B.A.: 1st Year
Mathematics (Core Course)

Course Code: MATH101TH
New Syllabus : CBCS

# Differential Calculus

## Units 1 to 20

Centre for Distance and Online Education (CDOE)
Himachal Pradesh University,
Gyan Path, Summer Hill, Shimla - 171005

## B.A.-1st Mathematics MATH101TH

## Differential Calculus

**CONTENTS**

# SYLLABUS
# BA 1st Year Mathematics
# MATH101TH CBCS New Syllabus
## Himachal Pradesh University
## B.A. with Mathematics (Annual System)
## Syllabus and Examination Scheme

| | |
|---|---|
| Course Code | MATH101TH |
| Name of the Course | Differential Calculus |
| Type of the Course | Core Course |
| Assignments | Max. Marks:30 |
| Yearly Based Examination | Max Marks: 70 |
| | Maximum Times: 3 hrs. |

## Instructions

**Instructions for Candidates:** Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

## Core 1.1 : Differential Calculus

### Unit-I

Limit and Continuity (epsilon and delta definition), Types of discontinuities, Differentiability of functions, Successive differentiation, Leibnitz's theorem.

### Unit-II

Indeterminate forms, Rolle's theorem, Lagrange's & Cauchy Mean Value theorems, Taylor's theorem with Lagrange's and Cauchy's form of remainder, Taylor's series. Maclaurin's series of sin x, cos x, $e^x$, log (1+x), $(1+x)^m$.

### Unit-III

Concavity, Convexity & Points of Inflexion, Curvature, Radius of curvature, center of curvature, Asymptotes, Singular points, Double point, Polar coordinates, Relation between Cartesian and polar coordinates.

### Unit-IV

Function of several variable (upto three variables) : Limit and Continuity of these functions Partial differentiation. Euler's theorem on homogeneous functions, Maxima and Minima with Lagrange Multipliers Method (two variables), Jacobian (upto three variables).

**Books Recommended**


1. H. Anton, I. Birens and S. Davis, *Calculus*, John Wiley and Sons, Inc., 2002.
2. G.g. Thomas and R.L. Finney, *Calculus*, Pearson Education, 2007.

## Unit - 1

# Some Basic Concepts

**Structure**



### 1.1 Introduction

Dear students, let us briefly recapitulate about the idea of a set, real numbers and functions which we feel is important to understand the concept of limit, continuity etc. The understanding of these concepts will surely help you to learn and to build a strong foundation for the theory of calculus.

### 1.2 Learning Objectives:

The main objectives of this unit are

(i) to revisit the concept sets.

(ii) to study different types of sets, notation for describing sets.

(iii) to revisit the concept of number system

(iv) to study order relation and geometric representation of rational number.

(v) to revisit the concept of functions

(vi) to study different types of functions

### 1.3 A Set.

Intuitively speaking a set is just a collection of well defined distinct objects. By this we mean that given a set A and an object x, then we are sure that either

(I) x is in the collection A.

or

(II) x is not in the collection of A.

That is there is no ambiguity about the element of A

**Examples**

(i) The collection of all natural number is a set. We will denote it by N.

(ii) The collection of all integers is a set. We generally denote it by Z.

(iii) The collection of all the letters of English alphabets.

(iv) The collection of all the integers n, $n \geq 0$ is a set.

Set A be a set. The any object x of A is also called an element of A or a member of A, and we write $x \in A$ and read it as "x is an element of A" or "x belongs to A"

If an element x is not a member of A, we write $x \notin A$ and read as "x does not belong to A"

**Notation for Describing Sets**

Sets are generally denoted by capital letter A, B, C, ...... The elements of sets are denoted mostly by small letters a, b, c, ......

In our discussion we have to describe various sets within a given universal set X. Let us first understand what do we mean by universal set.

A set containing all the possible elements of concern in each particular context or application from which sets can be formed is called Universal Set. It is denoted by X or U.

Now, a set can be described in two ways.

**1. Tabular or Roster Form**

The eariest way available to us is by listing all the objects of the set within a curly bracket { } separated by commas. This method can be used for finite sets as well as for infinite sets.

For example, A set A whose members are $a_1$, $a_2$, .....$a_n$ are usually written as

(i) $A = \{a_1, a_2, .....a_n\}$

(ii) A set of vowels : A = {a, e, i, o, u}.

In those cases where the number of elements are infinite but where listing a few elements should suffice to know all the elements of the set under consideration, we can use the above notation with slight modification, e.g.

N = {1, 2, 3, .... } denotes the set of natural number.

Z = {.... -3, -2, -1  0, 1, 2, 3, ......} denotes the set of all integers. The dots in these cases indicates the existence of other objects in these sets.

**2. Set Builder Form**

In this form a set is defined by a property satisfied by its members. A common notation expressing this method is

A = {x | p (x) is true} or simply A = {x | p(x)}, where the symbol | (:) denotes the phrase "such that" and p is a proposition and A is the set of all those elements for which proposition p is true. For example.

(i) $A = \{ a \in R \mid a^3 - 1 = 0\}$

(ii) A = {x | x is a vowel in English alphabet}

**Some Illustrative Examples**

**Ex. 1** Which of the following collections are sets?

(i) The collection of positive multiple of 10.

(ii) The collection of five talented students in India.

**Solution :** (i) Positive multiple of 10 are 10, 20, 30, .....

$\therefore$ given Collection is a set.

(ii) Since there is no definite rule to decide the talent of a student.

$\therefore$ the given collection is not a set.

**Ex. 2** Write the set $\left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6}\right\}$ in the set builder form

**Solution :** Let $X = \left\{\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},\frac{5}{6}\right\}$

$X = \{ x \mid x = \frac{n}{n+1}$, n is natural number and $n \in [1, 5]\}$

**1.4 Self Check Exercise-1**

Q. 1 Which of the following collection are set?

(i) the collection of all girls in your class

(ii) the collection of all even integers.

Q. 2 Write the solution set of the equation

$x^2 - 4x + 3 = 0$ in the roster form

Q. 3 Write the set $\left\{\frac{6}{7},\frac{7}{8},\frac{8}{9},\frac{9}{10}\right\}$ in the set builder form.

**Different Types of Sets.**

**Finite Set :** A set consisting of finite number of elements is called a finite set.

For example

(i) $A = \{1, 3, 5, 7\}$

(ii) $B = \{x \mid x \in N \text{ and } 3 < x < 9\}$

**Infinite Set :** A set having an infinite number of elements is called an infinite set.

For example

(i) $X = \{1, 2, 3, .....\}$

$Y = \{x \mid x \text{ is an even integer}\}$

**Singleton Set :** A set is said to be a singleton set (or unit set) if it has only one element in it.

For example

$A = \{x : x \text{ is perfect square and } 20 \leq x \leq 30\}$

$= \{ 5 \}$

**Empty set (Null Set or Void Set)**

A set containing no element in it is referred to as empty set. It is denoted by $\phi$ (read as phi) or { }

For example

(i) $X = \{x : x \text{ is a positive integer satisfying } x^2 = -1\}$

(ii) $X = \{x : x \text{ is a fraction satisfying } x^2 = 16\}$

Consequently, a set consisting of atleast one element is called a non-empty set.

**Order (Cardinal Number) of a Finite Set**

The number of distinct element of a finite set is called the order of a set. It is denoted by 0(A), A is a finite set.

For example

If $A = \{1, 2, 3, 4, 5\}$, then

$0(A) = 5$

**Equivalent Sets**

Two finite sets A and B are said to be equivalent if they same cardinal numbers i.e. $0(A) = 0(B)$.

For Example

Set $A = \{1, 2, 3, 4,\}$ and $B = \{3, 4, 5, 6,\}$

Here $0(A) = 4 = 0(B)$

Therefore A and B are equivalent sets and we write $A \sim B$.

**Equality of Set**

Two sets A and B are said to be equal if both A and B have same elements.

For example  $A = \{1,2,3,4\}$

and  $B = \{x : x$ is natural number and $1 < x < 4\}$ we note here that $A = B$.

**Some Illustrative Examples**

**Example 1:** Which of the following sets are empty? Give reason.

(i) $A = \{Y : Y + 2 = 2\}$

(ii) $B = \{x : x^2 + 2 = 0, x \text{ is real}\}$

**Solution:** (i) Here $A = \{Y : Y + 2 = 0\}$

$= \{Y : Y = 0\}$

$= \{0\} \neq \phi$

$\therefore$ A is not an empty set

(ii) $B = \{x : x^2 + 2 = 0, x \in R\} = \phi$

Since there is no $x \in R$ s.t. $x^2 = -2$ and $x^2 + 2 = 0$

$\therefore$ $B = \phi$

**Example 2:** State which of the sets are finite or infinite.

(i) $A = \{x : x \in N, (x - 2)(x - 3) = 0\}$

(ii) $B = \{x : x \in N \text{ and } x \text{ is even}\}$

**Solution:** (i) $A = \{x : x \in N, (x - 2)(x - 3) = 0\}$

$= \{2, 3\}$ = Finite set

(ii) $B = \{x : x \in N \text{ and } x \text{ is even}\}$

$= \{2, 4, 6, \ldots\ldots..\}$ = Infinite set

**Subset.** Let A and B be two sets. If every member of a set A is also a member of the set B, then A is called subset of B and we write $A \subset B$. In subset of A, we write $A \not\subset B$.

Note: (i) $A \subset B$ is also written as $A \subseteq B$

(ii) Since every element of A is contained in A of $A \subset A$ i.e. every set is subset of itself.

(iii) The empty set $\phi$ is subset of every set.

For example Let $A = \{1,2,3,4,5,6\}$

and $B = \{2,4,6\}$

$\Rightarrow$ $B \subset A$ since every element of set B is also a member of the set A.

Further if $C = \{2,6,7\}$

Clearly $C \not\subset A$, since $7 \notin A$.

**Proper Subset.** If $A \subseteq B$ and $A \neq B$ we say that A is a proper subset of A. Clearly $\phi$ and A are regarded as improper subset of A.

**Note:** 1. If A has n elements, then number of subsets of A is $2^n$.

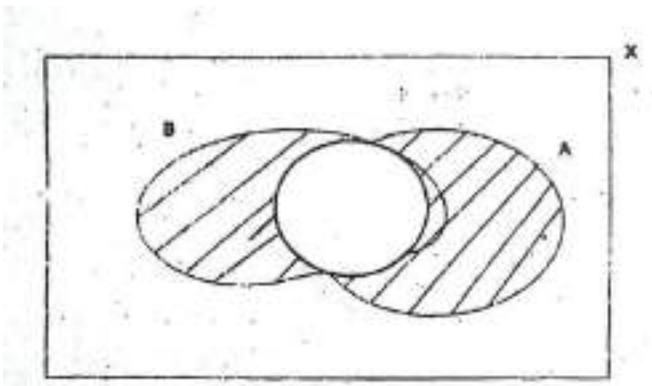
2. N, the set of natural numbers, Z, the set of all integers and Q, the set of rational numbers, are some important subset of R (real numbers).

**Union and Intersection of Sets.**

Let A and B are two sets. Then A union B denoted by AUB is the set of all those elements belonging to atleast one of A or B.

Symbolically $A \cup B = \{x | x \in A \text{ or } x \in B\}$

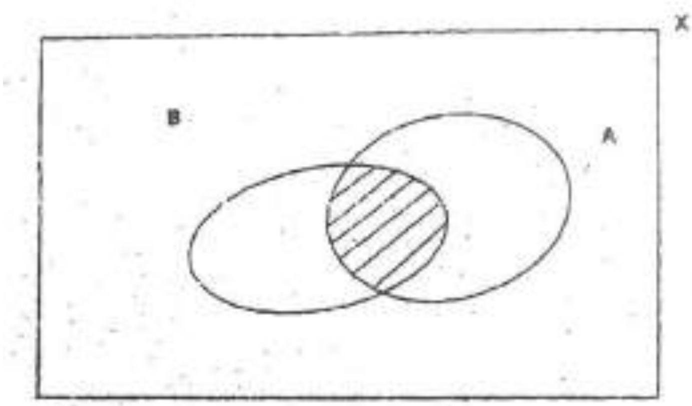
Note: If $x \in A$ and $x \in B$ then definitely $x \in A \cup B$:




$A \cup B$ = Shaded area

The set A intersection B denoted by $A \cap B$ is defined to be set of those elements which are in A as well as in B i.e.

$A \cap B = \{x | x \in A \text{ and } x \in B\}$




$A \cap B$ = shaded area

If $A \cap B = \phi$ Then A and B are said to be disjoint.

**Examples** (i) Let A be the set of even integers and B be the set of odd integers. Then

$A \cup B = Z$

and $A \cap B = \phi$

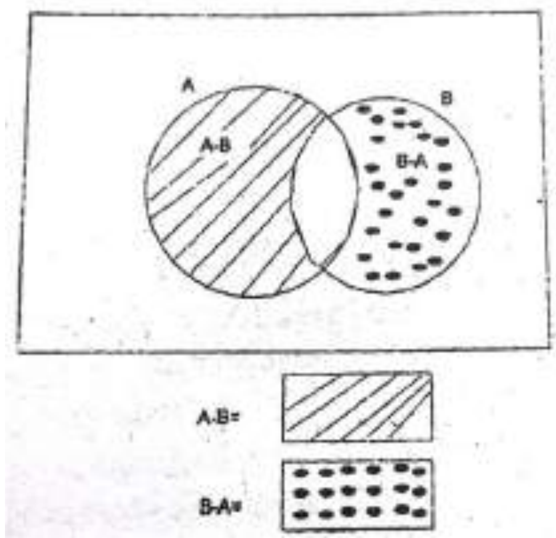
(ii) Let A = (1,2,3,4,6,12} = positive division of 12.

Let B = set of all even integers = {2,4,6,.....,}

and $A \cap B = \{2,4,6,12\}$

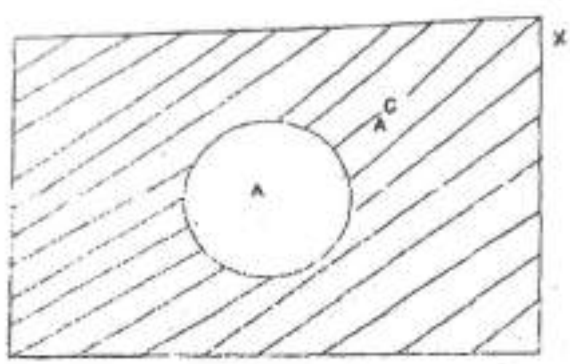
**Complement of a Set:** Let A and B sets we define A-B, complement of B in A or complement of B w.r.t. A to be the set of all those elements of A which are not in B, i.e.

A - B $\{x|x \in A, x \notin B\}$




If all the sets under consideration are subsets of a particular set called universal set then $A^c$ will denote the complements of A in X.

Namely X-A




$A^c = X - A$ = Shaded Area

**Examples** (1) Let $A = \{2,4,6,8\}$

$B = \{1,3,4\}$

Then $A - B = \{2,6,8\}$

and $B - A = \{1,3\}$

(ii) Let $A = Z$; $B = N$

Then $A - B = \{0 - 1, -2, -3, .......\}$

$B - A = \phi$

**Exercises**

1. Find $A \cup B$, $A \cap B$, $A - B$ and $B - A$ where

(i) $A = \left\{0, 1, 2, 3, \tfrac{1}{2}\right\}$

$B = \left\{1, \frac{2}{5}, 3, \tfrac{1}{2}, 7, 8\right\}$

(ii) $A \{x|x \in z, x > 0\}$

$B = \{x|x \in z, x < 0)$

2. Prove the following:

(i) $A \cup A = A$

(ii) $A \cap A = A$

(iii) $A \cup B = B \cup A$

(iv) $A \cap B = B \cap A$

(v) $A \cup A^c = X$; $A \cap A^c = \phi$

(vi) $B \subseteq A \Leftrightarrow A \cap B = B$

## 1.4 Number System

The simplest numbers are the positive integers or natural numbers, 1,2,3,..... used for counting. We denote this set by N, i.e.

$N = \{1,2,3,......\}$

This system of numbers is inadequate for the arithmetical operations of addition, multiplication, substraction and division for example the sum of two natural numbers is again a natural number and so is the product of two natural numbers. However the difference of two natural numbers may not be a natural number. For example

$2 - 7 = -5 \notin N$

In order to overcome this difficulty, we are forced to extend the system of Natural numbers to a larger system which include all Natural numbers, their negatives and the number zero,. We denote this new system by

Z = {....., -2,-1,0,1,2,........} and we call this the set of integers.

The difficulty that we faced in Natural numbers has been removed as the sum, product and difference of two integers is always an integer. But when we apply division the outcome may or may not belong to Z. For example 1+2 = $1/2 \notin$ Z. So there is a need to extend this system so that the operation of division is possible. The collection of all these numbers is called the set of rational numbers and is denoted by Q.

Q = $\left\{ \frac{P}{q} \mid p, q \in Z, q \neq 0 \right\}$

This set is closed under the operation of addition, substraction and multiplication, i.e. if x, y $\in$ Q then x+y, x-y, and xy $\in$Q and also if y $\neq$ 0 then $\frac{x}{y} \in$ Q {Note $\frac{x}{y}$ has meaning only when y $\neq$ 0}.

Since the rational numbers p/q is the same as the rational number $-p/-q$ (q $\neq$ 0). So every rational number can be expressed can be expressed in the form $p/q$, where p an q are integers and q is a positive integer. So with this assumption, rational number of the form $p/q$, where p is a positive integer, (q is assumed to be positive) will be called positive rational number, and when p is negative, will be called negative rational number.

We recall the following facts

(i) if x, y are positive rational numbers, then x+y, xy and $\frac{1}{x}$ are also positive rational numbers.

(ii) If x is any rational number then either x is positive or x = 0 or x negative.

**Order-relation:-** Given two rational numbers x and y, we say that x is greater than y and write x > y if x - y is a positive rational number. This is equivalently also written as y < x and say that y is less than x.

**Theorem:** If a and b are two rational numbers.

(i) If a < b than a + c < b + c for all rational c.

