Due to this, a set of n linearly independent solutions of (1) is known as fundamental set of (1). This is a linear combination of n linearly independent solutions and this is called a general solution. Thus, we have the following definition:-

Def: If $f_1$, $f_2$,......,$f_n$ are n linearly independent solutions of the nth-order homogeneous linear differential equation.

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots + a_{n-1}(x) \frac{dy}{dx} + a_n(x)\, y = 0 \qquad \dots\dots(6)$$

on $a \le x \le b$, then the set $f_1$, $f_2$,......., $f_n$ is called a fundamental set of solutions of (6) and the function $f$ defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \dots\dots + c_n f_n(x), \; a \le x \le b \qquad \dots..(7)$$

where $c_1$, $c_2$,....., $c_n$ are arbitrary constants, is called a general solution of (6) on

$a \le x \le b$.

Therefore, if we can find n linearly independent solutions of (6), we can at once write the general solution of (6) as a general linear combination of these n solutions. In particular for the second-order homogeneous linear differential equation.

$$a_0(x) \frac{d^2 y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0 \qquad \dots.(8)$$

fundamental set consists of two linearly independent solutions. If $f_1$ and $f_2$are a fundamental set of (8) on $a \le x \le b$, then a general solution of (8) on $a < x < b$ is defined by $c_1 f_1(x) + c_2 f_2(x)$, $a \le x \le b$, where $c_1$ and $c_2$ are arbitrary constants.

To clarify what we have just said, consider the following examples:-

Example 3: We know that sin x and cos x are solutions of

$$\frac{d^2 y}{dx^2} + y = 0$$

for all x, $-\infty < x < \infty$. Further, we can show that these two solutions are linearly independent. Thus, they constitute a fundamental set of solutions of the given differential equation and its general solution may be expressed as the linear combination $c_1 \sin x + c_2 \cos x$,

where $c_1$ and $c_2$ are arbitrary constants. We write this as $y = c_1 \sin x + \cos x$.

**Example 4:** The solutions $e^x$, $e^{-x}$ and $e^{2x}$ of

$$\frac{d^3y}{dx^3} = 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

can be shown to be linearly independent for all x, $-\infty < x < \infty$.

Thus, $e^x$, $e^{-x}$ and $e^{2x}$ constitute a fundamental set of the given deferential equation, and its general solution may be expressed as the linear combination $c_1e^x + c_2e^{-x} + c_3e^{2x}$,

where $c_1$, $c_2$ and $c_3$ are arbitrary constants. We write this as $y = c_1e_x + c_2e_{-x} + c_3e_{2x}$.

Particular Cases of Theorem 1, 2, 3

**Theorem 4:** If $y_1(x)$ and $y_2(x)$ are any two solutions of

$a_0(x)\, y^n(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$

Then the linear combination $c_1y_1(x) + c_2y_2(x)$,

Where $c_1$ and $c_2$ are arbitrary constants, is also a solution of the given equation.

**Proof:** Since $y_1(x)$ and $y_2(x)$ are solutions of

$a_0(x)\, y^n(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$ ..... (1)

$\therefore$ $a_0(x)\, y_1''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x) = 0$ ......(2)

and $a_0(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_2(x)\, y_2(x) = 0$ ......(3)

$u(x) = c_1y_1(x) + c_2y_2(x)$ .......(4)

From (4) $\left.\begin{aligned} u'(x) &= c_1y_1'(x) + c_2y_2'(x) \\ u''(x) &= c_1y_1''(x) + c_2y_2''(x) \end{aligned}\right]$ .......(5)

Then $a_0(x)u''(x) + a_1(x)\, u'(x) + a_2(x)\, u(x)$

$= a_0(x)[c_1y_1''(x) + c_2y_2''(x)] + a_1(x)\, [c_1y_1'(x) + c_2y_2'(x)]$

$+ a_2(x)\, [c_1y_1(x) + c_2y_2(x)]$ [By (4) and (5)]

$= c_1[a_0(x)\, y''(x) + a_1(x)\, y_1'(x) + a_2(x)\, y_1(x)]$

$+ c_2\, [a_0(x)\, y_2''(x) + a_1(x)\, y_2'(x) + a_2\, y_2(x)]$

$= c_1(0) + c_2(0) = 0$

Thus $a_0(x)\, u''(x) + a_1(x)\, u'(x) + a_2(x)\, u(x) = 0$

$\Rightarrow$ $y = u(x)$ is a sol. of (1)

i.e. $y = c_1y_1(x) + c_2y_2(x)$ is a sol. of (1).

**Theorem 5:** There exists two linear independent solutions, $y_1(x)$ and $y_2(x)$ of the equation

$a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$

such that its every solution y (x) may be written as

$y(x) = c_1y_1(x) + c_2y_2(x), x \in (a, b)$

Where $c_1$ and $c_2$ are suitable chosen constants

**Proof:** Given equation is

$a_0(x) y''(x) + a_1(x) y'(x) + a_2(x) y(x) = 0$ .....(1)

Let $x_0 \in (a, b)$ and $y_1(x)$ and $y_2(x)$ be two solutions of (1) satisfying

$y_1(x_0) = 1$ and $y_1'(x_0) = 0$ .....(2)

and $y_2(x_0) = 0$ and $y_2'(x_0) = 1$ .....(3)

To prove that $y_1(x)$ and $y_2(x)$ are linearly independent

Let if possible, $y_1(x)$ and $y_2(x)$ are linearly dependent.

$c_1y_1(x) + c_2y_2(x) = 0$ for each $x \in [a, b]$ ......(4)

$\therefore$ by def, there must exist constants $c_1$ and $c_2$ not both zero, such that

$c_1y_1(x) + c_2y_2(x) = 0$ for each $x \in [a, b]$

Now (4) $\Rightarrow$ $c_1y_1'(x) + c_2y_2'(x) = 0$ for each $x \in [a, b]$ .....(5)

Since $x_0 \in [a, b]$

$\therefore$ (4) and (5) give

$c_1y_1(x_0) + c_2y_2(x_0) = 0$ .....(6)

$c_1y_1'(x_0) + c_2y_2'(x_0) = 0$ .....(7)

$$\Rightarrow \left.\begin{array}{l} c_1(1)+c_2(0)=0 \Rightarrow c_1 = 0 \\ c_1(0)+c_2(1)=0 \Rightarrow c_2 = 0 \end{array}\right]$$

This is a contradiction to the fact that $c_1$, $c_2$ are not both zero.

$\therefore$ our assumption that $y_1(x)$ and $y_2(x)$ are linear dependent is wrong.

$\therefore$ $y_1(x)$ and $y_2(x)$ are linearly independent.

For the remaining part, let $y(x)$ be any solution of (1) satisfying

$y(x_0) = c_1$ and $y'(x_0) = c_2$ .....(8)

Let $u(x) = u(x) - c_1y_1(x) + c_2y_2(x)$ .....(9)

(9) shows that $u(x)$ is a linear combination of solution $y(x)$, $y_1(x)$ and $y_2(x)$ of (1).

$\therefore$ $u(x)$ is also solution of (1) satisfying $u(x_0) = 0$ and $u'(x_0) = 0$

Hence $u(x) = 0$ for all $x$ on $(a, b)$.

$\therefore$ by (9), we have

$y(x) - c_1 y_1(x) - c_2 y_2(x) = 0$

or $y(x) = c_1 y_1(x) + c_2 y_2(x)$

Where $c_1$ and $c_2$ are suitable chosen constants and are given by (8)

## 2.6 Wronskian And Its Properties

Def: Let $f_1, f_2, \ldots\ldots, f_n$ be n real functions each of which has a derivative of order (n-1) on a real interval [a, b], then the determinant

$$\begin{vmatrix} f_1 & f_2 & \ldots\ldots & f_n \\ f_1' & f_2' & \ldots\ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix}$$

denoted by $W(f1, f2, \ldots\ldots, fn)$ is called Wronskian of $f_1, f_2, \ldots\ldots, f_n$.

The wronskian$W(f_1, f_2, \ldots\ldots, f_n)$ is itself a real function defined on $a \leq x \leq b$ and its value is denoted by $W(f_1, f_2, \ldots\ldots, f_n)(x)$ or by $W[f_1(x), f_2(x), \ldots\ldots, f_n(x)]$.

Properties

**Property 1:** If the wronskian of the functions $f_1, f_2, \ldots\ldots, f_n$ over an interval I is non-zero, then these functions are linearly independent over I.

**Proof:** Consider the relation

$c_1 f_1 + c_2 f_2 + \ldots\ldots + c_n f_n = 0 \qquad \ldots\ldots(1)$

where $c_1, c_2, \ldots\ldots, c_n$ are constants.

Differentiating (1) successively n-1 times w.r.t. x, we get

$c_1 f_1' + c_2 f_2' + \ldots\ldots + c_n f_n' = 0 \qquad \ldots\ldots(2)$

$c_1 f_1'' + c_2 f_2'' + \ldots\ldots + c_n f_n'' = 0 \ldots\ldots(3)$

......................................................

.....................................................

$c_1 f_1^{(n-1)} + c_2 f_2^{(n-1)} + \ldots\ldots + c_n f_n^{(n-1)} = 0 \qquad \ldots\ldots(n)$

These n equations can be written as

$$\begin{vmatrix} f_1 & f_2 & \ldots\ldots & f_n \\ f_1' & f_2' & \ldots\ldots & f_n' \\ \ldots & \ldots & \ldots & \ldots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Now we know that the matrix equation AX = 0 has a trivial solution if $|A| \neq 0$

$\therefore$ for $c_1 = c_2 = \ldots\ldots = c_n = 0$, we have

$$\begin{vmatrix} f_1 & f_2 & \cdots\cdots & f_n \\ f_1' & f_2' & \cdots\cdots & f_n' \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots\cdots & f_n^{(n-1)} \end{vmatrix} \neq 0$$

i.e. $W(f_1, f_2, \ldots\ldots, f_n) \neq 0$

$\therefore$ If $W(f_1, f_2, \ldots\ldots, f_n) \neq 0$, then $c_1 = c_2 = \ldots\ldots\ldots = c_n = 0$

$\therefore$ the functions $f_1, f_2, \ldots\ldots, f_n$ are linearly independent over I.

**Property 2:** If $f_1, f_2, \ldots\ldots, f_n$ are linearly dependent over I, then

$W(f_1, f_2, \ldots\ldots, f_n) = 0 \quad \forall\ x \in I$.

**Proof:** If $f_1, f_2, \ldots\ldots, f_n$ are linearly independent over I, then there exists constants $c_1, c_2, \ldots\ldots, c_n$, not all zero, such that

$c_1 f_1'(x) + c_2 f_2'(x) + \ldots\ldots + c_n f_n'(x) = 0$

$c_1 f_1''(x) + c_2 f_2''(x) + \ldots\ldots + c_n f_n''(x) = 0$

..........................................................

..........................................................

$c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + \ldots\ldots + c_n f_n^{(n-1)}(x) = 0$

Putting $x = x_0$, where $x_0 \in I$, we get

$c_1 f_1(x_0) + c_2 f_2(x_0) + \ldots\ldots + c_n f_n(x_0) = 0$

$c_1 f_1'(x_0) + c_2 f_2'(x_0) + \ldots\ldots + c_n f_n'(x_0) = 0$

..........................................................

..........................................................

$c_1 f_1^{(n-1)}(x_0) + c_2 f_2^{(n-1)}(x_0) + \ldots\ldots + c_n f_n^{(n-1)}(x_0) = 0$

These n equations can be written as

$$\begin{bmatrix} f_1(x_0) & f_2(x_0) & \cdots\cdots & f_n(x_0) \\ f1'(x_0) & f_2(x_0) & \cdots\cdots & f_n'(x_0) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \cdots\cdots & f_n^{(n-1)}(x_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \mathrm{M} \\ c_n \end{bmatrix} = 0$$

Now we know that the matrix equation A X = 0 has a non-zero solution if |A| = 0

$$\therefore \quad \begin{bmatrix} f_1(x_0) & f_2(x_0) & \ldots\ldots & f_n(x_0) \\ f1'(x_0) & f_2(x_0) & \ldots\ldots & f_n'(x_0) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f_1^{(n-1)}(x_0) & f_2^{(n-1)}(x_0) & \ldots\ldots & f_n^{(n-1)}(x_0) \end{bmatrix} = 0$$

$\Rightarrow$ W($f_1$, $f_2$,......., $f_n$) ($x_0$) = 0

But $x_0$ is an arbitrary point of I.

$\therefore$ W($f_1$, $f_2$,......, $f_n$) ($x_0$) = 0 $\quad \forall$ x $\in$ I

$\Rightarrow$ W($f_1$, $f_2$,......, $f_n$) (x) = 0 $\quad \forall$ x $\in$ I.

**Theorem 6:** Two solutions $f_1$ (x) and $f_2$ (x) of the equation.

$a_0$ (x) y'' + $a_1$ (x) y' + $a_2$ (x) y = 0, $a_0$(x) $\neq$ 0, x $\in$ [a, b] are linearly dependent if and only if their wronskian is identically zero.

**Proof:** Condition is necessary

Let $f_1$(x) and $f_2$(x) be linearly dependent. Then by definition, there must exist two constants $c_1$ and $c_2$, not both zero, such that

$c_1 f_1(x) + c_2 f_2(x) = 0$ for each x $\in$ [a, b] .....(1)

From (1), we have

$c_1 f_1'(x) + c_2 f_2'(x) = 0$ for each x $\in$ [a, b] ......(2)

Since $c_1$ and $c_2$ cannot be zero simultaneously, the system of simultaneous equations (1) and (2) possess non-zero solutions for which the conditions is

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \equiv 0 \text{ on } [a, b]$$

$\Rightarrow$ W(x) $\equiv$ 0 on [a, b] i.e. Wronskian is identically zero.

**Condition is sufficient**

Suppose that wronskian of $f_1$(x) and $f_2$(x) is identically zero on [a, b] i.e. let

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} \equiv 0 \text{ on } [a, b] \qquad .....(3)$$

Let x = $x_0 \in$ [a, b], then from (3), we get

$$\begin{vmatrix} f_1(x_0) & f_2(x_0) \\ f_1'(x_0) & f_2'(x_0) \end{vmatrix} = 0 \qquad ..... (4)$$

Now (4) is the condition for existence of two constants $k_1$ and $k_2$, both not zero, such that

$k_1 f_1(x_0) + k_2 f_2(x0) = 0$ .....(5)

and $k_1 f_1'(x_0) + k_2 f_2'(x0) = 0$ .....(6)

Let $f(x) = k_1 f_1(x) + k_2 f_2(x)$ .......(7)

Then $f(x)$ being a linear combination of solutions $f_1(x)$ and $f_2(x)$ is also a solution of the given equation.

Again from (7), $f'(x) = k_1 f_1'(x) + k_2 f_2'(x)$ .......(8)

Now (7) gives $f(x_0) = k_1 f_1(x_0) + k_2 f_2(x_0) = 0$, by (5)

and (8) gives $f'(x_0) = k_1 f_1'(x_0) + k_2 f_2'(x_0) = 0$, by (6)

Thus, we find that $f(x)$ is a solution of the given equation such that $f(x_0) = 0$ and $f'(x_0) = 0$.

Hence, $f(x) \equiv 0$ on (a, b)

and therefore by (7), we have

$k_1 f_1(x) + k_2 f_2(x) = 0$ for each $x \in (a, b)$,

Where $k_1$ and $k_2$ are constants, both not zero.

Hence by definition, $f_1(x)$ and $f_2(x)$ are linearly dependent.

**Theorem 7:** The Wronskian of two solutions of the equation

$a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$, $a_0(x) \neq 0$, $x \in [a, b]$ is either identically zero or never zero on [a, b].

**Proof:** Given $a_0(x) y'' + a_1(x) y' + a_2(x) y = 0$, $a_0(x) \neq 0$, $x \in [a, b]$ .....(1)

Let $f_1(x)$ and $f_2(x)$ be two solutions of (1). Then their wronskian W(x) is given by

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = f_1(x)\, f_2'(x) - f_2(x)\, f_1'(x). \qquad ......(2)$$

Differentiating both sides of (2) w.r.t. x, we have $W'(x) = \frac{d}{dx}[f_1(x)\, f_2'(x)] - \frac{d}{dx}[f_2(x)\, f_1'(x)]$

or $W'(x) = [f_1'(x)\, f_2'(x) + f_1(x)\, f_2''(x)] - [f_2'(x)\, f_1'(x) + f_2(x)\, f_1''(x)]$

or $W'(x) = f_1(x)\, f_2''(x) - f_2(x)\, f_1''(x)$ .....(3)

Since $a_0(x) \neq 0$, dividing by $a_0(x)$ and rewriting, (1) becomes.

$$f''(x) = -\left(\frac{a_1}{a_0}\right) f'(x) = \left(\frac{a_2}{a_0}\right) f(x) \qquad .....(4)$$

Since $f_1(x)$ and $f_2(x)$ are solutions of (1) i.e. (4), we have

$f''(x) = -\left(\frac{a_1}{a_0}\right) f_1'(x) - \left(\frac{a_2}{a_0}\right) f_1(x)$ ......(5)

and $f_2''(x) = -\left(\frac{a_1}{a_0}\right) f_2'(x) - \left(\frac{a_2}{a_0}\right) f_2(x)$ ......(6)

Putting the values of $f_1''(x)$ and $f_2''(x)$ from (5) and (6) in (3), we have

$$W'(x) = f_1(x)\left[-\left(\frac{a_1}{a_0}\right) f_2'(x) - \left(\frac{a_2}{a_0}\right) f_2(x)\right] - f_2(x)\left[-\left(\frac{a_1}{a_0}\right) f_1'(x) - \left(\frac{a_2}{a_0}\right) f_1(x)\right]$$

$\Rightarrow$ $W'(x) = -\left(\frac{a_1}{a_0}\right)\left[f_1(x) f_2'(x) - f_2(x) f_1'(x)\right]$

$\Rightarrow$ $W'(x) = -\left(\frac{a_1}{a_0}\right) W(x)$, using (2) ......(7)

$\Rightarrow$ $a_0(x)\, W'(x) + a_1(x)\, W(x) = 0$ .......(8)

$\Rightarrow$ $\frac{W'(x)}{W(x)} = -\frac{a_1(x)}{a_0(x)}$

Integrating we get

$$\log W(x) = -\int \frac{a_1(x)}{a_0(x)}\, dx$$

$\Rightarrow$ $W(x) = e^{-\int \frac{a_1(x)}{a_0(x)} dx}$ is a solution of (8)

Now, the following two cases arise:

Case I :- Let $W(x) \neq 0$ on [a, b]. This prove second part of the theorem.

Case II :- If possible, let $W(x_0) = 0$ for some $x_0 \in [a, b]$ Then (7) gives.

$$W'(x_0) = -\left(\frac{a_1}{a_0}\right) W(x_0) = 0$$

Thus, W(x) is a solution of (8) such that $W(x_0) = 0$

and $W'(x) = 0$. Hence $W(x) = 0$ on [a, b]

i.e. Wronskianis identically zero on [a, b]

This proves the first part of the theorem. Now, to clarify what we have just said, consider the following examples:-

**Example 5:** Prove by Wronskian that the functions 1, x, $x^2$..... $x^{n-1}$ are linearly independent over reals ($n \in N$).

**Sol.** Let $y_1 = 1,\ y_2 = x,\ y_3 = x_2,\ \ldots\ldots,\ y_n = x^{n-1}$

$y'_1 = 0$, $y'_2 = 1 = \underline{|1}$, $y'_3 = 2x$,......, $y'_n = (n - 1)\, x^{n-2}$

$y''_1 = 0$, $y''_2 = 0$, $y'''_3 = 2 = \underline{|2}$,......., $y''_n = (n-2)\,(n-1)\, x^{n-1}$

..........................................................................

..........................................................................

$y_1^{(n-1)} = 0$. $y_2^{(n-1)} = 0$, $y_3^{(n-1)} = 0$, ........, $y_n^{(n-1)} = \underline{|n-1}$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & y_3(x).....y_n(x) \\ y'_1(x) & y'_2(x) & y'_3(x).....y'_n(x) \\ y''_1(x) & y''_2(x) & y''_3(x).....y''_n(x) \\ ..... & ..... & ..... \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & y_3^{(n-1)}(x).....y_n^{(n-1)}(x) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2......x^{n-1} \\ 0 & \underline{|1} & 2x.....(n-1)^{x^{n-1}} \\ 0 & 0 & \underline{|2}...(n-1)(n-2)^{x^{n-3}} \\ ..... & ..... & ..... \\ 0 & 0 & 0..................\underline{|n-1} \end{vmatrix}$$

$= 1.\underline{|1}\,\underline{|2}\,...........\underline{|n-1}$ [product of diagonal elements]

$\neq 0$ for all real x

Hence the given functions are linearly independent over reals.

**Example 6:** Show that x, $e^x$, $xe^x$, $(3 + 2x)\, e^x$ are linearly dependent.

**Sol.** Let $f_1(x) = x$; $f_2(x) = e^x$; $f_3(x) = xe^x$, $f_4(x) = (3 + 2x)\, e^x$

$$\text{Now } W(f_1, f_2, f_3, f_4)(x) = \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) & f_4(x) \\ f'_1(x) & f'_2(x) & f'_3(x) & f'_4(x) \\ f''_1(x) & f''_2(x) & f''_3(x) & f''_4(x) \\ f'''_1(x) & f'''_2(x) & f'''_2(x) & f'''_4(x) \end{bmatrix}$$

$$= \begin{vmatrix} x & e^x & xe^x & 3e^x + 2xe^x \\ 1 & e^x & xe^x + e^x & 3e^x + 2xe^x + 2e^x \\ 0 & e^x & xe^x + 2e^x & 3e^x + 2xe^x + 4e^x \\ 0 & e^x & xe^x + 3e^x & 3e^x + 2xe^x + 6e^x \end{vmatrix}$$

$$= \begin{vmatrix} x & 0 & -3e^x & -6e^x \\ 1 & 0 & -2e^x & -4e^x \\ 0 & 0 & -e^x & -2e^x \\ 0 & e^x & xe^x+3e^x & 9e^x+2xe^x \end{vmatrix}$$

$$= e^x \begin{vmatrix} x & -3e^x & -6e^x \\ 1 & -2e^x & -4e^x \\ 0 & -e^x & -2e^x \end{vmatrix} = 2e^x \begin{vmatrix} x & 3e^x & 3e^x \\ 1 & 2e^x & 2e^x \\ 0 & e^x & e^x \end{vmatrix}$$

$= 2\ e^x\ (0)$ [$\because$ $c_1$, $c_2$ are identical]

**Example 7:** Show that sin x and cos x are linearly independent solution of $\frac{d^2y}{dx^2} + y = 0$

Also write down their general solution.

**Sol.** Since W(sin x, cos x) = $\begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$ $= -\sin^2 x - \cos^2 x = -(\sin^2 x + \cos^2 x)$

$= -1 \neq 0$

$\therefore$ sin x and cos x are linearly independent.

Again let $y = \sin x \ \therefore \frac{dy}{dx} = \cos x$

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\sin x = -y$

$\therefore$ $\frac{d^2y}{dx^2} + y = 0$

$\therefore$ sin x is a solution of $\frac{d^2y}{dx^2} + y = 0$

Again $y = \cos x \quad \Rightarrow \quad \frac{dy}{dx} = -\sin x$

$\Rightarrow$ $\frac{d^2y}{dx^2} = -\cos x = -y$

$\frac{d^2y}{dx^2} + y = 0$

$\therefore$ cos x is a solution of $\frac{d^2y}{dx^2}$ + y = 0

Thus sin x, cos x are linearly independent solutions of $\frac{d^2y}{dx^2}$ + y = 0

General solution is y = $c_1$ sin x + $c_2$ cos x, where $c_1$, $c_2$ are arbitrary constants.

**Example 8:** If $y_1(x)$ and $y_2(x)$ are two functions for which

W ($y_1$, $y_2$) = $\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$ = $y_1$ $y'_2$ - $y_2$ $y'_1$ = 0

for each x is an interval I, then each sub interval $I_1$ of I contains a sub interval $I_2$ over which $y_1$ (x) and $y_2$ (x) are dependent.

**Sol.** Let I1 be a sub interval of I. If $y_1$ (x) = 0 for each x in $I_1$, then $y_1$ (x) and $y_2$ (x) must be dependent over $I_1$, because $c_1y_1$ (x) + $c_2y_2$ (x) = 0 when $c_1$ = 1 and $c_2$ = 0.

If $y_1$ (x) $\neq$ 0 for all x in $I_1$, there exists at least one $x_0 \in$ I, such that $y_1(x) \neq 0$. Since $y_1(x)$ is differentiable and hence continuous, we can choose sub-interval $I_2$ of $I_1$ such that $y_1(x)$

$\neq$ 0 for all x $\in$ $I_2$

When x $\in$ $I_2$, $\frac{y_1y'_2 - y_2y'_1}{y^21}$ = 0 (Given)

$\frac{y_2}{y_1}$ = constant = c (say) $\therefore$ $y_2$ (x) = c $y_1$ (x) when x $\in$ $I_2$

However, different intervals can require different values of c, so $y_1$ (x), and $y_2$ (x) may fail to be dependent over the whole interval I.

**Example 9:** Define the Wronskian of three functions $f_1$, $f_2$, $f_3$ over an interval I. Show by Wronskian that the following functions are linearly independent for all reals:

$e^x$, sin x, cos x

**Sol.** If $f_1$, $f_2$, $f_3$ are real functions each of which has a derivative of order 2 on an interval (a, b), then the determinant

W ($f_1$, $f_2$, $f_3$) = $\begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix}$

is called the Wronskian of these three functions.

For second part:-

Let $f_1$ (x) = $e^x$, $f_2$ (x) = sin x, $f_3$ (x) = cos x

$\therefore$ W ($f_1$, $f_2$, $f_3$) (x) = $\begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix} = \begin{vmatrix} e^x & \sin x & \cos x \\ e^x & \cos x & -\sin x \\ e^x & -\sin x & -\cos x \end{vmatrix}$

Operate $R_3 + R_1$ ;

W ($f_1$, $f_2$, $f_3$) (x) = $\begin{vmatrix} e^x & \sin x & \cos x \\ e^x & \cos x & -\sin x \\ 2e^x & 0 & 0 \end{vmatrix}$ = $2 e^x (- \sin^2 x - \cos^2 x)$

= $- 2e^x \neq 0$ for all $x \in R$.

Hence the given functions are linearly independent over R.

**Example 10:** If $y_1$ (x) = sin 3x and $y_2$ (x) = cos 3x are two solutions of differential equation y'' + 9y = 0, show that $y_1$ (x) and $y_2$ (x) are linearly independent solutions.

**Sol.** The Wronskian of $y_1$ (x) and $y_2$ (x)

W (x) = $\begin{vmatrix} y_1(x) & y_2(x) \\ y'_1(x) & y'_2(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$

= $-3 \sin^2 3x - 3 \cos^2 3x$

= $-3 (\sin^2 3x + \cos^2 3x) = -3 \neq 0$

Since W (x) $\neq 0$, $\therefore$ $y_1$ (x) and $y_2$ (x) are linearly independent solutions of y'' + 9y = 0

**Example 11:** Show by Wronskian that the functions $e^x$, $e^{2x}$, $e^{3x}$ are linearly independent.

**Sol.** Let $f_1$ (x) = $e^x$, $f_2$(x) = $e^{2x}$, $f_3$ = $e^{3x}$

$\therefore$ $f_1'(x) = e^x$, $f_2'(x) = 2e^{2x}$, $f_3'(2x) = 3e^{3x}$

$f_1''(x) = e^x$, $f_2''(x) = 4e^{2x}$, $f_3''(x) = 9e^{3x}$

$\therefore$ W ($f_1$, $f_2$, $f_3$) = $\begin{vmatrix} f_1 & f_2 & f_3 \\ f'_1 & f'_2 & f'_3 \\ f''_1 & f''_2 & f''_3 \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$

= $e^x . e^{2x} . e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix}$ [BP $R_2 \to R_2 - R_1$ ; $R_3 \to R_3 - R_1$]

= $e^{6x}$ [1 (8 - 6) = $2e^{6x} \neq 0$ $\forall x \in R$

$\therefore$ $f_1$, $f_2$, $f_3$ are L.I. over all reals.

**Example 12:** Show that the solutions $e^x$, $e^{-x}$ and $e^{2x}$ of

$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$ are linearly independent for all real numbers. Also write down their general solution.

**Sol.** $W(e^x, e^{-x}, e^{2x}) = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$

$$= e^x e^{-x} e^{+2x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= e^{2x}\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{vmatrix}$$ [By operating $R_3 \to R_3 - R_1$; $R_2 \to R_2 - R_1$]

$= -6e^{2x} \neq 0$ for all real x

$\therefore$ $e^x$, $e^{-x}$, $e^{2x}$ are linearly independent solutions.

$\therefore$ general solution is

$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$, where $c_1$, $c_2$, $c_3$ are arbitrary constants.

**Example 13:** If $\propto_1$, $\propto_2$, $\propto_3$ are all distinct, prove by Wronskian that the functions

$e^{\propto_1 x}$, $e^{\propto_2 x}$, $e^{\propto_3 x}$ are linearly independent over reals.

**Sol.** Let $f_1(x) = e^{\propto_1 x}$, $f_2(x) = e^{\propto_2 x}$, $f_3(x) = e^{\propto_3 x}$

$\therefore$ $f_1'(x) = \propto_1 e^{\propto_1 x}$, $f_2'(x) = \propto_2 e^{\propto_2 x}$, $f_3'(x) = \propto_3 e^{\propto_3 x}$

and $f_1''(x) = \propto_1^2 e^{\propto_1 x}$, $f_2''(x) = \propto_2^2 e^{\propto_2 x}$, $f_3''(x) = \propto_3^2 e^{\propto_3 x}$

Now

$$W(f_1, f_2, f_3) = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} e^{\propto_1 x} & e^{\propto_2 x} & e^{\propto_3 x} \\ \propto_1 e^{\propto_1 x} & \propto_2 e^{\propto_2 x} & \propto_3 e^{\propto_3 x} \\ \propto_1^2 e^{\propto_1 x} & \propto_2^2 e^{\propto_2 x} & \propto_3^2 e^{\propto_3 x} \end{vmatrix}$$

$$= e^{\propto_1 x} e^{\propto_2 x} e^{\propto_3 x}\begin{vmatrix} 1 & 1 & 1 \\ \propto_1 & \propto_2 & \propto_3 \\ \propto_1^2 & \propto_2^2 & \propto_3^2 \end{vmatrix}$$

$$= e^{\propto_1 x} e^{\propto_2 x} e^{\propto_3 x} \begin{vmatrix} 1 & 1 & 1 \\ \propto_1 & \propto_2 & \propto_3 \\ \propto_1^2 & \propto_2^2 & \propto_3^2 \end{vmatrix}$$

$$= e^{(\propto_1 + e^{\propto_2 +} e^{\propto_3})x} \ (\propto_1 - \propto_2)\ (\propto_2 - \propto_3)\ (\propto_3 - \propto_1)$$

$\neq 0 \qquad \forall\ x \in R$

$\therefore$ $f_1$, $f_2$, $f_3$ are L.I. over all reals.

**Example 14:** Show by Wronskian that the functions $1 - x^3$, $x^2 + x^3$, $- 4 + 3x^2$, $x^3 - x^2$ are linearly dependent dependent $\forall\ x \in R$.

**Solution:** The given functions $1 - x^3$, $x^2 + x^3$, $- 4 + 3x^2$, $x^3 - x^2$ are linearly dependent if there exist constants $c_1$, $c_2$, $c_3$, $c_4$ not all zero, such that

c1 (1 - x3) + c2 (x2 + x3) + c3 (-4 + 3x2) + c4 (x3 - x2) = 0

Taking $c_1 = 4$, $c_2 = \frac{1}{2}$, $c_3 = 0$, $c_4 = \frac{7}{2}$, we find the above relation satisfied.

$\therefore$ given functions are linearly dependent.

**Example 15:** Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$.

**Sol.** Given equation is

$y'' - 2y' + 2y = 0$

Let $y_1(x) = e^x \sin x$ and $y_2(x) = e^x \cos x$ ....(2)

From (2), $y1'(x) = e^x \sin x + e^x \cos x = e^x (\sin x + \cos x)$ ......(3)

From (3),

$y_1''(x) = e^x (\sin x + \cos x) + e^x (\cos x - \sin x) = 2 e^x \cos x$ ......(4)

Now, $y_1''(x) - 2y_1'(x) = 2 e^x \cos x - 2e^x (\sin x + \cos x) + 2 e^x \sin x = 0$

This shows that $y_1(x) = e^x \sin x$ is a solution of (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x(\sin x + \cos x) & e^x(\cos x - \sin x) \end{vmatrix}$$

$= e^{2x} (\sin x \cos x - \sin^2 x) - e^{2x} (\sin x \cos x + \cos^2 x)$

$= -e^{2x} \neq 0$

$\therefore$ $W(x) \neq 0$ for all $x$ in $(-\infty, \infty)$ and hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

The general solution of (1) is given by

$y(x) = c_1y_1 (x) + c_2y_2 (x)$

$= e^x (c_1 \sin x + c_2 \cos x)$ .....(5)

Where $c_1$ and $c_2$ are arbitrary constants

From (5), $y'(x) = e^x (c_1 \sin x + c_2 \cos x) + e^x (c_1 \cos x - c_2 \sin x)$ .....(6)

Putting x = 0 in (5) and using the given result y(0) = 2, we get

$y (0) = c_2$ or $c_2 = 2$

Putting x = 0 in (6) and using the given result y'(0) = - 3, we get

$y'(0) = c_2 + c_1$ or $- 3 = 2 + c_1$

or $c_1 = - 5$

∴ From (5), the solution of given equation satisfying the given properties is

$y = e^x (2 \cos x - 5 \sin x)$

**Self-Check Exercise-1**

Q.1 Show by Wronskian that the functions $e^{2x}$, $e^{3x}$, $e^{4x}$ are linearly independent.

Q.2 Show that x and $xe^x$ are linearly independent on any interval.

Q.3 Show that sin x, cos x, 2 sin x + cos x are linearly dependent.

Q.4 Show that e2x and e3x are linearly independent solutions of y'' -5y' + 6y = 0.

Find the solution with the property that y(0) = 0 and y'(0) = 1

Q.5 Are the following sets of functions defined on $-\infty < x < \infty$ linearly independent or dependent?

$f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^3$.

## 2.7 Linear Deferential Operator

**Def.** If D denotes the differential operator $\frac{d}{dx}$, then the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots\dots\dots + P_{n-1} \frac{dy}{dx} + P_n y = X \qquad \dots\dots(1)$$

can be written as $[P_0D^n + P_1 D^{n-1} \dots\dots + P_{n-1} D + P_n] y = X$.

IF X = 0 i.e. (1) is a homogeneous linear equation and assuming that P0 ≠ 0 at any point of interval then (1) takes the form

$$\frac{d^n y}{dx^n} + Q_1 \frac{d^{n-1} y}{dx^{n-1}} + Q_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + Q_{n-1} \frac{dy}{dx} + Q_n y = 0 \qquad \dots\dots\dots(2)$$

Where $Q_r = \frac{P_r}{P_0}$, r = 1,2,......n.

i.e. $[D^n + Q_1 D^{n-1} + Q_2 D^{n-2} + ........+ Q_{n-1} D + Q_n] (y) = 0$ .......(3)

Denoting

$L[y] = [D^n + Q_1 D^{n-1} + Q_2 D^{n-2} + ........+ Q_{n-1} D + Q_n' (y)$.

(3) reduces to L (y) = 0

Here L is called a linear differential operator.

Two basic laws of linear differential operator

$L[cy] = c L[y]$ and $l[y_1 + y_2] = L[y_1] + L[y_2]$, where c is any constant.

Note. By using the above properties, it can be easily proved that $L[c_1y_1 + c_2y_2]$

$= c_1 L[y_1] + c_2 L[y_2]$, where $c_1$, $c_2$ are any constants. It can be further extended so that

$L\left[\sum_{i=1}^{n} c_i y_i\right] = \sum_{i=1}^{n} c_i L[y_i]$, where $c_i$' s are constants.

The sum and product of linear differential operators

(i) The sum $L_1 + L_2$ of two differential operators, $L_1$ and $L_2$ is obtained by adding corresponding coefficients after expressing each of them in the form

$P_0D^n + P_1 D^{n-1} + ........+ P_{n-1} D + P_n$

e.g. If $L_1 = x^2 D^2 + 2xD + 7$ and $L_2 = D^3 - x D + 1$ be two differential operators, then $L_1 + L_2 =$

$D^3 + x^2D^2 + x D + 8$ is also a differential operator.

(ii) The Product, $L_1L_2$ of two differential operators as $(L_1L_2) [y]$

$= L_1 (L_2 [y])$ i.e. the differential operator $L_1L_2$ produces the same result as is obtained by first using the operator $L_2$ and then applying the operator $L_1$.

e.g. if $L_1 = 2D + 1$ and $L_2 = 1 - 3D$, then

$(L_1L_2) [y] = L_1 (L_2 [y]) = L_1 \{1 - 3D)[y]\} = (2D + 1)\left(y - 3\frac{dy}{dx}\right)$

$= 2\frac{d}{dx}\left(y - 3\frac{dy}{dx}\right) + y - 3\frac{dy}{dx} = 2\frac{dy}{dx} - 6\frac{d^2y}{dx^2} + y - 3\frac{dy}{dx}$

$= -6\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = (-6D^2 - D + 1) y \Rightarrow L_1L_2 = -6D^2 - D + 1$.

**Important Properties of Differential Operators**

**Property 1:** If $f(D)$ is any polynomial in D, then

$f(D) e^{ax} = e^{ax} f(a)$

**Proof:** Since $f(D)$ is any polynomial in D,

$\therefore$ Let $f(D) = P_0D^n + P_1 D^{n-1} + \dots\dots\dots + P_n$

Now $f(D)\, e^{ax} = (P_0D^n + P_1 D^{n-1} + \dots\dots\dots + P_n)\, e^{ax}$

$= P_0D^n (e^{ax}) + P_1 D^{n-1} (e^{ax}) + \dots\dots\dots + P_n \{e^{ax})$

$= P_0\, a^n e^{ax} + P_1\, a^{n-1}e^{ax} + \dots\dots\dots + P_n\, (e^{ax})$

$= e^{ax} (P_0\, a^n + P_1\, a^{n-1} + \dots\dots\dots + P_n)$

Hence $f\,(D)\, e^{ax} = e^{ax} f(a)$

**Con:-** If $\propto$ is a $f(a) = 0$, then $e^{\propto x}$ is a solution of the equation $f(d)\, y = 0$

**Proof:** Since $\propto$ is a root of $f(a) = 0$

$\therefore$ $f(\propto) = 0$ ......(1)

Also $f(D)\, e^{\propto x} = e^{\propto x} f(\propto)$

$= e^{\propto x}\,(0)$ [Q of (1)]

$\therefore$ $f(D)\, e^{\propto x} = 0$

$\Rightarrow$ $e^{\propto x}$ satisfies $f(D)\, y = 0$

Hence $e^{\propto x}$ is a solution of $f(D)\, y = 0$

**Property 2: Exponential Shift**

If $f(D)$ is any polynomial in D with constant coefficients, then

$$e^{\propto x} f(D) = f(D - \propto)\,(e^{\propto x}\, y)$$

where y is any function of x.

**Proof:** Since $f(D)$ is any polynomial in D with constant coefficients.

$\therefore$ Let $f(D) = P_0D^n + P_1 D^{n-1} + \dots\dots\dots + P_n$

Now $(D - \propto)(e^{\propto x}y) = \dfrac{d}{dx}(e^{\propto x}y) - \propto(e^{\propto x}y)$

$= e^{\propto x}\dfrac{dy}{dx} + e^{\propto x}\propto y - \propto e^{\propto x}\, y$

$= e^{\propto x}\dfrac{dy}{dx}$

$= e^{\propto x}\, D(y)$

and $(D - \propto)^2\, (e^{\propto x}\, y) = (D - \propto)\, [(D - \propto)\, (e^{\propto x}\, y)]$

$= (D - \propto)\, [e^{\propto x}\, D(y)] = e^{\propto x}\, D^2(y)$

In general,

$(D - \propto)^n (e^{\propto x} y) = e^{\propto x} D^n (y)$

Hence $e^{\propto x} f(D)\, y = f(D - \propto)(e^{\propto x} y)$

[$\because$ differential operators are linear operators and $f(D)$ is a polynomial in D]

**Let us do some examples:-**

**Example 16:** Given an example to show that

$L_1L_2 = L_2L_1$

**Sol:** Let $L_1 = D + 2$, $L_2 = D^2 - 3D + 5$

$\therefore \quad L_1L_2 (y) = (D + 2) ((D^2 - 3D + 5))y$

$$= (D + 2)\left(\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 5y\right)$$

$$= \left(\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx}\right) + \left(2\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 10y\right)$$

$$= \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + 10\,y$$

$$= (D^3 - D^2 - D + 10)\,y \qquad .....(1)$$

Thus $\quad L_1L_2 = D^3 - D^2 - D + 10$

Again $\quad L_2L_1 (y) = (D^2 - 3D + 5) ((D + 2))\, y$

$$= (D^2 - 3D + 5)\left(\frac{dy}{dx} + 2y\right)$$

$$= \left(\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\right) - 3\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) + 5\left(\frac{dy}{dx} + 2y\right)$$

$$= \frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + 10\,y$$

$$= (D^3 - D^2 - D + 10)\,y$$

Thus $\quad L_2L_1 = D^3 - D^2 - D + 10 \qquad .......(2)$

(1) and (2) $\Rightarrow L_1L_2 = L_2$

**Example 17:** Let $L_1$, $L_2$ be two different operators and if

$L_1 = x\,D + 3$, $L_2 = D + x$

then show that $L_1L_2 \neq L_2L_1$.

**Sol:** $L_1L_2$ (y) = (x D + 3) ((D + x)) y

$$= (xD + 3)\left(\frac{dy}{dx} + xy\right)$$

$$= x\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + 3\frac{dy}{dx} + 3xy$$

$$= [xD^2 + (x^2 + 3) D + 3x]y$$

$\Rightarrow$ $L_1L_2 = xD^2 + (x^2 + 3) D + 3x$ ......(1)

Again $L_2L_1$ (y) = (D + x) ((xD + 3)) y

$$= (D + x)\left[x\frac{dy}{dx} + 3y\right]$$

$$= \frac{d}{dx}\left(x\frac{dy}{dx}\right) + 3\frac{dy}{dx} + x^2\frac{dy}{dx} + 3xy$$

$$= x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3\frac{dy}{dx} + x^2\frac{dy}{dx} + 3xy$$

$$= [xD^2 + (x^2 + 4) D + 3x] y$$

$\Rightarrow$ $L_2L_1 = xD^2 + (x^2 + 4) D + 3x$ ......(2)

(1) and (2) $\Rightarrow L_1L_2 \neq L_2L_1$

**Example 18:** Find the general solution of $(D + 3)^2 y = 0$

**Sol:** Given equation is $(D + 3)^2 y = 0$ .....(1)

Multiplying by $e^{3x}$, $e^{3x} (D + 3)^2 y = 0$ ......(2)

Since $e^{\propto x} f(D) y = f(D - \propto) (e^{\propto x} y)$ [Exponential shift]

$\therefore$ $D^5 (e^{3x}y) = 0$

[Q $e^{3x} (D + 3)^5 y = (D + 3 - 3)^5 (e^{3x} y)$

$= D^5 (e^{3x} y)$

Integrating five times, we get

$e^{3x} y = c_1 + c_2 + c_3 x_2 + c_4 x^3 + c5 x^4$

where c's are arbitrary constants.

$\Rightarrow$ $y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4) e^{-3x}$,

which is the required solution.

Now try the following exercise

**Self-Check Exercise-2**

Q.1 Let $L_1$, $L_2$ be two different operators, and if

$L_1 = 2D^2 - 3$, $L_2 = 3D + y$,

then show that $L_1L_2 = L_2L_1$

Q.2 Given an example to show that

$L_1L_2 \neq L_2L_1$

## 2.8 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined linear combination and discussed theorems related to it.
2. Defined linear dependence and linear independence and proved theorems related to them.
3. Wronskian and its properties discussed in detail.
4. Defined differential operators. Its properties discussed in detail.

## 2.9 Glossary:

1. A differential equation is said to be linear if the unknown function and all of its derivatives occurring in the equation occur only in the first degree and are not multiplied together.
2. The n functions $f_1$, $f_2$,......,$f_n$ are called linearly dependent on $a \leq x \leq b$ if there exists constants $c_1$, $c_2$,......., $c_n$ not all zero, such that $c_1f_1(x) + c_2f_2(x) + \ldots\ldots + c_nf_n(x) = 0$ for all x such that $a \leq x \leq b$.

## 2.10 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 Functions are L.I.

Hint:- $2e^{9x} \neq 0$ as $x \neq 0$

Ans.2 Functions are L.I.

Hint- $x^2e^x \neq 0$ on any interval

Ans.3 L.D.

Hint:- Taking $c_1 = -2$, $c_2 = -3$, $c_3 = 1$

Ans.4 $y = e^{3x} - e^{2x}$ as the required solution.

Ans.5 L.I.

Hint:- $6x \neq 0$ for $x \neq 0$

**Self-Check Exercise-2**

Ans.1 Hint: $L_1L_2 = 3D^3 + 8D^2 - 9D - 12$

and $L_2L_1 = 6D^3 + 8D^2 - 9D - 12$

Ans.2 Hint: $L_1 = D-1$ and $L_2 = x\,D+2$

## 2.11 References/Suggested Readings


1. Boyce, w. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
2. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
3. Wylie, C.R., Differential Equations, McGraw-Hill, New York, 1979.


## 2.12 Terminal Questions

1. Show by Wronskian that he functions

   $e^{ax}\cos bx,\ e^{ax}\sin bx$

   are linearly independent over all reals.

2. Show that $x^2$ and $\frac{1}{x^2}$ are linearly independent solutions of

   $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$$

   in the interval $0 < x < \infty$. Also write down their general solution.

3. If $\propto_1, \propto_2, \propto_3, ......, \propto_n$ are all distinct, prove by Wronkian that the functions.

   $e^{\propto_1 x}, e^{\propto_2 x}, e^{\propto_3 x}, ........, e^{\propto_n x}$ are L.I. over all reals.

4. Show by Wronskian that the functions sin x, cos x, 3 sin x - 4 cos x are linearly dependent $\forall\ x \in R$,

5. Are the following sets of functions defined on $-\infty < x < \infty$ linearly independent or dependent?

   $f_1(x) = e^{ix}$, $f_2(x) = \sin x$, $f_3(x) = 2\cos x$

6. Let $L_1$, $L_2$ be two linear operators and if

   $L_1 = x\,D + 2$, $L_2 = x\,D - 1$

   then prove that $L_1L_2 = L_2L_1$

7. By using the exponential show, find the general solution of the following equation.

   $(D + 2)^2\, y = 0$

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## Unit - 3

# Exact Differential Equations

**Structure**



## 3.1 Introduction

The theory of exact differential equations is a branch of differential equations that deals with a specific type of differential equation known as an exact equation. Exact differential equations have several important properties. Firstly, the solution to an exact differential equation is independent of the path taken, only depending on the initial and final points. This property is known as path independence. Additionally, exact equations have a conservation property, meaning that the total differential of the potential function is zero along any solution curve.

The theory of exact differential equations provides useful techniques for solving a specific class of differential equations. However, not all first order ordinary differential equations are exact, and in such cases, other methods like integrating factors or solving techniques specific to the particular equation type may be required.

## 3.2 Learning Objectives

After studying this unit, you should be able to:-

- Define first order exact differential equation.
- Discuss first order exact differential equations.
- Find solutions of first order exact differential equations.

## 3.3 First Order Exact Differential Equations

### A. Standard Forms of First-Order Differential Equations:-

The first-order differential equations may be expressed in either the derivative form

$$\frac{dy}{dx} = f(x, y) \qquad .....(1)$$

or the differential form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \qquad .....(2)$$

An equation in one of these forms may readily be written in the other form e.g., the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$$

is of the form (1). It may be written as $(x^2 + y^2)\, dx + (y - x)\, dy = 0$, which is of the form (2).

In the form (1) it is clear form the notation itself that y is regarded as the dependent variable and x as the independent one; but in the form (2) we may actually regard either variable as the dependent one and the other as independent.

**B. Exact Differential Equations:-**

Def: Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula.

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\, dx + \frac{\partial F(x, y)}{\partial y}\, dy \text{ for all } (x, y) \in D.$$

Def: The expression

$$M(x, y)\, dx + N(x, y)\, dy \qquad ......(3)$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential dF(x, y) for all $(x, y) \in D$ i.e., expression (3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \text{ and } \frac{\partial F(x, y)}{\partial y} + N(x, y) \text{ for all } (x, y) \in D.$$

If M(x, y) dx + N(x, y) dy is an exact differential, then the differential equation

M(x, y) dx + N(x, y) dy = 0

is called an exact differential equation.

e.g. x dy + y dx is exact

∴ it can be obtained from xy = c directly by differentiation.

Similarly, sin x cos y dy + cos x sin y dx is exact

Q it can be obtained from sin x sin y = c

directly by differentiation.

**Theorem 1:** Find the necessary and sufficient conditions that the equation M dx + N dy = 0 (where M and N are functions of x and y with the condition that M, N, $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial x}$ are continuous functions of x and y) may be exact.

**Sol: Necessary Condition**

Let the equation M dx + N dy = 0 be exact.

Then, by def, M dx + N dy = dF, (By def.)

where F is a function of x and y

$\therefore$ Mdx + Ndy = $\frac{\partial F}{\partial x}$ dx + $\frac{\partial F}{\partial y}$ dy [Total Differential]

Equating coeffs. of dx on both sides, M = $\frac{\partial F}{\partial x}$ .....(1)

Equating coeffs of dy on both sides, N = $\frac{\partial F}{\partial y}$ .......(2)

From (1), $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial F \partial x}$ .......(3)

From (2), $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial^2 F}{\partial y \partial x}$ .......(4)

Since $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$

$\therefore$ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ [From (3) and (4)]

Which is the required necessary condition.

**Sufficient Condition**

Given $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To prove M dx + N dy = 0 is exact.

Let $\int M\,dx = \phi$ (x, y) ......(5)

where integration has been performed w.r.t. x while treating y as constant.

So that $\frac{\partial}{\partial x}\left[\int M\,dx\right] = \frac{\partial \phi}{\partial x}$ i.e. M = $\frac{\partial \phi}{\partial x}$ ......(6)

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial^2 \phi}{\partial y \partial x}$ ....(7)

But $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ [Given] and $\frac{\partial^2 \phi}{\partial y \partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$ [Assuming]

$\therefore$ From (7), $\frac{\partial N}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)$

Integrating both sides w.r.t. x, treating y as constant, we get

N = $\frac{\partial \phi}{\partial y}$ + a function of y

= $\frac{\partial \phi}{\partial y}$ + $f$(y) (say) .......(8)

From (6) and (8)

M dx + N dy = $\frac{\partial \phi}{\partial x}$ dx + $\left[\frac{\partial \phi}{\partial y} + f(y)\right]$dy

= $\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right)$ + $f$(y) dy

= d$\phi$ + $f$(y) dy ......(9)

which is an exact differential.

$\left[\because \text{f (y) dy is an exact differential as f(y) dy=d}\left(\int f\ y\right)dy\right]$

Hence M dx + N dy = 0 is exact

**Cor:** If the condition is satisfied, solve the equation

The equation is

M dx + N dy = 0

i.e. d$\phi$ + $f$(y) dy = 0 [From (9)]

Integrating, $\phi + \int f(y)dy = 0$ ......(10)

But from (5) and (6)

$$\phi = \int_{y\,\text{constant}} Mdx$$

and from (8), $f(y)$ = terms of N not containing x

$\therefore$ From (10), $\int_{x \supset dy = \subset} Mdx + \int$ terms of N not containing which is the required solution.

**Working Rule**

1. If for an equation of form M dx + N dy = 0. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then it is exact.

2. Its solution

$$\int Mdx + \int \text{(terms of N not containing x)}\, dy = c$$

y constant

Note 1: If there is no term in N independent of x, then the solution is

$$\int Mdx = c$$

y constant

Note2: If the equation M dx + N dy = 0 is exact, then on regrouping its terms, it can be written as d($f$(x, y)) = 0

$\therefore$ its solution is $f(x, y) = c$,

To clarify what we have just said, consider the following examples:-

**Example 1:** (a) When does M(x, y) dx + N(x, y)dy = 0 becomes an exact differential equation. Explain method to solve it.

(b) Solve: $(x^2 - 2xy - y^2)\, dx - (x + y)^2 dy = 0$

**Sol:** (a) M (x, y) dx + N(x, y) dy = 0 becomes an exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

**Method of solution:**

Required solution is $\int Mdx + \int \text{(terms of N not containing x)}\, dy = c$

y constant

(b) Here $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 - 2xy - y^2) = 2x - 2y$

and $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\{-(x + y)^2\} = -2(x + y)(1 + 0) = -2x - 2y$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given equation is exact.

$\therefore \quad$ Solution is $\int (x^2 - 2xy - y^2)\, dx + \int (-y^2)\, dy = c$

y constant

[$\because$ N = $-(x + y)^2 = -x^2 - y^2 - 2xy$ and term without x is $-y^2$]

$\Rightarrow \quad \left(\frac{x^3}{3} - 2y\frac{x^2}{2} - y^2 x\right) - \frac{y^3}{3} = c \Rightarrow \frac{x^3}{3} - x^2y - y^2x - \frac{y^3}{3} = c$, is the reqd. sol.

**Example 2:** Solve: $(x^2 - yxy - 2y^2)\, dx + (y^2 - 4xy - 2x^2)\, dy = 0$.

**Sol:** This is of the form Mdx + Ndy = 0

Here $M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$.

$\therefore \quad \frac{\partial M}{\partial y} = -4x - 4y$ and $\frac{\partial N}{\partial x} = -4y - 4x$.

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ the given equation is exact.

$\therefore \quad$ the solution is:

$\int (x^2 - 4xy - 2y^2)\, dx + \int y^2 dy = c$

y constant

or $\quad \int x^2 dx - 4y \int x dx - 2y^2 \int 1.dx + \int y^2 dy = c$

or $\quad \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$, is the reqd. sol.

**Example 3:** Solve $xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$

**Sol:** The given equation is $xdx + ydy = \frac{a^2(xdy - ydx)}{x^2 + y^2}$

or $\quad \left[x + \frac{a^2 y}{x^2 + y^2}\right] dx + \left[y + \frac{a^2 y}{x^2 + y^2}\right] dy = 0,$

which is the form M dx + N dy = 0

Here M = x + $\frac{a^2y}{x^2+y^2}$ and N = y - $\frac{a^2y}{x^2+y^2}$

$\therefore \quad \frac{\partial M}{\partial y} = 0 + a^2 . \frac{\left(x^2+y^2\right)(1)-y(2y)}{\left(x^2+y^2\right)^2} = \frac{a^2\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$

and $\quad \frac{\partial N}{\partial x} = 0 - a^2 . \frac{\left(x^2+y^2\right)(1)-x(2x)}{\left(x^2+y^2\right)^2} = \frac{a^2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ the given equation is exact.

$\therefore$ the solution is:

$$\underset{y \text{ constant}}{\int} \left(x+\frac{a^2y}{x^2+y^2}\right)dx + \int y\,dy = c$$

or $\quad \int x\,dx + a^2y \int \frac{1}{x^2+y^2}\,dx + \int y\,dy = c$

or $\quad \frac{x^2}{2} + a^2y.\frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c$

or $\quad x^2 + y^2 + 2a^2 \tan^{-1}\frac{x}{y} = 2c$, is the reqd. sol.

**Example 4:** Solve: [cos x tan y + cos(x + y)] dx + [sin x $\sec^2$y + cos(x + y)]dy = 0.

**Sol:** The given equation is of the form Mdx + Ndy = 0

Here M = cos x tan y + cos (x + y) and N = sin x, $\sec^2$y + cos (x + y)

$\therefore \quad \frac{\partial M}{\partial y}$ = cos x $\sec^2$ y - sin (x + y) and $\frac{\partial N}{\partial x}$ = cos x $\sec^2$ y - sin (x + y)

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ the given equation is exact

$\therefore$ the solution is:

$\int$ [cos x tan y + cos (x + y)] dx + 0 = c

y constant

or  tan y $\int$ cos x dx + $\int$ cos (x + y) dx = c

y constant

or tan y sin x + sin(x + y) = c, is the reqd. sol.

**Example 5:** Solve: (y cos x + sin y + y) dx + (sin x + x cos y + x) dy = 0

**Sol:** Here M = y cos x + sin y + y $\Rightarrow \frac{\partial M}{\partial y}$ = cos x + cos y + 1

and  N = sin x + x cos y + x $\Rightarrow \frac{\partial N}{\partial x}$ = cos x + cos y + 1

$\therefore$  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given equation is exact.

$\therefore$  Solution is $\int$ (y cos x + sin y + y) dx = c

[$\because$ there is no term in N without x]

y constant

$\Rightarrow$  y sin x + x in y + xy = c, is the reqd. sol.

**Example 6:** (a) Solve: $(x^2 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$

(b) Solve: $(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$.

**Sol:** (a) Here M = $x^2 + e^{x/y} \Rightarrow \frac{\partial M}{\partial y} = e^{x/y}\left\{\frac{-x}{y^2}\right\}$

and  N = $e^{x/y}\left(1 - \frac{x}{y}\right) \Rightarrow \frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1 - \frac{x}{y}\right)\left(e^{x/y}\,\frac{1}{y}\right) = \frac{x}{y^2}.e^{x/y}$

$\therefore$  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the given equation is exact.

$\therefore$  Solution is $\int (x^2 + e^{x/y})\,dx = c_1$  [Note that there is no term without x in N]

y constant

$\Rightarrow \quad \frac{x^3}{3} + \frac{e^{x/y}}{\left(\frac{1}{y}\right)} = c_1 \Rightarrow \frac{x^3}{3}$ + ye$^{x/y}$ = $c_1 \Rightarrow$ $x^3$ + 3ye$^{x/y}$ = 3 $c_1$ = c $\Rightarrow$ $x^3$ + 3y e$^{x/y}$ = c

(b) Here M = 1 + e$^{e/y} \Rightarrow \frac{\partial M}{\partial y}$ = e$^{x/y}\left(\frac{-x}{y^2}\right)$

and N = e$^{x/y}\left(1-\frac{x}{y}\right) \Rightarrow \frac{\partial N}{\partial x}$ = e$^{x/y}\left(\frac{-1}{y}\right)$ + $\left(1-\frac{x}{y}\right)$. e$^{x/y}$ . $\frac{1}{y}$ = - $\frac{x}{y}$ . e$^{x/y}$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so given equation is exact.

$\therefore$ Solution is $\int (1+e^{x/y})$ dx = c $\Rightarrow$ x + $\frac{e^{x/y}}{\left(\frac{1}{y}\right)}$ = c $\Rightarrow$ x + y + y. e$^{x/y}$ = c

y constant

**Example 7:** Solve: $\left(y^2e^{xy^2}+4x^3\right)$dx + 3xy$e^{xy^2}$ - 3y$^2$dy = 0

**Sol:** Here M = y$^2e^{xy^2}$ + 4x$^3 \Rightarrow \frac{\partial M}{\partial y}$ = y$^2$ ($e^{xy^2}$ 2xy) + $e^{xy^2}$ . (2y) + 0

= 2y$e^{xy^2}$ (xy$^2$ + 1)

and N = 2xy$e^{xy^2}$ - 3y$^2 \Rightarrow \frac{\partial N}{\partial x}$ = 2xy $e^{xy^2}$ .y$^2$ + ($e^{xy^2}$ . 2y) = 2y$e^{xy^2}$ (xy$^2$ + 1)

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so given equation is exact.

$\therefore$ Solution is $\int \left(y^2e^{xy^2}+4x^3\right)$dx + $\int$ (-3y$^2$) dy = c

y constant

$\Rightarrow \quad y^2\left(\frac{e^{xy^2}}{y^2}\right) + 4\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right)$ = c $\Rightarrow e^{xy^2}$ + x$^4$ - y$^3$ = c

**Example 8:** Solve: $\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$

OR

Prove that (ax + hy + g) dx + (hx + by + $f$) dy = 0, represents a family of conics.

**Sol:** $\dfrac{dy}{dx} = \dfrac{ax+hy+g}{hx+by+f}$

or (ax + hy + g) dx + (hx + by + $f$)dy = 0 which is of the form Mdx + Ndy = 0

Here M = ax + hy + g and N = hx + by + $f$

$\therefore$ $\dfrac{\partial M}{\partial y}$ = h and $\dfrac{\partial N}{\partial x}$ = h $\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ the given equation is exact.

$\therefore$ the solution is

$$\int (ax+hy+g)dx + \int (by+f)dy = c$$

y constant

or a $\int$ x dx + (hy + g) $\int$ 1.dx + b$\int$ y dy +$f\int$ 1.dy = c

or a.$\dfrac{x^2}{y}$ + (hy + g) x + b. $\dfrac{y^2}{2}$ + $f$y = c' or $ax^2 + 2hxy + 2gx + by^2 + 2fy = 2c$

[Let or $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

2c' = c']

which, being of second degree in x and y, represent a family of conics.

**Example 9:** Solve: $(y^4 + 4x^3y + 3x)\,dx + (x^4 + 4xy^3 + y + 1)\,dy = 0$

**Sol:** Here $M = y^4 + 4x^3y + 3x \Rightarrow \dfrac{\partial M}{\partial y} = 4y^3 + 4x^3$

and $N = x^4 + 4xy^3 + y + 1 \Rightarrow \dfrac{\partial N}{\partial x} = 4x^3 + 4y^3$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ and so given equation is exact.

$\therefore$ Solution is $\int (y^4 + 4x^3y + 3x)dx + \int (y + 1)dy = c$

y constant

$\Rightarrow$ $xy^4 + 4y\dfrac{x^4}{4} + 3\dfrac{x^2}{2} + \dfrac{y^2}{2} + y = c$

$\Rightarrow$ $xy^4 + x^4y + \dfrac{3x^2}{2} + \dfrac{y^2}{2} + y = c$

**Example 10:** Solve: $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$

**Sol:** Here M = $x^4 - 2xy^2 + y^4 \Rightarrow \frac{\partial M}{\partial y} = -4xy + 4y^3$

and N = $-(2x^2y - 4xy^3 + \sin y) = -2x^2y + 4xy^3 - \sin y \Rightarrow \frac{\partial N}{\partial x} = -4xy + 4y^3$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so given equation is exact.

$\therefore$ Solution is $\int (x^4 - 2xy^2 + y^4)dx + \int (-\sin y)dy = c$

y constant

$\Rightarrow \quad \frac{x^5}{5} - 2y^2\left(\frac{x^2}{2}\right) + y^4x + \cos y = c \Rightarrow \frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = c$

**Example 11:** Solve. $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$

**Sol:** Here M = $3x^2 + 4xy$ and N = $2x^2 + 2y \Rightarrow \frac{\partial M}{\partial y} = 4x \Rightarrow \frac{\partial N}{\partial x}$

$\Rightarrow$ Given equation is exact $\therefore$ Solution is $\int (3x^2 + 4xy)dx + \int 2y\, dy = c$

y constant

$\Rightarrow \quad x^3 + 2x^2y + y^2 = c$, is the required solution.

**Example 12:** Solve the differential equation (cos x cos y - cot x) dx - (sin x sin y) dy = 0

**Sol:** Given differential equation is (cos x cos y - cof x) dx - (sin x sin y)dy = 0 comparing it with M dx + N dy = 0, we get M = cos x cos y - cot x, N = -sin x sin y

$\Rightarrow \quad \frac{\partial M}{\partial y}$ - cos x sin y and $\frac{\partial N}{\partial x}$ = - cos x sin y

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given equation is exact and its solution is

$$\int M\,dx + \int \text{(terms in N not containing x) } dy = c$$

y constant

or $\quad \int (\cos x \cos y - \cot x)dx + 0 = c$

y constant

or $\quad (\cos y)\left(\int \cos x\, dx\right) - \int \cot x\, dx = c$

or $\quad \cos y \sin x - \log|\sin x| = c$

**Example 13:** Solve

(a) $(a^2 - 2xy - y^2)\, dx = (x + y)^2 dy$

(b) $\left(\frac{2x-1}{y}\right)dx + \left(\frac{x-x^2}{y^2}\right)dy = 0$

**Solve:** (a) Given differential equation is

$(a^2 - 2xy - y^2)dx = (x + y)^2 dy$

or $\quad (a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0 \qquad ......(1)$

Comp axing it with M dx + N dy = 0, we get

$M = a^2 - 2xy - y^2,\ N = -(x + y)^2$

$\therefore \quad \frac{\partial M}{\partial y} = -2x - 2y,\ \frac{\partial N}{\partial x} = -2(x + y) = -2x - 2y$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation (1) is exact and its solution is

$\int M\, dx + \int (\text{terms of N not containing x})\, dy = c$

y constant

or $\quad \int (a^2 - 2xy - y^2)dx + \int (-y^2)dy = c$

y constant

$\therefore \quad a^2x - 2y\frac{x^2}{2} - xy^2 - \frac{y^3}{3} = c$

or $\quad a^2x - x^2y - xy^2 - \frac{y^3}{3} = c$

(b) Given differential equation is

$\left(\frac{2x-1}{y}\right) dx + \left(\frac{x-x^2}{y^2}\right)dy = 0 \qquad ......(1)$

Comparing it with M dx + N dy = 0, we get

$M = \frac{2x-1}{y}$, $N = \frac{x-x^2}{y^2}$

$\therefore \quad \frac{\partial M}{\partial y} = -\frac{2x-1}{y} = \frac{1-2x}{y^2}$ and $\frac{\partial N}{\partial x} = \frac{1-2x}{y^2}$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation (1) is exact and its solution is

$\int M\,dx + \int$ (terms of N not containing x) dy = c

y constant

$\therefore \quad \int \left(\frac{2x-1}{y}\right)$dx + 0 = cor $\quad \frac{1}{y}\int$ (2x - 1) dx = c

y constant

$\therefore \quad \frac{1}{y}\left(2.\frac{x^2}{2}-x\right)$= c $\quad$ or $\quad \frac{1}{y}$ ($x^2$ - x) = c

$\therefore \quad x^2 - x = cy.$

**Example 14:** Solve the following differential equation.

$$(1 + 6y^2 - 3x^2y)\frac{dy}{dx} = 3xy^2 - x^2$$

**Sol. :** Given differential equation is

$$(1 + 6y^2 - 3x^2y)\frac{dy}{dx} = 3xy^2 - x^2$$

or $\quad (3xy^2 - x^2)dx + (3x^2y - 1 - 6y^2)dy = 0 \quad$ ......(1)

Comparing it with M dx + N dy = 0, we get

$M = 3xy^2 - x^2$, $N = 3x^2y - 1 - 6y^2$

$\therefore \quad \frac{\partial M}{\partial y} = 6xy \quad , \quad \frac{\partial N}{\partial x} = 6xy$

$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given equation is exact and its solution is $\int M\,dx + \int$ (terms of N not containing x) dy = c

y constant

or $3y^2 \int x\,dx - \int x^2\,dx - \int (1 + 6y^2)\,dy = c$

or $\dfrac{3x^2y^2}{2} - \dfrac{x^3}{3} - (y + 2y^3) = c$

**Example 15:** Solve the following differential equations:

(a) $\text{Sec}^2 x \tan y\,dx + \tan x \sec^2 y\,dy = 0$

(b) $\left[y\left(1+\dfrac{1}{x}\right)+\cos y\right]dx + [x + \log x - x \sin y]\,dy = 0$

**Sol:** (a) Given differential equation is $\sec^2 x \tan y\,dx + \tan x \sec^2 y\,dy = 0$ ......(1)

Comparing it with M dx + N dy = 0, we get

$M = \sec^2 x \tan y,\ N = \tan x \sec^2 y$

$\therefore$ $\dfrac{\partial M}{\partial y} = \sec^2 x \sec^2 y,\ \dfrac{\partial N}{\partial x} = \sec^2 x \sec^2 y$

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ Given equation is exact and its solution is

$\int M\,dx + \int$ (terms of N not containing x) dy = c

y constant

or $\tan y \int \sec^2 x\,dx = c$

or $\tan y \tan x = c$

(b) Given differential equation is

$\left[y\left(1+\dfrac{1}{x}\right)+\cos y\right]dx + [x + \log x - x \sin y]\,dy = 0$

Comparing it with M dx + N dy = 0, we get ......(1)

$M = y\left(1+\dfrac{1}{x}\right) + \cos y,\ N = x + \log x - x \sin y$

$\therefore$ $\frac{\partial M}{\partial y}$ = 1 + $\frac{1}{x}$ - sin y, $\frac{\partial N}{\partial x}$ = 1 _ $\frac{1}{x}$ - sin y

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given equation is exact and its solution is

$\int M\,dx + \int$ (terms of N not containing x) dy = c

y constant

or y$\int \left(1+\frac{1}{x}\right)$dx + cos y$\int$ dx = c

or y(x + log |x|) + x cos y = c

**Example 16:** Verify that the differential equation (2x + $e^x$ sin y) dx + $e^x$ cos y dy = 0 is exact and find its solution when y(0) = $\pi/2$ .

**Sol:** Given differential equation is

(2x + $e^x$ sin y) dx + $e^x$ cos y dy = 0 .......(1)

Comparing it with M dx + N dy = 0, we get

M = 2x + $e^x$ sin y, N = $e^x$ cos y

$\therefore$ $\frac{\partial M}{\partial y}$ = ex cos y, $\frac{\partial N}{\partial x}$ = ex cos y

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation (1) is exact and its solution is

$\int M\,dx + \int$ (terms of N not containing x) dy = c

y constant

$\therefore$ $\int$ (2x + ex sin y)dx + 0 = c

y constant

or 2 $\frac{x^2}{2}$ + $e^x$ sin y = c

$\therefore$ $x^2$ + ex sin y = c ......(2)

Now $y(0) = \pi/2 \Rightarrow y = \pi/2$ when $x = 0$

$\therefore$ from (2),

$0 + e^0 \sin \pi/2 = c \Rightarrow 1.1 = c \Rightarrow c = 1$

Putting c = 1 in (2), we get

$x^2 + e^x \sin y = 1$,

which is the required solution.

**Example 17:** For what value of k, the differential equation.

$$\left(1+e^{\frac{kx}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0$$

is exact.

**Sol:** Given differential equation is

$$\left(1+e^{\frac{kx}{y}}\right)dx + e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy = 0 \qquad .......(1)$$

Comparing it with M dx + N dy = 0, we get

$$M = 1 + e^{\frac{kx}{y}}, \; N = e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{\frac{kx}{y}} \cdot \left(-\frac{kx}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{\frac{x}{y}}\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right) e^{\frac{x}{y}} \cdot \frac{1}{y}$$

$$= e^{\frac{x}{y}}\left[\frac{1}{y}-\frac{x}{y^2}-\frac{1}{y}\right]$$

Now (1) will be exact

if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

i.e. if $e^{\frac{kx}{y}}\left(-\dfrac{kx}{y^2}\right) = e^{\frac{x}{y}}\left[\dfrac{1}{y}-\dfrac{x}{y^2}-\dfrac{1}{y}\right]$

i.e. if - $\frac{kx}{y^2}$ e$\frac{kx}{y}$ = $e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$

i.e. if k $e^{\frac{kx}{y}}$ = $e^{\frac{x}{y}}$, which is true when k = 1.

∴ required value of k is 1.

**Self-Check Exercise**

Q.1 Show that the differential equation

$2x \sin 3y\, dx + 3x^2 \cos 3y\, dy = 0$

is exact.

Q.2 Solve the differential equation

$(x^2 + y^2 - a^2)\, x dx + (x^2 - y^2 - b^2)\, y\, dy = 0.$

Q.3 Solve the differential equation

$$\frac{dy}{dx} = \frac{x-4y+7}{4x+y-8}$$

Q.4 Solve the following differential equation

$$\left[y^2 e^{xy^2} + 4x^3\right] dx + \left[2xye^{xy^2} - 3y^2\right] dy = 0$$

Q.5 Solve the initial value problem

$e^x (\cos y\, dx - \sin y\, dy) = 0;\ y(0) = 0$

## 3.4 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed form of first order differential equation
2. Defined exact differential equations.
3. Discussed with theorem necessary and sufficient condition for first order differential equation to be exact.
4. Did some examples to verify for the first order differential equations to be exact.

## 3.5 Glossary:

1. Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\delta x} dx + \frac{\partial F(x, y)}{\delta y} dy$$

for all (x, y) $\in$ D

2. The first order differential equations may be expressed in either the derivative form $\frac{dy}{dx} = f(x, y)$ or the differential form

   $M(x, y)\, dx + N(x, y) dy = 0$

## 3.6 Answer to Self Check Exercise

Ans.1 Equation is exact and Solution is $x^2 \sin 3y = c$

Hint: $\frac{\partial M}{\partial y} = 6x \cos 3x = \frac{\partial N}{\partial x}$

Ans.2 Equation is exact and Solution is

$$\frac{x^4}{4} + \frac{x^2 y^2}{2} - \frac{a^2 x^2}{2} - \frac{y^4}{4} - \frac{b^2 y^2}{2} = c$$

Ans.3 Hint:- $\frac{\partial M}{\partial y} = 2xy = \frac{\partial N}{\partial x}$

Equation is exact and Solution is

$x^2 - 8xy + 14x + y^2 - 16y = c$

Ans.4 Equation is exact and its solution is

$$e^{xy^2} + x^4 - y^3 = c$$

Hint: $\frac{\partial M}{\partial y} = 2y e^{xy^2} (xy^2 + 1) = \frac{\partial N}{\partial x}$

Ans.5 Equation is exact and its solution is

$$e^x \cos y = 1$$

Hint:- $\frac{\partial M}{\partial y} = e^x \sin y = \frac{\partial N}{\partial x}$ and c = 1

## 3.7 References/Suggested Readings

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Sneddon, I., Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

## 3.8 Terminal Questions

1. Solve the differential equation

   $(3x^2 + 8xy - 3y^2 - 5)dx + (4x^2 - 6xy + 3y^2 + 6)dy = 0$

2. Solve the differential equation

   $\left(x\sqrt{x^2 - y^2} - y\right)dx + \left(y\sqrt{x^2 - y^2} - x\right)dy = 0$

3. Solve the following differential equation

   $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

4. Solve the following differential equation

   $x\,dx + y\,dy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

5. Show that the differential equation

   $x\, e^{x^2+y^2}\, dx + y\, (e^{x^2+y^2} + 1)dy = 0$

   is exact and find its solution when $y(0) = 0$

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## Unit - 4

# Integrating Factors

**Structure**



### 4.1 Introduction

Integrating factors are a fundamental concept in differential equations. When solving certain types of differential equations, integrating factors play a crucial role in transforming the equation into a more manageable form. They allow us to simplify the equation or make it solvable by standard methods.

Integrating factors are particularly useful when dealing with non-exact differential equations. A non-exact equation is one where the left-hand side is not an exact derivative of a function. By multiplying the equation by an integrating factors, it is possible to convert it into an exact equation, which can then be solved using standard techniques.

Integrating factors can simplify complex differential equations by reducing their order or making them separable. They can transform equations into more manageable forms, making them easier to analyze and solve. The simplification often leads to closed-form solutions, providing valuable insights into the behavior of the system.

They also allow us to extend the range of methods available for solving differential equations. They enable us to use techniques such as separation of variables, integrating factors, or linearization, which may not be directly applicable to the original equation. By introducing an integrating factor, we can manipulate the equation to match the form that suits a chosen solution method.

Integrating factors find widespread applications in various areas of physics and engineering. They are commonly used in fields such as fluid dynamics, electrical circuits, heat transfer, quantum mechanics, and many other disciplines. Integrating factors provide a powerful tool for modeling and analyzing real-world systems, allowing us to make predictions and optimize designs.

Dear students, understanding and effectively civilizing integrating factors are key skills for anyone working with differential equations.

## 4.2 Learning Objectives

After studying this unit, you should be able to:-

- Define integrating factors
- Discuss rules for finding the integrating factors
- Explain each rule for finding the integrating factors with examples.

## 4.3 Integrating Factor-Definition

Consider the differential equation M(x, y)dx + N (x, y) dy = 0. .....(1)

If

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

then the equation is exact and we can obtain a one-parameter family of solutions by one of the procedures explained in unit-3. But if

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

then the equation (1) is not exact and the procedures discussed in Unit-3 do not apply. What shall we do in such a case? Perhaps we can multiply the non-exact equation by some expression that will transform it into an essentially equivalent exact equation. If so, we can proceed to solve the resulting exact equation.

**Let us consider the differential equation**

y dx + 2x dy = 0 .....(2)

We observed that this equation is not exact. However, if we multiply, equation (2) by y, it is transformed into the essentially equivalent equation.

$y^2$dx + 2x y dy = 0 ......(3)

Which is exact since this resulting exact equation (3) is integrable, we call y an integrating factor of equation (2). In general, we have the following definition:-

**Def:** If the differential equation

M(x, y)dx + N(x, y)dy = 0 ......(4)

is not exact in a domain D but the differential equation.

$\mu$ (x, y) M(x, y)dx + $\mu$ (x, y) N(x, y)dy = 0 ......(5)

is exact in D, then $\mu$ (x, y) is called an integrating factor of the differential equation (4).

Let us do some examples:-

**Example 1:** Show that $\frac{1}{y^2}$ is an integrating factor of the equation y dx - x dy = 0

**Sol:** Given differential equation is

y dx - x dy = 0 .....(1)

$\therefore$ M = y and N = - x

$\Rightarrow$ $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$

$\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so (1) is not exact.

Multiplying (1) by $\frac{1}{y^2}$, we get

$$\frac{y}{y^2} dx - \frac{x}{y^2} dy = 0$$

$$\Rightarrow \quad \frac{1}{y} dx - \frac{x}{y^2} dy = 0$$

Here M = $\frac{1}{y} \Rightarrow \frac{\partial M}{\partial y} = -\frac{1}{y^2}$

and N = $-\frac{x}{y^2} \Rightarrow \frac{\partial N}{\partial x} = -\frac{1}{y^2}$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ (1) becomes exact when it is multiplied by $\frac{1}{y^2}$. Therefore, $\frac{1}{y^2}$ is an integrating factor of (1).

**Example 2:** Show that $\frac{1}{x^2 + y^2}$ is an I.F. of x dx + y dy + $(x^2 + y^2)x^2$dx = 0.

**Sol:** Given equation is x dx + y dy + $(x^2 + y^2)x^2$ dx = 0 .....(1)

$\Rightarrow$ $(x + x^4 + x^2y^2)dx + y\,dy = 0 \therefore M = x + x^4 + x^2y^2 \Rightarrow \frac{\partial M}{\partial y} = 2x^2y$

and $N = y \Rightarrow \frac{\partial N}{\partial x} = 0 \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so (1) is not exact.

Multiply (1) by $\frac{1}{x^2+y^2}$, we get

$$\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy + x^2dx = 0 \Rightarrow \left(\frac{x}{x^2+y^2} + x^2\right)dx + \frac{y}{x^2+y^2}dy = 0$$

$\therefore$ $M = \frac{x}{x^2+y^2} + x^2$ and $N = \frac{y}{x^2+y^2}$

$\Rightarrow$ $\frac{\partial M}{\partial y} = \frac{(-x)(2y)}{\left(x^2+y^2\right)^2} = \frac{-2xy}{\left(x^2+y^2\right)^2}$ and $\frac{\partial N}{\partial x} = \frac{(-y)(2x)}{\left(x^2+y^2\right)^2} = \frac{-2xy}{\left(x^2+y^2\right)^2}$

$\therefore$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ (1) becomes exact when multiplied by $\frac{1}{x^2+y^2}$

$\therefore$ $\frac{1}{x^2+y^2}$ is I.F. of given equation (1).

**Example 3:** Shows that $\frac{1}{xy}$ is an I.F. of $y\,dx - x\,dy = 0$

**Sol:** Given equation is $ydx - xdy = 0$

$\therefore$ $M = y$ and $N = -x \Rightarrow \frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$

$\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so (1) is not exact

Multiplying (1) by $\frac{1}{xy}$, we get, $\frac{1}{x}dx - \frac{1}{y}dy = 0$

Here, $M = \frac{1}{x}$ and $N = -\frac{1}{y} \Rightarrow \frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ (1) becomes exact when multiplied by $\frac{1}{xy}$

$\therefore$ $\frac{1}{xy}$ is I.F. of given equation (1)

**Example 4:** Show that $\frac{1}{x^2}$ is an I.F. of y dx - x dy = 0

**Sol:** Given differential equation is y dx - x dy = 0 ......(1)

$\Rightarrow$ M = y and N = -x $\Rightarrow \frac{\partial M}{\partial y} = 1 \; \alpha\nu\delta \; \frac{\partial N}{\partial x} = -1$

$\Rightarrow$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ and so (1) is not exact

Multiplying (1) by $\frac{1}{x^2}$, we get, $\frac{y}{x^2}$ dx - $\frac{1}{x}$ dy = 0

Here M = $\frac{y}{x^2}$ and N = -$\frac{1}{x} \Rightarrow \frac{\partial M}{\partial y} = \frac{1}{x^2}$ and $\frac{\partial N}{\partial x} = \frac{1}{x^2} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\Rightarrow$ (1) becomes exact when multiplied by $\frac{1}{x^2} \Rightarrow \frac{1}{x^2}$ is I.F. of given equation (1).

## 4.4 Number of Integrating Factors of a Differential Equation

The number of integrating factors of a differential equation are infinite

Let us prove it as follows:

Let $\mu$ be an I.F. of the equation

M dx + N dy = 0

$\therefore$ $\mu$ (M dx + N dy) = 0 is exact. .....(1) [Q $\mu$ is an I.f.]

Let

$\mu$ (M dx + N dy) = du, ......(2)

where u is a function of x and y.

$\therefore$ u = c is a solution.

Let $f$(u) be any function of u. Multiplying (1) by M$f$(u),

$\mu$ $f$(u) (M dx + N dy) = 0 .....(3)

Now $\mu$ $f$(u) (M dx + N dy) = $f$( $\mu$ ) du [by (2)]

which is an exact differential.

$\therefore \qquad \mu f(u)$ is also an I.F. of (1)

But $f(u)$ is an arbitrary function of u, therefore, the number of integrating factors of () are infinite.

### 4.5 Integrating Factors By Inspection

If given differential equation M dx + N dy = 0 is not exact, then sometimes its I.F. can be found by inspection. Following is the list of integrating factors of certain groups of terms, which are parts of an exact differential equations.

| | Groups of terms | I.F. | Exact Differential |
|---|---|---|---|
| 1. | xdy - y dx | $\frac{1}{x^2}$ | $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$ |
| | x dy-y dx | $\frac{1}{y^2}$ | $\frac{y\,dy - x\,dx}{-y^2} = d\left(-\frac{x}{y}\right)$ |
| | x dy-y dx | $\frac{1}{xy}$ | $\frac{dy}{y} - \frac{dx}{x} = d\left(\log\frac{y}{x}\right)$ |
| | x dy - y dx | $\frac{1}{x^2+y^2}$ | $\frac{x\,dy - y\,dx}{x^2+y^2} = \frac{\frac{x\,dy - y\,dx}{x^2}}{1+\left(\frac{y}{x}\right)^2} = d\left(\tan^{-1}\frac{y}{x}\right)$ |
| 2. | x dy + y dx | $\frac{1}{xy}$ | $\frac{x\,dy - y\,dx}{xy}$ = d [log (xy)] |
| | x dy + y dx | $\frac{1}{(xy)^n}$ | $\frac{x\,dy + y\,dx}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$ |
| 3. | x dx + y dy | $\frac{1}{x^2+y^2}$ | $\frac{x\,dy + y\,dx}{x^2+y^2} = \frac{1}{2}$ d [log $(x^2 + y^2)$] |
| | x dx + y dy | $\frac{1}{(x^2+y^2)^n}$ | $\frac{x\,dx + y\,dy}{(x^2+y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$ |

Let us improve our understanding of these results by looking at some following examples:-

**Example 5:** Find integrating factor by inspection and solve :ydx - xdy + log x. dx = 0. .....(1)

**Sol:** Here log x. dx is exact differential and an I.F. is required for ydx - xdy. Obviously this I.F. should be a function of x, so it is $\frac{1}{x^2}$. Multiply (1) by $\frac{1}{x^2}$, we get $\frac{y\,dy - x\,dx}{x^2} + \frac{\log x}{x^2}$ dx = 0

Now $\int \frac{\log x}{x^2}$ dx = $\int$ log x. $\frac{1}{x^2}$ dx (Integrate by parts)

= log x.$\frac{-1}{x}$ - $\int \frac{1}{x}.\frac{-1}{x}$dx = - $\frac{\log x}{x}$ - $\frac{1}{x}$

$\therefore$ $\int \frac{\log x}{x^2}$ dx = - $\frac{\log x + 1}{x}$; $\therefore$ $\frac{\log x}{x^2}$ = -d $\left(\frac{\log x+1}{x}\right)$

$\therefore$ -d $\left(\frac{y}{x}\right)$ -d $\left(\frac{\log x+1}{x}\right)$ = 0, i.e., d. $\left(\frac{y}{x}+\frac{\log x+1}{x}\right)$ = 0 which is exact.

Integrating, we get $\frac{y}{x}$ + $\frac{\log x+1}{x}$ = c $\Rightarrow$ y + log x + 1 = cx, is the reqd. solution.

**Example 6:** Find an I.F. for ($x^4$ ex - 2m$xy^2$)dx + 2m$x^2$y dy = 0 and hence solve.

**Sol:** Given equation ($x^4$ex - 2m$xy^2$)dx + 2m$x^2$y dy = 0

Here I.F. = $\frac{1}{x^4}$

Multiplying (1) by $\frac{1}{x^4}$, we get, $\left(e^x - \frac{2my^2}{x^3}\right)$dx + $\frac{2my}{x^2}$dy = 0

From Mdx + Ndy = 0 and it is exact

$\therefore$ solution is $\int \left(e^x - \frac{2my^2}{x^3}\right)$dx = c $\Rightarrow$ $e^x$ - 2m$y^2\left(\frac{x^{-3+1}}{-3+1}\right)$ = c

y constant

$\Rightarrow$ $e^x$ + $\frac{my^2}{x^3}$ = c $\Rightarrow$ $x^2e^x$ + m$y^2$ = c$x^2$, is the reqd. solution.

**Example 7:** Find an I.F. for y(2xy + $e^x$)dx = $e^x$dy and hence solve.

**Sol:** Given equation is y(2xy + ex) dx = $e^x$dy .....(1)

Here I.F. = $\frac{1}{y^2}$

Multiplying (1) by $\frac{1}{y^2}$, we get, $\left(2x + \frac{e^x}{y}\right)$dx - $\frac{e^x}{y^2}$ dy = 0

From Mdx + Ndy = 0 and it is exact.

$\therefore$ solution

y constant is $\int \left(2x+\frac{e^x}{y}\right)$dx = c $\Rightarrow$ 2$\frac{x^2}{2}$ + $\frac{e^x}{y}$ = c $\Rightarrow$ $x^2$y + ex = cy, is the reqd. solution

**Example 8:** Find an I.F. for ydx + x(1 - 3$x^2y^2$)dy = 0 and hence solve.

**Sol:** Given equation is ydx + x(1 - 3$x^2y^2$)dy = 0 .....(1)

Here I.F. = $\frac{1}{(xy)^3}$

$\left[\because \text{ydx+xdy present in given eqn. and the term-3x}^3 y^2 dysuggeststheI.F. = \frac{1}{(xy)^3}\right]$

Multiplying (1) by $\frac{1}{(xy)^3}$, we get, $\frac{y\,dx + x\,dy}{x^3y^3}$ - $\frac{3}{y}$ dy = 0 $\Rightarrow \left(\frac{1}{x^2y^3} - \frac{3}{y}\right)$dy = 0

From Mdx + Ndy and it is exact.

$\therefore$ Solution is $\int \frac{1}{x^3y^2}$ dx + $\int \frac{-3}{y}$ dy = c $\Rightarrow \frac{1}{y^2}\left(\frac{x^{-2}}{-2}\right)$- 3 log |y| = c

y constant

$\Rightarrow$ $\frac{-1}{2x^2y^2}$ - 3 log |y| = c, is the reqd. solution.

**Example 9:** Find an I.F. for ($e^y$ + x$e^y$)dx + x$e^y$dy = 0 and hence solve.

**Sol:** Given equation is $e^y$ + x$e^y$)dx + x$e^{ydy}$ = 0 .....(1)

Here I.F. = $\frac{1}{xe^y}$ [It is suggested by last term]

Multiplying (1) by $\frac{1}{xe^y}$, we get

$\left(\frac{1}{x}+1\right)$dx + 1 dy = 0

From M dx + N dy = 0 and it is exact.

$\therefore$ solution is

$\int \left(\frac{1}{x}+1\right)$dx + $\int$ 1 dy = c

y constant

$\Rightarrow$ $\log|x| + x + y = c$

**Example 10:** Find an integrating factor by inspection and hence the differential equation

$$y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$$

**Sol:** Given differential equation is

$$y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$$

Since $3x^2e^{x^3} = d(e^{x^3})$, the term $3x^2y^2e^{x^3}\,dx$.

Should not involve $y^2$

This suggests that $\frac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\frac{1}{y^2}$, we have

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}\,dx = 0$$

or $d\left(\frac{x}{y}\right) + d(e^{x^3}) = 0$, which is an exact expression.

Integrating, we have

$\frac{x}{y} + e^{x^3} = c$, which is the required solution.

**Example 11:** Find the integrating factor by inspection of the differential equation

$(x^2 + y^2 + 2x)dx + 2y\,dy = 0$ and hence solve it.

**Sol:** Given differential equation is

$$(x^2 + y^2 + 2x)dx + 2y\,dy = 0 \qquad ......(1)$$

Comparing (1) with mdx + Ndy = 0, we get

$M = x^2 + y^2 + 2x$, $N = 2y$

$\therefore$ $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 0$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation (1) is not exact.

Now, (1) can be written as

$$(x^2 + y^2)dx + 2x\,dx + 2y\,dy = 0 \qquad .....(2)$$

By inspection, $\frac{1}{x^2+y^2}$ is an I.F.

Multiplying both sides of (2) by $\frac{1}{x^2+y^2}$, we get

$$\frac{(x^2+y^2)dx}{x^2+y^2} + \frac{2xdx+2ydy}{x^2+y^2} = 0$$

$\Rightarrow$ $dx + d[\log(x^2 + y^2)] = 0$

or $d[x + \log(x_2 + y_2)] = 0$

Integrating, we get

$$x + \log(x^2 + y^2) = c,$$

which is the required solution.

**Example 12:** Solve $(x^3 + xy^2 + k^2y)\,dx + (y^3 + yx^2 - k^2x)dy = 0$

**Sol:** Given differential equation is

$$(x^3 + xy^2 + k^2y)\,dx + (y^3 + yx^2 - k^2x)dy = 0 \qquad ......(1)$$

Comparing it with $Mdx + Ndy = 0$, we get

$M = x^3 + xy^2 + k^2y$ and $N = y^3 + yx^2 - k^2x$

$\therefore$ $\frac{\partial M}{\partial y} = 2xy + k^2$ and $\frac{\partial N}{\partial x} = 2xy - k^2$

$\therefore$ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given differential equation (1) is not exact.

The given differential equation (1) can be written as

$$k^2(x\,dy - y\,dx) - x(x^2 + y^2)\,dx - y(x^2 + y^2)dy = 0$$

Now the expression $xdy - y\,dx$ suggest that $\frac{1}{x^2+y^2}$ is an I.F.

$\therefore$ Multiplying (1) by $\frac{1}{x^2+y^2}$, we get

$$k^2\left[\frac{x\,dy - y\,dx}{x^2+y^2}\right] - x\,dx - y\,dy = 0$$

or $\quad k^2\, d\left[-\tan^{-1}\frac{x}{y}\right]$ - x dx - y dy = 0

Integrating, we get

$$k^2\left[-\tan^{-1}\frac{x}{y}\right] - \frac{x^2}{2} - \frac{y^2}{2} = c'$$

or $\quad x^2 + y^2 + 2k^2 \tan^{-1}\frac{x}{y}$ = c, where c = -2c'

or $\quad x^2 + y^2 + 2k^2 \tan^{-1}\frac{x}{y}$ = c, where c = -2c'

which is the required solution

**Example 13:** Solve $(x^2y - 1 - x^2y^2)dx + x3\, dy = 0$

**Sol:** Given differential equation is

$$(x^2y - 1 - x^2y^2)dx + x^3dy = 0$$

or $\quad x^2(y\, dx + x\, dy) - (1 + x^2y^2)\, dx = 0$

Multiplying it by $\frac{1}{x^2(1+x^2y^2)}$, we get

$$\frac{y\,dx + x\,dy}{1+x^2y^2} - \frac{dx}{x^2} = 0$$

or $\quad \frac{d(xy)}{1+(xy)^2} - \frac{dx}{x^2} = 0$

Integrating, we get

$$\tan^{-1}(x\, y) + \frac{1}{x} = c$$

which is the required solution

**Self-Check Exercise-1**

Q.1 Find the integrating factor by inspection and solve the differential equation.

$y(1 - xy)\, dx - x\,(1 + xy)\, dy = 0$

Q.2 Solve the differential equation

$x\, dx + y\, dy + 4y^3\,(x^2 + y^2)\, dy = 0$

Q.3 Find the integrating factor of the differential equation $(y - 1)\, dx - x\, dy = 0$ and

| | | |
|---|---|---|
| | | hence solve it. |
| | Q.4 | Solve the differential equation<br>y dx - x dy + (1 + $x^2$) dx + $x^2$ sin y dy = 0 |
| | Q.5 | Find the integrating factor by inspection and hence solve the differential equation<br>(x + $y^3$) dy = y (dx + dy), y > 0. |

## 4.6 Rules For Finding The Integrating Factors of the Equation Mdx + Ndy = 0

If M dx + N dy = 0 is not exact and it is difficult to find integrating factor by inspection, then five rules help us in finding integrating factors.

**Rule 1: Homogeneous Equation :**

If the equations Mdx + Ndy = 0 is homogenous in x and y i.e. if M, N are homogeneous functions of the same degree in x and y, then $\frac{1}{Mx+Ny}$ is an integrating factor provided Mx + Ny $\neq$ 0

**Proof:** The equation is Mdx + Ndy = 0 .....(1)

Q (1) is homogeneous, $\therefore$ M and N are homogeneous functions of x and y of the same degree, say n; so by the Euler's theorem on homogeneous functions

$$x\frac{\partial M}{\partial x} + y\frac{\partial M}{\partial y} = nM; \text{ and } x\frac{\partial N}{\partial x} + y\frac{\partial N}{\partial y} = nN. \quad .....(2)$$

We are to show that $\frac{1}{Mx+Ny}$ is an I.F. of (1), i.e.

$$\frac{M}{Mx+Ny}dx + \frac{N}{Mx+Ny}dy = 0, \quad ......(3) \text{ is exact. } (Mx + Ny \neq 0)$$

We are to prove that

$$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{\partial}{\partial x}\left(\frac{M}{Mx+Ny}\right)$$

Now $$\frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial M}{\partial y} M\left(\frac{\partial M}{\partial y}x + N + \frac{\partial M}{\partial y}y\right)}{(Mx+Ny)^2} = \frac{Ny\frac{\partial M}{\partial y} - MN - My\frac{\partial N}{\partial y}}{(Mx+Ny)^2}$$

and $$\frac{\partial}{\partial x}\left(\frac{N}{Mx+Ny}\right) = \frac{(Mx+Ny)\frac{\partial N}{\partial x} N\left(M + \frac{\partial M}{\partial x}x + \frac{\partial N}{\partial x}y\right)}{(Mx+Ny)^2} = \frac{Mx\frac{\partial M}{\partial x} - MN - Mx\frac{\partial M}{\partial x}}{(Mx+Ny)^2}$$

$\therefore \quad \frac{\partial}{\partial y}\left(\frac{M}{Mx+Ny}\right) - \frac{M}{\partial x}\left(\frac{M}{Mx+Ny}\right) = \frac{N\left(x\frac{\partial M}{\partial x}+y\frac{\partial M}{\partial y}\right)-M\left(x+\frac{\partial N}{\partial x}+y\frac{\partial N}{\partial y}\right)}{(Mx+Ny)^2}$

$= \frac{N(nM)-M(nN)}{(Mx+Ny)^2} = 0$ ......(4) [by (2)]

$\therefore$ (4) is true; and so (3) is an exact equation.

$\therefore$ 1/(Mx + Ny) is an I.F. of (1).

**Case of failure.** The above rule fails when Mx + Ny = 0

When Mx + Ny = 0 $\Rightarrow$ N = -Mx/y. so (1) becomes

Mdx - $\frac{Mx}{y}$dy = 0 or $\frac{dx}{y} - \frac{dy}{y} = 0$

Integrating log x - log y = log c, $\therefore$ x = cy which is the required solution.

Let us do some examples:-

**Example 14:** Solve: $(x^2y - 2xy^2)$ dx - $(x^3 - 3x^2y)$ dy = 0 .....(1)

**Sol:** This is homogeneous in x and y.

Comparing with Mdx + Ndy = 0, we have

M = $x^2$ y - $2xy^2$, N = $-(x^3 - 3x^2y)$.

$\therefore$ I.F. = $\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ Multiplying (1) by $\frac{1}{x^2y^2}$

We have, $\left(\frac{x}{y^2}-\frac{3}{y}\right)$dy = 0 .....(2)

This is an exact equation of the form: Mdx + Ndy = 0

$\therefore \quad \left[\text{Sol. is} \int Mdx + \int (\text{terms of N not containing x}) \text{ dy} = \text{c}\right]$

$\therefore$ Sol. of (2) is : $\int \left(\frac{1}{y}-\frac{2}{x}\right)$dx + $3\int \frac{1}{y}$ dy = c

$\Rightarrow \quad \frac{x}{y}$ - 2 log x + 3 log y = c, where c is an arbitrary constant.

**Example 15:** Solve: (i) $x^2y$ dx - $(x^3 + y^3)$ dy = 0

(ii) $(3xy^2 - y^3)$ dx + $(xy^2 - 2x^2y)$ dy = 0

**Sol.** (i) Given equation is $x^2y$dx - $(x^3 + y^3)$ dy = 0

Here M and N are homogeneous

$\therefore$ by Rule I, I.F. = $\dfrac{1}{Mx+Ny} = \dfrac{1}{(x^2y)x+(-(x^3+y^3))y} = \dfrac{-1}{y^4}$

Multiplying (1) by I.F. i.e. $\dfrac{-1}{y^4}$, we get $\dfrac{-x^2y}{y^4}$ dx + $\dfrac{1}{y^4}$ ($x^3 + y^3$) dy = 0

$\Rightarrow$ $\dfrac{-x^2}{y^3}$ dx + {$\dfrac{x^3}{y^4}$ + $\dfrac{1}{y}$} dy = 0, which is exact.

$\therefore$ Solution is $\int \dfrac{-x^2}{y^3}$ dx + $\int \dfrac{1}{y}$ dy = c $\Rightarrow \left(\dfrac{-1}{y^3}\right)\left(\dfrac{x^3}{3}\right)$ + log |y| = c, is the reqd. solution.

y constant

(ii) Given equation is

($3xy^2 - y^3$) dx + ($xy^2 - 2x^2y$) dy = 0 ......(1)

Here M and N are homogeneous

$\therefore$ by Rule I, I.F. = $\dfrac{1}{Mx+Ny} = \dfrac{1}{3x^2y^2 - xy^3 + xy^3 - 2x^2y^2} = \dfrac{1}{x^2y^2}$

Multiplying (1) by I.F. i.e. $\dfrac{1}{x^2y^2}$, we get $\left(\dfrac{3}{x} - \dfrac{y}{x^2}\right)$dx - $\left(\dfrac{2}{y} - \dfrac{1}{x}\right)$dy = 0, which is exact

$\therefore$ Solution is $\int \left(\dfrac{3}{x} - \dfrac{y}{x^2}\right)$dx + $\int \dfrac{-2}{y}$ dy = c

$\Rightarrow$ 3 log |x| + $\dfrac{y}{x}$ - 2 log |y| = c $\Rightarrow$ log $\left|\dfrac{x^3}{y^2}\right|$ + $\dfrac{y}{x}$ = c, is the reqd. solution.

**Example 16:** Solve the differential equation

($x^4 + y^4$) dx - $xy^3$dy = 0

**Sol:** Given differential equation is

($x^4 + y^4$) dx - $xy^3$dy = 0 .....(1)

Comparing (1) with Mdx + Ndy = 0, we get

M = $x^4 + y^4$, N = $-xy^3$

$\therefore$ $\dfrac{\partial M}{\partial y}$ = 0 + $4y^3$ = $4y^3$, $\dfrac{\partial N}{\partial x}$ = $-y^3$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given equation (1) is not exact.

Now equation (1) is homogeneous.

$\therefore \quad$ I.F. = $\frac{1}{Mx + Ny} = \frac{1}{x^5 + xy^4 - xy^4} = \frac{1}{x^5}$

Multiplying (1) by $\frac{1}{x^5}$, we get

$\left(\frac{x^4 + y^4}{x^5}\right)$dx - $\frac{y^3}{x^4}$dy = 0, which is exact.

$\therefore$ its solution is

$\int \frac{x^4 + y^4}{x^5}$ dx + 0 = c

y constant

$\therefore \quad \int \frac{1}{x}$ dx + $y^4 \int x^{-5}$ dx = c

or $\quad$ log x + $y^4\left(\frac{x^{-4}}{-4}\right)$ = c

$\therefore \quad$ log x - $\frac{y^4}{4x^4}$ = c, which is the required.

Solution:

**Self-Check Exercise-2**

Q.1 Solve the differential equation

$(x^2 + y^2)$ dx - 2xy dy = 0

Q.2 Solve the following differential equation

$(3xy^2 - y^3)$ dx - $(2x^2y - xy^2)$ dy = 0

**Rule II.** If the equation Mdx + Ndy = 0 is of the form $yf(xy)$ dx + xy(xy) dy = 0,

then $\frac{1}{Mx + Ny}$ is an I.F. (provided Mx - Ny $\neq$ 0)

**Proof:** Here $x.\frac{\partial f}{\partial x} = y.\frac{\partial f}{\partial y}$, because each of these = xy$f'$(u), where u = xy .....(1)

[$f$ = $f$(u), where u = xy. $\therefore \frac{\partial f}{\partial x} = f'(u).\frac{\partial u}{\partial x} = yf'(u).c; \frac{\partial f}{\partial y} = f'(u).\frac{\partial u}{\partial y} = xf'(u)$]

Similarly, $x.\frac{\partial g}{\partial x} = y.\frac{\partial g}{\partial y}$ .....(2)

The equation Mdx + Ndy = 0 is of the form y. $f$(xy)dx + xg (xy) dy = 0 .....(3)

Here M = y.$f$, N = xg; $\therefore$ Mx - Ny = xy ($f$ - g). ....(4)

We have to show that $\frac{1}{Mx+Ny} = \frac{1}{xy(f-g)}$ is an I.F. of (4), i.e.

$$\frac{y}{xy(f-g)}dx + \frac{xg}{xy(f-g)}dy = 0,$$

i.e. $\frac{f}{x(f-g)}dx + \frac{g}{y(f-g)}dy$ is exact (Mx - Ny $\neq$ 0) .....(5)

We are to prove that

$$\frac{\partial}{\partial y}\left|\frac{f}{x(f-g)}\right| = \frac{\partial}{\partial x}\left|\frac{g}{y(f-g)}\right| \quad .....(6)$$

Now $$\frac{\partial}{\partial y}\left|\frac{f}{x(f-g)}\right| = \frac{(f-g)\frac{\partial f}{\partial y} - f\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial y}\right)}{x(f-g)^2} = \frac{f.\frac{\partial g}{\partial y} - g\frac{\partial f}{\partial y}}{x(f-g)^2}$$

Similarly $$\frac{\partial}{\partial x}\left|\frac{g}{y(f-g)}\right| = \frac{f.\frac{\partial g}{\partial x} - g\frac{\partial f}{\partial x}}{y(f-g)^2}$$

$$\therefore \quad \frac{\partial}{\partial y}\left|\frac{f}{x(f-g)}\right| - \frac{\partial}{\partial x}\left|\frac{g}{y(f-g)}\right| = \frac{f\left(y\frac{\partial g}{\partial x} - x\frac{\partial g}{\partial x}\right) + g\left(x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right)}{xy(f-g)^2} = 0$$

[By (1) and (2)

$\therefore$ (6) is true, and so (5) is an exact equation.

$\therefore$ $\frac{1}{(Mx+Ny)}$ is an I.F of (4).

**Case of failure.** If Mx - Ny = 0 i.e. xy ($f$ - g) = 0, $\therefore f$ = g.

$\therefore$ (4) becomes ydx + xdy = 0; $\therefore$ d(xy) = 0, or xy = c, which is the required solution.

**Example 17:** Solve: $y(xy + 2x^2y^2)\, dx + x\, (xy - x^2y^2)\, dy = 0$

**Sol:** The given equation is $y(xy + 2x^2y^2)\, dx + x\, (xy - x^2y^2)\, dy = 0$

This is of the form $yf\,(xy)\, dx + xg\,(xy)\, dy = 0$

Comparing with Mdx + Ndy = 0, we have

$M = y\,(xy + 2x^2y^2),\ N = x\,(xy - x^2y^2)$

$\therefore$ I.F. = $\dfrac{1}{(Mx+Ny)} = \dfrac{1}{3x^3y^3}$

Multiplying (1) by $\dfrac{1}{3x^3y^3}$, we get

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right) dx + \left(\frac{1}{3xy^2}+\frac{1}{3y}\right) dy = 0 \qquad ....(2)$$

This is an exact equation of the form Mdx + Ndy = 0

Now Solution is: $\int$ Mdx + $\int$ [terms of N not containing x] dy = c

y constant

$\therefore$ Solution of (2) is, y constant $\displaystyle\int \left(\frac{1}{3x^2y}+\frac{2}{3x}\right) dx - \int \frac{1}{3y}\, dy = c$

or $\displaystyle -\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = 0$, where c is an arbitrary constant.

**Example 18:** Solve

$(2 + x^2y^2)\, y\, dx + (2 - x^2y^2)\, x\, dy = 0$

**Sol:** Given equation is

$(2 + x^2y^2)\, y\, dx + (2 - x^2y^2)\, x\, dy = 0 \qquad ......(1)$

It is of the form $yf(x, y)\, dx + x\, g\,(xy)\, dy = 0$

$\therefore$ by rule II, I.F. = $\dfrac{1}{Mx-Ny} = \dfrac{1}{(2+x^2y^2)xy-(2-x^2y^2)xy} = \dfrac{1}{2x^3y^3}$

Multiplying (1) $\dfrac{1}{2x^3y^3}$, we get

$$\left(\frac{1}{x^3y^2}+\frac{1}{2x}\right) dx + \left(\frac{1}{x^2y^3}-\frac{1}{2y}\right) dy = 0$$

Flow it is an exact equation

$\therefore$ solution is

$$\int \left(\frac{1}{x^3y^2} + \frac{1}{2x}\right) dx + \int \frac{-1}{2y} = 0$$

y constant

$$\Rightarrow \quad \frac{-1}{2x^2y^2} + \frac{1}{2} \log (x) - \frac{1}{2} \log |y| = c_1$$

$$\Rightarrow \quad \log \frac{|x|}{|y|} = 2c_1 + \frac{1}{x^2y^2}$$

$$\Rightarrow \quad \frac{|x|}{|y|} = e^{2c_1} \cdot e^{1/x^2y^2} \Rightarrow |x| = e^{2c_1} \cdot |y|\, e^{1/x^2y^2}$$

$\Rightarrow$ $|x| = c\,|y|\, e^{1/x^2y^2}$, where $c = e^{2c_1}$, is the required Solution

Now, try the following exercises:-

Q.3 Solve $(1 + xy)\, y\, dx + (1 - xy)\, x\, dy = 0$

Q.4 Solve $(x^2y^2 + xy + 1)\, y\, dx + (x^2y^2 - xy + 1)\, x\, dy = 0$

**Rule III.** When $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ is function of x alone, say $f(x)$, then $e^{\int f(x)dx}$ is an integrating factor of equation $Mdx + Ndy = 0$ ....(1)

**Proof:** Now $e^{\int f(x)dx}$ is an integrating factor of (1), if the equation

$$M e^{\int f(x)dx} . dx + N e^{\int f(x)dx} dy = 0 \qquad ......(2)$$

is exact, which is so

if $$\frac{\partial}{\partial y}\left[M e^{\int f(x)dx}\right] = \frac{\partial}{\partial x}\left[N e^{\int f(x)dx}\right]$$

or if $$\frac{\partial M}{\partial y} e^{\int f(x)dx} = \frac{\partial N}{\partial x} + N. e^{\int f(x)dx} . f(x)$$

or if $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + N f(x)$$

or $\qquad f(x) = \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$, which is true (given).

Hence $e^{\int f(x)dx}$ is an I.F. of (1),

**Example 19:** Solve: $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$

**Sol:** The given equation is $(x^2 + y^2 - 2x)\,dx + 2y\,dy = 0$ .....(1)

Comparing (1) with mdx + Ndy = 0, we have

$M = x^2 + y^2 + 2x$, $N = 2y$

$\therefore \qquad \dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 2y - 0 = 2y \; \therefore \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = 1 = f(x)$ (say)

$\therefore \qquad$ I.F. of (1) = $e^{\int f(x)dx} = e^{\int f(x)dx} = e^x$.

Multiplying (1) by $e^x$, $e^x (x^2 + y^2 + 2x)\,dx + e^x\, 2y dy = 0$ ....(2)

or $\qquad d\, e^x (x^2 + y^2) = 0$ which is exact.

Integrating, we get $e^x (x^2 + y^2) = c$ is the reqd. solution.

**Example 20:** Solve: (i) $\left(xy^2 - e^{1/x^3}\right)dx = x^2y\,dy$. (ii) $(x^2 + y^2)\,dx - 2xy\,dy = 0$

**Sol:** (i) Given equation is $\left(xy^2 - e^{1/x^3}\right)dx = x^2y\,dy$

$\Rightarrow \qquad \left(xy^2 - e^{1/x^3}\right)dx - x2ydy = 0$ [Type M dx + N dy = 0]

$\therefore \qquad \dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{2xy - (2xy)}{-x^2 y} = \dfrac{4xy}{-x^2 y} = \dfrac{-4}{x}$

$\therefore \qquad$ I.F. = $e^{\int \frac{-4}{x}dx} = e^{-4\log|x|} = e^{\log|x|^{-4}} = |x|^{-4} = \dfrac{1}{|x|^4} = \dfrac{1}{x^4}$

Multiplying (1) by I.F. i.e. $\dfrac{1}{x^4}$, we get $\left(\dfrac{xy^2 - e^{1/x^3}}{x^4}\right) dx - \dfrac{x^2 y}{x^4} dy = 0$

$\Rightarrow$ $\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)$dx - $\frac{y}{x^2}$ dy = 0, which is exact.

$\therefore$ Solution is, y constant $\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)$dx = $c_1 \Rightarrow$ $y^2\left(\frac{x^{-2}}{-2}\right)$ - $\int \frac{e^{1/x^3}}{x^4}$dx = $c_1$ .....(2)

Let I = $\int \frac{e^{1/x^3}}{x^4}$dx $\left[put\ \frac{1}{x^3} = t \Rightarrow x^{-3} = t \Rightarrow -3x^{-4}dx = dt \Rightarrow \frac{dx}{x^4} = \frac{dt}{-3}\right]$

= $\int$ $e^t \frac{dt}{-3}$ = $\left(\frac{1}{-3}\right)$ $(e^t)$ = $\frac{1}{-3}e^{1/x^3}$, putting in (2), We get

$\frac{-y^2}{2x^2}$ - $\left\{\frac{1}{-3}e^{1/x^3}\right\}$ = c1 $\Rightarrow \frac{-3y^2}{x^2}$ + 2$e^{1/x^3}$ = $ec_1 \Rightarrow$ $\frac{-3y^2}{x^2}$ + 2$e^{1/x^3}$ = c,

where $6c_1$ = c, which is the reqd. sol.

(ii) Given equation is $(x^2 + y^2)$ dx - 2xy dy = 0

Here M = $x^2 + y^2$ and N = -2xy

$\therefore$ $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ = $\frac{2y-(2y)}{-2xy}$ = $\frac{4y}{-2xy}$ = $\left(\frac{-2}{x}\right)$

$\therefore$ I.F. = $e^{\int \frac{-2}{x}dx}$ = $e^{-2\log x}$ = $e^{\log x^{-2}}$ = $x^{-2}$ = $\frac{1}{x^2}$

Multiplying (1) by I.F. i.e.. $\frac{1}{x^2}$, we get $\left(1+\frac{y^2}{x^2}\right)$dx - $\frac{2y}{x}$ dy = 0, which is exact.

$\therefore$ Solution is $\int \left(1+\frac{y^2}{x^2}\right)$dx = c

$\Rightarrow$ x + $y^2\left(\frac{x^{-1}}{-1}\right)$ = c $\Rightarrow$ x - $\frac{y^2}{x}$ = c $\Rightarrow$ x - $\frac{y^2}{x}$ = c $\Rightarrow$ $x^2 - y^2$ = cx is the reqd. solution.

**Example 21:** Find an integrating factor for

cos x cos y dx - 2 sin x sin y dy = 0 and solve it.

**Sol:** Given differential equation is

cos x cos y dx - 2 sin x sin y dy = 0 .....(1)

Comparing with Mdx + Ndy = 0, we get

M = cos x cos y, N = -2 sin x sin y

$\therefore$ $\frac{\partial M}{\partial y}$ = - cos x sin y, $\frac{\partial N}{\partial x}$ = - 2 cos x sin y

$$\therefore \quad \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-\cos x \sin y + 2\cos x \sin y}{-2\sin x \sin y}$$

$$= \frac{\cos x \sin y}{-2\sin x \sin y}$$

$$= -\frac{1}{2}\cot x = f(x)$$

I.F. = $e^{\int f(x)dx} = e^{-\frac{1}{2}\int \cot x dx} = e^{-\frac{1}{2}\log \sin x}$

$$= e^{\log(\sin x)^{-1/2}}$$

$$= (\sin x)^{-1/2}$$

$$= \frac{1}{\sqrt{\sin x}}$$

Multiplying both sides of (1) by $\frac{1}{\sqrt{\sin x}}$, we get

$$\frac{\cos x \cos y}{\sqrt{\sin x}} dx - 2\sqrt{\sin x} \sin y\, dy = 0$$

which is exact and its solution is

$$\cos y \int \frac{\cos x}{\sqrt{\sin x}} dx = c$$

or $\quad \cos y \int (\sin x)^{-1/2} \cos x\, dx = c$

or $\quad \cos y \frac{(\sin x)^{-1/2}}{1/2} = c$

or $\quad 2\cos y \sqrt{\sin x} = c$

Try the following exercises:-

Q.5 Solve the following differential equation

$(x^2 + y^2 + x)\,dx + xydy = 0,\ x > 0$

Q.6 Solve the following differential equation

$(x^4e^x - 2\,mx\,y^2)\,dx + 2\,mx^2y\,dy = 0$

Q.7 Solve the differential equation

$\left(y+\frac{y^3}{3}+\frac{x^2}{2}\right)dx + \frac{1}{4}(x + xy^2)\,dy = 0,\ x > 0$

**Rule IV.** When $\dfrac{\frac{\partial M}{\partial x}-\frac{\partial N}{\partial y}}{M}$ is a function of y alone, say f(y), then $e^{\int f(y)dy}$ is an integrating factor of the equation Mdx + Ndy = 0

**Proof:** Similar to the proof of Rule III.

**Example 22:** Solve: $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - - 4x)\,dy = 0$.

**Sol:** The given equation is $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dx = 0$

Comparing (1) with mdx + Ndy = 0, we have

$M = y^4 + 2y,\ N = xy^3 + 2y^4 - 4x$

$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (y^3 - 4) - (4y^3 + 2) = -3\,(y^3 + 2)$

$\therefore \quad \dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = -\frac{3}{y} = f(y)$ (say)

$\therefore \quad$ I.F. of (1) $= e^{\int f(y)dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3}$

Multiplying (1) by $y^{-3}$, we have

$$\left(y+\frac{2}{y^2}\right)dx + \left(x+2y-\frac{4x}{y^3}\right)dy = 0$$

$\Rightarrow \quad (ydx + xdy) + 2\left(\frac{dx}{y^2}-\frac{2x}{y^3}dy\right) + 2ydy = 0$

or $\quad d(xy) + 2d\left(\frac{x}{y^2}\right) + d(y^2) = 0$. which is exact

Integrating we get xy + $\frac{2x}{y^2}$ + $y^2$ = 0. which is exact.

Integrating, we get xy + $\frac{2x}{y^2}$ + $y^2$ = c is the required solution.

**Example 23:** Solve: (i) $(3x^2y^4 + 2xy)\, dx + (2x^3y^3 - x^2)dy = 0$

(ii) $(2xy^4e^y + 2xy^3 + y)\, dx + (x^2y^4e^y - x^2y^2 - x^2y^2 - 3x)\, dy = 0$

**Sol:** (i) Given equation is ......(1)

$(3x^2y^4 + 2xy)\, dx + (2x^3y^3 - x^2)\, dy = 0$

Here $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(6x^2y^3 - 2x) - (12x^2y^3 + 2x)}{3x^2y^4 + 2xy} = \frac{6x^2y^3 - 4x - (12x^2y^3 + 2x)}{y(3x^2y^3 + 2x)}$$

$$= \frac{-2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)} = \frac{-2}{y}$$

$\therefore$ I.F. = $e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

Multiplying (1) by I.F. i.e. $\frac{1}{y^2}$, we get, $\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(3x^3y - \frac{x^2}{y^2}\right)dy = 0$, which is exact.

$\therefore$ Solution is $\int\left(3x^2y^2 + \frac{2x}{y}\right)dx = c \Rightarrow 3y^2\left(\frac{x^3}{3}\right) + \left(\frac{2}{y}\right)\left(\frac{x^2}{2}\right) = c$

y constant

$\Rightarrow$ $x^3y^2 + \frac{x^2}{y} = c \Rightarrow x^3y^3 + x^2 = cy$, is the reqd. solution.

(ii) Given equation is

$(2xy^4ey + 2xy^3 + y)\, dx + (x^2y^4ey - x^2y^2 - 3x)\, dy = 0$ .....(1)

Here $M = 2xy^4e^y + 2xy^3 + y$ and $N = x^2y^4ey - x^2y^2 - 3x$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{(2xy^4e^y - 2xy^2 - 3) - (8xy^3e^y + 2xy^4e^y + 6xy^2 + 1)}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{(2xy^2e^y + 2xy^2 + 1)} = \frac{-4(2xy^3e^y + 2xy^2 + 1)}{(2xy^3e^y + 2xy^2 + 1)y} = \frac{-4}{y}$$

$\therefore$ I.F. = $e^{\int -\frac{-4}{y}dy}$ = $e^{-4 \log y}$ = $e^{\log y^{-4}}$ = $y^{-4}$ = $\frac{1}{y^4}$

Multiplying (1) by I.F. i.e. $\frac{1}{y^4}$, we get

$$\left[2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right]dx + \left[x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right]dy = 0$$

It is exact and so its solution is

$$\int \left[2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right]dx = c$$

y constant

$\Rightarrow$ $x^2ey + \frac{x^2}{y} + \frac{x}{y^3}$ = c, is the required solution

Dear Students, now try the following exercises:-

Q.8 Solve the differential equation

$(xy^3 + y)\,dx + 2\,(x^2y^2 + x + y^4)\,dy = 0$

Q.9 Solve the differential equation

$(x + 2y^3)\frac{dy}{dx} = y$

**Rule V.** If $\frac{a+u+1}{m} = \frac{b+v+1}{n}$, $\frac{a'+u+1}{m} = \frac{a'+u+1}{m}$

then $x^uy^v$ is an integrating factor of the equation

$x^ay^b\,(mydx + nxdy) + x^ay^b\,(m'\,ydx + n'\,xdy) = 0$ ......(1)

**Proof:** Multiplying (1) by $x^uy^v$, we have

$(mx^{a+u}y^{b+v+1}\,dx + nx^{a+u+1}y^{b+v})dy$

$+ m'x^{a'+u}y^{b'+v+1}\,dx + n'\,x^{a'+u+1}y^{b'+v}dy = 0$ ....(2)

Now $x^uy^v$ is an I.F. of (1), if (2) is exact; so both the parts of (2) must be exact.

The first part of (2) is exact if

$$\frac{\partial}{\partial y}(mx^{a+u}y^{b+v+1}) = \frac{\partial}{\partial x}(nx^{a+u+1}y^{b+v})$$

or if $m(b + v + 1) x^{a+u}y^{b+v} = n(a + u+ 1) x^{a+u}y^{b+v}$

or if $m(b +v + 1) = n(a + u + 1)$

or if $\frac{a+u+1}{m} = \frac{b+v+1}{n}$, which is true (given)

Similarly, the second part of (2) is exact if

$\frac{a+u+1}{m} = \frac{b+v+1}{n}$, which is true (given)

Q $x^u y^v$ is an integrating factor of (1)

**Example 24:** Solve: $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$

**Sol:** The given equation is $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$ .....(1)

It can be written as

$$x^2y(2ydx - xdy) + x^0y^0 (ydx + 3xdy) = 0$$

Compare with $x^ay^b (mydx + nxdy) + x^{a'}y^{b'} (m'ydx + n'xdy) = 0$

$a = 2, b = 1, m = 2, n = -1 : a' = 0, b' = 0, m' = 1, n' = 3$

$\therefore$ $\frac{a+u+1}{m} = \frac{b+v+1}{n}$ and $\frac{a'+u+1}{m'} = \frac{b'+v+1}{n'}$

$\therefore$ $\frac{u+3}{2} = \frac{2+v}{-1}$ and $\frac{u+1}{1} = \frac{v+1}{3}$

or $u + 2v = -7$ and $3u - v = -2$ $\therefore$ $u = -\frac{11}{7}$, $v = -\frac{19}{7}$

$\therefore$ the integrating factor is $x^u y^v = x^{-11/7} y^{-19/7}$

Multiplying (2) by $x^{-11/7}$, $y^{-19/7}$ we have

$$\left(2x^{\frac{3}{7}} - y^{-\frac{5}{7}}dx = x^{\frac{10}{7}}y - ^{\frac{12}{7}} dy\right) + \left(x^{-\frac{11}{7}} - y^{-\frac{12}{7}}dx = 3x^{-\frac{4}{7}}y^{-\frac{19}{7}}dy\right) = 0$$

or $\frac{7}{5} d \left(x^{\frac{10}{7}} y^{-\frac{5}{7}}\right) - \frac{7}{4} d \left(x^{-\frac{4}{7}} y^{-\frac{12}{7}}\right) = 0$

Integrating we get, $\frac{7}{5} x^{-\frac{10}{7}} y^{-\frac{5}{7}-\frac{7}{4}} x^{-\frac{4}{7}} y^{-\frac{12}{7}}$ = c as the reqd. solution.

**Example 25:** Solve: $(20x^2 + 8xy + 4y^2 + 3y^3) ydx + 4 (x^2 + xy + y^2 + y^3) xdy = 0$

**Sol:** Here M = $20x^2y + 8xy^2 + 4y^3 + 3y^4$ and N = $4x^3 + 4x^2y + 4y^2x + 4y^3x$

$\therefore \quad \frac{\partial M}{\partial y}$ = $20x^2 + + 16xy + 12y^2 + 12y^3$ and $\frac{\partial N}{\partial x}$ = $12x^2 + 8xy + 4y^2 + 4y^3$

$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Given equation is not exact.

Rewrite given equation as

$x2y^0$ (20ydx + 4xdy) + xy(8ydx + 4xdy) + $y^2x^0$ (4ydx + 4xdy) + $y^3x^0$ (3ydx + 4xdy) = 0

Compare it with

$x^ay^b$ [mydx + nxdy] + $x^cy^d$ [pydx + qxdy] + $x^ey^f$ [rydx + sxdy] + $x^gy^h$ [kydx + [kydx + lxdy] = 0

$\therefore$ a = 2, b = 0, m = 20, n = 4; c = 1, d = 1, p = 8, q = 4;

e = 0, f = 2, r = 4, s = 4; g = 0, h = 3, k = 3, l = 4

Let $x^u.y^v$ be I.F.

$$\therefore \quad \frac{a+u+1}{m} = \frac{b+v+1}{n}; \frac{c+1+u}{p} = \frac{d+1+v}{q};$$

$$\frac{e+1+u}{r} = \frac{f+1+v}{s}; \frac{g+1+u}{k} = \frac{h+1+v}{l}$$

$$\Rightarrow \quad \frac{30+u}{20} = \frac{1+v}{4}; \frac{2+u}{8} = \frac{2+v}{4}; \frac{1+u}{4} = \frac{3+v}{4}; \frac{1+u}{3} = \frac{4+v}{4}$$

$\Rightarrow$ 3 + u = 5 + 5v; 2 + u = 4 + 2v, 1 + u = 3 + v, 4 + 4u = 12 + 3v

$\Rightarrow$ u = 2 + 5v;u = 2 + v;u = 2 + v, 4u = 8 + 3v

$\Rightarrow$ u = 2 and v = 0

$\Rightarrow$ I.F. = $x^u$. $y^v$ = $x^2y^0$ = $x^2$

Multiply given equation by I.F. = $x^2$, we get

$(20x^4y + 8x^3y^2 + 4x^2y^3 + 3x^2y^4)$ dx + $(4x^5 + 4x^4y + 4x^3y^2)$ dy = 0

It is now exact equation and therefore its solution is

$\int (20x^4y + 8x^3y^2 + 4x^2y^3 + 3x^2y^4)dx = c$

y constant

$$\Rightarrow \quad \frac{20x^5y}{5} + \frac{8x^4y^2}{4} + \frac{4x^2y^3}{3} + \frac{3x^3y^4}{3} = c \Rightarrow 4x^5y + 2x^4y^2 + \frac{4}{3}x^3y^3 + x^3y^4 = c$$

Now, try the following exercises:-

Q.10 Solve the differential equation

$(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

Q.11 Solve the differential equation

$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

## 4.7 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined integrating factor
2. Discussed the number of integrating factors of a differential equation.
3. Discussed integrating factors found by inspection.
4. Discussed in detail different rules for finding the integrating factors and give examples in support of each rule.

## 4.8 Glossary:

1. If the differential equation

   M (x, y) dx + n (x, y) dy = 0

   is not exact in a domain D but the differential equation

   $\mu$ (x, y) M (x, y) dx + $\mu$ (x, y) N (x, y) dy = 0

   is exact in D, then $\mu$ (x, y) is called an integrating factor of the differential equation.

2. If the equation Mdx + Ndy = 0 is homogeneous in x and y, then $\dfrac{1}{Mx+Ny}$ is an integrating factor provided Mx + Ny $\neq$ 0

3. If the equation Mdx + Ndy = 0 is of the form y$f$(x, y) dx + xg (x, y) dy = 0, then $\dfrac{1}{Mx+Ny}$ is an integrating factor provided Mx - Ny $\neq$ 0

4. When $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f$(x), then $e^{\int f(x)dx}$ is an integrating factor of equation Mdx + Ndy = 0

## 4.9 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 I.F. = $\dfrac{1}{xy}$ and solution is x = cye$^{xy}$

Ans.2 I.F. = $\frac{1}{x^2y^2}$ and solution is log ($x^2$ + $y^2$) + $2y^4$ = c,

where c is an arbitrary constant.

Ans.3 I.F. = $\frac{1}{x^2}$ and solution is y = 1 - cx.

Ans.4 I.F. = $\frac{1}{x^2}$ and solution is $\frac{-y}{x}$ = $\frac{1}{x}$ + x - cos y = c.

Ans.5 I.F. = $\frac{1}{y^2}$ and solution is $\frac{x}{y}$ + log y $\frac{-y^2}{2}$ = c

**Self-Check Exercise-2**

Ans.1 $x^2$ - $y^2$ = cx is the required solution.

Hint: Given equation is not exact but it is homogeneous in x and y. $\frac{1}{Mx+Ny}$ i.e. $\frac{1}{x(x^2-y^2)}$ is an I.F.

Ans.2 3 log |x| + $\frac{y}{x}$ - 2 log |y| = c is the required solution

Hint: Given differential equation is homogeneous in x and y. $\frac{1}{Mx+Ny}$ i.e. - $\frac{1}{x^2y^2}$ is an I.F.

Ans.3 Log |x| - log |y| - $\frac{1}{xy}$ = c, where $2c_1$ = c is the required solution.

Hint: I.F. = $\frac{1}{Mx+Ny}$ = $\frac{1}{2x^2y^2}$

Ans.4 xy + log $\left|\frac{x}{y}\right|$ - $\frac{1}{xy}$ = c, si the required solution.

Hint: I.F. = $\frac{1}{Mx+Ny}$ = $\frac{1}{2x^2y^2}$

Ans.5 $\frac{x^4}{4}$ + $\frac{x^2y^2}{2}$ + $\frac{x^3}{3}$ = c is the required solution

Hint: I.F. is x

Ans.6 $e^x + m\frac{y^2}{x^2} = c$ is the required solution

Hint: I.F. is $\frac{1}{x^4}$

Ans.7 $3x^4y + x^4y^3 + x^6 = c$, where c = 12c' is the required solution

Hint: I.F. is $x^3$

Ans.8 $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$, is the required solution

Hint: I.F. is $|y|$

Ans.9 $x - y^3 = cy$, is the required solution.

Hint: I.F. is $\frac{1}{y^2}$

Ans.10 $-\frac{1}{xy} + 2\log|x| - \log|y+| = c$, is the required solution

Hint: I.F. is $\frac{1}{x^3y^3}$

Ans.11 $x^3y^2 + \frac{x^2}{y} = c$, is the required solution

Hint: I.f. is $\frac{1}{y^2}$

## 4.10 References/Suggested Readings


1. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
2. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.


## 4.11 Terminal Questions

1. Find the integrating factor by inspection and hence solve the following differential equation

   $(x + x^4 + 2x^2y^2 + y^4)\,dx + y\,dy = 0$

2. Find the integrating factor for

   $\cos x \cos y\,dx - 4\sin x \sin y\,dy = 0$

   and solve it.

3. Find the integrating factor of the equation

$xdy - ydx = x(1 - 2x^2)\, dx, x > 0$

by inspection method and hence solve it.

4. Solve the following differential equation

$(y^3 - 2yx^2)\, dx + (2xy^2 - x^3)\, dy = 0$

5. Solve the differential equation

$y(2xy + 1)\, dx + x(1 + 2xy - x^3y^3)\, dy = 0$

6. Solve the differential equation

$(x^3y^4 + x^2y^3 + xy^2 + y)\, dx + (x^4y^3 - x^3y^2 + x^2y + x)\, dy = 0$

7. Solve the following differential equation

$(e^y + xe^y)\, dx + xe^y dy = 0$

8. Solve the differential eqution

$(x^3 + xy^4)\, dx + 2y^3dy = 0$

9. Solve the differential equation

$y(x + y + 1)\, dx + x(x + 3y + 2)\, dx = 0, y > 0$

10. Solve the differential equation

$(20x^2 + 8xy + 4y^2 + 3y^3)\, y\, dx + 4(x^2 + xy + y^2 + y^3)\, x\, dy = 0$

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## Unit - 5

# Equations of First Order and Higher Degree

**Structure**



## 5.1 Introduction

A differential equation is an equation that relates an unknown function to its derivatives. It expresses how the rate of change of the function depends on the values of the function itself and its derivatives. The order of a differential equation is determined by the highest derivative that appears in the equation.

A differential equation of form $f\left(x, y, \frac{dy}{dx}\right) = 0$, where degree of $\frac{dy}{dx}$ is greater than 1 is called a non-linear differential equation of first order and higher degree. For convenience, we shall write $\frac{dy}{dx} = p$. The general form of the first order and $n^{th}$ degree equation can be written as

$$p^n + P_1(x, y)\, p^{n-1} + P_2(x, y)\, p^{n-2} + \ldots\ldots + P_{n-1}(x, y)\, p + P_n(x, y) = 0$$

There is no general method for solving first-order non-linear differential equations. In fact, the determination of such solutions is often difficult if not impossible. In the present unit, we shall discuss a few techniques for finding solutions of certain particular types of non-linear equation of first order and higher degree.

## 5.2 Learning Objectives

After studying this unit, you should be able to:-

- Define equation of first order and higher degree.
- Discuss and find solutions of equations solvable for p
- Discuss and find solutions of equations solvable for y
- Discuss and find solutions of equations solvable for x
- Discuss and find solutions of equations not containing x
- Discuss and find solutions of equations not containing y
- Discuss and find solution of clairaut equation

## 5.3 Equations Solvable Fro p

Let $f(x, y, p) = 0$ ......(1)

be the given differential equation of the first order and degree n (> 1)

Since it is solvable for p,

$\therefore$ $(p - f_1)(p - f_2) \ldots\ldots\ldots (p - f_n) = 0$

where $f_1, f_2, \ldots\ldots, f_n$ are functions of x and y.

$\therefore$ Equation (1) reduces to a problem of solving n differential equations of first order and first degree, viz.

$\frac{dy}{dx} = f_1(x, y)$, $\frac{dy}{dx} = f_2(x, y)$, ........., $\frac{dy}{dx} = f_n(x, y)$, which can be solved by the methods already known to us

Let the solutions of these n first degree equations be

$f_1(x, y, c_1) = 0$, $f_2(x, y, c_2) = 0$, ........, $F_n(x, y, c_n) = 0$

Since the given equations is of first order, therefore, it cannot have more than one independent arbitrary constant.

Let $c1 = c2 = \ldots\ldots\ldots = cn = c$ (say)

Hence the general solution of (1) is the product $F_1(x, y, c)\, F_2(x, y, c) \ldots\ldots\ldots F_n(x, y, c) = 0$ of n solutions.

**Method to Solve**

(i) Resolve the given equation into linear factor of p.

(ii) Equate each factor to zero, which will give a differential equation

(iii) Combine these equations to get the required solution

Let us look at some examples:-

**Example 1:** Solve: $p^2 + p - 6 = 0$

**Sol:** The given equation is

$p^2 + p - 6 = 0$ or $(p - 2)(p - 3) = 0 \therefore p = 2, -3$

i.e. $\frac{dy}{dx} = 2$ and $\frac{dy}{dx} = -3$

Integrating, we get $y = 2x + c$ and $y = -3x + c$

or $2x - y + c = 0$ and $3x + y - c = 0$

Hence the complete solution is $(2x - y + c)(3x + y - c) = 0$

**Example 2:** Solve: $p^2 - 5p + 6 = 0$

**Sol:** $p^2 - 5p + 6 = 0 \Rightarrow (p - 2)\, p - 3) = 0 \Rightarrow p = 2, 3$

$\Rightarrow \frac{dy}{dx} = 2$ and $\frac{dy}{dx} = 3$

On integration, we get $y = 2x + c$ and $y = 3x + c \Rightarrow (y - 2x - c) = 0$ and $(y - 3x - c) = 0$

Hence the complete solution is $(y - 2x - c)(y - 3x - c) = 0$

**Example 3:** Solve: $p^2 - (x + y)p + xy = 0$

**Sol:** Given equation is $p^2 - (x + y)p + xy = 0 \Rightarrow (p - x)(p - y) = 0 \Rightarrow p = x, y$

$\Rightarrow \frac{dy}{dx} = x$ and $\frac{dy}{dx} = y \Rightarrow dy = xdx$ and $\frac{dy}{y} = dx$

On integration we get, $y = \frac{x^2}{2} + c_1$ and $\log y = x + c_1$

$\Rightarrow 2y = x^2 + 2c_1$ and $y = e^{x+c_1} = ex.\ e^{c_1} \Rightarrow 2y = x^2 + c$ and $y = e^x.\, c$

$\Rightarrow 2y - x^2 - c = 0$ and $y - ce^x = 0$

Hence the complete solution is $(2y - x^2 - c)(y - ce^x) - 0$

**Example 4:** Solve: $x^2p^2 + xyp - 6y^2 = 0$

**Sol:** Given equation is $x^2 p^2 + xyp - 6y^2 = 0$

$\Rightarrow p = \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2} = \frac{-xy \pm 5xy}{2x^2} = \frac{4xy}{2x^2}, \frac{-6xy}{2x^2} = \frac{2y}{x}, \frac{-3y}{x}$

$\Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ and $\frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{y} = \frac{2dx}{x}$ and $\frac{dy}{y} = -\frac{3dx}{x}$ ]

On integration, we get,

$\log y = 2 \log x + \log c$ and $\log y \ -3 \log x + \log c$

$\Rightarrow \quad \log y = \log cx^2$ and $\log y = \log cx^{-3}$

$\Rightarrow \quad y = cx^2$ and $y = cx^{-3} \Rightarrow y = cx^2$ and $yx^3 = c$

$\Rightarrow \quad y - cx^2 = 0$ and $yx^3 - c = 0$

Hence the complete solution is $(y - cx^2)(yx^3 - c) = 0$

**Example 5:** Solve: $p^3 + 3xp^2 - y^2p^2 - 3xy^2p = 0$

**Sol:** Given equation can be rewritten as

$p[p^2 + 3px - py^2 + 3xy^2] = 0 \Rightarrow p(p + 3x)(p - y^2) = 0$

$\Rightarrow \quad p = 0,\ p + 3x = 0$ and $p - y^2 = 0 \Rightarrow \frac{dy}{dx} = 0,\ \frac{dy}{dx} = -3x$ and $\frac{dy}{dx} = y^2$

$\Rightarrow \quad y = c \int dy = -3\int xdx + c, \int \frac{dy}{y^2} = \int dx + c$

$\Rightarrow \quad y = c,\ y = -3\frac{x^2}{2} + c,\ \frac{-1}{y} = x + c$

$\therefore$ the required solution is $(y - c)(2y + 3x^2 - c)(y^{-1} + x + c) = 0$

**Example 6:** Solve: $y - \left(1+p^2\right)^{-\frac{1}{2}} = b$.

**Sol:** $y - \left(1+p^2\right)^{-\frac{1}{2}} = b$

$\Rightarrow \quad (y - b)^2 = ((1 + p^2)^{-1} = \frac{1}{1+p^2}$

$\Rightarrow \quad (y - b)^2 + p^2 (y - b)^2 = 0$

$\Rightarrow \quad p (y - b)^2 = 1 - (y - b)^2$

$\Rightarrow \quad p^2 = \frac{1-(y-b)^2}{(y-b)^2}$ [solvable for p]

$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{1-(y-b)^2}}{y-b}$

$\Rightarrow \quad \int \frac{y-b}{\sqrt{1-(y-b)^2}} dy = \int dx + c$

Put $\quad y - b = \sin\phi \Rightarrow dy = \cos\phi\, d\phi$

$\therefore \quad \int \frac{\sin\phi.\cos\phi d\phi}{\sqrt{1-\sin^2\phi}}$ = x + c $\Rightarrow \int$ sin $\phi$ d$\phi$ = x + c

$\Rightarrow$ - cos $\phi$ = x + c $\Rightarrow$ - $\sqrt{1-(y-b)^2}$ = x + c

$\Rightarrow$ 1 - (y - b)$^2$ = (x + c)$^2$ $\Rightarrow$ (x + c)$^2$ + (y - b)$^2$ = 1, is the complete solution, where c is an arbitrary constant.

**Example 7:** Solve: $p^2 - 2px + x^2 - y^2 = 0$

**Sol:** Given equation is $p^2 - 2px + x^2 - y^2 = 0$ ......(1)

$\Rightarrow$ p = $\frac{2x \pm \sqrt{4x^2 - 4(x^2-y^2)}}{2}$ = x $\underline{+}$ y $\Rightarrow$ p = x + y, x - y

Now p = x + y

$\Rightarrow \quad \frac{dy}{dx}$ + (-1)y = x

I.F. = $e^{\int -1dx}$ = $e^{-x}$

$\therefore$ solution is

y. $e^{-x}$ = $\int$ x. $e^{-x}$ dx = c

$\Rightarrow \quad \frac{y}{e^x} = x\left(\frac{e^{-x}}{-1}\right) - \int \frac{e^{-x}}{-1}$ dx + c

$\Rightarrow \quad \frac{y}{e^x} = \frac{-x}{e^x} + \frac{e^{-x}}{-1}$ + c

$\Rightarrow$ y = -x - 1 + ce$^x$

$\Rightarrow$ y - ce$^x$ + x + 1 = 0

p = x - y

$\Rightarrow \quad \frac{dy}{dx}$ + 1y = x

I.F. = $e^{\int 1dx}$ = $e^x$

$\therefore$ solution is

$y.\, e^x = \int x.\, e^x\, dx + c$

$\Rightarrow$ $y.\, e^x = x.\, e^x - \int e^x\, dx + c$

$\Rightarrow$ $ye^x = xe^x - e^x + c$

$\Rightarrow$ $y = x - 1 + ce^{-x}$

$\Rightarrow$ $y - ce^{-x} - x + 1 = 0$

$\therefore$ complete solution is $(y - ce^x + x + 1)(y - ce^{-x} - x + 1) = 0$

**Example 8:** Solve: $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

**Sol:** Given equation is $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

$\Rightarrow$ $p(p^2 + 2xp - y^2p - 2xy^2) = 0 \Rightarrow p[p(p + 2x) - y^2 (p + 2x)] = 0$

$\Rightarrow$ $p(p + 2x)(p - y^2) = 0 \Rightarrow p = 0, -2x, y^2$

$\Rightarrow$ $\frac{dy}{dx} = 0, -2x, y^2 \Rightarrow \frac{dy}{dx} = 0,\ dy = -2xdx,\ \frac{dy}{y^2} = dx$

$\Rightarrow$ On integration, $y = c$, $y = -x^2 + c$, $\frac{-1}{y} = x + c$

$\Rightarrow$ $y - c = 0,\ y + x^2 - c = 0,\ xy + cy + 1 = 0$

$\therefore$ Complete solution of (1) is $(y - c)(y + x^2 - c)(xy + cy + 1) = 0$

**Example 9:** Solve the following differential equation

$yp^2 + (x - y)\, p = x$

**Sol:** Given differential equation is

$yp^2 + (x - y)\, p = x$

or $py(p - 1) + x(p - 1) = 0$

or $(py + x)(p - 1) = 0$

$\therefore$ $py + x = 0$ and $p = 1$

or $y\frac{dy}{dx} + x = 0$ and $\frac{dy}{dx} = 1$

or $ydy + xdx = 0$ and $dy = dx$

Integrating, we get

$\frac{y^2}{2} + \frac{x^2}{2} = \frac{c}{2}$ (say) and $y = x + c$

$\therefore$ $(x^2 + y^2 - c)(y - x - c) = 0$ is the required solution

**Example 10:** Solve the following differential equations:-

(i) $p^2 + 2py \cot x = y^2$

(ii) $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0$

**Sol:** (i) Given differential equation is

$$p^2 + 2py \cot x = y^2 \quad .......(1)$$

or $p^2 + 2py \cot x - y^2 = 0$

$$\therefore \quad p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$\Rightarrow \quad p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$\Rightarrow$ $p = -y \cot x \pm y \operatorname{cosec} x$

$\therefore$ its component equation are

$$p = -y \cot x + y \operatorname{cosec} x \quad .....(2)$$

and $p = -y \cot x - y \operatorname{cosec} x \quad ......(3)$

Equation (2) can be written as

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

Separating the variables, we get

$$\frac{1}{y}\,dy = (-\cot x + \operatorname{cosec} x)\,dx$$

or $\frac{1}{y}\,dy = \left[-\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right]dx$

or $\frac{1}{y}\,dy = \frac{1-\cos x}{\sin x}\,dx$

or $\frac{1}{y}\,dy = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$

or $\quad \frac{1}{y}$ dy = $\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$ dx

or $\quad \frac{1}{y}$ dy = tan $\frac{x}{2}$ dx

Integrating both sides, we get

$$\log|y| = -\frac{\log\left|\cos\frac{x}{2}\right|}{\frac{1}{2}} + \log c$$

or $\quad \log|y+| + 2\log\left|\cos\frac{x}{2}\right| = \log c$

or $\quad \log|y| + \log\left|\cos^2\frac{x}{2}\right| = \log c$

or $\quad \log\left|y\cos^2\frac{x}{2}\right| = \log c$

or $\quad \left|y\cos^2\frac{x}{2}\right| = c$

or $\quad y\cos^2\frac{x}{2} = \pm c$

or $\quad y\cos^2\frac{x}{2} = A$, where $A = \pm c$

$\therefore$ Solution of (2) is

$$y\cos^2\frac{x}{2} = A$$

Equation (3) can be written as

$$\frac{dy}{dx} = -y\cot x - y\operatorname{cosec} x$$

or $\quad \frac{dy}{dx} = -y\left[\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right]$

Separating the variables, we get

$$\frac{1}{y}\,dy = -\,\frac{1+\cos x}{\sin x}\,dx$$

or $$\frac{1}{y}\,dy = -\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx$$

Integrating, we get

$$\int\frac{1}{y}\,dy = -\int\cot\frac{x}{2}\,dx$$

or $$\log|y| = -2\log\left|\sin\frac{x}{2}\right| + \log c'$$

or $$\log|y| + \log\left|\sin^2\frac{x}{2}\right| = \log c'$$

$\therefore$ $$\left|y\sin^2\frac{x}{2}\right| = c'$$

$\therefore$ $$y\sin^2\frac{x}{2} = +\,c'$$

$\therefore$ $$y\sin^2\frac{x}{2} = A$$

$\therefore$ general solution of (1) is

$$\left|y\cos^2\frac{x}{2}-A\right|\left|y\sin^2\frac{x}{2}-A\right| = 0$$

**(ii) Given equation is**

$$x^2p^2 + xyp - 6y^2 = 0, \text{ where } p = \frac{dy}{dx}$$

or $(xp + 3y)(xp - 2y) = 0$

$\Rightarrow$ $xp + 3y = 0$ or $xp - 2y = 0$

Now $xp + 3y = 0$ $\Rightarrow$ $x\frac{dy}{dx} + 3y = 0$

or $\dfrac{dy}{y} + 3\,\dfrac{dx}{x} = 0$

Integrating, we get,

$\log y + 3 \log x = \log c$

or $x^3y = c$

Also $xp - 2y = 0 \quad \Rightarrow \quad x\,\dfrac{dy}{dx} - 2y = 0$

or $\dfrac{dy}{y} - 2\,\dfrac{dx}{x} = 0$

Integrating, we get

$\log y - 2 \log x = \log c$

or $\dfrac{y}{x^2} = c$ or $y = c\,x^2$

$\therefore$ general solution of the given equation is

$(x^3y - c)(y - cx^2) = 0$

**Self-Check Exercise-1**

Q.1 Solve the following differential equation

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

Q.2 Solve $x^2 = p^2 (a^2 - x^2)$

Q.3 Solve $p^3 (x + 2y) + 3p^2 (x + y) + (y + 2x) p = 0$

Q.4 Solve $p^3x^2 + p^2y + p^2x^2y^2 + py^3 = 0$

Q.5 Solve $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

## 5.4 Equations Solvable For y

Let the given differential equation be

$f(x, y, p) = 0$ .....(1)

Since (1) is solvable for y,

$\therefore$ (1) can be expressed as

$y = g(x, p)$ .....(2)

Differentiating (2) w.r.t. x,

$$\frac{dy}{dx} = p = h\left(x, p, \frac{dp}{dx}\right) \qquad .....(3)$$

which is an equation in two variables x and p.