(ii) If a < b and c is a positive rational number, then ac < bc.

(iii) If a < b and c is a negative rational number, then ac > bc.

(iv) If a < b and b < c then a < c.

**Proof:** (i) Q (b+c) - (a+c) = b - a > 0 [Q a < b] $\Rightarrow$ a+c < b + c

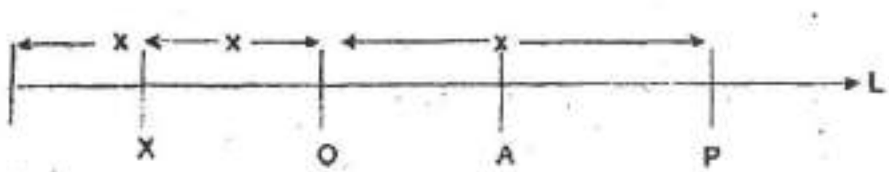
(ii) Q (bc - ac) = (b-a) c > 0 [Q b - a and c are both positive]

(iii) Q bc - ac = (b - a) c < 0, [Q b - a is positive and c is negative]

(iv) Q (c - a) = (c - b) + (b - a) > 0 as both c - b and b - a are positive)

**Geometric Representation of rational numbers**

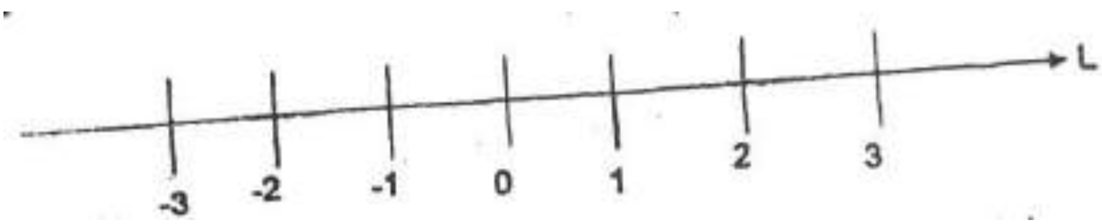
Let L be a straight line. We choose two arbitrary points O and A on L, we associate the rational number o with the point O and rational number 1 with the point A. We use the distance between the points O and A as a unit of measurement, and define the direction from O to A as positive.




If x is a positive rational number, we associate with x the point P on L which is at a distance of x units to the right of O. If x is a negative rational numbers, we associate with x the point Q on L which is at a distance of -x units to the left of O.

Thus we have a geometric representation of rational numbers as points on the number axis L.

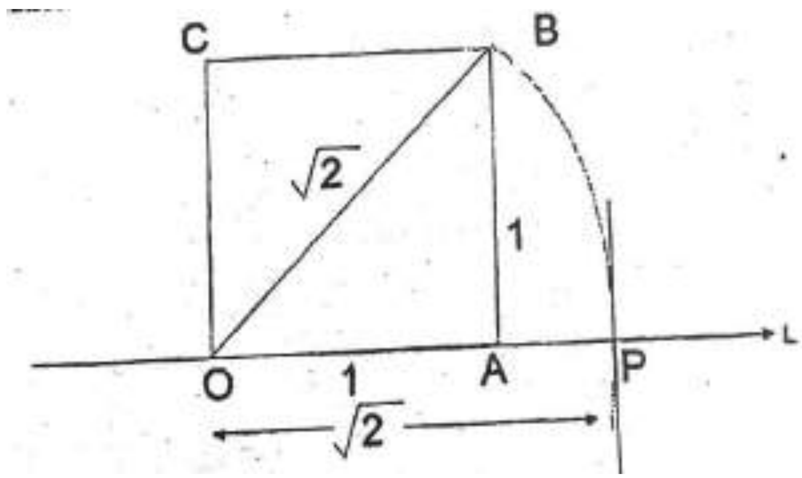
The points corresponding to the rational numbers 0, $\pm$ 1, $\pm$ 2,...... subdivide the numbers axis L into intervals of unit length. If




We further subdivide every interval into q equal parts we obtain a subdivision of L into intervals of length $\frac{1}{q}$ by rational points. Now every point P on L is either a rational point or lies between two successive rational points $\frac{p}{q}$ and $\frac{p+1}{q}$.

Since successive points of subdivision are $\frac{1}{q}$ units apart, it follows that we can find a rational point $\frac{p}{q}$ whose distance from the point P does not exceed $\frac{1}{q}$ units. This number $\frac{1}{q}$ can be made as small as we please by choosing q sufficiently large positive integer. This means that given any point P on L, we can find a rational point which is arbitrarily close to the point P. This property of rational numbers is also stated as that the rational points are dense on the number line.

My dear students this may lead you to think that every point on the line L corresponds to a rational number. But we will show that this is not the case i.e. there are points on the line which do not correspond to any rational number i.e. there are gaps in rational numbers.




We construct a square O ABC of side unit lengths. Then by Pythagoras theorem

$OB^2 = OA^2 + AB^2 = 1^2 + 1^2 = 2$

$\Rightarrow$ OB = $\sqrt{2}$

Let P be a point on L to the right of O such that OP = OB. We shall show the P does not correspond to any rational number. Suppose P corresponds to rational number $\frac{p}{q}$ i.e. OP = $\frac{p}{q}$ where p & q are positive integers we may assume that p & q have no common factors.

Q $\frac{p}{q}$ = OP = OB = $\sqrt{2}$

$\therefore \left(\frac{p}{q}\right)^2 = \left(\sqrt{2}\right)^2 = 2$

that is $p^2 = 2q^2$ ......1

Now $2q^2$ is even integer

$\Rightarrow$ $p^2$ is an even integer

$\Rightarrow$ p is an even integer [Q square of an odd integer is odd]

Let p = 2m, where m is an integer .....(2)

Hence $2q^2 = p^2 = (2m)^2 = 4m^2$

$\Rightarrow \quad q^2 = 2m^2$

$\Rightarrow \quad q^2$ is an even integer.

$\Rightarrow \quad q$ is an even integer. ......(3)

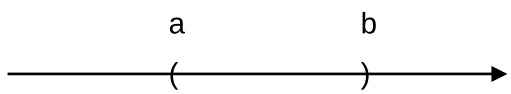
Thus p & q have 2 as a common factor contradicting the hypothesis that they do not have any common factor.

The conclusion, therefore, is that there is no rational number p/q whose square is 2. Or there is no rational number which correspond to the point P on L.

Thus there is need to extend the rational numbers to a larger system in a way that the points of the new system when plotted on the number line, no gaps are left. Such a system exists and is called the system of real numbers and will be denoted by R.

Real Numbers :- The set of all points on a directed line is called the set of real numbers. So we assume that to each point P on a directed line corresponds a real number and conversely. The line is called the real lime.

Real numbers to the right of O will be called positive real numbers and those to the left negative real numbers.




If x is a real number then either x is positive or zero or negative.

Real numbers which are not rational numbers are called irrational numbers (i.e. $\sqrt{2}$ )

Intervals Let a and b be two real numbers and a < b.

The set of all real numbers between a and b is called then open interval from a to b and is denoted by (a, b) i.e.

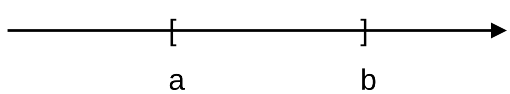
$(a, b) = \{x \in R| \ a < x <b\}$ The number a is called the left end point of the interval and b the right end point.

Note that both end points do not belong to the open interval

The closed interval from a to b denoted by [a, b] consists of all reals satisfying

$[a. b] = x \in R| \ a \leq x \leq b\}$

The closed interval is represented on the real line as follows.



Besides open and closed intervals we also have following semi closed intervals.

[a, b) = $\{x \in R \mid a \leq x < b\}$

(a, b] = $\{x \in R \mid a < x \leq b\}$

All above intervals were finite. We also have infinite intervals.

$(a, \infty) = \{x \in R \mid x > a\}$

$[a, \infty) = \{x \in R \mid x \geq a\}$

or $(-\infty, a) = \{x \in R \mid x < a\}$

$(-\infty, a] = \{x \in R \mid x \leq a\}$

The set R of real number is also denoted by $(-\infty, \infty)$

Absolute value of a real number

For any real number x we define the absolute value |x| of x by

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

**Proposition :** For all real number x

1) |x| = max (x, -x)
2) $|x| \geq 0$
3) |x| = 0 if and only if x = 0
4) |-x| = |x|
5) $|x| \geq x$.

These are simple results which follow from the definition and are left for you to prove.

**Note :** By $\sqrt{\ }$ , we always mean the positive square root. That is each real number $a \geq 0$ $\sqrt{a}$ denotes the unique non-negative real number x such that x2 = a. Also note that $\sqrt{a}$ is not defined (as a real number) if a < 0.

From this note it follows that

$$|x| = \sqrt{x^2}$$

**Theorem :** For any two real number a & b.

$$|ab| = |a||b| \ \& \ \left|\frac{a}{b}\right| = \left|\frac{a}{b}\right| \text{ if } b \neq 0.$$

**Proof :** $|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b|$

Also $\left|\frac{a}{b}\right| = \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}}\ \frac{\sqrt{a^2}}{\sqrt{b^2}} = \left|\frac{a}{b}\right|$

**Exercises :** 1) Find the absolute value of

0, 3, -3, 6 - $5/2$, $\pi$ - 3

(ii) Is it true that

$\sqrt{x^2}$ = x for every real number x?

(iii) If x is rational and y is irrational than what can you say about x + y and xy.

## 1.5 Functions

It is believed that the term function was first used by Leibnitz to refer to certain kind of mathematical formulae. Since then the meaning of the word function has undergone many changes. Here we give the modern meaning of the word function.

**Definition :** A function f from a non-empty set X to a set Y is a rule which associates to each element x in X a unique element y in Y.

The unique element of Y which is associated with x in X is denoted by f(x). The symbol

$f$ : X $\rightarrow$ Y or $\rightarrow$ X $\xrightarrow{f}$ Y

are usually used to denote that f is a function from X to Y.

The set X is called the Domain of the function f and set Y is called the co-domain or range space of f.

The subset of Y defined by $f$(X) = [y|y = $f$(x) for some x $\in$ X] is called the range of f.

The range of f is a subset of Y which may or may not be equal to Y.

The element $f$(x) which f associates with x is called the image of x under f. The element x is called the pre-image of y = f(x).

The function f : X $\rightarrow$ Y is also denoted by y = f (x)/ x $\in$ X is called the independent variable and y $\in$ Y called the dependent variable.

The functions whose domain and co domain are both subsets of R, the set of real numbers, are called real valued function of a real variable.

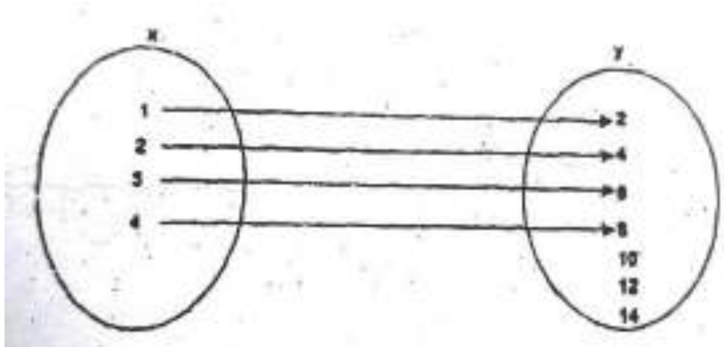
Examples (1) Let X = {1, 2, 3, 4,}

& Y = {2, 4, 6, 8, 10, 12, 14}

$f$ : X $\rightarrow$ Y

x $\rightarrow$ y = $f$ (x) = 2x

is a function from X to Y.




(2) Let X = Y = Z, the set of integers

$f : X \to Y$

$x \to f(x) = x^2$

Then $f : X \to Y$ is a function from X to Y.

e.g. $f(0) = 0$

$f(-2) = (-2)^2 = 4$

$f(z) = (2)^2 = 4$

and so on

**Graph of a Function :** By the graph of a function f we mean the set $\{(x, f(x))\ x \in \text{domain of } f\}$.

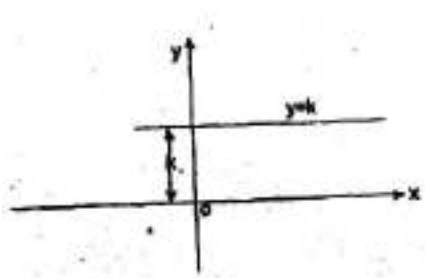
**Note :** For real function given by a formula, we shall consider the set of all real number x for which f (x) is defined to be the domain of f, unless domain is explicitly defined.

Let us study some frequently used real functions and also their graphs.

**Constant function :-** A function $f : X \to Y$, X, Y $\subseteq$, R is said to a constant function if there exists a real number $k \in y$ s.t. $f(x) = k\ \forall\ x \in X$.

let a, b, c $\in$ X

| x | a | b | c |
|---|---|---|---|
| f(x) | k | k | k |




graph of y = k (k > 0)

Domain = Xi range = {k}

**Identity function :-** This is a function defined by

$f$ : R → R

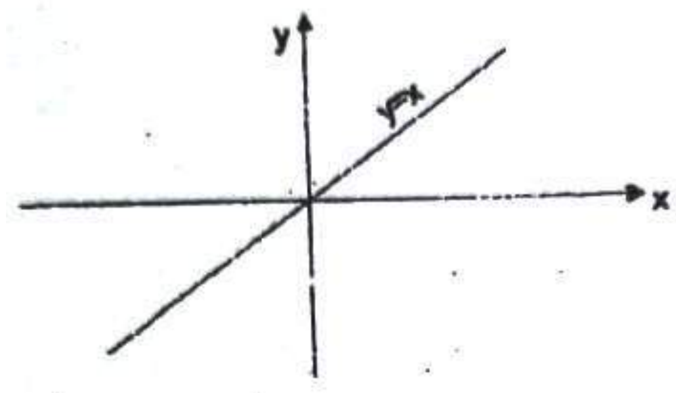
x → $f$ (x) = x ∀ x ∈ R

i.e. it takes the element to itself

This is called identity function and is usually denoted by I

i.e. I : R → R

x → I (x) = x, ∀ x ∈ R




graph of y = x

Domain = R

Range = R

| x | 0 | 1 | 2 |
|---|---|---|---|
| f(x) | 0 | 1 | 2 |

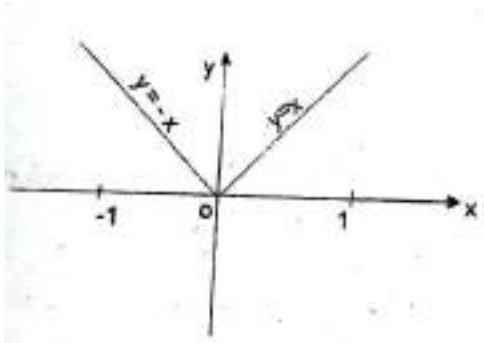
Modulus or absolute value function the function f : R → R defined by

$$f(x) = |x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if y } x < 0 \end{cases}$$

Domain = R

Range $R_0^+$ = {x | x is a non-negative real number. To plot its graph

| x | -1 | 0 | 1 |
|---|---|---|---|
| f(x) | 1 | 0 | 1 |




graph of y = y = |x|

**Square root function :** If x is a positive real number, then there are two square roots for x, of which one is positive and is denoted by $\sqrt{x}$ .

A function $f$ $R_0^+ \to R$

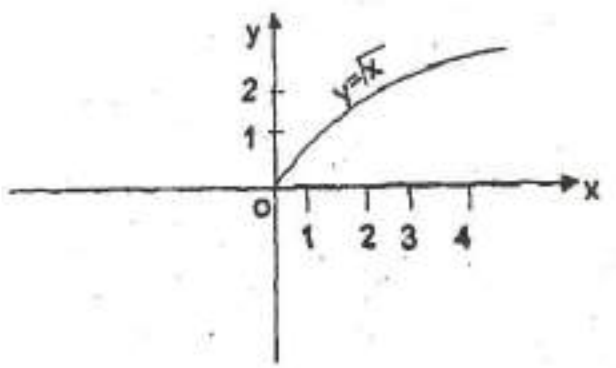
$x \to f(x) = \sqrt{x}$

is called the square root function.

To plot its graph

| X | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|
| $\sqrt{x}$ | 1 | 1 | 1.4 | 1.7 | 2 | |

Plotting these value we get its graph




**The greatest integer function :**

For a real number x, we denote by [x], the greatest integer less than or equal to x

For example [4.5] = 4, [1] = 1, [.5] = 0, [-4.5] = -5

The function $f : R \to R$ defined by

$x = f(x) = [x], x \in R$

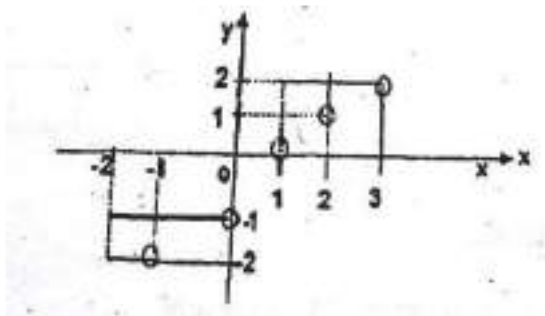
is called the greatest integer function.

It is also called a step function because of the shape of its graph.