Integrating (3), let its solution be F(x, y, c) = 0 .....(4)

The elimination of p between (2) and (4) gives the general solution of (1)

If the elimination of p between (2) and (4) is not possible, the values of x and y may be obtained in terms of the parameter p; say

$$x = f_1(p, c),\ y = f_2(p, c)$$

These two equations together constitute the solution of (1) in the parametric form.

## 5.5 Equations Solvable For x

Let the given differential equation be

$$f(x, y, p) = 0 \qquad .....(1)$$

Since (1) is solvable for x,

∴ (1) can be expressed as

$$x = g(y, p) \qquad ......(2)$$

Differentiating (2) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{p} = h\left(y, p, \frac{dp}{dy}\right) \qquad ......(3)$$

which is an equation in two variables y and p.

Integrating (3), let its solution be F(x, y) = 0 .....(4)

The elimination of p between (2) and (4) gives the general solution of (1)

If the elimination of p between (2) and (4) gives the general solution of (1)

If the elimination of p between (2) and (4) is not possible, the values of x and y may be obtained in terms of the parameter p; say

$$x = f^1(p, c),\ y = f^2(p, c)$$

These two equations together constitute the solution of (1) in parametric form.

**Example 11:** Solve $y = p^2x + p$

**Sol:** Given equation is

$$y = p^2x + p \qquad ..... (1)$$

Differentiating w.r.t. x, we have

$\frac{dy}{dx}$ = $p^2 + 2xp\ \frac{dp}{dx} + \frac{dp}{dx}$

or $\quad p = p^2 + 2xp\ \frac{dp}{dx} + \frac{dp}{dx}$

or $\quad p(1 - p) = 2xp\ \frac{dp}{dx} + \frac{dp}{dx}$

$\qquad = (2xp + 1)\ \frac{dp}{dx}$

or $\quad \frac{dx}{dp} . p(1 - p) = 2xp + 1$ or $\frac{dx}{dp} - \frac{2x}{1-p} = \frac{1}{p(1-p)}$, which is linear in x

Here $P = \frac{-2}{1-p}$ , $Q = \frac{1}{p(1-p)}$

$\therefore \quad e^{\int P dp} = e^{-\int \frac{2}{1-p} dp} = e^{2 \log (1-p)} = e^{\log (1-p)^2} = (1 - p)^2$

$\therefore \quad$ Sol is $x (1 - p)^2 = \int \frac{1}{p(1-p)} . (1 - p)^2 dp + c = \int \frac{1-p}{p} dp + c = \int \left(\frac{1}{p} - 1\right) dp + c$

i.e. $(p - 1)^2 x = \log p - p + c$ ...(2)

$\therefore \quad$ from (1), $\quad y = p^2\left[\frac{\log p}{(p-1)^2} - \frac{p}{(p-1)^2} + \frac{c}{(c-1)^2}\right] + p$

or $\quad (p - 1)^2 y = p^2 [\log p - p + c] + p(p - 1)^2 = p^2 [\log p - p + c + p - 2] + p$

i.e. $\quad (p - 1)^2 y = p^2 [c - 2 + \log p] + p$ ...(3)

Hence (2) and (3) give the solution

**Example 12 :** Solve : $y + px = x^4 p^2$

**Sol.** The given equation is $y = -px + x^4 p^2$

Differentiate w.r.t., $\frac{dy}{dx} = -p - x\ \frac{dp}{dx} + x^4\left(2p\frac{dp}{dx}\right) + p^2(4x^3)$

i.e. $\quad p = -p - x\ \frac{dp}{dx} + 2px^4 \frac{dp}{dx} + 4p^2x^3$

or $\quad 2p + x\frac{dp}{dx} - 2px^3\left(x\frac{dp}{dx} + 2p\right) = 0$ or $\left(2p + x\frac{dp}{dx}\right)(1 - 2px^3) = 0$

$\therefore \quad 2p + x\dfrac{dp}{dx} = 0$ or $(1 - 2px^3) = 0$

Now $\quad 2p + x\dfrac{dp}{dx} = 0$ or $2\dfrac{dx}{x} + \dfrac{dp}{p} = 0$

Integrating, 2 log x + log p = log c or log $px^2$ = log c or $px^2 = c$ $\therefore$ $p = \dfrac{c}{x^2} + c^2$

Putting in (1), we get $y = -x\left(\dfrac{c}{x^2}\right) + x^4\left(\dfrac{c^2}{x^4}\right)$ or $y = \dfrac{c}{x} + c^2$

or $xy = -c + c^2 x$, which is the reqd. solution.

**Example 13.** Solve : $y = x + p^3$

**Sol.** The given equation is $y = x + p^3$

Differentiate w.r.t. x, $\dfrac{dy}{dx} = 1 + 3p^2\dfrac{dp}{dx}$ i.e. $p = 1 + 3p^2\dfrac{dp}{dx}$

or $\dfrac{p-1}{3p^2} = \dfrac{dp}{dx}$ or $dx = \dfrac{3p^2}{p-1}dp$

Integrating we get

$x = 3\displaystyle\int\frac{p^2}{p-1}dp + c = 3\int\frac{p^2-1+1}{p-1}dp + c$ or $x = 3\displaystyle\int\left(p+1+\frac{1}{p-1}\right)dp + c$

or $x = \dfrac{3}{2}p^2 + 3p + 3\log(p-1) + c$ ...(2)

From (1), $y = p^3 + \dfrac{3}{2}p^2 + 3p + 3\log(p-1) + c$ ...(3)

Hence (2) and (3) give the solution.

**Example 14.** Solve : $y = 2px - xp^2$

**Sol.** Given equation is $y = 2px - xp^2$ ...(1)

Differentiate (1) w.r.t. x, we get $\dfrac{dy}{dx} = \left(2p + 2x\dfrac{dp}{dx}\right) - \left\{x.2p\dfrac{dp}{dx} + p^2\right\}$

$\Rightarrow \quad p = 2p - p^2 + 2x\dfrac{dp}{dx}(1-p) \Rightarrow -p(1-p) = 2x(1-p)\dfrac{dp}{dx}$

$\Rightarrow$ -p = 2x $\frac{dp}{dx} \Rightarrow \frac{2dp}{p} = \frac{-dx}{x} \Rightarrow$ 2 log p = 1 log x + log c

$\Rightarrow$ log $p^2$ = log $\frac{c}{x} \Rightarrow p^2 = \frac{c}{x} \Rightarrow$ p = $\sqrt{\frac{c}{x}}$ ...(2)

Using (2) in (1), we get y = 2 $\sqrt{\frac{c}{x}}$ .x - x $\left(\frac{c}{x}\right) \Rightarrow$ y = $2\sqrt{cx}$ - c $\Rightarrow$ y + c = $2\sqrt{cx}$

$\Rightarrow$ (y + c)2 = 4cx, which is read solution.

**Example 15.** Solve : x - yp = $ap^2$

**Sol.** Given equation is x - yp = $ap^2$

$\Rightarrow$ y = $\frac{x - ap^2}{p}$, Differentiate w.r.t.x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{p\left[1 - 2ap\frac{dp}{dx}\right] - (x - ap^2).\frac{dp}{dx}}{p^2}$$

$\Rightarrow$ $p^2\frac{dy}{dx}$ = p - $2ap^2\frac{dp}{dx}$ - x $\frac{dp}{dx}$ + $ap^2\frac{dp}{dx} \Rightarrow p^3$ - p = - $\frac{dp}{dx}$ [x + $ap^2$]

$$\Rightarrow \quad \frac{dp}{dx} = \frac{p - p^3}{x + ap^2} \Rightarrow \frac{dx}{dp} = \frac{x + ap^2}{p(1 - p^2)}$$

$$\Rightarrow \quad \frac{dx}{dp} = \frac{x}{p(1 - p^2)} + \frac{ap^2}{p(1 - p^2)}$$

$$\Rightarrow \quad \frac{dx}{dp} + \left\{\frac{-1}{p(1 - p^2)}\right\} x = \frac{ap}{1 - p^2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{-1}{p(1-p^2)}dp} = e^{\int \frac{-1}{p(1-p)(1+p)}dp}$$

$$= e^{\int\left[\frac{-1}{p} - \frac{1}{2(1-p)} + \frac{1}{2(1+p)}\right]dp} = e^{\left[-\log p + \frac{1}{2}\log(1-p) + \frac{1}{2}\log(1+p)\right]}$$

$$= e^{\left[\log\sqrt{1-p} + \log\sqrt{1+p} - \log p\right]} = e^{\left[\log\frac{\sqrt{1-p}\sqrt{1+p}}{p}\right]} = \frac{\sqrt{1-p^2}}{p}$$

$\therefore$ solution of (2) is

$(x)\left(\frac{\sqrt{1-p^2}}{p}\right) = \int \frac{ap}{1-p^2} \cdot \frac{\sqrt{1-p^2}}{p}\,dp + c = \int \frac{a}{\sqrt{1-p^2}}\,dp + c$

$\Rightarrow \frac{x\sqrt{1-p^2}}{p} = a \sin^{-1} p + c \Rightarrow x\sqrt{1-p^2} = (a \sin^{-1} p + c)$

$\Rightarrow \quad x = \left(\frac{p}{\sqrt{1-p^2}}\right)(a \sin^{-1} p + c)$ ...(3)

(1) Can be written as $y = \frac{x}{p} - ap$

$\Rightarrow y = -ap + \frac{1}{p} \cdot \left(\frac{p}{\sqrt{1-p^2}}\right)(a \sin^{-1} p + c)$ [using (3)]

$\Rightarrow y = -ap + \frac{1}{\sqrt{1-p^2}}(a \sin^{-1} p + c)$ ...(4)

(3) and (4) represent the solution of given equation

**Example 16 :** Solve : y = 3x + log p

**Sol.** Given equation is y = 3x + log p

$\Rightarrow \quad \frac{dy}{dx} = 3 + \frac{1}{p}\frac{dp}{dx} \Rightarrow p = 3 + \frac{1}{p}\frac{dp}{dx} \Rightarrow (p - 3) = \frac{1}{p}\frac{dp}{dx}$

$\Rightarrow \quad p(p - 3) = \frac{dp}{dx} \Rightarrow dx = \frac{dp}{p(p-3)}$

$\Rightarrow \quad \int 1dx = \int \frac{1}{p(p-3)}\,dp = \int\left[\frac{1}{3(p-3)} - \frac{1}{3p}\right]$

$\Rightarrow \quad x = \frac{1}{3}\log(p - 3) - \frac{1}{3}\log p - \frac{1}{3}\log c \Rightarrow 3x = \log\left(\frac{p-3}{cp}\right) \Rightarrow e^{3x} = \frac{p-3}{cp}$

$\Rightarrow \quad ce^{3x} = 1 - \frac{3}{p} \Rightarrow \frac{3}{p} = 1 - c^{3x}$

$\Rightarrow \quad p = \frac{3}{1-ce^{3x}}$; putting in (1), we get y = 3x log $\left(\frac{3}{1-ce^{3x}}\right)$, is the reqd. solution.

**Example 17 :** Solve x = 4p + 4p$^3$

**Sol.** the given equation is x = 4p + 4p$^3$ ...(1)

Differentiate w.r.t.y, $\frac{dx}{dy}$ = 4 $\frac{dp}{dy}$ + 12p$^2$ $\frac{dp}{dy}$ i.e. $\frac{1}{p}$ = 4 $\frac{dp}{dy}$ + 12p$^2$ $\frac{dp}{dy}$

or dy = (4p + 12p$^3$) dp

integrating, y = 2p$^2$ + 3p$^4$ + c ...(2)

Hence (1) and (2) give the solution.

**Example 18.**Solve : p = tan $\left(x-\frac{p}{1+p^2}\right)$

**Sol.** The given equation can be written as tan$^{-1}$ p = x - $\frac{p}{1+p^2}$

$\therefore$ x = $\frac{p}{1+p^2}$ + tan$^{-1}$ p

Differentiate w.r.t. y, $\frac{dy}{dx}$ = $\left[\frac{(1+p^2)-2p^2}{(1+p^2)^2}+\frac{1}{1+p^2}\right]\frac{dp}{dy}$

or $\frac{1}{p}$ = $\frac{1-p^2+1+p^2}{(1+p^2)^2}$ . $\frac{dp}{dy}$ or dy = $\frac{2p}{(1+p^2)^2}$ dp

Integrating, y = $\int(1+p^2)^{-2}(2p)dp+c$

or y = $\frac{(1+p^2)^{-1}}{-1}$ + c or y c - $\frac{1}{1+p^2}$ ....(2)

Hence (1) and (2) give the solution.

**Example 19.** Solve x = y p$^2$

**Sol.** $\frac{dy}{dx}$ = 1 + 2p $\frac{dp}{dy}\Rightarrow\frac{1}{p}$ - 1 = 2p $\frac{dp}{dy}\Rightarrow\frac{2p^2}{1-p}$ dp = dy

$\Rightarrow\int dy$ = $\int 2\left[-p-1+\frac{1}{1-p}\right]$ dp

$\Rightarrow$ y $= 2\left[\frac{-p^2}{2} - p - \log|1-p|\right]$ + c; putting it in given differential equation.

We get, x = c - [$p^2$ + 2p + log |1 - p|] + $p^2$

$\Rightarrow$ x = c - [2p + 2 log |1 - p|], which is the regd. solution.

**Example 20 :** Solve $y^2$ log y = xyp + $p^2$

**Sol. :** Given equation is $y^2$ log y = xyp + $p^2$ ....(1)

$\Rightarrow$ xyp = $y^2$ log y = $p^2$

$\Rightarrow$ x = $\frac{y\log y}{p} - \frac{p}{y}$

$$\Rightarrow \frac{dx}{dy} = \frac{p\left\{y.\frac{1}{y}+(\log y).1\right\}-(y\log y)\left(\frac{dp}{dy}\right)}{p^2} - \frac{y\frac{dp}{dx}-p-1}{y^2}$$

$$\Rightarrow \frac{1}{p} = \frac{1}{p} + \frac{\log y}{p} - \frac{y\log y}{p^2}\frac{dp}{dy} - \frac{1}{y}\frac{dp}{dy} + \frac{p}{y^2}$$

$$\Rightarrow \left(\frac{1}{y}\frac{dp}{dy}\right)\left\{1+\frac{y^2}{p^2}\log y\right\} = \left(\frac{p}{y^2}\right)\left\{1+\frac{y^2\log y}{p^2}\right\}$$

$$\Rightarrow \frac{1}{y}\frac{dp}{dy} = \frac{p}{y^2} \Rightarrow \frac{dp}{dy} = \frac{p}{y}$$

$$\Rightarrow \frac{dp}{p} = \frac{dy}{y}$$

$$\Rightarrow \int\frac{dp}{p} = \int\frac{dy}{y} + \log c$$

$\Rightarrow$ log p = log y + log c

$\Rightarrow$ log p = log cy $\Rightarrow$ p = cy ...(2)

Using (2) in (1), we get

$y^2$ log y = xy (cy) + $c^2y^2$, which is the required solution.

**Self Check Exercise-2**

Q. 1 Solve y = x - $p^2$

| | |
|---|---|
| Q. 2 | Solve $y = x + a \tan^{-1} p$ |
| Q. 3 | Solve $y = 3px + 4p^2$ |
| Q. 4 | Solve $x = y + a \log p, p > 0$ |
| Q. 5 | Solve $p^2y + 2px = y$ |

## 5.6 Equations not Containing

A differential equation which does not contain x is of form

$$f(y, p) = 0$$

Two cases arise:

**Case I.** When (1) is solvable for p

Then $p = g(y)$ i.e. $\frac{dy}{dx} = g(y)$

$$\Rightarrow \quad \frac{dy}{g(y)} = dx$$

Integrating, $\int \frac{dy}{g(y)} = x + c$, which is the solution.

**Case II.** When (1) is solvable for y

Then $y = h(p)$, and we can proceed as distressed earlier.

## 5.7 Equations Not Containing y

The differential equation, not containing y, is of the form

$$f(x, p) = 0 \qquad .....(1)$$

Two cases arise:

Case I. When (1) is solvable for p

Then $p = g(x)$ i.e. $\frac{dy}{dx} = g(x)$

$$\Rightarrow \quad dy = g(x)\, dx$$

Integrating, $y + c = \int g(x)dx$, which is the solution.

Case II. When (1) is solvable for x

Then $x = h(p)$, and we can proceed as discussed earlier

Let us do some examples:-

**Example 21:** Solve $x^2 = p^2 (a^2 - x^2)$

**Sol:** Given equation is

$$x^2 = p^2 (a^2 - x^2)$$

$$\Rightarrow \quad p^2 = \frac{x^2}{a^2 - x^2}$$

$$\Rightarrow \quad p = \pm \frac{x}{\sqrt{a^2 - x^2}} \qquad \Rightarrow \quad \frac{dy}{dx} \pm \frac{x}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x}{\sqrt{a^2 - x^2}} \qquad \text{or} \qquad \frac{dy}{dx} = - \frac{x}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow \quad dy = \frac{x}{\sqrt{a^2 - x^2}}\, dx \qquad dy = - \frac{x}{\sqrt{a^2 - x^2}}$$

$$\Rightarrow \quad y + c = \frac{x}{\sqrt{a^2 - x^2}}\, dx \qquad y + c = -\int \frac{x}{\sqrt{a^2 - x^2}}\, dx$$

$$= - \frac{1}{2} \int \left(a^2 - x^2\right)^{-1/2} (-2x)\, dx \qquad = \frac{1}{2} \int \left(a^2 - x^2\right)^{-1/2} (-2x)\, dx$$

$$= - \frac{1}{2} \frac{\left(a^2 - x^2\right)^{1/2}}{1/2} \qquad = \frac{1}{2} \frac{\left(a^2 - x^2\right)^{1/2}}{1/2}$$

$$= - \sqrt{a^2 - x^2} \qquad = \sqrt{a^2 - x^2}$$

$$\Rightarrow \quad (y + c)^2 = a^2 - x^2 \qquad \Rightarrow \quad (y + c)^2 = a^2 - x^2$$

Hence the required solution is

$$(y + c)^2 + x^2 = a^2$$

**Example 22:** Solve $y^2 (1 - p^2) = b$

**Sol:** Given equation is $y^2 (1 - p^2) = b$

$$\Rightarrow \quad 1 - p^2 = \frac{b}{y^2} \Rightarrow 1 - \frac{b}{y^2} = p^2 \Rightarrow p^2 = \frac{y^2 - b}{y^2}$$

$$\Rightarrow \quad p = \frac{\sqrt{y^2 - b}}{y} \Rightarrow \frac{dy}{dx} = \pm \frac{\sqrt{y^2 - b}}{y}$$

$\Rightarrow \quad \frac{\pm\, y}{\sqrt{y^2 - b}}$ dy = dx $\Rightarrow \pm \frac{1}{2} \int$ (2y) $\left(y^2 - b\right)^{-1/2}$ dy = $\int$ 1 dx

$\Rightarrow \quad \pm \frac{1}{2} \frac{\left(y^2 - b\right)^{1/2}}{\left(1/2\right)}$ = x + c $\Rightarrow \quad y^2 - b = (x + c)^2$

$\Rightarrow \quad y^2 = b(x + c)^2$, which is the required solution.

**Self Check Exercise-3**

Q. 1 Solve $x(1 + p^2) = 0$

Q. 2 Solve $y^2 + p^2 = a^2$

## 5.8 The Clairaut Equation

Def. The equation

$$y = px + f(p) \qquad .......(1)$$

of first degree in x and y is called the Clairaut equation. It is after the name of Alexis Claude Clairaut.

Differentiating (1) w.r.t. 'x', we get

$$\frac{dy}{dx} = p.1 + x\frac{dp}{dx} + f'(p)\,\frac{dp}{dx} \Rightarrow p = p + \frac{dp}{dx}(x + f'(p))$$

$\Rightarrow \quad (x + f'(p))\,\frac{dp}{dx} = 0 \Rightarrow$ either $x + f'(p) = 0$ or $\frac{dp}{ax} = 0$

Now $\frac{dp}{dx} = 0 \Rightarrow p = c$ (constant)

Putting this value of p in the given equation, we get

$$y = cx + f(c)$$

as the complete solution of the given equation,

Elimination of p between

$$x + f'(p) = 0 \text{ and } y = px + f(p)$$

will give us a solution free from arbitrary constant and this is called Singular Solution of the given equation.

Elimination of p between

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Will give us a solution free from arbitrary constant and this is called Singular Solution of the given equation.

Rule to solve Clairaut's Equation. Put p = c

To clarify what we have just said, consider the following examples:-

**Example 23:** Solve: $y = px + \sin^{-1} p$

**Sol:** Clearly $y = px + \sin^{-1} p$ is in the Clairaut's form.

$\therefore$ its complete solution is $y = cx + \sin^{-1} c$, where c is an arbitrary constant.

**Example 24:** Solve $p = \log(px - y)$

**Sol:** Given equation is $p = \log(px - y)$

$\Rightarrow$ $px - y = e^p$

$\Rightarrow$ $y = px - e^p$ is in Clairaut's form

$\therefore$ its complete solution is $y = cx - e^c$.

**Example 25:** Solve $(y - px)(p - 1) = p$

**Sol:** Given equation is $(y - px)(p - 1) = p$

$\Rightarrow$ $y - px = \dfrac{p}{p-1}$

$\Rightarrow$ $y = px + \dfrac{p}{p-1}$ is in Clairaut's form

$\therefore$ its complete solution is $y = cx + \dfrac{c}{c-1}$ (Replacing p by c)

**Example 26:** Solve: $p^2 (x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

**Sol:** Given equation is $p^2 (x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

$\Rightarrow$ $(px - y)^2 = a^2 p^2 + b^2 \Rightarrow px - y = + \sqrt{a^2 p^2 + b^2}$

$\Rightarrow$ $y = px \, \mathrm{m} \sqrt{a^2 p^2 + b^2}$, which is in Clairaut;s from

$\therefore$ its complete solution is $y = cx \, \mathrm{m} \sqrt{a^2 p^2 + b^2}$

Note: It may be sometimes possible to reduce a given equation to Clairaut's form by change of variable in a suitable manner.

(1) If an equation involves $y^2$, $2py$ put $y^2 = Y$

(2) If an equation involves $x^2$, $\frac{-P}{2x}$ put $x^2 = X$.

(3) If an equation involves $x^2$, $y^2$; put $x^2 = X$, $y^2 = Y$

(4) If an equation involves $e^{ax}$, $e^{by}$; put ex = X, ey = Y

or Sometimes it is useful to put $e^{ax} = X$, $e^{by} = Y$.

**Example 27:** Solve: $x^2 (y - px) = p^2y$

**Sol:** The given equation is $x^2 (y - px) = p^2y$ ......(1)

Put $x^2 = X$ and $y^2 = Y$

$\therefore$ 2xdx = dX and 2ydy = dY

Divide, $\frac{y\,dy}{x\,dx} = \frac{dY}{dX}$ or $\frac{y}{x}$ p = $\frac{dY}{dX}$ = P (say) or p = $\frac{x}{y}$ p

Putting in (1), we get $x^2\left(y - \frac{x}{y}p.x\right) = \frac{x^2}{y^2}$ $(p)^2$. y

or $(y^2 - x^2\, p) = p^2$ or $Y - XP = P^2$

or $Y = PX + P^2$, which is in Clairaut's form.

$\therefore$ the solution is $Y = cX + c^2$ or $y^2 = cx^2 + c^2$

**Example 28:** Using the substitution X = $\frac{1}{x}$, Y = $\frac{1}{y}$, solve the equation $y^2 (y - px) = x^4p^2$.

**Sol:** The given equation is $y^2 (y - px) = x^4p^2$. .......(1)

Put X = $\frac{1}{x}$ and Y = $\frac{1}{y}$ $\therefore$ dX = - $\frac{1}{x^2}$ dx and dY = - $\frac{1}{y^2}$ dy.

Dividing, $\frac{dY}{dX} = \frac{x^2}{y^2} \cdot \frac{dy}{dx}$ = p $\frac{x^2}{y^2}$ i.e. p = $\frac{y^2}{x^2}$ p $\left(\text{Where } P = \frac{dY}{dX}\right)$

Dividing (1) by $y^4$, we get $\frac{1}{y} - \frac{1}{x}\frac{px^2}{y^2} = p^2\frac{x^4}{y^4}$

or $\frac{1}{y} - \frac{1}{x}$ P = $p^2$ or $Y - XP = P^2$ or $Y = PX + p^2$, which is in Clairaut's form.

$\therefore$ the sol is $Y = cX + c^2$

or $\frac{1}{y}$ - $\frac{c}{x}$ + $c^2$ or $c^2xy + cy - x = 0$.

**Example 29:** Solve: $(px - y)(py + x) = h^2p$

**Sol:** Given equation is $(px - y)(py + x) = h^2p$

$$\Rightarrow \quad (px - y)(x)\left(\frac{py}{x}+1\right) = h^2p \Rightarrow y(px - y)\left(\frac{py}{x}+1\right) = \frac{h^2py}{x}$$

$$\Rightarrow \quad (pxy - y^2) = \frac{\left(\frac{h^2py}{x}\right)}{\left(\frac{py}{x}+1\right)} \Rightarrow y^2 = pxy - \frac{\left(\frac{h^2py}{x}\right)}{\left(\frac{py}{x}+1\right)} \Rightarrow y^2 = \frac{py}{x}x^2 - \frac{\left(\frac{h^2py}{x}\right)}{\left(\frac{py}{x}+1\right)} \qquad .....(1)$$

Put $x^2 = X$, $y^2 = Y$ $\therefore$ $2x\,dx = dX$, $2y\,dy = dY$

$$\therefore \quad \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} \Rightarrow P = \frac{yp}{x}, \text{ where } P = \frac{dY}{dX},\ p = \frac{dy}{dx}$$

$\therefore$ (1) takes the form

$$Y = PX - \frac{h^2p}{P+1} \qquad \text{(Clairaut's form)}$$

$$\therefore \quad \text{solution is } Y = cX - \frac{h^2c}{c+1} \Rightarrow y^2 = cx^2 - \frac{h^2c}{c+1}$$

**Example 30:** Solve: $y = 2px + y^2p^3$

**Sol:** Given equation is $y = 2px + y^2p^3$

$$\Rightarrow \quad y^2 = 2pxy + y^3p^3 \Rightarrow y^2 = (2py)x + \frac{(2py)^3}{8}$$

Put $y^2 = Y \Rightarrow 2y\frac{dy}{dx} = \frac{dY}{dX} \Rightarrow 2yp = P$, where $P = \frac{dY}{dx}$

$$\therefore \quad \text{(1) takes the form } Y = Px + \frac{P^3}{8} \qquad \text{(Clairaut's equation)}$$

$$\therefore \quad \text{Solution is } Y = cx + \frac{c^3}{8} \Rightarrow y^2 = cx + \frac{c^3}{8}$$

**Self Check Exercise-4**

Q. 1 Solve $y^2 + x^2\left(\frac{dy}{dx}\right)^2 - 2x\frac{dy}{dx} = 4\left(\frac{dx}{dy}\right)^2$

Q. 2 Solve $(p - 1) e^{4x} + p^2 e^{2y} = 0$

Q. 3 Solve $(x^2 + y^2)(1 + p)^2 = -2(x + y)(1 + p)(x + py) + (x + py)^2 = 0$

## 5.9 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined equation of first order and higher degree.
2. Solutions of first order higher degree equations solvable for x, y, p.
3. Discussed and find solutions of Clairaut equation

## 5.10 Glossary:

1. A differential equation of form $f\left(x, y, \frac{dy}{dx}\right) = 0$, where degree of $\frac{dy}{dx}$ is greater than 1 is called a non-linear differential equation of first order and higher degree.
2. A differential equation which does not contain x is of form

   $f(y, p) = 0$
3. The equation

   $y = px + f(p)$

   of first degree in x and y is called the Clairaut equation.

## 5.11 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 General Solution is $(y^2 - x^2 - c)(xy - c) = 0$

Ans.2 General Solution is

$x^2 + (y + c)^2 = a^2$

Ans.3 General Solution is

$(y - c)(x + y - c)(3xy + x^2 - c) = 0$

Ans.4 General Solution is

$(y - c)(\log|y| - x^{-1} - c)\left(x - \frac{1}{y} - c\right) = 0$

Ans.5 General Solution is

$(y - c)(y^{-1} + x + c)(y + x^2 - c) = 0$

**Self-Check Exercise-2**

Ans.1 $x^2 = -2p - 2 \log |1 - p| + c$

and $y = -2p - 2 \log 1 - p/-p2 + c$, constitute the solution.

Ans.2 $x = \frac{a}{2}\left[\log\left(\frac{p-1}{\sqrt{p^2+1}}\right) - \tan^{-1} p\right] + c$

and $y = \frac{a}{2}\left[\log\left(\frac{p-1}{\sqrt{p^2+1}}\right) + \tan^{-1} p\right] + c$,

constitute the required solution.

Ans.3 $x = -\frac{8}{5}p + c\, p^{-\frac{3}{2}}$

and $y = 3c\, p^{-1/2} - \frac{4}{5} p^2$,

constitute the required solution.

Ans.4 $y = c - a \log |1 - p|$

and $x = c + a \log\left(\frac{p}{|1-p|}\right)$,

constitute the required solution.

Ans.5 $y^2 = 2cx + c^2$, is the required solution

**Self-Check Exercise-3**

Ans.1 $y = \pm\left[\sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}}\right] + c$, is the required solution.

Ans.2 $x = 2 \log p \mid 6p + c$

and $y = 2p + 3p^2$, give the complete solution.

**Self-Check Exercise-4**

Ans.1 $y = cx + \frac{2}{c}$ is the required solution.

Ans.2 $e^{2y} = c\, e^{2x} + c^2$ is the required solution

Ans.3 $x^2 + y^2 = c\,(x + y) - \frac{c^2}{4}$ is the required solution.

## 5.12 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Wylie, C.R., Differential Equations, McGraw-Hill, New York, 1979.


## 5.13 Terminal Questions

1. Solve $x + y\,p^2 = p(1 + xy)$
2. Solve $xp^2 + (y - x)\,p - y = 0$
3. Solve $p^3(x + 2y) + 3p^2\,(x + y) + (y + 2x)\,p = 0$
4. Solve $p^2 + 2py \cot x = y^2$
5. Solve $4p^2x - (9x - k)^2 = 0$, where k is constant.
6. Solve $p^3 + mp^2 = a(y + mx)$, where $p = \frac{dy}{dx}$
7. Solve $y - 2px = \tan^{-1}(xp^2)$
8. Solve $xp^2 - 3yp + 9x^2 = 0,\ x > 0$
9. Solve $y = 2px + y^2p^3$
10. Solve $p^3 + 8y^2 = 4pxy$
11. Solve $x + \frac{p}{\sqrt{1+p^2}} = a$
12. Solve $y = px + \sin^{-1} p$
13. Solve $(px - y)\,(py + x) = h^2p$
14. Solve $2x^2y = px^3 + p^2$, by reducing the Clairaut's form.

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## Unit - 6

# Linear Homo Geneous Equations With Constant Coefficients

**Structure**



## 6.1 Introduction

A differential equation is said to be linear if the unknown function and all of its derivatives occurring in the equation only in the first degree and are not multiplied together e.g. the differential equation $\frac{d^2y}{dx^2}$ + 2y = 0, $\frac{dy}{dx}$ = cos x are linear whereas $\left(\frac{d^2y}{dx^2}\right)^3 + x^2\left(\frac{dy}{dx}\right)^2 = 0$ is non-liner. It should be noted that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear e.g. the differential equation.

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y^2 = 0$$ is not linear though its degree is 1.

A linear differential equation with constant coefficient is one in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together. The most general form of the linear differential equation with constant coefficients of nth order is

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + P_ny = Q \qquad \ldots(1)$$

(where $P_0, P_1, P_2, \ldots\ldots, P_n$ are constants and Q is a function of x) Also $P_0 \neq 0$

The corresponding homogeneous linear differential equation is

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + ....... + P_n y = 0 \qquad ...(2)$$

## 6.2 Learning Objectives

After studying this unit, you should be able to :-

- Defined symbolic form of the equation and Auxiliary equation.
- Define product of operators
- Discuss and find the solutions of the linear homogeneous equation with constant coefficients.

## 6.3 Some Definitions

### I. Operators

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ is called an operator. Also $\frac{d^2}{dx^2}$, $\frac{d^3}{dx^3}$, ......., $\frac{d^n}{dx^n}$ are also operators.

### II. Symbolic Form

In symbolic form, we write

$$\frac{d}{dx} = D,\ \frac{d^2}{dx^2} = D^2,\ .......,\ \frac{d^n}{dx^n} = D^n$$

$\therefore$ The symbolic form of the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + ....... + P_n y = Q \qquad ...(1)$$

is $P_0D^ny + P_1 D^{n-1}y + P_2 D^{n-2}y + ...... + P_ny = Q$

i.e. $(P_0D^n + P_1 D^{n-1} + P_2 D^{n-2} + ...... + P_n)\, y = Q \qquad ...(2)$

or $f(D)\, y = Q$

where $f(D) = P_0D^n + P_1 D^{n-1} + ...... + P_n$

The form (2) is the symbolic form of (1).

### III Auxiliary Equation (A.E.)

Auxiliary equation is that equation which can be obtained by equating the symbolic coefficients of y to zero.

Thus the A.E. of the above equation is

$P_0D^n + P_1 D^{n-1} + P_2 D^{n-2} + ...... + P_n = 0$

i.e. $f(D) = 0$

## IV. Characteristic Polynomial

The polynomial $f(D)$ is called the characteristic polynomial of the given differential equation. This can be regarded as an operator.

## 6.4 Product of Operators

If $f_1(D)$ and $f_2(D)$ be two operators, then $f_1(D)$, $f_2(D)$ is also an operator.

**Theorem 1 :** Prove that $(D - \alpha)(D - \beta)y = (D - \beta)(D - \alpha)y$,

where $\alpha$, $\beta$ are two arbitrary constants.

**Proof :** Now $(D - \beta)y = Dy - \beta y = \dfrac{dy}{dx} - \beta y$ $\left[\because D = \dfrac{d}{dx}\right]$

$$\therefore \quad (D - \alpha)(D - \beta)y = (D - \alpha)\left(\frac{dy}{dx} - \beta y\right)$$

$$= D\left(\frac{dy}{dx} - \beta y\right) - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d}{dx}\left(\frac{dy}{dx} - \beta y\right) - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d^2y}{dx^2} - \beta\frac{dy}{dx} - \alpha\frac{dy}{dx} + \alpha\beta y$$

$$= \frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y$$

$$= [D^2 - (\alpha + \beta)D + \alpha\beta]y$$

Similarly $(D - \beta)(D - \alpha)y = [D^2 - (\alpha + \beta)D + \alpha\beta]y$

Hence $(D - \alpha)(D - \beta)y = (D - \beta)(D - \alpha)y$

## 6.5 Some Important Theorems

**Theorem 2 :** If $y = y_1$, $y = y_2$, ...... $y = y_n$ are linearly independent particular solutions of the equation.

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ....... + P_ny = 0$$

where $P_0, P_1, P_2, ......, P_n$ are constants, then the complete solution is

$y = c_1y_1 + c_2y_2 + ......c_ny_n$

**Proof :** Given Equation is

$$P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots. + P_n y = 0 \qquad \ldots.(1)$$

Since $y = y_1$, $y = y_2$, ...... $y = y_n$ are solutions of (1) [Given]

$$\left.\begin{aligned}
&P_0\frac{d^n y_1}{dx^n} + P_1\frac{d^{n-1}y_1}{dx^{n-1}} + P_2\frac{d^{n-2}y_1}{dx^{n-2}} + \ldots\ldots. + P_n y_1 = 0\\
&P_0\frac{d^n y_2}{dx^n} + P_1\frac{d^{n-1}y_2}{dx^{n-1}} + P_2\frac{d^{n-2}y_2}{dx^{n-2}} + \ldots\ldots. + P_n y_2 = 0\\
&\text{------------------------------------------------}\\
&\text{------------------------------------------------}\\
&P_0\frac{d^n y_n}{dx^n} + P_1\frac{d^{n-1}y_n}{dx^{n-1}} + P_2\frac{d^{n-2}y_n}{dx^{n-2}} + \ldots\ldots. + P_n y_n = 0
\end{aligned}\right\}$$

Putting $y = c_1y_1 + c_2y_2 + \ldots\ldots c_ny_n$ in (1), we get

$$P_0\frac{d^n}{dx^n}(c_1y_1 + c_2y_2 + \ldots\ldots c_ny_n) + P_1\frac{d^{n-1}}{dx^{n-1}}(c_1y_1 + c_2y_2 + \ldots\ldots c_ny_n) + \ldots\ldots\ldots..$$

$$+P_n(c_1y_1 + c_2y_2 + \ldots\ldots c_ny_n) = 0$$

$\Rightarrow$ $$c_1\left(P_0\frac{d^n y_1}{dx^n} + P_1\frac{d^{n-1}y_1}{dx^{n-1}} + P_2\frac{d^{n-2}y_1}{dx^{n-2}} + \ldots\ldots. + P_n y_1\right) +$$

$$c_2\left(P_0\frac{d^n y_2}{dx^n} + P_1\frac{d^{n-1}y_2}{dx^{n-1}} + P_2\frac{d^{n-2}y_2}{dx^{n-2}} + \ldots\ldots. + P_n y_2\right) +$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots +$$

$$+ c_n\left(P_0\frac{d^n y_n}{dx^n} + P_1\frac{d^{n-1}y_n}{dx^{n-1}} + P_2\frac{d^{n-2}y_n}{dx^{n-2}} + \ldots\ldots. + P_n y_n\right) = 0$$

$\Rightarrow$ $c_1(0) + c_2(0) + \ldots\ldots + c_n(0) = 0$ [using (2)]

$\Rightarrow$ $0 = 0$, which is true

Thus $y = c_1y_1 + c_2y_2 + \ldots\ldots c_ny_n$ is a solution of (1).

Since it contains n arbitrary constants and (1) is of the nth order,

$\therefore$ It is the complete solution of (1)

**Theorem 3 :** If $f_1(D)$ is a factor of $f(D)$ and $y = y_1$ is a solution of $f_1(D)\,y = 0$, then $y = y_1$ is also a solution of $f(D)y = 0$

Proof : Let $f(D) = f_1(D)\, f_2(D)$

$f(D)y_1 = f_1(D)\, f_2(D)y_1$

$= f_2(D)\, f_1(D)y_1$

$= f_2(D)\,(0)$ [$\therefore$ $y_1$ is a solution of $f_1(D)y = 0$]

$= 0$

$\therefore$ $y = y_1$ is a solution of $f(D)\, y = 0$

## 6.6 Solution of the Linear Homogeneous Equation with Constant Coefficients

Here we have to

Solve the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + ....... + P_n y = 0 \qquad ....(1)$$

where $P_0, P_1, P_2, ......, P_n$ are constants

The equation (1) in the symbolic form is

$$(P_0D^n + P_1D^{n-1} + ......+ P_n)y = 0 \text{ i.e. } f(D)\, y = 0 \qquad ...(2)$$

Where $f(D) = P_0D^n + P_1D^{n-1} + ......+ P_n$

The A.E. of (1) is

$$f(D) = P_0D^n + P_1D^{n-1} + ......+ P_n = 0 \qquad ...(3)$$

This is an algebraic equation of degree n (with real coefficients) and has exactly n roots, real or complex. Following cases arise

**Case I.** All roots of A.E. are real and distinct.

Let $\alpha_1, \alpha_2$ ....... $\alpha_n$ be the real and distinct roots of A.E. (3)

$\therefore$ $f(D)\, y = \phi(D)\,(D - \alpha_1)y$ ...(4)

Where $\phi(D)$ is a polynomial of degree (n - 1) in D.

Now we know that the solution of

$$(D - \alpha_1)y = 0 \qquad ...(5)$$

is also a solution of $f(D)\, y = 0$ i.e. of the given equation.

$$\text{Now } (D - \alpha_1)\, y = 0 \Rightarrow \frac{dy}{dx} - \alpha_1 y = 0 \Rightarrow \frac{dy}{y} = \alpha_1\, dx$$

$\Rightarrow \log y = \alpha_1 x \Rightarrow y = e^{\alpha_1 x}$

Where we have omitted the constant of integration.

$\therefore$ $e^{\alpha_1 x}$ is a solution of (1)

Similarly $e^{\alpha_2 x}$, $e^{\alpha_3 x}$, ....., $e^{\alpha_n x}$ are solution of (1)

$\therefore$ C.S. of (1) is

$$y = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x} + ..... + c_n e^{\alpha_n x} \qquad ....(6)$$

where $c_1, c_2$ ... ... $c_n$ are n arbitrary constants.

**Case II. A. E. having two real and equal roots.**

Let the two real and equal roots of A.E. be $\alpha$ (Here $\alpha_1 = \alpha_2 = \alpha$)

Then (6) becomes:

$$y = c_1 e^{\alpha x} + c_2 e^{\alpha x} + c_3 e^{\alpha_3 x} + ..... + c_n e^{\alpha_n x} = c\, e^{\alpha x} + c_3 e^{\alpha_3 x} + ...... + c_n e^{\alpha_n x}$$

where $c = c_1 + c_2$

Since this solution contains (n - 1) arbitrary constants

$\therefore$ it is not the general solution of (1)

$\therefore$ In order to find the general solution of (1)

We proceed as follows:

Since $x = \alpha, \alpha$ be two equal roots

$\therefore$ $f(D) = \psi(D)\,(D - \alpha)^2$

$\therefore$ $f(D)y \Rightarrow \psi(D)\,(D - \alpha)^2\, y = 0$ .....(7)

$\therefore$ Solution of $(D - \alpha)^2 y = 0$ is also a solution of $f(D)\, y = 0$

Consider $(D - \alpha)^2 y = 0$. Put $(D - \alpha)\, y = z$

$\therefore$ $(D - \alpha)\, z = 0$

$$\Rightarrow \frac{dz}{dx} - \alpha z = 0 \Rightarrow \frac{dz}{z} = \alpha\, dx$$

$$\Rightarrow \log z = \alpha x + c \Rightarrow z = e^{\alpha x + c} = e^{\alpha x}.\, e^{c} = c_1 e^{\alpha x}$$

where $c_1$ is an arbitrary constant

Again $\because$ $(D - \alpha)y = z \therefore \frac{dy}{dx} - \alpha y = c_1 e^{\alpha x}$

which is linear in y. $\left[Type: \frac{dy}{dx} + Py = Q\right]$

Here $P = -\alpha$, $Q = c_1 e \alpha x$ $\therefore \int Pdx = \alpha \int -\alpha\, dx = -\alpha x$

$\therefore$ Solution is

$y\, e^{-\alpha x} = \int c_1 e^{\alpha x}.\, e^{-\alpha x}\, dx + c_2 = c_1 \int dx + c_2$

$= c_1 x + c_2$ $\qquad$ [$\because e^{\alpha x}.\, e^{-\alpha x} = e^0 = 1$]

$\therefore$ $\quad y = e^{\alpha x}(c_1 x + c_2)$ is the general solution of (7).

Changing $c_1$ to $c_2$ and $c_2$ to $c_1$, the part of the complete solution corresponding to two equal and real roots $\alpha$ of the A.E. is $(c_1 + c_2 x)\, e^{\alpha x}$

Hence C.S. in this case is

$$y = (c_1 + c_2 x)e^{\alpha x} + c_3 e^{\alpha_3 x} + \ldots\ldots + c_n e^{\alpha_n x}$$

Similarly if A.E. has r equal and real roots, then C.S of (1) is

$$y = (c_1 + c_2 x + c_3 x^2 + \ldots\ldots + c_r x^{r-1})\, e^{\alpha x} + c_{r+1}\, x^{r+1} + \ldots\ldots + c_n x^n$$

**Case III. A.E. having a conjugate pairs of imaginary roots** (non-repeated)

Let $\alpha \pm i\beta$ be a pair of imaginary roots of the A.E.

$\therefore$ $\quad$ the corresponding part of C.S. of (1)

$= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha+i\beta)x} = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{-i\beta x}$

$= c_1 e^{\alpha x}[\cos \beta x + i \sin \beta x] + c_2 e^{\alpha x}[\cos \beta x - i \sin \beta x]$

[Note. $e^{i\theta} = \cos \theta + i \sin \theta$]

$= e^{\alpha x}[c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x] = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$

where $A = c_1 + c_2$, $B = i(c_1 - c_2)$ are arbitrary constants.

$\therefore$ $\quad$ C.S. of (1) becomes

$$y = e^{\alpha n}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{\alpha_3 x} + \ldots\ldots. + c_n e^{\alpha_n x}$$

where $c_1, c_2, \ldots\ldots., c_n$ are arbitrary constants $\qquad$ [Replacing A by $c_1$ and B by $c_2$]

**Case IV. A.E. having a conjugate pair of imaginary roots occurring r times**

If the A.E. has the pair of imaginary roots $\alpha \pm i\beta$ occurring r times, we can similarly prove the corresponding part of (1) is

$$e^{\alpha x}(c_1 + c_2 x + \ldots\ldots. + c_r x^{r-1}) \cos \beta x + (c_{r+1} + c_{r+2} x + \ldots\ldots. + c_{2r\ x^{r-1}}) \sin \beta x$$

Working rule for solving the equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots. + P_n y = 0 \qquad \ldots\ldots(1)$$

where $P_0, P_1, P_2, \ldots\ldots, P_n$ are real constants.

**Step I. Write the equation in the symbolic form i.e. of the form $f(D)y = 0$**

**Step II. Write A.E. i.e. $f$(D) = 0**

**Step III. Solve this equation for D.**

**Step IV. From the roots of A.E. write the corresponding part of the general solution of (1) as follows:**

| **Roots of A.E** | **Corresponding part of general solution** |
|---|---|
| One real root $\alpha$(when other roots are distinct) | $c_1e^{\alpha x}$ |
| Two equal real roots $\alpha$, $\alpha$ | $(c_1 + c_2x)e^{\alpha x}$ |
| r equal roots $\alpha$ each | $(c_1 + c_2x +......+ c_rx^{r-1})\ e^{\alpha x}$ |
| One pair of complex roots $\alpha \pm i\beta$ | $e^{\alpha x}\ (c_1 \cos \beta x + c_2 \sin \beta x)$ |
| Complex roots $\alpha \pm i\beta$ repeated r times | $e^{\alpha x}\ [(c_1 + c_2x +.......+ c_rx^{r-1}) \cos \beta x$ <br> $+ (c_{r+1} + c_{r+2}x +......+ c_{2r}\ x^{r-1}) \sin \beta x]$ |

To Clarify what we have just said, consider the following examples:-

**Example 1:** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

**Sol:** The given equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$

The equation in the symbolic form is $(D^2 - 4D + 3)\ y = 0$ $\qquad \left(D = \frac{d}{dx}\right)$

A.E. is $D^2 - 4D + 3 = 0 \Rightarrow (D - 3)\ (D - 1) = 0$

$\Rightarrow$ D = 3, 1 (Real and different)

$\therefore$ Complete solution is $y = c_1\ e^{3x} + c_2\ e^x$. where $c_1$, $c_2$ are arbitrary constants.

**Example 2:** Solve: (i) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

(ii) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

(iii) $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

(iv) $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} - 12y = 0$

**Sol:** (i) Given equation in symbolic form is $(D^3 - 3D^2 + 4)y = 0$

A.E. is $D^3 - 3D^2 + 4 = 0 \Rightarrow (D - 2)(D^2 - D - 2) = 0$

$\Rightarrow$ $(D - 2)(D + 1)(D - 2) = 0 \Rightarrow D = -1,2,2$

$\therefore$ Complete solution is $y = c_1e^{-x} + (c_2 + c_3x) e^{2x}$, where $c_1$, $c_2$, are arbitrary constants.

(ii) Given equation in symbolic form is $(D^4 - D^3 - 9D^2 - 11D - 4) y = 0$

A.E. is $D^4 - D^3 - 9D^2 - 11D - 4 = 0 \Rightarrow (D + 1)(D^3 - 2D^2 - 7D - 4) = 0$

$\Rightarrow$ $(D + 1)(D + 1)(D^2 - 3D - 4) = 0$

$\Rightarrow$ $(D + 1)(D + 1)(D + 1)(D - 4) = 0$

$\Rightarrow$ $(D - 4)(D + 1)^3 = 0 \Rightarrow D = 4, -1, -1, -1$

$\therefore$ Complete solution is

$y = c_1e^{4x} + (c_2 + c_3x + c_4x_2) e^{-x}$, where $c_1$, $c_2$, $c_3$, $c_4$ are arbitrary constants.

(iii) Given equation in symbolic form is $(D^3 - 9D^2 + 23D - 15)y = 0$

A.E. is $D^3 - 9D^2 + 23D - 15 = 0 \Rightarrow (D - 1)(D^2 - 8D + 15) = 0$

$\Rightarrow$ $D = 1,3,5 \Rightarrow (D - 1)(D - 3)(D - 5) = 0$

Complete solution is $y = c_1^{ex} + c_2e^{3x} + c_3e^{5x}$, where $c_1$, $c_2$, $c_3$ are arbitrary constants.

(iv) Given equation in symbolic form is $(D^3 - 13D - 12)y = 0$

A.E. is $D^3 - 13D - 12 = 0 \Rightarrow (D + 1)(D^2 - D - 12) = 0$

$\Rightarrow$ $(D + 1)(D - 4)(D + 3) = 0 \Rightarrow D = -1, -3, 4$

Complete solution is $y = c_1e^{-x} + c_2e^{-3x} + c_3e^{4x}$, where $c_1$, $c_2$, $c_3$ are arbitrary constants

**Example 3:** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$, given that y = 1 and $\frac{dy}{dx} = 2$ when x = 0

**Sol:** Given equation in symbolic form is $(D^2 - 4D + 5)y = 0$

A.E. is $D^2 - 4D + 5 = 0 \Rightarrow D = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm i2}{2} = 2 \pm i$

$\therefore$ complete solution is $y = e^{2x}.(c_1 \cos x + c_2 \sin x)$ .....(1)

$\therefore$ $\frac{dy}{dx} = e^{2x}(-c_1 \sin x + c_2 \cos x) + (c_1 \cos x + c_2 \sin x)(2e^{2x})$ ......(2)

Given y = 1, x = 0: putting in (1), we get

$1 = c_1$ .......(3)

Given $\frac{dy}{dx} = 2$, x = 0; putting in (2), we get

$2 = c_2 + 2c_1 = c_2 + 2(1)$ [Using (3)]

$\Rightarrow \quad c_2 = 0$ ......(4)

Putting $c_1 = 1$, $c_2 = 0$ in (1), we get, $y = e^{2x}$, cos x

**Example 3:** Solve y'' - y = 0 and find the solution with initial condition y(0) = 0, y'(0) = 1

**Sol:** Given equation is

$y'' - y = 0$ ......(1)

$\Rightarrow \quad (D^2 - 1)\, y = 0$

A.E. is $D^2 - 1 = 0$

$\Rightarrow \quad D = \pm 1$

$\therefore$ Complete solution is

$y = c_1 e^x + c_2 e^{-x}$ ......(2)

$\therefore \quad y' = c_1 e^x - c_2 e^{-x}$ ......(3)

Given $y(0) = 0 \Rightarrow x = 0, y = 0$

Putting in (2), we get

$0 = c_1 + c_2$ .......(4)

Given $y'(0) = 1 \Rightarrow x = 0, y' = 1$

Putting in (3), we get

$1 = c_1 - c_2$ ........(5)

Adding (4) and (5), we get

$2c_1 = 1 \Rightarrow \quad c_1 = \frac{1}{2}$

(4) - (5) given $2c_2 = -1 \Rightarrow c_2 = -\frac{1}{2}$

Put $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}$ in (2), we get

$y = \frac{1}{2} e^x - \frac{1}{2} e^{-x} = \frac{1}{2} (e^x - e^{-x})$

= sin h x

$\Rightarrow \quad y = \sinh x$ is the solution

**Example 4:** If $\frac{d^2y}{dx^2}$ - $m^2y = 0$, show that

$y = c_1e^{mx} + c_2e^{-mx} = A \cosh mx + B \sinh mx$

**Sol:** Given equation is

$$\frac{d^2y}{dx^2} - m^2y = 0$$

The equation in the symbolic form is

$(D^2 - m^2)y = 0$

A.E. is $D^2 - m^2 = 0 \Rightarrow D^2 = m^2 \Rightarrow D = \underline{+}\, m$

$\therefore$ C.S. is $y = c_1e^{mx} + c_2e^{-mx}$, where $c_1$, $c_2$, are arbitrary constants.

Since $e^{\theta} = \cosh\theta + \sinh\theta$, $e^{-\theta} = \cosh\theta - \sinh\theta$

$\therefore$ $y = c_1 (\cosh mx + \sinh mx) + c_2 (\cosh mx - \sinh mx)$

i.e. $y = (c_1 + c_2) \cosh mx + (c_1 - c_2) \sinh mx$

$= A \cosh mx + B \sinh mx$

where $A = c_1 + c_2$, $B = c_1 - c_2$ are arbitrary constants

**Example 5:** Solve y'' + y = 0 and find the solution with initial conditions

$y(2) = 2;\ y\left(-\frac{\pi}{2}\right) = -2$

**Sol:** Given equation is

$y' + y = 0$ .......(1)

$\Rightarrow$ $(D^2 + 1)y = 0$

A.E. is $D^2 + 1 = 0$

$\Rightarrow$ $D = \underline{+}i$

$\therefore$ complete solution is

$y = c_1 \cos x + c_2 \sin x$ ......(2)

Given $y(2) = 2 \Rightarrow x = 2, y = 2;$

Putting these values in (2), we get

$2 = c_1 \cos 2 + c_2 \sin 2$ ......(3)

Given $y\left(-\frac{\pi}{2}\right) = -2 \Rightarrow x = -\frac{\pi}{2}, y = -2$

Putting these values in (2), we have

$-2 = 0 - c_2 \Rightarrow c_2 = 2$

Put $c_2 = 2$ in (3), we get

$2 = c_1 \cos 2 - 2 \sin 2$

$\Rightarrow \quad c_1 = \frac{2 - 2\sin 2}{\cos 2}$

Now put $c_1 = \frac{2 - 2\sin 2}{\cos 2}$ and $c_2 = 0$ in (2), we get

$y = \left(\frac{2 - 2\sin 2}{\cos 2}\right) \cos x + 2 \sin x$, as the required solution

**Example 6:** If $\frac{d^4y}{dx^4}$ - m4y = 0, then show that the solution $y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$, where $c_1, c_2, c_3, c_4$ are arbitrary constants.

**Sol:** Given equation is

$$\frac{d^4y}{dx^4} - m^4y = 0$$

The equation in the symbolic form is

$(D^4 - m^4)y = 0$

Now, A.E. is $D^4 - m^4 = 0$

$\Rightarrow \quad (D^2 + m^2)(D^2 - m^2) = 0$

$\Rightarrow \quad (D^2 + m^2)(D - m)(D + m) = 0$

$\Rightarrow \quad D = \pm im, m, -m$

$\therefore \quad$ C.S. is $y = c_1 \cos mx + c_2 \sin mx + Ae^{mx} + Be^{-mx}$

where $c_1, c_2, A, B$ are arbitrary constants.

$\therefore \quad y = c_1 \cos mx + c_2 \sin mx + A(\cosh mx + \sinh mx) + B(\cosh mx - \sinh mx)$

$= c_1 \cos mx + c_2 \sin mx + (A + B) \cosh mx + (A - B) \sinh mx$

Putting $A + B = c_3$, $A - B = c_4$ - we get

$y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$,

where $c_1, c_2, c_3, c_4$ are arbitrary constants.

Hence the required result

**Example 7:** Solve $l\,\dfrac{d^2\theta}{dt^2}$ + g$\theta$ = 0, given that $\theta = \propto$ and $\dfrac{d\theta}{dt}$ = 0 when t = 0

**Sol:** Given equation is

$$l\,\frac{d^2\theta}{dt^2} + g\theta = 0$$

Equation in the symbolic form is

$$(l\,D^2 + g)\theta = 0$$

A.E. is $l\,D^2 + g = 0$ or $D^2 = \dfrac{-g}{l}$

$\therefore$ $D = \pm\sqrt{\dfrac{g}{l}}\,i = 0 \pm \sqrt{\dfrac{g}{l}}\,i$

$\therefore$ C.S. is $\theta = e^{ox}\left(c_1 \cos\sqrt{\dfrac{g}{l}}\,t + c_2 \sin\sqrt{\dfrac{g}{l}}\,t\right)$

or $\theta = c_1 \cos\sqrt{\dfrac{g}{l}}\,t + c_2 \sin\sqrt{\dfrac{g}{l}}\,t$ .....(1)

Differentiating,

$$\frac{d\theta}{dt} = -c_1\sqrt{\frac{g}{l}}\,\sin\sqrt{\frac{g}{l}}\,t + c_2\sqrt{\frac{g}{l}}\,\cos\sqrt{\frac{g}{l}}\,t \qquad ....(2)$$

When t = 0, $\theta = \propto$, $\dfrac{d\theta}{dt} = 0$,

$\therefore$ from (1), $\propto = c_1$ or $c_1 = \propto$

and from (2), $0 = c_2\sqrt{\dfrac{g}{l}}$

or $c_2 = 0$

Putting in (1), we get

$\theta = \propto \cos\sqrt{\dfrac{g}{l}}\,t$, as the required solution.

**Example 8:** Solve y''' - 6y'' + 11y' - 6y = 0

y(0) = 0, y' (0) = 0, y''(0) = 2

**Sol:** Given differential equation is

$y''' - 6y'' + 11y' - 6y = 0$

or $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Equation in S.F. is

$(D^3 - 6D^2 + 11 D - 6)y = 0$

A.E. is $D^3 - 6D^2 + 11D - 6 = 0$

or $(D - 1)(D^2 - 5D + 6) = 0$

or $(D - 1)(D - 2)(D - 3) = 0$

$\therefore$ $D = 1, 2, 3$

$\therefore$ C.S. is $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ .....(1)

Now $y' = c_1e^x + 2c_2e^{2x} + 3c_3e^{3x}$ .....(2)

$\Rightarrow$ $y'' = c_1e^x + 4c_2e^{2x} + 9c_3e^{3x}$ ......(3)

Now $y(0) = 0 \Rightarrow 0 = c_1 + c_2 + c_3$ .....(4)

$y'(0) = 0 \Rightarrow 0 = c_1 + 2c_2 + 3c_3$ ......(5)

$y''(0) = 2 \Rightarrow 2 = c_1 + 4\, c_2 + 9\, c_3$ ......(6)

Subtracting (4) from (5), we get

$c_2 + 2c_3 = 0$ .......(7)

Subtracting (4) from (6), we get

$3c_2 + + 8c_3 = 2$ .......(8)

Solving (7) and (8), we get $c_2 = -2$, $c_3 = 1$

Putting values of $c_2$ and $c_3$ in (4), we get $c_1 = 1$

$\therefore$ from (1), $y = e^x -2e^{2x} + e^{3x}$,

which is the required solution.

**Self Check Exercise**

Q. 1 Solve $\frac{d^4y}{dx^4} - a^4y = 0, a > 0$

Q. 2 Solve $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

Q. 3 Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Q. 4 Solve $\frac{d^2y}{dx^2} - 3\frac{dx}{dt} + 2x = 0$, given that t = 0, x = 0, $\frac{dx}{dt} = 0$

Q. 5 Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$, given that y = 1 and $\frac{dy}{dx} = 2$ when x = 0

Q. 6 Solve $\frac{d^2y}{dx^2} + \mu x = 0$, $\mu > 0$ given that x = a and $\frac{dx}{dt} = 0$ when t = $\frac{\pi}{2\sqrt{\mu}}$

## 6.7 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined symbolic form of the equation
2. Defined Auxiliary equation
3. Discussed in detail product of operators
4. Discussed and find solution of the linear homogeneous equation with constant coefficients.

## 6.8 Glossary:

1. The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ is called an operator.
2. In symbolic form, we write

   $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$, ......., $\frac{d^n}{dx^n} = D^n$
3. Auxiliary equation is that equation which can be obtained by equating the symbolic coefficients of y to zero.
4. If $f^1(D)$ and $f^2(D)$ be two operators, then $f^1(D)$. $f^2(D)$ is also an operator.

## 6.9 Answer to Self Check Exercise

Ans.1 $y = c_1e^{ax} + c_2e^{-ax} + c_3 \cos ax + c_4 \sin ax$, where $c_1, c_2, c_3, c_4$ are arbitrary constants, is the required solution.

Ans.2 $y = c_1e^{x} + c_2e^{3x} + c_{3e}{}^{5x}$, where $c_1, c_2, c_3$ are arbitrary constants, is the required solution.

Ans.3 $y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$, where $c_1, c_2$ are arbitrary constants, is the required solution.

Ans.4 x = 0 is the required solution.

Ans.5 $y = e^{2x} \cos x$ is the required solution.

Ans. 6 $x = a \sin \sqrt{\mu}\ t$ is the required

## 6.10 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.


## 6.11 Terminal Questions

1. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$
2. Solve $\frac{d^2y}{dx^2} + w^2y = 0$
3. Solve $(D^4 + 8D^2 + 16)\, y = 0$, where $D = \frac{d}{dx}$
4. Solve $y'' + y = 0$ and find the solution with initial conditions $y(0) = 1$, $y'(0) = 0$
5. Solve $y'' + ay' + by = 0$, where a, b are +ve constants and

   (i) $a^2 > 4b$ (ii) $a^2 = 4b$ (iii) $a^2 < 4b$

   show that all solutions tend to zero as $x \to \infty$.

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## Unit - 7

# Solution of Non-Homo Geneous Equation with Constant Coefficients - I

**Structure**



## 7.1 Introduction

Non-homogeneous equations with constant coefficients are a type of differential equation where the equation involves a function and its derivatives, along with a non-zero function on the right-hand side. These equations are called non-homogeneous because they do not satisfy the property of homogeneity, which states that if $f$(x) is a solution to the equation, then k$f$(x) (where k is a constant) is also a solution.

To solve non-homogeneous equations with constant coefficients, one typically follows a two-stop process. First, the associated homogeneous equation is solved to find the complementary Solution (also known as the homogeneous solution). then, a particular solution is found by guessing a form of the solution that satisfies the non-homogeneous equation and substituting it into the equation. The general solution is obtained by adding the complementary solution and the particular solution.

## 7.2 Learning Objectives

After studying this unit, you should be able to:-

- Prove theorem on complete solution of the non-homogeneous equation with constant coefficients.
- Define complementary function, particular integral and inverse operator.
- Discuss method to solve the non-homogeneous equation with constant coefficients
- Discuss the method to evaluate $\frac{1}{f(D)}e^{ax}$ with examples.

## 7.3 Some Useful Theorems

**Theorem 1 :** If y = Y is the complete solution of the equation

$$P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots. + P_n y = 0 \qquad \ldots.(1)$$

and y = u is a particular solution (containing no arbitrary constants) of the equation

$$P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots. + P_n y = Q \qquad \ldots(2)$$

where Q is a function of x,

then the complete solution of (2) is y = Y + u.

**Proof :** Since y = Y is the c.s. of (1),

$$\therefore \quad P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots. + P_n y = 0 \qquad \ldots.(3)$$

Again, since y = u is a solution of (2),

$$\therefore \quad P_0\frac{d^n u}{dx^n} + P_1\frac{d^{n-1}u}{dx^{n-1}} + P_2\frac{d^{n-2}u}{dx^{n-2}} + \ldots\ldots. + P_n u = Q \qquad \ldots(4)$$

Putting y = Y + u in (2), we get

$$P_0\frac{d^n}{dx^n}(y + u) + P_1\frac{d^{n-1}}{dx^{n-1}}(y + u) + P_2\frac{d^{n-2}}{dx^{n-2}}(y + u) + \ldots\ldots. + P_n(y + u) = Q$$

$$\Rightarrow \quad P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots. + P_n y +$$

$$+ P_0\frac{d^n u}{dx^n} + P_1\frac{d^{n-1}u}{dx^{n-1}} + P_2\frac{d^{n-2}u}{dx^{n-2}} + \ldots\ldots. + P_n u = Q$$

i.e. 0 + Q = Q, which is true. [using (3) and (4)]

Thus y = Y + u is a solution of (2)

Since it contains n arbitrary constants and (2) is of the nth order,

$\therefore$ it is the complete solution of (2)

**Definitions (i) Complementary Function (C.F.)**

The portion y is known as complementary function.

**(ii) Particular Integral (P.I.) :** The portion u is known as particular solution.

(iii) Complete Solution of (2) = C.F. + P.I.

(iv) Inverse Operator $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ X is that function of x independent of arbitrary constants, which when operated on by $f$(D) gives X i.e.$f$(D). $\frac{1}{f(D)}$ X = X

Hence $\frac{1}{f(D)}$ is the inverse operator of $f$(D)

**Theorem 2 :** Prove that $\frac{1}{D}$ X = $\int$ X dx, no arbitrary constant being added.

**Proof.** Let $\frac{1}{D}$ X = z

Operating both sides by D, we get D. $\frac{1}{D}$X = Dz $\Rightarrow$ X = $\frac{dz}{dx}$

Integrating, we get z = $\int$ X dx, no arbitrary constant being added.

Hence $\frac{1}{D}$ X = $\int$ X dx, no arbitrary constant being added.

Note $\frac{1}{D}$ stands for integration.

**Theorem 3 :** Prove that $\frac{1}{f(D)}$X is the particular integral of the equation $f$(D) y = X.

**Proof :**$f$(D)y = X ...(1)

Put y = $\frac{1}{f(D)}$X $\therefore f$(D).$\frac{1}{f(D)}$X = X i.e. X = X, which is true.

$\therefore \quad \frac{1}{f(D)}$ X is a solution of (1).

Since it contains no arbitrary constants, $\therefore$ it is the particular integral.

**Theorem 4 :** Prove that $\frac{1}{D-a}$ X

$= e^{ax} \int X.e^{-ax}\, dx$, no arbitrary constant being added.

**Proof :** Let $\frac{1}{D-a}$ X = y ...(1)

Operating both sides by (D - a), we get (D - a) $\frac{1}{D-a}$ X = (D - a) y

or $\quad x = (D - a)\, y = \frac{dy}{dx} - ay$

or $\quad \frac{dy}{dx} - ay = X$ [Which is linear in y]

Here P = a and Q = X

Now $\int P\, dx = \int a\, dx = -ax$. $\therefore e^{\int P\, dx} = e^{-ax}$

$\therefore$ the solution is

$y \,.\, e^{-ax} = \int X.e^{-ax}\, dx$, no arbitrary constant being added.

or $\quad y = e^{ax} = \int X.e^{-ax}\, dx$, no arbitrary constant being added.

or $\quad \frac{1}{D-a} X = e^{ax} = \int X.e^{-ax}\, dx$, no arbitrary constant being added.

Let us do some examples to clarify what we have just said :-

**Example 1 :** Evaluate : (i) $\frac{1}{D} x^2$ (ii) $\frac{1}{D^2} x^4$ (iii) $\frac{1}{(D-1)(D-2)} e^x$

**Sol.** (i) $\frac{1}{D} x^2 = \int x^2\, dx = \frac{x^3}{3}$

(ii) $\frac{1}{D^2} x^4 = \frac{1}{D} \,.\, \frac{1}{D} x^4 = \frac{1}{D} \int x^4\, dx = \frac{1}{D} \frac{x^5}{5} = \frac{1}{5} \int x^5\, dx = \frac{1}{5} \,.\, \frac{x^6}{6} = \frac{x^6}{30}$

(iii) $\frac{1}{(D-1)(D-2)}$ e$^x$ = $\frac{1}{D-1}$ . $\frac{1}{D-2}$ e$^x$ = $\frac{1}{D-1}$ . e$^{2x}\int$ e$^{-2x}$ . e$^x$ dx

= $\frac{1}{D-1}$ . e$^{2x}\int$ e$^{-x}$ dx

= $\frac{1}{D-1}$ . e$^{2x}\frac{e^{-x}}{-1}$ = -$\left(\frac{1}{D-1}\right)$ e$^x$ = -e$^x\int$ e$^{-x}$ . e$^x$ dx = -e$^x\int$ 1 dx = -e$^x$.x

**Example 2 :** Show that

(i) $\frac{1}{D-a}$ x = e$^{ax}\int$ e$^{-ax}$ x dx, where x is a function of x.

(ii) Hence prove that

$\frac{1}{(D-a)^2}$ e$^{ax}$ = $\frac{x^2}{2}$ e$^{ax}$

(iii) Find the particular integral of

(D - e)$^2$ y = e$^{2x}$

**Sol. :** (i) Same as Theorem 4.

(ii) by first part,

$\frac{1}{D-a}$ e$^{ax}$ = e$^{ax}\int$ e$^{-ax}$e$^{ax}$ dx = -e$^{ax}\int$ 1 dx =xe$^{ax}$

Now $\frac{1}{(D-a)^2}$ e$^{ax}$ = $\frac{1}{D-a}\left[\frac{1}{D-a}e^{ax}\right]$= $\frac{1}{D-a}$ (xe$^{ax}$)

Here X = x e$^{ax}$

$\therefore$ by first part,

$\frac{1}{(D-a)^2}$ e$^{ax}$ = e$^{ax}\int$ e$^{-ax}$ x e$^{ax}$ dx = e$^{ax}\int$ x dx

= $\frac{x^2}{2}$ e$^{ax}$

(iii) $\frac{1}{(D-a)^2}$ e$^{2x}$ = e$^{2x}\int$ e$^{-2x}$ x e$^{2x}$ dx = e$^{2x}\frac{x^2}{2}$

**Self-Check Exercise-1**

Q1. Evaluate $\frac{1}{D}$(x sin x)

Q.2 Evaluate $\frac{1}{D^2}$ $(2x + 1)^2$

Q.3 Evaluate $\frac{1}{D}$ (x cos x)

## 7.4 Method To Solve The Equation

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots + P_n y = Q \qquad \ldots\ldots.(1)$$

where $P_0$, $P_1$, $P_2$,........., $P_n$ are constants and Q is a function of x

**Seep I:** write the equation in the symbolic form as

$(P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots\ldots. + P_n)\ y = Q$

**Seep II:** Write down the auxiliary equation (A.E.) as

$P_0D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots\ldots. + P_n = 0$

Solve it for D.

**Step III:** Write down complementary function (C.F.) by the same method as for writing down C.S. if R.H.S. is zero instead of Q.

**Step IV:** Find particular integral (P.I.)

[For P.I., see different eases)

**Step V:** Then C.S. is y = C.F. + P.I.

Now we will discuss further in this unit and next two units of finding a particular solution of (1)

## 7.5 Method To Evaluate $\frac{1}{f(D)} e^{ax}$

Prove that $\frac{1}{f(D)} e^{ax} = \frac{1}{f(D)} e^{ax}$ provided $f(a) \neq 0$

**Proof:** We know that

$D (e^{ax}) = ae^{ax}$, $D^2(e^{ax}) = a^2e^{ax}$, ......, $D^n (e^{ax}) = a^ne^{ax}$.

It is clear that D is replaced by a in each case.

$\therefore \quad f(D)e^{ax} = f(a)\ e^{ax}$

$\therefore \quad \frac{1}{f(D)} . f(D)e^{ax} = \frac{1}{f(D)} . f(a)\, e^{ax}$

or $\quad e^{ax} = \frac{1}{f(D)} f(a)e^{ax}$

$\therefore \quad \frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$

Hence $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$, where $f(a) \neq 0$

Rule: to evaluate $\frac{1}{f(D)} e^{ax}$, put D = a provided $f(a) \neq 0$

Let us do some examples

**Example 3:** Solve: $4 \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$

**Sol:** Equation in the symbolic form is (4D2 + 4D - 3) y = e2x

A.E. is $4D^2 + 4D - 3 = 0$ or $(2D - 1)(2D + 3) = 0 \therefore D = \frac{1}{2}, -\frac{3}{2}$

$\therefore \quad$ C.F. = $c_1 e^{x/2} + c_2 e^{-3x/2}$

P.I. = $\frac{1}{4D^2 + 4D = 3} e^{2x}$

$= \frac{1}{4(4)+4(2)-3} e^{2x}$ [Put D = 2]

$= \frac{1}{21} e^{2x}$

Hence C.S. is y = C.F. + P.I. i.e. $y = c_1e^{x/2} + c_2e^{-3x/2} + \frac{1}{21} e^{2x}$ is the reqd. solution

**Example 4:** Solve: (a) $(D^3 - 3D^2 + 4)\, y = e^{3x}$. (b) $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = e^{2x}$

**Sol:** (a) Given equation is $(D^3 - 3D^2 + 4)\, y = e^{3x}$

A.E. is $D^3 - 3D^2 + 4 = 0 \Rightarrow (D + 1)(D^2 - 4D + 4) = 0$

$\Rightarrow \quad (D + 1)(D - 2)^2 = 0 \Rightarrow D = -1, 2, 2 \therefore$ C.F. = $c_1 e^{-x} + (c_2 + c_3x)\, e^{2x}$

A.E. is $D^2 - 1 = 0 \Rightarrow D = +1 \therefore$ C.F. is $c_1 e^t + c_2 e^{-t}$

$$\text{P.I.} = \frac{1}{D^2-1} e^t + \frac{1}{D^2-1} e^{-t}$$

$$= t.\ \frac{1}{2D} e^t + t.\ \frac{1}{2D} e^{-t} \text{ (Case of failure)}$$

$$= \frac{t}{2} . \frac{1}{1} e^t + \frac{t}{2} . \frac{1}{-1} e^{-t} = \frac{t}{2} [e^t - e^{-t}]$$

$\therefore$ complete solution is

x = C.F. + P.I.

$$= c_1 e^t + c_2 e^{-t} + \frac{t}{2} [e^t - e^{-t}]$$

Thus $x = c_1 e^t + c_2 e^{-t} + t. \sinh t$, where $c_1$, $c_2$ are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4} . e^{3x} = \frac{1}{(3)^3 - 3(3^2) + 4} e^{3x} = \frac{e^{3x}}{4}$$

$\therefore$ C.S. is $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x} + \frac{e^{3x}}{4}$, where $c_1$, $c_2$, $c_3$ are arbitrary constants.

(b) given equation in symbolic form is $(D^3 - 3D + 2)y = e^{2x}$

A.E. is $D^3 - 3D + 2 = 0 \Rightarrow (D - 1)(D^2 + D - 2) = 0$

$$\Rightarrow \quad D = 1, \frac{-1 \pm \sqrt{1+8}}{2} = 1, \frac{-1 \pm 3}{2} = 1, 1 - 2$$

$\therefore$ C.F. $= (c_1 + c_2 x) e^x + c_3 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^3 - 3D + 2} . e^{2x} = \frac{1}{(2)^3 - 3(2) + 2} . e^{2x} = \frac{e^{2x}}{4}$$

$\therefore$ C.S. is $y = (c_1 + c_2 x) e^x + c_3 e^{-2x} + \frac{e^{2x}}{4}$, where $c_1$, $c_2$, $c_3$ are arbitrary constants.

**Case of Failure.**

Rule to evaluate: $\frac{1}{f(D)} e^{ax}$ when $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x. \frac{1}{\frac{d}{dD}[f(D)]} e^{ax} = x. \frac{1}{\text{diff.coeff.of denom.w.r.t.D}} e^{ax}$$

Note: If by using this rule, the denominator again vanishes, repeat the rule.

**Example 5:** Solve: $3\frac{d^2y}{dx^2} + \frac{dy}{dx} - 14y = 13e^{2x}$

**Sol:** Equation in the symbolic form is $(3D^2 + D - 14)\, y = 13e^{2x}$

A.E. is $3D^2 + D - 14 = 0$ or $(3D + 7)(D - 2) = 0 \therefore D = 2, -\frac{7}{3}$

$\therefore$ C.F. $= c_1e^{2x} + c_2 e^{-\frac{7x}{3}}$

P.I. $= \frac{1}{3D^2 - D - 14} 13e^{2x} = 13. \frac{1}{3D^2 - D - 14} e^{2x}$

$= 13. \frac{1}{3(4)+2-14} e^{2x}$ [Put D = 2]

$= 13. \frac{1}{0} e^{2x}$ (This is a case of failure)

$\therefore$ P.I. $= x. \frac{1}{\frac{d}{dD}(3D^2 - D - 14)} .13\, e^{2x}$

$= 13x. \frac{1}{6D+1} e^{2x} = 13x. \frac{1}{6(2)+1} e^{2x} = 13x. \frac{1}{13} e^{2x}. = xe^{2x}$

C.S. is y = C.F. + P.I. i.e. $y = c_1e^{2x} + c_2 e^{-\frac{7x}{3}} + xe^{2x}$, is the reqd solution.

**Example 6:** Solve: $\frac{d^2x}{dt^2} = x + e^t + e^{-t}$

**Sol:** The given differential equation can be written as : $\frac{d^2x}{dt^2} - x = e^t + e^{-t}$

Equation in symbolic form is $(D^2 - 1)\, x = e^t + e^{-t}$ where $D = \frac{d}{dt}$

**Example 7:** Solve $\frac{d^2y}{dx^2}$ - 7 $\frac{dy}{dx}$ + 6y = $e^{2x}$, given that y = 0 when x = 0

**Sol:** Equation in symbolic form is

$(D^2 - 7D + 6)\, y = e^{2x}$

A.E. is $D^2 - 7D + 6 = 0$

or $(D - 1)(D - 6) = 0$

$\therefore$ D = 1, 6

$\therefore$ C.F. = $c_1e^x + c_2e^{6x}$

P.I. = $\frac{1}{D^2 - 7D + 6} e^{2x}$

= $\frac{1}{4-14+6} e^{2x}$ [Put D = 2]

= $-\frac{1}{4} e^{2x}$

$\therefore$ C.S. is y = C.F. + P.I. = $c_1e^x + c_2e^{6x} - \frac{1}{4} e^{2x}$

When x = 0, y = 0, then $0 = c_1 + c_2 - \frac{1}{4}$

or $c_2 = \frac{1}{4} - c_1$

$\therefore$ $y = c_1e^x + \left(\frac{1}{4} - c_1\right)e^{6x} - \frac{1}{4} e^{2x}$

= $c_1(e^x - e^{6x}) + \frac{1}{4}(e^{6x} - e^{2x})$, is the required solution, where $c_1$, $c_2$ are arbitrary constants.

**Example 8:** find the P.I. of the differential equation $\frac{d^2y}{dx^2}$ - 5 $\frac{dy}{dx}$ + 6y = $e^x$

**Sol:** Given equation is $\frac{d^2y}{dx^2}$ - 5 $\frac{dy}{dx}$ + 6y = $e^x$

or in S.F. $(D^2 - 5D + 6)\, y = e^x$

P.I. = $\frac{1}{D^2 - 5D + 6} e^x$

$= \dfrac{1}{(1)^2 - 5(1) + 6} e^x$ [Put D = 1]

$= \dfrac{1}{1\text{-}5+6} e^x$

$= \dfrac{1}{2} e^x$

**Example 9:** Solve $(D - 1)^3 y = 16 e^{3x}$

**Sol:** Given equation is

$(D - 1)^3 y = 16e^{3x}$

A.E. is $(D - 1)^3 = 0$

$\therefore$ D - 1, 1, 1

$\therefore$ C.F. = $(c_1 + c_2x + c_3x^2) e^x$

P.I. = $\dfrac{1}{(D-1)^3} (16e^{3x}) = 16 \dfrac{1}{(D-1)^3} e^{3x}$

$= 16 \dfrac{1}{(3-1)^3} e^{3x}$

$= 16 \dfrac{1}{8} e^{3x}$

$= 2 e^{3x}$

$\therefore$ C.S. = C.F. + P.I.

$= (c_1 + c_2x + c_3x^2) e^x + 2e^{3x}$

**Example 10:** Solve: $(D^3 - 5D^2 + 7D - 3) y = e^{2x}\cosh x$

**Sol:** Given equation is

$(D^3 - 5D^2 + 7D - 3) y = e^{2x}\cosh x$

A.E. is $D^3 - 5D^2 + 7D - 3 = 0$

$\Rightarrow$ $(D - 1) (D^2 - 4D + 3) = 0$

$\Rightarrow$ $(D - 1) (D - 1) (D - 3) = 0$

$\Rightarrow$ D = 1, 1, 3

$\therefore$ C.F. = $(c_1 + c_2x) e^x + c_3 e^{3x}$

P.I. = $\frac{1}{D^3-5D^2+7D-3}$ $e^{2x}\cosh x$

$$= \frac{1}{D^3-5D^2+7D-3} e^{2x}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{D^3-5D^2+7D-3}\left(\frac{e^{3x}+e^x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^3-5D^2+7D-3}e^{3x}+\frac{1}{D^3-5D^2+7D-3}e^x\right]$$

$$= \frac{1}{2}\left[(x)\frac{1}{3D^2-10D+7}e^{3x}+(x)\frac{1}{3D^2-10D+7}e^x\right]$$

$$= \frac{1}{2}\left[x\frac{1}{3(3)^2-10(3)+7}e^{3x}+x.x\frac{1}{6D-10}e^x\right]$$

$$= \frac{1}{2}\left[x\frac{1}{4}e^{3x}+x^2\frac{1}{6(1)-10}e^x\right]$$

$$= \frac{xe^{3x}}{8} + \frac{x^2e^x}{-8}$$

$$= \frac{xe^{3x}}{8} - \frac{x^2e^x}{8}$$

$\therefore$ C.S. is y = C.F. + P.I.

$$= (c_1 + c_2x)\, e^x + c_3e^{3x} + \frac{xe^{3x}}{8} - \frac{x^2e^x}{8}$$

**Example 11:** Solve 4y'' - 4y' + y = $e^{x/2}$

**Sol:** Given differential equation is

4y'' - 4y' + y = $e^{x/2}$

Equation in S.F. is

$(4D^2 - 4D + 1)\, y = e^{x/2}$

A.E. is

$4D^2 - 4D + 1 = 0$

$\therefore$ $(2D - 1)^2 = 0 \Rightarrow$ $D = \frac{1}{2}, \frac{1}{2}$

$\therefore$ C.F. is $e^{x/2}$ $(c_1 + c_2x)$

Now P.I. is

$$= \frac{1}{(2D-1)^2} e^{x/2}$$

$$= \frac{1}{\left(2\times\frac{1}{2}-1\right)^2} e^{x/2} = \frac{1}{\partial} e^{x/2}$$ [Case of failure]

$$= x. \frac{1}{2(2D-1)\times 2} e^{x/2}$$

$$= \frac{x}{4} \frac{1}{\left(2\times\frac{1}{2}-1\right)} e^{x/2}$$

$$= \frac{x}{4} \frac{1}{\partial} e^{x/2}$$ [Again case of failure]

$$= x^2 \frac{1}{4(2)} e^{x/2}$$

$$= \frac{x^2}{8} e^{x/2}$$

$\therefore$ C.S. is $y = e^{x/2} (c_1 + c_2x) + \frac{x^2}{8} e^{x/2}$

**Example 12:** Solve: $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 2$

**Sol:** Given differential equation is

$$\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 2$$

Equation in S.F. is written as

$(D^4 - D^2)\, y = 2$

A.E. is $D^4 - D^2 = 0$

$\Rightarrow$ $D^2(D^2 - 1) = 0$

$\Rightarrow$ $D^2 = 0, 1$

$\Rightarrow$ $D = 0, 0, -1, 1$

$\therefore$ C.F. $= e^{0x} (c_1 + c_2x) + c_3e^{-x} + c_4^{ex}$

$= c_1 + c_2x + c_3e^{-x} + c_4e^x$

P.I. $= \frac{1}{D^4 - D^2}(2)$

$= 2 \frac{1}{D^4 - D^2}(1)$

$= 2 \frac{1}{D^4 - D^2} e^{ox}$

$= 2 \frac{1}{0} e^{ox}$ [case of failure]

$= 2x \frac{1}{4D^3 - 2D} e^{ox}$

$2x \frac{1}{0} e^{ox}$ [Again case of failure]

$= 2x^2 \frac{1}{12D^2 - 2} e^{ox}$

$= 2x^2 \frac{1}{0-2} e^{ox}$

$= 2x^2 \frac{1}{-2} e^{ox}$

$= -x^2$

$\therefore$ C.S. is $y = c_1 + c_2x + c_3e^{-x} + c_4e^x - x^2$

**Example 13:** Solve $\frac{d^2y}{dx^2} - y = \cosh x$

**Sol:** Given differential equation is

$$\frac{d^2y}{dx^2} - y = \cosh x$$

Equation in S.F. is

$(D^2 - 1)\, y = \cosh x$

A.E. is $D^2 - 1 = 0$

$\therefore \quad (D + 1)(D - 1) = 0$

$\Rightarrow \quad D = -1, 1$

$\therefore \quad$ C.f. is $c_1e^{-x} + c_2e^{x}$

P.I. = $\frac{1}{(D^2-1)} \cosh x$

$= \frac{1}{(D^2-1)}\left[\frac{e^x+e^{-x}}{2}\right]$

$= \frac{1}{2}\,\frac{1}{D^2-1} e^x + \frac{1}{2}\,\frac{1}{D^2-1} e^{-x}$

Now $\quad \frac{1}{D^2-1} e^x = \frac{1}{1-1} e^x = \frac{1}{0} e^x$ [Case of failure]

$= x\,\frac{1}{2\mathrm{D}} e^x$

$= x\,\frac{1}{2} e^x$

$= \frac{1}{2} xe^x$

Also $\quad \frac{1}{D^2-1} e^{-x} = \frac{1}{1-1} e^{-x} = \frac{1}{0} e^{-x}$ [case of failure]

$= x\frac{1}{2\mathrm{D}} e^{-x}$

$= x\frac{1}{-2} e^{-x}$

$= -\frac{1}{2} xe^{-x}$

$\therefore \quad$ P.I. = $\frac{1}{4} xe^x - \frac{1}{4} xe^{-x}$

$\therefore \quad$ C.S. is

$$y = c_1e^{-x} + c_2e^{x} + \frac{1}{4}xe^{x} - \frac{1}{4}xe^{-x}$$

or $$y = c_1e^{-x} + c_2e^{x} + \frac{1}{2}x\left(\frac{e^{x}-e^{-x}}{2}\right)$$

$\therefore$ $$y = c_1e^{-x} + c_2e^{x} + \frac{1}{2}x \sinh x$$

**Example 14:** Solve $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = 1 + e^x$

**Sol:** Given differential equation is

$$2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = 1 + e^x$$

Equation in S.F. is

$$(2D^3 - 3D^2 + 1)\,y = 1 + e^x$$

A.E. is $2D^3 - 3D^2 + 1 = 0$

or $(D - 1)(2D^2 - D - 1) = 0$

$\therefore$ $D = 1, \frac{1\pm\sqrt{1+8}}{4} = 1, \frac{1\pm3}{4} = 1, 1, -1/2$

$\therefore$ C.F. is

$$(c_1 + c_2x)\,e^x + c_3e^{-x/2}$$

P.I. $= \frac{1}{2D^3-3D^2+1}(1 + e^x)$

Now $\frac{1}{2D^3-3D^2+1} = \frac{1}{2D^3-3D^2+1}e^{ox}$

$= \frac{1}{0-0+1}e^{ox} = 1$

Also $\frac{1}{2D^3-3D^2+1}e^x = \frac{1}{2-3+1}e^x = \frac{1}{0}e^x$ [case of failure]

$= x\frac{1}{6D^2-6D}e^x$

$= x \dfrac{1}{6-6} e^x$

$= x \dfrac{1}{0} e^x$ [Again case of failure]

$= x^2 \dfrac{1}{12D-6} e^x$

$= x^2 \dfrac{1}{12-6} e^x$

$= \dfrac{1}{6} x^2 e^x$

$\therefore$ C.S. is $y = (c_1 + c_2 x) e^x + c_3 e^{-x/2} + 1 + \dfrac{1}{6} x^2 e^x$

**Self Check Exercise-2**

Q. 1 Solve $(D^2 + 4D + 3) y = e^{-3x}$

Q. 2 Solve $(D^2 - a^2) y = e^{ax} + e^{nx}$

Q. 3 Solve $(D^3 - 1) y = (e^x + 1)^2$

Q. 4 Solve $D^2y - 3Dy + 2y = \cosh x$

Q. 5 Solve $(D + 2) (D - 1)^2 y = e^{-2x} + 2 \sinh x$

## 7.6 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Proved theorem related to complete solution.
2. Defined complementary function, Particular integral and Inverse operator.
3. Discussed method to solve non-homogeneous equation with constant coefficients.
4. Discussed method to evaluate $\dfrac{1}{f(D)} e^{ax}$ with exampled. Also discussed the case of failure.

## 7.7 Glossary:

1. Complete solution of non-homogeneous equation with constant coefficients is the sum of complementary function and particular integral.

2. $\frac{1}{f(D)} \times$ is that function of x independent of arbitrary constants, which when operated on by $f$(D) gives x i.e. $f$(D) $\frac{1}{f(D)}$ = X.

## 7.8 Answer to Self Check Exercise-1

Ans.1 -x cos x + sin x

Ans.2 $\frac{1}{48}$ $(2x + 1)^4$

Ans.3 x sin x + cos x

**Self-Check Exercise-2**

Ans.1 $y = c_1e^{-x} + \left(c_2 - \frac{x}{2}\right)e^{-3x}$, where $c_1$, $c_2$ are arbitrary constants.

Ans.2 $y = c_1e^{ax} + c_2e^{-ax} + \frac{xe^{ax}}{n^2-a^2} + \frac{e^{nx}}{n^2-a^2}$

Ans. 3 $y = c_1e^{x} + e^{-x/2}\left\{c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right\} + \frac{e^{2x}}{7} + \frac{2xe^{x}}{3} - 1$

Ans. 4 $y = c_1e^{x} + c_2e^{2x} - \frac{1}{2}xe^{x} + \frac{1}{12}e^{-x}$

Ans. 5 $y = c_1e^{-2x} + (c_2 + c_3x)\,e^{x} + \frac{xe^{-2x}}{9} + \frac{x^2e^{x}}{6} - \frac{e^{-x}}{4}$

## 7.9 References/Suggested Readings


1. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
2. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.


## 7.10 Terminal Questions

1. Prove that

$$\frac{1}{(D-a)^n}e^{ax} = \frac{x^n}{n!}e^{ax}$$

2. Evaluate $\frac{1}{D^2}$ $(4x + 5)^3$

3. Find the P.I. of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x$$

4. Solve $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$

5. Solve the differential equation

$y'' + 3y' + y = 30e^{-x}$

6. Solve the differential equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}$$

7. Solve the differential equation

$$\frac{d^3y}{dx^3} - 5\frac{d^2y}{dx^2} + 7\frac{dy}{dx} -3 = e^{2x}\cosh x$$

8. Prove that $F(D)\, e^{ax} = e^{ax} F(a)$

9. Solve the differential equation

$$\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$$

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## Unit - 8

# solution of Non-Homogeneous Equation With Constant Coefficients-II

**Structure**



**8.1 Introduction**

Non-homogeneous equations with constant coefficients are a particular type of linear differential equations of the form

$P_0\, y^{(n)} + P_1\, y^{(n-1)} + ......+ P_{n-1}\, y^1 + P_n\, y = Q(x)$

where y is the unknown function of x, y' denotes the first derivative of y with respect to x, $y^{(n)}$ represents the $n^{th}$ derivative of y with respect to x and $P_0, P_1, ....., P_n$ are constants.

The term Q(x) on the right-hand side of the equation is the non-homogeneous part, which distinguishes these equations from homogeneous equation where Q(x) = 0

The solution of a non-homogeneous equation with constant coefficients involves finding the particular solution that satisfies the equation, as well as the general solution of the corresponding homogeneous equation (where Q(x) = 0). The general solution of the non-homogeneous equation is the sum of the particular solution and the general solution of the homogeneous equation.

In UNIT-7, we have discussed the solution of non-homogeneous equation with constant coefficient in detail and also discussed the method to evaluate $\frac{1}{f(D)}$ $e^{ax}$.

In the present unit, we will discuss further two types alongwith case a failures.

## 8.2 Learning Objectives

After studying this UNIT, you should be able to :-

- Discuss the method to evaluate $\frac{1}{f(D^2)}$ sin ax or $\frac{1}{f(D^2)}$ cos ax alongwith their case of failures.
- Solve non-homogeneous equations with constant coefficient and non-homogeneous part is of the form $\frac{1}{f(D^2)}$ sin ax or $\frac{1}{f(D^2)}$ cos ax.
- Discuss the method to evaluate $\frac{1}{f(D)}$ $x^m$ and solve equations related to it.

## 8.3 Method to Evaluate $\frac{1}{f(D^2)}$ sin ax or $\frac{1}{f(-a^2)}$ cos ax.

(a) Prove that $\frac{1}{f(D^2)}$ sin ax = $\frac{1}{f(D^2)}$ cos ax, provided $f(-a^2) \neq 0$

**Proof :** We know that

D cos ax = - sin (ax) . a

= -a sin ax

$D^2$ cos ax = -a cos ax . a = $-a^2$ cos ax

$D^3$ cos ax = $-a^2$ (-sin ax) . a = $-a^3$ sin ax

$D^4$ cos ax = $a^3$ (cos ax) . a = $a^4$ cos ax

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i.e. $D^2$ cos ax = $(-a^2)$ cos ax

$(D^2)^2$ cos ax = $(-a^2)^2$ cos ax, and so on

$\therefore$ $(D^2)^n$ cos ax = $(-a^2)^n$ cos ax

$\therefore$ $f(D^2)$ cos ax = $f(-a^2)$ cos ax

[$\therefore f(D^2)$ is a polynussia in $D^2$]

Operating on both sides by $\frac{1}{f(D^2)}$, we get

$$\frac{1}{f(D^2)} f(D^2) \cos ax = \frac{1}{f(D^2)} [f(-a^2) \cos ax]$$

or $\cos ax = f(-a^2) \frac{1}{f(D^2)} \cos ax$

Dividing both sides by $f(-a^2)$, we get

$$\frac{1}{f(-a^2)} \cos ax = \frac{1}{f(D^2)} \cos ax$$

or $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax, \; f(-a^2) \neq 0$

Similarly $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \; f(-a^2) \neq 0$

Note : If $f(D)$ contains odd powers of D also, it can be put in the form

$f(D) = f_1(D^2) + D\, f_2(D^2)$,

Then

$$\frac{1}{f(D)} \cos ax = \frac{1}{f_1(D^2) + f_2(D^2)} \cos ax$$

$$= \frac{1}{f_1(-a^2) + Df_2(-a^2)} \cos ax$$

$$= \frac{1}{p + qD} \cos ax,$$

where $p = f_1(-a^2)$ and $q = f_2(-a^2)$

$$= (p - qD) \frac{1}{(p + qD)} \frac{1}{p + qD} \cos ax$$

$$= (p - qD) \frac{1}{p^2 + q^2 D^2} \cos ax$$

$$= (p - qD) \frac{1}{p^2 + q^2 (a^2)} \cos ax$$

$$= \frac{1}{p^2 + q^2a^2} \text{ (p - qD) cos ax}$$

$$= \frac{1}{p^2 + q^2a^2} \text{ [p cos ax - q D cos ax]}$$

$$= \frac{1}{p^2 + q^2a^2} \text{ [p cos ax q a sin ax]}$$

Similarly we can deal with $\frac{1}{f(D)}$ sin ax

Rule to evaluate $\frac{1}{f(D^2)}$ sin ax or $\frac{1}{f(D^2)}$ cos as

Put $D^2 = -a^2$ provided $f(-a^2) \neq 0$

(b) Case of Failure

Prove that $\frac{1}{f(D^2)}$ cos ax = x $\frac{1}{\frac{d}{dD}[f(D^2)]}$

**Proof :** $\frac{1}{f(D^2)}$ cos ax = Real part of $\frac{1}{f(D^2)} e^{iax}$

= Real part of x $\frac{1}{\frac{d}{dD}[f(D^2)]} e^{iax}$ $[\therefore f(\text{-a2}) = 0]$

= Real part of x $\frac{1}{\frac{d}{dD}[f(D^2)]}$ (cos ax + i sin ax)

= x $\frac{1}{\frac{d}{dD}[f(D^2)]}$ cos ax

Similarly, we can prove that

$\frac{1}{f(D^2)}$ sin ax = x $\frac{1}{\frac{d}{dD}[f(D^2)]}$ sin ax, when $f(-a^2) = 0$

**Note :** If by using this rule, the denominator again vanishes, repeat the rule.

To clarify what we have just said, consider the following examples :-

**Example 1 :** Solve : $\frac{d^2y}{dx^2}$ - 3 $\frac{dy}{dx}$ + 2y = sin 3x.

**Sol.** Equation in the symbolic form is $(D^2 - 3D + 2)\, y = \sin 3x$

$\therefore$ A.E. is $D^2 - 3D + 2 = 0$ or $(D - 1)(D - 2) = 0 \Rightarrow D = 1.2$

$\therefore$ C.F. = $c_1 e^x + c_2 e^{2x}$

Again P.I. is = $\frac{1}{D^2 - 3D + 2} \sin 3x$

$= \frac{1}{-9 - 3D + 2} \sin 3x$ [put $D^2 = -9$]

$= \frac{1}{7 + 3D} \sin 3x = -\frac{7 - 3D}{(7 + 3D)(7 - 3D)} \sin 3x$

$= \frac{-(7 - 3D)}{49 - 9D^2} \sin 3x = -\frac{7 - 3D}{49 - 9(-9)} \sin 3x$

$= \frac{-7 + 3D}{130} \sin 3x = -\frac{7}{130} \sin 3x + \frac{3}{130} D \sin 3x$

$= \frac{7}{130} \sin 3x + \frac{3}{130} (\cos 3x)(3) = -\frac{7}{130} \sin 3x + \frac{9}{130} \cos 3x$

$\therefore$ C.S. is y = C.f. + P.I.

i.e. $y = c_1 e^x + c_2 e^{2x} - \frac{7}{130} \sin 3x + \frac{9}{130} \cos 3x$, is the complete solution.

**Example 2:** Solve: (i) $(D^3 + D^2 - D - 1)\, y = \cos 2x$ (ii) $(D^3 + D^2 - D - 1)\, y = \sin 2x$

**Sol:** (i) Given equation is $(D^3 + D^2 - D - 1)\, y = \cos 2x$

A.E. is $D^3 + D^2 - D - 1 = 0 \Rightarrow D^2(D + 1) - 1(D + 1) = 0$

$\Rightarrow$ $(D + 1)(D^2 - 1) = 0 \Rightarrow (D - 1)(D + 1)^2 = 0 \Rightarrow D = 1, -1, -1$

C.F. = $c_1 e^x + (c_2 + c_3 x) e^{-x}$

P.I. $= \frac{1}{D^2 + D^2 - D - 1} \cos 2x = \frac{1}{D^2.D + D^2 - D - 1} \cos 2x$

$= \frac{1}{-4D - 4 - D - 1} \cos 2x = \frac{1}{-5D - 5} \cos 2x = \frac{-1}{5D + 5} \cos 2x$

$= \frac{-1}{5} . \frac{(D - 1)}{(D + 1)(D - 1)} \cos 2x = \frac{-1}{5} \frac{D - 1}{D^2 - 1} . \cos 2x$

$= \frac{-1}{5} \frac{1}{-4-1}$ (D - 1) cos 2x = $\frac{1}{25}$ (D cos 2x - cos 2x)

$= \frac{1}{25}$ [-2 sin 2x - cos 2x] = $\frac{-1}{25}$ (2sin 2x + cos 2x)

$\therefore$ C.S. is y = $c_1e^x + (c_2 + c_3x)e^{-x} - \frac{1}{25}$ (2 sin 2x + cos 2x)

(ii) Given equation is $(D^3 + D^2 - D - 1)$ y = sin 2x

A.E. is $D^3 + D - 1 = 0 \Rightarrow D^2(D + 1) -1 (D + 1) = 0$

$\Rightarrow$ $(D + 1)(D^2 - 1) = 0 \Rightarrow (D + 1)^2 (D - 1) = 0 \Rightarrow D = -1, -1, 1$

C.F. = $(c_1 + c_2x)e^{-x} + c_3e^x$

P.I. $= \frac{1}{D^3+D^2-D-1}$ sin 2x = $\frac{1}{D.D^2+D^2-D-1}$ sin 2x

$= \frac{1}{D(-4)+(-4)-D-1}$ sin 2x = $\frac{-1}{5}\frac{1}{D+1}$ sin 2x

$= \frac{-1}{5}\frac{(D-1)}{(D-1)(D+1)}$ sin 2x = $\frac{-1}{5}\frac{(D-1)}{(D^2-1)}$ sin 2x

$= \frac{-1}{5}\frac{D-1}{-4-1}$ sin 2x = $\frac{1}{25}$ [D sin 2x - sin 2x] = $\frac{1}{25}$ [2 cos 2x - sin 2x]

C.S. is y = $(c_1 + c_2x)e^{-x} + c_3e^x + \frac{1}{25}$ [2 cos 2x - sin 2x]

**Example 3:** Solve $\frac{d^2y}{dx^2} + a^2y = \cos ax$.

**Sol:** Equation in the symbolic form is $(D^2 + a^2)$ y = cos ax.

$\therefore$ A.E. is $D^2 + a^3 = 0$ or $D^2 = -a^2$

$\Rightarrow$ D = $\pm$ia or D = 0 $\pm$ia

$\therefore$ C.F. = $e^{0x}(c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$

Again P.I. $= \frac{1}{D^2+a^2}$ cos ax = $\frac{1}{-a^2+a^2}$ cos ax [Put $D^2 = -a^2$]

$= \frac{1}{0}$ . cos ax (This is a case of failure)

$= x. \dfrac{1}{\dfrac{d}{dD}(D^2+a^2)}$ cos ax = x. $\dfrac{1}{2D}$cos ax

$= \dfrac{x}{2}\int$ cos ax dx = $\dfrac{x}{2} . \dfrac{\sin ax}{a} = \dfrac{x\sin ax}{2a}$

∴ C.S. is y = C.F. + P.I.

i.e. $y = c_1 \cos ax + c_2 \sin ax + \dfrac{x\sin ax}{2a}$ is the complete solution.

**Example 4:** Solve: $\dfrac{d^2x}{dt^2} + b^2x = k\cos bt$, given that x = 0 and $\dfrac{dx}{dt} = 0$ when t = 0

**Sol:** The equation in symbolic form is $(D^2 + b^2)\, x = k \cos bt$

∴ A.E. is $D^2 + b^2 = 0$ or $D^2 = -b^2$

⇒ $D = \pm ib$ or $D = 0 \pm ib$

∴ C.F. = $e^{0x}(c_1 \cos bt + c_2 \sin bt) = c_1 \cos bt + c_2 \sin bt$.

P.I. = $\dfrac{1}{D^2+b^2}$ k cos bt

$= k\dfrac{1}{D^2+b^2}$ cos bt

$= k\dfrac{1}{-b^2+b^2}$ cos bt

$= k\dfrac{1}{0}$ cos bt [case of failure]

$= k.t. \dfrac{1}{\dfrac{d}{dD}(D^2+b^2)}$ cos bt

$= kt\dfrac{1}{2D}$ cos bt

$= \dfrac{kt}{2}\int$ cos bt dt

$= \dfrac{kt}{2b}$ sin bt

∴ C.S. is x = C.F. + P.I.

i.e. $\quad x = c_1 \cos bt + c_2 \sin bt + \dfrac{kt}{2b} \sin bt$

**Example 5:** Solve $(D^2 - 4)\, y = 2 \sin \dfrac{x}{2}$

**Sol:** Given equation is

$$(D^2 - 4)\, y = 2 \sin \frac{x}{2}$$

A.E. is $D^2 - 4 = 0 \Rightarrow D^2 = 4$

$\Rightarrow \quad D = \pm 2$

$\therefore \quad$ C.F. $= c_1e2^x + c_2e^{-2x}$

P.I. $\quad = \dfrac{1}{D^2 - 4}\left(2\sin\dfrac{x}{2}\right)$

$$= 2\, \frac{1}{-\frac{1}{4} - 4} \sin \frac{x}{2}$$

$$= \frac{-8}{17} \sin \frac{x}{2}$$

C.S. is $y = c_1e^{2x} + c_2e^{-2x} - \dfrac{8}{17} \sin \dfrac{x}{2}$

**Example 6:** Solve $(D^3 + 1)\, y = \sin 3x - \cos^2 \dfrac{x}{2}$

**Sol:** Given equation is

$$(D^3 + 1) = \sin 3x - \cos^2 \frac{x}{2}$$

$\Rightarrow \quad (D^3 + 1)\, y = \sin 3x - \left(\dfrac{1 + \cos x}{2}\right)$

A.E. is $D^3 + 1 = 0$

$\Rightarrow \quad (D + 1)(D^2 - D + 1) = 0$

$\Rightarrow \quad D = -1, 1 \pm \dfrac{\sqrt{1-4}}{2}$

$= -1,\ \frac{1 \pm i\sqrt{3}}{2}$

$= -1,\ \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

C.F. is $c_1e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$

P.I. is

$= \frac{1}{D^3+1}\left[\sin 3x - \frac{1+\cos x}{2}\right]$

$= \frac{1}{D.D^2+1}$ sin 3x - $\frac{1}{D^3+1}.\frac{1}{2}$ - $\frac{1}{D.D^2+1}.\frac{1}{2}$ cos x

$= \frac{1}{D(-9)+1}$ sin 3x - $\frac{1}{2}\ \frac{1}{D^3+1}e^{ox}$ - $\frac{1}{D(-1)+1}\left(\frac{\cos x}{2}\right)$

$= \frac{1+9D}{(1+9D)(1-9D)}$ sin 3x - $\frac{1}{2}\ \frac{1}{0+1}$ - $\frac{1+D}{(1+D)(1-D)}\left(\frac{\cos x}{2}\right)$

$= \frac{1+9D}{1-81D^2}$ sin 3x - $\frac{1}{2}$ - $\frac{1+D}{1-D^2}\left(\frac{1}{2}\cos x\right)$

$= \frac{1+9D}{1-81(-9)}$ sin 3x - $\frac{1}{2}$ - $\frac{1}{2}\ \frac{1+D}{1-(-1)}$ cos x

$= \frac{1}{730}$ [sin 3x + 9D sin 3x] - $\frac{1}{2}$ - $\frac{1}{4}$ (cos x + D cos x)

$= \frac{1}{730}$ [sin 3x + 9(3 cos 3x)] - $\frac{1}{2}$ - $\frac{1}{4}$ (cos x - sin x)

C.S. is

$y = c_1e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{730}$ [sin 3x + 27 cos 3x] - $\frac{1}{2}$ - $\frac{1}{4}$ (cos x - sin x)

**Example 7:** Solve $(D^2 + D + 1)\ y = (1 + \sin x)^2$

**Sol:** Given equation is

$(D^2 + D + 1)\ y = (1 + \sin x)^2$

A.E. is $D^2 + D + 1 = 0$

$\Rightarrow$ D = $\dfrac{-1 \pm \sqrt{1-4}}{2}$ = $-\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$

C.F. = $e^{-\frac{x}{2}}\left[c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \sin\dfrac{\sqrt{3}}{2}x\right]$

P.I. = $\dfrac{1}{D^2+D+1}$ $(1 + \sin x)^2$

= $\dfrac{1}{D^2+D+1}$ $(1 + 2 \sin x + \sin^2 x)$

= $\dfrac{1}{D^2+D+1}\left[1+2\sin x+\dfrac{1-\cos 2x}{2}\right]$

= $\dfrac{1}{2(D^2+D+1)}$ [2 + 4 sin x + 1 - cos 2x]

= $\dfrac{1}{2}\dfrac{1}{D^2+D+1}\,3 + \dfrac{1}{2}\dfrac{1}{D^2+D+1}\,4 \sin x - \dfrac{1}{2}\dfrac{1}{D^2+D+1}\cos 2x$

= $\dfrac{3}{2}\dfrac{1}{D^2+D+1}e^{ox} + \dfrac{4}{2}\dfrac{1}{D^2+D+1}4 \sin x - \dfrac{1}{2}\dfrac{1}{D^2+D+1}\cos 2x$

= $\dfrac{3}{2}\dfrac{1}{D^2+D+1}e^{ox} + \dfrac{4}{2}\dfrac{1}{-1+D+1}\sin x - \dfrac{1}{2}\dfrac{1}{-4+D+1}\cos 2x$

= $\dfrac{3}{2}\dfrac{1}{0+0+1} + 2\dfrac{1}{D}\sin x - \dfrac{1}{2}\dfrac{(D+3)}{(D-3)(D+3)}\cos 2x$

= $\dfrac{3}{2} + 2(-\cos x) - \dfrac{1}{2}\dfrac{D+3}{D^2-9}\cos 2x$

= $\dfrac{3}{2} - 2\cos x - \dfrac{1}{2}\dfrac{D+3}{-4-9}\cos 2x$

= $\dfrac{3}{2} - 2\cos x + \dfrac{1}{26}(D\cos 2x + 3\cos 2x)$

= $\dfrac{3}{2} - 2\cos x + \dfrac{1}{26}(-2\sin 2x + 3\cos 2x)$

$\therefore$ C.S. is y = $e^{-\frac{x}{2}}\left[c_1 \cos\dfrac{\sqrt{3}}{2}x + c_2 \sin\dfrac{\sqrt{3}}{2}x\right] + \dfrac{3}{2} - 2\cos x + \dfrac{1}{26}(3\cos 2x - 2\sin 2x)$

**Example 8:** Solve $\frac{d^2y}{dx^2}$ - 5 $\frac{dy}{dx}$ + 2y = sin 3x

**Sol:** Given differential equation is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 2y = \sin 3x$$

S.F. of equation is

$$(D^2 - 5D + 2)\, y = \sin 3x$$

A.E. is $D^2 - 5D + 2 = 0$

$\therefore$ $D = \frac{5 \pm \sqrt{25-8}}{2} = \frac{5 \pm \sqrt{17}}{2}$

$\therefore$ C.F. $= c_1 e^{\left(\frac{5+\sqrt{17}}{2}\right)} n + c_2 e^{\left(\frac{5-\sqrt{17}}{2}\right)} x$

$$= e^{\frac{5}{2}x}\left[c_1 e^{\frac{\sqrt{17}}{2}x} + c_2 e^{-\frac{\sqrt{17}}{2}x}\right]$$

P.I. $= \frac{1}{D^2 - 5D + 2} \sin 3x$

$$= \frac{1}{-9 - 5D + 2} \sin 3x$$

$$= -\frac{1}{5D + 7} \sin 3x$$

$$= -\frac{(5D-7)}{(5D-7)(5D+7)} \sin 3x$$

$$= -\frac{5D-7}{25D^2 - 49} \sin 3x$$

$$= -\frac{5D-7}{25(-9) - 49} \sin 3x$$

$$= \frac{1}{274} (5D \sin 3x - 7 \sin 3x)$$

$$= \frac{1}{274} (15 \cos 3x - 7 \sin 3x)$$

$\therefore$ C.S. is

$$y = e^{\frac{5x}{2}}\left[c_1 e^{\frac{\sqrt{17}}{2}x} + c_2 e^{-\frac{\sqrt{17}}{2}x}\right] + \frac{1}{274}(15 \cos 3x - 7 \sin 3x)$$

**Example 9:** Solve the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x \cos 2x$$

**Sol:** Given differential equation is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \sin 3x \cos 2x$$

S.F. of equation is

$$(D^2 - 4D + 3) = \sin 3x \cos 2x$$

A.E. is $D^2 - 4D + 3 = 0$

or $(D - 1)(D - 3) = 0$

$\Rightarrow$ $D = 1, 3$

$\therefore$ C.F. $= c_1e^x + c_2e^{3x}$

$$\text{P.I.} = \frac{1}{D^2-4D+3}\sin 3x \cos 2x$$

$$= \frac{1}{2}\frac{1}{D^2-4D+3}(2 \sin 3x \cos 2x)$$

$$= \frac{1}{2}\frac{1}{D^2-4D+3}(\sin 5x + \sin x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+3}\sin 5x + \frac{1}{D^2-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}.\frac{1}{2D+11}\sin 5x - \frac{1}{2}.\frac{1}{2D-1}\sin x\right]$$

$$= \frac{1}{4}\left[\frac{2D-11}{(2D-11)(2D+11)}\sin 5x + \frac{2D+1}{(2D+1)(2D-1)}\sin x\right]$$

$$= -\frac{1}{4}\left[\frac{2D-11}{4D^2-121}\sin 5x + \frac{2D+1}{4D^2-1}\sin x\right]$$

$$= -\frac{1}{4}\left[\frac{2D-11}{4(-25)-121}\sin 5x + \frac{2D+1}{4(-1)-1}\sin x\right]$$

$$= -\frac{1}{4}\left[-\frac{1}{221}(2D\sin 5x - 11\sin 5x) - \frac{1}{5}(2D\sin x + \sin x)\right]$$

$$= \frac{1}{4}\left[\frac{1}{221}(10\cos 5x - 11\sin 5x) + \frac{1}{5}(2\cos x + \sin x)\right]$$

$$= \frac{1}{884}\ (10 \cos 5x - 11 \sin 5x) + \frac{1}{20}\ (2 \cos x + \sin x)$$

$\therefore$ C.S. is

$$y = c_1e^x + c_2e^{3x} + \frac{1}{884}\ (10 \cos 5x - 11 \sin 5x) + \frac{1}{20}(2 \cos x + \sin x)$$

**Example 10:** Solve $\frac{d^2s}{dt^2}$ + $b^2s$ = k cos bt,

given that s = 0, $\frac{ds}{dt}$ = 0 when t = 0.

**Sol:** Given differential equation is

$$\frac{d^2s}{dt^2} + b^2s = k \cos bt$$

Equation is S.F. is

$(D^2 + b^2)$ s = k cos $bt^2$, where D = $\frac{d}{dt}$

A.E. is $D^2 + b^2 = 0$

$\therefore$ $D^2 = -b^2$

$\Rightarrow$ D = $\underline{+}$ib = 0 $\underline{+}$ib

$\therefore$ C.F. is = $e^{ot}$ ($c_1$ cos bt + $c_2$ sin bt)

= $c_1$ cos bt + $c_2$ sin bt

P.I. = $\frac{1}{D^2+b^2}$ (k cos bt)

= k $\frac{1}{D^2+b^2}$ cos bt

$= k \frac{1}{-b^2 + b^2} \cos bt$ [Case of Failure]

$= kt \frac{1}{2D} \cos bt$

$= \frac{kt}{2} \int \cos bt.\, dt$

$= \frac{kt}{2} \frac{\sin bt}{2}$

$= \frac{k}{2b}\, t \sin bt$

$\therefore$ C.S. is $s = c_1 \cos bt + c_2 \sin bt + \frac{k}{2b}\, t \sin bt$ ......(1)

$\therefore$ $\frac{ds}{dt} = -b\, c_1 \sin bt + bc_2 \cos bt + \frac{k}{2b}(bt \cos bt + \sin bt)$ ......(2)

Now $\frac{ds}{dt} = 0$ when $t = 0$

$\therefore$ from (2), $0 = 0 + bc_2 + 0 \Rightarrow c_2 = 0$

Again $s = 0$ when $t = 0$

$\therefore$ from (1), $0 = c_1 + 0 + 0$

$\Rightarrow$ $c_1 = 0$

Putting $c_1 = 0$, $c_2 = 0$ in (1), we get

$s = \frac{k}{2b}\, t \sin bt$

**Self-Check Exercise-1**

Q.1 Solve $(D^2 + D + 1)\, y = \sin 2x$

Q.2 Solve $(D^4 - m^4)\, y = \cos mx + \cosh mx$

Q.3 Solve $(D^3 + 1)\, y = \cos 2x$

Q.4 Solve $(D^2 - 4D + 3)\, y = \sin 2x \cos x$

Q.5 Solve the differential equation

$(D^2 - 3D + 2)\, y = 6\, e^{-3x} + \sin 2x$

## 8.4 Method to Evaluate

$\frac{1}{f(D)}$ $x^m$, where m is a positive integer

Expand $\frac{1}{f(D)}$ in ascending powers of D by the binomial theorem, taking the expansion upto the terms containing $D^m$ and operating on $x^m$ with each term of the expansion of $\frac{1}{f(D)}$

Consider $\frac{1}{D-a} x^m$

Now $\frac{1}{D-a} x^m = e^{ax} \int x^m e^{-ax}\, dx$

$$= e^{ax}\left[x^m\left(\frac{e^{-ax}}{-a}\right) - mx^{m-1}\left(\frac{e^{-ax}}{a^2}\right) + \left\{m(m-1)x^{m-2}\right\}\left(\frac{e^{-ax}}{-a^3}\right)........\right] \qquad ......(1)$$

9Integration by parts]

The last term of the bracket being

$= (-1)^m \int u^{(m)} v_m\, dx$, where $u = x^m$, $v = e^{-ax}$

$$= (-1)^m \int \underline{\lfloor m}\, \frac{e^{-ax}}{(-a)^m}\, dx$$

$$= \frac{(-1)^m \underline{\lfloor m}}{(-a)^m} \int e^{-ax}\, dx$$

$$= \frac{\underline{\lfloor m}\, e^{-ax}}{a^m(-a)} = -\frac{\underline{\lfloor m}}{a^{m+1}} e^{-ax}$$

$\therefore$ from (1)

$$\frac{1}{D-a} x^m = e^{ax} = \left\{-\frac{x^m}{a}e^{-ax} - \frac{mx^{m-1}}{a^2}e^{-ax} - \frac{m(m-1)}{a^3}x^{m-2}e^{-ax}..... - \frac{\underline{\lfloor m}}{a^{m+1}}e^{-ax}\right\}$$

$$= -\frac{1}{a}\left[x^m + \frac{1}{a}mx^{m-1} + \frac{1}{a^2}m(m-1)x^{m-2} + ......... + \frac{1}{a^m}\underline{\lfloor m}\right]$$

If we expand $\frac{1}{D-a}$ in ascending powers of D, $\frac{1}{D-a}x^m = \frac{1}{-a+D}x^m = \frac{1}{-a\left(1+\frac{D}{a}\right)}x^m = -\frac{1}{a}\left(1-\frac{D}{a}\right)^{-1}x^m$

$$= -\frac{1}{a}\left[1+\frac{D}{a}+\frac{D^2}{a^2}+.........+\frac{D^m}{a^m}+\frac{D^{m+1}}{a^{m+1}}+........\right]x^m$$

$$= -\frac{1}{a}\left[x^m+\frac{1}{a}mx^{m-1}+\frac{1}{a^2}m(m-1)x^{m-2}+.........+\frac{1}{a^m}\underline{\lfloor m}+0\right]$$

$$= -\frac{1}{a}\left[x^m+\frac{m}{a}x^{m-1}+\frac{m(m-1)}{a^2}x^{m-2}+.........+\frac{\underline{\lfloor m}}{a^m}\right]$$

which is the same as (2)

Thus, to evaluate $\frac{1}{D-a}x^m$, expand $\frac{1}{D-a}$ in ascending powers of D as far as the term $D^m$ and operate on x by each term of the expansion.

Thus to evaluate $\frac{1}{f(D)}x^m$, we proceed as follows:-

(i) From $f$(D), take the lowest degree term outside. Then the remaining factor will be of the type [1 $\pm\phi$(D)]

(ii) Take [1 $\pm\phi$(D)] to the numerator and expand it by Binomial Theorem as far as $D^m$. [See Note]

(iii) Operate on $x^m$ with each term

**Note:** $\because$ $D^{m+1}(x^m) = 0$ ;etc

Following results are useful for solving problems:-

(i) $(1 - D)^{-1} = 1 + D + D2 + .........$ to $\infty$

(ii) $(1 + D)^{-1} = 1 + 2D + 3D2 + ............$ to $\infty$

(iii) $(1 - D)^{-2} = 1 + 2D + 3D2 + ..............$ to $\infty$

(iv) $(1 - D)^{-3} = 1 + 3D + 6D2 + ................$ to $\infty$

To clarify what we have just said, consider the following examples:-

**Example 11 :** Solve $\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 12 = x$.

**Sol.** Equation in the symbolic form is $(D^2 - 13D + 12)\,y = x$

A.E. is D2 - 13D + 12 = 0 or (D - 1)(D - 12) = 0 $\therefore$ D = 1,12

$\therefore$ C.F. = $c_1 e^x + c_2 e^{12x}$.

Again P.I. = $\dfrac{1}{D^2-13D+12}$ x = $\dfrac{1}{12\left(1-\dfrac{13}{12}D+\dfrac{D^2}{12}\right)}$ x = $\dfrac{1}{12}\left[1-\left(\dfrac{13}{12}D-\dfrac{D^2}{12}\right)\right]^{-1}$ x

$$= \frac{1}{12}\left[1-\left(\frac{13}{12}D-\frac{D^2}{12}\right)+\dots\right]x$$

$$= \frac{1}{12}\left[1+\frac{13}{12}D\right]x \qquad \text{[Expand upto D]}$$

$$= \frac{1}{12}\left[x+\frac{13}{12}D(x)\right] = \frac{1}{12}\left[x+\frac{13}{12}(1)\right] = \frac{1}{144}(12x+13).$$

$\therefore$ C.S. is y = C.F. + P.I.

i.e. $y = c_1e^x + c_2e^{12x} + \dfrac{1}{144}(12x+13)$ is the reqd solution.

**Example 12 :** Solve $(D^3 + 3D^2 + 2D)y = x^2$

**Sol.** Given Equation is $(D^3 + 3D^2 + 2D)y = x^2$

A.E. is $D^3 + 3D^2 + 2D = 0 \Rightarrow D(D^2 + 3D + 2) = 0$

$\Rightarrow$ D(D + 1) (D + 2) = 0 $\Rightarrow$ D = 0 - 1, -2

C.F. = $c_1 e^{0x} + c_2 e^{-x} + c_3 e^{-2x} = c_1 + c_2 e^{-x} + c_3 e^{-2x}$

P.I. = $\dfrac{1}{D^2-3D^2+2D}.x^2 = \dfrac{1}{2D\left(1+\dfrac{3D}{2}+\dfrac{D^2}{D}\right)}x^2 = \dfrac{1}{2D}\left[1+\left(\dfrac{3D}{2}+\dfrac{D^2}{D}\right)\right]^{-1}.x^2$

$$= \frac{1}{2D}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\frac{(-1)(-2)}{2}\left(\frac{3D}{2}+\frac{D^2}{2}\right)^2+\dots\right]$$

$$= \frac{1}{2D}\left[1-\left(\frac{3D}{2}+\frac{D^2}{2}\right)+\frac{9D^2}{4}.\dots\right]x^2 = \frac{1}{2D}\left[1-\frac{3D}{2}+\left(\frac{9}{4}-\frac{1}{2}\right)D^2+\dots\right]x^2$$

$$= \frac{1}{2D}\left[1-\frac{3D}{2}+\frac{7}{4}D^2.\dots\right] = \frac{1}{2D}\left[x^2-\frac{3}{2}Dx^2+\frac{7}{4}D^2x^2\right]$$

$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}.2x + \frac{7}{4}2\right] = \frac{1}{2}\left[\frac{1}{D}x^2 - \frac{1}{D}3x + \frac{1}{D}\frac{7}{2}\right] = \frac{1}{2}\left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{7x}{2}\right]$$

C.S. is $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3}{2}x^2 + \frac{7x}{2}\right]$

**Example 13 :** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \sin 2x$

**Sol.** Equation in the symbolic form is $(D^2 - 4D + 4)\,y = x^2 + e^x + \sin 2x$

A.E. is $D^2 - 4D + 4 = 0 \Rightarrow (D-2)^2 = 0$

$\Rightarrow \quad D = 2, 2$

$\therefore \quad$ C.F. $= (c_1 + c_2x)\,e^{2x}$

Again P.I. $= \frac{1}{(D-2)^2}[x^2 + e^x + \sin 2x]$

$$= \frac{1}{(D-2)^2}x^2 + \frac{1}{(D-2)^2}e^x + \frac{1}{(D-2)^2}\sin 2x$$

$$= \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2}x^2 + \frac{1}{(1-2)^2}e^x + \frac{1}{D^2 - 4D + 4}\sin 2x$$

$$= \frac{1}{4}\left(1 - 2\left(-\frac{D}{2}\right) + \frac{(2)(-3)}{1.2}.\left(-\frac{D^2}{2}\right) + \ldots\right)[x^2 + e^x + \frac{1}{-4 - 4D + 4}\sin 2x$$

$$= \frac{1}{4}\left(1 + D + \frac{3}{4}D^2 \ldots\right)x^2 + e^x - \frac{1}{4D}\sin 2x$$

$$= \frac{1}{4}\left(x^2 + 2x + \frac{3}{4}(2)\right) + e^x - \frac{1}{4}\left(-\frac{\cos 2x}{2}\right)$$

$$= \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x + \frac{1}{8}\cos 2x$$

$\therefore$ complete solution is given by = C.F. + P.I.

i.e., $\quad y = (c_1 + c_2x)\,e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x + \frac{1}{8}\cos 2x$

**Example 14 :** Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8(x^2 + e^x + \sin 2x)$$

**Sol.** Given equation in the symbolic form is

$$(D^2 - 4D + 4)\, y = 8(x^2 + e^x + \sin 2x) \qquad \text{where } D = \frac{d}{dx}$$

A.E. is $D^2 - 4D + 4 = 0$

i.e. $(D - 2)^2 = 0$

$\Rightarrow$ $D = 2, 2$

$\therefore$ C.F. is $e^{2x}(c_1 + c_2x)$

$$\text{P.I.} = 8\frac{1}{(D-2)^2}\, x^2 + 8\frac{1}{(D-2)^2}\, e^{2x} + 8\,.\frac{1}{(D-2)^2}\sin 2x$$

$$= \frac{8}{4}\left(1-\frac{D}{2}\right)^{-2} x^2 + 8x\,\frac{1}{2(D-2)}\, e^{2x} + 8\,\frac{1}{D^2-4D+4}\,\sin 2x$$

$$= 2\left[1+\frac{2D}{2}+\frac{(-2)(-3)}{1.2}\frac{D^2}{4}\right][x^2 + 4x\,.\,xe^{2x} + 8\,\frac{1}{-4-4D+4}\sin 2x$$

$$= 2\left[1+D+\frac{3}{4}D^2 ...\right]x^2 + 4x^2 e^{2x} - \frac{2}{D}\sin 2x$$

$$= 2\left(x^2+2x+\frac{3}{4}.2\right) + 4x^2 e^{2x} + \frac{2\cos 2x}{2}$$

$$= 2x^2 + 4x + 3 + 4x^2 e^{2x} + \cos 2x$$

$\therefore$ C.S. is $y = e^{2x}(c_1 + c_2x) + 2x^2(1 + 2e^{2x}) + 4x + \cos 2x + 3$

**Example 15 :** Solve the following differential equation

$$(D - 1)^2 (D + 1)^2\, y = \sin^2\frac{x}{2} + e^x + x$$

**Sol. :** Given equation is S.F. is

$$(D - 1)^2 (D + 1)^2\, y = \sin^2\frac{x}{2} + e^x + x$$

A.E. is $(D - 1)^2 (D + 1)^2 = 0$

$\therefore$ $D = 1, 1, -1, -1$

$\therefore$ C.F. is $(c_1 + c_2x)e^x + (c_3 + c_4x)e^{+x}$

$$\text{P.I. is } = \frac{1}{(D-1)^2(D+1)^2}\left[\sin^2\frac{x}{2}+e^x+x\right]$$

$$= \frac{1}{(D-1)^2(D+1)^2}\left[\frac{1-\cos x}{2}+e^x+x\right]$$

$$= \frac{1}{2}\ \frac{1}{(D-1)^2}\ .\ 1 - \frac{1}{2}\ .\ \frac{1}{(D-1)^2}\cos x + \frac{1}{(D-1)^2}\ ex + \frac{1}{(D-1)^2}\ x \qquad ...(1)$$

Now $\frac{1}{2}\cdot\frac{1}{(D^2-1)^2}\ .\ 1 = \frac{1}{2}\ \frac{1}{(D^2-1)^2} + e^{0x} = \frac{1}{2}\ .\ \frac{1}{(0-1)^2} = \frac{1}{2}$

Also $\frac{1}{2}\ \frac{1}{(D^2-1)^2}\ \cos x = \frac{1}{2}\ .\ \frac{1}{(-1-1)^2}\ \cos x = \frac{1}{8}\ \cos x$

and $\frac{1}{(D^2-1)^2}\ e^x = \frac{1}{D^4-2D^2+1}e^x = \frac{1}{1-2+1}e^x = \frac{1}{0}\ e^x$ [Case of failure):

$$= x\frac{1}{4D^3-4D}e^x$$

$$= x\ \frac{1}{4-4}e^x$$ [Again case of failure]

$$= x^2\frac{1}{12D^2-4}e^x$$

$$= x^2\frac{1}{12-4}e^x$$

$$= \frac{x^2e^x}{8}$$

$$\frac{1}{(D^2-1)^2}\ x = \frac{1}{(1-D^2)^2}\ x = (1 - D^2)^{-2}\ x = (1 + 2D^2 + .......)x$$

$\therefore$ from (1), P.I. = $\frac{1}{2} - \frac{1}{8}\ \cos x + \frac{x^2e^x}{8} + x$

$$= \frac{1}{2}(1 + 2x) - \frac{1}{8}\ \cos x + \frac{x^2e^x}{8}$$

$\therefore$ C.S. is

$$y = (c_1 + c_2x)\ e^x + (c_3 + c_4x)\ e^{-x} + \frac{1}{2}(1 + 2x) - \frac{1}{8}\cos x + \frac{x^2e^x}{8}$$

**Self Check Exercise-2**

| | |
|---|---|
| Q. 1 | Solve the differential equation $(D^3 - 5D^2 + 6D)\, y = x^2$ |
| Q. 2 | Solve $(D^2 - 1)\, y = 2 + 3x$ |
| Q. 3 | Solve $(D^2 - 5D + 6)\, y = x + e^{mx}$ |
| Q. 4 | Solve $y'' - y' - 2y = x^2 + \cos x$ |

## 8.5 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed the method to evaluate non-homogeneous equations with constant coefficients and containing terms sin ax or cos ax on R.H.S. Also by using this method find the solutions of differential equation.

2. Discussed the method to evaluate $\frac{1}{f(D)} x^m$, and also solved equations by using this method.

## 8.6 Glossary:

1. $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$, provided $f(-a^2) \neq 0$

2. $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$, provided $f(-a^2) \neq 0$

3. $\frac{1}{f(D^2)}$ sin ax or cos ax = x - $\frac{1}{\frac{d}{dx}[f(D^2)]}$ sin ax or cos ax when $f(-a^2) = 0$

## 8.7 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 $y = e^{-x/2}\left\{c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right\} - \frac{1}{13}\{2 \cos 2x + 3 \sin 2x\}$

Ans.2 $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3}(\sinh mx - \sin mx)$

Ans.3 $y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{65}[\cos 2x - 8 \sin 2x]$

Ans.4 $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{60}(2 \cos 3x - \text{sn } 3x + 3 \sin x + 6 \cos x)$

Ans.5 $y = c_1e^x + c_2e^{2x} + \frac{3}{10}$e-3x + $\frac{1}{20}$(3 cos 2x - sin 2x)

**Self-Check Exercise-2**

Ans.1 $y = c_1 + c_2\, e^{2x} + c_3\, e^{3x} + \frac{1}{18}\left[x^3 + \frac{5x^2}{2} + \frac{19x}{6}\right]$

Ans.2 $y = c_1e^x + c_2e^{-x} - (2 + 3x)$

Ans. 3 $y = c_1e^{2x} + c_2\, e^{3x} + \frac{1}{6}\left(x + \frac{5}{6}\right) + \frac{e^{mx}}{m^2 - 5m + 6}$

Ans. 4 $y = c_1e^{2x} + c_2e^{-x} - \frac{1}{2}\left(x^2 - x + + \frac{3}{2}\right) - \frac{1}{10}$(sin x + 3 cos x)

## 8.8 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
3. Zill, D.A. A First Course in Differential Equations with Applications, 2nd Ed., Prindle, Weber & Schmidt, Boston, 1982.


## 8.9 Terminal Questions

1. Solve $(D^2 + 4) = e^x + \sin 2x$
2. Solve $\frac{d^4y}{dx^4} + 2x^2\frac{d^2y}{dx^2} + n^4y = \cos mx$
3. Solve $(D^2 - 4D + 4)\, y = e^{-4x} + 5\cos 3x$
4. Solve $(D^4 - 1)\, y = \cos ax \cos bx$
5. Solve $(D^4 - a^4)\, y = x^4$
6. Solve $(D^3 - 13D + 12)\, y = x$
7. Solve the following differential equation

   $$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$$
8. Solve the following differential equation

   $(D^3 + 8)\, y = x^4 + 2x + 1$

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## Unit - 9

# Solution of Non-Homogeneous Equation with Constant Coefficients-III

**Structure**



## 9.1 Introduction

Dear students, in the last two Units we have discussed solution of non-homogeneous equation with constant coefficients. The solution of non-homogeneous equations with constant coefficients is of significant importance in various areas of mathematics and physics. The solutions of linear non-homogeneous equations with constant coefficients exhibit a useful property known as the superposition principle. It states that if two particular solutions of the non-homogeneous equation are known, then any linear combination of these solutions is also a solution. This property allows us to construct more general solutions by adding or subtracting specific solutions. Overall, the ability to solve non-homogeneous equations with constant coefficients is essential for understanding and analyzing a wide range of phenomena in both theoretical and applied mathematics, as well as in various scientific and engineering disciplines.

## 9.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss the method to evaluate $\frac{1}{f(D)}(e^{ax}X)$ and solve equations by using this method.
- Discuss the method to evaluate $\frac{1}{f(D)}(x\,V)$ and solve questions by using this method.

## 9.3 Method to Evaluate $\frac{1}{f(D)}(e^{ax}X)$, where X is any function of x

Prove that

$$\frac{1}{f(D)}(e^{ax}X) = e^{ax}\frac{1}{f(D+a)}X$$

**Proof:** $D^n(e^{ax}X) = D^n(X e^{ax})$

$= D^n X e^{ax} + n_{c_1} D^{n-1} x\, e^{ax} a + n_{c_2} D^{n-2}X . e^{ax} . a^2 + ....... + X. e^{ax} . a^n$

[$\because$ by Leibnitz Theorem, $(uv)_n = u_n v + n_{c_1} u_{n-1} V_1 + ....... + uV_n$]

$= e^{ax}[D^n X + n_{c_1} D^{n-1}X. a + n_{c_2} D^{n-2}X . a^2 + ........ + X. a^n]$

$= e^{ax}(D + a)^n X$

$\therefore$ $D^n(e^{ax}X) = e^{ax}(D + a)^n X$

$\therefore$ $f(D)(e^{ax}X) = e^{ax}f(D + a)X$ [$\because f$(D is a polynomial in D]

Operating on both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}f(D)(e^{ax}X) = \frac{1}{f(D)}[e^{ax}f(D+a)X]$$

or $e^{ax}X = \frac{1}{f(D)}[e^{ax}f(D+a)X]$

or $\frac{1}{f(D)}[e^{ax}f(D+a)X] = e^{ax}X$ .......(1)

Put $f(D+a)X = X_1$ .......(2)

Operating on both sides by $\frac{1}{f(D+a)}$, we get

$$\frac{1}{f(D+a)} f(D + a) = \frac{1}{f(D+a)} X_1$$

or $\quad X = \frac{1}{f(D+a)} X_1 \qquad .......(3)$

Substituting from (2) and (3) in (1), we get

$$\frac{1}{f(D)} (e^{ax} X_1) = e^{ax} \frac{1}{f(D+a)} X_1$$

Changing $X_1$ to X, we get

$$\frac{1}{f(D)} (e^{ax} X_1) = e^{ax} \frac{1}{f(D+a)} X$$

Hence the result

**In other words:-** Take $e^{ax}$ outside and in $f$(D) write (D + a) for every D so that $f$(D) becomes $f$(D + a) and operate $\frac{1}{f(D+a)}$ with X alone by the previous methods.

To clarify what we have just said, consider the following examples:-

**Example 1:** Solve: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2 e^{3x}$.

**Sol:** Equation in the symbolic form is $(D^2 - 2D + 1) y = x^2 e^{3x}$

A.E. is $D^2 - 2D + 1 = 0$ or $(D - 1)^2 = 0 \Rightarrow D = 1, 1$

$\therefore \quad$ C.F. = $(c_1 + c_2x)e^x$.

Again P.I. = $\frac{1}{D^2-2D+1} x^2e^{3x} = \frac{1}{D^2-2D+1} (e^{3x} x^2)$

$$= e^{3x}. \frac{1}{(D+3)^2-2(D+3)+1} x^2 = e^{3x}. \frac{1}{D^2+6D+9-2D-6+1} x^2$$

$$= e^{3x}. \frac{1}{4+4D+D^2} x^2 = e^{3x}. \frac{1}{4\left(1+D+\frac{D^2}{4}\right)} x^2$$

$$= \frac{e^{3x}}{4}\left[1+\left(D+\frac{D^2}{4}\right)\right]^{-1}x^2 = \frac{e^{3x}}{4}\left[1-\left(D+\frac{D^2}{4}\right)+\left(D+\frac{D^2}{4}\right)^2+......\right]x^2$$

$$= \frac{e^{3x}}{4}\left[1-\left(D+\frac{D^2}{4}\right)+D^2\right]x^2 \qquad \text{[Expand upto } D^2\text{]}$$

$$= \frac{e^{3x}}{4}\left[1-D+\frac{3}{4}D^2\right]x^2$$

$$= \frac{e^{3x}}{4}\left[x^2-D(x^2)+\frac{3}{4}D^2x^2\right]$$

$$= \frac{e^{3x}}{4}\left[x^2-(2x)+\frac{3}{4}(2)\right] = \frac{e^{3x}}{8}(2x^2 - 4x + 3)$$

$\therefore$ y = C.f. + P.I. i.e. $y = (c_1 + c_2x)\, e^x + \frac{e^{3x}}{8}(2x^2 - 4x + 3)$

**Example 2:** Solve: $(D^2 + 3D + 2)\, y = e^{2x} \sin x$

**Sol:** Given equation is $(D^2 + 3D + 2)\, y = e^{2x} \sin x$

A.E. is $D^2 + 3D + 2 = 0 \Rightarrow (D + 1)(D + 2) = 0 \Rightarrow D = -1, -2$

C.F. = $c_1e^{-x} + c_2e^{-2x}$

P.I. = $\frac{1}{D^2+3D+2}(e^{2x}\sin x) = e^{2x}.\frac{1}{(D+2)^2+3(D+2)+2}\sin x$

$$= e^{2x}\frac{1}{D^2+7D+12}\sin x = e^{2x}\frac{1}{(-1)+7D+12}\sin x$$

$$= e^{2x}\frac{(7D-11)}{(7D+11)(7D-11)}\sin x = e^{2x}\frac{7D-11}{49D^2-121}\sin x$$

$$= e^{2x}\frac{7D-11}{49(-1)-121}\sin x = e^{2x}\frac{7D-11}{-170}\sin x$$

$$= \left(\frac{-e^{2x}}{170}\right)[7D\sin x - 11\sin x] = \left(\frac{-e^{2x}}{170}\right)[7\cos x - 11\sin x]$$

C.S. is $y = c_1e^{-x} + c_2e^{-2x} + \left(\frac{-e^{2x}}{170}\right)[7\cos x - 11\sin x]$

**Example 3:** Solve $(D^2 + 2)\, y = x^2e^{3x} + e^x \cos 2x$

**Sol:** Given equation is

$$(D^2 + 2)\, y = x^2e^{3x} + e^x \cos 2x$$

A.E. is $D^2 + 2 = 0 \Rightarrow D = \underline{+}i\sqrt{2}$

C.f. is = $c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x$

$$\text{P.I.} = \frac{1}{D^2+2}\,(x^2e^{3x} + e^x \cos 2x)$$

$$= \frac{1}{D^2+2}(x^2\, e^{3x}) + \frac{1}{D^2+2}(e^x \cos 2x)$$

$$= e^{3x}\frac{1}{(D+3)^2+2}x^2 + e^x\frac{1}{(D+1)^2+2}\cos 2x$$

$$= e^{3x}\frac{1}{D^2+6D+11}x^2 + e^x\frac{1}{D^2+2D+3}\cos 2x$$

$$= e^{3x}\frac{1}{11\left[1+\frac{6D+D^2}{11}\right]}x^2 + e^x\frac{1}{-4+2D+3}\cos 2x$$

$$= \frac{e^{3x}}{11}\left[1+\frac{6D+D^2}{11}\right]^{-1}x^2 + e^x\frac{(2D+1)}{(2D-1)(2D+1)}\cos 2x$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D+D^2}{11}+\left(\frac{6D+D^2}{11}\right)^2+........\right]x^2 + e^x\frac{(2D+1)}{4D^2-1}\cos 2x$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D}{121}\right]x^2 + e^x\frac{(2D+1)}{4(-4)-1}\cos 2x$$

$$= \frac{e^{3x}}{11}\left[1-\frac{6D}{11}-\frac{25D^2}{121}\right]x^2 + \left(\frac{e^x}{-17}\right)[2D \cos 2x + \cos 2x]$$

$$= \frac{e^{3x}}{11}\left[x^2-\frac{6}{11}(2x)+\frac{25}{121}.2\right] - \frac{e^x}{-17}[-4 \sin 2x + \cos 2x]$$

$$= \frac{e^{3x}}{11}\left[x^2-\frac{12x}{11}+\frac{50}{121}\right] - \frac{e^x}{17}[\cos 2x - 4 \sin 2x]$$

$\therefore$ C.S. is

$$= c_1 \cos\sqrt{2}\, x + c_2 \sin\sqrt{2}\, x + \frac{e^{3x}}{11}\left[x^2 - \frac{12x}{11} + \frac{50}{121}\right] - \frac{e^x}{17}[\cos 2x - 4 \sin 2x].$$

**Example 4:** Solve $(D^3 + 3D^2 + 2D)\, y = xe^x$

**Sol:** Given equation is

$(D^3 + 3D^2 + 2D) = xe^x$

A.E. is $D(D^2 + 3D + 2) = 0$

$\Rightarrow \quad D(D + 1)(D + 2) = 0$

$\Rightarrow \quad D = 0, -1, -2$

$\therefore \quad$ C.F. is $c_1e^{ox} + c_2e^{-x} + c_3 e^{-2x}$

$= c_1 + c_2e^{-x} + c_3 e^{-2x}$

$$\text{P.I.} = \frac{1}{D(D+1)(D+2)} xe^x$$

$$= e^x \frac{1}{(D+1)(D+2)(D+3)} x$$

$$= e^x \frac{1}{6+11D+6D^2+D^3} x$$

$$= \frac{e^x}{6}\left[1+\frac{11D+6D^2+D^3}{6}\right]^{-1} x$$

$$= \frac{e^x}{6}\left(1-\frac{11D}{6}\right)^{-1} x$$

$$= \frac{e^x}{6}\left(x-\frac{11}{6}\right)$$

$$= \frac{e^x}{36}(6x - 1)$$

$\therefore \quad$ C.S. is

$$y = c_1 + c_2e^{-x} + c_3 e^{-2x} + \frac{e^x}{36}(6x - 11)$$

**Example 5:** Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2e^{2x}$

**Sol:** Given differential equation is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2e^{2x}$$

Equation in S.F. is

$$(D^2 - 4D + 4)\, y = x^2e^{2x}$$

A.E. is $D^2 - 4D + 4 = 0$

$\Rightarrow \quad (D - 2)^2 = 0$

$\Rightarrow \quad D = 2, 2$

$\therefore \quad$ C.F. $= (c_1 + c_2x)\, e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}(x^2\, e^{2x})$$

$$= \frac{1}{(D-2)^2}(x^2e^{2x})$$

$$= e^{2x}\frac{1}{(D+2-2)^2}\, x^2$$

$$= e^{2x}\frac{1}{D^2}\, x^2$$

$$= e^{2x}\frac{1}{D}\left(\frac{1}{D}x^2\right)$$

$$= e^{2x}\frac{1}{D}\left(\frac{x^3}{3}\right)$$

$$= e^{2x}\frac{x^4}{12}$$

$$= \frac{1}{12}x^4\, e^{2x}$$

$\therefore \quad$ C.S. is

$$y = (c_1 + c_2x)\, e^{2x} + \frac{1}{12}\, x^4\, e^{2x}$$

**Example 6:** Solve the differential equation

$$\frac{d^4y}{dx^4} - y = e^x \cos x$$

**Sol:** Given equation is

$$\frac{d^4y}{dx^4} - y = e^x \cos x$$

Equation in S.F. is

$$(D^4 - 1)\, y = e^x \cos x$$

A.E. is $D^4 - 1 = 0$

or $(D - 1)(D + 1)(D^2 + 1) = 0$

$\therefore$ $D = 1, -1, \pm i$

$\therefore$ C.F. $= c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$

P.I. $= \frac{1}{D^4 - 1}(e^x \cos x)$

$$= \frac{1}{(D-1)(D+1)(D^2+1)} e^x \cos x$$

$$= e^x \frac{1}{(D+1-1)(D+1+1)[(D+1)^2+1]} \cos x$$

$$= e^x \frac{1}{D(D+2)(D^2+2D+2)} \cos x$$

$$= e^x \frac{1}{(D^2+2D)(D^2+2D+2)} \cos x$$

$$= e^x \frac{1}{(-1+2D)(-1+2D+2)} \cos x \qquad [\because D^2 = -1^2]$$

$$= e^x \frac{1}{(-1+2D)(2D+1)} = e^x \frac{1}{4D^4-1} \cos x$$

$$= e^x \frac{1}{-4-1} \cos x$$

$$= -\frac{1}{5} e^x \cos x$$

$\therefore$ C.S. is $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} e^x \cos x$

**Example 7:** Solve the differential equation

$$(D^2 - 4D + 4)\, y = e^{2x} \cos^2 x$$

**Sol:** Given equation in S.F. is

$(D^2 - 4D + 4)\ y = e^{2x} \cos^2 x$

A.E. is $D^2 - 4D + 4 = 0$

or $(D - 2)^2 = 0$

$\therefore$ $D = 2, 2$

$\therefore$ C.F. is $(c_1 + c_2 x)\ e^{2x}$

P.I. = $\dfrac{1}{D^2 - 4D + 4}\ e^{2x} \cos^2 x$

$= e^{2x} \dfrac{1}{(D+2)^2 - 4(D+2) + 4} \cos^2 x$

$= e^{2x} \dfrac{1}{D^2} \left( \dfrac{1 + \cos 2x}{2} \right)$

$= \dfrac{1}{2}\ e^{2x} \left[ \dfrac{1}{D^2} + \dfrac{1}{D^2} \cos 2x \right]$

$= \dfrac{1}{2}\ e^{2x} \left[ \dfrac{x^2}{2} + \dfrac{\cos 2x}{-4} \right]$

$= \dfrac{1}{8}\ e^{2x}\ (2x^2 - \cos 2x)$

$\therefore$ C.F. is $y = (c_1 + c_2 x)\ e^{2x} + \dfrac{1}{8}\ e^{2x}\ (2x^2 - \cos 2x)$

**Self Check Exercise-1**

Q. 1 Solve the differential equation

$(D^2 + 1)\ y = xe^{2x}$

Q. 2 Solve the following differential equation

$(D^2 - 2D + 1)\ y = x^2\ e^{3x}$

Q. 3 Solve the differential equation

$(D^4 - 2D^3 - 3D^2 + 4D + 4)\ y = x^2\ e^x$

Q. 4 Solve $(D^2 - 4D + 4)y = e^{2x} \cos 2x$

### 9.4 Method to Evaluate $\dfrac{1}{f(D)}$ (xv), where v is any function of x.

Prove that

$$\frac{1}{f(D)}(x\,v) = x\,\frac{1}{f(D)}v + \frac{d}{dD}\left[\frac{1}{f(D)}\right]v$$

**Proof :** We have

$D^n (xv) = D^n (vx)$

$= D^n v \,.\, x + n\,c_1\, D^{n-1}\, v.1$

[By Leibnitz Theorem]

$= x\,D^n\, v + D^{n-1}\, v$

$= x\,D^n\, v + \frac{d}{dD}(D^n)\, v$

$\therefore \quad f(D)\,(x\,v) = x\,f(D)\,v + \frac{d}{dD}[f(D)]v$

[$\therefore f$(D) is a polynomial in D]

or $f(D)\,(x\,v) = x\,f(D)\,v + f'(D)\,v$

Operating on both sides by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}f(D)\,(x\,v) = \frac{1}{f(D)}\left[x\,f(D)\,v + \frac{1}{f(D)}f'(D)\,v\right]$$

or $x\,v = \frac{1}{f(D)}[x\,f(D)\,v] + \frac{1}{f(D)}f'(D)\,v$

$\therefore \frac{1}{f(D)}[x\,f(D)\,v] = x\,v - \frac{1}{f(D)}f'(D)\,v$ ...(1)

Put $f(D)\,v = v_1$ ...(2)

$\therefore \frac{1}{f(D)}f(D)\,v = \frac{1}{f(D)}\,v_1$

or $\quad v = \frac{1}{f(D)}\,v_1$ ...(3)

Substituting from (2) and (3) in (1), we get

$$\frac{1}{f(D)}(x\,v_1) = x\,\frac{1}{f(D)}\,v_1 - \frac{1}{f(D)}f'(D)\,\frac{1}{f(D)}\,v_1$$

$$= \frac{1}{f(D)} v_1 - \frac{1}{[f(D)]^2} f'(D) v_1$$

Changing $v_1$ to $v_1$

$$\frac{1}{f(D)} (x v) = x \frac{1}{f(D)} v - \frac{1}{[f(D)]^2} f'(D) v$$

Or $$\frac{1}{f(D)} (x v) = x \frac{1}{f(D)} v + \frac{d}{dD}\left[\frac{1}{f(D)}\right] v$$

**Note :** The above rule in some cases fails if by using the usual rule for evaluating $\frac{1}{f(D)}$ v, we get zero in the denominator.