Domain = R

Range = Z

| x | $-2\leq x<1$ | $-1\leq x<0$ | $0\leq x<1$ | $1\leq x<2$ | $2\leq x<3$ |
|---|---|---|---|---|---|
| [x] | -2 | -1 | 0 | 1 | 2 |

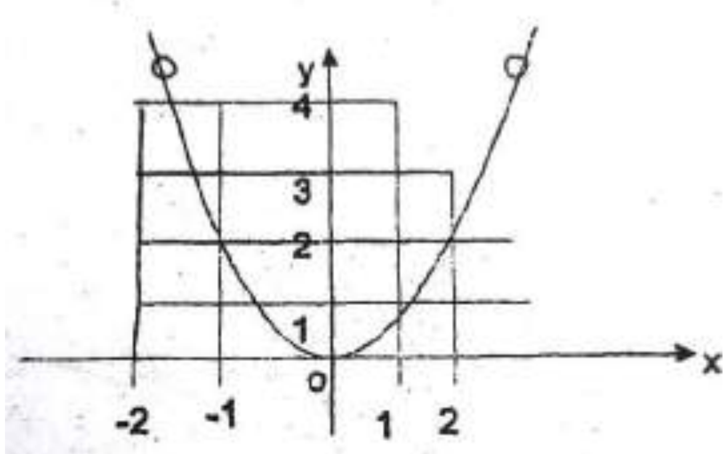



graph of y = [x]

**Polynomial function :** A function $f$ : R → R is said to be polynomial function if for each x $\in$ R f(x) is a polynomial in x. For example

$f(x) = x^3 + 3x^2 - 2$, $g(x) + x^4 + 3x$ are example of polynomial functions. We will plot the graph of the famous polynomial function $y = x^2$ which is a parabola.

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $x^2$ | 4 | 1 | 0 | 1 | 4 |




graph of $y = x^2$

**Signum function :** The function f defined by

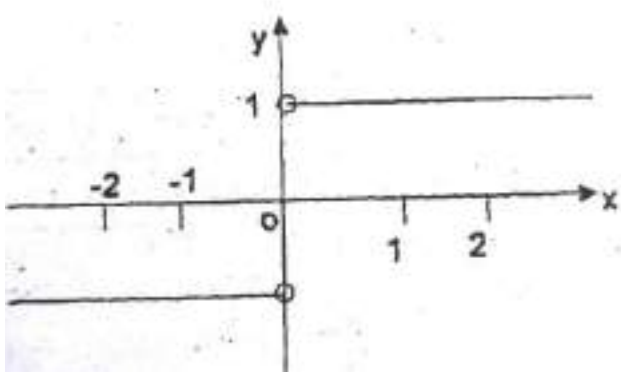
$$f\,(x)\begin{cases}1 \text{ if } x>0\\ 0 \text{ if } x=0\\ -1 \text{ if } <0\end{cases}$$

is called the signum function

Domain = R

Range {-1, 0, 1}

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| f(x) | -1 | -1 | 0 | 1 | 1 |




**Even Function :** Function f satisfying

f(-1) = f (x) is called an even function, For example

$f(x) = x^2$, $f(x) = x^4 - 3x^2 + y$

**Odd function :** function satisfying f (-x) = -f(x)

is called an odd function

For example $y = x$, $y = x^3$

**One-one function or Injective function :** A function f : X → Y is said to be one-one- or injective if $x^1 \neq x^2 \Rightarrow f(x^1) \neq f(x^2)$ i.e. different elements in the domain have different images.

or equivalently images are same only when elements are same.

i.e. $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

For example f(x) = x, f(x) = 3x + 5 are 1 - 1 functions where as $f(x) = x^2$ is not 1 - 1.

**Onto or Subjective Function:**

A function $f : X \to Y$ is said to be onto or subjective if for $y \in Y$, there exists $x \in X$ s.t. $f(x) = y$. Clearly a function $f : X \to Y$ is onto if every element of Y is the image of some element in X.

i.e. range of f = Y

**Algebra of Real functions**

Let us first define the equality of two functions.

**Equal functions:-** Two function $f : X \to Y$ from a set of X to a set Y are said to be equal if f(x) = g(x) for every $x \in X$.

Thus, for two functions f and g to be equal, both f and g should have same domain and their values at each point of the domain are identical.

In other words if $f : X \to Y$ & $g : X \to Y$

Then we say that f = g if (1) domain f = domain g (ii) $f(x) = g(x)\ \forall\ x \in$ domain.

**Addition of two real function:** Let $f : X \to R$ and $g : X \to R$ be any two real function where $X \subseteq R$.

Then we define

$f + g : X \to R$ by

$(f + g)(x) = f(x) + g(x),\ \forall x \in X.$

**Subtraction of real function from another**

Let $f : X \to R$ and $g : X \to R$ be any two real functions then we define

$f - g : X \to R$ by

$(f - g)(x) = f(x) - g(x),\ \forall x \in X$

**Multiplication by a scalar.**

Let $f : X \to R$ be any real valued function and $\alpha$ be any scalar i.e. real number. Then the product $\alpha f$ is a function from X to R defined by

$\alpha f : X \to R$

$(\alpha f)(x) = \alpha f(x),\ \forall \in R$

**Multiplication of two functions:-**

The product or multiplication of two real function $f : X \to R$ and $g : X \to R$ is a function

$g : X \to R$ is a function

$g : X \to R$ defined by

$(fg)(x) = f(x)\, g(x),\ \forall\ x \in X$

**Quotient of two real functions:**

Let $f$: X → R and g : X → R be two functions. Then the quotient of f by g is a function.

$\frac{f}{g}$ : X - (x : g(x) = (0) → R defined by -

$\left(\frac{f}{g}\right)$ (x) = $\frac{f(x)}{g(x)}$

**Composition of two functions:-**

Let f and g be two functions such that he domain of f contains the range of g. Thus it makes sense to talk of f(g(x)) for each x in the domain of G. In this case we can define a function with domain X. We denote this function by fog, and call it the composite of f and g. In other words if $f$ : Y → Z and g : X → Y. Then fog : X → Z defined by

(fog) (x) = f(g(x)) for x $\in$ X. composite of two functions is also called function of a function.

**Note:** given two function f & g, it is important to note that fog is defined only when domain f contains range of g. and Similarly g of will be defined only when domain of g contains range of f.

**Example 1.** Let $f$ : N → N $f$ (n) = 2n, $\forall$n $\in$ N

g : N → N

and

g(n) = $n^2$, $\forall$ n $\in$ N

Then Q domain of g contains the range of f

Q gof : N → N

(gof) (n) = g(f(n)

= g (2n) = $(2n)^2$ = $4n^2$, $\forall$ n $\in$ N

also domain f contains range of g

Q fog : N → N

(fog) (n) = $f$(g(n))

= $f(n^2)$ = $2n^2$, $\forall$ n $\in$ N

This example shows that gof $\neq$ fog

**Inverse of a function:** A function $f$ : X → y is said to be invertible if there exists a function g : Y → X s.t.

fog = $I_y$ and gof = $I_x$, $I_y$ are respectively identity function on X and Y. The function g is called the inverse of f and is denoted by g = $f^1$.

**Theorem:** A function $f$: x → Y $f$ will have an inverse if $f$ is one-one and onto. Moreover when $f$ has an inverse, it is unique, we assume this result.

**Some Illustrated Examples**

**Example 1:** If A and B are two sets such that A∪B = 60 members, A has 25 members and B has 17 members.. Then find A∩B.

**Solution:** We shall use the result

n (A∪B) = n (A) + n (B) - n (A∩B) ....(1)

Here n (A∪B) = number of members in A∪B = 60

n (A) = 25, n (B) = 47

∴ From (1), we have

60 = 25 + 47 - n (A∩B)

⇒ n (A∩B) = 25 + 47 - 60 = 12

**Example 2.** How many subsets can be formed of the set X = {l, m, n}? List all these subset so obtained.

**Solution:** We have X = {l, m, n}

⇒ X has 3 distinct elements

⇒ number of subsets of X is $2^3$ i.e. B.

The subsets of X so obtained are as follows

ϕ, X, {l}, {m}, {n}, {l,m}, {m,n}, {n,l}.

**Example 3:** Write the following as intervals:

(i) A = {x| x ∈ R, -3 < x ≤ 5}

(ii) B = {y, y ∈ R, -2 ≤ y < 4}

(iii) C = {x| x ∈ R, 1 < x < 3}

(iv) D = {y| y ∈ R, 5 ≤ y ≤ 9}

**Solution:** (i) A = [-3, 5]

(ii) B = 9-2, 4]

(iii) C = [1, 3]

(iv) D = [5, 9]

**Example 4:** Find the domain and range of the function

$$f(x) = \frac{x+2}{2x+1}$$

**Solution:** $f(x) = \frac{x+2}{2x+1}$

For $f$ (x) to be defined, $2x + 1 \neq 0 \Rightarrow x \neq \frac{-1}{2}$

$\therefore$ Domain of $f$ = D$f$ = Set R of all reals except $\frac{-1}{2}$ = R - $\left\{\frac{-1}{2}\right\}$

To find Range, we put $f(x) = y$

$\Rightarrow \quad \frac{x+2}{2x+1} = y$

$\therefore \quad 2xy + y = x + 2$

$\Rightarrow \quad x(2y - 1) = 2 - y$

$\Rightarrow \quad x = \frac{2-y}{2y-1}$

Now $x \in$ R except $\frac{-1}{2}$

$\therefore \quad \frac{2-y}{2y-1} \in$ R except $y = \frac{1}{2}$

$\therefore \quad R_f$ = Range of $f$ = R - $\left\{\frac{1}{2}\right\}$

## 1.7 Self Check Exercise

Q.1 Let U = {1,2,3,4,5,6} and A = {3,4,5}, Find A' compliment of A.

Q.2 Find A∪B and A∩B if

A = {1,2,3,4,5,6}

B = {2,3,4,5,6,7}

Q.3 Find the domain and range of the function $f(x) = \log(x - 1)$

## 1.8 Summary

In this unit we learnt the following

(i) concept of a set, different types of ets, union and intersection of sets, Venn's digrem etc.

(ii) concept of number system, intervals (open and closed intervals)

(iii) functions, domain and range of a function, types of functions etc.

## 1.9 Glossary:

1. **cardinal number or order of a finite set -** The number of distinct or different element of a finite set is called order of a set

2. **Length of interval -** If (a, b), [a, b], [a, b] and (a, b] are intervals then b - a is called length of any of the above intervals.

3. Comparable and non-comparable sets- Two sets are said to be comparable if one of the two sets is a subset of the other.

4. Applications of sets in some important problems.

   (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

   (b) 4 A, B are disjoint sets i.e. $A \cap B = \phi$ then $n(A \cup B) = n(A) + n(B)$

   (c) $n(A^c \cup B^c) = n\left(\left(A \cap B\right)^c\right) = n(\cup) - n(A \cap B)$

   (d) $n(A) = n(A - B) + n(A \cap B)$.

## 1.10 Answer to Self Check Exercise

**Self Check Exercise - 1**

Ans.1 a set

Ans.2 {1, 3}

Ans.3 }x} $x = \frac{n}{n+1}$, $n \in N$ and $6 < n < 9$}

**Self Check Exercise - 2**

Ans.1 A = {1, 2, 6}

Ans. 2 $A \cup B = \{1,2,3.4,5,6,7\}$, $A \cap B = \{2,3,4,5,6\}$

Ans. 3 $D_f = (1, \infty)$, $R_c$ = Set of real = R.

## 1.11 References/Suggested Readings


1. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002.

2. G.B. Thomos and R.L. Finney, Calculus, Pearson Education, 2007.


## 1.12 Terminal Questions

1. Which of the following are sets? Give reason

   (i) The collection of all natural numbers less than 50

   (ii) The collection of all odd integers

   (iii) The collection of all months of a year beginning with letter M.

2. Which of these sets are equal?

{a, c, b}, {b, c, a}, {a, b, c}

3. Write the following intervals in set builder form:

(i) (-2, 1) (ii) {5, 10] (iii) (-6, 1)

4. For A, B two sets if

n (A) = 15, n (B) = 20, n ($A \cup B$) = 40, find n ($A \cap B$).

5. Find the domain and range of the following functions

(i) $f$(x) = $\sqrt{2-x}$

(ii) $f$(x) = $2^{3x}$

(iii) $f$(x) = $\frac{|x|}{x}$

(iv) g(x) = log (-x)

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**Unit - 2**

# Limit of A Function

# (Epsilon and Delta Definition)

**Structure**



## 2.1 Introduction

Dear students, we have learnt some basic concepts in unit I. Surely the idea of these concepts will help you to learn and to build a strong foundation for the theory of calculus. Limit of a function is one of the most fascinating idea in whole calculus and analysis.

## 2.2 Learning Objectives

The main objectives of this limit are

(i) to understand the concept of neighborhood of a point

(ii) to define limit of a function ($\in$-definition)

(iii) to find the limit of some important functions

(iv) to study Cauchy criterion for the existent of limit.

(v) to study algebra of limit

(vi) to study squeeze principle

(vii) to study infinite limits.

## 2.3 Neighborhood of a Point

Let a, b $\in$ R and c be any real number such that a < c < b then the open interval (a, b) is called a neighborhood (nhd) of c. Thus any open interval (a, b) containing the point c $\in$ R is called the nhd. of c. The length of a nhd (a, b) is measured by |b - a|. The nhd. (a, b) is called symmetric nhd of c iff |c - a|. For a given $\delta$ (delta) > 0, $(c - \delta, c + \delta) = \{x \in R|. |x - c| < \delta\}$ is called $\delta$-nhd of c and is symmetrical about the point c.

**Deleted Nhd.-** The set

$$\{x \in R|\ o < |x - c| < \delta\} = \{x \in R|\ |x - c| < \delta \text{ and } x \neq c\}.$$

$$= (c - \delta, c) \cup (c, c + \delta) \text{ is a deleted nhd of c.}$$

Thus any nhd. of c which does not contain the point c is called a deleted nhd. of c. That is, if c $\in$ (a, b), then (a, c) $\cup$ (c, b) is a deleted nhd. of the point c.

**Remark:** Let c $\in$ R and a, b $\in$ R s.t. a < c < b then [a, c] is called a left nhd of c and [c, b] is called a right nhd. of c.

## 2.4 Limit of a Function ($\in$- $\delta$ definition)

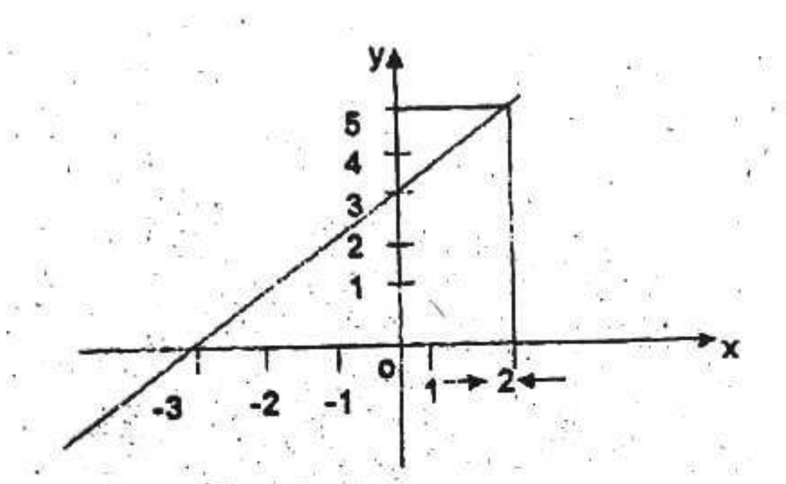
Let us first understand what do we mean by $x \to a$ (x approaches to a).

$x \to a$ means that $x \neq a$ but |x - a| is very small. So $x \to a$ means that there exists a positive number $\delta > 0$, however small such that $0 < |x - a < \delta$. In other words $x \in (a - \delta, a + \delta)$ and $x \neq a$ i.e. x 6 $(a - \delta, a) \cup (a, a + \delta)$.

Now Let us explain the limit of a function with the help of an example

**Example 1 :** Consider the function f(x) = x + 3

The graph of this function is shown. It is clear that as $x \to 2$ from left or right, the function is approaching 5. That is when x is near to 2, f(x) is close to 5.



**Example 2:** Consider the function $f(x) = x^2$ we have seen its graph earlier. It is plotted here also.

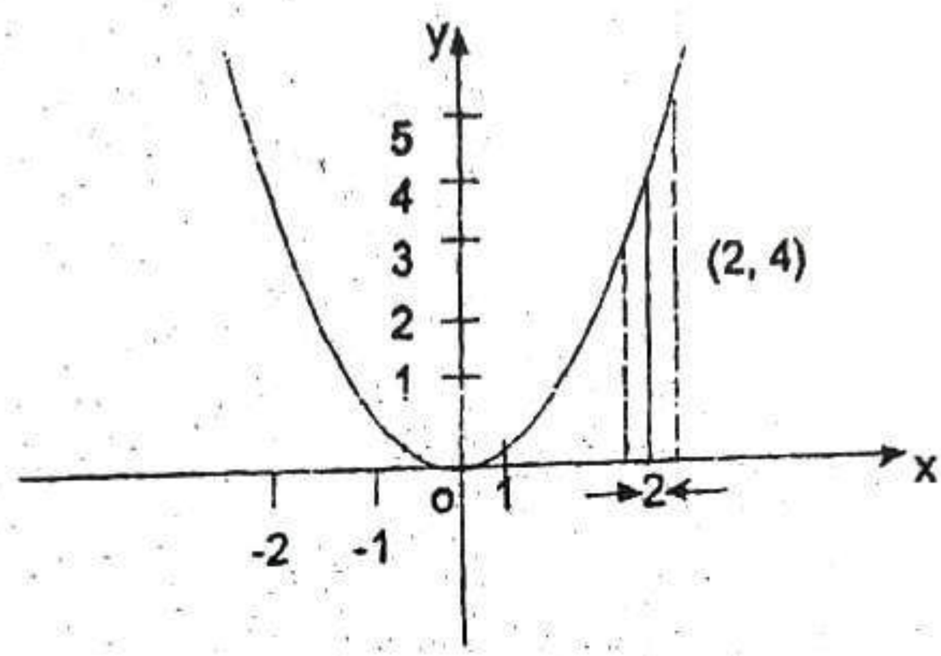



Table $x \to 2$ from left $x \to 2$ from right

| x | f(x) | | X | f(x) |
|---|---|---|---|---|
| 1.5 | 2.25 | | 2.5 | 6.25 |
| 1.6 | 2.56 | | 2.4 | 5.76 |
| 1.7 | 2.89 | | 2.3 | 5.29 |
| 1.8 | 3.24 | | 2.4 | 4.84 |
| 1.9 | 3.61 | | 2.1 | 4.41 |
| 1.95 | 3.80 | | 2.05 | 4.20 |
| 1.99 | 3.96 | | 2.02 | 4.08 |
| 1.996 | 3.996 | | 2.001 | 4.004 |

Please observe that as x approaches 2 from left i.e. approaching 2 always remaining less than 2, the graph of $f(x) = x^2$ approaches the point (2, 4) and the value of f(x) approaches 4.