To clarify what we have just said, consider the following examples :-

**Example 8 :** Solve $(D^2 + 4) y = x \sin x$

**Sol. :** A.E. is $D^2 + 4 = 0$

$\therefore \quad D = 0 \neq 2i$

$\therefore \quad$ C.F. $= (c_1 \cos 2x + c_2 \sin 2x) e^{0x}$

$= c_1 \cos 2x + c_2 \sin 2x$

Now P.I. $= \frac{1}{D^2+4} x \sin x$

$$= x \frac{1}{D^2+4} \sin x + \frac{d}{dD}\left[\frac{1}{D^2+4}\right] \sin x$$

$$= x \frac{1}{D^2+4} \sin x - 2D \frac{1}{(D^2+4)^2} \sin x$$

$$= x \frac{1}{-1+4} \sin x - 2D \frac{1}{(-1+4)^2} \sin x \qquad [\text{Put } D^2 = -1]$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} D \sin x$$

$$= \frac{1}{3} x \sin x - \frac{2}{9} \cos x$$

Now C.S. = C.F. + P.I.

i.e. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{9} \cos x$ , is the complete solution.

**Example 9 :** Solve $(D^4 - 1)\, y = x \sin x$.

**Sol. :** Given equation is

$(D^4 - 1)\, y = x \sin x$

A.E. is $D^4 - 1 = 0$

$\Rightarrow \quad (D^2 - 1)(D^2 + 1) = 0$

$\Rightarrow \quad c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$

P.I. $= \dfrac{1}{D^4+1} x \sin x$

I. P. of $\dfrac{1}{D^4-1} x\, e^{ix}$

$$= e^{ix} \text{ I.P. of } e^{ix} \left\{ \frac{1}{(D+i)^4 - 1} \right\} x$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ \frac{1}{1 + \frac{6D^2}{4Di} \ldots\ldots} \right\} x$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ 1 + \frac{3D}{2i} \right\}^{-1} . x$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ 1 - \frac{3D}{2i} \ldots\ldots \right\} x$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ x - \frac{3D}{2i} \right\}$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ \frac{2i-3}{2i} \right\}$$

$$= \text{I.P. of } \frac{e^{ix}}{-4Di} \left\{ x - \frac{3}{3i} \right\}$$

$= \text{I.P. of } \frac{e^{ix}}{-4Di}\left\{\frac{2ix-3}{2i}\right\}$

$= \text{I.P. of } \frac{e^{ix}}{8} \cdot \frac{1}{D}$ (2 ix - 3)

$= \text{I.P. of } \left(\frac{\cos x + i\sin x}{8}\right)$ ($ix^2$ - 3x)

$= \frac{1}{8}$ ($x^2$ cos x - 3 x sin x)

**Example 10 :** Solve ($D^2$ + 3D +2) y = $xe^x$ sin x

**Sol. :** Given equation is

($D^2$ + 3D +2) y = $xe^x$ sin x

A.E. is $D^2$ + 3D +2) = 0

$\Rightarrow$ (D + 1) (D + 2) = 0

$\Rightarrow$ D = -1, -2

C.F. $c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{D^2+3D+2} xe^x \sin x$

$= e^x\left[\frac{1}{(D+1)^2+3(D+1)+2}\right](x \sin x)$

$= e^x \frac{1}{D^2+2D+1+3D+3+2} x \sin x$

$= e^x \frac{1}{D^2+5D+6} x \sin x$

$= e^x\left[x\frac{1}{D^2+5D+6}\sin x + \frac{d}{dD}\left\{\frac{1}{D^2+5D+6}\right\}\sin x\right]$

$= e^x\left[x\frac{1}{-1+5D+6}\sin x - \frac{2D+5}{(D^2+5D+6)^2}\sin x\right]$

$= e^x\left[x\frac{1}{5(D+1)}\sin x - \frac{2D+5}{(-1+5D+6)^2}\sin x\right]$

$$= e^x\left[\frac{x}{5}\frac{D-1}{D^2-1}\sin x - \frac{2D+5}{\{5(D+1)\}^2}\sin x\right]$$

$$= e^x\left[\frac{x}{5}\frac{D-1}{-1-1}\sin x - \frac{2D+5}{25(D^2+2D+1)}\sin x\right]$$

$$= e^x\left[\frac{x}{5(-2)}(D\sin x - \sin x) - \frac{2D+5}{25(-1+2D+1)}\sin x\right]$$

$$= e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\frac{2D+5}{2D}\sin x\right]$$

$$= e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\left(1+\frac{5}{2D}\right)\sin x\right]$$

$$= e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\left(\sin x + \frac{5}{2}\int \sin x\,dx\right)\right]$$

$$= e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\left\{\sin x + \frac{5}{2}(-\cos x)\right\}\right]$$

$$= e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$$

$\therefore$ C.S. is

$$y = c_1e^{-x} + c_2\,e^{-2x} + e^x\left[\frac{-x}{10}(\cos x - \sin x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$$

**Self-Check Exercise-2**

Q.1 Solve $(D^2 + 4)\,y = x \cos x$

Q.2 Solve the differential equation

$(D^2 - 1)\,y = x^2 \cos x$

Q.3 Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$

Now we do some typical problems:-

**Example 11:** Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$

**Sol:** Given equation in S.F. is

$(D^2 + a^2)\ y = \sec ax$

A.E. is $D^2 + a^2 = 0$

$\Rightarrow$ $D = \underline{+}\ ai = 0\ \underline{+}i\ a$

$\therefore$ C.F. is $(c_1 \cos ax + c_2 \sin ax)e^{ox}$

$= c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2+a^2}\sec ax$$

$$= \frac{1}{(D-ia((D+ia)}\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}\sec ax-\frac{1}{D+ia}\sec ax\right] \qquad \text{.......(1)}$$

Now $\frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax\ e^{iax}\,dx$ $\qquad \left[\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx\right]$

$$= e^{iax}\int \sec ax\ (\cos ax - i \sin ax)\,dx$$

$$= e^{iax}\int (1 - i \tan ax)\,dx$$

$$= (\cos ax + i \sin ax)\left[x+\frac{i}{a}\log(\cos ax)\right]$$

$$= \left[x\cos ax-\frac{\sin ax\log(\cos ax)}{a}\right] + i\left[x\sin ax+\frac{\cos ax}{a}\log(\cos ax)\right]$$

Similarly by changing i to -i, we get

$$\frac{1}{D-ia}\sec ax = \left[x\cos ax-\frac{\sin ax\log(\cos ax)}{a}\right] - i\left[x\sin ax+\frac{\cos ax}{a}\log(\cos ax)\right]$$

Putting in (1), we get

$$\text{P.I.} = \frac{1}{2ia}\left[2i\left\{(x\sin ax)+\frac{\cos ax}{a}\log(\cos ax)\right\}\right]$$

$$= \frac{x}{a} \sin ax + \frac{\cos ax}{a^2} \log (\cos ax)$$

Hence C.S. is given by

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x \sin ax}{a} + \frac{\cos ax}{a^2} \log (\cos ax)$$

**Example 12:** Solve $(D^2 + 4)\, y = 4 \tan 2x$

**Sol:** Given equation is $(D^2 + 4)\, y = 4 \tan 2x$

A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

C.F. is $c_1 \cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2 + 4} (4 \tan 2x)$$

$$= \frac{4}{(D+2i)(D-2i)} (\tan 2x)$$

$$= \frac{1}{i}\left[\frac{1}{D-2i} - \frac{1}{D+2i}\right] \tan 2x \qquad \text{......(1)}$$

Now $\frac{1}{D-2i} \tan 2x = e^{2ix} \int e^{-2ix} \tan 2x \, dx$

$$= e^{2ix} \int (\cos 2x - i \sin 2x) \frac{\sin 2x}{\cos 2x} dx$$

$$= e^{2ix}\left[\int \sin 2x \, dx - i \int \frac{1-\cos^2 2x}{\cos 2x} dx\right]$$

$$= e^{2ix}\left[\frac{-\cos 2x}{2} - i\left\{\frac{1}{2} \log \tan\left(\frac{\pi}{4} + x\right) - \frac{\sin 2x}{2}\right\}\right]$$

$$= e^{2ix}\left[-\frac{1}{2}(\cos 2x - i \sin 2x) - \frac{i}{2} \log \tan\left(\frac{\pi}{4} + x\right)\right]$$

$$= e^{2ix}\left[\frac{-e^{-2ix}}{2} - \frac{i}{2} \log \tan\left(\frac{\pi}{4} + x\right)\right]$$

$$= -\frac{1}{2} - \frac{i}{2} e^{2ix} \log \tan\left(\frac{\pi}{4} + x\right) \qquad \text{.....(2)}$$

Changing i to -i in (2), we get

$$\frac{1}{(D+2i)} \tan 2x = \frac{1}{2} + \frac{i}{2} e^{2ix} \log\tan\left(\frac{\pi}{4}+x\right) \qquad \text{......(3)}$$

Using (2) and (3) in (1), we get

$$\text{P.I.} = \frac{1}{i}\left[\left\{-\frac{i}{2}-\frac{1}{2}e^{2ix}\log\tan\left(\frac{\pi}{4}+x\right)\right\}-\left\{-\frac{1}{2}+\frac{ie^{-2ix}}{2}\log\tan\left(\frac{\pi}{4}+x\right)\right\}\right]$$

$$= \frac{1}{i}\left[\left\{-\frac{i}{2}\log\tan\left(\frac{\pi}{4}+x\right)\right\}\left\{e^{2ix}+e^{-2ix}\right\}\right]$$

$$= \left[\left\{-\log\tan\left(\frac{\pi}{4}+x\right)\right\}\left\{\frac{e^{2ix}+e^{-2ix}}{2}\right\}\right]$$

$$= -\left\{\log\tan\left(\frac{\pi}{4}+x\right)\right\}\left\{\cos 2x\right\}$$

∴ C.S. is

$$y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log\tan\left(\frac{\pi}{4}+x\right)$$

Q.4 Solve the differential equation

$y'' + 16y = \sec 4x$

Q.5 Solve the differential equation

$(D^2 + 1)\, y = \text{cosec } x$

## 9.5 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed the method to evaluate $\frac{1}{f(D)}(e^{ax}X)$, where X in any function of x.
2. Solved differential equations questions by the method to evaluate $\frac{1}{f(D)}(e^{ax} X)$.
3. Discussed the method to evaluate $\frac{1}{f(D)}(x V)$, where V is any function of x.
4. Solved questions by this method i.e. method to evaluate $\frac{1}{f(D)}$ (x V).

## 9.6 Glossary:

1. $\dfrac{1}{f(D)}$ ($e^{ax}X$) = $e^{ax}\dfrac{1}{f(D+-a)}$ X

2. $\dfrac{1}{f(D)}$(X V) = x $\dfrac{1}{f(D)}$V + $\dfrac{d}{dD}\left[\dfrac{1}{f(D)}\right]$V

## 9.7 Answer to Self Check Exercise

**Self-Check Exercise-1**

Ans.1 $y = c_1 \cos x + c_2 \sin x + \dfrac{e^{2x}}{5}\left(x-\dfrac{4}{5}\right)$

Ans.2 $y = (c_1 + c_2x)\, e^x + \dfrac{e^{3x}}{4}\left[x^2-2x+\dfrac{3}{2}\right]$

Ans.3 $y = (c_1+ c_2x)e^{2x} + (c_3 + c_4x)e^{-x} + \dfrac{e^x}{8}[2x^2 + 4x + 7]$

Ans.4 $y = (c_1+ c_2x)\, e^{2x} - \dfrac{e^{2x}\cos 2x}{4}$

**Self-Check Exercise-2**

Ans.1 $y = c_1 \cos 2x + c_2 \sin 2x + \dfrac{x\cos x}{3} + \dfrac{2\sin x}{9}$

Ans.2 $y = c_1e^x + c_2e^{-x} + \dfrac{1}{2}(- - x^2) \cos x + x \sin x$

Ans. 3 $y = (c_1 + c_2\, x)e^x = e^x (x \sin x + 2 \cos x)$

Ans. 4 $y = c_1\cos 4x + c_2 \sin 4x + \dfrac{1}{4}\left[x\sin 4x+\dfrac{\cos 4x}{4}\log\cos 4x\right]$

Ans. 5 $y = c_1 \cos x + c_2 \sin x + \left[\dfrac{\sin x}{1}\log\sin x - x\cos x\right]$

## 9.8 References/Suggested Readings


1. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
2. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.


## 9.9 Terminal Questions

1. Solve the following differential equation

   $(D^2 - 5D + 6) = xe^{4x}$

2. Solve the following differential equation

$(D^2 - 2D + 4)\, y = (x + 1)\, e^x$

3. Solve the differential equation

$(D^2 - 4D + 3)\, y = e^x \cos 2x + \cos 3x$

4. Solve $(D^2 - 2D + 2)\, y = e^x \sin x$

5. Solve $(D^2 - 1)\, y = x \sin x$

6. Solve $(D^2 - 4)\, y = x \cos 2x$

7. Solve $(D^4 + 2D^2 + 1)\, y = x^2 \cos x$

8. Solve $y'' + y = x\, e^x \cos 2x$

9. Solve the differential equation

$(D^2 + 1)\, y = \cot x$

10. Solve the differential equation

$(D^2 + 3D + 2)\, y = \sin e^x$

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## Unit - 10

# Reduction Of Order

**Structure**



### 10.1 Introduction

Differential equations playa a fundamental role in various fields of science and engineering, describing the relationships between variables and their rates of change. Solving differential equations is essential in understanding and predicting the behavior of dynamic systems. One common approach to solving differential equations is by reducing their order. This technique simplifies the problem by transforming a higher-order differential equation into a system of lower-order equations. By doing so, we can often obtain explicit solutions or numerical approximation that are more manageable and easier to interpret.

Reducing the order of a differential equation offers several advantages and is of significant importance in various aspects. Higher order differential equations can be complex and challenging to solve directly. By reducing the order, we can break down the problem into a series of simpler equations, making the solution process more feasible. Lower-order differential equations often have well-known solution techniques and formulas. Reducing the order allows us to apply these established methods, thus providing a wider range of tools for finding solutions.

### 10.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss how to solve a differential equation by reducing its order.
- Discuss to find a particular solution by inspection.

## 10.3 Solving A Differential Equation By Reducing Its Order

**Theorem:** Let $f$(x) be a non-trivial solution of the $n^{th}$ order homogeneous linear differential equation.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)\,y = 0 \qquad \ldots\ldots(1)$$

Then the transformation y = $f$(x) $v$(x) reduces (1) to a (n - 1)st order homogeneous linear differential equation in the dependent variable, z = $\frac{dv}{dx}$ [Without Proof]

This theorem states that if one non-zero solution of the nth-order homogeneous linear differential equation is known, then by making the suitable transformation, we may reduce the equation to another homogeneous linear differential equation, which is one order lower than the original.

We shall discuss the theorem if n = 2.

Let differential $f$ be a known non-trivial solution of the second order homogeneous linear equation.

$$a_0(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)\,y = 0 \qquad \ldots\ldots(2)$$

Put y = $f$(x). $v$ ......(3)

when $f$(x) is the known solution of (2) and $v$ is a function of x to be found out.

Differentiating, we get

$$\frac{dy}{dx} = f(x)\frac{dv}{dx} + f(x)\,v. \qquad \ldots..(4)$$

$$\frac{d^2 y}{dx^2} = f(x)\frac{d^2 v}{dx^2} + f(x)\frac{dv}{dx} + f(x)\frac{dv}{dx} + f(x)\,v. \qquad \ldots\ldots.(5)$$

Putting the values of $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$ in (2), we get

$$a_0(x)\left[f(x)\frac{d^2 y}{dx^2} + 2'(x)\frac{dv}{dx} + f''(x)\,v\right] + a_1(x)\left[f(x)\frac{dv}{dx} + f'(x)\,v\right] + a_2(x)\,f(x)\,v = 0$$

$$\Rightarrow \qquad a_0(x)\,f(x)\frac{d^2 y}{dx^2} + v\frac{dv}{dx} + [a_0(x)\,f''(x) + a_1(x)\,f'(x) + a_2(x)\,f(x)]\,v = 0$$

Since $f$(x) is a solution of (2), the co=eff. of $v$ is zero.

$$\therefore \qquad \text{we get a0 } (x)\frac{d^2 y}{dx^2}[2\,a_0(x)\,f'(x) + a_1(x)\,f'(x)]\frac{dv}{dx} = 0$$

Put $\frac{dv}{dx} = z \therefore \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\therefore$ $a_0(x)\frac{dz}{dx} + [2a_0(x)f(x) + a_1(x)f(x)]z = 0$

This is a first order homogeneous linear differential equation in the dependent variable z.

The equation is variable separable type.

Assuming $f(x) \neq 0$ and $a_0(x) \neq 0$, we can write

$$\frac{dz}{z} = -\left[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}\right]dx$$

Integrating, we get

$$\log|z| = -\log(f(x))^2 - \int \frac{a_1(x)}{a_0(x)}dx + \log|c|$$

where c is an arbitrary constant

$$\Rightarrow \quad z = \frac{c\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{(f(x))^2}$$

This is the general solution of (6)

If we choose c = 1 and replace z by $\frac{dv}{dx}$, we get

$$\frac{dv}{dx} = \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{(f(x))^2}$$

Integrating we get, $v = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{(f(x))^2}dx$

from (3), we get

$y = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}dx\right]}{(f(x))^2}dx$ is a solution of (2), Call this function as g(x)

We shall prove that $f(x)$, g(x) are linearly independent.

Now

$$W(f. g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix}$$

$$= \begin{vmatrix} f(x) & f(x)+(x) \\ f'(x) & f(x)(x)+f'(x)v(x) \end{vmatrix} = (f(x))^2 v'(x)$$

$$= (f(x))^2 \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right.}{(f(x))^2}$$

$$= \exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right]$$

$$= \exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right] \neq 0$$

Thus the linear combination $c_1 f(x) + c_2\, g(x)$ is the general solution of (2)

Hence we have:

Let $f(x)$ be a non-trivial solution of the second-order homogeneous linear differential equation

$$a_0(x) \frac{d^2y}{dx^2} + a_1(x) \frac{dy}{dx} + a_2(x)\, y = 0 \qquad ......(1)$$

Then (i) The transformation $y = f(x)\, v(x)$ reduces (1) to the first order homogeneous linear differential equation

$a_0(x)\, f(x) \frac{dz}{dx} + [2a_0(x)\, f'(x) + a_1(x)\, f(x)'\, z = 0$ in the dependent variable , where z = dvldx.

..... (2)

(ii) The particular solution

$$z = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right]}{[f(x)]^2}$$

gives rise to the function v(x), where

$$v(x) = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)} dx\right]}{[f(x)]^2} dx.$$

The function g defined by g(x) = $f$(x) v(x) is then a solution of the second-order differential equation.

(iii) The original known solution $f$(x) and the "new" solution g(x) are linearly independent solutions (1) and hence the general solution of (1) may be expressed as the linear combination.

$c_1 f(x) + c_2\, g(x)$

Alternatively, Let any linear of second order be

$$\frac{d^2y}{dx^2} + P\,\frac{dy}{dx} + Qy = R$$

or $y'' + Py' + Qy = R$ .....(1)

where P, Q, R are function of x

Let u be a particular solution of the homogeneous differential equation.

$y'' + Py' + Qy = 0$, then

$u'' + Pu' + Qu = 0$ .......(2)

Let y = uv, where u and v are function of x.

∴ $y' = u'v + uv'$

$y' = u''v + u'v' + u'v' + uv''$

$= u''v + 2u'v' + uv''$

Substituting values of y, y', y'' in (1), we get

$u''v + 2\,u'\,v' + u\,v'' + P(u'v + uv') + Quv = R$

or $uv'' + (2u' + P\,u)\,v' + (u'' + Pu' + Qu)\,v = R$

or $uv'' + (2u' + Pu)\,v' = R$ [Q of (2)]

Dividing both sides by u, we get

$$v'' + \left(\frac{2}{u}v' + P\right) = \frac{R}{u}$$

or $v'' + P_1 = \frac{2}{u}v' + P,\ R_1 = \frac{R}{u}$

Let $v' = w,\ v'' = w'$

Then (3) becomes, $w' + P_1\,w = R_1$

This is a linear equation of order one in w and can be solved for w. After finding w, we can find v by integrating w'.

Therefore, the solution of (1) is given by y = uv.

## 10.4 Solution By Inspection

Find a particular solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$ by inspection, where P, Q, R are functions of x

$\therefore$ Given equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ......(1)$$

The corresponding homogeneous equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ..... (2)$$

Case I: We will find m so that $e^{mx}$ is a solution of (2)

Putting $y = e^{mx}$, $\frac{dy}{dx} = m\, e^{mx}$, $\frac{d^2y}{dx^2} = m^2e^{mx}$ in (2), we get

$m^2e^{mx} + Pm\, e^{mx} + Q\, e^{mx} = 0$

or $\quad m^2 + m\,P + Q = 0 \qquad ..... (3)$

Putting m = 1, -1, 2, -2, .......... in (3), we get

(i) $e^x$ is a solution of (2) if $1 + P + Q = 0$

(ii) $e^{-x}$ is a solution of (2) if $1 - P + Q = 0$

(iii) $e^{2x}$ is a solution of (2) if $4 + 2P + Q = 0$

(iv) $e^{-2x}$ is a solution of (2) if $4 - 2P + Q = 0$

and so on.

Case II. We find m so that $x^m$ is a solution of (2).

Putting $y = x^m$, $\frac{dy}{dx} = m\, x^{m-1}$, $\frac{d^2y}{dx^2} = m\,(m-1)\, x^{m-2}$ in (2), we get

$m(m-1)\, x^{m-2} + Pm\, x^{m-1} + Q\, x^m = 0$

or $\quad m(m-1) + m\,px + Q\,x^2 = 0 \qquad ......(4)$

Putting m = 1, -1, 2, -2,....... in (4), we get

(i) $y = x$ is a solution of (2) if $P + Qx = 0$

(ii) $y = x^{-1}$ is a solution of (2) if $2-Px + Qx^2 = 0$

(iii) $y = x^2$ is a solution of (2) if $2 + 2Px + Qx^2 = 0$

and so on.

**Working rule to solve $\frac{d^2y}{dx^2}$ + P$\frac{dy}{dx}$ + Qy = R, where P, Q, and R are functions of x, by change of dependent variable**

Step I: Find by inspection as explained above or otherwise, a particular solution u (say) of the corresponding homogeneous equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$

Step II: Put if = uv, where v is a function of x. We will get a linear equation of order one which can be solved.

Step III: After finding v, we get y = uv as required solution of given differential equation.

Let us do some examples to clarify what we have just said.

**Example 1:** Given that y = x is a solution of $(x^2 + 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$,

Find another linearly independent solution by reducing its order.

**Sol:** Put y = vx

$$\therefore \quad \frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$

$$= x\frac{dv}{dx} + v$$

and $$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + \frac{dv}{dx}.1 + \frac{dv}{dx}$$

$$= x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Putting the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in the given equation, we get

$$(x^2 + 1)\left[x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right] - 2x\left[x\frac{dv}{dx} + v\right] + 2vx = 0$$

$$\Rightarrow \quad x(x^2 + 1)\frac{d^2v}{dx^2} + 2x^2\frac{dv}{dx} + 2\frac{dv}{dx} - 2x^2\frac{dv}{dx} - 2xv + 2xv = 0$$

$$\Rightarrow \quad x(x^2 + 1)\frac{d^2v}{dx^2} + 2\frac{dv}{dx} = 0$$

Putting $\frac{dv}{dx} = z, \therefore \quad \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\therefore$ $x(x^2+1)\dfrac{dz}{dx} + 2z = 0$

$\Rightarrow$ $\dfrac{dz}{z} + \dfrac{2dx}{x(x^2+1)} = 0$

$\Rightarrow$ $\dfrac{dz}{z} + 2\left[\dfrac{1}{x} - \dfrac{x}{x^2+1}\right]dx = 0$

Integrating, we get the general solution as

log z + (2 log x - log ($x^2$ + 1) = log c

$\Rightarrow$ log z + log $x^2$ - log ($x^2$ + 1) = log c

$\Rightarrow$ $\log\left(\dfrac{zx^2}{x^2+1}\right) = \log c$

$\Rightarrow$ $\dfrac{zx^2}{x^2+1} = c$, where c is an arbitrary constant

$\Rightarrow$ $z = c\left[\dfrac{x^2+1}{x^2}\right]$

Choosing c = 1, we find that $z = 1 + \dfrac{1}{x^2}$

i.e. $\dfrac{dv}{dx} = 1 + \dfrac{1}{x^2}$

Integrating, we get

$v = x - \dfrac{1}{x}$

$\therefore$ $y = vx = \left(x - \dfrac{1}{x}\right)x$

= $x^2$ - 1 is also a solution

**Example 2:** Given that $y = e^2x$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Find the general solution of the differential equation by reducing its order.

**Sol:** Given differential equation is

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \qquad ......(1)$$

Here $e^2x$ is solution of (1)

Put $y = ve^{2x}$, where v is a function of x.

$\therefore$ $y' = v.\, 2e^{2x} + v'\, e^{2x}$

$= (2v + v')\, e^{2x}$

and $y'' = (2v + v').\, 2e^{2x} + (2v' + v'')\, e^{2x}$

$= (4v + 4v' + v'')\, e^{2x}$

Substituting these values of y, y', y'' in (1), we get

$(4v + 4v' + v'')\, e^{2x} - 5\,(2v + v')\, e^{2x} + 6ve^{2x} = 0$

or $(4v + 4v' + v'') - 5\,(2v + v') + 6v = 0$

or $v'' - v' = 0 \qquad .....(2)$

Put $v' = w$, $v'' = w'$ in (2),

$\therefore$ $w' - w = 0$

or $\frac{dw}{dx} - w = 0$

$\therefore$ $\frac{dw}{dx} = w$

Separating the variable, we get

$$\frac{1}{w}\, dw = dx$$

Integrating , $\int \frac{1}{w} \int dx$

$\therefore$ $\log w - \log c_1 = x$

or $\log\left(\frac{w}{c_1}\right) = x$

or $\frac{w}{c_1} = e^x$

$\therefore$ $w = c_1\, e^x$

or $\quad \frac{dv}{dx} = c_1 e^x$

Integrating, we get

$$v = c_1\int e^x \, dx + c_2$$

$\therefore \quad v = c_1 e^x + c_2$

$\therefore \quad y = v e^{2x}$

or $y = c_1 e^{3x} + c_2 e^{2x}$ is the required general solution of $c_1x$.

**Example 3 :** Given that $y = x^2$ is a solution of

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 4y = 0$$

find a linearly independent solution by reducing the order write the general solution.

**Sol. :** Put $y = vx^2$

$$\therefore \quad \frac{dy}{dx} = v.2x + x^2 \frac{dv}{dx}$$

and $\quad \frac{d^2y}{dx^2} = v.2 + 2x \frac{dv}{dx} + 2x \frac{dv}{dx} + x^2 \frac{d^2v}{dx^2}$

Putting these values in the given equation, we get

$$x^2\left[x^2 \frac{d^2v}{dx^2} + 4x\frac{dv}{dx} + 2v\right] - 3x \left[x^2 \frac{dv}{dx} + 2v\right] + 4v\, x^2 = 0$$

$\Rightarrow \quad x^4 \frac{d^2v}{dx^2} + (4x^3 - 3x^3) \frac{dv}{dx} + 2\, v\, x^2 - 6v\, x^2 + 4v\, x^2 = 0$

$\Rightarrow \quad x^4 \frac{d^2v}{dx^2} + x^3 \frac{dv}{dx} = 0$

$\Rightarrow \quad \frac{d^2v}{dx^2} + \frac{1}{x} \frac{dv}{dx} = 0$

Put $\quad \frac{dv}{dx} = z$

$\therefore \quad \frac{d^2v}{dx^2} = \frac{dz}{dx}$

$\therefore \quad \frac{dz}{dx} + \frac{1}{x} . z = 0$

$\Rightarrow \quad \frac{dz}{z} + \frac{dx}{x} = 0$

Integrating, we get

$\log z + \log x = \log c$

or $\quad \log (zx) = \log c$

$\therefore \quad zx = c$

Take e = 1, $\therefore$ z x = 1

$\Rightarrow \quad z = \frac{1}{x}$

$\Rightarrow \quad \frac{dv}{dx} = \frac{1}{x}$

$\Rightarrow \quad dv = \frac{dx}{x}$

$\Rightarrow \quad v = \log x$

$\Rightarrow \quad y = vx^2 = x^2 \log x$ is the second solution.

Hence general solution is

$y = c_1 x^2 + c_2 x^2 \log x$ for all $x \in (0,\infty)$

[$\therefore$ for log x, x > 0]

**Example 4 :** Given that $y = x^3$ is a solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 6x \frac{dy}{dx} + 12y = 0$$

Find the general solution of the differential equation by reducing the order.

**Sol. :** Given differential equation is

$$x^2 \frac{d^2 y}{dx^2} - 6x \frac{dy}{dx} + 12y = 0 \qquad ...(1)$$

Here $x^3$ is a solution of (1)

Put $\quad y = vx^3$, where v is a function of x

$\therefore \quad y' = v. 3x^2 + v' x^3$

$= x^2 (3v + v'x)$

and $y'' = 2x (3v + v'x) + x^2 (3v' + v''x + v')$

Substituting the values of y, y', y'' is (1), we get

$$x^2[2x (3v + v'x) + x^2 (4v + v'' x)] - 6x [x^2 (3v + v'x)] + 12v x^3 = 0$$

$\therefore$ $2 (3v + v'x) + x (4v' + v''x) - 6(3v + v'x) + 12v = 0$

or $6v + 2v'x + 4v' x + v''x^2 = 18v - 16v'x + 12v = 0$

or $x^2 v'' = 0$

or $v'' = 0$

$\Rightarrow$ $v' = c_1$

$\therefore$ $\dfrac{dv}{dx} = c_1$

$\Rightarrow$ $v = c_1x + c_2$

$\therefore$ $y = x^3 (c_1 x + c_2)$ is the general solution of (1)

**Example 5 :** Given that y = x +1 is a solution of

$$(x+1)^2 \frac{d^2y}{dx^2} - 3 (x + 1) \frac{dy}{dx} + 3y = 0$$

Find a linearly independent solution by reducing its order. Write also the general solution.

**Sol. :** Given differential equation is

$$(x+1)^2 \frac{d^2y}{dx^2} - 3 (x + 1) \frac{dy}{dx} + 3y = 0 \qquad ...(1)$$

Put $y = (x + 1)$

$\therefore$ $\dfrac{dy}{dx} = (x + 1) \dfrac{dv}{dx} + V$

and $\dfrac{d^2y}{dx^2} = (x + 1) \dfrac{d^2v}{dx^2} + \dfrac{dv}{dx} + \dfrac{dv}{dx}$

Putting in (1), we get

$$(x + 1)^2\left[(x+1)\frac{d^2v}{dx^2}+2\frac{dv}{dx}\right] - 3 (x + 1) \left[(x+1)\frac{dv}{dx}+v\right] + 3v (x + 1) = 0$$

$\Rightarrow \quad (x + 1)^3 \dfrac{d^2v}{dx^2} + 2 (x + 1)^2 \dfrac{dv}{dx} - 3 (x + 1)^2 \dfrac{dv}{dx} - 3v (x + 1) + 3v (x + 1) = 0$

$\Rightarrow \quad (x + 1)^3 \dfrac{d^2v}{dx^2} - (x + 1)^2 \dfrac{dv}{dx} = 0$

$\Rightarrow \quad (x + 1) \dfrac{d^2v}{dx^2} - \dfrac{dv}{dx} = 0$

Put $\dfrac{dv}{dx} = z$

$\therefore \quad \dfrac{d^2v}{dx^2} = \dfrac{dz}{dx}$

$\therefore \quad (x + 1) \dfrac{dz}{dx} - z = 0$

$\Rightarrow \quad \dfrac{dz}{z} = \dfrac{dx}{x+1}$

Integrating, we get

$\log z = \log (x + 1) + \log c$

$\Rightarrow \quad z = c (x + 1)$

Take $c = 2$

$\therefore \quad z = 2 (x + 1)$

$\Rightarrow \quad \dfrac{dv}{dx} = 2 (x + 1)$

$\Rightarrow \quad \int dv = \int 2(x + 1)\, dx$

$\Rightarrow \quad v = (x + 1)^2$

$\Rightarrow \quad y = v(x + 1) = (x +1)^3$ is also a solution.

$\therefore$ the two linearly independent solutions are $(x + 1)$, $(x + 1)^3$

$\therefore$ general solution is

$y = c_1(x + 1) + c_2(x + 1)^3$, where $c_1$, $c_2$ are arbitrary constants

**Example 6:** Solve the following equation by the method of reduction of order

$$x^2 \frac{d^2y}{dx^2} - 2x (1 + x) \frac{dy}{dx} + 2 (1 + x) y = x^3$$

**Sol:** Given equation is

$$x^2 \frac{d^2y}{dx^2} - 2x\,(1 + x)\ \frac{dy}{dx} + 2\,(1 + x)\ y = x^3$$

or $$\frac{d^2y}{dx^2} - 2\left(\frac{1+x}{x}\right)\frac{dy}{dx} + \frac{2(1+x)}{x^2}\,y = x \qquad .....(1)$$

Comparing it with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$P = -\frac{2(1+x)}{x}$, $Q = \frac{2(1+x)}{x^2}$, $R = x$

Now $\quad P + Q\,x = 0$

$\Rightarrow \quad -\frac{2(1+x)}{x} + \frac{2(1+x)}{x^2}\,x = 0$

$\Rightarrow \quad 0 = 0$

$\therefore \quad y = x$ is a solution of the equation

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}\,y = 0$$

Put $\quad y = vx$, where v is a function of x

$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{d^2y}{dx^2} - \frac{dv}{dx} + x\frac{d^2v}{dx^2} + \frac{dv}{dx}$$

Substituting values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1), we get

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 2\left(\frac{1+x}{x}\right)\left(v + x\frac{dv}{dx}\right) + 2\left(\frac{1+x}{x^2}\right)vx = x$$

or $$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\,v - 2\,(1 + x)\frac{dv}{dx} + \frac{2(1+x)}{x}\,v = x$$

or $$x\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} = x$$

or $\quad \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \qquad .....(2)$

Put $\quad \frac{dv}{dx} = w, \frac{d^2v}{dx^2} = w'$ in (2) $\qquad ......(3)$

$\therefore \quad w' - 2w = 1$

This is linear equation in w.

I.F. = $e^{\int -2dx}$

$\therefore \quad$ Solution of (3) is

w. $e^{-2x} = \int 1.e^{-2x}\, dx + c_1$

or $\quad v'\, e^{-2x} = \frac{e^{-2x}}{-2} + c_1$

or $\quad \frac{dv}{dx} = -\frac{1}{2} + c_1e^{2x}$

Integrating w.r.t. x, we get

$v = \frac{1}{2}x + \frac{c_1}{2}e^{2x} + c_2$

$\therefore \quad$ General solution of (1) is $y = vx$

or $\quad y = -\frac{1}{2}x^2 + \frac{c_1}{2}x\, e^{2x} + c_2x$

or $\quad y = -\frac{1}{2}x^2 + \frac{c_1}{2}x\, e^{2x} + c_2x$

**Example 7:** Solve the following differential equation by the method of reduction of order

$$x\frac{d^2y}{dx^2} - (2x - 1)\frac{dy}{dx} + (x - 1)\, y = 0$$

**Sol:** Given equation is

$$x\frac{d^2y}{dx^2} - (2x - 1)\frac{dy}{dx} + (x - 1)\, y = 0$$

or $\quad \frac{d^2y}{dx^2} - \frac{(2x+1)}{x}\frac{dy}{dx} + \frac{1-x}{x}y = 0 \qquad ......(1)$

Comparing (1) with $\frac{d^2y}{dx^2}$ + P $\frac{dy}{dx}$ + Qy = R, we get

P = - $\frac{(2x+1)}{x}$, Q = $\frac{x-1}{x}$, R = 0

$\therefore$ 1 + P + Q = 1 - $\frac{2x-1}{x}$ + $\frac{x-1}{x}$ = $\frac{x-2x+1+x-1}{x}$

or 1 + P + Q = $\frac{0}{x}$ = 0

$\therefore$ y = $e^x$ is a solution of (1)

Put y = $ve^x$, where v is a function of x

$\therefore$ $\frac{dy}{dx}$ = $ve^x + e^x \frac{dv}{dx}$

and $\frac{d^2y}{dx^2} = ve^x + e^x\frac{dv}{dx} + e^x\frac{dv}{dx} + e^x\frac{d^2y}{dx^2}$

Substituting values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (1), we get

$$vex + 2e^x\frac{dv}{dx} + e^x\frac{d^2v}{dx^2} - \frac{(2x-1)}{x}\left[ve^x + e^x\frac{dv}{dx}\right] + \frac{x-1}{x}ve^x = 0$$

or $ve^x\left[1-\frac{2x-1}{x}+\frac{x-1}{x}\right] + e^x\frac{dv}{dx}\left[2-\frac{2x-1}{x}\right] + e^x\frac{d^2v}{dx^2} = 0$

or $\frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0$

Putting $\frac{dv}{dx}$ = w and $\frac{d^2y}{dx^2}$ = w' in (2), we get

$w' + \frac{1}{x}w = 0$

or $\frac{dw}{dx} + \frac{1}{x}w = 0$

or $\frac{1}{w}dw + \frac{1}{x}dx = 0$

Integrating, we get

$$\int \frac{1}{w}\,dw + \int \frac{1}{x}\,dx = \log|c_1|, \; c_1 \text{ is arbitrary}$$

or $\quad \log|w| + \log|x| = \log|c_1|$

or $\quad \log|w\,x| = \log|c_1|$

or $\quad wx = c_1$

or $\quad \frac{dv}{dx}x = c_1$

or $\quad dv = c_1 \frac{1}{x}\,dx$

Integrating, we get

$$\int dv = c_1 \int \frac{1}{x}\,dx + c_2$$

or $\quad v = c_1 \log|x| + c_2$

$\therefore$ the general solution of (1) is

$y = [c_1 \log|x| + c_2]e^x$ $\qquad$ [Q $y = ve^x$]

**Self-Check Exercise**

Q.1 If $y = e^x$ is a solution of $xy'' - (x + 1)y' + y = 0$, then find another linearly independent solution by reducing its order.

Q.2 Given that $y = x$ is a solution of $(x^2 - 1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

find a linearly independent solution by reducing its order. Write the general solution.

Q.3 Given that $y = e^{2x}$ is a solution of $y'' - 6y' + 8y = 0$, find a linearly independent solution by reducing its order. Write the general solution.

Q.4 Solve the differential equation by the method of reduction of order:

$$\frac{d^2y}{dx^2} - \cot x\frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$$

Q.5 Solve the following differential equation by the method of reduction of order:

$$y'' + 2y' + y = (ex - 1)^{-2}$$

## 10.5 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed a method to solve a differential equation by reducing its order.
2. Discussed method to find a particular solution by inspection
3. Find solutions of differential equations by reducing its order.

## 10.6 Glossary:

1. By reducing the order, we can break down the problem into a series of simpler equations, making the solution process more feasible.
2. y = x is a solution of

   $\frac{d^2y}{dx^2}$ + P $\frac{dy}{dx}$ + Qy = 0 if P + Q x = 0
3. $y = x^{-1}$ is a solution if

   $2 - Px + Qx^2 = 0$
4. y = x2 is a solution if

   $2 + 2\,Px + Qx^2 = 0$

## 10.7 Answer to Self Check Exercise

**Self-Check Exercise**

Ans.1 Another solution is -(x + 1) and complete solution is $y = c_1e^x - c_2\,(x + 1)$, where $c_1$, $c_2$ are arbitrary constants.

Ans. 2 Another solution is $\frac{x^3}{2}$ - x log x and general solution is

$y = c_1\,x + c_2\left(\frac{x^3}{2} - x\log x\right)$, where $c_1$, $c_2$ are arbitrary constants.

Ans. 3 Another independent solution is $e^{4x}$ and general solution is

$y = c_1e^{2x} + c_2e^{4x}$, where $c_1$, $c_2$ are arbitrary constants.

Ans. 4 $y = -\frac{1}{2}e^x \cos x - \frac{c_1}{5}e^{-x}(\cos x + 2\sin x) + c_2\,e^x$

Ans. 5 $y = -e^{-x}\log(1 - e^{-x}) + c_1xe^{-x} + c_2e^{-x}$

## 10.8 References/Suggested Readings

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.

2. I. Sneddon, Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

## 10.9 Terminal Questions

1. Given that y = x is a solution of

$$x^2 \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + 4y = 0,$$

find a linearly independent solution by reducing the order. Write the general solution.

2. Given that $y = e^x$ is a solution of y'' - 5y' + 4y = 0, find a linearly independent solution by reducing its order. Write the general solution.

3. Given that $y = e^{4x}$ is one solution of y'' - 6y' + 8y = 0. Use reduction of order to find a second linear independent solution. Also find the general solution.

4. Solve the following differential equation by the method of reduction of order:

$$(1 - x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x (1 - x^2)^{3/2}$$

5. Solve the following differential equation by the method of reduction of order:

$$y'' - \frac{x}{x-1} y' + \frac{y}{x-1} = x - 1$$

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**Unit - 11**

# Variation of Parameters

**Structure**



## 11.1 Introduction

The variation of parameter method is a technique used in solving non-homogeneous linear differential equations. It provides a systematic approach to finding particular solutions by assuming that they can be expressed as a linear combination of known functions multiplied by unknown coefficients.

By using the variation of parameter method, one can find a particular solution that satisfies the non-homogeneous equation, while also considering the complementary solution that satisfies the associated homogeneous equation. This method allows for the complete solution of the differential equation, incorporating both the general solution of the homogeneous equation and a particular solution of the non-homogeneous equation.

Compared to other techniques, the variation of parameter method is more versatile and applicable to a wider range of non-homogeneous equations. It does not required making specific assumptions about the form of the particular solution, which can be limiting in certain cases. Instead, the method allows for the flexibility of choosing the basis functions for the particular solution, resulting in a more general solution.

## 11.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss method of variation of parameters to solve a second order linear differential equation with constant coefficients.
- Solve differential equations by the method of variation of parameters.

## 11.3 Method of Variation of Parameters To Solve A Second Order Linear Differential Equation With Constant Coefficients

Consider the second order linear differential equation

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = Q$$

where $P_1$, $P_2$ are constants and Q is a function of x.

Let $y_1$ and $y_2$ be two particular L.I. solutions of the equation

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2y = 0$$

$$\Rightarrow \quad \frac{d^2y_1}{dx^2} + P_1\frac{dy_1}{dx} + P_2 y_1 = 0$$

and $$\frac{d^2y_2}{dx^2} + P_1\frac{dy_2}{dx} + P_2 y_2 = 0$$

Now Let $\quad y = Ay_1 + By_2$

where A and B are functions of x

Differentiating (5) w.r.t. x, we get

$$\frac{dy}{dx} = A\frac{dy_1}{dx} + B\frac{dy_2}{dx} + y_1\frac{dA}{dx} + y_2\frac{dB}{dx}$$

Choose A and B so that

$$y_1\frac{dA}{dx} + y_2\frac{dB}{dx} = 0$$

Then (6) becomes

$$\frac{dy}{dx} = A\frac{dy_1}{dx} + B\frac{dy_2}{dx}$$

Differentiating (8) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = A\frac{d^2y}{dx^2} + B\frac{d^2y_2}{dx^2} + \frac{dA}{dx}\frac{dy_1}{dx} + \frac{dB}{dx}.\frac{dy_2}{dx}$$

(1) becomes

$$A\frac{d^2y_1}{dx^2} + B\frac{d^2y_2}{dx^2} + \frac{dA}{dx}\frac{dy_1}{dx} + \frac{dB}{dx}.\frac{dy_2}{dx} + P_1\left(A\frac{dy_1}{dx} + B\frac{dy_2}{dx}\right) + P_2(Ay_1 + By_2) = A$$

$$\Rightarrow \quad A\left(\frac{d^2y_1}{dx^2}+P_1\frac{dy_1}{dx}+P_2y_1\right) + B\left(\frac{d^2y_2}{dx^2}+P_1\frac{dy_2}{dx}+P_2y_2\right) + \frac{dA}{dx}\frac{dy_1}{dx} + \frac{dB}{dx}\frac{dy_2}{dx} - Q = 0$$

$$\Rightarrow \quad A.0 + B.0 + \frac{dA}{dx}\frac{dy_1}{dx} + \frac{dB}{dx}.\frac{dy_2}{dx} - Q = 0 \qquad \text{[Using (3) and (4)]}$$

$$\Rightarrow \quad \frac{dy_1}{dx}\frac{dA}{dx} + \frac{dy_2}{dx}\frac{dB}{dx} - Q = 0 \qquad ......(9)$$

Solving (4) and (9) for $\frac{dA}{dx}$ and $\frac{dB}{dx}$, we get

$$\frac{dA/dx}{-y_2} = \frac{dB/dx}{-y_1} = \frac{+Q}{y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}}$$

$$\Rightarrow \quad \frac{dA}{dx} = -\frac{y_2Q}{y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}} \quad \frac{dB}{dx} = \frac{y_1Q}{y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}}$$

$$\Rightarrow \quad A = -\int \frac{y_2Q}{W(y_1,y_2)}dx + c_1,\ B = \int \frac{y_1Q}{W(y_1,y_2)}dx + c_2$$

where $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} \end{vmatrix} = y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} \neq 0$ [$\because$ $y_1, y_2$ are L.I.]

$c_1$, $c_2$ are arbitrary constants.

Substituting these values of A and B in (5) we get the general solution of (1).

Let us do some examples to clarify this method:

**Example 1:** Solve the following equation by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + y = \tan x$$

**Sol:** Given equation is

$$\frac{d^2y}{dx^2} + y = \tan x$$

Equation in S.F. is $(D^2 + 1)\, y = \tan x$ ......(1)

The corresponding homogeneous equation is

$$(D^2 + 1)\, y = 0$$

A.E. is $D^2 + 1 = 0$ ......(2)

A.E. is $D^2 + 1 = 0$

or $D = \pm i = 0 \pm i$

$\therefore$ the complementary solution of (1) is

$y_c = c_1 \cos x + c_2 \sin x$

Now we seek a particular solution of (1) by variation of parameters.

Let $y = A \cos x + B \sin x$ .....(3)

Differentiating (3) w.r.t. x, we get

$y' = A' \cos x + B' \sin x - A \sin x + B \cos x$ .......(4)

Choose $A' \cos x + B' \sin x = 0$ ......(5)

$\therefore$ (4) becomes

$y' = - A \sin x + B \cos x$ ......(6)

Diff. w.r.t. x, we get

$y'' = A' \sin x + B' \cos x - A \cos x - B \sin x$ ......(7)

Substituting the values of y, y'' from (3) and (7) in (1), we get

$- A' \sin x + B' \cos x - B \sin x - A \cos x + A \cos x + B \sin x = \tan x$

or $- A' \sin x + B' \cos x = \tan x$ .......(8)

Now we try to find values of A' and B' from (5) and (8)

Multiplying (5) by cos x, (8) by sin x and subtracting, we get

$A' (\cos^2 x + \sin^2 x) = 0 - \sin x. \tan x$

$\therefore$ $A' = - \sin x \tan x$

Again multiplying (5) by sin x, (8) by cos x and adding, we get

$B' (\sin^2 x + \cos^2 x) = \cos x \tan x$

or $B' = \sin x$

Now $A' = - \sin x \tan x = -\sin x. \dfrac{\sin x}{\cos x}$

$= - \dfrac{\sin^2 x}{\cos x}$

$= - \dfrac{1 - \sin^2 x}{\cos x}$

$= - \left[ \frac{1}{\cos x} - \cos x \right]$

= cos x - sec x

$\therefore$ Integrating w.r.t. x, we have

$A = \sin x - \log \left| \tan\left( \frac{\pi}{4} + \frac{x}{2} \right) \right|$

Also B' = sin x $\Rightarrow$ B = - cos x

Putting values of A and B in (3), we get

$y = \left[ \sin x - \log \left| \tan\left( \frac{\pi}{4} + \frac{x}{2} \right) \right| \right]$ cos x - cos x . sin x,

which is a particular solution of (1)

$\therefore$ General solution of (1) is

$y = c_1 \cos x + c_2 \sin x + \left[ \sin x - \log \left| \tan\left( \frac{\pi}{4} + \frac{x}{2} \right) \right| \right] \cos x - \cos x . \sin x$

**Example 2:** Solve $\frac{d^2y}{dx^2}$ + y = cosec x, by the method of variation of parameters.

**Sol:** Given equation is

$\frac{d^2y}{dx^2} + y = \text{cosec } x$ .......(1)

Equation in the symbolic form is $(D^2 + 1)$ y = cosec x

A.E. is $D^2 + 1 = 0$ $\therefore$ D = 0 + i ......(2)

$\therefore$ C.F. is y = A cos x + B sin x where A, B are constants.

Let (2) be a sol. of (1) where A, B are functions of x.

Differentiate (2), $\frac{dy}{dx} = - A \sin x + \cos x \frac{dA}{dx} + B \cos x + \sin x \frac{dB}{dx}$

Put $\cos x \frac{dA}{dx} + \sin x \frac{dB}{dx} = 0$ .....(3)

then $\frac{dy}{dx} = - A \sin x + B \cos x$

Differentiate $\frac{d^2y}{dx^2}$ = - A cos x - sin x $\frac{dA}{dx}$ - B sin x + cos x $\frac{dB}{dx}$

Putting in (1), - A cos x - sin x $\frac{dA}{dx}$ - B sin x + cos x $\frac{dB}{dx}$

+ A cos x + B sin x = cosec x

or - sin x $\frac{dA}{dx}$ + cos x $\frac{dB}{dx}$ - cosec x = 0 ......(4)

From (3), cos x $\frac{dA}{dx}$ + sin x $\frac{dB}{dx}$ + 0 = 0 .......(5)

Solving, $$\frac{\frac{dA}{dx}}{1} = \frac{\frac{dB}{dx}}{-\cot x} = \frac{1}{-1}$$

$\therefore$ $\frac{dA}{dx}$ = -1 and $\frac{dB}{dx}$ = cos x

Integrating, A = $c_1$ - x and B = $c_2$ + log sin x

Hence the sol. is y = ($c_1$ - x) cos x + ($c_2$ + log sin x) sin x

**Example 3:** Solve: $\frac{d^2y}{dx^2}$ + 4y = 4 tan 2x by the method of variation of parameters.

**Sol:** The given equation is $\frac{d^2y}{dx^2}$ + 4y = 4 tan 2x ......(1)

Equation in the symbolic form is ($D^2$ + 4) y = 4 tan 2x

A.E. is $D^2$ + 4 = 0 $\therefore$ D = 0 $\pm$ 2i

$\therefore$ C.F. is y = A cos 2x + B sin 2x where A, B are constants .....(2)

Let (2) be a sol. of (1) where A, B are functions of x.

Differentiate (2), $\frac{dy}{dx}$ = -2A sin 2x + cos 2x $\frac{dA}{dx}$ + 2B cos 2x + sin 2x $\frac{dB}{dx}$

Put cos 2x $\frac{dA}{dx}$ + sin 2x $\frac{dB}{dx}$ = 0 .....(3)

then $\frac{dy}{dx}$ = - 2A sin 2x + 2B cos 2x

Differentiate $\frac{d^2y}{dx^2}$ = -4A cos 2x - 2 sin 2x $\frac{dA}{dx}$ - 4B sin 2x + 2 cos 2x $\frac{dB}{dx}$

Putting in (1), - 4A cos 2x - 2 sin 2x $\frac{dA}{dx}$ - 4B sin 2x + 2 cos 2x $\frac{dB}{dx}$

+ 4A cos 2x + 4B sin 2x = 4 tan 2x

or -2 sin 2x $\frac{dA}{dx}$ + 2 cos 2x $\frac{dB}{dx}$ - 4 tan 2x = 0 .....(4)

or sin 2x $\frac{dA}{dx}$ - cos 2x $\frac{dB}{dx}$ + 2 tan 2x = 0

From (3), cos 2x $\frac{dA}{dx}$ + sin 2x $\frac{dB}{dx}$ + 0 = 0 ..... (5)

$\frac{dA}{dx}$ = $\frac{dB}{dx}$ = 1

Solving, -2 tan 2x sin 2x = 2 tan 2x cos 2x = 1

$\frac{dA}{dx}$ = -2 tan 2x sin 2x and $\frac{dB}{dx}$ = 2 tan 2x cos 2x

or $\frac{dA}{dx}$ = - $\frac{2\sin^2 2x}{\cos 2x}$ = -2 $\left(\frac{1-\cos^2 2x}{\cos 2x}\right)$

= -2 (sec 2x - cos 2x) and $\frac{dB}{dx}$ = 2 sin 2x

Integrating, A = -log tan $\left(\frac{\pi}{4}+x\right)$ + sin 2x + $c_1$ and B = - cos 2x + $c_2$

Hence the sol. is

y = $\left[c_1 - \log\tan\left(\frac{\pi}{4}+x\right) + \sin 2x\right]$ cos 2x + [$c_2$ - cos 2x] sin 2x

= $\left[c_1 - \log\tan\left(\frac{\pi}{4}+x\right)\right]$ cos 2x + $c_2$ sin 2x

**Example 4:** Solve: $\frac{d^2y}{dx^2}$ - y = $\frac{2}{1+e^x}$ (by the method of variation of parameters)

**Sol:** The given equation is $\frac{d^2y}{dx^2}$ - y = $\frac{2}{1+e^x}$ ......(1)

Equation in the symbolic form is $(D^2 - 1)\, y = \frac{2}{1+e^x}$

A.E. is $D^2 - 1 = 0$ $\therefore$ $D = \underline{+}\,1$

$\therefore$ C.f. is $y = Ae^x + Be^{-x}$ .....(2), where A, B are constants

Let (2) be a sol. of (1), where A, B are functions of x

Differentiate (2), $\frac{dy}{dx} = Ae^x + e^x\frac{dA}{dx} - Be^{-x}, e^{-x}\frac{dB}{dx}$

Put $e^x\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0$ ......(3)

then $\frac{dy}{dx} = Ae^x - Be^{-x}$

Differentiate $\frac{d^2y}{dx^2} = Ae^x + e^x\frac{dA}{dx} + Be^{-x} - e^{-x}\frac{dB}{dx}$

Putting in (1), we get $Ae^x + e^x\frac{dA}{dx} + Be^{-x} - e^{-x}\frac{dB}{dx} - Ae^x - Be^{-x} = \frac{2}{1+e^x}$

or $e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} - \frac{2}{1+e^x} = 0$ ......(4)

From (3), $e^x\frac{dA}{dx} - e^{-x}\frac{dB}{dx} - 0 = 0$ ......(5)

Solving, $\dfrac{\frac{dA}{dx}}{\frac{2e^{-x}}{1+e^x}} = \dfrac{\frac{dB}{dx}}{\frac{-2e^{-x}}{1+e^x}} = \frac{1}{1+1}$

$\therefore$ $\frac{dA}{dx} = \frac{e^{-x}}{1+e^x}$ and $\frac{dB}{dx} = -\frac{e^{-x}}{1+e^x}$

Integrating, we get $A = c_1 + \int \frac{e^{-x}}{1+e^x}\,dx = c1 + \int \frac{e^{-2x}}{e^{-x}+1}\,dx$

$= c_1 + \int \frac{(1+e^{-x})e^{-x} - e^{-x}}{1+e^{-x}}\,dx = c_1 + \int e^{-x}\,dx - \int \frac{e^{-x}}{1+e^{-x}}\,dx$

$= c_1 - e^{-x} + \log(1 + e^{-x})$ and $B = c_2 - \log(1 + e^x)$

Hence the sol. is $y = [c_1 - e^{-x} + \log(1 + e^{-x})\, e^x + [c_2 - \log(1 + e^x)]\, e^{-x}$

**Example 5:** Solve the following differential equation by the method of variation of parameters

$$\frac{d^2y}{dx^2} + y = \sec x \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

**Sol:** Given equation is

$(D^2 + 1)\, y = \sec x$ ......(1)

The corresponding homogeneous equation is

$(D^2 + 1)\, y = 0$ ......(2)

A.E. is $D^2 + 1 = 0$

or $D = \pm i$

$\therefore$ the complementary solution of (1) is

$y_c = c_1 \cos x + c_2 \sin x$

Now let us try to find a particular solution of (1) by variation of parameters.

Let $y = A \cos x + B \sin x$ ......(3)

$\therefore$ $y' = A' \cos x - A \sin x + B' \sin x + B \cos x$ .....(4)

Choose $A' \cos x + B' \sin x = 0$ .....(5)

So that (4) becomes

$y' = -A \sin x + B \cos x$

$\therefore$ $y'' = - A' \sin x - A \cos x + B' \cos x - B \sin x$

Putting values of y, y'' in (1), we get

$- A' \sin x - A \cos x + B' \cos x - B \sin x + A \cos x + B \sin x = \sec x$

$\therefore$ $- A' \sin x + B' \cos x = \sec x$

Also $A' \cos x + B' \sin x = 0$ [From (5)]

Solving simultaneously, we get $A' = \tan x$, $B' = 1$

$\therefore$ $A = -\int \tan x \, dx$, $B = \int 1 \, dx$

$\therefore$ $A = \log |\cos x|$, $B = x$

$\therefore$ a particular solution of (1) is

$y_p = \cos x \log |\cos x| + x \sin x$

$\therefore$ General solution of (1) is $y = y_c + y_p$

or $y = c_1 \cos x + c_2 \sin x + \cos x \log |\cos x| + x \sin x$

**Example 6:** Solve the equation by the method of variation of parameters

$(D^2 + 1)\ y = \text{cosec } x \cot x$

**Sol:** Given differential equation is

$(D^2 + 1)\ y = \text{cosec } x \cot x$ ......(1)

Corresponding homogeneous equation is

$(D^2 + 1) = 0$ ......(2)

A.E. is $D^2 + 1 = 0$

or $D = \underline{+}i$

$\therefore$ Complementary solution of (1) is

$y_c = c_1 \cos x + c_2 \sin x$

Now let us try to find a particular solution of (1) by variation of parameters.

Let $y = A \cos x + B \sin x$ ......(3)

$\therefore$ $y' = A' \cos x + B' \sin x - A \sin x + B \cos x$ ......(4)

Choose $A' \cos x + B' \sin x = 0$, ......(5)

So that (4) becomes

$y' = -A \sin x + B \cos x$

$\therefore$ $y'' = -A' \sin x + B' \cos x - A \cos x - B \sin x$

Now putting values of y, y'' in (1), we get,

$- A' \sin x + B' \cos x = \text{cosec } x \cot x$ [From (5)]

Also $A' \cos x + B' \sin x = 0$

$\therefore$ Solving simultaneously, we get

$A' = - \cot x,\ B' = \cot^2 x$

$\therefore$ $A = -\int \cot x\, dx,\ B = \int \cot^2 x\, dx$

or $A = -\log |\sin x|,\ B = \int (\text{cosec}^2 x - 1)\, dx$

$= - \cot x - x$

$\therefore$ a particular solution of (1) is

$y_p = (- \cot x - x) \sin x + (- \log |\sin x|) \cos x$

$\therefore$ General solution of (1) is $y = y_c + y_p$

or $y = c_1 \cos x + c_2 \sin x - \cos x \log |\sin x| - (\cos x + x \sin x)$

or $\quad y = c_1 \cos x + c_2 \sin x - \cos x \log|\sin x| - x \sin x \quad [c_1 = c_1 - 1]$

**Example 7:** Solve $(D^2 - 3D + 2)y = \cos(e^{-x})$ by the variation of parameters method

**Sol:** Given differential equation is

$$(D^2 - 3D + 2)\, y = \cos(e^{-x}) \quad .....(1)$$

Corresponding homogeneous equation is

$$(D^2 - 2D + 2)\, y = 0 \quad .....(2)$$

A.E. is $D^2 - 3D + 2 = 0$

or $\quad (D - 1)(D - 2) = 0$

$\therefore \quad D = 1, 2$

$\therefore \quad$ the complementary solution of (1) is

$$y_c = c_1 e^x + c_2 e^{2x}$$

Now let us try to find a particular solution of (1) by variation of parameters.

Let $\quad y = A e^x + B e^{2x} \quad ......(3)$

$\therefore \quad y' = A'e^x + B'e^{2x} + A e^x + 2 B e^{2x} \quad ......(4)$

Choose $\quad A' e^x + B' e^{2x} = 0 \quad .......(5)$

So that (4) becomes

$$y' = A e^x + 2 B e^{2x}$$

$\therefore \quad y'' = A' e^x + 2B' e^{2x} + A e^x + 4 B e^{2x}$

Now putting the values of y, y', y'' in (1), we get

$$A' e^x + 2B' e^{2x} = \cos(e^{-x})$$

Also $\quad A' e^x + B' e^{2x} = 0 \quad$ [From (5)]

$\therefore \quad$ Solving simultaneously, we get

$$A' = -e^{-x} \cos(e^{-x}) \text{ and } B' = e^{-2x} \cos(e^{-x})$$

$\therefore \quad A = -\int e^{-x} \cos(e^{-x})\, dx \text{ and } B = \int e^{-2x} \cos(e^{-x})\, dx$

$\therefore \quad A = \sin(e^{-x}),\ B = -\int t \cos t\, dt, \text{ where } t = e^{-x} = -(t \sin t + \cos t)$

$\therefore \quad A = \sin(e^{-x}),\ B = -(e^{-x} \sin e^{-x} + \cos e^{-x})$

$\therefore \quad$ a particular solution of (1) is

$$y_p = (\sin e^{-x})\, e^x - (e^{-x} \sin e\text{-}x + \cos e^{-x})\, e^2x$$

or $\quad y_p = e^x \sin e^{-x} - e^x \sin e^{-x} - e^2x \cos e^{-x}$

$= -e^{2x} \cos e^{-x}$

$\therefore$ General solution of (1) is $y = y_c + y_p$

or $y = c_1 e^x + c_2 e^{2x} - e^{2x} \cos e^{-x}$

**Example 8:** Find the general solution of following non-homogenous differential equation using method of variation of parameters:

$$(D^2 + 6D + 9)\, y = \frac{16 e^{-3x}}{x^2 + 1}$$

**Sol:** The given differential equation is $(D2 + 6D + 9)\, y = \dfrac{16 e^{-3x}}{x^2 + 1}$ .....(1)

It's A.E. is $D^2 + 6D + 9 = 0$

$\therefore$ $(D + 3)^2 = 0 \Rightarrow D = -3, -3$

$\therefore$ complementary solution of (1) is

$y_c = (c_1 + c_2 x) e^{-3x} = c_1 e^{-3x} + c_2 x e^{-3x}$

Now we seek a particular solution of (1) by method of variation of parameters

Let $y = A e^{-3x} + B x e^{-3x}$ ......(2)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = -3 A e^{-3x} + A' e^{-3x} + B(e^{-3x} - 3 x e^{-3x}) + B' x e^{-3x}$$

Choose $A' e^{-3x} + B' x e^{-3x} = 0$ .....(3)

$\therefore$ $\dfrac{dy}{dx} = -3 A e^{-3x} + B e^{-3x} - 3 B x e^{-3x}$ ......(4)

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 3A' e^{-3x} + 9 A e^{-3x} + B' e^{-3x} - 3 B e^{-3x}$$

$- 3 B(e^{-3x} - 3 x e^{-3x} e^{-3x}) - 3 B' x e^{-3x}$

$\therefore$ $\dfrac{d^2y}{dx^2} = -3 A' e^{-3x} + B' e^{-3x} + 9 A e^{-3x} - 6 B e^{-3x} + 9 B x e^{-3x}$

$- 3 B' x e^{-3x}$ ......(5)

Putting values of $y. \dfrac{dy}{dx} . \dfrac{d^2y}{dx^2}$ from (2), (4), (5) in (1), we get

$-3 A' e^{-3x} + B' e^{-3x} + 9 A e^{-3x} - 6 B e^{-3x} + 9 B x e^{-3x} - 3 B' x e^{-3x}$

$+ 6(-3 A e^{-3x} + Be^{-3x} - 3 B x e^{-3x}) + 9 (A e^{-3x} + B e^{-3x}) = \dfrac{16e^{-3x}}{x^2+1}$

or $\quad -3 A' e^{-3x} + B' e^{-3x} - 3 B' x e^{-3x} = \dfrac{16e^{-3x}}{x^2+1}$

or $\quad -3 A' -3 B'x + B' = \dfrac{16}{x^2+1}$ ......(6)

From (3), $A' + B x = 0$ ...... (7)

Solving (6) and (7), $B = \dfrac{16}{x^2+1}$ and $A' = \dfrac{-16}{x^2+1}$

$\therefore \quad B = 16\displaystyle\int \frac{dx}{x^2+1} = 16 \tan^{-1} x$ and $A = -8\displaystyle\int \frac{2x}{x^2+1} dx = -8 \log (x^2 + 1)$

Putting values of A and B in (2), we get

$y_p = - 8 e^{-3x} \log (x^2 + 1) + (16 \tan^{-1} x) (x e^{-3x})$

$= [- 8 \log (x^2 + 1) + 16 x \tan^{-1} x] e^{-3x}$

which is particular solution of (1)

$\therefore$ general solution of (1) is given by

$y = y_c + y_p = (c_1 + c_2 x) e^{-3x} + [-8 \log (x^2 + 1) + 16x \tan^{-1} x] e^{-3x}$

i.e. $\quad y = [c_1 + c_2 - 8 \log (x^2 - 1) + 16 x \tan^{-1} x] e^{-3x}$

**Self-Check Exercise**

Q.1 Solve the following differential equation by the method of variation of parameters:-

$$\frac{d^2y}{dx^2} + 4y = \cos x$$

Q.2 Solve the following differential equation by the method of variation of parameters.

$$(D^2 - 1) y = e^x$$

Q.3 Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin e^x$$

Q.4 Solve by the method of variation of parameters:

$$(D^2 + a^2)\, y = \sin ax$$

Q.5 Solve by the method of variation of parameters:

$$y'' + 2y' + y = 2e^{2x}$$

## 11.4 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed in detail the method of variation of parameters to solve a second order linear differential equation with constant coefficients.
2. Find solutions of second order linear differential equations with constant coefficients by using method of variation of parameters.

## 11.5 Glossary:

1. The variation of parameter method is a technique used in solving non-homogeneous linear differential equations.

## 11.6 Answer to Self Check Exercise

**Self-Check Exercise**

Ans.1 $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} \cos x$

Ans. 2 $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{2} x e^{x}$, where

$c_1 = c_1 - \frac{1}{4}$ and $c_2 = c_2$

Ans. 3 $y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin e^{x}$

Ans. 4 $y = c_1 \cos ax - \frac{x}{2a} \cos ax + c_3 \sin ax$

where $c_3 = c_2 + \frac{1}{4a^2}$

Ans. 5 $y = c_1 e^{-x} + c_2 x e^{-x} + \frac{2}{9} e^{2x}$

## 11.7 References/Suggested Readings


1. Boyce, w. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.
2. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.


## 11.8 Terminal Questions

1. Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 9\,y = \sin x$$

2. Solve the following differential equation by the method of variation of parameters

$$\frac{d^2y}{dx^2} + 9\,y = \sec 3x$$

3. Solve by the method of variation of parameters:

$$\frac{d^2y}{dx^2} + 4\,y = \tan x$$

4. Solve by the method of variation of parameters:

$(D^2 - 6D + 9)\,y = x^{-2}\,e^{3x}$

5. Solve by the method of variation of parameters:

$(d^2 - 2D + 2)\,y = e^x \tan x$

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## Unit - 12

# Cauchy-Euler's Homogeneous Linear Equation

**Structure**



## 12.1 Introduction

A differential equation of the form $P_0\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + ....... + P_n y = Q$, where $P_0$, $P_1$, $P_2$,.....,$P_n$ and Q are functions of x is called linear differential equation with variable coefficients.

A homogeneous linear equation of the form

$$P_0x_n\frac{d^n y}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ......... + P_ny = Q(x)$$

where $P_0$, $P_1$, $P_2$,......., $P_n$ are all real constants and Q(x) is a function of x, is called courtly - Euler's linear equation. It is named after the mathematicians Augustin-Louis Cauchy and Leonhard Euler, who made significant contributions to the field of differential equation.

## 12.2 Learning Objectives

After studying this unit, you should be able to:-

- Define Cauchy-Euler's linear equation.
- Discuss method to solve Cauchy-Euler's equation.
- Find solutions by this method.

## 12.3 Method to Solve Cauchy-Euler's Equation

A linear equation of the form

$$P_0x_n \frac{d^n y}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \text{.........} + P_n y = Q(x) \qquad \text{.....(1)}$$

where $P_0$, $P_1$,.......,$P_n$ are real constant and Q(x) is a function of x, is called linear equation.

The equation (1) can be written in the symbolic form as

$$(P_0x^nD^n + P_1\, x^{n-1}\, D^{n-1} + \text{.........}P_n)\, y = Q\,(x) \qquad \text{......(2)}$$

where $D = \frac{d}{dx}$

Put $x = e^z$ i.e., $z = \log x$, $x > 0$

$$\frac{dz}{dx} = \frac{1}{x}$$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} = \frac{1}{x}\frac{dz}{dx}$

Let $\frac{d}{dz} = \theta$

$\therefore$ $D\,y = \frac{1}{x}\theta\, y$. i.e., $x\,D\,y = \theta\,y$ or $x\,D = \theta$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$

$$= \frac{1}{x}\frac{d^2y}{dx^2}\frac{dz}{dx} + \frac{dy}{dz}\left(-\frac{1}{x^2}\right) + \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz} \qquad \left[\because \frac{dz}{dx} = \frac{1}{x}\right]$$

$\therefore$ $\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$

$\therefore$ $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

or $x^2\,D^2\,y = \theta^2\,y - \theta\,y$

or $x^2\,D^2\,y = (\theta^2 - \theta)\,y \quad \Rightarrow \quad x^2\,D^2 = \theta^2 - \theta$

$\Rightarrow$ $x^2\,D^2 = \theta\,(\theta - 1)$

Similarly $x^3\,D^3 = \theta\,(\theta - 1)\,(\theta - 2)$

and so on

In general, $x^nD^n = \theta(\theta - 1)(\theta - 2) \ldots\ldots. (\theta - \overline{n-1})$

Substituting the values of $x^nD^n$ ......... $x^2 D^2$, $x D$ in (2), we get,

$[P_0\theta(\theta - 1)(\theta - 2) \ldots. (\theta - \overline{n-1}) + P_1\theta(\theta - 1) \ldots.. (\theta - \overline{n-2}) + \ldots\ldots + P_{n-1}\theta + P_n]\, y = Q(e^z)$ .....(3)

or $f(\theta)\, y = \phi(z)$

where $f(\theta)$ is a polynomial in $\theta$ with real coeffs. and $\phi(z)$ is a function of z.

Now (3) can be solved for y in terms of z by the methods already known to us. Let its general solution be

$y = \psi(z)$

$\therefore$ general solution of (1) is

$y = \psi(\log x)$, $x > 0$.

Note: Working rule to solve Cauchy's linear equation

Step 1: Put $x = e^z$ i.e., $z = \log x$, $x > 0$

Step 2: Put $\frac{d}{dz} = \theta$ so that

$x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$,......, $x^nD^n = \theta(\theta - 1)(\theta - 2) \ldots (\theta - \overline{n-1})$

Step 3: Putting these in the given equation, we get,

$[P_0\theta(\theta - 1) \ldots. (\theta - \overline{n-1}) + P_1\theta(\theta - 1) \ldots. (\theta - \overline{n-2}) + \ldots.. + P_n]\, y = Q(e^z)$ which is

linear equation with constant coeffs. and solve for y in terms of z.

Step 4: Put $z = \log x$ to get the required solution.

To clarify what we have just said, consider the following examples:-

**Example 1:** Convert the following differential equation by substituting $x = e^z$ into another whose coefficients are constant:

$$x^2 \frac{d^2y}{dx^2} + 9x \frac{dz}{dx} + 25 y = 0$$

Sol. : The given differential equation is

$$x^2 \frac{d^2y}{dx^2} + 9x \frac{dz}{dx} + 25 y = 0 \qquad \ldots\ldots(1)$$

or, in S.F. $(x^2 D^2 + 9x D + 25)\, y = 0$ .......(2)

Put $x = e^z$ i.e., $z = \log x$, $x > 0$

and $x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$ in (2)

$\therefore$ $[\theta(\theta - 1) + 9\theta + 25]\, y = 0$

or $(\theta^2 + 8\theta + 25]\, y = 0$, which is required differential equation.

**Example 2:** Solve the following differential equation

$$\frac{d^3y}{dx^3} = \frac{6y}{x^3},\ x > 0$$

**Sol:** Given differential equation is

$$\frac{d^3y}{dx^3} = \frac{6y}{x^3}$$

or $x^3 \frac{d^3y}{dx^3} = 6y$

or $(x^3 D^3 - 6)\, y = 0$ .....(1)

Put $x = e^z$, so that $z = \log x$

and $x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$, $x^3 D^3 = \theta(\theta - 1)(\theta - 2)$

where $\theta = \frac{d}{dz}$

Putting these values in (1), we get

$$[\theta(\theta - 1)(\theta - 2) - 6]\, y = 0$$

or $[\theta^3 - 3\theta^2 + 2\theta - 6]\, y = 0$

A.E. is $\theta^3 - 3\theta^2 + 2\theta - 6 = 0$

or $(\theta - 3)(\theta^2 + 2) = 0$

$\Rightarrow$ $\theta = 3, \pm\sqrt{2}\, i$

$\therefore$ General solution is

$$y = c_1 e^{3z} + c_2 \cos\sqrt{2}\, z + c_3 \sin\sqrt{2}\, z$$

or $y = c_1 x3 + c_2 \cos(\sqrt{2} \log x) + c_3 \sin(\sqrt{2} \log x),\ x > 0$

**Example 3:** Solve the following differential equation

$$x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} + 25\, y = 50$$

**Sol:** Given differential equation is

$$x^2 \frac{d^2y}{dx^2} + 9x \frac{dy}{dx} + 25\, y = 50 \quad ......(1)$$

Equation is S.F. is

$(x^2 D^2 + 9 x D + 25) y = 50$ ......(2)

Put $x = e^z$ i.e. $z = \log x, x > 0$

and $x D = \theta$, $x^2 D^2 = \theta (\theta - 1)$ in (2), we get

$[\theta (\theta - 1) + 9\theta + 25] y = 50$

or $(\theta^2 + 80 + 25) y = 50$

A.E. is $\theta^2 + 80 + 25 = 0$

$\therefore$ $\theta = \dfrac{-8 \pm \sqrt{64-100}}{2} = \dfrac{-8 \pm \sqrt{-36}}{2} = -4 \underline{+} 3i$

C.f. is $= e^{-4x} (c_1 \cos 3z + c_2 \sin 3z)$

$= x^{-4}[c_1 \cos (3 \log x) + c_2 \sin (3 \log x)]$

P.I. $= \dfrac{1}{\theta^2 + 8\theta + 25} (50)$

$= 50 \dfrac{1}{\theta^2 + 8\theta + 25} (1)$

$= 50 \dfrac{1}{\theta^2 + 8\theta + 25} e^{0.z}$

$= 50 \dfrac{1}{0 + 0 + 25} e^{0z}$

$= 2$

$\therefore$ C. S. is

$y = 2 + x^{-4} [c_1 \cos (3 \log x) + c_2 \sin (3 \log x)]$

**Example 4:** Solve the following differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2$$

**Sol:** Given differential equation is

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 2x^2 \quad .....(1)$$

Equation in S.F. is

$(x^2 D^2 - 3 x D + 4) y = 2 x^2$ .....(2)

Put $x = e^z$ i.e. $z = \log x, x > 0$

and $\quad x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$, where $\theta = \frac{d}{dz}$

$\therefore$ from (2), we have

$[\theta(\theta - 1) - 3\theta + 4]\, y = 2e^{2z}$

or $\quad (\theta^2 - 4\theta + 4)\, y = 2e^{2z}$

A.E. is $\theta^2 - 4\theta + 4 = 0$

or $\quad (\theta - 2)^2 = 0$

$\therefore \quad \theta = 2, 2$

$\therefore \quad$ C.F. is $y_c = (c_1 + c_2 z)\, e^{2z}$

P.I. $= \frac{1}{\theta^2 + 4\theta + 4}\, 2e^{2z}$

$= 2\, \frac{1}{(2)^2 - 4(2) + 4}$

$= 2\, \frac{1}{0}\, e^{2z}$ [case of failure]

$= z.\, 2\, \frac{1}{2\theta - 4}\, e^{2z}$

$= 2z\, \frac{1}{2(2) - 4}\, e^{2z}$

$= 2z.\, \frac{1}{0}.e^{2z}$ [Again case of failure]

$= z.2z.\, \frac{1}{2}\, e^{2z}$

$= z^2 e^{2z}$

$\therefore$ General solution is $y = y_c + y_p$

or $\quad y = (c_1 + c_2 z)\, e^{2z} + z^2 e^{2z}$

or $\quad y = (c_1 + c_2 \log x)\, x^2 + (\log x)^2 x^2$

**Example 5:** Solve $x^2 \frac{d^2y}{dx^2} - x\, \frac{dy}{dx} + y = 2 \log x$

**Sol:** Given differential equation is

$$x^2 \frac{d^2y}{dx^2} - x \frac{dy}{dx} + y = 2 \log x \qquad .....(1)$$

Equation in S.F. is

$$(x^2 D^2 - x D + 1) y = 2 \log x \qquad ......(2)$$

Put $x = e^z$ i.e. $\log x = z, x > 0$

and $x D = \theta, x^2 D^2 = \theta (\theta - 1)$, where $\theta = \frac{d}{dz}$ in (2)

We have $[\theta (\theta - 1) - \theta + 1] y = 2z$

$\Rightarrow$ $(\theta^2 - 2\theta + 1) y = 2z \Rightarrow (\theta - 1)^2 y = 2z$

$\therefore$ A.E. is $(0 - 1)^2 = 0 \Rightarrow \theta = 1, 1$

$\therefore$ C.F. = $(c_1 + c_2 z) e^z$

$$\text{P.I.} = \frac{1}{(\theta-1)^2} 2z = 2 \frac{1}{(1-\theta)^2} z = 2(1 - \theta)^{-2} z = 2 (1 + 2\theta) z$$

$$= 2(z + 2\theta z) = 2(z + 2) = 4 + 2z$$

$\therefore$ C.S. is $y = (c_1 + c_2 z) e^z + 4 + 2z$

i.e. $y = (c_1 + c_2 \log x) x + 4 + 2 \log x$, is the reqd. solution.

**Example 6:** Solve: $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 2y = 4x^3$

**Sol:** Put x = ez i.e., $z = \log x$ and $x D = \theta, x^2 D^2 = \theta (\theta - 1)$, where $\theta = \frac{d}{dz}$

$\therefore$ the given equation becomes $(\theta(\theta - 1) - 2\theta + 2) y = 4e^{3z}$

i.e., $(\theta^2 - 3\theta + 2) y = 4e^{3z}$

$\therefore$ A.E. is $\theta^2 - 3\theta + 2 = 0$

$\Rightarrow$ $(\theta - 1) (\theta - 2) = 0 \Rightarrow \theta = 1, 2$

C.F. = $c_1e^z + c_2 e^{2z}$

$$\text{P.I.} = \frac{1}{\theta^2 - 3\theta + 2} 4e^{3z} = 4 \frac{1}{(3)^2 - 3(3) + 2} e^{3z}$$

$$= 4.\frac{1}{9 - 9 + 2} e^{3z} = 2e^{3z}$$

$\therefore$ C. S is $y = c_1e^z + c_2e^{2z} + 2e^{3z}$

$\therefore$ $y = c_1x + c_2x^2 + 2x^3$, is the reqd. solution.

**Example 7:** Solve: $x^3 \frac{d^3y}{dx^3}$ + $3x^2 \frac{d^2y}{dx^2}$ + $x \frac{dy}{dx}$ + 8y = 65 cos (log x)

**Sol:** Put $x = e^z$ i.e. z = log x, the given equation becomes

$[\theta[\theta - 1) (\theta - 2) + 3\theta(\theta - 1) + \theta + 8] y = 65 \cos z$

A.E. is $\theta[\theta^2 - 3\theta + 2) + 3\theta^2 - 3\theta + \theta + 8 = 0$

i.e. $\theta^3 - 3\theta^2 + 2\theta + 3\theta^2 - 3\theta + \theta + 8 = 0$ i.e$\theta^3 + 8 = 0$

$\Rightarrow$ $(\theta + 2) (\theta^2 - 2\theta + 4) = 0$

$\Rightarrow$ $\theta = -2, \frac{2 \pm \sqrt{4-16}}{2} = -2, 1 \underline{+} \sqrt{3}\, i$

$\therefore$ C.F. is $c_1e^{-2z} + e^z[c_2 \cos \sqrt{3}\, z]$

P.I. = $\frac{1}{\theta^3+8}$ 65 cos z = $65.\frac{1}{\theta^2.\theta+8}$ cos z = $65 \frac{1}{-\theta+8}$ cos z

= $65 (8 + \theta) \frac{1}{64-\theta^2}$ cos z = $65 (8 + \theta) \frac{1}{64+1}$ cos z = $(8 + \theta)$ cos z

= 8 cos z - sin z

$\therefore$ C.s. is $y = c_1e^{-2z} + e^z (c_2 \cos \sqrt{3}\, z + c_3 \sin \sqrt{3}\, z) + 8 \cos z - \sin z$

i.e. $y = c_1x^{-2} x [c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) - \sin (\log x)$, is the complete solution.

**Example 8:** Solve: $\{x^4D^4 + 2x^3D^3 + x2D^2 - xD + 1\} y = x \log x$

**Sol:** The given equation is $(x^4D^4 + 2x^3D^3 + x^2D^2 - xD + 1\} y = x \log x$ ......(1)

Putting x = ez$\Rightarrow$ z = log x, the given equation becomes

$[\theta(\theta - 1) (\theta - 2) (\theta - 3) + 2\theta(\theta - 1) (\theta - 2) + \theta(\theta - 1) - \theta + 1] y = e^z, z$

$\Rightarrow$ $[\theta(\theta^3 - 6\theta2 + 11\theta - 6\_ + 2\theta(\theta^2 - 3\theta + 2) + \theta2 - \theta - \theta + 1] y = ze^z$

$\Rightarrow$ $9\theta^4 - 4\theta^3 + 6\theta^2 - 4\theta + 1] y = ze^z$

$\Rightarrow$ $(1 - \theta)^4 y = ze^z$ .....(2)

A.E. is $(1 - \theta)^4 = 0 \Rightarrow \theta = 1, 1, 1, 1$ C.F. is $(c_1 + c_2 z + c_3z^2 + c_4z^3)e^z$

P.I. is $\frac{1}{(1+\theta)^4} z e^z$

$= e^z \frac{1}{[1-(\theta+1)]^4} z$ $\left[\because \frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V\right]$

$= e^z \frac{1}{\theta^4} z = e^z \frac{z^5}{5!}$

$\therefore$ the general solution of (2) is

$y$ = C.F. + P.I. = $(c_1 + c_2 z + c_3 z^2 + c_4 z^3)\, e^z + \frac{z^5 e^z}{5!}$

Hence $y = [c_1 + c_2 \log x + c_3 (\log x)^2 + c_4 (\log x)^3]\, x + \frac{x}{5!} (\log x)^5$, is the general solution of the given equation.

**Example 9:** Solve: $x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + xy = 1 + x$

**Sol:** Given differential equation is

$$x^4 \frac{d^3y}{dx^3} + 2x^3 \frac{d^2y}{dx^2} - x^2 \frac{dy}{dx} + x\,y = 1 + x$$

or $(x^3 D^3 + 2x^2D^2 - x D + 1)\, y = 1 + x$ .....(1)

Put $x = e^z$ i.e. $z = \log x$

and $x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$, $x^3D^3 = \theta (\theta - 1) (\theta - 2)$ in (1), we get

$[\theta(\theta - 1) (\theta - 2) + 2\theta (\theta - 1) - \theta + 1]\, y = 1 + e^{-z}$

or $[\theta^3 - \theta^2 - \theta + 1)\, y = 1 + e^{-z}$

A.E. is $\theta^3 - \theta^2 - \theta + 1 = 0$

or $(\theta - 1)(\theta^2 - 1) = 0$

$\Rightarrow$ $(\theta - 1)^2 (\theta + 1) = 0$

$\therefore$ $\theta = 1, 1, -1$

$\therefore$ C.F. is $= c_1e^{-z} + (c_2 + c_3z)e^z$

$= c_1x^{-1} + (c_2 + c_3 \log x)\, x$

P.I. = $\frac{1}{\theta^3 - \theta^2 - \theta + 1} (1 + e^{-z})$

$= \frac{1}{\theta^3 - \theta^2 - \theta + 1} .1 + \frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{-z}$

Now $\frac{1}{\theta^3 - \theta^2 - \theta + 1} .1 = \frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{oz} = \frac{1}{0 - 0 - 0 + 1} e^{oz}$

Also $\frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{-z} = \frac{1}{-1 - 1 + 1 + 1} e^{-z} = \frac{1}{0} e^{-z}$ [case of failure]

$$= z.\frac{1}{3\theta^2 - 2\theta - 1} e^{-z}$$

$$= z.\frac{1}{3 + 2 - 1} e^{-z}$$

$$= \frac{1}{4} z\, e^{-z} = \frac{1}{4}\, x^{-1} \log x$$

$\therefore$ C.S. is =

$$y = c_1 x^{-1} + (c_2 + c_3 \log x) + \frac{1}{4} x^{-1} \log x$$

**Example 10:** Solve the following differential equation

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$$

**Sol:** The given differential equation is

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = x^2 + \frac{1}{x^2}$$

or in, S.F., $(x^2 D^2 + 4 x D + 2) y = x^2 + \frac{1}{x^2}$ ......(1)

Put $x = e^z$, or $z = \log x, x > 0$

and $xD = \theta, x^2D^2 = \theta(\theta - 1)$, where $\frac{d}{dz} = \theta$

$\therefore$ (1) becomes $[\theta(\theta - 1) + 4\theta + 2] y = e^{2z} + e^{-2z}$

or $(\theta^2 + 3\theta + 2) y = e^{2z} + e^{-2z}$

$\therefore$ A.E. is $\theta^2 + 3\theta + 2 = 0$ or $(\theta + 1)(\theta + 2) = 0$

$\therefore$ $\theta = -1, -2$

$\therefore$ C.F. $= c_1e^{-z} + c_2e^{-2z}$

$$\text{P.I.} = \frac{1}{\theta^2 + 3\theta + 2} (e^{2z} + e^{-2z})$$

$\therefore$ $\text{P.I.} = \frac{1}{\theta^2 + 3\theta + 2} e^{2z} + \frac{1}{\theta^3 + 3\theta + 2} e^{-2z}$ .....(2)

Now $\frac{1}{\theta^2 + 3\theta + 2} e^{2z} = \frac{1}{4 + 6 + 2} e^{2z} = \frac{1}{12} e^{2z}$

and $\frac{1}{\theta^3+3\theta+2}$ $e^{-2z}$ = $\frac{1}{4+6+2}$ $e^{-2z}$ = $\frac{1}{0}$ $e^{-2z}$

∴ $\frac{1}{\theta^2+3\theta+2}$ $e^{-2z}$ = $\frac{1}{2\theta+3}$ $e^{-2z}$ = z. $\frac{1}{-4+3}$ $e^{-2z}$ = $-ze^{-2z}$

∴ from (2), P.I. = $\frac{1}{12}$ $e^{2z}$ - $ze^{-2z}$

∴ C.S. is

$$y = c_1e^{-z} + c_2\, e^{-2z} + \frac{1}{12}e^{2z} - z\, e^{-2z}$$

or $y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{12}x^2 - \frac{1}{x^2}\log x.$

**Example 11:** Solve the differential equation

$$x^3\frac{d^3y}{dx^3} + 6x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = (\log x)^2$$

**Sol:** The given differential equation is

$$x^3\frac{d^3y}{dx^3} + 6x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 4y = (\log x)^2$$

In S.F., $(x^3 D^3 + 6x^2D^2 + 4x D - 4)\, y = (\log x)^2$ .....(1)

Put $x = e^z$ or $z = \log x$

$x D = \theta$, $x^2 D^2 = \theta(\theta - 1)$, $x^3 D^3 = \theta(\theta - 1)(\theta - 2)$ in (1)

∴ $[\theta(\theta - 1)(\theta - 2) + 6\theta(\theta - 1) + 4\theta - 4]\, y = z^2$

∴ $(\theta^3 - 3\theta^2 + 2\theta + 6\theta^2 - 6\theta + 4)\, y = z^2$

∴ $(\theta^2 + 3\theta^2 - 4)\, y = z^2$

The A.E. is $\theta^3 + 3\theta^2 - 4 = 0$

or $(\theta - 1)(\theta^2 + 4\theta + 4) = 0 \Rightarrow (\theta - 1)(\theta + 2)^2 = 0$

∴ $\theta = 1, -2, -2$

C.f. = $c_1e^z + (c_2 + c_3z)\, e^{-2z} = c_1x + (c_2 + c_3 \log x)\, x^{-2}$

$$\text{P.I.} = \frac{1}{\theta^3+3\theta^2-4}\, z^2$$

$$= \frac{1}{-4\left(1-\frac{3.\theta^2+\theta^3}{4}\right)}\, z^2 = -\frac{1}{4}\left(1-\frac{3\theta^2+\theta^3}{4}\right)^{-1} z^2$$

$= -\frac{1}{4}\left(1+\frac{3\theta^2+\theta^3}{4}+....\right)z^2 = \frac{1}{4}\left(1+\frac{3}{4}\theta^2\right)z^2 = -\frac{1}{4}\left(z^2+\frac{3}{4}\theta^2 z^2\right)$

$= -\frac{1}{4}\left(z^2+\frac{3}{2}\right) = -\frac{1}{4}\left[(\log x)^2+\frac{3}{2}\right]$

$\therefore$ C.S. is $y = c_1 x + (c_2 + c_1 \log x)\, x^{-2} - \frac{1}{4}\left[(\log x)^2+\frac{3}{2}\right]$

**Example 12:** Solve: (x2D2 - 3x D + 5) y = x2 (log x), x > 0.

**Sol:** The given equation is

$(x^2D^2 - 3x D + 5)\, y = x^2 (\log x)$ ......(1)

Put $x = e^z$ i.e. $z = \log x$, $x > 0$

and $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ in (1), we get

$[\theta(\theta - 1) - 3\theta + 5]\, y = ze^{2z}$

or $(\theta^2 - 4\theta + 5)y = ze^{2z}$

A.E. is $\theta^2 - 4\theta + 5 = 0$

$\Rightarrow \theta = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4\pm 2i}{2}$ $2 \pm i$

C.F. is $y = e^{2z}(c_1 \cos z + c_2 \sin z)$

P.I. = $\frac{1}{\theta^2 - 4\theta + 5} ze^{2z} = e^{2z}\frac{1}{(\theta+2)^2 - 4(\theta+2)+5} z$

$= e^{2z}\frac{1}{\theta^2+4\theta+4-4\theta-8+5} z = e^{2z}\frac{1}{\theta^2+1} z = e^{2z}(1+\theta^2)^{-1}z$

$= e^{2z}(1 - \theta^2)z = z\, e^{2z}$ [Expanding by Binomial Theorem]

C.S. is $y = e^{2z}(c_1 \cos z + c_2 \sin z) + e^{2z}$

or $y = x^2(c_1 \cos(\log x) + c_2 \sin(\log x)) + x^2 \log x$

**Example 13:** Solve the following differential equation

$(x^2D^2 - xD + 4)\, y = \cos(\log x) + x \sin(\log x)$

**Sol:** Given differential equation is

$(x^2D^2 - xD + 4)\, y = \cos(\log x) + x \sin(\log x)$ ....(1)

Put $x = e^z$ i.e. $z = \log x$, $x > 0$

and $xD = \theta$, $x^2D^2 = \theta(\theta - 1)$ in (1), we have

$[\theta(\theta - 1) - \theta + 4]\, y = \cos z + e^z \sin x$

or $[\theta^2 - 2\theta + 4]\, y = \cos z + e^z \sin z$

A.E. is $\theta^2 - 2\theta + 4 = 0$

$\therefore \quad \theta = \dfrac{2 \pm \sqrt{4-16}}{2} = \dfrac{2 \pm \sqrt{-12}}{2} = \dfrac{2 \pm 2i\sqrt{3}}{2}$

$= 1 \pm i\sqrt{3}$

$\therefore$ C.S. is $= e^z [c_1 \cos(\sqrt{3}\, z) + c_2 \sin(\sqrt{3}\, z)]$

$$\text{P.I.} = \frac{1}{\theta^2 - 2\theta + 4} (\cos z + e^z \sin z)$$

$$= \frac{1}{\theta^2 - 2\theta + 4} \cos z + \frac{1}{\theta^2 - 2\theta + 4} e^z \sin z$$

$$= \frac{1}{-1 - 2\theta + 4} \cos z + e^z \frac{1}{(\theta+1)^2 - 2(\theta+1) + 4} \sin z$$

$$= \frac{1}{3 - 2\theta} \cos z + e^z \frac{1}{\theta^2 + 3} \sin z$$

$$= \frac{3 + 2\theta}{(3 - 2\theta)(3 + 2\theta)} \cos z + e^z \frac{1}{-1 + 3} \sin z$$

$$= \frac{3 + 2\theta}{9 - 4\theta^2} \cos 2 + \frac{1}{2} e^z \sin z$$

$$= \frac{3 + 2\theta}{9 - 4(-1)} \cos z + \frac{1}{2} e^z \sin z$$

$$= \frac{1}{13} [3 \cos z + 2\theta \cos z] + \frac{1}{2} e^z \sin z$$

$$= \frac{1}{13} [3 \cos z - 2 \sin z] + \frac{1}{2} e^z \sin z$$

$\therefore$ General solution of (1) is

$$y = e^z [c_1 \cos(\sqrt{3}\, z) + c_2 \sin(\sqrt{3}\, z)] + \frac{1}{13} (3 \cos z - 2 \sin z) + \frac{1}{2} e^z \sin z$$

or
$$y = e^{\log x} [c_1 \cos(\sqrt{3}\, x) + c_2 \sin(\sqrt{3}\, x)] + \frac{1}{13} (3 \cos(\log x) - 2 \sin(\log x))$$

$$+ \frac{1}{2} e^{\log x} \sin(\log x)$$

or $y = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)] + \frac{1}{13}(3\cos(\log x) - 2\sin(\log x)]$

$+ \frac{1}{2} x \sin(\log x)$

**Self-Check Exercise**

Q.1 Solve the following differential equation

$(x^2 D^2 + 2x D - 2) y = 0$

Q.2 Solve the differential equation

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x$$

Q.3 Solve the following differential equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x+1)^2$$

Q.4 Solve $(x^3 D^3 - x^2 D^2 + 2x D - 2) y = x^3 + 3x$

Q.5 Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

## 12.4 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined Cauchy-Euler's homogeneous linear equation.
2. Discussed the method to solve Cauchy-Euler's equation.
3. Find solutions of differential equations by this method.

## 12.5 Glossary:

1. A differential equation of the form

   $P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots\dots + P_n y = Q$, where $P_0, P_1, \dots\dots, P_n$ and Q are functions of x is called linear differential equation with variable coefficients.

2. A homogeneous linear equation of the form $P_0x^n\frac{d^ny}{dx^n} + P_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + P_n y = Q(x)$, is called Cauchy-Euler's linear equation.

## 12.6 Answer to Self Check Exercise

Ans.1 $y = c_1 x + c_2 x^{-2}$

Ans. 2 $y = c_1 + c_2 \log x + \frac{1}{4} x^2$ , $x > 0$

Ans. 3 $y = c_1 x^4 + c_2 x^5 - \left(\frac{x^2}{14} + \frac{x}{9} + \frac{1}{20}\right)$

Ans. 4 $y = (c_1 + c_2 \log x) x + (3x^2 + \frac{1}{4} x^3 - \frac{3}{2} x (\log x)^2$

Ans. 5 $y = c_1 x^{-1} + c_2 x^2 + \frac{1}{3}(x^2 - x^{-1}) \log x$

## 12.7 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Boyce, w. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.


## 12.8 Terminal Questions

1. Solve the following differential equation

$$x^3 \frac{d^3y}{dx^3} + 6x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} - 4y = 0$$

2. Solve the following differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 3y = x^2$$

3. Solve $\frac{d^3y}{dx^3} - \frac{4}{x} \frac{d^2y}{dx^2} + \frac{5}{x^2} \frac{dy}{dx} - \frac{2y}{x^3} = 1$

4. Solve the differential equation

$$(x^3D^3 + 2x^2D^2 + 2) y = 10\left(x + \frac{1}{x}\right)$$

5. Solve the differential equation

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = x \log x$$

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**Unit - 13**

# Legender's Linear Equation

**Structure**



## 13.1 Introduction

There are some equations which can be easily reducible to the homogeneous linear form and hence also to the linear equation with constant coefficients.

Any equation of the form

$$P_0\ (0 + bx)^n \frac{d^n y}{dx^n} + P_1\ (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + .......+ P_n\ y = Q\ (x),$$

Where $P_0$, $P_1$, $P_2$,....., $P_n$ are real constants and Q(x) is a function of x, is transformed into the homogeneous linear equation, is called Legendre's Homogeneous linear Differential Equation.

## 13.2 Learning Objectives

After studying this unit, you should be able to:-

- Define Legendre's homogeneous linear differential equation.
- Discuss method to solve Legendre's homogeneous linear differential equation
- Solve homogeneous linear differential equations of Legendre's type by this method.

## 13.3 Method to Solve Legendre's Linear Equation

A linear equation of the form

$$P_0 (a + bx)^n \frac{d^n y}{dx^n} + P_1 (a + bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + ......+ P_n y = Q (x), \quad ......(1)$$

where $P_0$, $P_1$, $P_2$,......, $P_n$ are real constants and Q(x) is a function of x, is called Legendre's linear equation.

The equation (1) can be written in the symbolic form as

$[P_0 (a + b x)^n D^n + P_1 (a + b x)^{n-1} D^{n-1} + .... P_n]y\ Q(x)$

where $D = \frac{d}{dx}$

Put a + b x = ez, i.e., z = log (a + b x), a + b x > 0

$$\therefore \quad \frac{dz}{dx} = \frac{b}{a+b\ x}$$

$$\text{Now} \quad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}.\frac{b}{a+b\ x} = \frac{b}{a+b\ x}\frac{dy}{dz}$$

$$\text{Let} \quad \frac{d}{dz} = \theta$$

$$\therefore \quad D\,y = \frac{b}{a+b\ x}\theta y, \qquad \text{i.e., } (a + b\ x)\ d\ y = b\ \theta\ y$$

$$\Rightarrow \quad (a + b\ x)\ D = b\ \theta$$

$$\text{Also} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+b\ x}\frac{dy}{dz}\right)\frac{b}{a+b\ x}\frac{d}{dx}\left(\frac{dy}{dz}\right) + \frac{dy}{dz}.\frac{d}{dx}\left(\frac{b}{a+b\ x}\right)$$

$$= \frac{b}{a+b\ x}\frac{d^2y}{dz^2}\frac{dz}{dx} + \frac{dy}{dz}.\left[\frac{-b^2}{(a+b\ x)^2}\right]$$

$$= \frac{b}{a+b\ x}\frac{d^2y}{dz^2}.\frac{b}{a+b\ x} - \frac{b^2}{(a+b\ x)^2}\frac{dz}{dx} \qquad \left[\because \frac{dz}{dx} = \frac{b^2}{a+b\ x}\right]$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{b^2}{(a+b\ x)^2}\frac{d^2y}{dz^2} - \frac{b^2}{(a+b\ x)^2}\frac{dy}{dz}$$

$$\Rightarrow \quad (a + b\ x)^2\frac{d^2y}{dx^2} = b^2\frac{d^2y}{dz^2} - b^2\frac{dy}{dz}$$

$$\Rightarrow \quad (a + b\ x)^2 D^2\ y = b^2\theta^2\ y - b^2\theta\ y$$

$$\Rightarrow \quad (a + b\ x)^2 D^2\ y = b^2\theta\ (\theta - 1)y$$

$\Rightarrow$ $(a + b x)^2 D^2 = b^2 \theta (\theta - 1)$

Similarly $(a + b x)^3 D^3 = b^3 \theta (\theta - 2)$

and so on.

In general, $(a + b x)^n D^n = b^n \theta (\theta - 1 (\theta - 2) \ldots (\theta - \overline{n-1})$

Substituting the values of $(a + b x)^n D^n$, $(a + b x)^{n-1} D^{n-1}$, ...., $(a + b x) D$ in (2),

are get,

$$[P_0 b^n \theta(\theta - 1) \ldots (\theta - \overline{n-1}) + P_1 b^{n-1} \theta(\theta - 1) \ldots (\theta - \overline{n-2}) + \ldots$$

$$+ P_{n-1} b \theta + P_n] y = Q \left( \frac{e^z - a}{b} \right)$$

or $f(\theta) y = \phi(z)$ .....(3)

where $f(\theta)$ is a polynomial in $\theta$ with real coefficients and $\phi(z)$ is a function of z.

Now (3) can be solved for y in terms of z by the methods already known to us. Let its general solution be

$y = \psi(z)$

$\therefore$ general solution of (1) is $y = \psi[\log (a + b x)]$, $a + b x > 0$.

Note. Working rule to solve Legendre's linear equation

Step 1. Put $a + b x = e^z$, i.e., $z = \log (a + b x)$, $a + b x > 0$

Step 2. Put $\frac{d}{dx} = \theta$, so that

$(a + b x) D = b \theta$, $(a + b x)^2 D^2 = b^2 \theta(\theta - 1)$, ...... $(a + b x)^n D^n$

$$= b^n \theta(\theta - 1)(\theta - \overline{n-1})$$

Step 3. Putting in the given equation, we get,

$$[P_0 b^n \theta(\theta - 1) \ldots (\theta - \overline{n-1}) + P_1 b^{n-1} \theta(\theta - 1) \ldots (\theta - \overline{n-2}) + \ldots$$

$$+ P_n] y = Q \left( \frac{e^z - a}{b} \right)$$

which is linear equation with constant coefficients and solve for y in terms of z.

Step 4 Put $z = \log (a + b x)$ to get the required solution.

To clarify what we have just said consider the following examples :-

By putting $5 + 2 x = e^z$, i.e., $z = \log (5 + 2 x)$, $5 + 2 x > -$

and $(5 + 2 x) D = 2 \theta$, $(5 + 2 x)^2 D^2 = 2^2 \theta (\theta - 1) = 4 \theta (\theta - 1)$, The given equation (1) becomes,

$$[4\,\theta\,(\theta - 1) - 6\,.\,2\,\theta + 8]\,y = 0$$

or $\quad [4\,\theta^2 - 4\theta - 12\,\theta + 8]\,y = 0$

or $\quad (4\,\theta^2 - 16\,\theta + 8)\,y = 0$

or $\quad (\theta^2 - 4\,\theta + 2)\,y = 0$

A.E. is $\theta^2 - 4\,\theta + 2 = 0$

$$\therefore \quad \theta = \frac{4 \pm \sqrt{1-8}}{2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$\therefore$ general solution of (1) is

$$y = c_1 e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z} = e^{2z}\left[c_1 e^{\sqrt{2}z} + c_2 e^{-\sqrt{2}z}\right]$$

or $\quad y = (5 + 2\,x)2\,[c_1\,(5 + 2\,x)^{\sqrt{2}} + c_2\,(5 + 2\,x)^{-\sqrt{2}}]$.

**Example 2.** Solve the following differential equations :

(i) $\quad (x + 1)^2 \dfrac{d^2y}{dx^2} - (x + 1)\dfrac{dy}{dx} + y = 0.$

(ii) $\quad (2\,x - 1)^3 \dfrac{d^3y}{dx^3} + (2x - 1)\dfrac{dy}{dx} - 2y = 0.$

**Sol.** (i) The given differential equation is

$$(x + 1)^2 \frac{d^2y}{dx^2} - (x + 1)\frac{dy}{dx} + y = 0$$

or, in S.F., $[x + 1)^2 D^2 - (x + 1)\,D + 1]\,y = 0$ ...(1)

Put $x + 1 = e^z$ i.e. $z = \log(x + 1)$

and $(x + 1)\,d = \theta$, $(x + 1)^2 D^2 = \theta\,(\theta - 1)$

$\therefore$ the given equation (1) becomes

$$[\theta\,(\theta - 1) - \theta + 1]\,y = 0$$

or $\quad (\theta^2 - \theta - \theta + 1]\,y = 0$

$\therefore \quad (\theta^2 - 2\,\theta + 1)\,y = 0$

$\therefore \quad$ A.E. is $\theta^2 - 2\,\theta + 1 = 0$

$\therefore \quad (\theta - 1)^2 = 0 \Rightarrow \theta = 1, 1$

$\therefore \quad$ C.S. is $y = (c_1 + c_2\,z)$

or $\quad y = [c_1 + c_2 \log(x + 1)]\,e^{\log(x+1)}$

or $\quad y = [c_1 + c_2 \log (x + 1)] (x+1)$

(ii) The given differential equation is

(ii) $\quad (2x - 1)^3 \frac{d^3y}{dx^3} + (2x - 1) \frac{dy}{dx} - 2y = 0.$

or, in S.F., $[(2x - 1)^3 D^3 - (2x - 1) D - 2] y = 0$ ...(1)

Put $2x - 1 = e^z$ or $z = \log (2x - 1)$

and $(2x - 1) D = 2\theta$, $(2x - 1)^3 D^3 = 2^3\theta (\theta - 1)(\theta - 2)$ in (1).

$\therefore \quad [2^3\theta(\theta - 1)(\theta - 2) + 2\theta - 2] y = 0$

$\therefore \quad [4\theta(\theta - 1)(\theta - 2) + \theta - 1] y = 0$

$\therefore \quad (4\theta^3 - 12\theta^2 + 8\theta + \theta - 1) y = 0$

$\therefore$ A.E. is $4\theta^3 - 12\theta^2 + 9\theta - 1 = 0$

$\therefore \quad (\theta - 1)(4\theta^2 - 8\theta + 1) = 0$

$\therefore \quad \theta = 1, \frac{8 \pm \sqrt{64-16}}{8} = 1, \frac{8 \pm 4\sqrt{3}}{8}$

$\therefore \quad \theta = 1, 1 \pm \frac{\sqrt{3}}{2}$

$\therefore$ C.S. is $y = c_1 e^z + c_2 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$

or $\quad y = e^z \left[ c_1 + c_2 e^{\frac{\sqrt{3}}{2}z} + c_3 e^{-\frac{\sqrt{3}}{2}z} \right]$

or $\quad y = (2x - 1) \left[ c_1 + c_2 e^{\frac{\sqrt{3}}{2}\log(2x-1)} + c_3 e^{-\frac{\sqrt{3}}{2}\log(2x-1)} \right]$

or $\quad y = (2x - 1) \left[ c_1 + c_2 (2x-1)^{\frac{\sqrt{3}}{2}} + \frac{c_3}{(2x-1)^{\frac{3}{2}}} \right]$

**Example 3 :** Solve

$[(3x + 2)^2 D^2 + 3(3x + 2) D - 36] y \; 3x^2 + 4x + 1$

**Sol.** The given differential equation in S.F. is

$[(3x + 2)^2 D^2 + 3(3x + 2) D - 36] y \; 3x^2 + 4x + 1$ ...(1)

Put $3x + 2 = e^z$, or $z = \log(3x + 2)$, $x > -\frac{2}{3}$

and $(3x + 2) = 3^1\theta$ and $(3x + 2)^2 D^2 = 3^2\theta(\theta - 1)$ where $\theta = \frac{d}{dz}$

From (1), we get, $[9\theta(\theta - 1) + 3.\ 3\theta - 36]\, y = 3\left(\frac{e^z - 2}{3}\right)^2 + 4\left(\frac{e^z - 2}{3}\right)^2 + 1$

or $[9\theta^2 - 36]\, y = \frac{1}{3}[e^{2z} - 1]$ ......(2)

$\therefore$ the A.E. is $9\theta^2 - 36 = 0$

or $9(\theta - 2)(\theta + 2) = 0$

$\therefore$ $\theta = 2, -2$

$\therefore$ C.F. is $y_c = c_1e^{2z} + c_2 e^{-2z}$

and P.I. is $y_p = \frac{1}{9(\theta^2 - 4)}\left[\frac{1}{3}(e^{2z} - 1)\right] = \frac{1}{27}\, \frac{1}{\theta^2 - 4}.e^{2z} - \frac{1}{27}\, \frac{1}{\theta^2 - 4}.1$

Now $\frac{1}{\theta^2 - 4}e^{2z} = \frac{1}{(2)^2 - 4}e^{2z} = \frac{1}{0}e^{2z}$ which is a case of failure.

$\therefore$ $\frac{1}{\theta^2 - 4}e^{2z} = z.\frac{1}{2\theta}e^{2z} = z.\ \frac{1}{2(2)}e^{2z} = \frac{1}{4}ze^{2z}$

and $\frac{1}{\theta^2 - 4}.1 = \frac{1}{(0)^2 - 4}.1 = -\frac{1}{4}$

$\therefore$ $y_p = \frac{1}{27}.\frac{1}{4}\ z^{e2z} + \frac{1}{27}\ \frac{1}{4} = \frac{1}{108}(z\, e^{2z} + 1)$

$\therefore$ the general solution of (1) is

$y = y_c + y_p$

or $y = c_1e^{2z} + c_2e^{-2z} + \frac{1}{108}(z\, e^{2z} + 1)$

or $y = c_1(3x + 2)^2 + c_2(3x + 2)^{-2} + \frac{1}{108}[(3x + 2)^2 \log(3x + 2) + 1]$

**Example 4:** Solve the following differential equation

$$(x + 3)^2\frac{d^2y}{dx^2} - 4(x + 3)\frac{dy}{dx} + 6y = x$$

Sol. The given differential equation is

$$(x + 3)^2 \frac{d^2y}{dx^2} - 4 (x + 3) \frac{dy}{dx} + 6y = x$$

or, in S.F. $[(x + 3)^2 D^2 - 4 (x + 3) D + 6] y = x$ ......(1)

By putting $x + 3 = e^z$, or $z = \log (x + 3), x > - 3$

and $(x + 3) D = \theta$, $(x + 3)^2 D^2 = \theta (\theta - 1)$ where $\frac{d}{dz} = \theta$ in (1), we get,

$$[\theta(\theta - 1) - 4 \theta + 6] y = e^z - 3$$

or $[\theta^2 - 5\theta + 6] y = e^z - 3$ ......(2)

The A.E. is $\theta^2 - 5\theta + 6 = 0$

or $(\theta - 2) (\theta - 3) = 0$

or $\theta = 2, 3$

$\therefore$ C.F. is $y_c = c_1 e^{2z} + c_2 e^{3z}$

and P.I. is $y_p = \frac{1}{\theta^2 - 5\theta + 6} (e^z - 3) = \frac{1}{\theta^2 - 5\theta + 6} e^z. - 3 \frac{1}{\theta^2 - 5\theta + 6} .1$

$$= \frac{1}{1^2 - 5(1) + 6} e^z - 3 \frac{1}{(0)^2 - 5(0) + 6} = \frac{1}{2} e^z = \frac{3}{6}$$

or $y_p = \frac{1}{2} e^z - \frac{1}{2}$

$\therefore$ General solution of (2) is

$$y = y_c + y_p$$

or $y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2} (e^z - 1)$

or $y = c_1 (x + 3)^2 + c_2 (x + 3)^3 + \frac{1}{2} (x + 3 - 1)$

or $y = c_1 (x + 3)^2 + c_2(x + 3)^3 + \frac{1}{2} x + 1$

**Example 5:** Solve $(1 + x)^2 y_2 + (1 + x) y_1 = 2 \cos [\log (1 + x)]$

**Sol.** The given differential equation is

$$((1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} = 2 \cos [\log (1 + x)]$$

or, in S.F., $[(1 + x)^2 D^2 + (1 + x) D] = 2 \cos [\log (1 + x)]$ .....(1)

Put $1 + x = e^z$, or $z = \log (1 + x)$, $x > -1$

and $(1 + x) D = \theta$, $(1 + x)^2 D^2 = \theta (\theta - 1)$ where $\frac{d}{dz} = \theta$

From (1), we get,

$[\theta(\theta - 1) + \theta]y = 2 \cos z$

or $\theta^2 y = 2 \cos z$

The A.E. is $\theta^2 = 0$ .....(2)

$\therefore$ $\theta = 0, 0$

$\therefore$ C.F. $= (c_1 + c_2 z) e^{0z} = c_1 + c_2 z$

P.I. $= \frac{1}{\theta^2}$ $(2 \cos z) = 2 \frac{1}{\theta^2}$ $\cos z = 2 \frac{1}{-1} \cos z = -2 \cos z$

$\therefore$ C.S. is

$y = c_1 + c_2 z - 2 \cos z = c_1 + c_2 \log (1 + x) -2 \cos [\log (1 + x)]$

**Example 6:** Solve the following differential equations:

$$(1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 2 \sin [\log (1 + x)]$$

**Sol:** The given differential equation is

$$(1 + x)^2 \frac{d^2 y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 2 \sin [\log (1 + x)]$$

or, in S.F. $[(1 + x)^2 D^2 + (1 + x) D + 1] y = 2 \sin [\log (1 + x)]$ .....(1)

Put $1 + x = e^z$, or $z = \log (1 + x)$, $x > - 1$

and $(1 + x) D = \theta$ and $(1 + x)^2 D^2 = \theta (\theta - 1)$ where $\frac{d}{dz} = \theta$

From (1), we get, $[\theta (\theta - 1) + \theta + 1] y = 2 \sin z$

or $[\theta^2 + 1] y = 2 \sin z$ ......(2)

The A.E. is $\theta^2 + 1 = 0$

or $\theta = \pm i$

$\therefore$ C.F. is $y_c = c_1 \cos z + c_2 \sin z$

and P.I. is $y_p = \frac{1}{\theta^2 + 1}$ $(2 \sin z) = 2. \frac{1}{-1+1}$ $\sin z$ $(\because \theta^2 = -1)$

$= 2. \frac{1}{0} \sin z$ which is a case of failure

$\therefore$ $y_p = 2z. \frac{1}{2\theta} \sin z = - z \cos z$

$\therefore$ the general solution of (2) is

$y = y_c + y_p$

or $y = c_1 \cos z + c_2 \sin z - z \cos z$

or $y = c_1 \cos [\log (1 + x)] + c_2 \sin [\log (1 + x)] - \log (1 + x) \cos [\log(1 + x)]$

or $y = [c_1 - \log (1 + x)] \cos [\log(1 + x)] + c_2 \sin [\log (1 + x)]$.

**Example 7:** Solve

$$(x + 1)^2 \frac{d^2y}{dx^2} + (x + 1)\frac{dy}{dx} = (2x + 3)(2x + 4)$$

**Sol:** Given equation is S.F. is

$$[(x + 1)^2D^2 + (x + 1) D] y = [2(x + 1) + 1] [2(x + 1) + 2] \quad .....(1)$$

Put $x + 1 = e^z \Rightarrow z = \log (x + 1),\ x + 1 > 0$

$\therefore$ (1) $\Rightarrow [\theta(\theta - 1) + \theta] y = [2e^z + 1] [2e^z + 2]$

$\Rightarrow \theta^2 y = 2 (2e^{2z} + 3e^z + 1) \quad .....(2)$

A.E. is $\theta^2 = 0 \ \Rightarrow \theta = 0, 0$

$\therefore$ C.F. is $= (c_1 + c_2 z) e^{0z} = c_1 + c_2 z$

and P.I. $= \frac{1}{\theta^2} [2(2e^{2z} + 3e^z + 1)]$

$$= 2\left[2\frac{1}{\theta^2}e^{2z} + 3\frac{1}{\theta^2}e^z + \frac{1}{\theta^2} - 1\right]$$

$$= \frac{4}{4}e^{2z} + \frac{6}{1}e^z + 2\frac{z^2}{2}$$

$$= e^{2z} + 6e^z + z^2$$

$\therefore$ complete solution of (1) is

$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 z + e^{2z} + 6e^z + z^2$

$= c_1 + c_2 \log (x + 1) + (x + 1)^2 + 6 (x + 1)) + [\log (x + 1)]^2$

$\Rightarrow$ $y = c_1 + c_2 \log (x + 1) + x^2 + 8x + 7 + [\log (x + 1)]^2$

$\Rightarrow$ $y = c_1 + c_2 \log (x + 1) + x^2 + 8x + [\log (x + 1)]^2$, where $c_1 = c_1 + 7$

**Self-Check Exercise**

Q.1 Solve the following differential equation

$$(2x-1)^3 \frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = x$$

Q.2 Solve

$$(3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2 + x + 1$$

Q.3 Solve the following differential equation

$$(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = 4x^2 + 14x + 12$$

## 13.4 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined Legendre's homogeneous linear differential equation.
2. Discussed method to solve Legendre's homogeneous linear differential equation.
3. Find solutions of homogeneous linear differential equations of Legendre's type by this method.

## 13.5 Glossary:

1. An equation of the form

$P_0(a+bx)^n \frac{d^ny}{dx^n} + P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots.. + P_n y = Q(x)$, is called Legendre's homogeneous linear equation, where $P_0, P_1, \ldots\ldots., P_n$ are real constants and $Q(x)$ is a function of x.

## 13.6 Answer to Self Check Exercise

Ans.1 $y = (2x-1)\left[c_1 + c_2(2x-1)^{\frac{\sqrt{3}}{2}} + c_3(2x-1)^{\frac{\sqrt{3}}{2}} - \frac{1}{12}\log(2x-1)\right] - \frac{1}{4}$

Ans. 2 $y = c_1(3x+2)^{-1} + c_2(3x+2)^{\frac{1}{3}} + \frac{1}{27}\left[\frac{(3x+2)^2}{15} - \frac{3x+2}{4} - 7\right]$

Ans. 3 $y = c_1 + c_2 \log(x+1) + x^2 + 8x + [\log(x+1)]^2$, where $c_1 = c_1 = 7$

## 13.7 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.

2. Boyce, w. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.

## 13.8 Terminal Questions

1. Solve

$$(x + a)^2 \frac{d^2y}{dx^2} - 4(x + a) \frac{dy}{dx} + 6y = x$$

2. Solve the following differential equation

$$(1 + 2x)^2 \frac{d^2y}{dx^2} - 6(1 + 2x) \frac{dy}{dx} + 16y = 8 (1 + 2)^2$$

3. Solve the following differential equation

$$(x + 1)^2 \frac{d^2y}{dx^2} + (x + 1) \frac{dy}{dx} = (2x + 3) (2x + 4)$$

4. Solve the following differential equation

$$(3x + 2)^2 \frac{d^2y}{dx^2} + 5 (3x + 2) \frac{dy}{dx} - 3y = x^2 + x + 1$$

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## Unit - 14

# Simultaneous Differential Equations

**Structure**



## 14.1 Introduction

Simultaneous differential equations, also known as systems of differential equations, are a collection of equations that describe the relationships between several unknown functions and their derivatives with respect to one or more independent variables. Unlike single differential equations that involve only one unknown function, simultaneous differential equations involve multiple unknown functions and their interactions.

A general system of simultaneous differential equations can be written in the following form:

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \ldots\ldots, x_n, t)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \ldots\ldots, x_n, t)$$

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$\frac{dx_n}{dt} = f_n(x_1, x_2, \ldots\ldots.., x_n, t)$

Here $x_1, x_2, \ldots\ldots, x_n$ represent the unknown functions, t is the independent variable (often representing time), and $f_1, f_2, \ldots\ldots, f_n$ are functions that describe the relationships between the unknown functions and their derivatives. Each equation in the system represents the rate of change of a particular unknown function with respect to the independent variable. The functions $f_1, f_2, \ldots\ldots, f_n$describe how the unknown functions depend on each other and on the independent variable. There are different methods for solving simulkineous differential equations, depending on the nature of the system. In this unit, we shall discuss differential equation in which there is one independent variable and two or more than two dependent variables. In order to solve such equations completely, we need as many simultaneous equations as are the number of dependent variables.

## 14.2 Learning Objectives

After studying this unit, you should be able to:-

- Define simultaneous differential equations
- Discuss method of solving simultaneous differential equations with constant coefficients.
- Find solutions of simultaneous differential equations with constant coefficients
- Discuss method of solving simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

## 14.3 Simultaneous Linear Differential Equations with Constant Coefficients

There are two methods for solutions of these types of equations

First Method Symbolic Method (use of operator D)

Let $f_1(D)x + f_2(D)y = T_1$ .....(1)

and $g_1(D)x + g_2(D)y = T_2$ ......(2)

where $D = \frac{d}{dt}$, $T_1$, $T_2$ are functions of t (independent variable) and $f_1(D)$, $g_1(D)$, $g_2(D)$ are all rational integral functions of D with constant coefficients and x and y are dependent variables.

Operating on both sides of (:) by $g_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting, we get

$(f_1(D)\, g_2(D) - g_1(D)(f_2(D))x = g_2(D)T_1 - f_2(D)\, T_2$

which is a linear equation in x and can be solved to give the value of x.

Putting this value of x in (1) or (2), we shall get y.

Note. Since $f_2(D)$ and $g_2(D)$ are functions with constants co - efficients

$\therefore \quad f_2(D)\, g_2(D) = g_2(D)\, f_2(D)$

Let us see what the method is with the help of following examples:-

**Example 1:** Solve: $Dy - z = 0$ ......(1)

$(D - 1)y - (D + 1)z = 0$ ......(2)

**Sol:** First of all we shall eliminate z, where $D \equiv \frac{d}{dx}$

From (1), $z = Dy$ ......(3)

Putting in (2), we get $(D - 1)\, y - (D + 1)Dy = 0$

or $\quad (D - 1 - D^2 - D)y = 0$ or $-(D^2 + 1)y = 0$ or $(D^2 + 1)y = 0$

A.E. is $D^2 + 1 = 0$ or $D^2 = -1$ $\therefore$ $D = \underline{+}i$ or $D = 0 \underline{+} i$

$\therefore \quad$ C.S. is $y = e^{0x}(c_1 \cos x + c_2 \sin x)$ or $y = c_1 \cos x + c_2 \sin x$

$\therefore \quad$ From (3), $z = Dy = \frac{d}{dx}\; c_1 \sin x + c_2 \cos x$

Hence $y = c_1 \cos x + c_2 \sin x$, $z = - c_1 \sin x + c_2 \cos x$ is the required solution.

**Example 2:** Solve $(D + 1)y = z + e^x$ ....(1)

$(D + 1)z = y + e^x$ ......(2)

**Sol:** First of all we shall eliminate z where $D \equiv \frac{d}{dx}$

From (1), $z = (D + 1)y = e^x$ .......(3)

Putting in (2), we get $(D + 1)[(D + 1)y - e^x$

or $\quad (D + 1)^2\, y - (D + 1)e^x = y + e^x$

or $\quad [(D + 1)^2 - 1]y = (D + 2)e^x$ or $(D^2 + 2D)y = (D + 2)e^x$

or $\quad D(D + 2)y = De^x + 2e^x = e^x + 2e^x$ or $D(D + 2)y = 3e^x$

A.E. is $D(D + 2) = 0$ $\therefore$ $D = 0, -2$

$\therefore \quad$ C.F. $= c_1e^{ox} + c_2e^{-2x} = c_1 + c_2\, e^{-2x}$

Again P.I. $= \frac{1}{D^2 + 2D} 3e^x = 3\frac{1}{1+2} e^x = e^x$

$\therefore \quad$ C.S. is $y =$ C.F. + P.I. $= c_1 + c_2e^{-2x} + e^x$

Putting in (3), we get

$z = (D + 1)[c_1 + c_2e^{-2x} + e^x] - e^x$

$= D(c_1) + D(c_2e^{-2x}) + D(e^x) + c_1 + c_2e^{-2x} + e^x - e^x$

$= 0 + c_2(-2)\, e^{-2x} + e^x + c_1 + c_2e^{-2x} = e^x + c_1 - c_2e^{-2x}$

Hence $y = c_1 + c_2e^{-2x} + e^x$, $z = e^x + c_1 - c_2e^{-2x}$, is the reqd solution.

**Example 3:** Solve: $\frac{dx}{dt} - 7x + y = 0$

$\frac{dy}{dt} - 2x - 5y = 0$

**Sol:** Given equations in symbolic form are $(D - 7)x + y = 0$ .....(1)

$-2x + (D - 5)y = 0$ ......(2)

Multiply (1) by - 2 and operate (2) by D - 7, we get, $-2(D - 7)x - 2y = 0$ .....(3)

$-2(D - 7)x + (D - 7)(D - 5)y = 0$ .....(4)

(4)- (3) gives, $[(D - 7)(D - 5) + 2]y = 0$

$\Rightarrow \quad (D^2 - 12D + 37)y = 0$

A.E. is $D^2 - 12D + 37 = 0 \Rightarrow D = \frac{12 \pm \sqrt{144-148}}{2} = \frac{12 \pm 2i}{2} = 6 \pm i$

$\therefore \quad$ C.S. is $y = e^{6t}[c_1 \cos t + c_2 \sin t]$ ......(5)

$\Rightarrow \quad \frac{dy}{dt} = e^{6t}[-c_1 \sin t + c_2 \cos t] + [c_1 \cos t + c_2 \sin t](6e^{6t})$

$\Rightarrow \quad \frac{dy}{dt} = e^{6t}[(6c_2 - c_1) \sin t + (6c_1 + c_2) \cos t]$ .....(6)

From (2), $2x = (D - 5)y = \frac{dy}{dt} - 5y$

$\Rightarrow \quad 2x = e^{6t}[(6c_2 - c_1) \sin t + (6c_1 + c_2) \cos t]$

$-5e^{6t}[c_1 \cos t + c_2 \sin t]$ [Using (5) and (6)]

$\Rightarrow \quad 2x = e^{6t}[(6c_2 - c_1 - 5c_2) \sin t + (6c_1 + c_2 - 5c_1) \cos t]$

$\Rightarrow \quad x = \frac{e^{6t}}{2}[(c_2 - c_1) \sin t + (c_1 + c_2) \cos t]$ .....(7)

$\therefore \quad$ (7) and (5) form the solution

**Example 4:** Solve: $\frac{dx}{dt} = ax + by$

$\frac{dy}{dt} = a'x + b'y$

**Sol:** Given equation in symbolic form are $(D - a)x - by = 0$ ......(1)

$-a'x + (D - b')y = 0$ .....(2)

Multiply (1) by (D - b') and (2) by (-b), we get, .......(3)

$(D - a)(D - b')x - b(D - b')y = 0$ .....(4)

$a'bx - b(D - b')y = 0$

(3) - (4) gives $[(D - a)(D - b') - a'b]x = 0$

$\Rightarrow$ $[D^2 - (a + b')D + (ab' - a'b)]x = 0$

$\therefore$ A.E. is $D^2 - (a + b')D + (ab' - a'b) = 0$

$$\Rightarrow \quad D = \frac{(a+b') \pm \sqrt{(a+b')^2 - 4(ab' - a'b)}}{2}$$

$$\Rightarrow \quad D = \frac{(a+b') \pm \sqrt{\{(a+b')^2 - 4ab'\} + 4a'b)}}{2}$$

$$\Rightarrow \quad D = \frac{(a+b') \pm \sqrt{(a-b')^2 - 4ab'}}{2} \Rightarrow D = m_1, m_2$$

where $m_1 = \dfrac{(a+b') \pm \sqrt{(a-b')^2 - 4ab'}}{2}$ and $m_2 = \dfrac{(a+b') - \sqrt{(a-b')^2 - 4ab'}}{2}$

$\therefore$ Sol is $x = c_1 em\, e^{m_1 t} + c_2 e^{m_2 t}$ ......(5)

$\therefore$ $\dfrac{dx}{dt} = c_1 m_1 e^{m_1 t} + c_2 e^{m_2 t}$ ......(6)

From (1), $by = -ax + \dfrac{dx}{dt}$

$\Rightarrow$ $by = -a[c_1 e^{m_1 t} + c_2 e^{m_2 t}] + (c_1 m_1 e^{m_1 t} + c_2 e^{m_2 t})$

$\Rightarrow$ $by = -(ac_1 - m_1 c_1) e^{m_1 t} - (ac_2 - m_2 c_2) e^{m_2 t}$

$\Rightarrow$ $y = \dfrac{1}{b}[c_1(m_1 - a) e^{m_1 t} + c_2(m_2 - a) e^{m_2 t}]$ .....(7)

$\therefore$ (5) and (7) form the solution.

**Example 5:** Solve the Simultaneous differential equations

$$\frac{d^2x}{dt^2} - 3x - 4y = 0; \quad \frac{d^2y}{dt^2} + x + y = 0$$

**Sol:** Write D for $\dfrac{d}{dt}$, the differential equations are

$(D^2 - 3)x - 4y = 0$ ....(1)

$x + (D^2 + 1)\,y = 0$ .....(2)

Operating on both sides of (1) by $(D^2 + 1)$ and on both sides of (2) by

-4 and subtracting, we get

$[(D^2 - 3)(D^2 + 1) + 4]\,X = 0 \Rightarrow (D^4 - 2D^2 + 1)x = 0$

$\Rightarrow \quad (D^2 - 1)^2\,x = 0$

A.E. is $(D^2 - 1)^2 = 0$

$\Rightarrow \quad D = \underline{+}\,1, \underline{+}\,1$ i.e. 1, 1, -1, -1

$\therefore \quad x = (c_1 + c_2t)e^t + (c_3 + c_4t)e^{-t}$ .....(3)

$\therefore \quad \dfrac{dx}{dt} = (c_1 + c_2t)e^t + c_2e^t - (c_3 + c_4t)e^{-t} - c_4e^{-t}$

and $\quad \dfrac{d^2x}{dt^2} = (c_1 + c_2t)e^t + 2c_2e^t + (c_3 + c_4t)e^{-t} - 2c_4e^{-t}$

Putting in (1), we get

$(c_1 + c_2t)e^t + 2c_2e^t + (c_3 + c_4t)e^{-t} - 2c_4e^{-t} - (3c_1 + 3c_2t)e^t$

$-(3c_3 + 3c_4t)e^{-t} = 4y$

$\therefore \quad 4y = (c_1 + c_2t + 2c_2 - 3c_1 - 3c_2t))e^t + (c_3 + c_4t - 2c_4 - 3c_3 - 3c_4t)e^{-t}$

$= (-2c_1 + 2c_2 - 2c_2t)e^t + (-2c_3 - 2c_4 - 2c_4t)e^{-t}$

$\therefore \quad y = -\dfrac{1}{2}[(c_1 - c_2 + c_2t)e^t + (c_3 + c_4\,t)e^{-t}]$

Hence $x = (c_1 + c_2t)e^t + (c_3 + c_4t)e^{-t}$

$\therefore \quad y = -\dfrac{1}{2}[(c_1 - c_2 + c_2t)e^t + (c_3 + c_4\,t)e^{-t}]$

Form the complete solution of the given simultaneous equations.

**Example 6:** Solve: $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0$

$\dfrac{dy}{dt} + 5x + 3y = 0.$

**Sol:** The Given equations are $\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 0$ .....(1)

$\dfrac{dy}{dt} + 5x + 3y = 0$ ....(2) $\Rightarrow 5x + (D + 3)y = 0$ .......(3)

(1) - (2) $\Rightarrow \frac{dx}{dt}$ - 3x - 2y = 0 .....(4) $\Rightarrow$ (D - 3) x - 2y = 0 ......(5)

Multiply (4) by 2 and (5) by (D + 3), we get,

$$\left.\begin{aligned} 10x + 2(D+3)y = 0 \\ (D^2 - 9)x - 2(D+3)y = 0 \end{aligned}\right\}$$

Adding we get,

$(D^2 - 9 + 10)x = 0 \Rightarrow (D^2 + 1)x = 0$ .....(6)

A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ Solution of (6) is, $x = c_1 \cos t + c_2 \sin t$ ......(7)

$\Rightarrow$ $\frac{dx}{dt} = -c_1 \sin t + c_2 \cos t \Rightarrow Dx = -c_1 \sin t + c_2 \cos t$ .....(8)

From (5); $y = \frac{1}{2}(Dx - 3x)$

$\Rightarrow$ $y = \frac{1}{2}[(-c_1 \sin t + c_2 \cos t)$

$-3 (c_1 \cos t + c_2 \sin t)]$ [Using (7) and (8)]

$\Rightarrow$ $y = \frac{1}{2}(c_2 - 3c_1) \cos t - \frac{1}{2}(c_1 + 3c_2) \sin t$ .....(9)

$\therefore$ (7) and (9) form the required solution

**Example 7:** Solve $\frac{dx}{dt} = 2y; \frac{dy}{dt} = 2z; \frac{dz}{dt} = 2x$

**Sol:** Given equations are

$\frac{dx}{dt} = 2y \quad \Rightarrow \quad y = \frac{1}{2}\frac{dx}{dt}$ ....(1)

$\frac{dy}{dt} = 2z \quad \Rightarrow \quad z = \frac{1}{2}\frac{dy}{dt}$ ...(2)

$\frac{dz}{dt} = 2x \quad \Rightarrow \quad x = \frac{1}{2}\frac{dz}{dt}$ ...(3)

From (3)

$$x = \frac{1}{2}\frac{dz}{dt} = \frac{1}{2}\frac{d}{dt}\left\{\frac{1}{2}\frac{dy}{dt}\right\} = \frac{1}{4}\frac{d^2y}{dt^2} = \frac{1}{4}\left\{\frac{1}{2}\frac{dx}{dt}\right\}$$

$\Rightarrow \quad x = \frac{1}{8}\frac{d^3x}{dt^3} \Rightarrow \quad 8x = \frac{d^3x}{dt^3}$

$\Rightarrow \quad \frac{d^3x}{dt^3}$ - 8x = 0 $\Rightarrow$ ($D^3$ - 8) x = 0 where D = $\frac{d}{dt}$

$\therefore$ A.E. is $D^3$ - 8 = 0 $\Rightarrow$ (D - 2) $D^2$ + 2D + 4) = 0

$\Rightarrow$ D = 2, $\frac{-2 \pm \sqrt{4-16}}{2}$ = 2, $\frac{-2 \pm i2\sqrt{3}}{2} \Rightarrow$ D = 2, -1 $\pm$i$\sqrt{3}$

$\therefore$ C. Sol is

$x = c_1 e^{2t} + e^{-t} (c_2 \cos \sqrt{3}\, t + c_3 \sin \sqrt{3}\, t)$ ...(4)

From (1), $y = \frac{1}{2}\frac{dx}{dt} = \frac{1}{2}[2c_1 e^{2t} + e^{-t} (-\sqrt{3}\, c_2 \sin \sqrt{3}\, t + \sqrt{3}\, c_3 \cos \sqrt{3}\, t)$

$+ (c_2 \cos \sqrt{3}\, t + c_3 \sin \sqrt{3}\, t)(-e^{-t})]$ ...(5)

and $z = \frac{1}{2}\frac{dy}{dt}$ ...(6)

$\therefore$ (4), (5), (6) give the required result.

**Self-check Exercise-1**

Q. 1 Solve

$(5D + 4) y - (2D + 1) z = e^{-x}$,

$(D + 8) y - 3z = 5e^{-x}$

Q. 2 Solve the simultaneous differential equations

$\frac{dx}{dt} = 3x + 2y$

$\frac{dy}{dt} = 5x + 3y$

Q. 3 Solve

$\frac{dx}{dt} + 2x - 3y = t$

$\frac{dy}{dt} - 3x + 2y = e^{2t}$

Second Method. Method of Differentiation

Sometimes x or y can be eliminated if we differentiate (1) or (2). From the resulting equations, we can find the second variable and then the value of the first Variable can be found from (1) or (2).

**Example 8.** Solve : $\frac{dx}{dt} + wy = 0$

$\frac{dy}{dt} - wx = 0$

**Sol.** The given equations can be written as

$Dx + wy = 0$ ...(1)

$Dy - wx = 0$ ...(2)

Diff. (1) w.r.t. t we get $D^2 x + wDy = 0$ ...(3)

$\Rightarrow$ $D^2 x + w(wx) = 0$ [by (2) $Dy = wx$]

$\Rightarrow$ $D^2 x + w^2 x = 0 \Rightarrow (D^2 + w^2) x = 0$

A.E. is $D^2 + w^2 = 0 \Rightarrow D = \pm wi$

$\therefore$ $x = c_1 \cos wt + c_2 \sin wt$ ...(4)

$\therefore$ (1) gives $c_1 w \sin wt + c_2 w \cos wt + wy = 0$

$\therefore$ $y = c_1 \sin wt - c_2 \cos wt$ ...(5)

(4) and (5) form the solution

Example 9. Solve the simultaneous differential equations

$x \frac{dy}{dx} + z = 0$

$x \frac{dz}{dx} + y = 0$

[Use Method 1]

Sol. Diff. (1) w.r.t. 'x', we get

$x \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dz}{dx} = 0$

$\Rightarrow$ $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + x\frac{dz}{dx} = 0$ ....(3)

(3) - (2) gives

$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ which is a homogeneous equation.

Putting $x = e^t$, we get A.E. as $[\theta (\theta - 1) + \theta - 1] y = 0$

$\therefore$ A.E. is $\theta^2 - 1 = 0 \Rightarrow \theta = \pm 1$

$\therefore$ $y = c_1 et + c_2 e^{-t}$

i.e. $y = c_1 x + c_2 x^{-1}$ ...(4)

$\therefore$ $\frac{dy}{dx} c_1 - \frac{c_2}{x^2}$

$\therefore$ (1) gives $xc_1 - \frac{c_2}{x} + z = 0$

$\therefore$ $z = -c_1 x + c_2 x^{-1}$ ...(5)

Hence (4) and (5) give the solution

## 14.6 SIMULTANEOUS EQUATIONS IN DIFFERENT FORM

If the equations are given in the form

$$P_1\, dx + Q_1 dy + R_1 dz = 0 \qquad ...(1)$$

$$P_2\, dx + Q_2 dy + R_2 dz = 0 \qquad ...(2)$$

where $P_1$, $P_2$ etc. are function of x, y , z then (1) and (2) can be written in the form

$$P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0$$

$$P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0$$

$$\therefore \quad \frac{\frac{dx}{dz}}{Q_1R_2 - Q_2R_1} = \frac{\frac{dy}{dz}}{R_1P_2 - R_2P_1} = \frac{1}{P_1Q_2 - P_2Q_1}$$

$$\Rightarrow \quad \frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

which is of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

where P, Q, R are function of x, y, z. Hence (1) and (2) can be put in the form (3).

## 14.5 Method of Solving Simultaneous equation of the form : $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

**Sol.** Method I. The given equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ...(1)

First take any two members of equations are (1)

$$\frac{dx}{P} = \frac{dz}{R} \text{ (say)} \qquad ...(2)$$

Integrating (2), we get an equation

Again take other two members of (1)

$$\frac{dy}{Q} = \frac{dz}{R} \text{ (say)} \qquad ...(3)$$

Integrating (3), we get another equation.

These two equations form the complete solutions.

Note. One solution so obtained can be used to simplify the other differential equation in the integrable form.

**Method II**

We may be able to find two sets of multiplier (not necessarily constants) l, m , n and L, M, N such that one of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx+mdy+ndz}{lP+mQ+nR} = \frac{Ldx+Mdy+Ndz}{IP+MQ+NR}$$

can be easily integrated.

If l, m, n are such that iP + mQ + nR = 0, and LP + MQ + NR = 0 then we get

l dx + m dy + n dz = 0 and Ldx + M dy + N dz = 0

which gives two equations on integration

These two equation so obtained form the complete solution.

Dear Students, lets us see what the method is with the following examples :-

**Example 10.** Solve : $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

**Sol.** The given equations are $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$ ...(1)

From the first two members of (1), we have

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$\Rightarrow \quad \log \frac{x}{y} = \log c_1 \Rightarrow \frac{x}{y} = c_1 \qquad ...(2)$$

From the last two members of (1), we have

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$\Rightarrow$ $\log \frac{y}{z} = \log c_2 \Rightarrow \frac{y}{z} = c_2$ ...(3)

Then (2) and (3) form the complete solution.

**Example 11.** Solve : $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

**Sol.** Given equation is $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$

From first two, $\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow xdx = ydy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + \frac{c_1}{2}$

$\Rightarrow$ $x^2 - y^2 = c_1$ ....(1)

From first and last, $\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow xdx = zdz \Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + \frac{c_2}{2}$

$\Rightarrow$ $x^2 - z^2 = c_2$ ....(2)

(1) and (2) form the complete solution.

**Example 12.** Solve $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$

**Sol.** The given equation are $\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$ ..(1)

Choosing l, m, n as multipliers, we have each fraction = $\frac{ldx + mdy + ndz}{0}$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

Integrating, we get $lx + my + nz = c_1$ ...(2)

Again choosing x, y, z as multipliers, we have each fraction = $\frac{xdx + ydy + zdz}{0}$

$\Rightarrow$ $x\,dx + y\,dy + z\,dz = 0 \Rightarrow x^2 + y^2 + z^2 = c_2$ ...(3)

Then (2) and (3) form the solution of (1).

**Example 13 :** Solve ; $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2 + y^2}$

**Sol.** The given equation are $\frac{dx}{z(x+y)} = \frac{dy}{z(x-y)} = \frac{dz}{x^2+y^2}$ ...(1)

From the first two members of (1), we have $\frac{dx}{x+y} = \frac{dy}{x-y}$

$\Rightarrow$ x dx - y dx = x dy + y dy

$\Rightarrow$ x dx = (y dx + x dy) + y dy $\Rightarrow$ x dx = d(xy) + y dy

Integrating, we have $\frac{x^2}{2}$ = xy + $\frac{y^2}{2}$ + $c_1$

$\Rightarrow$ $x^2 - y^2 - 2xy = c_1$ ...(2)

Again each traction

$$= \frac{xdx - ydy - zdz}{xz(x+y) - yz(x-y-z(x^2+y^2)} = \frac{xdx+ydy+zdz}{0}$$

$\therefore$ x dx - y dy - z dz = 0

Integrating, we get $\frac{x^2}{2} - \frac{y^2}{2} - \frac{z^2}{2}$ = constant

$\Rightarrow$ $x^2 - y^2 - z^2 = c_2$ ...(3)

(2) and (3) give the complete solution of (1).

**Example 14**Solve : $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$

**Sol.** Given equation is $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z)+(z-x)+(x-y)}$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$\Rightarrow$ $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \Rightarrow$ log x + log z = log $c_1$

$\Rightarrow$ log xyz = log $c_1 \Rightarrow xyz = c_1$ ...(1)

Again $\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{dx+dy+dz}{0}$

$\Rightarrow$ dx + dy + dz = 0 $\Rightarrow$ x + y +z = $c_2$ ...(2)

(1) and (2) form complete solution.