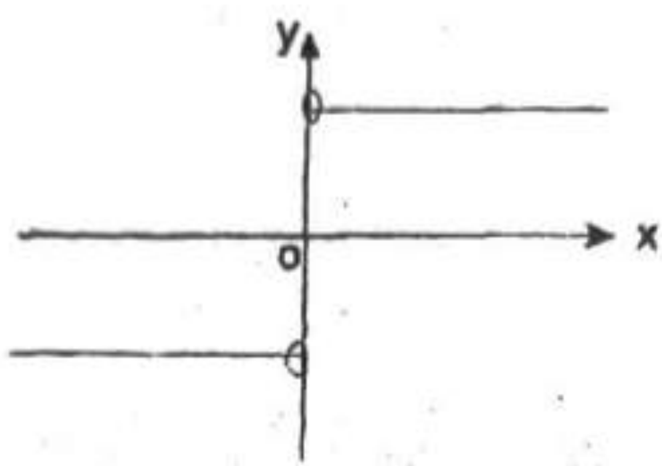
Also as x approaches 2 from right, i.e. approaching 2 always remaining greater than 2, the graph of the function $f(x) = x^2$ again approaches the point (2, 4) and f(x) approaches 4.

**Example 3.** Consider the function $f$(x) = $\begin{cases} 1, & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

This is signum function. We know its graphs

Table x → 0 from left x → 0 from right

| X | f(x) | | X | f(x) |
|---|---|---|---|---|
| -1 | -1 | | 1 | 1 |
| -0.5 | -1 | | 0.5 | 1 |
| -0.1 | -1 | | 0.1 | 1 |
| -0.01 | -1 | | 0.01 | 1 |
| -0.001 | -1 | | 0.001 | 1 |




We obwerve here

(1) as x approaches o from left f(x) remains at -1 here we say that f(x) approaches -1 as x approaches o from left.

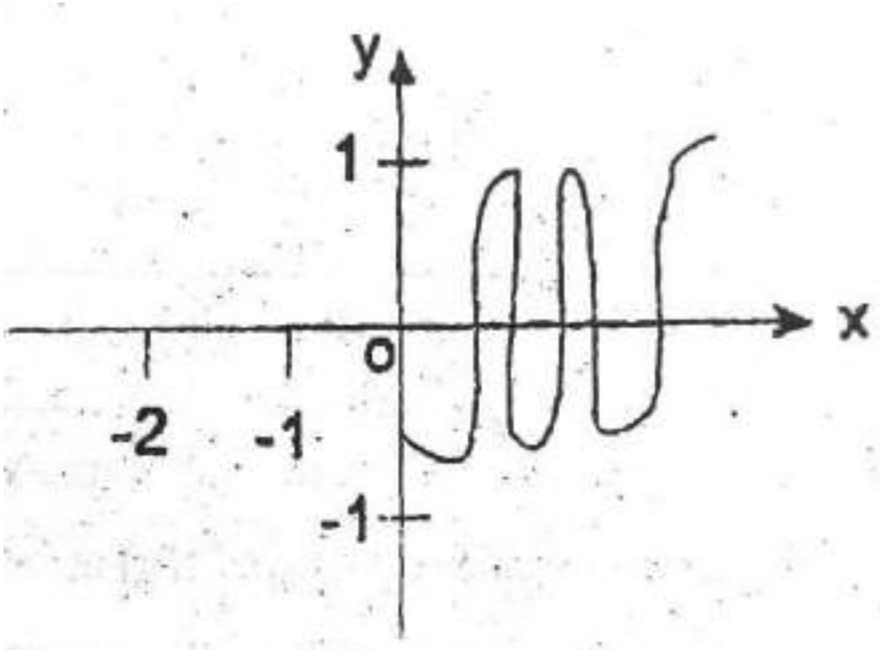
(2) as x approaches o from right f(x) remains at 1.

Here we say that f(x) approaches 1 as x approaches zero from right.

**Example 4.** Consider the function

$f$(x) = $\begin{cases} 0 \; if \; x \le 0 \\ \sin \frac{1}{x} if \;\; x > 0 \end{cases}$

graph of the function is




is Here we see that as x approaches zero from left f(x) approaches zero. However as x approaches zero from right, the graph f(x) oscillates too much between -1 & 1 i.e. f(x) does not stay close to any number.

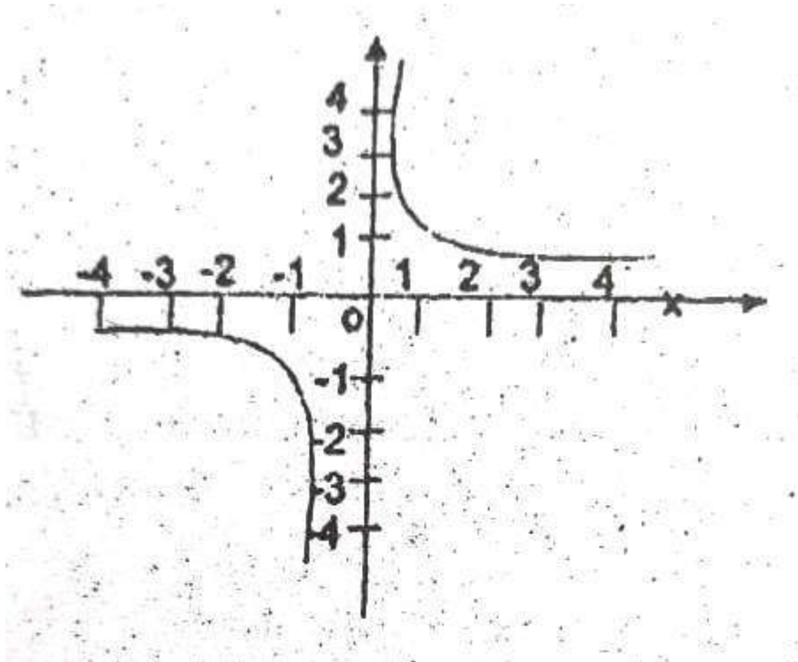
**Example 5:** Let us consider another function $f(x) = \frac{1}{x}$ its graph is shown here

Table

x → 0 from left x ← 0 from right

| X | $f(x) = \frac{1}{x}$ | | x | $f(x) = \frac{1}{x}$ |
|---|---|---|---|---|
| -1 | -1 | | 1 | 1 |
| -0.1 | -10 | | 0.1 | 10 |
| -0.01 | -100 | | 0.01 | 100 |
| -0.001 | -1000 | | 0.0001 | 1000 |
| -0.0001 | -10000 | | 0.0001 | 10000 |
| -0.00001 | -100000 | | 0.00001 | 1.00000 |

Here we note that




(1) as x approaches zero from left, we do not get a point of the type (0, y) on the graph of f. Rather f(x) goes on decreasing as we near zero from left.

(2) as x approaches zero from right, the function f(x) goes on increasing, it does not approach a finite number.

In conclusion, we see

(i) that in example 1 and 2 f(0x approaches a definite number whether we approach the point from left, or right.

(ii) In example 3 f(x) approaches definite numbers when we approach the point from left and right but these numbers are different.

(iii) In example 4 f(x) approaches a definite number when we approach the point from left but does not approach any number when we approach the point from right.

(iv) In example 5 f(x) does not approach a definite point when we approach the point either from left or from right

The above discussion shows that it is not only useful but also necessary to consider the limiting behaviour of a function separately for points lying on the left and right of a given point. Let us define these concepts now.

**Definition:** Let f be a real valued function and an arbitrary point in the domain of definition of f.

**Left hand limit at a:-** The function is said to have a left hand limit 1 (a real number) at a if given $\in > 0$ there exists a real number $\delta > 0$ ( $\delta$ depending on $\in$) such that f(x) is defined for all x in (a- $\delta$ , a) and

$$|f(x)-l| < \in$$

The symbol lim f(x) = L or f(x) $\to \alpha^{-}$ (x $\to \alpha$ - 0) are used to express that 1 is a left hand

$x \to \alpha^-$

limit of f(x) at a. this is also called limit from below at a, and is denoted as $f(\alpha^-)$ or $f(\alpha - 0)$

**Right hand limit at a:-**

The function f(x) is said to have right hand limit 1 (a real number) at a if given $\in > 0$ there exists a real number $\delta > 0$ ($\delta$ depending on $\in$) such that f(x) is defined for all x in (a, a + $\delta$ ) and $|f(x) - l| < \in$

The symbol $\lim f(x) = 1$ and $f(x) \to l$ as $x \to a+$ $(x \to a + 0)$ are used to express that 1 is a right

$x \to a^+$

hand limit of f(x) at a. This limit is also called limit from above at a and is often denoted by $f(a^+)$ or $f(a + 0)$

**Limit of a function at a:-**

A function f(x) is said to approach a limit 1 (1 a real number) as x approaches a if for each $\in>0$, there exists a positive number $\delta$ ($\delta$ depending on $\in$) such that when ever $0 < |x - a| < \delta$, $f$ (x) is defined and $|f(x) - l| < \in$.

The symbol $\lim f(x) = 1$ and $f(x) \to l$as $x \to a$ are used to denote the fact that f(x) approaches

$x \to a$

1 as x approaches or tends to a.

If there exists no such 1, we say that f(x) does not have a limit as x approaches a.

**Note:** Since the statement "h approaches o" is equivalent to the statement "a +h approaches a" if follows that lime $f(x) \to l \Leftrightarrow f(a + h) \to l$ as $h \to 0$ We will use this fact many a time.

The following theorem follows obviously from the above definitions.

Theorem $\lim_{x \to a} f(x) = l \Leftrightarrow \lim_{x \to a^+} f(x) = l$ and $\lim_{x \to a^-} f(x) = l$

Therefore in order to prove that $\lim_{x \to a} f(x)$ does not exist, it is enough to prove that either one of the one sided limits does not exist or both are not equal, in case they exist.

If $f(x) \to l$ as $x \to a$, then theoretically, given $\in>0$ it should always be possible to find $\delta > 0$ satisfying the requirements of the definition of limit. But in practice most of the time for a given $\in>0$ it is not at all easy to find the required $\delta$. In other words while computing the limit it is not easy to apply the definition directly. So we state below a theorem, without proof, which will simplify our effort in finding the limit in a large number of cases without explicit use of the definition.

## 2.4 Algebra of Limits

Theorem: Let f and g be two real valued function such that

$\lim_{x \to a} f(x) = l$ $\lim_{x \to a} g(x) = m$

Then

(i) $\lim_{x \to a} (l f)(x) = \alpha L$, where $\alpha$ is a real number

(ii) $\lim_{x \to a} (f \pm g)(x) = l \pm m$

(iii) $\lim_{x \to a} (fg)(x) = l m$

(iv) $\lim \left(\frac{1}{f}(x)\right) = \frac{1}{l}$ if $l \neq 0$

We are assuming here that both f and g are defined in some deleted neighborhood of a From (iii) & (iv) it follows

$$\lim_{x \to a} \frac{f(x)}{g(x)} \quad \lim_{x \Rightarrow a} f(x) \quad \lim_{x \to a} \frac{1}{g(x)} = \frac{m}{l} \ (1 \neq 0)$$

We will consider examples using above theorem.

**Example 5:** $\lim_{x \to a} (\alpha + \beta x) = \alpha + \beta a$, where $\alpha$, $\beta$ are cons tan ts

**Solution:** We will first prove that

$\lim_{x \to a} x = a$ and $\lim_{x \to a} \alpha = \alpha$

This we will prove by using the definition of lmit.

In the first case f(x) = x

Here f(x) = x

Let $\in > 0$ be any real number however small. We choose $\delta = \in$. Then For all x,

$0 < |x - a| < \delta = \in$

$|f(x) - a| = |x - a| < \in (= \delta)$

$\therefore$ By definition of lim it $\Rightarrow \lim_{x \to a} x = a$ .....(1)

In the second case $f(x) = \alpha$, Let $\in > 0$ be any real here $|f(x) - \alpha| = |\alpha - \alpha| = 0 < \in$ for any number $\delta \to 0$

$\therefore \lim_{x \to a} \alpha = \alpha$ ......(1)

Now by applying the above theorem

$\lim_{x \to a} \alpha + \beta x$ $\lim_{x \to a} \alpha + \lim_{x \to a}$ $\beta x$

$= \alpha + \beta \lim x$

$x \to a$

$= \alpha + \beta a$

**Example 6:** Show $\lim x^n = a^n$ for all integer $n \geq 1$.

$x \to a$

**Solution:** We will prove by using the principle of mathematical induction for n = 1

We have

$\lim_{x \to a} x^1 = \lim_{x \to a} x$ = a (from previous example

Let it hold for n = m. That is we assume that $\lim_{x \to a}$ $x^m = a^m$ ....(1)

Consider $\lim_{x \to a} x^{m+1} = \lim_{x \to a}$ $(x^m\ x)$

$= \lim_{x \to a} x^m \lim_{x \to a}$

$= a^m\ a = a^{m+1}$

therefore by the principle of mathematical induction result hold for all $n \geq 1$.

i.e. $\lim_{x \to a} x^n = a^n$

**Example 7:** Limit of polynomial

Let $\alpha_0, \alpha_1, \alpha_2, .......\alpha$ be real number. Then

$$\lim_{x \to a} \left( \sum_{j=0}^{n} a_j x^j \right) = \sum_{j=0}^{n} a_j x^j$$

**Solution:** Using the above stated theorem

$$\therefore \quad \lim_{x \to a} \left( \sum_{j=0}^{n} a_j x^j \right) = \sum_{j=0}^{n} \lim_{x \to j} (a_j x^j)$$

$$= \sum_{j=0}^{n} \alpha_j \quad \lim_{x \to a} x^j$$

$$= \sum_{j=0}^{n} \alpha_j a^j \qquad \text{.................... (1)}$$

Thus what we have proved it

i.e. if $f$(x) = $\sum_{j=0}^{n} \alpha_j x^j$ is a polynomial

Then $f$(a) = $\sum_{j=0}^{n} \alpha_j a^j$

$\therefore$ from (1) we have

$$\lim_{x\to a}\overset{x\to a}{f}(x) \qquad = \overset{x\to a}{\lim_{x\to a}} \qquad \left(\sum_{j=0}^{n} \alpha_j x^j\right) = \sum_{j=0}^{n} \alpha_j a^j = f(\mathrm{a})$$

**Example 8:** Limit of rational function Let f and g be two polynomials and g(a) $\neq$ 0. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$$

**Solution:** This follows from the I$f$(x)ous example and using the theorem as

lim $f$(x)

$$= \frac{x \to a}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)}$$

**Example 9:** Evaluate (1) $\lim_{x\to 2} x$ (x - 1)

(ii) $\lim_{x\to 2} \dfrac{x^2+4x}{x+2}$

(iii) $\lim_{x\to 3} \dfrac{x^2+4x}{x-2}$

(iv) $\lim_{x\to 1} \dfrac{x^2-3x+2}{x^2-1}$

**Solution:** $\lim_{x\to 2} x(x-1)$ = $\lim_{x\to 2} x \quad \lim_{x\to 2}(x-1)$

= 2(2 - 1) = 2.1

= 2

(ii) $\lim_{x\to 2} \dfrac{x^2+4x}{x+2} = \dfrac{x \to 2}{\lim_{x\to 2}(x+2)} = \dfrac{2^2+4.2}{2+2} = \dfrac{4+8}{4} = \dfrac{12}{4} = 3$

(iii) $\lim_{x\to 3} \frac{x^2+4x}{x-2} = \frac{x\to 3}{\lim\limits_{x\to 3}(x-2)} = \frac{3^2-4.3}{3-2} = \frac{9-12}{1} \quad \frac{-3}{1} = -3$

(iv) $\lim_{x\to 1} \frac{x^2-3x+2}{x^2-1} = \lim_{x\to 1} \frac{(x-2)(x-1)}{5,(x-1)(x+1)} = \lim_{x\to 1} \frac{(x-2)}{(x+1)} = \frac{1-2}{1+1} = \frac{1}{2}$

**Example 10:** Evaluate

(i) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

(ii) $\lim_{x\to 0} \frac{(2x-3)\sqrt{x}-1}{3x^2+3x-6}$

**Solution:** (i) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \quad \lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} \text{ X } \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$

$= \lim \frac{(1+x)-(1-x)}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$

$= \lim_{x\to 0} \frac{2x}{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}$

$= \lim_{x\to 1} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1+0}+\sqrt{1-0}} = \frac{2}{2}$

(ii) $\lim_{x\to 1} \frac{(2x-3)\sqrt{x}+1}{(3x^2+3x-6} = \lim_{x\to 1} \frac{(2x-3)\sqrt{x}-1}{3\left(x^2+x-2\right)}$

$= \lim_{x\to 1} \frac{(2x-3)\sqrt{x}-1}{3(x+2)(x-1)}$

$= \lim_{x\to 1} \frac{(2x-3)\sqrt{x}-1}{3(x+2)\left(\sqrt{x}-1\right)\left(\sqrt{x}+1\right)}$

$= \lim_{x\to 1} \frac{(2x-3)}{3(x+2)\left(\sqrt{x}+1\right)} = \frac{(2.1-3)}{3(1+2)\left(\sqrt{1}+1\right)} = \frac{-1}{3.3.2} = \frac{-1}{18}$

**Theorem:** Let n be any positive integer. Then

$$\lim_{h\to 0} \frac{x^n - a^n}{x-a} = na^{n-1}$$

**Proof:** Let x = a + h then x → a iff h → O. Then

$$\frac{x^n - a^n}{x-a} = \frac{(a+h)^n - a^n}{a+h-a} = \frac{1}{h}\left[(a+h)^n - a^n\right]$$

$$= \frac{1}{h}\left[a^n + n_{c_1}a^{n-1}h + n_{c_2}a^{n-2}h^2 + .......h^n\right] - a^n \text{ [} \because \text{ Using binomial Theorem]}$$

$$= \frac{1}{h}\left[n_{c_1}a^{n-1} + n_{c_2}a^{n-1} + n_{c_2}a^{n-2}h + ....... + g^n\right]$$

$$= n_{c_1}a^{n-1} + n_{c_2}a^{n-2}h + ....... + h^{n-1}$$

$$\Rightarrow \quad \lim_{x\to 1} \frac{x^n - a^n}{x-a} = \lim_{h\to 0}\left\lfloor n_{c_1}a^{n-1} + n_{c_2}a^{n-2}h + ....... + h^{n-1}\right\rfloor$$

$$= n_{c_1}a^{n-1} + n_{c_2}a^{n-2}0 + ....... + 0$$

$$= n_{c_1}a^{n-1} + 0 + ....... + 0$$

$$= na^{n-1} \qquad \left[\because n_{c_1} = n\right]$$

Note: This theorem holds when a is any rational number and a is positive. We shall assume this.