Example 15 : Solve $\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{z(xy-2x^2)}$

Sol. From first two members of given equation

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

or $$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, we get

log x = log y + log c

$\Rightarrow$ log x = log cy

$\Rightarrow$ x = cy ...(1)

From last two members of given equation

$$\frac{dy}{y^2} = \frac{dz}{z(xy-2x^2)}$$

$\Rightarrow$ $$\frac{dy}{y^2} = \frac{dz}{z(cyy-2c^2y^2)} \quad \text{[using (1)]}$$

$$= \frac{dz}{y^2z(c-2c^2)}$$

or $(c - 2c^2)\, dy = \frac{dz}{z}$

Integrating, we get

$(c - 2c^2)y = \log z + c_1$

$\Rightarrow$ $\left(\frac{x}{y} - 2\frac{x^2}{y^2}\right) y = \log z + c_1$

$\Rightarrow$ $x - \frac{2x^2}{y^2} = \log z + c_1$ ...(2)

(1) and (2) together form a complete solution

**Example 16 :** Solve $\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$

**Sol. :** From given equation, we have

$$\frac{dx-dy}{(x^2-y^2)+(zx-yz)} = \frac{dy-dz}{(y^2-z^2)+(xy-zx)}\ \frac{dz-dx}{(z^2-x^2)+(yz-xy)}$$

$\Rightarrow$ $$\frac{dx-dy}{(x-y)(z+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} = \frac{dz-dx}{(z-x)(z+x+y)}$$

$\Rightarrow$ $$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z} = \frac{dz-dx}{z-x}$$

$\Rightarrow$ log (x - y) = log (y - z) + log $c_1$ and log (y - z) = log (z - x) + log $c^2$

$\Rightarrow$ log $\left(\frac{x-y}{y-z}\right)$ = log $c_1$ and log $\left(\frac{y-z}{z-x}\right)$ = log $c_2$

$\Rightarrow$ $\frac{x-y}{y-z} = c_1$ and $\frac{y-z}{z-x} = c_2$

which together form the complete solution.

**Example 17 :** Solve

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

Sol. : Given differential equations are

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \qquad ...(1)$$

Taking x, y, z as multipliers, we get

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{xdx+ydy+zdz}{0}$$

[As $x^2 (y^2 - z^2) + y^2 (z^2 - x^2) + z^2 (x^2 - y^2) = 0$]

$\therefore$ x dx + y dy + z dz = 0

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1}{2}$$

$\therefore$ $x^2 + y^2 + z^2 = c_1$ ...(2)

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

[As $y^2 - z^2 + z^2 - x^2 + x^2 - y^2 = 0$]

$\therefore$ $\frac{1}{x}$ dx + $\frac{1}{y}$ dy + $\frac{1}{z}$ dz = 0

Integrating, we get

log x + log y + log z = log $c_2$

$\therefore$ log (x y z) = log $c_2$

$\therefore$ xyz = $c_2$ ...(3)

$\therefore$ complete integral of (1) is

$x^2 + y^2 + z^2 = c_1$, $xyz = c_2$, where $c_1$, $c_2$ are arbitrary constants.

**Example 8 :** Solve $\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)}$

**Sol. :** Given differential equations are

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)} \quad ....(1)$$

Taking 1, 1, 0 as multipliers, we have

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)} = \frac{dx+dy}{x^2+y^2+2xy} \quad ...(2)$$

Taking last two members, we have

$$\frac{dz}{z(x+y)} = \frac{dx+dy}{(x+y)^2}$$

or $\frac{dz}{z} = \frac{dx+dy}{x+y}$

Integrating,

log z + log a = log (x + y)

$\therefore$ x + y = az ...(3)

Taking 1, -1, 0 as multipliers, we get

$$\frac{dx}{x^2+y^2} = \frac{dy}{2xy} = \frac{dz}{z(x+y)} = \frac{dx-dy}{x^2+y^2-2xy} \quad ...(4)$$

From (2) and (4), we have

$$\frac{dx-dy}{(x-y)^2} = \frac{dx+dy}{(x+y)^2}$$

Integrating,

$$-\frac{1}{x-y} = \frac{1}{x+y} - b$$

$$\therefore \quad \frac{1}{x-y} - \frac{1}{x+y} = b$$

or $$\frac{x+y-x+y}{x^2-y^2} = b$$

$$\therefore \quad \frac{2y}{x^2-y^2} = b$$

or $\quad 2y = b(x^2 - y^2) \qquad \qquad ...(5)$

From (3) and (5), complete integrals of (1) are

$x + y = dz,\ 2y = b(x^2 - y^2)$

where a, b are arbitrary constants

**Self-check Exercise- 2**

Q. 1 Solve

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$$

Q. 2 Solve

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

Q. 3 Solve

$$\frac{dx}{x^2+y^2+yz} = \frac{dy}{x^2+y^2-yz} = \frac{dz}{z(x+y)}$$

## 14.6 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined different types of simultaneous differential equations.
2. Symbolic method and method of differentiation discussed for solving simultaneous linear differential equations with constant coefficients.

3. Find the solutions of simultaneous differential equations with constant coefficients by symbolic method and method of differentiation.

4. Discussed the method of solution of simultaneous equations of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and find the complete solution of the differential equations.

## 14.7 Glossary:

Simultaneous differential equations, also known as systems of differential equations, are a collection of equations and describe the relationship between several unknown functions and their derivatives with respect to one or more independent variables.

## 14.8 Answer to Self-Check Exercise

### Self-Check Exercise-1

Ans.1 $y = c_1 e^x + c_2 e^{-2x} + 2 e^{-x}$

and $z = 3c_1 e^x + 2c_2 e^{-2x} + 3 e^{-x}$,

given the required results

Ans. 2 $x = \left[c_1 e^{(3+\sqrt{10})t} - c_2 e^{(3-\sqrt{10})t}\right]$

and $y = \frac{1}{2}\sqrt{10}\left[c_1 e^{(3+\sqrt{10})t} - c_2 e^{(3-\sqrt{10})t}\right]$

Ans. 3 $x = c_1 e^t + c_2 e^{-5t} - \frac{2t}{5} - \frac{13}{25} + \frac{3}{7} e^{2t}$

and $y = c_1 e^t - c_2 e^{-5t} - \frac{12}{25} + \frac{4}{7} e^{2t} - \frac{3t}{5}$

### Self-Check Exercise-2

Ans.1 $xy = c_1$

and $x^4 - (z^2 + xy)^2 = c_2$

Ans. 2 $\frac{y-x}{z-y} = c_1$

and $(x + y + z)(x - y)^2 = c_2$

Ans. 3 $x - y - z = c_1$

and $x^2 + y^2 = z^2 c_2$

## 14.9 References/Suggested Readings


1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.

2. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.

3. Zill, D., A First Course in Differential Equation with Applications, 2nd Ed., Prindle, Weber, Schmidt, Boston, 1982.

## 14.10 Terminal Questions

1. Solve

   (D - 17) y + (2D - 8) z = 0

   (13D - 53) y = 2z.

2. Solve

   $$\frac{dx}{dt} + 4x + 3y = t$$

   $$\frac{dy}{dt} + 2x + 5y = e^t$$

3. Solve

   $$z\frac{d^2y}{dx^2} - \frac{dz}{dx} - 4y - 2x = 0,$$

   $$z\frac{dy}{dx} + 4\frac{dz}{dx} - 3z = 0$$

4. Solve

   $$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2\sin(y+2x)}$$

5. Solve

   $$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{x^2+(x+y)^2}$$

6. Solve

   $$\frac{dx}{-x(x+y)} = \frac{dy}{y(x+y)} = \frac{dz}{(x-y)(2x+2y+z)}$$

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## Unit - 15

# Total Differential Equations

**Structure**



## 15.1 Introduction

An equation of the form Pdx + Qdy + Rdz = 0, where P, Q, R are function of x, y, z is called a total differential equation. It is also known as single differential equation. The differential equation Pdx + Qdy + Rdz = 0 is said to be integrable if there exists a relation of the form u(x, y, z) = c, where differentiation gives the given differential equation. The relation u(x, y, z) = c is called the complete integral or solution of the given differential equation.

## 15.2 Learning Objectives

After studying this unit, you should be able to:-

- Define Pfaffion differential form, Pfaffian differential equation and total differential equation.

- Find the necessary and sufficient condition for the integrability of single differential equation Pdx + Qdy + Rdz = 0
- Discuss Inspection Method, Method of Auxiliary equations and by taking one variable constant to solve equation of the form Pdx + Qdy + Rdz = 0

## 15.3 Some Definitions

**Pfaffian Differential form**

The expression

$\sum_{i=1}^{n} F_i (x_1, x_2, .....x_n)\, dx_i$ in which Fi, i = 1, 2, ...., n are functions of some or all of the n independent variables $(x_1, x_2, .....x_n)$ is called a Pfaffian differential form in n variables.

**Pfaffian Differential Equation**

An equation of the form

$$\sum_{i=1}^{n} F_i \, dx_i = 0$$

is called a Pfaffian differential equation in n variables $x_1, x_2, .....x_n$, which Fi, i = 1, 2, ...., n are functions of some or all of the n independent variables $x_1, x_2, .....x_n$.

**Total Differential Equation (or Pfaffian Differential Equation in Three Variables)**

An equation of the form

Pdx + Qdy + Rdz = 0,

where P, Q, R, are functions of x, y, z is called the total differential equation or single differential equation or Pfaffian differential equation in three variables x, y, z.

If we are given a relation of the form

$$f(x, y, z) = \text{constant} \qquad ....(1)$$

then, we have

$$\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ have a common factor, it may be cancelled throughout i.e. we can express it in the form

$$Pdx + Qdy + Rdz = 0 \qquad ...(2)$$

where P, Q, R are functions of x, y, z. Thus form any given relation of the form (1), we always get a relation of the form (2), known as total differential equation.

**Integrating Factor**

An integrating factor (abbreviated as I.F.) $\mu$ of a differential equation Pdx + Qdy + Rdz = 0 is a factor such that if the equation is multiplied by it, the resulting equation is exact.

### 15.4 Necessary and Sufficient Condition for the Integrability of Single Differential Equations Pdx + Qdy + Rdz = 0 where P, Q, R are Functions of x, y, z

**Sol.** Necessary Condition

Let us take up the single$\alpha$ differential equation Pdx + Qdy + Rdz = 0 ....(1)

where P, Q, R are functions of x, y, z

Let u(x, y, z) = 0 be integral of (1) ...(2)

$\therefore$ du = Pdx + Qdy + Rdz or a multiple of it.

But we know that du = $\frac{\partial u}{\partial x}$ dx + $\frac{\partial u}{\partial y}$ dy + $\frac{\partial u}{\partial z}$ dz ...(3)

Since (2) is an integral of (1)

$\therefore$ P, Q, R must be proportional to $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ , $\frac{\partial u}{\partial z}$ respectively.

$$\therefore \quad \frac{\frac{\partial u}{\partial x}}{P} = \frac{\frac{\partial u}{\partial y}}{Q} = \frac{\frac{\partial u}{\partial z}}{R} = \mu\,(x, y, z)\ [\text{say}]\ (\neq 0)$$

Then, we have
$$\left.\begin{aligned} \mu\ P &= \frac{\partial u}{\partial x} \\ \mu\ Q &= \frac{\partial u}{\partial y} \\ \mu\ R &= \frac{\partial u}{\partial z} \end{aligned}\right] \qquad ...(4)$$

From the first two equations of (4), we get

$$\frac{\partial}{\partial y}(\mu\,P) = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\mu\,Q)$$

$$\Rightarrow \quad \mu\frac{\partial p}{\partial y} + P\,\frac{\partial u}{\partial y} = \mu\frac{\partial Q}{\partial x} + Q\,\frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \mu\left(\frac{\partial p}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\,\frac{\partial u}{\partial x} - P\frac{\partial u}{\partial y} \qquad ...(5)$$

Similarly, we have $\mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)$ = R $\frac{\partial u}{\partial y}$ - Q $\frac{\partial u}{\partial z}$ ...(6)

$\mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)$ = P $\frac{\partial u}{\partial z}$ - R $\frac{\partial u}{\partial x}$ ....(7)

Multiply (5) by R, (6) by P and (7) by Q and then adding, we have

$$\mu\left(P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial p}{\partial y}-\frac{\partial Q}{\partial x}\right)\right)$$

$$= RQ\frac{\partial u}{\partial x} - RP\frac{\partial u}{\partial y} + PR\frac{\partial u}{\partial y} - PQ\frac{\partial u}{\partial z} + PQ\frac{\partial u}{\partial z} - QR\frac{\partial u}{\partial x} = 0$$

$\Rightarrow$ $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial p}{\partial y}-\frac{\partial Q}{\partial x}\right) = 0$ [Q $\mu \neq 0$] .....(8)

This is the necessary condition.

**The condition is sufficient**

Let us suppose that the co-efficientsP,Q,R of (1) satisfy (8). Then we prove that (1) is integrable.

Let z be treated as constant so that dz = 0

$\therefore$ (1) becomes Pdx + Qdy = 0 .....(9)

Now Pdx + Qdy may be regarded as an exact differential

[Q if it is not so, then with the help of an integrating factor, we can make it exact.]

Thus without loss of generality, we can take Pdx + Qdy = 0 as an exact differential equation.

$\therefore$ $\frac{\partial p}{\partial y} = \frac{\partial Q}{\partial x}$ .....(10)

Let $\int$ (P dx + Q dy) = V .....(11)

$\Rightarrow$ P dx + Q dy = dV = $\frac{\partial V}{\partial x}$ dx + $\frac{\partial V}{\partial y}$ dy

Then we have $\left.\begin{array}{l} P=\frac{\partial V}{\partial x} \\ Q=\frac{\partial V}{\partial y} \end{array}\right]$ [On comparing] .....(12)

From (12), we have $\frac{\partial P}{\partial z} = \frac{\partial^2 V}{\partial z\,\partial x}$ and $\frac{\partial Q}{\partial z} = \frac{\partial^2 V}{\partial z\,\partial y}$

By using (10) and (12), (8) becomes

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z\,\partial x}\right)+ R(0) = 0$$

$$\Rightarrow \frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial x\,\partial z}\right)= 0$$

$$\Rightarrow \frac{\partial V}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}-R\right)-\frac{\partial V}{\partial y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}-R\right) = 0$$

$$\Rightarrow \begin{vmatrix} \frac{\partial V}{\partial x} & \frac{\partial}{\partial x} & \left(\frac{\partial V}{\partial z}-R\right) \\ \frac{\partial V}{\partial y} & \frac{\partial}{\partial y} & \left(\frac{\partial V}{\partial z}-R\right) \end{vmatrix} = 0$$

$\Rightarrow$ there exists a relation between V and $\frac{\partial V}{\partial z}$ - R independent of x and y.

$$\therefore \qquad \frac{\partial V}{\partial z} - R = \phi(z, V)$$

Now Pdx + Qdy + Rdz = $\frac{\partial V}{\partial x}$ dx + $\frac{\partial V}{\partial y}$ dy + $\left(\frac{\partial V}{\partial z}-\phi\right)$dz

$$= \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz - \phi dz$$

$$= dV - \phi dz$$

$\therefore$ (1) becomes dV - $\phi$dz = 0 which is an equation in two variables. Hence its integration will give an integral of the form F(V, z) = 0

The condition (8) is sufficient.

Hence (8) is the necessary and sufficient condition for the inerrability of (1)

Note. Condition of exactness of Pdx + Qdy + Rdz = 0 can be written as $\begin{vmatrix} P & Q & R \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ = 0

This is the same as (8)

To Clarify what we have just said, consider the following examples:-

**Example 1:** Shows that $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ is integrable.

**Sol:** The given equation is $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ .....(1)

Compare the given equation with Pdx + Qdz = 0, we get

$P = 2x + y^2 + 2xz$ ; $Q = 2xy$ ; $R = x^2$

Now $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (2x + y^2 + 2xz)(0 - 0) + 2xy(2x - 2x) + x^2 (2y - 2y) = 0$

$\therefore$ given equation is integrable.

### 15.5 Method to Solve Pdx + Qdy + Rdz = 0 [Inspection Method]

**Sol:** If Pdx + Qdy + Rdz = 0 is integrable, inspection method can be applied using standard results like.

(i) $d(xy) = xdy + ydx$ (ii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(iii) $d\left(\log\frac{y}{x}\right) = \frac{xdy - ydx}{xy}$ (iv) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2 + y^2}$

(v) $d(xyz) = yz\,dx + zxdy + xydz$ etc.

**Example 2:** Solve : $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$

**Sol:** The given equation is $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$ .....(1)

Compare the given equation with Pdx + Qdy + Rdz = 0

$\therefore$ $P = yz + xyz$. $Q = zx + xyz$, $R = xy + xyz$

$\therefore$ $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (yz + xyz)[x + xy - x - xz] + (zx + xyz)(y + yz - y - xy) + (xy + xyz)(z + xz - z - yz)$

$= (yz + xyz)(xy - yz) + (zx + xyz)(yz - xy) + (xy + xyz)(xz - yz)$

$= xyz (1 + x)(y - z) + xyz (1 + y) (z - x) + xyz (1 + z) (x - y)$

$= xyz [y - z + z - x + x - y + x(y - z) + y(z - x) + z(x - y)]$

$= xyz (0 + 0) = 0$ $\therefore$ given equation is integrable.

Now $(yz + xyz)dx + (zx + xyz)dy + (xz + xyz)dz = 0$

Dividing throughout but by xyz, we get $\left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} + 1\right)dy + \left(\frac{1}{z} + 1\right)dz = 0$

$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}$ + dx + dy + dz = 0

Integrating we get

log x + log y + log z + x + y + z = c

or log(xyz) + x + y + z = c, where c is an arbitrary constant, is the required solution.

**Example 3:** Verify that the equation is integrable and find the solution of

$zydx = zxdy + y^2dz$

**Sol:** The given equation is $zy\,dx = zxdy + y^2dz$

i.e. $\quad zydx - zxdy - y^2dz = 0 \qquad$ .....(1)

Compare (1) with

$$Pdx + Qdy + Rdz = 0 \qquad ......(2)$$

$\therefore \quad P = zy, Q = -zx, \quad R = -y^2$

$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= zy(-x + 2y) - zx(0 - y) - y^2(z + z) = -xyz + 2zy^2 + xyz - 2y^2z = 0$

$\therefore$ given equation (1) is integrable.

(1) can be written as

$$\frac{y\,dx - x\,dy}{y^2} - \frac{dz}{z} = 0$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right) - \frac{dz}{z} = 0$$

Integrating, we get

$$\frac{x}{y} - \log z = -\log c$$

$$\Rightarrow \quad \log z - \log c = \frac{x}{y}$$

$$\Rightarrow \quad \log\frac{z}{c} = e^{x/y}$$

$\Rightarrow \quad z = c\,e^{x/y}$ as the required solution.

**Example 4:** Verify that the equation is integrable and find the solution of

$(e^x y + \cos x)\, dx + (e^x + e^y z)\, dy + e^y dz = 0$

**Sol:** Given equation is

$(e^x y + \cos x)\, dx + (e^x + e^y z)dy + e^y dz = 0$ ......(1)

Comparing (1) with Pdx + Qdy + Rdz = 0, .....(2)

we get

$P = e^x y + \cos x,\ Q = e^x + e^y z,\ R = e^y$

We can check that (1) is integrable by proving

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0$$

Again (1) can be written as

$(e^x y\, dx + e^x dy) + \cos x\, dx + (e^y z dy + e^y dz) = 0$

$\Rightarrow \quad d(e^x y) + d(\sin x) + d(e^y z) = 0$

Integrating, we get

$e^x y + \sin x + e^y z = c$, is the required solution.

**Self-Check Exercise**

Q.1 Show that $zdx + zdy + 2(x + y + \sin z)dz = 0$ is integrable.

Q.2 Solve $(y^2 + z^2 - x^2)\, dx - 2xy\, dy - 2xz\, dz = 0$

Q.3 Verify that the equation is integrable and find the solution of
$xdy - ydx - 2x^2 zdz = 0$

## 15.6 Method to Solve Pdx + Qdy + Rdz = 0

(Method of Auxilliary Equations)

Let Pdx + Qdy + Rdz = 0 .....(1)

be the given equation.

Its condition of inerrability is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \quad .....(2)$$

Comparing (1) and (2), we obtain simultaneous equations, known as Auxiliary Equations (A.E.) as

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}} \qquad ......(3)$$

Let $u = c_1$ and $v - c_2$ be two independent integrals of (3). With these we formulate the following equation.

$$A\,du + B\,dv = 0 \qquad ....(4)$$

Compare (1) and (4), we get values of A and B, putting in (4), and integrate the resulting equation. Thus substitute values of u and v in the relation so obtained after integration. This will give us the required general solution.

**Note.** This method will fail if $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$, and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

This method is generally appeared when method of inspection is not applicable.

Let us see what the method is with the help of following examples:-

**Example 5:** Solve

$$xz^3\,dx - z dy + 2y dz = 0$$

**Sol:** Given differential equation is

$$xz^3 dx - z dy + 2y dy = 0 \qquad .....(1)$$

comparing it with Pdx + Qdy + Rdz = 0,we get

$$P = xz^3,\ Q = -z,\ R = 2y$$

$$\therefore \qquad P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$= xz^3(-1 - 2) + (-z)(0 - 3z^2x) + 2y(0 - 0)$$

$$= -3xz^3 + 3xz^3$$

$$= 0$$

$\therefore$ Condition of integrability of (1) is satisfied.

Auxilliary equations are

$$\frac{dx}{\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}}$$

or $$\frac{dx}{-1-2} = \frac{dy}{0-3xz^2} = \frac{dz}{0-0}$$

$$- \frac{dx}{1} = \frac{dy}{xz^2} = \frac{dz}{0} \qquad .....(2)$$

From (2), by taking the third member,

$$dz = 0$$

$\therefore \quad z = a = u$ (say) $\qquad ......(3)$

Taking first and second members of (2), we get

$$\frac{dx}{1} = \frac{dy}{xz^2}$$

or $\quad xdx = \frac{1}{z^2} dy$

Integrating,

$$\frac{x^2}{2} = \frac{1}{z^2} y + \frac{b}{2}$$

$$x^2z^2 - 2y = b = v \text{ (say)} \qquad ....(4)$$

Now we find A and B in such a way that

$$A\, du + B\, dv = 0 \qquad ......(5)$$

becomes identical with given differential equation (7).

Putting the values of u, v in (5), we get

$$Adz + Bd(x^2z^2 - 2y) = 0$$

or $\quad Adz + B(x^2\, 2zdz + z^2\, 2xdx - 2dy) = 0$

or $\quad 2B\, z^2xdx - 2\, B\, dy + (A + 2B\, x^2z)dz = 0 \qquad .....(6)$

comparing (6) and (1), we get

$$2B = z \Rightarrow B = \frac{z}{2} = \frac{u}{2}$$

$$A + 2Bx^2z = 2y \text{ or } A = 2y - x^2z^2 = -v$$

Putting values of A and B in (5), we get

$$- vdu + \frac{u}{2} dv = 0$$

or $\quad 2\frac{du}{u} - \frac{dv}{v} = 0$

Integrating,

$$2 \log u - \log v = \log c$$

or $\frac{u^2}{v}$ = c

$\therefore$ $\frac{z^2}{x^2z^2 - 2y}$ = c

or $z^2 = c(x^2z^2 - 2y)$, which is the required solution.

**Example 6:** Solve

$(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$

**Sol:** Given differential equation is

$(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0$ .....(1)

Comparing it with Pdx + Qdy + Rdz = 0, we get

$P = 2xz - yz,\ Q = 2yz - zx,\ R = -(x^2 - xy + y^2)$

$\therefore$ $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (2xz - yz)(2y - x - x + 2y) + (2yz - zx)(-2x + y - 2x + y)(x^2 - xy + y^2)(-z + z)$

$= (2xz - yz)(-2x + 4y) + (2yz - zx)(-4x + 2y)$

$= -4x^2z + 8xyz + 2xyz - 4y^2z - 8xyz + 4y^2z + 4x^2z - 2xyz$

$= 0$

$\therefore$ Condition of inerrability is satisfied

Auxiliary equations are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

or $$\frac{dx}{(2y - x)(x - 2y)} = \frac{dy}{(-2x + y) - (2x - y)} = \frac{dz}{-z - (-z)}$$

or $$\frac{dx}{2y - x - x + 2y} = \frac{dy}{-2x + y - 2x + y} = \frac{dz}{-z + z}$$

$\therefore$ $$\frac{dx}{2(2y - x)} = \frac{dy}{2(y - 2x)} = \frac{dz}{0}$$ .....(2)

From (2), dz = 0 or z = a = u (say) .....(3)

Taking first and second members of (2), we get

$$\frac{dx}{2(2y-x)} = \frac{dy}{2(y-2x)}$$

or $$\frac{dx}{2y-x} = \frac{dy}{y-2x}$$

$\therefore$ ydx - 2xdx = 2ydy - xdy

or (x dy + ydx) - 2xdx - 2y dy = 0

$\therefore$ d(xy) -2xdx - 2ydy = 0

Integrating, we get

$xy - x^2 - y^2 = b = v$ (say) .....(4)

Now We find A and B in such a way that A du + B dv = 0 .....(5)

becomes identical with the given differential equation (1).

Putting the values of u, v in (5), we get

$A\, d(z) + B\, d(xy - x^2 - y^2) = 0$

or A dz + B(xdy + ydx - 2xdx - 2ydy) = 0

or (By - 2xB)dx + (Bx - 2By)dy + A dz = 0

or (y - 2x) B dx + (x - 2y) B dy + A dz = 0 ......(6)

Comparing (6) with (1), we get

(y - 2x) B = 2xz - yz

i.e. (y - 2x) B = -z (y - 2x)

$\therefore$ B = - z = - u

and $A = xy - (x^2 + y^2)$

i.e. A = v

Putting values of A and B in (5), we get

v du - u dv =

or $$\frac{du}{u} - \frac{dv}{v} = 0$$

Integrating,

log u - log v = log c

$\therefore$ $$\frac{u}{v} = c$$

or $$\frac{z}{xy - x^2 - y^2} = c$$

or $\quad c(xy - x^2 - y^2) = z$, which is the required solution.

**Example 7:** Solve

$$(yz - 1)dx + (z - x)\, x\, dy + (1 - xy)dz = 0$$

**Sol:** Given differential equation is

$$(yz - 1)\, dx + (z - x)\, x\, dy + (1 - xy)dz = 0 \qquad .....(1)$$

Comparing it with Pdx + Qdy + Rdz = 0, we get

$$P = yz - 1,\ Q = z\, x - x^2,\ R = 1 - xy$$

$$\therefore \quad P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= (yz - 1)(x + x) + (zx - x^2)(-y - y) + (1 - xy)(z - z + 2x)$$

$$= 2x\,(yz - 1) - 2y\,(zx - x^2) + 2x\,(1 - xy)$$

$$= 2xyz - 2x - 2xyz + 2x^2y + 2x - 2x^2y$$

$$= 0$$

$\therefore$ Condition of integrability of (1) is satisfied.

Auxilliary equations are

$$\frac{dx}{\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}} = \frac{dy}{\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}} = \frac{dz}{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}$$

$$\therefore \quad \frac{dx}{-2x} = \frac{dy}{-2y} = \frac{dz}{2x}$$

$$\therefore \quad \frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x} \qquad .....(2)$$

Taking first and second members of (2), we get

$$\frac{dx}{x} = \frac{dy}{-y}$$

or $\quad \log x = -\log y + \log a$

$$\therefore \quad xy = a = u \text{ (say)} \qquad .....(3)$$

Taking first and third members of (3), we get

$$\frac{dx}{x} = \frac{dz}{z}$$

or $\quad dx = dz$

$\Rightarrow \quad x = z + b$

$\therefore \quad x - z = b = v$ (say) .....(4)

Now we find A and B in such a way that A du + B dv = 0 .....(5)

becomes identical with given differential equation (1)

Putting the values of u, v in (5), we have

$$A\, d(xy)\; B\, d(x - z) = 0$$

or $\quad A(xdy + ydx) + B(dx - dz) = 0$

or $\quad (Ay + B)\, dx + Axdy - Bdz = 0$ .....(6)

Comparing (6) with (1), we get

$$Ay + B = yz - 1,\; Ax = (z - x)x,\; B = xy - 1$$

$\therefore \quad A = z - x = -v$

$\quad B = xy - 1 = u - 1$

Putting values of A, B in (5), we get

$$- v\, du + (u - 1)\, dv = 0$$

or $\quad \dfrac{vdu - udv}{u^2} + \dfrac{1}{v^2}\, dv = 0$

or $\quad d\left(\dfrac{u}{v}\right) + \dfrac{1}{v^2}\, dv = 0$

Integrating, we get

$$\frac{u}{v} - \frac{1}{v} = c$$

or $\quad u - 1 = cv$

or $\quad xy - 1 = c(x - z)$, which is the required solution.

**Self-Check Exercise-2**

Q.1 Solve

$$(y^2 + yz + z^2)dx + (z^2 + zx + x^2)\, dy + (x^2 + xy + y^2)dz = 0$$

Q.2 Solve

$$z(z - y)dx + (z + x)zdy + x(x + y)dz = 0$$

## 15.7 Method of Solving Pdx + Qdy + Rdz = 0 By Taking One Variable Constant

1. First of all check the condition of integrability.

2. Treat one of the variables, say z, as a constant so that dz = 0

$\therefore$ given equation becomes Pdx + Qdy = 0 ...(1)

Let its solution be u (x, y) = $f$ (z) where $f$ (z) is an arbitrary function of z. ...(2)

[$\because$ z = constant $\Rightarrow f$(z) is constant]

Diff. (2) totally w.r.t. x, y, z and compare the result with the given equation

Pdx + Qdy + Rdz = 0, we shall get $f$ and z.

**Example 8.** Solve : $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2zdz = 0$

**Sol.** the given equation is $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2zdz = 0$ ...(1)

It is of the form Pdx + Qdy + Rdz = 0 where

$P = 2x^2 + 2xy + 2xz^2 + 1$, Q = 1, R = 2z

$$\therefore \quad P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$= (2x^2 + 2xy + 2xz^2 + 1)(0 - 0) + 1\,(1 - 4xz) + 2z\,(2x - 0)$

$= 0 - 4xz + 4zx = 0$ $\therefore$ given equation is integrable

Put x = constant so that dx = 0 $\therefore$ dy + 2zdz = 0

Integrating, we get

$y + z2 = \text{constant } f(x)$ (say)

$\Rightarrow$ $dy + 2zdz = f'(x)\,dx \Rightarrow -f'(x)\,dx + dy + 2zdz = 0$

Compare with the given equation, we have

$-f(x) = 2x^2 + 2xy + 2xz^2 + 1$

$= 2x^2 + 1 + 2x(y + z^2) = 2x^2 + 1 + 2x\,f(x)$

$$\Rightarrow \quad \frac{df}{dx} + 2x\,f = -(2x^2 + 1) \text{ which is linear in } f.$$

$$\text{I.F.} = e^{\int 2x dx} = e^{x^2}$$

$\therefore$ the solution is

$$f.\ e^{x^2} = \int e^{x^2}.\ (-2x^2 - 1)\,dx + c$$

$$= -\int x.\ (2x e^{x^2})\,dx - \int e^{x^2}\,dx + c \qquad ...(2)$$

Now for $\int 2xe^{x^2}\,dx$, Put $x^2 = t$ so that 2xdx = dt $\therefore \int 2xe^{x^2}\,dx = \int e^t\,dt = e^t = e^{x^2}$

$\therefore$ (2) becomes

$f\, e^{x^2}\left[x.e^{x^2} - \int 1.e^{x^2}dx\right]$ - $\int e^{x^2}$ dx + c = - x$e^{x^2}$ + c

$\Rightarrow$ (y + z$^2$) $e^{x^2}$ + x$e^{x^2}$ = c $\Rightarrow$ (x + y + z$^2$)$e^{x^2}$ = c is the required solution.

**Example 9.** Verify that the following equation is integrable and find its primititve

zydx + (x$^2$y - zx) dy + (x$^2$z - xy) dz = 0

**Sol.** The given equation is zydx + (x$^2$y - zx) dy + (x$^2$z - x$^2$) dz = 0 ...(1)

Compare equation (1) with Pdx + Qdy + Rdz = 0 ....(2), we get

P = zy, Q = x$^2$ y - zx, R x$^2$ z - xy.

$\therefore$ P $\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)$ + Q $\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)$ + R$\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$

= zy(-x - (-x)) + (x$^2$y - zx) 2xz - y - y) + (x$^2$z - xy) (z - 2xy + z)

= zy(0) + (x$^2$y - zx) 2xz - 2y) + (x$^2$z - xy) (2z - 2xy)

= 2x$^3$yz - 2x$^2$y$^2$ -2x$^2$z$^2$ + 2xyz + 2x$^2$z$^2$ - 2x$^3$yz - 2xyz + 2x$^2$y$^2$ = 0

$\therefore$ given equation is integrable.

Take x = constant so that dx = 0 $\therefore$ (1) becomes (x$^2$y - zx) dy + (x$^2$z - xy) dz = 0

$\Rightarrow$ x$^2$ (ydy + zdz) = x (zdy + ydz)

Integrating (2) (x is assumed to be constant), x$^2\left(\frac{y^2}{2}+\frac{z^2}{2}\right)$= xyz + constant

$\Rightarrow$ $\frac{1}{2}$ x$^2$ (y$^2$ + z$^2$) - xyz = $f$ (x)

[Q constant = $f$(x) since x is constant]

Diff. (3), we get

$\frac{1}{2}$ 2x dx(y$^2$ + z$^2$) + $\frac{1}{2}$ x$^2$ (2ydz + 2zdz) - (yzdx + zxdy + xydz) = $f'$ (x) dx.

$\Rightarrow$ (x(y$^2$ + z$^2$) - yz - $f'$ (x)) dx + (x$^2$y - zx) dy + (x$^2$z - xy) dz = 0 ....(4)

compare (1) and (4), we get

zy = x(y$^2$ + z$^2$) - yz - $f'$ (x) $\Rightarrow$ x(x$^2$ + z$^2$) - 2yz = $f'$ (x)

$\Rightarrow$ $\frac{1}{2}$ x$^2$(y$^2$ + z$^2$) - xyz = $\frac{1}{2}$ x $f'$ (x)

By (3) and (5), we get $f$ (x) = $\frac{1}{2}$ x $f'$ (x)

$\Rightarrow \quad \frac{f'(x)}{f(x)} = \frac{2}{x} \Rightarrow$ log $f$ (x) = 2 log x + log c = log $x^2$ + log c m = log $cx^2$

$\Rightarrow \quad f$ (x) = $cx^2$ where c is arbitrary constant.

Putting in (3), we get $\frac{1}{2}$ $x^2(y^2 + z^2)$ - xyz = cx2

i.e. $x^2(y^2 + z^2 - 2c) = 2xyz$ as the required solution

**Example 10.** Solve : $(x^2 + y^2 + z^2)$ dx - 2xy dy - 2xz dz = 0

**Sol.** Given equation is

$(x^2+ y^2 + z^2)$ dx - 2xy dy - 2xz dz = 0 ...(1)

First check that (1) is integrable by using condition of integrability

Let x = constant so that dx = 0

$\therefore$ (1) becomes - 2xydy - 2xzdz = 0

$\Rightarrow$ 2ydy + 2zdz = 0 ...(2)

Integrating, we get , $y^2 + z^2$ = constant $f$ (x) (say) ...(3)

Diff. (3), we get 2ydy + 2zdz = $f'$ (x) dx

$\Rightarrow$ x $f'$(x) dx - 2xydy - 2xzdz = 0

Compare (4) and (1), we get

x $f'$ (x) = $x^2 + y^2 + z^2 = x^2 + f$ (x) (by (3))

$\Rightarrow \quad x \frac{df}{dx} - f = x2 \Rightarrow \frac{df}{dx} - \frac{f}{x} = x$, linear in $f$

I.F. $= e^{\int -\frac{1}{x} dx}$ = . $e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

$\therefore$ Sol is

$$f \cdot \frac{1}{x} = \int \frac{1}{x} \cdot x dx + c = x + c$$

$\Rightarrow \quad f(x) = x^2 + cx \Rightarrow y^2 + z^2 = x^2 + cx$ is the read solution.

**Example 11.** Solve : $xdy - ydx - 2x^2 zdz = 0$

**Sol.** The given equation is

$xdy - ydx - 2x^2zdz = 0$ ....(1)

First of all check that (1) is integrable by using condition of integrability

Now put z = constant $\Rightarrow$ dz = 0

$\therefore$ (1) becomes

$$xdy - ydx = 0 \Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$$

$\Rightarrow$ log y - log x = constant

$\Rightarrow$ $\log \frac{y}{x}$ = constant $\Rightarrow \frac{y}{x}$ = constant = $f$ (z) (say) ...(2)

$\Rightarrow$ $\frac{xdy - ydx}{x^2} = f'$ (z) dz

$\Rightarrow$ xdy - ydx - $x^2 f'$ (z) dz = 0 ...(3)

Compare (1) and (3), we get

$$-x^2 f'(z) = -2x^2 z \Rightarrow f'(z) = 2z$$

$\Rightarrow$ $f'$ (z) = $z^2$ + c $\Rightarrow \frac{y}{x} = z^2 + c$

$\Rightarrow$ y = x($z^2$ + c), is the require sol.

**Example 12.** Solve : 2yzdx + zxdy - xy (1+ z) dz = 0

**Sol.** The given equation is

2yzdx + zxdy - xy (1+ z) dz = 0 ....(1)

First of all check that (1) is integrable by using condition of integrability

Now put z = constant $\therefore$ dz = 0

$\therefore$ (1) becomes

$$2yzdx + zxdy = 0 \Rightarrow 2ydx + xdy = 0$$

$\Rightarrow$ $\frac{2dx}{x} + \frac{dy}{y} = 0 \Rightarrow$ 2 log x + log y = log (const.)

$\Rightarrow$ log $x^2$y = log (constant) $\Rightarrow$ $x^2$ y = constant $f$ (z) (say) ...(2)

$x^2$dy + y. 2xdx = $f'$ (x) dz $\Rightarrow$ 2xydx + $x^2$dy = $f'$ (z) dz

$\Rightarrow$ 2y dx + x dy = $\frac{f'(z)}{x}$ dz

$\Rightarrow$ 2yzdx + xzdy = $\frac{z}{x} f'$ (z) dz ...(3)

Compare (1) and (3), we get

$\frac{z}{x} f'$ (z) = xy (1 + z)

$\Rightarrow$ $f'$ (z) = $\frac{x^2 y}{z}$ (1 + z) = $x^2y \left(\frac{1}{z}+1\right) = f$ (z) $\left(\frac{1}{z}+1\right)$

$\Rightarrow$ $\frac{f'(z)}{f(z)} = 1 + \frac{1}{z}$

Integrating, we get log $f$ (z) = z + log z + log c

$\Rightarrow$ log $\frac{f(z)}{cz}$ = z $\Rightarrow \frac{f(z)}{cz} = e^z \Rightarrow f$ (z) = $cze^z$

$\Rightarrow x^2y = cze^z$ is the required solution.

**Self-check Exercise-3**

Q. 1 Solve

$xz^3$ dx - zdy + 2 y dz = 0

Q. 2 Verify that the equation is

z (z + $y^2$) dx + z (z + $x^2$) dy - xy (x +y) = 0

is ingegrable and find its primitive

Q. 3 Solve

$z^2$ dx + $cz^2$ - 2yz) dy + ($2y^2$ - yz - xz) dz = 0

## 15.8 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined Pfaffian differential form, Pfaffian differential equation and total differential equation or Pfaffian differential equation in three variables.
2. Find the necessary and sufficient condition for the integrability of single differential equation Pdx + Qdy + Rdz = 0, where P, Q, R are functions of x, y, z.
3. Discuss Inspection method and method of Auxiliary equation to solve the equation Pdx + Qdy + Rdz = 0 and performed some questions related to it.
4. Discussed the method of solving Pdx + Qdy + Rdz = 0 by taking one variable constant and some examples related to it are also given.

## 15.9 Glossary:

1. The expression $\sum_{i=1}^{n} Fi$ ($x_1$, $x_2$, ....., $x_n$) $dx_i$, in which Fi, i = 1, 2, ....., n are functions of some or all of the n independent variables $x_1$, $x_2$, ....., $x_n$ is called Pfaffian differential form in n variables.

2. An equation of the form

$\sum_{i=1}^{n} Fi$ $dx_i = 0$, is called a Pfaffian differential equation in n variable $x_1, x_2, \ldots, x_n$ where Fi, i = 1, 2, .... , n, are function of some or all of the n independent variables $x_1, x_2, \ldots, x_n$.

3. An equation of the form

Pdx + Qdy + Rdz = 0

where P, Q, R are function of x, y, z is called the total differential equation or single differential equation or pfaffian differential equation in three variable x, y and z.

## 15.10 Answer to Self-Check Exercise

### Self-Check Exercise-1

Ans.1 $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right) = 0$

integrable

Ans. 2 $x^2 + y^2 + z^2 = cx$ is the required solution, where c is an arbitrary constant

Ans. 3 $\frac{y}{x} - z^2 = c$

### Self-Check Exercise-2

Ans.1 $xy + yz + zx = c(x + y + z)$

Ans. 2 $(x + y) z = c (x + z)$

### Self-Check Exercise-3

Ans. 1 $2y = x^2z^2 + 2cz^2$, where c is an arbitrary constant

Ans. 2 $x (x + y^2) = z (x + y) (1 - cy)$

Ans. 3 $z (x + y) - y^2 = cz^2$

## 15.11 References/Suggested Readings

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. Zill, D., A First Course in Differential Equation with Applications, 2nd Ed., Prindle, Weber, Schmidt, Boston, 1982.
3. Boyce, W. and Diprima, R., Elementary Differential Equations and Boundary Value Problems, 3rd Ed., Wiley, New York, 1977.

## 15.12 Terminal Questions

1. Check the integrability of the differential equation

$zydx - zxdy - y^2dz = 0$

2. Varify that the equation is integrable and find the solution of

   $2yzdx + zxdy - xy\,(1 + z)\,dz = 0$

3. Verify that the equation is integrable and find the solution of

   $(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0$

4. Solve

   $(y^2 + yz)\,dx + (z^2 + zx)\,dy + (y^2 - xy)\,dz = 0$

5. Solve

   $3x^2\,dx + 3y^2dy - (x^3 + y^3 + ez)\,dz = 0$

6. Find $f\,(y)$ if

   $f\,(y)\,dx - zxdy - xy \log y\,dz = 0$ is integrable. Find the corresponding integral.

7. Solve

   $3x^2\,dx + 3y^2dy - (x^3 + y^3 + e^{2z})\,dz = 0$

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## Unit - 16

# Basic Concepts and Formation of Partial Differential Equations

**Structure**



## 16.1 Introduction

Partial differential equation (PDEs) are mathematical equations that involve multiple variables and their partial derivatives. They are used to describe a wide range of physical phenomena and mathematical models in various scientific and engineering fields, including physics, engineering, finance and biology. In contrast to ordinary differential equations (ODEs), which involve only one independent variable, PDEs involve multiple independent variables. A PDE typically relates the rates of change of a dependent variable its partial derivatives with respect the independent variables. The dependent variable can be a scalar function, a vector function, or even a function of higher dimension. PDEs can be classified into different types based on their order, linearity, and the number of independent variables involved. Common types of PDEs include elliptic equations, parabolic equations and hyperbolic equations.

## 16.2 Learning Objectives

After studying this unit, you should be able to:-

- Define partial differential equation, order and degree of partial differential equations.

- Give Classification of first order partial differential equations.
- Discuss the method of formation of partial differential equation by elimination of arbitrary constants.

## 16.3 Some Definitions

**Partial Differential form**

If a differential equation contains one or more partial derivatives of an unknown function of two or more independent variables, then it is called a partial differential equation. In case of two independent variables and one dependent variable, we shall usually denote independent variables by x, y and dependent variable by z. If z is a function of more than two independent variables, then we denote them by $x_1, x_2, \ldots, x_n$.

The first order partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are denoted by p and q respectively. In case of n variables, these are denoted by $p_1, p_2, \ldots, p_n$. The second order partial derivatives $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ are denoted by r, s and t respectively.

The following are some of the examples of partial differential equations :-

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x + xy$

(ii) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\frac{\partial z}{\partial x}$

(iii) $z\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x$

(iv) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

(v) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/2}$

(vi) $y\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = z\frac{\partial z}{\partial y}$

**Order of a Partial Differential Equation**

The order of a partial differential equation is defined as the order of the highest partial derivative occurring in the partial differential equation.

Equations (i), (iii), (iv) and (vi) written above are of first order, (v) is of second order and (ii) is of the third order.

**Degree of a partial Differential Equation**

The degree of a partial differential equation is the power of the highest order derivative which occurs in it after making the equation free from radicals and fractions in its derivatives.

Equations (i), (ii), (iii) and (iv) written above are of first degree while equations (iv) and (vi) are of second degree.

## 16.4 Classification of First Order Partial Differential Equations.

**Linear Partial Differential Equation**

A partial differential equation is said to be Linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied together in the differential equation.

A first order partial differential equation $f$ (x, y, z, p, q) = 0 is said to be linear if it is linear in p, q and z i.e. if the given equation is of the form P(x, y) p + Q(x, y) q = R(x, y) z + S(x, y)

For example, the equation $x^2y\ p + xy^2\ q = xyz + x^2\ y^3$ and p + q = x + xy are both first order linear partial differential equations.

**Semi-linear Partial Differential Equation**

A first order partial differential equation $f$ (x, y, z, p, q) = 0 is said to be linear if it is linear in p, q and the coefficients i.e. if the given equation is of the form P(x, y) P + Q (x, y) q = R (x, y, z).

For example, one equations $xyp + x^2yq = x^2y^2z^2$ and $yp + xq = \frac{x^2z^2}{y^2}$ are both first order semi-linear partial differential equations.

**Quasi-Linear Partial Differential Equation**

A first order partial differential equation $f$(x, y, z, p, q) = 0 is said to be quasi-linear if it is linear in p and q i.e. if the given is of form.

P(x, y, z) p = Q (x, y, z) q = R (x, y, z)

For example, the equations $x^2z\ p + y^2zq = xy$ and $(x^2 - yz)\ p + (y^2 - zx)\ q = z^2 - xy$ are of first order quasi-linear partial differential equations.

**Non-linear Partial Differential Equation**

A first order partial differential equation $f$(x, y, z, p, q) = 0 which does not come under the above types is known as non-linear partial differential equation.

For example, the equations $p^2 + q^2 = 1$, pq = z and $x^2p^2 + y^2q^2 = z^2$ are all non-linear partial differential equations.

## 16.5 Formation of a Partial Differential Equation

In general, there are two method of forming a partial differential equation depending on the given relation between the variables. The two methods are : -

(i) By eliminating, arbitrary constants from the given relation between variables.

(ii) By eliminating arbitrary functions from the given relation between variables.

## 16.6 Method to Form a Partial Differential Equation by Elimination of Arbitrary Constants

Let z be a function of two independent variables x and y defined by

$f(x, y, z, a, b) = 0$ ...(1)

where a and b are arbitrary constants differentiating (1) partially w.r.t. x and y, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

i.e. $\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} = 0$ ...(2)

and $\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} = 0$ ...(3)

Eliminating a and b from (1), (2) and (3), we get an equation of the form

$F(x, y, z, p, q) = 0$

which is the required partial differential equation. The order of this partial differential equation is one.

**Note :** If the number of arbitrary constants is more than the number of independent variables, then the partial differential equation then obtained will be of hither order than the first.

But if the number of arbitrary constants is less than the number of independent variables, then we shall get more than one differential equation of order one.

For example, if z = K x + y, then we have

$$p = K = \frac{z - y}{x} \text{ i.e. } p = \frac{z - y}{x}$$

and q = 1

**Working Rule for solving problems**

1. Differential partially the given equation w.r.t. x and y.

2. Eliminate the arbitrary constants from the given equation and the two new relations. we shall get the required partial differential equation.

To clarify what we have just said, consider the following examples:-

**Example 1 :** Form a partial differential equation by eliminating a, b from

(i) $z = (x + a)(y + b)$

(ii) $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

**Sol. :** (i) We have

$$z = (x + a)(y + b) \qquad ...(1)$$

$$= xy + xb + ya + ab$$

Differentiating both sides partially w.r.t. x and y, we get

$$p = \frac{\partial z}{\partial x} = y + b \text{ and } q = \frac{\partial z}{\partial y} = x + a$$

Putting in the given equation (1), we get

$$z = qp$$

as the required partial differential eqution.

(ii) We have

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad ...(1)$$

Differentiating both sides partially w.r.t x and y, we get

$$q \frac{\partial z}{\partial x} = \frac{2x}{a^2}$$

$$\text{i.e. } \frac{\partial z}{\partial x} = \frac{x}{a^2}$$

$$\text{i.e. } p = \frac{x}{a^2}$$

$$\Rightarrow \quad a^2 = \frac{x}{p}$$

and

$$2 \frac{\partial z}{\partial y} = \frac{\partial y}{b^2} \quad \text{i.e. } \frac{\partial z}{\partial y} = \frac{y}{b^2}$$

$$\text{i.e.} \quad q = \frac{y}{b^2} \Rightarrow b^2 = \frac{y}{q}$$

Putting in (1), we get

$$2z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$\Rightarrow$ $2z = px + qy$

as the required partial differential equation.

**Example 2 :** Form a partial differential equation by eliminating a and b from the equation

$$(x - a)^2 + (y - b)^2 + z^2 = K^2$$

**Sol. :** We have

$$(x - a)^2 + (y - b)^2 + z^2 = K^2 \quad ...(1)$$

Diff. both sides partially w.r.t. x and y, we get

$$2(x - a) + 2z\frac{\partial z}{\partial x} = 0 \quad ...(2)$$

i.e. $x - a + zp = 0$

and $2(y - b) + 2z\dfrac{\partial z}{\partial y} = 0$ ... (3)

From (2) and (3), we have

$\therefore$ $x - a = -pz$

and $y - b = -qz$

Putting in (1), we get

$$p^2z^2 + q^2z^2 + z^2 = K^2$$

$\Rightarrow$ $z^2(1 + p^2 + q^2) = K^2$

is the required partial differential equation.

**Example 3 :** Form a partial differential equation by eliminating arbitrary constant from the equation $z = ax^2 + by^2 + ab$

**Sol. :** We have

$$z = ax^2 + by^2 + ab \quad ...(1)$$

Diff. both sides partially w.r.t. x and y, we get

$$p = \frac{\partial z}{\partial x} = 2ax$$

$\Rightarrow$ $a = \dfrac{p}{2x}$

and

$$q = \frac{\partial z}{\partial y} = 2by$$

$$\Rightarrow \quad b = \frac{q}{2y}$$

Putting in (1), we get

$$z = \frac{p}{2x} \cdot x^2 + \frac{q}{2y} \; y^2 + \frac{p}{2x}\frac{q}{2y}$$

$$= \frac{px}{2} + \frac{qy}{2} + \frac{pq}{4xy}$$

$\Rightarrow$ $4xyz = 2px^2y + 2qxy^2 + py$, as the required partial differential equation.

**Example 4 :** Form a partial differential equation by eliminating arbitrary constants from the equation

$$z = ax + by + c\,xy$$

**Sol. :** We have

$$z = ax + by + c\,xy \qquad ...(1)$$

Diff. partially both sides w.r.t. x and y, we get

$$p = \frac{\partial z}{\partial x} = a + cy \,, \quad q = \frac{\partial z}{\partial y} = b + cx$$

$$r = \frac{\partial^2 z}{\partial x^2} = 0 + 0, \qquad t = \frac{\partial^2 z}{\partial y^2} = 0,$$

$$S = \frac{\partial^2 z}{\partial x \partial y} = c$$

Now $z = ax + by + cxy$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = sxy = ax + cxy + by + cxy - cxy$$

$$= ax + by + cxy = z$$

$\therefore$ $z = x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} - xy\dfrac{\partial^2 z}{\partial x \partial y}$, as the required partial differential equation.

**Example 5 :** Show that the differential equation of all cones which have their vertex at the origin is $px + qy = z$

verify that $yz + zx + xy = 0$ is a surface satisfying the above equation.

**Sol. :** The equation of any cone with vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2\,hxy = 0 \qquad ...(1)$$

where a, b, c, g, h are constants

Diff. (1) partially w.r.t. x and y, we have

$$2ax + 2czp + 2fyp + 2g\,(px + z) + 2hy = 0$$

or $\quad ax + gz + hy + p\,(cz + gx + fy) = 0 \qquad ...(2)$

and $\quad 2by + 2(zq + 2f\,(yq + z) + 2gxq + 2\,hx = 0$

or $\quad by + fz + hx + q\,(cz + fy + gx) = 0 \qquad ...(3)$

Multiplying (2) by x and (3) by y and adding, we get

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)\,(px + 2y) = 0$$

$$-\,(cz^2 + fyz + gxz) + (cz + fy + gx)\,(px + qy) = 0$$

[By (1)]

or $\quad (cz + fy + gx)\,(px + qy - z) = 0$

or $\quad px + qy - z = 0 \qquad ...(4)$

which is required differential equation.

Given surface is

$$yz + zx + xy = 0 \qquad ...(5)$$

Diff. (5) partially w.r.t. x and y by turn, we get

$$up + px + z + y = 0 \text{ and } z + qy + xq + x = 0 \qquad ...(6)$$

Solving (6) for p and q,

$$p = -\,(z + y)/\,(x + y)\ ;\ q = -(z + x)/\,(x + y)$$

$\therefore \quad px + qy - z = -\dfrac{x(x+y)}{x+y} - \dfrac{y(z+x)}{x+y} - z$

$$= -\frac{2(xy+y+zx)}{x+y}$$

$$= 0\ ,\ \text{by (5)}$$

Hence (5) is a surface satisfying (4).

**Example 6 :** Find the partial differential equation of all planes which are at a constant distance 'a' from the origin

**Sol. :** Let $lx + my + nz = a \qquad ...(1)$

be the equation of a plane where l, m, n are d.c.'s of the normal to the plane.

Differentiating (1) partially w.r.t. x and y, we get l + np = 0   adnmn+ nq = 0

Also   $l^2 + m^2 + n^2 = 1$

$\therefore$   $(-np)^2 + (-nq)^2 + n^2 = 1$

Or   $(p^2 + q^2 + 1)\, x^2 = 1$

Or   $n = \frac{1}{\sqrt{p^2+q^2+1}}$   (Assuming n > 0)

$\therefore$   $l = -np = -\frac{p}{\sqrt{p^2+q^2+1}}$

$m = -nq = -\frac{q}{\sqrt{p^2+q^2+1}}$

Putting the values of l, m and n in (1), we get

$$-\frac{px}{\sqrt{p^2+q^2+1}} - \frac{qy}{\sqrt{p^2+q^2+1}} + \frac{z}{\sqrt{p^2+q^2+1}} = a$$

or   $z = px + qy + a\sqrt{p^2+q^2+1}$

**Self-check exercise**

Q. 1   Form a partial differential equation by eliminating a, b from

z = ax + by + ab

Q. 2   Form a partial differential equation by eliminating arbitrary constants from

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Q. 3   Find the partial differential equation of all sphere of radius 5 and having their centres in the xy - plane.

## 16.7 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Defined partial differential equation; order and degree of a partial differential equations with examples.
2. Discussed classification of first order partial differential equations i.e. discussed linear; Semi-linear; Quasi-linear and Non-linear partial differential equations.
3. Discussed the method of form a partial differential equation by elimination of arbitrary constants. Some examples are also given to clarify this method.

## 16.8 Glossary:

1. If a differential equation contains one or more partial derivatives of an unknown function of two or more independent variables, then it is called a partial differential equation.
2. A partial differential equation is said to be linear if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied together in the differential equation.

## 16.9 Answer to Self-Check Exercise

**Self-Check Exercise-1**

Ans.1 z = px + qy + pq

Ans. 2 $zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0$

and $xy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - \frac{\partial z}{\partial y} = 0$ are two

possible forms of the required equations.

Ans. 3 $Z^2\left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right] = 25$

## 16.10 References/Suggested Readings

1. Shepley L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984.
2. I, Sneddon, Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.
3. Sharma, J.N. and Singh, K., Partial Differential Equations for Engineers and Scientists, Narosa Publishing House.

## 16.11 Terminal Questions

1. Form a partial differential equation by eliminating arbitrary constants from

   $z = ax + a^2y^2 + b$
2. Find the partial differential equation of planes having equal x and y intercepts.
3. Find the differential equation of all spheres whose centre lies on z-axis.
4. Find the differential equation of the family of spheres of reduce 7 with centres on the planes x - y = 0.

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**Unit - 17**

# Formation of Partial Differential Equations by Elimination of Arbitrary Functions

**Structure**



## 17.1 Introduction

The formation of a partial differential equation involves determining the mathematical relationship between the variables and their partial derivatives that govern the behaviour of a system. This process typically involves combining physical laws, conservation principles and empirical observations to establish a mathematical model. By eliminating the arbitrary functions through additional equations or constraints, we can derive the desired partial differential equation that governs the behaviour of the system. The process of eliminating arbitrary functions from a PDE involves finding relationships among the partial derivatives such that the arbitrary functions can be expressed in terms of the known variables.

## 17.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss method to form a partial differential equation by elimination of arbitrary functions.
- Perform questions related to formation of partial differential equations by elimination of arbitrary functions.

## 17.3 Method To Form A Partial Differential Equation By Elimination Of Arbitrary Functions.

Let u and v be two functions of x, y, z which are connected by the relation

$f(u, v) = 0$ ......(1)

where $f$ is an arbitrary function.

Regarding z as a function of x, y and differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0 \quad ......(2)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

i.e. $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right) = 0 \quad ......(3)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}.q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right) = 0 \quad .....(4)$$

From (3), $$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial z}.p}{\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}.p}$$

From (4), $$\frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}.q}{\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}.q}$$

$$\therefore \quad \frac{\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial z}.p}{\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial z}.p} = \frac{\dfrac{\partial v}{\partial y}+\dfrac{\partial v}{\partial z}.q}{\dfrac{\partial u}{\partial y}+\dfrac{\partial u}{\partial z}.q}$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p + \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}.\frac{\partial x}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad Pp + Qq = R \quad .....(5)$$

where $P = \dfrac{\partial u}{\partial y}.\dfrac{\partial v}{\partial z} - \dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial z} = \dfrac{\partial(u,v)}{\partial(y,z)}$

$Q = \dfrac{\partial u}{\partial z}.\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial x} = \dfrac{\partial(u,v)}{\partial(z,x)}$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial v}{\partial x} = \frac{\partial(u, v)}{\partial(x, y)}$$

is the required partial differential equation.

To clarify what we have just said, consider the following examples:-

**Example 2:** Form partial differential equations by eliminating the arbitrary functions from the following equations:

(i) $z = F(x^2 + y^2)$ (ii) $z = f\left(\frac{y}{x}\right)$ (iii) $f(x^2 + y^2, z - xy) = 0$

(iv) $f(x^2 + y^2 + z^2) = ax + by + cz$ (v) $f(x + y + z, x^2 + y^2 - z^2) = 0$

**Sol:** (i) We have $z = F(x^2 + y^2)$ ....(1)

Differentiating (1) partially w.r.t. 'x'. and 'y', we get

$$\frac{\partial z}{\partial x} = F'(x^2 + y^2)(2x) \qquad .....(2)$$

$\Rightarrow$ $p = F'(x^2 + y^2)(2x)$

and $\frac{\partial z}{\partial y} = F'(x^2 + y^2)(2y)$

$\Rightarrow$ $q = F'(x^2 + y^2)(2y)$ .....(3)

Dividing (2) by (3), we get

$\frac{p}{q} = \frac{x}{y} \Rightarrow py - qx = 0$, is the required partial differential equation.

(ii) We have $z = f\left(\frac{y}{x}\right)$ .....(1)

Differentiating (1) w.r.t. 'x'. and 'y', partially, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) \qquad ....(2)$$

and $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$ .....(3)

Dividing (2) and (3), we get

$$\frac{\partial z / \partial x}{\partial z / \partial y} = -\frac{y}{x} \Rightarrow px = -qy$$

$\Rightarrow$ $px + qy = 0$ is the reqd. partial differential equation.

(iii) The given result is

$f(x^2 + y^2, z - xy) = 0$ ......(1)

Put $u = x^2 + y^2$ and $v = z - xy$

$\therefore$ We have

$f(u, v) = 0$ .....(2)

Differentiating (2) partially w.r.t. 'x'. and 'y', we get

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial u}.2x + \frac{\partial f}{\partial v}(-y + 1.p) = 0 \quad \Rightarrow \quad \frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{p-y}{2x} \qquad ....(3)$$

$$\text{and} \quad \frac{\partial f}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial y}\right) = 0 \Rightarrow \frac{\partial f}{\partial u}.2y + \frac{\partial f}{\partial v}(-x + 1.q) = 0$$

$$\Rightarrow \quad \frac{\partial f/\partial u}{\partial f/\partial v} = -\frac{q-x}{2y} \qquad .....(4)$$

From (3) and (4), we get

$$-\frac{p-y}{2x} = \frac{q-x}{2y} \Rightarrow py - y^2 = qx - x^2$$

$\Rightarrow$ $qx - py = x^2 - y^2$ is the required partial differential equation

(iv) We have

$f(x^2 + y^2 + z^2) = ax + by + cz$ .....(5)

Differentiating (1) both sides partially w.r.t. 'x' and 'y', we get

$$f'(x^2 + y^2 + z^2) = \left(2x + 2z\frac{\partial z}{\partial x}\right) = a + c\frac{\partial z}{\partial x}$$

$$f'(x^2 + y^2 + z^2) = \left(2y + 2z\frac{\partial z}{\partial y}\right) = b + c\frac{\partial z}{\partial y}$$

Dividing, we get

$$\frac{x+pz}{y+qz} = \frac{a+cp}{b+cq}$$

$\Rightarrow$ $bx + bpz + cqx + cpqz = ay + aqz + cpy + cpqz$

$\Rightarrow$ $(bz - ay)p + (cx - az)q = ay - bx$

is the regd. partial differential equation.

(v) The given result is

$$f(x + y + z, x^2 + y^2 - z^2) = 0 \qquad .....(1)$$

Put $u = x + y + z$, $v = x^2 + y^2 - z^2$ .....(2)

$\therefore$ We have $f(u, v) = 0$

Differentiating (2) partially w.r.t. 'x' and 'y' we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right) = 0$$

$\Rightarrow$ $\frac{\partial f}{\partial u}(1 + 1.p) + \frac{\partial f}{\partial v}(2x - 2zp) = 0$

Similarly $\frac{\partial f}{\partial u}(1 + 1.q) + \frac{\partial f}{\partial v}(2y - 2zq) = 0$

Eliminating $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$, we get $\begin{vmatrix} 1+p & 2x-2zp \\ 1+q & 2y-2zq \end{vmatrix} = 0$

$\Rightarrow$ $2y - 2zq + 2yp - 2pzq = 2x - 2zq - 2zpq$

Dividing throughout by 2 and cancelling - zpq from both sides, we get

$\Rightarrow$ $p(y + z) - (x + z)q = x - y$, which is the reqd. partial differential equation.

Find partial differential equations by eliminating arbitrary functions from the following relations: (Examples, 2 - 4)

**Example 3:** $z = xy + f(x^2 + y^2)$

**Sol:** $z = xy + f(x^2 + y^2)$ .....(1)

$\therefore$ $p = \frac{\partial z}{\partial x} = y + f'(x^2 + y^2)\,2x$ .....(2)

$q = \frac{\partial z}{\partial y} = x + f'(x^2 + y^2)\,2y$ .....(3)

From (2) and (3), we get $p - y = f'(x^2 + y^2)2x$ $q - x = f'(x^2 + y^2).2y$

Dividing, we get $\frac{p-y}{q-x} = \frac{x}{y} \Rightarrow py - y^2 = qx - x^2$

$\Rightarrow$ $py - qx = y^2 - x^2$ is the required partial differential equation.

**Example 3:** $z = x + y + f(xy)$

**Sol.** Since $z = x + y + f(xy)$ ...(1)

$\therefore \quad p = \dfrac{\partial z}{\partial x} = 1 + f'(xy) \,.\, y$ .....(2)

$q = \dfrac{\partial z}{\partial y} = 1 + f'(xy) \,.\, x$ .....(3)

i.e. $\quad p - 1 = f'(xy) \,.\, y \; q - 1 = f'(xy). x, \; \therefore \dfrac{p-1}{q-1} = \dfrac{y}{x}$

$\Rightarrow \quad px - x = qy - y \Rightarrow px - qy = x - y$, is the required partial differential equation.

**Example 4 :** $z = f(x^2 - y) + g(x^2 + y)$

**Sol.** Since $z = f(x^2 - y) + g(x^2 + y)$

$\therefore \quad \dfrac{\partial z}{\partial x} f'(x^2 - y)2x + g'(x^2 + y)\, 2x,$

$\dfrac{\partial z}{\partial y} f'(x - y)(-1) + g'(x^2 + y).1$

$\dfrac{\partial^2 z}{\partial x^2} = f'(x^2 - y).\, 2 + 2x\, f''(x^2 - y).\, 2x + g'(x^2 + y)\, 2 + 2x\, g''(x^2 + y)(2x)$

$= 2f'(x^2 - y) + 4x^2 f''(x^2 - y) + 2g'(x^2 + y)\, 4x^2\, g''(x^2 + y)$

$\dfrac{\partial^2 z}{\partial y^2} = (-1)\, f''(x^2 - y)(-1) + g''(x^2 + y) = f''(x^2 - y) + g''(x^2 + y)$

Now $\quad \dfrac{\partial^2 z}{\partial x^2} = 4x^2\, [f'(x^2 - y) + g''(x^2 + y)] + 2[\, f'(x^2 - y) + g'(x^2 + y)]$

$= 4x^2 \dfrac{\partial^2 z}{\partial y^2} + 2.\, \dfrac{\frac{\partial z}{\partial x}}{2x} = 4x^2 \dfrac{\partial^2 z}{\partial y^2} + \dfrac{1}{x}\, \dfrac{\partial z}{\partial x}$

$\therefore \quad x \dfrac{\partial^2 z}{\partial y^2} = 4x^3 \dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x}$

**Example 5** Form partial differential equations by eliminating arbitrary functions from the following relations :

(i) $\quad z = f(y + ax) + g(y + bx),\ a \neq b$

(ii) $\quad z = f(xy) + g(x / y)$

**Sol.** (i) We have

$z = f(y + ax) + g(y + bx)$ ...(1)

$\therefore \quad \dfrac{\partial z}{\partial x} = f'\,(y + ax)\,.\,a + g'\,(y + bx)\,.\,b$

$$\frac{\partial^2 z}{\partial x^2} = f''\,(y + ax)a\,.\,a + g''\,(y + bx)\,.b\,.\,b = a^2 f''\,(y + ax) + b^2\,g''\,(y + bx)$$

Again $\dfrac{\partial z}{\partial y} = f'\,(y + ax)\,.\,1 + g'\,(y + bx)\,.\,1$

and $\dfrac{\partial^2 z}{\partial y^2} = f''\,(y + ax) + g''\,(y + bx)$

$$\frac{\partial^2 z}{\partial y \partial x} = f''\,(y + ax).a + g''\,(y + bx)\,b$$

$\therefore \quad r = f''\,(y + ax).a^2 + g''\,(y + bx)\,b^2\,,\ s = f''\,(y + ax).a + g''\,(y + bx)b$

$t = f''\,(y + ax) + g''\,(y + bx)$

Eliminating $f''\,(y + ax)$, $g''\,(y + bx)$ from these three equations, we get

$$\begin{vmatrix} r & a^2 & b^2 \\ s & a & b \\ t & 1 & 1 \end{vmatrix} = 0 \Rightarrow r\,(a - b) - s\,(a^2 - b^2) + t\,(a^2 b - ab^2) = 0$$

$\Rightarrow \quad r\,(a - b) - s\,(a - b)\,(a + b) + t\,ab\,(a - b) = 0$

$\Rightarrow \quad r - (a + b)s + tab = 0$, is the required partial differential equation.

(ii) We have

$$z = f\,(xy) + g\left(\frac{x}{y}\right) \therefore \frac{\partial z}{\partial x} = f'\,(xy)\,.\,y + g'\left(\frac{x}{y}\right).\frac{1}{y}$$

$$\frac{\partial^2 z}{\partial x^2} = f''\,(xy)\,.\,y^2 + g''\left(\frac{x}{y}\right).\ \frac{1}{y^2}$$

Again $\dfrac{\partial z}{\partial y} = f'\,(xy)\,.\,x + g'\left(\dfrac{x}{y}\right) - \dfrac{x}{y^2}$

$$\frac{\partial^2 z}{\partial y^2} = f''\,(xy)\,.\,x^2 + g''\left(\frac{x}{y}\right).\ \frac{x^2}{y^4} + g'\left(\frac{x}{y}\right).\frac{2x}{y^3}$$

$\therefore \quad x^2 r = f''\,(xy)\,x^2\,y^2 + g''\left(\dfrac{x}{y}\right).\dfrac{x^2}{y^2}$

$$y^2t = f''(xy)\,x^2y^2 + g''\left(\frac{x}{y}\right).\frac{x^2}{y^2} + g'\left(\frac{x}{y}\right).\frac{2x}{y}$$

$$x^2r - y^2t = -\ g'\left(\frac{x}{y}\right)\frac{2x}{y}$$

Also, $xp - yq = xyf'(xy) + \frac{x}{y}\ g'\left(\frac{x}{y}\right) - xyf'(xy) + \frac{x}{y}\ g'\left(\frac{x}{y}\right) = 2\frac{x}{y}\ g'\left(\frac{x}{y}\right)$

$$x^2r - y^2t = -(xp - yq) \Rightarrow x^2r - y^2t + xp - yq = 0$$

**Example 6 :** Find partial differential equation by eliminating arbitrary functions from the relation.

$$z = f(x + iy)\ .\ + g(x - iy)$$

**Sol.** Since $z = f(x + iy)\ .\ + g(x - iy)$

$$\therefore \quad p = \frac{\partial z}{\partial x} = f'(x + iy) + g'(x - iy)$$

$$q = \frac{\partial z}{\partial y} = f'(x + iy)(i) + g'(x - iy)(-i)$$

$$r = \frac{\partial^2 z}{\partial x^2} = f''(x + iy) + g''(x - iy)$$

$$t = \frac{\partial^2 z}{\partial y^2} = f''(x + iy)(i)^2 + g''(x - iy)(-i)^2$$

$$= -f''(x + iy) - g''(x - iy)$$

$\therefore$ $r + t = 0 \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, is the required partial differential equation.

**Example 7 :** Find the partial differential equation from the relation by eliminating arbitrary functions.

$$z = f(x\cos x + y\sin x - at) + g(x\cos x + y\sin x + at)$$

**Sol. :** We have

$$z = f(x\cos x + y\sin x - at) + g(x\cos x + y\sin x + at) \qquad ...(1)$$

Diff. (1) on both sides partially w.r.t. x, y and t, we get

$$\frac{\partial z}{\partial x} = f'(x\cos x + y\sin x - at)\cos x +$$

$$+ g'(x\cos x + y\sin x + at)\cos x$$

$$\frac{\partial^2 z}{\partial x^2} = f'' \text{ (x cos x + y sin x - at) } \cos^2 x +$$

$$+ g'' \text{ (x cos x + y sin x + at) } \cos^2 x$$

Similarly,

$$\frac{\partial^2 z}{\partial y^2} = f'' \text{ (x cos x + y sin x - at) } \sin^2 x +$$

$$+ g'' \text{ (x cos x + y sin x + at) } \sin^2 x$$

$$\frac{\partial^2 z}{\partial t^2} = f'' \text{ (x cos x + y sin x - at) (-a) (-a) } +$$

$$+ g'' \text{ (x cos x + y sin x + at) (a) (a)}$$

$$= f'' \text{ (x cos x + y sin x - at) } a^2 +$$

$$+ g'' \text{ (x cos x + y sin x + at) } a^2$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} f'' \text{ (x cos x + y sin x - at) } (\cos^2 x +$$

$$+ \sin^2 x) + g'' \text{ (x cos x + y sin x + }$$

$$\text{at) } (\cos^2 x + \sin^2 x)$$

$$= f'' \text{ (x cos x + y sin x - at) } + g'' \text{ (x cos x + y sin x + at)}$$

Also $$\frac{\partial^2 z}{\partial t^2} = a^2 [f'' \text{ (x cos x + y sin x - at) } +$$

$$+ g'' \text{ (x cos x + y sin x + at)]}$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial t^2} = a^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{a^2}\frac{\partial^2 z}{\partial t^2}$$

**Example 8 :** Form partial differential equation by eliminating arbitrary functions from the relation $z = f(x + ay) + g(x - ay)$.