**Example 11:** Evaluate $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$

**Solution:** The given expression can be written as

$$\frac{x^3-2^3}{x^2-2^2} = \frac{x^3-2^3}{x-2} = \frac{x^2-2^2}{x-2}$$

$$\lim_{x\to 1} \frac{x^3-8}{x^2-4} \qquad \lim_{x\to 1} \frac{x^3-2^3}{x-2} + \lim_{x\to 1} \frac{x^2-2^2}{x-2}$$

$3.2^{3-1} + 2.2^{2-1}$

= 12 + 4 = 3 Ans.

**Example 12:** Evaluate $\lim_{x\to 0} \frac{(1+x)^n - 1}{x}$

**Solution :** Put y = 1 + x now Then as x → 0, y → 1

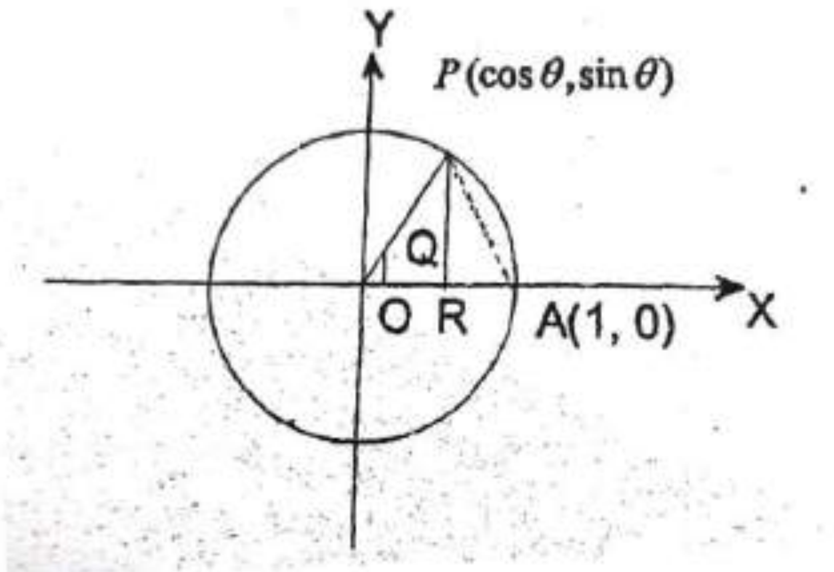
$$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} \qquad \lim_{y\to 0} \frac{y^n (1)^n}{y-1} = n(1)^{n-1} = n$$

**Limits of trigonometric function**

We shall need following reslults to evaluate the limits of trigonometrically functions

**Theorem :** i) For $0 < |\theta| \leq \frac{\pi}{2}$, $|\sin \theta| \leq |\theta|$

ii) For $0 < |\theta| < \frac{\pi}{2}$, $|\theta| < |\tan \theta|$




**Proof :** (1) The result is obvious $\theta = \frac{\pi}{2}$

For $0 < \theta < \frac{\pi}{2}$, Let $(\cos \theta, \sin \theta)$ be the coordinates of the point P on a unit circle.

Then $\sin \theta$ = length of PR < length of the chord PA which is less than the length of the are AP that equals $\theta$. Thus $\sin \theta < \theta$ for $0 < \theta < \frac{\pi}{2}$

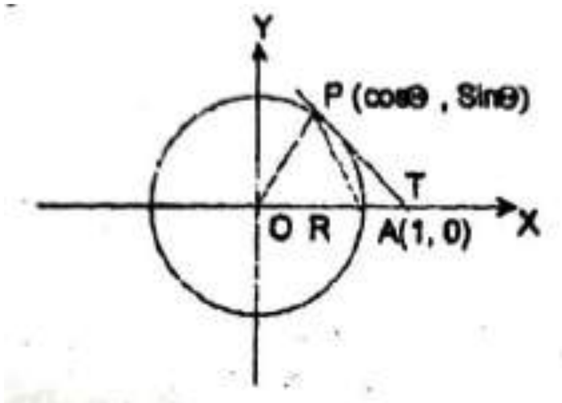
If $-\frac{\pi}{2} < \theta < 0$ then

$|\sin \theta| = -\sin \theta = \sin(-\theta) < -\theta = |\theta|$

completing the proof of (i)

(ii) for $0 < \theta < \frac{\pi}{2}$, Let $(\cos\theta, \sin\theta)$ be the coordinates of the point on the unit circle with centre (0, 0). Let PT be the tangent to the circle C at P. From the figure it is clear that Area of

$\Delta$OAP < area of sec oAP < area of $\Delta$OPT




i.e. $\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{\tan\theta}{2}$

$\Rightarrow$ $\sin\theta < \theta < \tan\theta$ for $o < \theta < \pi/2$ ...(1)

If $-\frac{\pi}{2} < \theta < o$, then $0 < \theta < \frac{\pi}{2}$ and hence

$\sin(-\theta) < -\theta < \tan(-\theta)$ ...(ii)

since $\sin(-\theta) = -\sin\theta = |\sin\theta|$, $|\theta| = -\theta$ and $\tan(-\theta) = -\tan\theta = |\tan\theta|$

$\therefore$ from (ii) it follows

$|\sin\theta| < |\theta| < |\tan\theta|$ for $-\pi/2 < \theta < 0$

From (1) & (ii) it follows that

for $0 < |\theta| < \frac{\pi}{2}$, we have

$|\sin\theta|\ |\theta| < |\tan\theta|$

**Example 13 :**

(i) $\lim_{x\to 0} \sin x = 0$, (ii) $\lim_{x\to 0} \cos x = 1$

(iii) $\lim_{x\to c} \sin x = \sin c$ (iv) $\lim_{x\to c} \cos x = \cos c$

**Solution :**

(i) Let $\in > 0$ be a given real number.

For $|x| < \in \leq \frac{\pi}{2}$ we have

$|\sin x| < |x| < \in$. For $\in > \frac{\pi}{2}$ we have $|\sin x| < x1 < \in$ for all n. Thus $\sin x \to 0$ as $x \to 0$

**(ii) We know that**

$\cos x = 1 - 2\sin^2 x/2$

$$\therefore \quad \lim_{x\to 0} \cos x = \lim_{x\to 0} \left(1 - 2\sin^2 \frac{x}{2}\right)$$

$$\lim_{x\to 0} 1 - 2\left[\lim_{x\to 0} \sin \frac{x}{2}\right]^2$$

$= 1 - 2.0 = 1$

**(iii) We know that**

$\sin(c+h) = \sin \cosh + \cos c \sin h$

$$\therefore \quad \lim_{h\to 0} \sin(c+h) = \sin c \lim_{h\to 0} \cosh + \cos c \lim_{h\to 0} \sin h$$

$= \sin c.1 + \cos c.0$

$= \sin c.$

Since $\lim_{x\to c} \sin x = \lim_{h\to 0} \sin(c+h)$

$$\therefore \quad \lim_{x\to c} \sin c = \lim_{h\to 0} \sin(c+h) = \sin \subset$$

**(iv) We know that**

$\cos(c+h) = \cos c \cosh - \sin c \text{ shin } h$

$$\therefore \quad \lim_{x\to c} \cos x = \lim_{h\to 0} \cos(c+h) = = \cos \lim_{h\to 0} \cosh - \sin c \lim_{h\to 0} \sinh$$

$= (\cos c) \times 1 - (\sin c) \times 0$

$= \cos c$

**Limit of composite function**

**Theorem:** Let f and g be two real valued functions such that domain of g contains range of f. If

$\lim_{x\to a} f(x) = b$ and $\lim_{x\to b} g(x) = 1$, Then

$$\lim_{x\to a}(gof)(x) = 1$$

We shall assume this result without proof.

**Example 14:** Prove that

(i) sin (ax + b) → sin(aα + b) as x → α

**Solution:** Let f(x) = ax + b and g(y) = sin y

Then (gof) (x) = g(f(x) = g(ax+b)

= sin (ax+b)

$\therefore$ $\lim_{x\to a}(ax+b) = a\alpha + b$

$and \lim_{y\to a\alpha+b}\sin y = \sin(a\alpha + b)$

so by composite rule sin(ax+b) = sin(aα + b) Ans

**Example 15:** Limit of Exponential functions

We shall prove that

i. $\lim_{x\to c} e^x = e^c$ and (ii) $\underset{h\to 0}{Lt} \dfrac{e^h - 1}{h} = 1$

**Solution:** We have a result which states that

$$1 \leq \frac{e^h - 1}{h} \leq \frac{1}{1-h} \text{ for } 0 < h < 1$$

Or $h \leq eh - 1 \leq \dfrac{h}{1-h}$ .....(1)

now $\lim_{x\to 0^+} h = 0$ also $\lim_{h\to 0} \dfrac{h}{1-h} = 0$

$\therefore$ by squeeze principle from (1)

$\lim(e^{h-0} - 1) = 0$ or $\lim e^h - 1$ .....(ii)

h → $0^+$ h → $0^+$

Since h → $0^-$ ⇔ -h → 0+, we have

$\lim_{h\to 0^-} e^h = \lim_{h\to 0^+} e^{-h} \lim_{x\to a} \dfrac{1}{e^n} = 1$ ......(iii)

$\therefore$ from (ii) and (iii)

$\lim\limits_{h \to 0}$ $e^h = 1$

Now $\lim\limits_{x \to c} e^x$ = $\lim\limits_{h \to 0} e^{c+h}$ $e^c$ $\lim\limits_{h \to 0} e^h$ = $e^c.1 = e^c$

(ii) For o < h < 1, we have

$1 \le \dfrac{e^{h-1}}{h} \le \dfrac{1}{1-h}$ .... (iv)

$\lim\limits_{h \to 0} \dfrac{1}{1-h} = 1$

therefore by squeeze principle, it follows from (iv) that

$\lim\limits_{h \to 0^+} \dfrac{e^h - 1}{h} = 1$ (v)

If - 1 < h < 0, then 0 < - h < 1 and

$\dfrac{e^h - 1}{h} = \dfrac{1}{e^{-h}} \dfrac{\left(e^h - 1\right)}{-h}$

$\therefore \lim\limits_{h \to 0^-} \dfrac{e^h - 1}{h} = \lim\limits_{h \to 0^+} \dfrac{1}{e^{-h}} \lim\limits_{x \to 0} \dfrac{e^h - 1}{h}$

= 1.1 = 1

from (v) & (vi) it follows that

$\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$

**Example 16:** Prove that

(i) $\lim\limits_{x \to 0}$ $(e^x - e^{-x}) = 0$

(ii) $\lim\limits_{x \to 1} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^2 - 1}{e^2 + 1}$

$(e^x + e^x + 2) = \dfrac{\left(e + 1\right)^2}{2}$

(iii) $\lim\limits_{x \to 1}$

**Solution:**

(1) $\lim\limits_{x \to 0}\left(e^x - e^{-x}\right)$

$$= \lim_{x\to 0} e^x - \lim_{x\to 0} e^{-x}$$

$$= 1 - \lim_{x\to 0} \left(\frac{1}{e^x}\right)$$

$$= 1 - \frac{1}{1} = 0 \text{ Ans.}$$

(ii) $$\lim_{x\to 1} \frac{e^x - e^{-x}}{e^x + e^x} = \lim_{x\to 0} \frac{e^{2x} - 1}{e^{2x} + 1}$$

(lim $e^x$)$^2$ - 1

$$= \frac{x \to 1}{\left(\lim_{x\to 1} e^x\right)^2 + 1} = \frac{e^2 - 1}{e^2 + 1} \text{ Ans.}$$

(iii) $$\lim_{x\to 1}\left(e^x + e^{-x} + 2\right) = \lim_{x\to 1} e^x + \frac{1}{\lim_{x\to 1} e^x} + \lim_{x\to 1} 2$$

$$= e + \frac{1}{e} + 2 = \frac{e^2 + 2e + 1}{e} = \frac{(e+1)^2}{e}$$

## 2.6 Self Check Exercise - 1

Q.1 Evaluate

$$\frac{Lim}{x \to 1} \frac{x^2 - 1}{x \to 1}$$

Q.2 Evaluate

$$\frac{Lim}{x \to 1} \frac{1}{x} [x]$$

Q. 3 Evaluate

$$\lim_{x\to 0} \frac{\sin ax}{x}, a \neq 0$$

## 2.7 Squeeze Principle

**Theorem:** Let $f_1(x) < g(x) < f_2(x)$ for all x in some deleted nhd of a, i.e., for all $x \in (a - \lambda, a + \lambda)$, x + a

If $\lim_{x\to a} f_1(x) = l = \lim_{x\to a} f_2(x)$, then

$$\lim_{x \to a} g(x) = l$$

**Proof:** Let $\in > 0$, then their exists $\delta_1, \delta_2 > 0$ s.t.

$0 < \delta_1, \delta_2 < \lambda$ and

$0 < |x - a| < \delta_1 \Rightarrow |f_1(x) - l| < \in$

$\Rightarrow \quad l - \in < f_1(x) < l + \in \quad$ .....(1)

and

$0 < |x - a| < \delta 2 \quad \Rightarrow \quad |f_2(x) - l| < \in$

$\Rightarrow \quad l - \in < f_2(x) < l + \in \quad$ ..... (2)

Let $\delta = \min\{\delta_1, \delta_2\}$, then $0 < |x - a) < \delta$.

Therefore, from (1) and (2), we have

$l - \in < f_1(x) < g(x) < f_2(x) < l + \in$

$\Rightarrow \quad |g(x) - l| < \in \quad$ for $\quad 0 < |x - a| < \delta$

i.e. $\lim_{x \to a} g(x) = l$

Hence the proof

**Some Illustrated Examples**

**Example 17:** Show that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

**Solution:** We confine our attention to those x for which $0 < |x| < \frac{\pi}{2}$

We know that

$|\sin x| < |x| < |\tan x|$

$\Rightarrow \quad 1 < \frac{|x|}{|\sin x|} < \frac{|\tan x|}{|\sin x|} \quad$ for $0 < |x| < \frac{\pi}{2}$

$\Rightarrow \quad 1 < \frac{|x|}{|\sin x|} < \left|\frac{1}{\cos x}\right|$

$\Rightarrow \quad |\cos x| < \left|\frac{\sin x}{x}\right| < 1$

Now since $|\cos x| \to 1$ as $x \to 0$ and $\lim_{x \to 0} 1 = 1$

$\therefore$ By Squeeze Principle

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

**Example 18:** Show that $\lim_{x \to 0} e^x = 1$

**Solution:** We know that

$$1 \leq \frac{e^x - 1}{x} \leq \frac{1}{1-x} \qquad \text{for} \qquad 0 < x < 1$$

$$\Rightarrow \qquad x \leq e^x - 1 \leq \frac{x}{1-x} \qquad \text{for } 0 < x < 1$$

$$\Rightarrow \qquad x + 1 \leq e^x \leq 1 + \frac{x}{1-x}$$

Now $\lim_{x \to 0} x + 1 = 1$

and $\lim_{x \to 0} \left(1 + \frac{x}{1-x}\right) = \lim_{x \to 0} + \lim_{x \to 0} \frac{x}{1-x}$

$\therefore$ By squeeze Principle

$$\lim_{x \to 0} e^x = 1$$

## 2.8 Infinite Limits

A function f is said to have limit l as $x \to \infty$ if for given $\in > 0$, however small, there exists $0 < M \in R$ s.t.

$|f(x) - l| < \in \forall$ (for all) $x > M$ and we write this as $\lim_{x \to \infty} f(x) = l$

**Some Illustrated Examples**

**Example 19:** Prove by using definition that

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

**Solution:** Let $f(x) = \frac{1}{x}$ and $\in > 0$, however small be given

then $|f(x) - 0| = \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| = \frac{1}{x} < \in$ whenever $x > \frac{1}{\in}$

Thus for given $\in$ > 0 E M = $\frac{1}{\in}$ > 0 s.t.

|f(x) - 0| < $\in$

$\therefore$ $\lim_{x\to\infty}$ f(x) = 0

**Example 20** Evaluate $\lim_{x\to\infty} \frac{5x^2+3x+1}{5x^2+2x+1}$

**Solution:** $\lim_{x\to\infty} \frac{5x^2+3x+1}{5x^2+2x+1}$

$= \lim_{x\to\infty} \frac{5+\frac{3}{x}+\frac{1}{x^2}}{5-\frac{2}{x}+\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{5+0+0}{5-0+0}$

$= \frac{5}{5} =$

## 2.9 Self Check Exercise-2

Q.1 Evaluate the limit

$$\lim_{x\to 0} \frac{\sin^{-1} x}{x}$$

Q.2 Find $\lim_{x\to 0} \frac{a^x-1}{x}$

Q.3 Evaluate $\lim_{x\to\infty} \left(1+\frac{1}{ax}\right)^{bx}$

Q.4 Evaluate $\lim_{x\to\infty} \frac{4x^3-3x+6}{5x^3+2x^2-3}$

Q.5 Using squeeze Principle, prove that

$$\lim_{x\to 0} \frac{e^x-1}{x} = 1$$

## 2.10 Summary

In this unit we learnt the following:

(i) The concept of limit of a function

(ii) The limit of rational function, limits of trigonometric functions, Limit of composite functions and limit of exponent rat functions.