**Sol. :** We are given

$$z = f(x + ay) + g(x - ay) \qquad \text{....(1)}$$

Diff. (1) partially w.r.t. x and y, we have

$$\frac{\partial z}{\partial x} = f'(x + ay)(1 + 0) + g'(x - ay)(1 - 0)$$

or $\dfrac{\partial z}{\partial x}$ = $f'$ (x + ay) + g' (x - ay) ...(2)

and $\dfrac{\partial z}{\partial y}$ = $f'$ (x + ay) (0 + a) + g' (x - ay) (0 - a)

or $\dfrac{\partial z}{\partial y}$ = a [$f'$ (x + ay) - g' (x - ay)] ....(3)

Again diff. (2) partially w.r.t. x and (3) w.r.t. y, we have

$\dfrac{\partial^2 z}{\partial x^2}$ $f''$ (x + ay) (1 + 0) + g'' (x - ay) (1 - 0)

or $\dfrac{\partial^2 z}{\partial x^2}$ $f''$ (x + ay) + g'' (x - ay) ...(4)

and $\dfrac{\partial^2 z}{\partial y^2}$ = a[$f''$ (x + ay) (0 + a) + g'' (x - ay) (0 - a)]

$\therefore$ $\dfrac{\partial^2 z}{\partial y^2}$ = $a^2$ [$f''$ (x + ay) + g'' (x - ay)]

or $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$ [Q of (4)]

$\therefore$ $\dfrac{\partial^2 z}{\partial y^2} = a^2 \dfrac{\partial^2 z}{\partial x^2}$

which is the required partial differential equation.

**Example 9 :** Form differential equation for cone with vertex at A ($\alpha$, $\beta$, $\gamma$) having equation of the form.

$$f\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right) = 0$$

**Sol. :** We are given

$$f\left(\frac{x-\alpha}{z-\gamma}, \frac{y-\beta}{z-\gamma}\right) = 0 \qquad ...(1)$$

Put $\dfrac{x-\alpha}{z-\gamma}$ = u and $\dfrac{y-\beta}{z-\gamma}$ = v

$$\therefore \qquad \frac{\partial u}{\partial x} = \frac{1}{z-y}, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial z} = \frac{x-\alpha}{(z-y)^2}$$

$$\text{and} \qquad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = \frac{1}{z-y}, \quad \frac{\partial v}{\partial z} = \frac{(y-\beta)}{(z-y)^2}$$

Now equation (1) becomes

$$f\,(u, v) = 0 \qquad (2)$$

Diff. (2) w.r.t. x and y, we get

$$\frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right] = 0$$

$$\text{and} \qquad \frac{\partial f}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] + \frac{\partial f}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right] = 0$$

$$\therefore \qquad \frac{\partial f}{\partial u}\left[\frac{1}{z-\gamma}-\frac{x-\alpha}{(z-\gamma)^2}.p\right] + \frac{\partial f}{\partial v}\left[0-\frac{y-\beta}{(z-\gamma)^2}\right] = 0$$

$$\text{and} \quad \frac{\partial f}{\partial v}\left[0-\frac{x-\alpha}{(z-\gamma)^2}\,p\right] + \frac{\partial f}{\partial v}\left[\frac{1}{z-\gamma}-\frac{y-\beta}{(z-\gamma)^2}\,q\right] = 0$$

$$\therefore \qquad \frac{\partial f}{\partial u}\left[\frac{(z-\gamma)-p(x-\alpha)}{(z-\gamma)^2}\right] = \frac{\partial f}{\partial v}\left[\frac{y-\beta}{(z-\gamma)^2}\,q\right] \qquad \text{...(3)}$$

$$\text{and} \qquad \frac{\partial f}{\partial v}\left[\frac{x-\alpha}{(z-\gamma)^2}\,p\right] = \frac{\partial f}{\partial v}\left[\frac{(z-\gamma)-q(y-\beta)}{(z-\gamma)^2}\right] \qquad \text{...(4)}$$

Dividing (3) by (4), we get

$$\frac{(z-\gamma)-p(x-\alpha)}{p(x-\alpha)} = \left[\frac{(y-\beta)q}{(z-\gamma)-q(y-\beta)}\right]$$

or $(z - \gamma)\,(y - \beta)\,(z - \gamma) = pq\,(x - \alpha)\,(y - \beta) - q\,(z - \gamma)$

$(y - \beta) + pq\,(x - \alpha)\,(y - \beta)$

or $p\,(x - \alpha)\,(z - \gamma) + q\,(z - \gamma)\,(y - \beta) = (z - \gamma)^2$

Dividing by $z - \gamma$, we get

$$p\,(x - \alpha) + q\,(y - \beta) = z - \gamma$$

which is the required partial differential equation.

**Self-Check Exercise**

Q. 1 Find partial differential equation by eliminating arbitrary function from the relation

$z = x f(x + t) + g(x + b)$.

Q. 2 The equation of any cone with vertex

at P ($x_0$, $y_0$, $z_0$) i≤ of the form

$$f\left(\frac{x-x_0}{z-z_0}, \frac{y-y_0}{z-z_0}\right) = 0$$

Find the partial differential equation corresponding to it.

Q.3 Form differential equation by eliminating arbitrary functions from the relation

$f(l x + my + nz, x^2 + y^2 + z^2) = 0$

## 17.4 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed in detail the method of formation of partial differential equation by elimination of arbitrary functions.
2. Give examples for the formation of partial differential equations by the method of elimination of arbitrary functions.

## 17.5 Glossary:

The formation of a partial differential equation involves determining the mathematical relationship between the variables and their partial derivatives that govern the behaviour of the system.

## 17.6 Answer to Self-Check Exercise

Ans. 1 $$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} - 2\frac{\partial^2 z}{\partial x \partial t} = 0$$

Ans. 2 $p(x - x_0) + q(y - y_0) = z - z_0$

Ans. 3 $(ny - mz)p + (lz - nx)q = mx - ly$

## 17.7 References/Suggested Readings


1. I, Sneddon, Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.
2. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley and Sons, New York.

## 17.8 Terminal Questions

1. Find partial differential equation by eliminating arbitrary functions from the relation

   $x = f(z) + g(y).$

2. Find the partial differential equation by eliminating arbitrary functions from the relation

   $z = y f(x) + x g(y)$

3. From differential equation by eliminating arbitrary functions from the relation

   $f(x - y), z + \cos x) = 0$

4. Form partial differential equation by eliminating arbitrary functions from the relation

   $f(x + y + z, xyz) = 0$

5. Form partial differential equation by eliminating arbitrary functions from the relation

   $z = f(x^2 - y) + g(x^2 + y)$

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**Unit - 18**

# Solution or Integral of a Partial Differential Equations

**Structure**



## 18.1 Introduction

In the last two units we have discussed method of forming partial differential equations. In this unit we shall discuss some methods to solve a partial differential equation i.e. we shall find a function which satisfies the given partial differential equation. A function which satisfies a partial differential equation is called its solution or integral. In this unit and in next unit, we shall discuss methods to find solutions of partial differential equations of the first order and also linear in p and q.

## 18.2 Learning Objectives

After studying this unit, you should be able to:-

- Discuss four types of solutions of partial differential equation.
- Define and discuss Lagrange's linear equation and Lagrange's method of solving linear partial differential equation of order one i.e. $Pp + Qq = R$.

## 18.3 Solution or Integral of a Partial Differential Equation

A solution or integral or a partial differential equation is a relation between the variables which satisfies the given partial differential equation.

There are four types of solution of partial differential equation.

1. Complete Solution or Complete Integral.

If from the partial differential equation $f$ (x, y, z, p, q) = 0, we can find a relation F (x, y, z, a, b) = 0 which contains as many arbitrary constants as there are independent variables, then the relation F (x, y, z, a, b) = 0 is called Complete Integral or Solution of the given partial differential equation.

e.g., z = (x + a) (y + b) is a complete solution of the partial differential equation

z = pq

**2. Particular Integral (or Solution).**

A solution obtained by giving some particular values to the arbitrary constants in the complete solution of a partial differential equation of the first order is called a particular solution of the partial differential equation.

e.g. z = (z + 2) (y + 3) is a particular solution of the partial differential equation

z = pq

3. Singular Integral (or Solution)

Let $f$ (x, y, z, a, b) = 0 be the complete solution of the partial differential equation F (x, y, z, p, q) = 0

The relation between x, y and z obtained by eliminating the arbitrary constants a and b between the equations.

$$f\,(x, y, z, a, b) = 0,\ \ \frac{\partial f}{\partial a} = 0,\ \ \frac{\partial f}{\partial b} = 0$$

is called Singular Solution of the partial differential equation

F (x, y, z, p, q) = 0, provided it satisfies the given equation.

This solution represents the envelope of the surface represented by the complete solution of the given equation.

The singular solution may or may not be contained in the complete solution of the given partial differential equation.

Rule to find singular integral

The singular integral of the given partial differential equation is obtained by eliminating a and b from

$$f = 0,\ \ \frac{\partial f}{\partial a} = 0,\ \ \frac{\partial f}{\partial b} = 0$$

where $f$ (x, y, z, a, b) = 0 is the complete integral of the given partial differential equation.

e.g. consider z = ax + by + ab as the complete solution of the partial differential equation

z = px + qy + pq

Let $f$ (x, y, z, a, b) = z -ax - by - ab = 0 ...(1)

$\therefore$ $\frac{\partial f}{\partial a}$ = -x - b = 0

i.e. x + b = 0 $\Rightarrow$ b = - x

and $\frac{\partial f}{\partial b}$ = - y - a = 0

i.e. y + a = 0

$\Rightarrow$ a = -y

$\therefore$ (1) gives

z - (-y)x - (-x) y - (-y) (-x) = 0

$\Rightarrow$ z + xy + xy - xy = 0

$\Rightarrow$ z + xy = 0

$\Rightarrow$ z = -xy is Singular Solution of z = px + qy + pq

4. General Integral (or Solution).

Let $f$ (x, y, z, a, b) = 0 be the complete solution of a partial differential equation F(x, y, z, p, q) = 0

Let b = $\phi$ (a)

$\therefore$ $f$[x, y, z, a, $\phi$ (a)] = 0

is a one-parameter family of the surfaces F (x, y, z, p, q) = 0. The relation between x, y and z obtained by eliminating arbitrary constant 'a' between the equation $f$[x, y, z, a, $\phi$ (a)] = 0 and $\frac{\partial f}{\partial a}$ = 0 is called a General Solution of the partial differential equation.

F (x, y, z, p, q) = 0, provided it satisfies the equation\.

This solution represents the envelop of the surfaces represented by the equation $f$(x, y, z, a, $\phi$ (a)] = 0.

**Note.** We can say that general solution of F (x, y, z, p, q) = 0 is the set of equation

$f$ [x, y, z, a, $\phi$ (a)] = 0 ...(1)

and $\frac{\partial f}{\partial a}$ = 0 ...(2)

where $f$ is an arbitrary function if the elimination of 'a' between (1) and (2) is not possible or not easy.

Again if u, v are independent function of x, y, z and $f$ (u, v) = 0 be an arbitrary function of u and v, then elimination of $f$ will give

$$Pp + Qq = R \qquad ...(1)$$

where
$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

(1) is partial differential equation of the first order.

Hence $f$ (u, v) = 0 is a solution of the equation Pp + Qq = R. Since $f$ (u, v) = 0 contains an arbitrary function '$f$'. $\therefore$ it is the general solution of the equation (1).

## 18.4 LAGRANGE LINEAR EQUATION

A partial differential equation of the form Pp + Qq = R

where P, Q, R are functions of x, y, z is called Lagrange's Linear Equation.

It is the standard form of the linear partial differential equation of order one.

e.g. (y + z) p + (z + x) q = x + y

is a Lagrange's Linear Equation.

**Lagrange's Method of Solving the Linear Partial Equation of Order One i.e. of Pp + Qq = R.**

The general solution of the linear partial differential equation

$$Pp + Qq = R \qquad ...(1)$$

(where P, Q and R are functions of x, y and z)

is $f$ (u, v) = 0 ...(2)

where $\phi$ is an arbitrary function and u (x, y, z) = a and v (x, y, z) = b ....(3)

from a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad ...(4)$$

**Proof.** We have already proved that by the elimination of $f$ from (2), we get (1)

[Give its proof here $f$ or complete proof of this]

So, we have to find the values of u, v.

Differentiating (3), we get

$$\frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0$$

and $$\frac{\partial v}{\partial x}\, dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0$$

Solving this $f$ or dx, dy and dz, we get

$$\frac{dx}{\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}}$$

i.e. $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad \text{...(5)}$$

Thus, (5) represents the differential equation whose solutions are u = a and v = b.

$\therefore$ u and v are found out.

Hence the solution of the Lagrange's Linear equation Pp + Qq = R is

$$f\,(u, v) = 0$$

where $f$ is an arbitrary function.

Note. The equation (3) is called Lagrange's auxiliary or subsidiary equation.

**Method to solve Pp + Qq = R. ...(1)**

1. Write down the auxiliary equation for (1) i.e. $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ ...(2)

2. Find two independent integrals of the auxiliary (2) say u = a, v = b

3. Thus general integral or solution of (1) is $f$ (u, v) = 0 or u = $\phi$ (v)

where $f$ or $\phi$ is an arbitrary function.

To clarify this method let us look at the following examples :-

Type - I

In this case, solution of Pp + Qq = R is obtained by taking two members of auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ at a time and then integrating to have two independent solutions in variables whose differentials are involved in equation.

**Example 1 :** Solve the following for general solution.

$$y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = x$$

**Sol. :** Given differential equation is

$$y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = x \qquad ...(1)$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x$$

or $\quad px + qy = x$

comparing it with Pp + Qq = R, we get

$P = x, Q = y, R = x$

$\therefore$ auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} \qquad ...(2)$$

From first two members of (2), we get

$$\frac{dx}{x} = \frac{dy}{y}$$

On integration, we get

$\log x = \log y + \log a$

or $\quad \log x = \log ay$

$\therefore \quad x = ay$

or $\quad \frac{x}{y} = a \qquad ....(3)$

From first and last members of (2), we get

$$\frac{dx}{x} = \frac{dz}{z}$$

or $\quad dz = dx$

On integration, we get

$z = x + b$

or $\quad z - x = b \qquad ...(4)$

From (3) and (4), we get u = a, and v = b where $u(x, y, z) = \frac{x}{y}$ and $v(x, y, z) = z - x$

The general solution of (1) is given by

$f\left(\frac{x}{y}, z-x\right)$ = 0, where $f$ is an arbitrary function.

**Example 2 :** Solve the following for general solution

$x^2 p + y^2 q + z^2 = 0$

**Sol. :** We are given

$x^2 p + y^2 q + z^2 = 0$

or $x^2 p + y^2 q = - z^2$

Comparing it with Pp + Qq = R, we get

$P = x^2, Q = y^2 , R = z^2$

$\therefore$ auxiliary equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2} \qquad ....(1)$$

Taking the first two members of (1), we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

or $x^{-1} dx = y^{-2}dy$

Integrating we get

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} = a$$

$$\therefore \quad \frac{1}{x} - \frac{1}{y} = a \qquad ...(2)$$

Taking the last two members of (1), we get

$$\frac{dy}{y^2} = \frac{dz}{-z^2}$$

or $y^{-1}dy = -z^{-2}dz$

Integrating, we get

$$\frac{y^{-1}}{-1} = \frac{z^{-1}}{-1} -b$$

$$\text{or} \quad \frac{1}{y} + \frac{1}{z} = b \qquad ...(3)$$

From (2) and (30, we get u = a and v = b

where u (x, y, z) = $\frac{1}{x}$ - $\frac{1}{y}$, v (x, y, z) = $\frac{1}{y}$ + $\frac{1}{z}$

The general solution is given by

$$f\left(\frac{1}{x}-\frac{1}{y},\frac{1}{y}+\frac{1}{z}\right) = 0, \text{ where } f \text{ is an arbitrary function.}$$

**Example 3:** Solve the following for general solution.

$(x - \alpha)\, p + (y - \beta)\, q = z - \gamma$

**Sol. :** We are given

$(x - \alpha)\, p + (y - \beta)\, q = z - \gamma$

Comparing it with Pp + Qq = R, we get

$P = x - \alpha$ , $Q - y - \beta$, $R = z - \gamma$

The auxiliary equations are

$$\frac{dx}{x-\alpha} = \frac{dy}{y-\beta} = \frac{dz}{z-\gamma} \qquad ...(1)$$

Integrating, $\log (x - \alpha) = \log (y - \beta) + \log a$

$$\therefore \quad \log\left(\frac{x-\alpha}{y-\beta}\right) = \log a$$

or $$\frac{x-\alpha}{y-\beta} = \alpha \qquad ...(2)$$

Taking first and last members of (1), we get

$$\frac{dx}{x-\alpha} = \frac{dz}{z-\gamma}$$

Integrating, $\log (x - \alpha) \log (z - \gamma) + \log b$

$$\therefore \quad \log\left(\frac{x-\alpha}{z-\gamma}\right) = \log b$$

or $$\frac{x-\alpha}{z-\gamma} = b \qquad ...(3)$$

From (2) and (3), we get u = a and v = b where u(x, y, z) = $\frac{x-\alpha}{y-\beta}$ and v (x, y, z) = $\frac{x-\alpha}{z-\gamma}$

$\therefore$ The general solution is given by

$$f\left(\frac{x-\alpha}{y-\beta}, \frac{x-\alpha}{z-\gamma}\right) = 0$$

or $\frac{x-\alpha}{z-\gamma} = f\left(\frac{x-\alpha}{y-\beta}\right)$, where $f$ is an arbitrary function.

**Example 4 :** Solve the following for general solution

p cos x + q cos y = cos z

**Sol. :** Given differential equation is

p cos x + q cos y = cos z

Comparing it with Pp + Qq = R, we get

P = cos x, Q = cos y, R = cos z

the auxiliary equations are

$$\frac{dx}{\cos x} = \frac{dy}{\cos y} = \frac{dz}{\cos z} \qquad ...(1)$$

Taking are first two members of (1), we get

$$\frac{dx}{\cos x} = \frac{dy}{\cos y}$$

or sec x dx = sec y dy

Integrating, we get

log | sec x + tan x | = log | sec y + tan y | +

or log | sec x + tan x | = log (c) | sec y + tan y |

or |sec x + tan x | = (c) | sec y + tan y |

or sec x + tan x = (± c) (sec y + tan y)

$\therefore$ $\frac{\sec x + \tan x}{\sec y + \tan y} = \alpha$, where ± c = a ...(2)

Taking first and last members of (1), we get

$$\frac{dx}{\cos x} = \frac{dz}{\cos z}$$

Integrating, we get

$$\int \sec x \, dx = \int \sec z \, dz + \log |d|$$

$\therefore$ $\log |\sec x + \tan x| = \log |\sec z + \tan z| + \log |d|$

or $\log |\sec x + \tan x| = \log |d| \, |\sec z + \tan z|$

$\therefore$ $|\sec x + \tan x| = |d| \, |\sec z + \tan z|$

or $\sec x + \tan x = \pm d (\sec z + \tan z)$

$= b (\sec z + \tan z)$

where $\pm d = b$

$\therefore$ $$\frac{\sec x + \tan x}{\sec y + \tan z} = b \qquad ...(3)$$

From (2) and (3), general solution is given by

$$f\left(\frac{\sec x + \tan x}{\sec y + \tan y}, \frac{\sec x + \tan x}{\sec z + \tan z}\right) = 0$$

Example 5 : Find the general solution of the linear partial differential equation

$y^2 p - xyq = x (z - 2y)$

Sol. : Given differential equation is

$$y^2 p - xyq = x (z - 2y) \qquad ...(1)$$

Lagrange's auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad ...(2)$$

Taking the first two members of (2), we have

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

or $$\frac{dx}{y} = \frac{dy}{-x}$$

$\therefore$ $x dx = -y \, dy$

or $2x \, dx + 2y \, dy = 0$

Integrating, we get

$$x^2 + y^2 = a \qquad ...(3)$$

Again, taking last two members of (2), we have

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

or $$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$\therefore$ $$\frac{dz}{dy} = \frac{-z+2y}{y}$$

or $$\frac{dz}{dy} = -\frac{y}{z} + 2$$

$\therefore$ $$\frac{dz}{dy} + \frac{1}{y} = 2$$

which is a linear differential equation of the form $\frac{dz}{dy} + P_2 = Q$

$\therefore$ $P = \frac{1}{y}$, $Q = 2$

I.F. $e^{\int pdy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$

$\therefore$ the solution is

$$e^{\int pdy} = \int Q\, e^{\int pdy}\, dy + b$$

$\therefore$ $zy = \int 2y\, dy + b$

or $zy = y^2 + b$ ...(4)

$\therefore$ $zy = y^2 = b$

From (3) and (4), we get

$u = a$ and $v = b$

where $u(x, y, z) = x^2 - y^2$

and $v(x, y, z) = xy - y^2$

Then general solution of (1) is given by

$f(x^2 + y^2, zy - y^2) = 0$, where $f$ is an arbitrary function.

**Self-check Exercise**

Q. 1 Solve the following for general solution

$p + q = \sin x$

Q. 2 Solve the following for general solution

$x^2 p + y^2 q = z^2$

Q. 3 Solve the following for general solution

$(x + y^2) p + y q = z + x^2$

## 18.5 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed different types of solution or integral of a partial differential equation.
2. Defined Lagrange's linear equation.
3. Discussed in detail Lagrange's method of solving the linear partial differential equation of order one i.e. of Pp + Qq = R.
4. Type-I solution of Pp + Qq = R is demonstrated with examples.

## 18.6 Glossary:

1. If from partial differential equation $f$(x, y, z, p. q) = 0, We can find a relation F(x, y, z, a, b) = 0 which contains as many constants as there are independent variables, then the relation F(x, y, z, a, b) = 0 is called complete integral or solution of the given partial differential equation.
2. A partial differential equation of the form

   Pp + Qq = R

   where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

## 18.7 Answer to Self-Check Exercise

Ans. 1 $f(x - y, z + \cos x) = 0$

Ans. 2 $f\left(\frac{1}{x}-\frac{1}{y},\frac{1}{y}-\frac{1}{z}\right)$ = 0, where $f$ is an arbitrary constant.

Ans. 3 $f\left(\frac{x}{y}-y,\frac{z}{y}-\frac{x^3}{zy^3}\right)$ = 0

## 18.8 References/Suggested Readings


1. Shepley, L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984

2. Sneddon, I., Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

## 18.9 Terminal Questions

1. Solve the following for general solution

   $$y \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$$

2. Solve the following for general solution $p \tan x + q \tan y = \tan z$

3. Solve the following for general solution

   $e^y p - e^x q = 0$

4. Find the general solution of the linear partial differential equation

   $(x^2 + 2y^2)p - xyq = xz.$

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**Unit - 19**

# Solution of Lagrange's Linear Equation

**Structure**



## 19.1 Introduction

A linear partial differential equation of the first order is an equation that relates a function and its partial derivatives of the first order. The Lagrange method also known as the method of characteristics, is a technique used to solve first-order linear partial differential equations. It involves transforming the given equation into a system of ordinary differential equations along characteristic curves. In last Unit, we discussed solutions of linear partial differential equations of first order (Type I). Now we go further for more type of equations to solve.

## 19.2 Learning Objectives

After studying this unit, you should be able to:-

- Define linear partial differential equation of first order
- Find the solution by Lagrange's method of the equation Pp + Qq = R
- Discuss Type-II solution, Type-III and Type-IV.

## 19.3 Solution of Lagrange's Linear Equation (Type-II)

The solution of Pp + Qq = R is obtained by taking two members of the auxiliary equation and integrate to have an equation (one independent solution) in the variables whose differentials are involved and another independent solution is obtained by making use of the first solution (integral)

Let us look at the following examples to clear the idea:-

**Example 1:-** Solve the following Lagrange's Linear equation for general solution

zp - zq = x + y

**Sol:** Given differential equation is

zp - zq = x + y

Compare it with Pp + Qq = R

Here P = z, Q = -z, R = x + y

The auxiliary equation are

$$\frac{dx}{p} = \frac{dy}{Q} = \frac{dz}{R}$$

or $$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{x+y} \qquad ......(1)$$

Taking first two members of (1), we get

dx = - dy

Integrating, x = -y + a or x + y = a ...(2)

Taking first and last members of (1), we get

$$\frac{dx}{z} = \frac{dz}{x+y}$$

or $$\frac{dx}{z} = \frac{dz}{a} \qquad [Q\ \text{of } (2)]$$

$\therefore$ adx = zdz

Integrating, ax = $\frac{z^2}{2}$ + b

or ax = $\frac{z^2}{2}$ = b

$\therefore$ (x + y) x $\frac{z^2}{2}$ = b ...(3)

From (2) and (3), we get u = a and v = b

where u(x, y, z) = x + y, v (x, y, z) = (x + y) x - $\frac{z^2}{2}$

$\therefore$ the general solution is given by

$$f\left(x + y, (x + y)\, x = \frac{z^2}{2}\right) = 0$$

**Example 2 :** Solve the following Lagrange's linear equation for general solution

$$px + qz = -y$$

**Sol.** Given differential equation is

$$px + qz = -y$$

Comparing it with Pp + Qq = R

Here P = x, Q = z, R = -y

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{-y} \qquad ...(1)$$

Taking last two members of (1), we have

$$y\, dy = -z\, dz$$

or $\quad 2y\, dy + 2z\, dz = 0$

Integrating, $y^2 + z^2 = a$ ...(2)

Taking first and second members of (1), we get

$$\frac{dx}{x} = \frac{dy}{z}$$

or $$\frac{dx}{x} = \frac{dy}{\sqrt{a - y^2}} \qquad [\because \text{ of } (2)]$$

Integrating, $\log x = \sin^{-1}\frac{y}{\sqrt{a}} + b$

$$\log x - \sin^{-1}\frac{y}{\sqrt{y^2 + z^2}} = b \qquad ...(3)$$

From (2) and (3), we get u = a, v = b

where $u(x, y, z) = y^2 + z^2$ and $v(x, y, z) = \log x - \sin^{-1}\frac{y}{\sqrt{y^2 + z^2}}$

$\therefore$ The general solution is given by

$$\left(y^2+z^2, \log x-\sin^{-1}\frac{y}{\sqrt{y^2+z^2}}\right) = 0, \text{ where}$$

$f$ is an arbitrary function

**Example 3 :** Solve the following Lagrange's linear equation for general solution.

p - q = log (x + y)

**Sol. :** Given differential equation is

p - q = log (x + y)

Compare it with Pp + Qq = R, we have

P = 1, Q = -1, R = log (x + y)

The auxiliary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Or $$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad ...(1)$$

Taking First two members of (1), we have

dx + dy = 0

Integrating, x + y = a ...(2)

Taking first and last members of (1), we get

$$\frac{dx}{1} = \frac{dz}{\log(x+y)}$$

or dx = $\frac{dz}{\log a}$ [Q of (2)]

$\therefore$ log a dx = dz

Integrating, (log a) x = z + b

or [log (x + y)] x = z + b

$\therefore$ x log (x + y) - z = b ...(3)

From (2) and (3), we get u = a and v = b

where u (x, y, z) = x + y, v (x, y, z) = x log (x + y) - z

$\therefore$ the general solution is $f$ (x + y, x log (x + y) -z) = 0

or x log (x + y) - z = $f$ (x + y), where $f$ is an arbitrary function.

**Example 4 :** Solve the following Lagrange's linear equation for general solution:

$xyp + y^2 q = xyz - 2x^2$

**Sol. :** Given differential equation is

$xyp + y^2 q = xyz - 2x^2$

Compare it with Pp + Qq = R, we get

P = xy, Q = y2, R = xyz - 2x2

The auxiliary equations are

$$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{xyz - 2x^2} \quad ...(1)$$

Taking the first two members of (1), we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

or $$\frac{dx}{x} = \frac{dy}{y}$$

Integrating, log |x| = log |y| + log |c|

or log |x| = log |y| |c|

or log |x| = log |y c|

$\therefore$ |x| = |yc|

or x = ± yc = ay (say)

$\therefore$ $$\frac{x}{y} = a \quad ...(2)$$

Taking the first and last members of (1), we have

$$\frac{dx}{xy} = \frac{dz}{xyz - 2x^2}$$

or $$\frac{dx}{xy} = \frac{dz}{xy\left(z - 2\frac{x}{y}\right)}$$

$\therefore$ $$\frac{dx}{1} = \frac{dz}{z - 2a} \quad [\because \text{ of } (2)]$$

Integrating, x = log |z - 2a| +b

$\therefore$ $$x - \log \left|z - 2\frac{x}{y}\right| = b \quad ....(3)$$

From (2) and (3), we get u = a and v = b

where u (x, y, z) = $\frac{x}{y}$ and v (x, y, z) = x - log $\left|z - \frac{2x}{y}\right|$

$\therefore$ The general solution is given by

$$f\left(\frac{x}{y}, x - \log\left|z - \frac{2x}{y}\right|\right) = 0,$$ where $f$ is an arbitrary function.

**Example 5 :** Find the general solution of the linear partial differential equation

px (z - 2y$^2$) = (z - qy) (z - y$^2$ - 2x$^3$)

**Sol. :** Given differential equation is

px (z - 2y$^2$) = (z - qy) (z - y$^2$ - 2x$^3$)

or x (z - 2y$^2$) p + y (z - y$^2$ - 2x$^3$) q = z(z - y$^2$ - 2x$^3$) ...(1)

Comparing it with Pp + Qq = R, we have

P = x (z - 2y$^2$), Q = y(z - y$^2$ -2x$^3$),

R = z (z - y$^2$ -2x$^3$)

The auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \quad ...(2)$$

Taking last two members of (2), we get

$$\frac{dy}{y} = \frac{dz}{z}$$

or $$\frac{dz}{z} = \frac{dy}{y}$$

Integrating, log z = log y + log a

= log (ay)

$\therefore$ z = ay or $\frac{x}{y}$ = a ...(3)

Taking first two members of (2), we have

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)}$$

or $$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)} \quad [\because \text{ of } (3)]$$

$\therefore \qquad \dfrac{dx}{ax-2xy} = \dfrac{dy}{ay-y^2-2x^3}$

or $\qquad (ay - y^2 - 2x^3)\, dx = (ax - 2xy)\, dy$

or $\qquad (ay - y^2 - 2x^3)\, dx + (2xy - ax)\, dy = 0 \qquad \text{...(4)}$

Comparing it with Mdx + ndy = 0, we get

$\qquad M = ay - y^2 - 2x^3 ,\ N = 2xy - ax$

$\therefore \qquad \dfrac{\partial M}{\partial y} = a - 2y ,\ \dfrac{\partial N}{\partial x} = 2y - a$

$\therefore \qquad \dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial x}$

$\therefore$ $\qquad$ equation (4) is not exact

Now $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = \dfrac{a-2y-2y+a}{x(2y-a)} = \dfrac{-4y+2a}{x(2y-a)}$

$\qquad = \dfrac{-2(2y-a)}{x(2y-a)} = -\dfrac{2}{x} = f(x) \qquad \text{(say)}$

I.F. $\qquad e^{\int f(x)dx} = e^{\int\left(-\frac{2}{x}\right)dx}$

$\qquad\qquad = e^{-2 \log x}$

$\qquad\qquad = e^{\log x^{-2}}$

$\qquad\qquad = x^{-2}$

$\qquad\qquad = \dfrac{1}{x^2}$

Multiplying (4) by $\dfrac{1}{x^2}$, we get

$$\left(\frac{ay-y^2-2x^3}{x^2}\right) dx + \left(\frac{2xy-ax}{x^2}\right) dy = 0$$

or $\qquad (ay\,x^{-2} - y^2\,x^{-2} - 2x)\, dx + \left(\dfrac{2y}{x}-\dfrac{a}{x}\right) dy = 0$

which is a exact differential equation.

$\therefore$ $\qquad$ Its solution is

$$\int (ay\, x^{-2} - y^2 x^{-2} - 2x)\, dx + 0 = b$$

y constant

Or $\quad ay \left(\frac{x^{-1}}{-1}\right) - y^2\left(\frac{x^{-1}}{-1}\right) - 2\left(\frac{x^2}{2}\right) = b$

Or $\quad -\frac{ay}{x} + \frac{y^2}{2} - x^2 = b$

$\therefore \quad \frac{y^2 - ay}{x} - x^2 = b \qquad .....(5)$

From (3) and (5), the required general solution of (1) is

$f\left(\frac{z}{y}, \frac{y^2 - ay}{x} - x^2\right) = 0$, where $f$ is an arbitrary function.

**Self-Check Exercise-1**

Q.1 Solve the following Lagrange's linear equation for general solution:

$xy^2p - y^3q = -\propto xz$

Q.2 Solve the following Lagrange's linear equation for general solution:

$z(z^2 + xy)(px - qy) = x^4$

Q.3 Find the general solution of the linear partial differential equation

$p + 3q = z + \cot(y - 3x)$

## 19.4 Solution of Lagrange's Linear Equation (Type-III)

Here, we find the solution of $Pp + Qq = R$ ......(1)

by the following formula (from algebra) i.e.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{Sdx + Tdy + Udz}{PS + QT + RU}$$

where S,T, U are some functions of x, y, z.

If S,T, U are chosen in such a manner that (known as multiploers) PS + QT + RU = 0,

Then we have Sdx + Tdy + Udz = 0

Now integrate it to get one independent solution of (1) as u(x, y, z) = a

And the other independent solution can be obtained either by selecting another set of multipliers

Or by taking two members of auxiliary equations.

Let us look at the following examples to clear the idea:-

**Example 6:** Find the general solution of the following Lagrange's linear equation:

x(y - z) p + y (z - x) q = z (x - y)

**Sol:** Given differential equation is

x(y - z) p + y) (z - x) q = z (x - y)

Compare it with Pp + Qq = R, we get

P = x(y - z), Q = y(z - x), R = z (x - y)

The auxiliary equations are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \qquad .....(1)$$

Taking 1, 1, 1 as multipliers; each of fraction of (1)

$$= \frac{dx+dy+dz}{1.x(y-z)+1.y(z-x)+1.z(x-y}$$

$$= \frac{dx+dy+dz}{0}$$

$\therefore$ dx + dy + dz = 0

Integrating, x + y + z = a ......(2)

Taking $\frac{1}{x}$, $\frac{1}{y}$, $\frac{1}{z}$ as multipliers; each of fraction of (1)

$$= \frac{\frac{1}{y}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\frac{1}{x}x(y-z)+\frac{1}{y}y(z-x)+\frac{1}{z}-z(x-y)}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$\therefore$ $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}$ = 0

Integrating,

log |x| + log |y| + log |z| = log |c|

Or log |x||y||z| = log |c|

Or log |xyz| = log |c|

Or xyz = $\pm$ c = b (say)

$\therefore$ xyz = b .....(3)

From (2) and (3), the general solution is given by

$f$(x + y + z, xyz) = 0

**Example 7:** Find the general solution of the following Lagrange's linear equation

$x(y^2 - z^2)\, p + y\,(z^2 - x^2)\, q = z\,(x^2 - y^2)$

**Sol:** Given differential equation is

$x(y^2 - z^2)\, p + y\,(z^2 - x^2)\, q = z\,(x^2 - y^2)$

Compare it with Pp + Qq = R, we get

$P = x\,(y^2 - z^2)$, $Q = y\,(z^2 - x^2)$, $R = z\,(x^2 - y^2)$

The auxiliary equations are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \qquad \text{......(1)}$$

Taking multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$; each of fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y^2 - z^2) + \frac{1}{y}y(z^2 - x^2) + \frac{1}{z}z(x^2 - y^2)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

Integrating, log |x| + log |y| + log |z| = log |c|

Or log |x| |y| |z| = log |c|

Or $\quad \log|xyz| = \log|c|$

Or $\quad |xyz| = \log|c|$

i.e. $\quad xyz = \underline{+}\, c = a$ (say)

$\therefore \quad xyz = a \qquad$ .....(2)

Taking multipliers as x, y, z ; each of fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$$

$$= \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0$

Or $\quad 2xdx + 2ydy + 2zdz = 0$

Integrating, we get $\quad x^2 + y^2 + z^2 = b \qquad$ ......(3)

From (2) and (3), we get u = a and v = b

where $\;u(x, y, z) = xyz$ and

$v(x, y, z) = x^2 + y^2 + z^2$

$\therefore \quad$ The general solution is given by

$f(xyz, x^2 + y^2 + z^2) = 0$, where $f$ is any arbitrary function.

**Example 8:** Find the general solution of the following Lagrange's linear equation:

$$\frac{\beta - \gamma}{\propto} yzp + \frac{\gamma - \propto}{\beta} xzq = \frac{\propto - \beta}{\gamma} xy$$

**Sol:** Given differential equation is

$$\frac{\beta - \gamma}{\propto} yzp + \frac{\gamma - \propto}{\beta} xzq = \frac{\propto - \beta}{\gamma} xy$$

Compare it with Pp + Qq = R, we get

$$P = \frac{\beta - \gamma}{\propto}(yz),\ Q = \frac{\gamma - \propto}{\beta}(xz),\ R = \frac{\propto - \beta}{\gamma}(xy)$$

The auxiliary equations are

$$\frac{dx}{\frac{\beta - \gamma}{\propto}(yz)} = \frac{dy}{\frac{\gamma - \propto}{\beta}(xz)} = \frac{dz}{\frac{\propto - \beta}{\gamma}(xy)} \qquad .....(1)$$

Taking multipliers as $\propto x$, $\beta y$, $\gamma z$ ; each of fraction of (1)

$$= \frac{\propto xdx + \beta ydy + \gamma zdz}{\frac{\propto(\beta-\gamma)}{\propto} yzx + \frac{\beta(\gamma-\propto)}{\beta} xyz + \frac{\gamma(\propto-\beta)}{\gamma} xyz}$$

$$= \frac{\propto xdx + \beta ydy + \gamma zdz}{(\beta-\gamma+\gamma-\propto+\propto-\beta)xyz}$$

$$= \frac{\propto xdx + \beta ydy + \gamma zdz}{0}$$

$\therefore \quad \propto xdx + \beta ydy + \gamma zdz = 0$

Or $\quad \propto(2x)dx + \beta(2y\,dy) + \gamma(2zdz) = 0$

Integrating, we get

$$\propto x^2 + \beta y^2 + \gamma z^2 = a \qquad \text{......(2)}$$

Taking multipliers as $\propto^2 x$, $\beta^2 y$, $\gamma^2 z$; each of fraction of (1)

$$= \frac{\propto^2 xdx + \beta^2 ydy + \gamma^2 zdz}{\propto^2\left(\frac{\beta-\gamma}{\propto}\right)xzy + \beta^2\left(\frac{\gamma-\propto}{\beta}\right)xzy + \gamma^2\left(\frac{\propto-\beta}{\gamma}\right)xyz}$$

$$= \frac{\propto^2 xdx + \beta^2 ydy + \gamma^2 zdz}{\left[\propto(\beta-\gamma)+\beta(\gamma-\propto)+\gamma(\propto-\beta)\right]xyz}$$

$$= \frac{\propto^2 xdx + \beta^2 ydy + \gamma^2 zdz}{0}$$

$\therefore \quad \propto^2 xdx + \beta^2 ydy + \gamma^2 zdz = 0$

Or $\quad \propto^2(2x)dx + \beta^2(2y)dy + \gamma^2(2z)dz = 0$

Integrating,

$$\propto^2 x^2 + \beta^2 y^2 + \gamma^2 z^2 = 0 \qquad \text{......(3)}$$

From (2) and (3), we get u = a and v = b

where $u(x, y, z) = \propto x^2 + \beta y^2 + \gamma z^2$

and $\quad V(x, y, z) = \propto^2 x^2 + \beta^2 y^2 + \gamma^2 z^2$

$\therefore$ The general solution is given by

$$f(\propto x^2 + \beta y^2 + \gamma z^2, \propto^2 x^2 + \beta^2 y^2 + \gamma^2 z^2) = 0$$

**Example 9:** Find the general solution of the following Lagrange's linear equation:

$$(xy^3 - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$$

**Sol:** Given differential equation is

$$(xy^3 - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3)$$

Comparing it with Pp + Qq = R, we get

$$P = xy^3 - 2x^4,\ Q = 2y^4 - x^3y,\ R = 9z(x^3 - y^3)$$

The auxiliary equations are

$$\frac{dx}{xy^3 - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)} \qquad .....(1)$$

Taking first and second members of (1), we get

$$(2y^4 - x^3y)dx = (xy^3 - 2x^4)dy$$

Dividing both sides by $x^3y^3$, we get

$$\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$$

Or $$\left(-\frac{1}{x^2}dy + \frac{2y}{x^3}\right)dx - \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$$

Or $$d\left(-\frac{y}{x^2}\right) - d\left(\frac{x}{y^2}\right) = 0$$

Integrating, we get

$$-\frac{y}{x^2} - \frac{x}{y^2} = -a$$

Or $$\frac{y}{x^2} + \frac{x}{y^2} = a$$

$\therefore$ $$\frac{x^3 + y^3}{x^2y^2} = a \qquad ......(2)$$

Taking multipliers as $\frac{1}{x}, \frac{1}{y}, \frac{1}{3z}$; each of fraction of (1)

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz}{\frac{1}{x}(xy^3 - 2x^4) + \frac{1}{y}(2y^4 - x^3y) + \frac{9z}{3z}(x^3 - y^3)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3}\frac{1}{z}dz}{y^3 - 2x^3 + 2y^3 - x^3 + 3x^3 - 3y^3}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3}\frac{1}{z}dz}{0}$$

$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{1}{3}\frac{dz}{z} = 0$

Integrating, we get

lot |x| + lot |y| + $\frac{1}{3}$ log |z| = log |c|

$\therefore$ log |x| |y| $|z|^{1/3}$ = log |c|

Or $|xyz^{1/3}| = |c|$

Or $xyz^{1/3} = \underline{+}\ c = b$ (say)

$\therefore$ $xyz^{1/3} = b$ ......(3)

From (2) and (3), the general solution is

$$f\left(\frac{x^3 + y^3}{x^2y^2}, xyz^{1/3}\right) = 0$$

**Self-Check Exercise - 2**

Q.1 Find the general solution of the following Lagrange's Linear equation:

(z - y) p + (x - z) q = y - x

Q.2 Find the general solution of the following Lagrange's linear equation:

$$\left(\frac{1}{z} - \frac{1}{y}\right)p + \left(\frac{1}{x} - \frac{1}{z}\right)q = \frac{1}{y} - \frac{1}{x}$$

## 19.5 Solution of Lagrange's Linear Equation (Type-IV)

Here we find the solution of Pp + Qq = R by the following formula

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{Sdx + Tdy + Udz}{PS + QT + RU} \qquad ......(1)$$

where S,T, U are some functions of x, y, z.

If the multipliers S,T, U are selected in such a manner that Sdx + Tdy + Udz is exact differential of a factor of PS + QT + RU.

Then we consider two members of (1) $\frac{Sdx + Tdy + Udz}{PS + QT + RU}$ and other suitably chosen to get one independent solution. And the other independent solution can be obtained either by selecting another set of multipliers or by taking two members of auxiliary equations.

Let us look at the following examples to clear the idea:-

**Example 10:** Find the general solution of the following Lagrange's linear equation

$px^2 + qy^2 = z (x + y)$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$

**Sol:** Given differential equation is

$$px^2 + qy^2 = z (x + y) \qquad .....(1)$$

Comparing it with Pp + Qq = R, we get

$P = x^2, Q = y^2, R = z (x + y)$

The auxiliary equations

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z(x+y)} \qquad .....(2)$$

Taking first two members of (2), we have

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Or $\quad x^{-2}\, dx = y^{-2} dy$

Integrating,

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + a$$

Or $\quad -\frac{1}{x} = -\frac{1}{y} + a$

$\therefore \quad \frac{1}{y} - \frac{1}{x} = a$

Taking multipliers as 1, -1, 0, each fraction of (2)

$$= \frac{dx - dy}{x^2 - y^2}$$

$$= \frac{dx - dy}{(x-y)(x+y)}$$

$\therefore$ $$\frac{dz}{z(x+y)} = \frac{dx - dy}{(x-y)(x+y)}$$

Or $$\frac{dz}{z} = \frac{dx - dy}{x - y}$$

$\therefore$ $$\frac{dz}{z} - \frac{dx - dy}{x - y} = 0$$

Integrating, we get

log z - log (x - y) = log b

Or $$\log\left(\frac{z}{x-y}\right) = \log b$$

$\therefore$ $$\frac{z}{x-y} = b \qquad .....(4)$$

From (3) and (4), we get u = a and v = b

where $u(x, y, z) = \frac{1}{y} - \frac{1}{x}$

and $v(x, y, z) = \frac{z}{x-y}$

$\therefore$ The general solution of (1) is given by $f\left(\frac{1}{y} - \frac{1}{x}, \frac{z}{x-y}\right) = 0$, where $f$ is arbitrary function.

**Example 11:** Find the general solution of the following Lagrange's linear equation:

p cos (x + y) + q sin (x + y) = z

**Sol:** Given differential equation is

p cos (x + y) + q sin (x + y) = z

Comparing it with Pp + Qq = R, we get

$P = \cos(x + y)$, $Q = \sin(x + y)$, $R = z$

The auxiliary equations are

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z} \qquad .....(1)$$

Taking multipliers as 1, 1, 0; each of fraction of (1) is

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} \qquad ......(2)$$

Taking multipliers as 1, -1, 0 ; each of members of (1)

$$= \frac{dx+dy}{\cos(x+y)-\sin(x+y)}$$

$$\therefore \qquad \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

Or $\quad \dfrac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx + dy) = dx - dy$

Or $\quad \dfrac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x + y) = d(x - y)$

Integrating, we get

$$\log\left|\frac{\sin(x+y)+\cos(x+y)}{c}\right| = x - y$$

Or $\quad \left|\dfrac{\sin(x+y)+\cos(x+y)}{c}\right| = e^{x-y}$

$\therefore \quad \sin(x + y) + \cos(x + y) = + c\, e^{x-y}$

Or $\quad \sin(x + y) + \cos(x + y) = ae^{x-y}$ (say)

$\therefore \quad e^{y-x}[\sin(x + y) + \cos(x + y)] = a \qquad ....(3)$

Now from (1) and (2), we have

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dz}{z}$$

Integrating, we get

$$\int \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \int \frac{dz}{z} + \log|k|$$

$\therefore \quad \int \dfrac{dt}{\cos t + \sin t}$ = log z + log |k| [By putting x + y = t]

$\therefore \quad \int \dfrac{dt}{\sqrt{2}\left[\dfrac{1}{\sqrt{2}}\cos t + \dfrac{1}{\sqrt{2}}\sin t\right]}$ = log |z| |k|

Or $\quad \dfrac{1}{\sqrt{2}} \int \dfrac{dt}{\cos\dfrac{\pi}{4}\cos t + \sin t \sin\dfrac{\pi}{4}}$ = log |z| |k|

$\therefore \quad \int \dfrac{dt}{\cos\left(t - \dfrac{\pi}{4}\right)} = \sqrt{2}$ log |z| |k|

Or $\quad \int$ sec $\left(t - \dfrac{\pi}{4}\right)$dt = log $\left(|z||k|\right)^{\sqrt{2}}$

Or $\quad \log\left|\sec\left(t - \dfrac{\pi}{4}\right) + \tan\left(t - \dfrac{\pi}{4}\right)\right|$ = log $\left(|z|\right)^{\sqrt{2}} \left(|k|\right)^{\sqrt{2}}$

Or $\quad \left|\sec\left(x + y - \dfrac{\pi}{4}\right) + \tan\left(x + y - \dfrac{\pi}{4}\right)\right|$ = log$\left(|z|\right)^{\sqrt{2}} \left(|k|\right)^{\sqrt{2}}$

Or $\quad \sec\left(x + y - \dfrac{\pi}{4}\right) + \tan\left(x + y - \dfrac{\pi}{4}\right)$ = + $z^{\sqrt{2}}\ k^{\sqrt{2}}$

$\therefore \quad z^{-\sqrt{2}}\left[\sec\left(x + y - \dfrac{\pi}{4}\right) + \tan\left(x + y - \dfrac{\pi}{4}\right)\right]$ = b .....(4)

From (3) and (4), the general solution is given by

$$f\left[e^{y-x}\left\{\sin(x+y) + \cos)x + y)\right\}, z^{-\sqrt{2}}\left\{\sec\left(x + y - \frac{\pi}{4}\right) + \tan\left(x + y - \frac{\pi}{4}\right)\right\}\right] = 0,$$

where $f$ is any arbitrary function

**Example 12:** Find the general solution of the following Lagrange's linear equation

$(x^2 - y^2 - z^2)$ p + 2 xyq = 2xz

**Sol:** Given differential equation is

$(x^2 - y^2 - z^2)$ p + 2 xyq = 2xz

Compare it with Pp, Qq = R, we get

$P = x^2 - y^2 - z^2$, $Q = 2\,xy$, $R = 2\,xz$

The auxiliary equations are

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \qquad \text{......(1)}$$

Taking multiplier as x, y, z; we get

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2}$$

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \qquad \text{......(2)}$$

$$\therefore \quad \frac{dz}{2xz} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

Or $$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2}$$

$$\therefore \quad \frac{dz}{z} = \frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2}$$

Integrating, we get

$\log z = \log (x^2 + y^2 + z^2) - \log a$ (say)

Or $\log (x^2 + y^2 + z^2) - \log z = \log a$

$$\therefore \quad \log\left(\frac{(x^2 + y^2 + z^2)}{z}\right) = \log a$$

$$\therefore \quad \frac{x^2 + y^2 + z^2}{z} = a \qquad \text{......(3)}$$

Taking last two factors of (1), we get

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating, we get

$\log |y| = \log |z| + \log |c|$

Or $\log |y| = \log |z|\,|c|$

$\therefore$ $|y| = |z|\,|c|$

Or $y = + z\,c = z\,b$ (say)

$\therefore$ $\frac{y}{z} = b$ .....(4)

From (3) and (4), we get

$u = a$ and $v = b$

where $u(x, y, z) = \frac{x^2 + y^2 + z^2}{z}$

and $v(x, y, z) = \frac{y}{z}$

$\therefore$ The general solution is

$$\int \left(\frac{x^2 + y^2 + z^2}{z}, \frac{y}{z}\right) = 0$$

**Self-Check Exercise-3**

Q.1 Find the general solution of the following Lagrange's linear equation:

$$(y + z)\,p + (z + x)\,q = x + y$$

Q.2 Find the general solution of the following Lagrange's linear equation:

$$(1 + y)\,p + (1 + x)\,q = z$$

## 19.6 Summary:

We conclude this unit by summarizing what we have covered in it:-

1. Discussed in detail Type-II, Type-III and Type-IV solutions by Lagrange's method of the equation of the type Pp + Qq = R
2. Discussed each type of solutions with examples.

## 19.7 Glossary:

1. A linear partial differential equation of the first order is an equation that relates a function and its partial derivatives of the first order.
2. The Lagrange method also known as the method of characteristics, is a technique used to solve first order linear partial differential equations.

## 19.8 Answer to Self-Check Exercise

**Self-Check Exercise-1**

Ans. 1 $f\left(xy, \frac{\propto x}{3y^2} + \log|z|\right)$ = 0, where $f$ is any arbitrary function.

Ans. 2 $f$(xy, $x^4 - z^4 - 2xyz^2$) = 0, where $f$ is any arbitrary function

Ans. 3 $f$(y - 3x, x - log [z + cot (y - 3x)] = 0, where $f$ is any arbitrary function.

**Self-Check Exercise-2**

Ans. 1 $f$(x + y + z, $x^2 + y^2 + z^2$) = 0, where $f$ is any arbitrary function

Ans. 2 $f$(x + y + z, xyz) = 0

**Self-Check Exercise-3**

Ans. 1 $f\left(\frac{x-y}{y-z}, (x+y+z)(x-y)^2\right)$

Ans. 2 $f\left((1+x)^2 - (1+y)^2, \frac{x+y+2}{z}\right)$ = 0,

where $f$ is any arbitrary function.

## 19.9 References/Suggested Readings


1. Sneddon, I., Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

2. Shepley, L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984


## 19.10 Terminal Questions

1. Solve the following Lagrange's linear equation for general solution:

   $z(p - q) = z^2 + (x + y)^2$

2. Solve the following Lagrange's linear equation for general solution:

   $py + qx = xyz^2 (x^2 - y^2)$

3. Solve the following Lagrange's linear equation for general solution:

   $py^3 + 2x^3 = \frac{x^2y^2(x^4 - y^4)}{5z^4}$

4. Find the general solution of the following Lagrange's linear equation:

   $(\beta z - \gamma y) p + (\gamma x - \propto z) q = \propto y - \beta x$

5. Find the general solution of the following Lagrange's linear equation:

$(3x + y - z) p + (x + y - z)q = 2 (z - y)$

6. Find the general solution of the following Lagrange's linear equation:

$(y^2 + yz + z^2) p + (z^2 + zx + x^2)q = (x^2 + xy + y^2)$

7. Find the general solution of the following Lagrange's linear equation:

$z - px - qy = k\sqrt{x^2 + y^2 + z^2}$

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## Unit - 20

# Partial Differential Equations of Second Order

**Structure**



## 20.1 Introduction

Partial differential equations (PDEs) are mathematical equations that involve multiple variables and their partial derivatives. Second-order partial differential equations are a specific type of PDEs where the highest derivative involved is of second order second-order PDEs can be classified based on their characteristics. The classification helps in understanding the nature of the equation and the methods used to solve them. The main classifications of second-order PDEs are:

Elliptic PDEs, Parabolic PDEs and Hyperbolic PDEs. The behavior and properties of second-order PDEs depend on the classification and the specific form of the coefficients. Each classification has its own set of solution techniques and boundary conditions associated with it. Analytical methods, such as separation of variables, Fourier series, and integral transforms, as well as numerical methods, like finite difference and finite element methods, are used to solve second-order PDEs and obtain solutions that satisfy the given boundary and initial conditions. In this Unit, we shall study solution of partial differential equations of second order. We shall also discuss classification of these types of differential equations.

## 20.2 Learning Objectives

After studying this unit, you should be able to:-

- Define partial differential equations of second order.
- Define linear and non-linear second order partial differential equations.
- Discuss solutions of partial differential equations of second order (Method of inspection)
- Discuss classification of linear partial differential equations of second order.

## 20.3 Some Definitions

Def. 1. A partial differential equation is said to be of second order when it contains atleast one of the differential coefficient r, s, t but none of higher order.

A general partial differential equation of second order can be written as

$$F(x, y, z, p, q, r, s, t) = 0 \qquad .....(1)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial z}{\partial y^2}$

Def. 2. A partial differential equation of second order is said to be linear, if it is linear relative to the required function and all its derivatives that enter into the equation.

Def. 3. A partial differential equation of second order is said to be non-linear, if it is not linear.

e.g. 1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

2. $x^2 t - y^2 t = px - qy$ are linear equations

3. $2y\frac{\partial^2 z}{\partial x^2} + 3x\frac{\partial z}{\partial y} + z^2 = 0$

4. $2z\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial z}{\partial x} + 4\frac{\partial z}{\partial y}$ are non-linear equations

## 20.4 Solutions of Partial Differential Equations of Second Order (Method of Inspection)

Type I : When the given equations can be reduced to linear equations

**Example 1:** Solve : $t - xq = x^2$

**Sol:** The given equation can be written as : $\frac{\partial q}{\partial y} - xq = x^2 \qquad .....(1)$

$$\left[\because t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}\right]$$

which is linear in q and y.

I.F. = $e^{-\int x dy} = e^{-xy}$

$\therefore$ its solution is

q. $e^{-xy}$ = $\int$ x$^2$ $e^{-xy}$ .dy + $f$(x)

[where $f$(x) is an arbitrary function of x]

$\frac{x^2 e^{-xy}}{-x}$ + $f$(x) = -x $e^{-xy}$ + $f$(x)

$\Rightarrow$ $\frac{\partial z}{\partial y}$ = $e^{xy}$ ($f$(x) - x $e^{-xy}$ ) = $e^{xy}$ $f$(x) - x

Integrating both sides w.r.t. 'y' (Keeping x as constant),

z = $f$(x). $\frac{e^{xy}}{x}$ - xy + constant

= $e^{xy}$ . $\frac{f(x)}{x}$ - xy + g(x)

Hence the reqd. solution is

z = $e^{xy}$ $\frac{f(x)}{x}$ - xy + g(x)

where $f$(x), g(y) are two arbitrary functions of x.

**Example 2:**Solve :yt - q = xy

**Sol:** The given equation can be written as $\frac{\partial q}{\partial y}$ - $\frac{1}{y}$ q = x, which is linear in q.

I.F. = $e^{\int -\frac{1}{y} dy}$ = e$^{-\log y}$ = e$^{\log}$y$^{-1}$ = $\frac{1}{y}$

$\therefore$ Solution is

q. $\frac{1}{y}$ = $\int$ x. $\frac{1}{y}$ dx + $f$(x) = x log y + $f$(x)

$\Rightarrow$ q = xy log y + y $f$(x) $\Rightarrow$ $\frac{\partial z}{\partial y}$ = xy log y + y $f$(x)

Integrating w.r.t. 'y', we get z = x $\int$ y log y dy + $f$(x) $\int$ y dy + F(x)

$$= x\left[\frac{y^2}{2}\log y - \int \frac{1}{y}.\frac{y^2}{2}dy\right] + \frac{y^2}{2} f(x) + F(x)$$

$$z = \frac{1}{2} xy^2 \log y - \frac{x}{4} y^2 + \frac{y^2}{2} f(x) + F(x)$$

**Self-Check Exercise-1**

Q.1 Solve xys = 1

Q.2 Solve rx = (n - 1)p

**Example 3:** Solve : $s - t = \frac{x}{y^2}$

**Sol:** The given equation can be written as

$$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial y} = \frac{x}{y^2}$$

Integrating both sides w.r.t. 'y' (keeping x as constant),

$$p - q = -\frac{x}{y} + f(x)$$

which is of the form Pp + Qq = R

$\therefore$ Lagrange's A.E. are $\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-\frac{x}{y} + f(x)}$

From the first two,

$$dx + dy = 0 \Rightarrow x + y = a$$

Again from the first and last, we get

$$dz = -\frac{x}{y} dx + f(x)\, dx$$

$$= \frac{x}{a-x} dx + f(x)dx \qquad \left(\because x + y = a \atop \Rightarrow y = a - x\right)$$

$$= \left(1 - \frac{x}{a-x}\right) dx + f(x)\, dx$$

Integrating we get

$$z = x + a \log(a - x) + g(x) + b$$

$\therefore$ z = a log(a - x) + h(x) + b [h(x) = x + g(x)]

$\Rightarrow$ z - a log(a - x) - h (x) = b

Hence the reqd. general solution is F[x + y, z - a log (a - x) - h(x)] = 0

where F is an arbitrary function.

**Example 4:** Solve: p + r + s = 1

**Sol:** The given equation is p + r + s = 1 $\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = 1$

Integrating both sides w.r.t. 'x', we get z + $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ = x + $f$ (y)

$\Rightarrow$ $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ = x + $f$ (y) - z $\Rightarrow$ p + q = x + $f$(x) - z $\Rightarrow$ 1. p + 1.q = x + $f$(y) - z

which is of the form Pp + Qq = R

$\therefore$ A.E. are $\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + f(y) - z}$

From the first two, dx = dy$\Rightarrow$ x - y = a (constant) .....(1)

Again from Second and third, we get $\frac{\partial z}{\partial y}$ = x + $f$(y) - z

$\Rightarrow$ $\frac{\partial z}{\partial y}$ + z = x + $f$ (y), which is linear in z.I.F. = $e^{\int 1.dy}$ = $e^y$

$\therefore$ Solution is $ze^y = \int [x + f(y)] e^y dy + b = xe^y + \int e^y f(y) dy + b$

$= xe^y + \phi(y) + b$ $\left[\phi(y) = \int e^y f(y) dy\right]$

$\therefore$ $ze^y - xe^y - \phi(y) = b$ (= constant)

= F(x - y) [Using (1)]

Type III. Equations easily integrable by Inspection

**Example 5:** Solve: s = 2x + 2y

**Sol:** The given equation can be written as

$$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y$$

Integrating both sides w.r.t. 'x', we get [Keeping y constant]

$$\frac{\partial z}{\partial y} = x^2 + 2xy + \text{constant} = x^2 + 2xy + f(y)$$

Integrating both sides w.r.t. 'y', (keeping x constant)

$$z = x^2y + 2x\frac{y^2}{2} + \int f(y)dy + \text{constant}$$

$$= x^2y + xy^2 + g(y) + h(x)$$

where g and h are arbitrary functions.

**Example 6:** Solve: log s = x + y

**Sol:** The given equation is log s = x + y $\Rightarrow$ s = $e^{x+y} \Rightarrow \frac{\partial^2 z}{\partial x \partial y} = e^{x+y}$

Integrating both sides w.r.t 'x' $\frac{\partial z}{\partial y} = e^{x+y} + f(y)$

Integrating both sides w.r.t. 'y' $z = e^{x+y} + y + \int f(y)dy + F(x)$

Hence, $z = e^{x+y} + \phi(y) + f(x)$ where $\phi$, $f$ are arbitrary functions.

| | |
|---|---|
| Q.4 | Solve: t = sin xy |
| Q.5 | Solve: yx + p = cos (x + y) - y sin (x + y) |

## 20.5 Reduction To Canonical Forms

We now consider the equation of the type

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \text{....(1)}$$

in which R,S,T are continuous functions of x and y possessing continuous partial derivatives of as high order as necessary.

By suitable change of independent variables, we shall show that the equation (1) can be transformed into one of the three canonical forms which can be integrated easily.

Let the independent variables x, y be changed to u, v by the transformation equations

$$u = u(x, y),\ v = v(x, y) \qquad \text{....(2)}$$

Now we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x},\ q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

This shows $\frac{\partial}{\partial x} \equiv \frac{\partial u}{\partial x}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}.\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}.\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}.\frac{\partial}{\partial v}$

r = $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}.\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}.\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}.\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}.\frac{\partial z}{\partial v}\right)$

= $\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\,\frac{\partial^2 z}{\partial u\partial v}.\frac{\partial u}{\partial x}\,\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}.\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}.\frac{\partial^2 v}{\partial x^2}$

s = $\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}.\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}.\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial v}\right)$

= $\frac{\partial^2 z}{\partial u^2}\,\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\partial v}\left(\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\,\frac{\partial v}{\partial x}\,\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\,\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial z}{\partial v}\,\frac{\partial^2 v}{\partial y\partial x}$

t = $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}.\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}.\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}.\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}\right)$

= $\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\,\frac{\partial^2 z}{\partial u\partial v}\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial y}\,\frac{\partial^2 v}{\partial y^2}$

Putting these values of p, q, r, s, t in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u\partial v} + C\frac{\partial^2 z}{\partial v^2} + F\left(u, v, z\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) = 0 \quad ......(3)$$

where

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\,\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2 \quad .....(4)$$

$$B = R\frac{\partial u}{\partial x}.\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right) + T\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial y} \quad .....(5)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\,\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \quad .....(6)$$

and F $\left(u, v, w\frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$ is the transformed form of $f$(x, y, z, p, q)

Now we determined u and v so that the equation (3) takes the simplest possible form. The equation (3) reduces to the simple integrable form when the discriminant $S^2$- 4 RT of the quadratic equation.

$$R\lambda^2 + S\lambda + T = 0 \quad .....(7)$$

is either positive, negative or zero everywhere.

Three cases are possible.

Case I. $S^2 - 4RT > 0$ In this case the two roots $\lambda_1$, $\lambda_2$ of equation (7) would be real and distinct.

We choose u and v such that

$$\frac{\partial u}{\partial x} = \lambda_1 \frac{\partial u}{\partial y} \qquad .....(8)$$

and $$\frac{\partial v}{\partial x} = \lambda_2 \frac{\partial v}{\partial y} \qquad .....(9)$$

$$A = (R\lambda^2_1 + S\lambda_1 + T)\left(\frac{\partial u}{\partial y}\right)^2 = 0$$

Q $R\lambda^2_1 + S\lambda_1 + T = 0$ as $\lambda$ is the root of equation (7). Similarly, C = 0

For differential equation (9), the Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-\lambda_1} = \frac{du}{0}$$

Given $du = 0 \therefore u = c_1$ (Constant)

and $$\frac{dy}{dx} + \lambda_1 = 0 \qquad .....(10)$$

Let $f_1(x, y) = c_2$ (constant) be the solution of equation (10)

$\therefore$ Solution of equation (8) can be taken as

$$u = f_1(x, y) \qquad .....(11)$$

Similarly, if $f_2(x, y)$ = constant, is a solution of

$$\frac{dy}{dx} + \lambda = 0$$

then solution of (9) can be taken as

$$v = f_2(x, y)$$

Now, it can be shown that

$$AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right)^2$$

Or $$B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}.\frac{\partial v}{\partial x}\right)^2 \qquad .....(13)$$

[$\because$ A = C = 0]

$\because$ $S^2 - 4RT > 0$, $\therefore$ $B^2 > 0$,

$\therefore$ We can divide both sides of the equation by it. Hence the equation (1) is reduced to the form

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad .....(14)$$

which is the canonical form of the equation (1)

Case II. $S^2 - 4RT = 0$. In this case the two roots of equation (7) would be equal (real).

Here we choose u, as in case 1.

i.e. $\dfrac{\partial u}{\partial x} = \lambda_1 \dfrac{\partial u}{\partial y}$

Given u $f$(x, y)

Also we take v to be any function of x and y, which is independent of u.

$\therefore$ As case I, A = 0

Also, from (13), $B^2 = 0$ $\because$ $S^2 - 4RT = 0$

i.e., B = 0

Putting A = 0, B = 0, in (3) and dividing by C ($\neq 0$), the equation (1) takes the form

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \qquad .....(15)$$

which is the other canonical form of the equation (1).

Case III. $S^2 - 4RT < 0$. In this case the two roots of equation (7) would be complex.

Proceeding as in case I, here the equation (1) will reduce to the same canonical form [equation (14)] as in case I but here the variables u and v are not real but are infact the complex conjugates.

To get a real canonical form

Let $u = \alpha + i\beta$, $v = \alpha - i\beta$

So that $\alpha = \dfrac{1}{2}(u + v)$ and $\beta = \dfrac{1}{2} i(u - v)$

Now we transform the independent variables u and v to $\alpha$ and $\beta$ with the help of these relations.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$$

Similarly, $\frac{\partial z}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$

and $\therefore$ $\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \frac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$

$$= \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right)$$

Putting in (14), the transformed Canonical form of the given equation is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta}\right)$$

## 20.6 Classification of Linear Partial Differential Equations of Second Order

A general linear partial differential equation of second order for a function of two independent variables x, y can be expressed as

$$Rr + Ss + Tt + f\,(x, y, z, p, q) = 0$$

where R, S, T are continuous functions of x, y defined in some domain D of the xy - plane.

The classification depends on the part Rr + Ss + Tt which is called the principal part of (1),

**Definition.** The linear partial differential equation (1) is said to be

I. Hyperbolic if S2 - 4 RT > 0

II. Parabolic if S2 - 4 RT = 0

III. Elliptic if S2 - 4 RT < 0

e.g., 1. The one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$$

is hyperbolic with canonical form

$$\frac{\partial^2 z}{\partial u\,\partial v} = 0$$

2. The one dimensional heat equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c}\,\frac{\partial z}{\partial t}$$

is parabolic

3. The two-dimensional Laplace's equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

is elliptic.

To Clarify what we have just said, consider the following examples:-

**Example 7:** Classify the following partial differential equation:-

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

**Sol:** Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad .....(1)$$

It can be written as

$r + 2s + t = 0$

Or $r(1) + s(2) + t(1) + 0 = 0$

Comparing it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = 2, T = 1, f = 0$

Now $S^2 - 4\,RT = (2)^2 - 4\,(1)\,(1)$

$= 0$

$\therefore$ Given partial differential equation is parabolic

**Example 8:** Classify the following partial differential equation:-

$$\frac{\partial^2 z}{\partial x^2} - 2\propto \frac{\partial^2 z}{\partial x \partial y} + \propto^2 \frac{\partial^2 z}{\partial y^2} = g(y + \propto x)$$

**Sol:** Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} - 2\propto \frac{\partial^2 z}{\partial x \partial y} + \propto^2 \frac{\partial^2 z}{\partial y^2} = g(y + \propto x) \qquad ......(1)$$

It can be written as

$1.\ r - 2\propto.s + \propto^2 t - g(y + \propto x) = 0$

Or $1.\ r - 2\propto.s + \propto^2 t - g(y + \propto x) = 0$

comparing it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = -2\propto, T = \propto^2, f = -g(y + \propto x)$

Now $S^2 - 4RT = (-2\propto)^2 - 4(1)(\propto)^2$

$= 4\propto^2 - 4\propto^2$

$= 0$

$\therefore$ Given partial differential equation (1) is parabolic

**Example 9:** Classify the following partial differential equation:

$$5\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

**Sol:** Given partial differential equation is

$$5\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

It can be written as

$5r + 4s + t = 0$

Comparing it with $rR + sS + tT + f = 0$, we get

$R = 5, S = 4, T = 1, f = 0$

$\therefore$ $S^2 - 4RT = (4)^2 - 4(5)(1)$

$= 16 - 20$

$= -4 < 0$

$\therefore$ Given partial differential equation is elliptic.

**Example 10:** Classify the following partial differential equation:

$r - 4s + 5t = 6 \cos(2x + 3y)$

**Sol:** Given partial differential equation is

$r - 4s + 5t = 6 \cos(2x + 3y)$ .....(1)

It can be written as

1. $r - 4s + 5t - 6 \cos(2x + 3y) = 0$

compare it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = -4, T = 5, f = -6 \cos(2x + 3y)$

Now $S^2 - 4RT = (-4)2 - 4(1)(5)$

$= 16 - 20$

$= -4 < 0$

∴ Given partial differential equation (1) is elliptic

**Example 11:** Classify the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = xy$$

**Sol:** Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = xy$$

It can be written as

$r - 5s + 6t - xy = 0$

comparing it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = -5, T = 6, f = -xy$

Now $S^2 - 4RT = (-5)^2 - 4(1)(6)$

$= 25 - 24$

$= 1 > 0$

∴ Given partial differential equation is hyperbolic

**Example 12:** Classify the following partial differential equation.

$(D_x^2 + 3D_x D_y + D_y^2)z = e^{2x+3y}$

**Sol:** Given partial differential equation is

$(D_x^2 + 3D_x D_y + D_y^2)z = e^{2x+3y}$ .....(1)

It can be written as

$r + 3s + t = e^{2x+3y}$

Or $r + 3s + t - e^{2x+3y} = 0$

Comparing it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = 3, T = 1, f = - e^{2x+3y}$

Now $S^2 - 4RT = 9 - 4(1)(1)$

$= 5 > 0$

∴ Given partial differential equation (1) is hyperbolic

**Example 13:** Classify the following partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = e^{2x+y}$$

**Sol:** Given partial differential equation is

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = e^{2x+y}$$

It can be written as

$r - t - 3p + 3q - e^{2x+y} = 0$

Or $r(1) + S(0) + t(-1) + (-3p + 3q - e^{2x+y}) = 0$

Comparing it with

$rR + sS + tT + f = 0$, we get

$R = 1, S = 0, T = -1$

$\therefore$ $S^2 - 4RT = (0)^2 - 4(1)(-1)$

$= 0 + 4$

$= 4 > 0$

$\therefore$ Given differential equation is hyperbolic

**Self-Check Exercise-2**

Q.1 Classify the following partial differential equation:

$$y^2r - 2xys + x^2t = \frac{y^2}{x}p + \frac{x^2}{y}q$$

Q.2 Classify the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = xy$$

Q.3 Classify the following partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 6\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x$$

## 20.7 Summary:

In this Unit we have introduced you to the concept of partial differential equations of second order. Specifically, we have covered in it:-

1. Definitions of second order partial differential equations, linear and non-linear second order partial differential equations.

2. Discussed in detail the solution of partial differential equations of second order by method of inspection for different types i.e. when the given equations can be reduced to linear equations; Equations integrable by Lagrange's method; Equations easily integrable by Ispection.

3. Discussed in detail classification of linear pde of second order.

4. Examples are given in support of each type.

## 20.8 Glossary:

1. A partial differential equation is said to be of second order when it contain atleast one of the differential coefficients r, s, t but none of higher order.

2. A partial differential equation of second order is said to be linear, if it is linear relative to the required function and all its derivatives that enter into the equation.

3. A partial differential equation of second order is said to be non-linear if it is not linear.

## 20.9 Answer to Self-Check Exercise

**Self-Check Exercise-1**

Ans. 1 $z = \log x \log y + \phi(y) + F(x)$

Ans. 2 $z = \frac{x^n}{n}\phi(y) + F(y)$

Ans. 3 $F[x - y, ze^x - \phi, (x)] = 0$, where F is an arbitrary function

Ans. 4 $z = -\frac{\sin(xy)}{x^2} + y\, f(x) + y(x)$,

where $f$, g are arbitrary functions

Ans. 5 $yz = y \sin (x + y) + \phi(y) + f(x)$,

where $\phi$, $f$ are arbitrary functions

**Self-Check Exercise-2**

Ans. 1 Parabolic

Ans. 2 Elliptic

Ans. 3 Hyperbolic

## 20.10 References/Suggested Readings

1. Sneddon, I., Elements of Partial Differential Equations, McGraw-Hill, International Edition, 1967.

2. Shepley, L. Ross, Differential Equations, 3rd Ed., John Wiley and Sons, 1984

3. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley and Sons, New York.

## 20.11 Terminal Questions

1. Solve $r = 2y^2$
2. Solve $xr + p = 9x^2y^3$
3. Solve $xyr + x^2s - yp = x^3e^y$
4. Solve $S = \frac{x}{y} + a$
5. Classify the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y}$$

6. Classify the following partial differential equation:

$$x^2r - 2\,xys + y^2t - xp + 3yq = \frac{8y}{x}$$

7. Classify the following partial differential equation:

$$2\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 u}{\partial x\,\partial y} + 5\frac{\partial^2 u}{\partial y^2} = 0$$

8. Classify the following partial differential equation:

$$\frac{\partial^2 z}{\partial x^2} - 6\frac{\partial^2 z}{\partial y^2} = 0$$

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