(iii) The Squeeze Principle

(iv) The infinite limits etc.

## 2.11 Glossary:

**i. Cauchy's Criterion -** The $\lim_{x \to a} f(x)$ exists if given $\in > 0$, however small, $\exists$ a positive real number $\delta(\in)$, s.t.

$|f(x_1) - f(x_2)| < \in$ for $0 < |x_i - a| < \delta$

where i = 1, 2

**ii. (a)** $\lim_{x \to \infty} [f(x) \pm g(x)] = \lim_{x \to \infty} f(x) \pm \lim_{x \to \infty} g(x)$

(b) $\lim_{x \to \infty} [f(x).\, g(x)] = \left[\lim_{x \to \infty} f(x)\right] \left[\lim_{x \to \infty} g(x)\right]$

(c) $\lim_{x \to \infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to \infty} f(x)}{\lim_{x \to \infty} g(x)}$, provided $\lim_{x \to \infty} g(x) \neq 0$.

## 2.12 Answer to Self Check Exercise

**Self Check Exercise - 1**

Ans.1 2

Ans.2 2

Ans.3 a

**Self Check Exercise - 2**

Ans.1 log a, a > 0

Ans. 2 $e^{b/a}$

Ans. 3 $4/5$

Ans. Use the result $1 \leq \dfrac{e^x - 1}{x} \leq \dfrac{1}{1-x}$

## 2.13 References/Suggested Readings

1. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002.

2. G.B. Thomos and R.L. Finney, Calculus, Pearson Education, 2007.

## 2.14 Terminal Questions

1. Using definition of limit, prove that

   $\lim_{x \to 2}$ (3 - 2x) = -1

2. Prove that $\lim_{x \to 2} \dfrac{x-2}{x^2-4} = \dfrac{1}{4}$

3. Evaluate $\lim_{x \to 1} \dfrac{|x-1|}{2(x-1)}$ (if exists)

4. Evaluate $\lim_{x \to \frac{\pi}{2}} \dfrac{1-\cos x}{1+\cos x}$

5. Using Squeeze Principle, prove that

   $\lim_{x \to 0} \dfrac{a^x - 1}{x}$ = log a : a > 0

6. Using definition, prove that

   $\lim_{x \to \infty} \dfrac{3x+4}{x-1}$ = 3

7. Evaluate $\lim_{x \to \infty} \dfrac{e^{\frac{1}{x}} + 4}{e^{\frac{1}{x}} - 4}$

8. Evaluate $\lim_{x \to \infty} \left(1 + \dfrac{2}{x}\right)^x$

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**Unit - 3**

# Continuity of Function
# &
# Types o Discontinuity

**Structure**



## 3.1 Introduction

Dear students, we have already discussed various types of functions and sketched their graph in Unit 2. We have seen that graphs of some function are continuous (without any break), whereas, some of them are discontinuous in nature for instance, sin x, cos x and |x| are continuous functions whereas [x], tan x, cot x etc are discontinuous functions for all x. The smoothness of a graph of a function varies about a point of its domain that predicts the behaviour of a function and is of great importance. Obviously, limit of a function helps us to study the above said course.

## 3.2 Learning Objectives

The main objectives of this unit are :

(i) To study the continuity of function

(ii) To know the types of discontinuity

(iii) To study the algebra of continuous function

## 3.3 Continuity of a Function

Definition : A function f is said to be continuous at a point $x_0$, if it is defined in some neighourhood of $x_0$ and

$$\lim_{x \to x_0} f(x) = f(x_0)$$

i.e. $f$ is continuous at $x = x_0$ if given $\in > 0$, however small, $\exists$ a +ve real number $\delta(\in)$ such that

$$|f(x) - f(x_0)| < \in \text{ for } |x - x_0) < \delta.$$

Now $\lim_{x \to x_0} f(x)$ exists iff

$$\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$$

Hence $f(x)$ is continuous at $x = x_0$ if it is left as well as right continuous at $x_0$. Consequently a function which is not continuous at a point is said to be discontinuous at the point.

**Note :** 1. A function $f$ is said to be continuous on a set A if it is continuous at every point of A. If A is the domain of f, then f is called a continuous function on A.

2. A function $f$ is discontinuous at $x = x_0$ in the following cases:

(i) $f$ is not defined at $x = x_0$ i.e. $f(x_0)$ does not exist.

(ii) $\lim_{x \to x_0} f(x)$ does not exist. This is due to following three cases.

Case 1. $\lim_{x \to x_0^-} f(x)$ does not exist

Case 2. $\lim_{x \to x_0^+} f(x)$ does not exist

Case 3. $\lim_{x \to x_0^-} f(x)$ and $\lim_{x \to x_0^+} f(x)$ both exist but are not equal.

(iii) $\lim_{x \to x_0} f(x)$ exists but $\lim_{x \to x_0} f(x) \neq f(x_0)$.

## 3.4 Type of Discontinuous

### I Removable Discontinuous

It may happen sometimes that $f$ is not defined at $x = x_0$ or $f(x_0)$ is defined in such a way that $\lim_{x \to x_0} f(x) \neq f(x_0)$.

In such cases, the discontinuity can be removed by redefining $f(x_0)$ in such a way that $\lim_{x \to x_0} f(x) \neq f(x_0)$.

This type of discontinuity is referred to as removable discontinuity

### II Discontinuity of First Type (kind)

If $\lim_{x \to x_0^-} f$ (x) or $\lim_{x \to x_0^+} f$ (x) both exist finitely but they are not equal, then $f$ (x) is said to have discontinuity of firs type or first kind or ordinary discontinuity at $x_0$.

**III Discontinuity of Second Type (kind)**

If $\lim_{x \to x_0^-} f$ (x) or $\lim_{x \to x_0^+} f$ (x) does not exist, then $f$ (x) is said to have discontinuity of second type or second kind.

Note : 1. A real valued function $f$ defined on an Open Interval (a, b) is said to be continuous on (a, b) iff $f$ is continuous at x = c for all c $\in$ (a, b).

2. A real valued function $f$ is continuous on the closed interval [a, b] iff

(i) $f$ is right continuous at x = a i.e. $\lim_{x \to a^+} f$ (a)

(ii) $f$ is right continuous at x = b i.e. $\lim_{x \to b^-} f$ (b)

(iii) $f$ is continuous at c $\forall$ c $\in$ [a, b] i.e. $\lim_{x \to c} f(x) = f$ (c) $\forall$ c $\in$ [a, b].

Importantly that such a function has continuous graph on [a, b]

3. The polynomial functions, the rational functions, constant function, sin x, cos x and $e^x$ are continuous functions.

**Some Illustrated Examples**

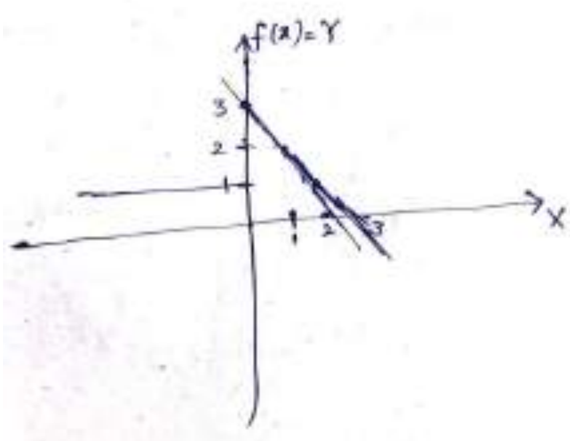
**Example 1 :** Examine the continuity of the function

$$f(x) \begin{cases} 1, x < 0 \\ 3 - x, x \geq 0 \end{cases}$$

at x = 0.

**Solution :** Let us sketch the graph of the function

$$f(x) \begin{cases} 1, x < 0 \\ 3 - x, x \geq 0 \end{cases}$$




In this case

$f$ $(0^-) = 1 \neq 3 = f$ $(0^+) = f$ $(0)$

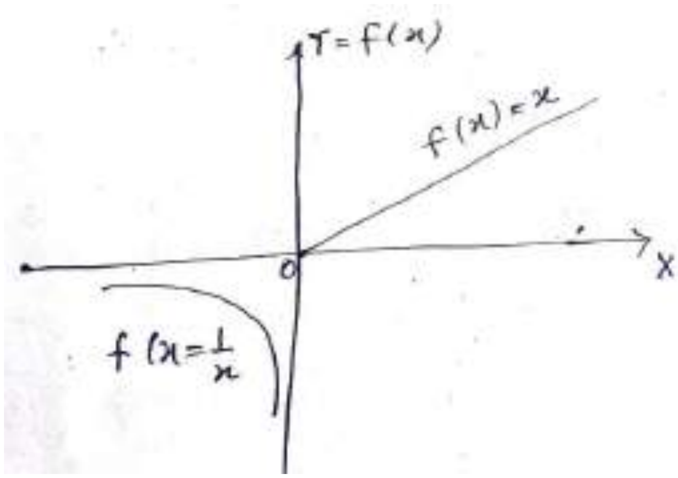
Therefore $f$ is not continuous at x = 0. We cannot do anything to remove this discontinuity.

**Example 2.** Examine the continuity of the function

$$f\ (x) = \begin{cases} x, x \geq 0 \\ \dfrac{1}{x}, x < 0 \end{cases}$$

at x = 0

**Solution :** The graph of the function is as follows :




Here $f(0^+) = 0$ $f$ $(0)$, but $f$ $(0^-)$ does not exist.

Hence $f$ is not continuous at x = 0.

Example 3 : Consider the function

$$f\,(x) = \begin{cases} \dfrac{x^2-1}{x-1} & x \neq 1 \\ 3 & x = 1 \end{cases}$$

Check the continuity of the function at x = 1.

Solution : Now, $\underset{x\to 1}{Lim} f(x)$

$$= \underset{x\to 1}{Lim} \frac{x^2-1}{x-1} = \underset{x\to 1}{Lim} \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}$$

= 1 + 1 = 2

But $f(1) = 3 \neq \underset{x\to 1}{Lim} f\,(x)$

$\therefore$ $f$ (x) is discontinuous at x = 1.

**Example 4 :** examine the continuity of the function $f\,(x) = \dfrac{\sin x}{x}$, at x = 0

**Solution :** We not that $f\,(x) = \dfrac{\sin x}{x}$ fails to continuous at x = 0 since it is not defined at x = 0, though $\underset{x\to 0}{Lim} \dfrac{\sin x}{x} = 1$.

However if we define $f$ (0) = 0, the function so obtained viz.

$$f\,(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is continuous at x = 0

**Self Check Exercise-1**

Q. 1 Examine the continuity of the function

$$f\,(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ \dfrac{-2}{1+x}, & x > 1 \end{cases} \quad \text{at } x = 1$$

Q. 2 Examine the continuity of the function

$$f\,(x) = \begin{cases} 1, & x \text{ rational} \\ -1, & x\ irrational \end{cases}$$

at any point c.

### 3.6 algebra of Continuous Functions

If $f$ and g are two continuous functions at a point a then following results hold.

(i) $\alpha f$ is continuous at x = a for any real $\alpha$.

(ii) $f$ + g is continuous at x = a

(iii) $f$ g is continuous at x = a

(iv) is continuous at x = 1 if g (a) $\neq$ 0

**Note :** 1. We assume, without proof, the above results to hold true. These results also hold good if continuity is replaced by left as well as right continuity.

2. Continuity of composite function let $f$ and g be two functions such that range of $f$ is a subset of the domain of g. Set $f$ is continuous at x = a and g be continuous at $f$(a). the gof is continuous at x = a.

3. Absolute Function : the absolute function of $f$, denoted by $|f|$ (x) = $|f$ (x)$|$ $\forall$ x $\in$ Df.

It $f$ is continuous at x = a, then $|f|$ is also continuous at x = a.

### 3.7 Self Check Exercise-2

Q. 1 Show that $|f$ (x) $|$ is continuous for all x $\in$ R if

$$f(x) = \begin{cases} -1 & x \le 1 \\ 1, & x > 1 \end{cases}$$

Q. 2 If $f(x) = \begin{cases} e^{\frac{1}{x}}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

Check whether $f$ is continuous at x = 0 or not.

**Some more illustrated Examples**

**Example 5 :** Examine the continuity of the function

$$f(x) = \begin{cases} 1+x & x \le 2 \\ 7-2x & x > 2 \end{cases}$$

at x = 2

**Solution :** Here

$$f(x) = \begin{cases} 1+x & x \le 2 \\ 7-2x & x > 2 \end{cases}$$

Now $\underset{x \to 2^-}{Lim} = f(x) = \underset{x \to 2^-}{Lim}(1+x)$

=3

and

$$\underset{x\to 2^+}{Lim} f(x) = \underset{x\to 2^+}{Lim} (7-2x) = 7 - 4 = 3$$

Also $f$ (2) = 1 + 2 = 3

$$\therefore \quad \underset{x\to 2^-}{Lt} f(x) = \underset{x\to 2^+}{Lim} f(2)$$

Hence $f$ (x) is a continuous function at x = 2

**Example 6 :** Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}, & x \neq 0 \\ 4, & x = 0 \end{cases}$$

at x = 0

**Solution :** We have

$$f(x) = \begin{cases} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}, & x \neq 0 \\ 4, & x = 0 \end{cases}$$

Now $\underset{x\to 0}{Lim} f(x) = \underset{x\to 0}{Lim} \dfrac{x}{\sqrt{1+x}-\sqrt{1-x}}$

$$= \underset{x\to 0}{Lim} \frac{x}{\sqrt{1+x}-\sqrt{1-x}} \times \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

$$= \underset{x\to 0}{Lim} \frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{(1+x)-(1-x)}$$

$$= \underset{x\to 0}{Lim} \frac{\cancel{x}\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2\cancel{x}}$$

$$= \frac{\sqrt{1+0}+\sqrt{1-0}}{2} = \frac{1+1}{2} = \frac{2}{2} = 1$$

But $\quad f(0) = 4 \neq \underset{x\to 0}{Lim} f(x)$

Hence $f$ is discontinuous at x = 0

**Example 7:** Let $f(x) = \begin{cases} 2, & x \le 3 \\ ax+b, & 3 < x < 5 \\ 4, & 5 \le x \end{cases}$

Find a and b so that $f$ become continuous.

**Solution:** Now $f$ is continuous $\forall$ x $\Rightarrow$ $f$ is continuous at x = 3,5.

$\underset{x \to 3^+}{Lim} f(x) = \underset{x \to 3^+}{Lim}$ (ax + b) = 3a + b = 2 = $f$ (3) ...... (1)

and $\underset{x \to 5^-}{Lim} f(x) = \underset{x \to 5^-}{Lim} f(x) = f(5)$

$\Rightarrow$ $\underset{x \to 5^-}{Lim}$ (ax+b) = 5a+b = 4 .....(2)

From (1) and (2) we have

3a + b = 2

and 5a + b = 4

On subtraction, we get

$$\begin{array}{r} 5a+b=4 \\ \underline{\underset{-}{3}a \underset{-}{+} b = 2} \\ 2a=2 \end{array} \quad \Rightarrow \quad a = 1$$

From (1) we get

3 (1) + b = 2 $\Rightarrow$ b = 2-3 = -1

Thus a = 1, b = -1 for $f$ to be continuous at all points.

**Example 8:** Examine the continuity of the function.

$$f(x) = \begin{cases} x \sin\frac{1}{x} - 1, x \ne 0 \\ 0, \quad x = 0 \end{cases}$$

at x = 0. Give the nature of discontinuity if otherwise

**Solution:** Here

$$f(x) = \begin{cases} x \sin\frac{1}{x} - 1, x \ne 0 \\ 0, \quad x = 0 \end{cases}$$

Let g(x) = x, h (x) = sin $\frac{1}{x}$

Now $\quad \lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} x = 0$

and $\quad |h(x)| = \left|\sin\frac{1}{x}\right| \leq 1 \qquad \forall\, x \in R,\ x \neq 0$

$\Rightarrow \quad$ h(x) is bounded in a deleted nhd of 0.

$\therefore \quad \lim\limits_{x\to 0} g(x)\, h(x) = 0$

$\Rightarrow \quad \lim\limits_{x\to 0} x \sin\frac{1}{x} = 0$

Thus $\quad \lim\limits_{x\to 0}\left(x\sin\frac{1}{x} - 1\right) = 0 - 1 = -1$

$\Rightarrow \quad \lim\limits_{x\to 0} f(x) = -1$ and also $f(0) = 0$

Hence $\lim\limits_{x\to 0} f(x) \neq f(0)$

$\Rightarrow \quad f$ is discontinuous at x = 0

The above discontinuity is removable discontinuity as $f$ becomes continuous if we define $f(0) = -1$

**Example 9:** Examine the continuity of the function

$$f(x) = \begin{cases} \dfrac{\sqrt{1-\cos 2x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at x = 0. State the kind of discontinuity of $f$ is discontinuous.

**Solution:** We have

$$f(x) = \begin{cases} \dfrac{\sqrt{1-\cos 2x}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Now $\quad \sqrt{1-\cos 2x} = \dfrac{\sqrt{2\sin^2 x}}{x} = \sqrt{2}\ \dfrac{\sin x}{x}$

$$\therefore \quad f(x) = \begin{cases} \dfrac{\sqrt{2\,|\sin x|}}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$\therefore \qquad \underset{x \to 0^-}{Lim} f$ (x) = $\underset{x \to 0^-}{Lim} \sqrt{2} \frac{\sin x}{x}$

Put $\qquad$ n = 0 - h, $\qquad \Rightarrow \qquad$ h → 0 as x → 0$^-$, h > 0

$\therefore \qquad \underset{x \to 0^-}{Lim} f$ (n) = $\underset{h \to 0}{Lim} \frac{\sqrt{2|\sin(-h)|}}{-h}$

$= -\sqrt{2} \underset{h \to 0}{Lim} \frac{\sin h}{h}$ = - $\sqrt{2}$ .1 = - $\sqrt{2}$

Similarly : $\underset{x \to 0^+}{Lim} f$(x) = $\underset{h \to 0}{Lim} \frac{\sqrt{2}|\sin(h)|}{h}$ $\qquad$ (Put x = 0 + h ∴ h → 0 is x → 0$^+$ h > 0)

= $\sqrt{2}$ .1 = $\sqrt{2}$

Thus $\quad \underset{x \to 0^-}{Lim} f$ (x) ≠ $\underset{x \to 0^+}{Lim} f$ (x)

$\therefore \qquad \underset{h \to 0}{Lim} f$(x) does not exist

⇒ $\qquad f$ is discontinuous at x = 0 and this discontinuity is of first kind.

## 3.8 Summary

In this unit we have learnt

(i) continuity of a function

(ii) discontinuity of a function

(iii) kind of discontinuity

(iv) continuity of a function in an interval

(v) Algebra of continuous function

(vi) continuity of composite functions

(vii) continuity of absolute value functions

## 3.9 Glossary

1. Domain of continuity - The set of are points where the function is continuous is refereed to as domain of continuity
2. A rational function is continuous at every point of its domain
3. A constant function is continuous everywhere.

## 3.10 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 Discontinuous

Ans. 2 discontinuous

Self Check Exercise - 2

Ans. 1 To be proved continuous

Ans. 2 discontinuous

## 3.11 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002


## 3.12 Terminal Questions

1. Examine the continuity of the function

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \quad \text{at } x = 0$$

2. Examine the continuity of the function

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases} \quad \text{at } x = 0$$

3. Show that the function

$$f(x) = \begin{cases} [x-2]+[2-x], x \neq 2 \\ 0, \quad x = 2 \end{cases} \text{ is discontinuous at } x = 2$$

4. Find λ for which

$$f(x) = \begin{cases} x^2+2x+\lambda, x \neq 0 \\ -3, \quad x = 0 \end{cases} \text{ is continuous at } x = 0$$

5. Find a, and b so that

$$f(x) = \begin{cases} 2, x = 2 \\ 2ax - bifx < 2 \\ ax^2 + bifx > 2 \end{cases} \text{ is continuous}$$

6. Find a if $f$ given by

$$f(x) = \begin{cases} 2x-1, & x < 2 \\ a, & x = 2 \\ n+1, & x > 2 \end{cases}$$

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## Unit - 4

# Differentiability of Functions

**Structure**



### 4.1 Introduction

Dear students, so far we have learnt the concept of limit and continuity of a function in Unit-1 and Unit-2 respectively. The definition of continuity and derivatives are based on unit of a function. You must have observed serial phenomenon where changing are taking place continuously, for example, temperature at a final point in a room, speed of railway train etc. To answer above questions we need the concept of derivatives (differentiation).

### 4.2 Learning Objectives

The main objectives of this unit are

(i) to define differentiability of a function

(ii) to give interpretation of derivative at a point

(iii) to give physical interpretation of a derivative

(iv) to study algebra of differentiable function

(v) to find derivatives of some important functions

(vi) to study successive differentiation

(vii) to prove Leibnitz's Theorem

## 4.3 Differentiability of a Function

**Definition:** Let $f$ be a real valued function defined on an open interval (a, b). Let c be any arbitrary point of (a, b). The function $f$ is said to be differentiable at the point c or is said to have the derivative at the point c if the limit.

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists and is finite} \qquad .....(1)$$

Note: The derivative of $f$ at the point x of its domain D is defined as the

$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, provided the limit exists and is finite. This derivative (or differential coefficient) is denoted by $f'(x)$ or $\frac{d}{dx}f(x)$.

If the above limit (1) does not exist, then the function $f$ is said to be non-differentiable at the point c.

The derivative or differential coefficient of $f$ at the point c, if exists will be denoted by

$f'(c)$ or $\left(\frac{df}{dx}\right)_{x=c}$

When we write y = $f(x)$ for the function, then the differential coefficient of $f$ at the point c is also denoted by $\left(\frac{df}{dx}\right)_{x=c}$

**Remark:** (1) It $x \to c + 0$, then the derivative is called the right handed derivative, denoted by R $f'(c)$

(2) If $x \to c - 0$, then the derivative is called left-handed derivative, denoted by L$f'(c)$

(3) The derivative at a point exists iff

RHD = LHD

(4) The process of finding the derivative of a function is called the differentiation.

(5) A function $f$ is said to be differentiable at a point c iff the derivative of $f$ exists at c.

**Example** (1) Let y = $f(x) = \alpha$, $\forall x$, $\alpha$ constant

Then for any c in the domain of definition of $f$

$$f'(c) = \lim_{x\to 0} \frac{f(c+h)-f(c)}{h}$$

$$= \lim_{x \to 0} \frac{\alpha - \alpha}{h} = 0 \qquad \text{(exists and is finite)}$$

Thus the derived function $f'$ is defined for all real numbers and $f'$ (x) = 0 $\forall$x

This is also expressed by saying that if y = $f$(x) = $\alpha$

then $\frac{dy}{dx} = \frac{df}{dx} = f'(x) = 0$

**Example 2:** Let y = $f$(x) = x Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h} \qquad \text{exists and is finite}$$

Thus if y = f(x) = x

Then $f'(x) = \frac{dy}{dx} = 1, \forall x$

Let $f$(x) = $x^2$ Then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2hx - x^2}{h}$$

$$= \lim_{h \to 0} (h + 2x) = 0 + 2x = 2x$$

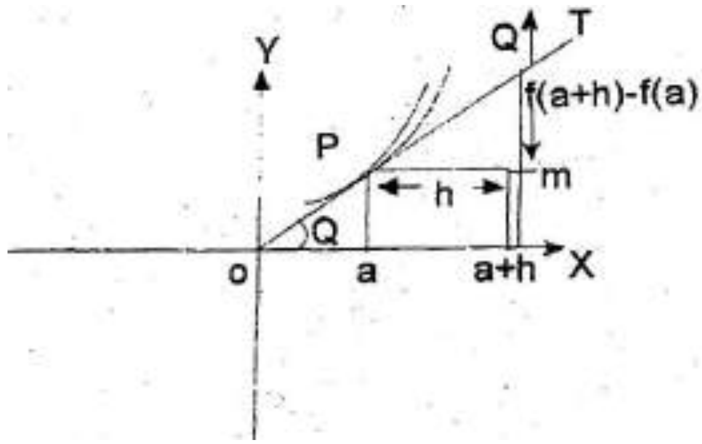
Thus if y = $f$(x) = $x^2$

Then $\frac{dy}{dx} = f'(x) = 2x \ \forall x.$

**Interpretation of derivative at a point**

We give the geometrical and physical interpretation of a derivative at a point

**Geometrical Interpretation:-** Let $f$ be a function defined by y = $f$(x). Let P and Q be point on the graph of the function corresponding to x = a and x = a + h respectively.

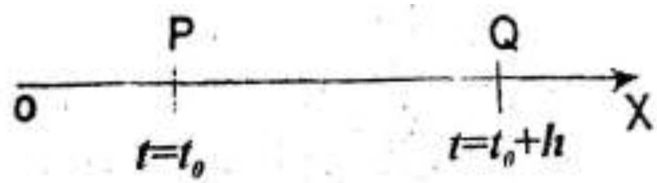



Draw PM $\perp$ to the ordinate through Q.

Then the ratio $\frac{f(a+h)-f(a)}{h}$

can be interpreted as tan $\angle$ MPQ which is the slop of the chord PQ. When h → Q will approach P and $\angle$ MPQ will eventually become the angel $\theta$ that the tangent TP at the point P makes with the x-axis, thus $f'$(a) represents the slop of the tangent to the curve y = $f'$(x) at the point a and is $f'$(a) = tan $\theta$ where $\theta$ is the inclinations of the tangent to the curve at (a, $f$(a)) with the x-axis.

Physical Interpretation: Let a particle move in a straight line OX starting from O towards X. Clearly, the distance of the particle from O will be some function $f$ of time.




Let at any time t = $t_0$, the particle is at P and after a further time h it is at Q so that OP = $f(t_0)$ and OQ = $f(t_0+h)$. Hence average speed of the particle during the journey from P to Q is

$\frac{PQ}{h}$ i.e. $\frac{f(t_0+h)-f(t_0)}{h}$

taking the limit h → 0, we get its instantaneous speed to be

$$\lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h} \qquad \text{which is } f'(t_0)$$

Thus if $f$(t) gives the distance of a moving particle at time t then the derivative of $f$ at t = $t_0$ represents the instantaneous speed of the particle at the point P i.e. at time t = $t_0$.

Differentiate $f$(x) with respect to (w.r.t.) x means find $f'$ (x) or $\frac{dy}{dx}$.

## 4.4 Algebra of differentiable function

**Theorem:** Let $f$ and g be real valued function defined in an interval containing the point a Let $f$'(a) and g'(a) exist. then

(i) If $\alpha$ is a constant, then ($\alpha$ $f$) is differentiable at a and ($\alpha f$)'(a) = $\alpha f$' (a).

(ii) The sum $f$+g is differentiable at a and ($f$+g)'(a) = $f$'(a) + g(a).

(iii) The product $f$g is differentiable at a and

($f$g)' (a) = $f$'(a)g(a) + $f$(a)g'(a)

(iv) If $f$ (c) $\neq$ 0 then $\frac{1}{f}$ is differentiable at a and

$$\left(\frac{1}{f}\right)(a) = \frac{-f'(a)}{\{f(a)\}^2}$$

**Proof:** (1) We note that

$$\frac{(\alpha f)(a+h)-(\alpha f)(a)}{h} = \underline{\alpha f}\ (a+h) - \alpha f\ (a) \quad \left\{\frac{f(a+h)-f(a)}{h}\right\}$$

Hence $\frac{(\alpha f)(a+h)-(\alpha h)(a)}{h}$ $\qquad \alpha\left\{\frac{f(a+h)-f(a)}{h}\right\}$

$\lim\limits_{h\to 0}$ $\qquad \lim\limits_{h\to 0}$ $\qquad$ [Q using algebra of limit]

i.e. ($\alpha f$)' (a) = $\alpha f$' (a)

(ii) Again, we have

$$\frac{(f+g)(a+h)-(f+g)(a)}{h} = \frac{[f(a+h)+g(a+h)-(f(a)+g(a))]}{h}$$

$$= \frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h}$$

Hence by algebra of limits

$$\frac{(f+g)(a+h)-(f+g)(a)}{h} \qquad \frac{f(a+h)-f(a)}{h} \qquad \frac{g(a+h)-g(a)}{h}$$

$\lim\limits_{h\to 0}$ $\qquad \lim\limits_{h\to 0}$ $\qquad$ i.e.) $\lim\limits_{h\to 0}$

i.e. ($f$+g)](a) = $f$' (a) + g' (a)

(iii)

$$\frac{(fg)(a+h)-(fg)(a)}{h} = \frac{f(a+h)g(a+b)-f(a)g(a)}{h}$$

$$= \frac{f(a+h)g(a+h)-g(a+h)f(a)+g(a+h)f(a)-f(a)g(a)}{h}$$

$$= \frac{f(a+g)-f(a)g(a+h)}{h} + \frac{f(a)(g(a+h)-g(a)\}}{h}$$

taking limit, using algebra of limits we get

$$\lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h} = \lim_{h\to 0} \frac{f(a+h)0f(a)}{h} \lim_{h\to 0} g(a+h) f(a)$$

$$\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

= $f'$(a)g(a) + $f$(a)g' (a)

[Q here we have used lim g(a+h) = g(a) h → 0 we will prove this later]

(v) We assume that $f(a) \neq 0$

We know that if $f$ is continuous at a and $f(a) \neq 0$ then there exists a neighborhood (a-δ ,a + δ ) of a in which $f$(x) has the same sign as $f$(a). Thus in the nbd. of a, $f(x) \neq 0$ Let us now $0 < |h| < \delta$. Then

$$\frac{\left(\frac{1}{f}(a+h)\right)-\frac{1}{f}(a)}{h} = \frac{\frac{1}{f(a+h)}-\frac{1}{f(a)}}{h}$$

$$= \frac{1}{f(a+h)f(a)} \left\{\frac{f(a+h)-f(a)}{h}\right\}$$

Hence $\dfrac{\left(\frac{1}{f}(a+h)\right)-\frac{1}{f}(a)}{h} = \dfrac{1}{f(a)\lim\limits_{h\to 0}(a+h)} + \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

$$= \frac{1}{[f(a)]^2} f'(a)$$

**Corollary:** Under the hypothesis of the above theorem

$$\left(\frac{g}{f}\right)(a) = \left(f \cdot \frac{1}{f}\right)(a) = g'(a)\left(\frac{1}{f}\right)(a) + g(a)\left(-\frac{f'(a)}{[f(a)]^2}\right)$$

$$= \frac{g(a) f(a) - g(a) f'(a)}{[f(a)]^2}$$

**Theorem:** (differentiability implies continuity)

If a real valued function $f$ has a derivative at a point a of the domain of $f$, then $f$ is continuous at a.

**Proof:** We are assuming that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We can write

$$f(a+h) - f(a) = h\left(\frac{f(a+h) - f(a)}{h}\right)$$

taking limit as h → 0 on both sides

$$\lim_{h \to 0} (f(a + h) - f(a)) = \lim_{h \to 0} h \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h}\right)$$

$$= 0.\ f'(a) = 0$$

Hence $\lim_{h \to 0} f(a+h) = f'(a)$

In other words $f$ is continuous at a.

**Note:-** This theorem gives us another way of checking lack of differentially at a point, namely, if a function is not continuous at a point it cannot be differentiable at that point.

**Derivative of some important functions**

(1) Derivative of $x^n$, where n is any integer

case (1) when x is a positive integer,

we will show that $\frac{d}{dx}(x^n) = nx^{n-1}$ .... (1)

We will prove this by the principle of mathematical induction. When n = 1.

We have $\frac{d}{dx} x^1 = \frac{d}{dx}(x) = 1 = 1.x^{1-l}$

∴ the result is true for n = 1

Let us assume that the result is true for n = m i.e. we are assuming that

$\frac{d}{dx}(x^m)\backslash mx^{m-1}$ .....(11)

consider

$$\frac{d}{dx} x^{m+1} = \frac{d}{dx} (x^m x)$$

$$= \left(\frac{d}{dx} x^m\right) x + x^m \frac{dx}{dx}$$

$$= mx^{m-1} x + x^m$$

$$= (m+1)x^m = (m+1)x^{(m+1)-1}$$

Thus result holds for n = m + 1

Hence by the principle of mathematical induction the result (1) is true for all n > 1.

(ii) When n is negative let n = -m, m > 0

Then $x^n = x - m = \frac{1}{x^m}$

$$\therefore \quad \frac{d}{dx} (x^n) = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{-\frac{d}{dx} x^m}{(x^m)^2} = - \frac{mx^{m-1}}{x^{2m}}$$

$$= -mx^{-m-1}$$

$$= nx^{n-1}$$

$\therefore \frac{d}{dx}(x^m) = nx^{n-1}$ for any integer.

Note: This result is true when n is any real exponent.

**Examples 3:** $\frac{d}{dx} x = \frac{d}{dx} (x^1) = 1x^{1-1} = 1$

(ii) $\frac{d}{dx} x^5 = 5x^{5-1} = 5x^4$

(iii) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = (-1)x^{-1-1} = -\frac{1}{x^2}$

(iv) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx} x^{1/2} = \frac{1}{2} x^{1/2} = \frac{1}{2} x^{1/2} = \frac{1}{2\sqrt{x}}$

**Derivative of exponential function.**

If $f$ (x) = $e^x$, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{e^{x+h} - e^{x}}{h}$$

$$= e^{x} \lim_{h\to 0} \frac{e^{h} - 1}{h}$$

$$= e^{x}\, 1 \qquad \left[\therefore \lim_{h\to 0} \frac{e^{h}-1}{h} = 1\right.$$

$$\therefore \qquad \frac{d}{dx}(e^{x}) = e^{x}$$

Derivative of logarithmic function

(i) Let $f(x) = \log_e x$ $(x > 0)$ Then

$$f'(x) = \lim_{h\to 0} \frac{\log_e (x+h) - \log_e x}{h}$$

$$\frac{\log_e\left(1+\frac{h}{x}\right)}{h}$$

$$= \lim_{h\to 0}$$

$$= \lim_{h\to 0} \frac{1}{x} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} \qquad \left.\because \lim_{h\to 0} \frac{\log_e(1+h)}{h} = 1\right]$$

$$= \frac{1}{x} \cdot 1$$

Thus $\frac{d}{dx} \log_e x = \frac{1}{x}$

(ii) We have $\frac{d}{dx}(\log_a x) = \frac{d}{dx}(\log_e x \log_a e)$

$$= \log_a e \frac{d}{dx}(\log_e x)$$

$$= \log_a e$$

Hence

$$\frac{d}{dx}\log_a x = \frac{1\log_a e}{x}$$

**Derivative of sin x**

Let y = $f$ (x) = sin x

Then $\frac{dy}{dx}$ = $\frac{df}{dx}$ = $f$' (x) = cos x

By definition

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x\cosh+\cos x\sinh=\sin x}{h}$$

$$= \lim_{h\to 0}\left[\sin x\left(\frac{\cosh-1}{h}\right)+\cos x\frac{\sin n}{h}\right]$$

$$= \sin x \lim_{h\to 0}\frac{\cosh-1}{h} + \cos x \lim_{h\to 0}\frac{\sin\ \mathrm{h}}{h}$$

= sin x , 0 + cos x.1

- cos x

$$\left[\begin{array}{c}\because \lim_{x\to 0}\frac{\cosh-1}{h}=0 \\ \& \lim_{x\to 0}\frac{\sin \mathrm{h}}{h}-1\end{array}\right]$$

**Derivative of see x**

Let y = cos x, $\frac{dy}{dx}$ = - sin x

By definition

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \frac{\sin x\mathrm{h}}{h}\quad\frac{\cos x\cosh-\sin x\sinh-\cos x}{h}$$

$$= \lim_{h\to 0}\left[\cos x\left(\frac{\cosh-1}{h}\right)-\sin x\frac{\sin\mathrm{h}}{h}\right]$$

$= \cos x \lim_{h\to 0} \frac{\cosh - 1}{h}$ - sin x $\frac{x}{\cos x}$ $\frac{\sin h}{h}$

= cos x. 0- sin x.1

= - sin x

Hence $\frac{d}{dx}$(cos x) = - sin x

**Derivative of tan x**

Let $f$ (x) = tan x. Then

$f'$ (x) $\lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$

$= \lim_{h\to 0} \left[\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos(x+h)\cos x}\right]$

$= \lim_{h\to 0} \left[\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}\right]$

$= \lim_{h\to 0} \frac{\sin h}{h}$ $\qquad = \lim_{h\to 0} \frac{1}{\cos(x+h)\cos x}$

= 1 $\frac{1}{\cos x \cos x}$ = $\frac{1}{\cos^2 x}$ = sec$^2$ x

Hence if $f$ (x) = tan x then $f'$ (x) = sec$^2$ x

i.e. $\frac{d}{dx}$ tan x = sec$^2$ x

**Derivative of Sec x**

Let f (x) = sec X. then

$f'$ (x) $\lim_{h\to 0} \frac{\sec(x+h) - \sec x}{h}$

$\lim_{h\to 0} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x}\right]$

$$= \lim_{h\to 0} \frac{\cos x - \cos x + h}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h\cos(x+h)\cos x}$$

$$= \lim_{h\to 0} \frac{\sin\left(x+\frac{h}{2}\right)}{\cos(x+h)\cos x} \lim_{h\to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}}$$

$= \frac{\sin x}{\cos^2 x}$ . 1 = sec x tan x

Let us do some examples now

**Example 4 :** Find $\frac{dy}{dx}$ if y = $x^4$ + sin x

**Solution :** We have y = $x^4$ + sin x

$\therefore \frac{dy}{dx} = \frac{d}{dx}(x^4 + \sin x)$

$= \frac{d}{dx} x^4 + \frac{d}{dx} \sin x$

$= 4x^3 + \cos x$

**Example 5 :** Find $\frac{dy}{dx}$ when y = 3 tan x + 5 $\log_a$ x + $\sqrt{x} + \frac{1}{x}$

Solution : Here y = 3 tan x + 5 $\log_a$ x + $\sqrt{x} + \frac{1}{x}$

Then $\frac{dy}{dx} = \frac{d}{dx}(3 \tan x + 5 \log_a x + \sqrt{x} + \frac{1}{x})$

$= \frac{d}{dx} 3 \tan x + \frac{d}{dx}(5 \log_a x) + \frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(x^{-1})$

$= 3\frac{d}{dx} \tan x + 5\frac{d}{dx} \log_a x + \frac{1}{2} x^{\frac{1}{2}-1} + (-1)\, x^{-1-1}$

$= 3 \sec^2 x + 5 \frac{1}{x} \log_a e + \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$

**Example 6 :** Differentiate $(x^2+7x+2)(e^x - \sin x)$

**Solution :** Let $y = (x^2+7x+2)(e^x - \sin x)$

Then $\frac{dy}{dx}[(x^2+7x+2)(e^x - \sin x)]$

$= (x^2+7x+2)\frac{d}{dx}(e^x - \sin x) + (e^x - \sin x)\frac{d}{dx}(x^2+7x+2)$

$= (x^2+7x+2)\left[\frac{d}{dx}e^x - \frac{d}{dx}\sin x\right] + (e^x - \sin x)\left[\frac{d}{dx}x^2 - \frac{d}{dx}(7x) + \frac{d}{dx}(2)\right]$

$= (x^2+7x+2)(e^x - \cos x) + (e^x - \sin x)(2x + 7.1+0)$

$= (e^x - \cos x)x^2 + (9e^x - 7\cos x - 2\sin x)x + 9e^x - 2\cos n - 7\sin x$

**Example 7 :** If $y = e^x \log \sqrt{x} \tan x$, Find $\frac{dy}{dx}$

**Solution :** Let $y = e^x \log \sqrt{x} \tan x$

$= \frac{1}{2} e^x \log x. \tan x$ $\qquad \left[\because \log x^{1/2} = 1/2 \log x\right]$

$\therefore \frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}[e^x \log x \tan x]$

$= \frac{1}{2}\left[e^x \log x \frac{d}{dx}\tan x + \tan x \frac{d}{dx}(e^x \log x)\right]$

$= \frac{1}{2}\left[e^x \log x.\sec^2 x + \tan x\left\{e^x \frac{d}{dx}\log x + \log x \frac{d}{dx}e^x\right\}\right]$

$= \frac{1}{2}\left[e^x \log x.\sec^2 x + \tan x\left(e^x \frac{1}{x} + \log x e^x\right)\right]$

$= \frac{1}{2} e^x \left[\log x.\sec^2 x + \frac{\tan x}{x} + \tan x \log x\right]$ Ans.

**Example 8:** Let $y = \frac{\sin x + \cos x}{\sin x - \cos x}$ Find $\frac{dy}{dx}$

**Solution:** Given $y = \frac{\sin x + \cos x}{\sin x - \cos x}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x + \cos x}{\sin x - \cos x}\right)$$

$$= \frac{(\sin - \cos x)\frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x)\frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{(\sin - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2}$$

$$= \frac{\sin \cos x - \cos^2 x - \sin^2 x + \sin x \cos x - (\sin^2 x + \cos^2 x + 2 \sin x \cos x)}{(\sin x - \cos x)^2}$$

$$= \frac{-2}{(\sin x - \cos x)^2} \quad \text{Ans.}$$

**Theorem:** (Chain Rule) It is a function at a point $x_0$ and let g be a function differentiable at $f(x_0)$. Then the composite functions gif is differentiable at $x_0$ and

$$(gof)'(x_0) = g'(f(x_0)\, f'(x_0)$$

Proof is omitted

**Note:** If we replace $f(x)$ by z in y = g ($f$(x)), the above theorem can be expressed as

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

We will use this chain rule in the following examples.

**Examples 9:** Consider the function

y = cos ($x^4$)

We put z = x4 so that y = cos z. Then

$$\frac{dy}{dz} = \frac{d}{dz}(\cos z) = \sin z \text{ and } \frac{dz}{dx} = \frac{d}{dx}(x^4) = 4x^3$$

Consequently

$$\frac{dy}{dx} = \frac{dy}{dx}\frac{dz}{dx} = (-4x^3)\sin x^4 \text{ Ans.}$$

**Example 10:** y = $e^{x^1}$

Let z = $x^3$ then y = $e^z$. Then

$$\frac{dy}{dz} = \frac{d}{dz}(e^x) = e^z \text{ and } \frac{dz}{dx} = (x^3) = 3x^2$$

$\therefore$ By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = (3x^2)\ e^{x^1} \qquad \text{Ans.}$$

**Example 11:** Let y = log sin x

put z = sin x, then y = log z. then

$$\frac{dy}{dz} = \frac{d}{dz} \log z = \frac{1}{z} \text{ and } \frac{dz}{dx} = \frac{d}{dx} (\sin x) = \cos x$$

$\therefore$ By Chain Rule

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \cos \frac{1}{\sin x} = \cot x \qquad \text{Ans.}$$

**Derivative of inverse trigonometric function**

Derivative of $\sin^{-1}x$

Let $y = \sin^{-1}x$ then $\sin y = x$

differentiating w.r.t. x we get

$$\cos y \frac{dy}{dx} = 1$$

$$\text{Therefore } \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{Thus } \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

**Exercise:** Show similarly that $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

**Derivative of $\tan^{-1}$ x**

Let $y = \tan^{-1} x$ Then $\tan y = x$

differentiating both sides w.r.t. x

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\text{or } \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\text{Thus } \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Show similarly that

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

**Derivative of sec$^{-1}$ x**

Let y = sec$^{-1}$ x, then sec y = x

differentiating both sides w.r.t. x

$$\sec y \tan y \frac{dy}{dx} = 1$$

or $\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y\sqrt{\sec^2 y - 1}} = \frac{1}{x\sqrt{x^2-1}}$

Thus $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

Show similarly $\frac{d}{dx}(\cos ec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

**Example 12:** Let y = sin$^{-1}$ (e$^x$), Find $\frac{dy}{dx}$

**Solution:** $\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}(e^x)) = \frac{1}{\sqrt{1-(e^x)^2}}\frac{d}{dx}(e^x) = \frac{e^x}{\sqrt{1-e^{zx}}}$ Ans.

**Example 13:** Differentiate $\cos^{-1}\left(\frac{x}{x+1}\right)$

**Solution:** Let y = $\cos^{-1}\left(\frac{x}{x+1}\right)$

Then $\frac{dy}{dx} = \frac{d}{dx}\cos^{-1}\left(\frac{x}{x+1}\right)$

$$= \frac{-1}{\sqrt{1-\left(\frac{x}{x+1}\right)^2}}\frac{d}{dx}\left(\frac{x}{x+1}\right)$$

$$= \frac{-1}{\sqrt{1=\left(\frac{x}{x+1}\right)^2}}\frac{x.1-(x+1).1}{(x+1)^2}$$

$$= \sqrt{(1+x)^2 - x^2}\ \frac{1}{(x+1)} \qquad \text{Ans.}$$

**Logarithmic differentiation**

To facilitate the differentiation of complicated function we use a device called logarithmic differentiation. which we illustrate below.

**Example 14:** If y = $x^\alpha$, $\alpha$ a real and x > 0

Then $\frac{dy}{dx} = \alpha x^{a-1}$

**Solution:** For $\alpha$ real and ....... is positive and hence log y has a meaning.

Taking log on both sides

log y = log $x^\alpha$ = $\alpha$ log x

differentiating with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{\alpha}{x}$$

or $\frac{dy}{dx} = \frac{\alpha}{x}$ y = $\frac{\beta}{x}$ $x^\alpha$ = $\alpha x^{\alpha-1}$

**Example 15:** For y = $a^x$, a > 0 .... (1)

$\frac{dy}{dx}$ = $a^z$ log a

**Solution:** Since $a^x$ is positive log y is defined taking log both sides of 1

log y = log $a^x$ = x log a

Differentiating w.r.t x we get

$\frac{1}{y}\frac{dy}{dx}$ = log $\alpha$

or $\frac{dy}{dx}$ = y log a = $a^z$ log a

**Example 16:** Let y = $x^x$, x > 0 Find $\frac{dy}{dx}$

**Solution:** Taking log on both sides as it is defined

log y = log $x^x$ = x log x

differentiating w.r.t x both sides

$$\frac{1}{y}\frac{dy}{dx} = \log x + \frac{x}{x} = \log x + 1$$

Therefore $\frac{dy}{dx} = x^z (\log x + 1)$ Ans.

## 4.5 Self Check Exercise - 1

Q.1 A function $f$ is defined as

$$f(x) = \begin{cases} 1+x, x \le 2 \\ 5-x, x < 2 \end{cases}$$

Show that $f$ is not differentiable at a = 2.

## 4.5 Successive Differentiation

### Derivatives of Higher Order and Notation

Let y = $f$(x) be a function of x, which is derivable at any point x of its domain. Then its derivative w.r.t. x.

generally, this derivative is again a function x and may be derivable itself at any point x of the domain of the original function $f$. The derivative of $\frac{dy}{dx}$ or $f'(x)$ is called second order derivative, and is denoted by $f''(x)$. The process of differentiation may be repeated again, if possible, and its derivative is called the third order derivative, and is denoted by $\frac{d^3y}{dx^3}$ or by $f'''(x)$. In general, suppose the n^th order derivative exists, n being a positive integer. Then it is denoted by $\frac{d^ny}{dx^n}$ or $f^{(n)}(x)$ and we say that the function $f$ is n-times differentiable or derivable. This process of repeated differentiation is know as successive differentiation.

Other notation for the successive derivatives of the function y=$f$ (x) are $y^1, y^2, y^3, \ldots\ldots. y^n$ or $y', y'', y''', \ldots\ldots\ldots\ldots\ldots. y^n$ or $\frac{df}{dx}, \frac{d^2f}{dx^2}, \frac{d^3f}{dx^3}, \ldots\ldots\ldots. \frac{d^nf}{dx^n}$ or

Dy, $D^2y$, $D^3y$,.........$D^ny$ where D = $\frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$,....... $D^n = \frac{d^n}{dx^n}$

**Example 17:** Let y = $x^n$, where x is a positive integer, then we know that $\frac{dy}{dx} = nx^{n-1}$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}(nx^{n-1}) = n(n-1)x^{n-2}$

By induction $\frac{d^my}{dx^m} = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-m+1)\, x^{n-m}$ if $m \le n$

obviously $\frac{d^n y}{dx^n}$ = n(n - 1)(n - 2) ...........(n - n + 1) $x^{n-n}$ for m = n = n! which is a constant

and therefore $\frac{d^{x+1}y}{dx^{x+1}} = \frac{d}{dx}$ (n!) = 0, $\frac{d^{x+2}y}{dx^{x+2}} = \frac{d}{dx}$ (0)

More generally $\frac{d^m y}{dx^m}$ = 0 if m $\geq$ n + 1

Hence if y = $x^n$, then

$$\frac{d^m y}{dx^m} = \begin{cases} n(n-1)(n-2).......(n-m+1)x^{n-m} & if\ m < n \\ n! & if\ m = n \\ 0 & if\ m > n \end{cases}$$

**Example 18:** Let y = $(ax + b)^n$, n being a positive integer and a and b are arbitrary constants.

Then $\frac{dy}{dx}$ = n$(ax + b)^{n-1}$ a

$\frac{d^2 n}{dx^2}$ = n$(ax + b)^{n-1}$, a/na $\frac{dy}{dx}$ $(ax + b)^{n-1}$

= na(n-1)$(ax+b)^{n-1}$.a

= n(n-1)$(ax+b)^{n-1}$.a

By induction

$\frac{d^m y}{dx^m}$ = n(n-1)(a+2)......................(n-m+1)$(ax+b)^{n-m}$ $a^m$. if m $\leq$ n

Obviously $\frac{d^n y}{dx^n}$ = n!$a^n$ $f$ or m = n

And $\frac{d^{x+1}y}{dx^{x+1}} = \frac{d}{dx}$ (n!$a^n$) = 0 [n! $a^n$ is constant]

$\frac{d^{n+2}y}{dx^{n+2}} = \frac{d}{dx}$ (0) = 0

More generally $\frac{d^m y}{dx^m}$ = 0 if m > n + 1

Hence if y = $(ax + b)^n$, then

$$\frac{d^m y}{dx^m} = \begin{cases} n(n-1)(n-2).......(n-m+1)(ax+b)^{n-m} a^m & if\ m < n \\ n!a^n & if\ m = n \\ 0 & if\ m > n \end{cases}$$

**Example 19:** Let y = $e^x$,

Then $\frac{dy}{dx}$ = $e^n$ = > $\frac{d^2y}{dx^2}$ = $\frac{d}{dx}$ ($e^x$) = $e^x$

By induction $\frac{d^n y}{dx^2}$ = $e^x$ for all positive integer n.

Or $\frac{d^n}{dx^n}$ ($e^x$) = $e^x$ for all positive integer n.

**Example 20:** I$f$ y = sin (ax + b).

Then $y_n$ = $a^n$ sin $\left(ax+b+\frac{\pi}{2}\right)$, n$\in$N.

**Solution:** We shall prove by induction

y = sin (ax + b)

$$\therefore \frac{dy}{dx} = \cos(ax+b)\, a = a\sin\left(ax+b+\frac{\pi}{2}\right)$$

[$\therefore$ cos θ = sin (π/2 + θ)

This proves the result for n = 1

Let it be true for n = k

i.e. yk = $a^k$ sin $\left(ax+b+k.\frac{\pi}{2}\right)$

Diff. both sides w.r.t. x. we have,

$$y_{k+1} = a^k \cos\left(ax+b+k.\frac{\pi}{2}\right).a$$

$$= a^{k+1} \cos\left(ax+b+\frac{k\pi}{2}\right)$$

$$= a^{k+1} \sin\left(ax+b+\frac{k\pi}{2}+\frac{\pi}{2}\right)$$

$= a^{k+1} \sin\left[ax+b+(k+1)\frac{\pi}{2}\right]$

This proves the result for n = k + 1 too

Hence by induction $y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$, $\forall$ n $\in$ N.

**Example 21:** if $y = e^{ax} \sin(bx + c)$, then

$$y_n = (a^2 + b^2)^{n/2} e^{an} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

**Solution:** We shall prove it by induction

$y = e^{ax} \sin (bx + c)$

Diff. w.r.t. x, we have.

$\frac{dy}{dx} = e^{ax} \cos (bx + c).b + \sin (bx + c).e^{ax}.a$

$= e^{ax} [a\sin (bx + c) + b \cos(bx + c)]$ ...... (1)

put a = r cos a, b = r sin a, r > 0

Squaring and adding, we have, a2 + b2 = r2 or r = $\sqrt{a^2+b^2}$ and on dividing, we have tan a = $\frac{b}{a}$ or a = $\tan^{-1} \frac{b}{a}$.

$\therefore$ from (1). $y = e^{ax}$ [r cos a sin (bx + c) + r sin a cos (bx + c)]

$= e^{ax}$ [sin (bx + c) cos a + cos(bx + c) sin a]

$= e^{ax}$ sin (bx + c + a)

Ory1 = $(a^2+b^2)^{\frac{1}{2}} e^{ax} \sin\left(bx+c+\tan^{-1}\frac{b}{a}\right)$

This proves the result for n = 1.

Let it be true for n = k

i.e., $y_k = (a^2+b^2)^{\frac{k}{2}} e^{ax} \sin\left(bx+c+k\tan^{-1}\frac{b}{a}\right)$

or$y_k = r^k e^{ax}$ sin (bx + c + ka)

Differentiating w.r.t. x, we have,

$y_{k+1} = r^k$ [cos(bx + c + ka).b + sin (bx + c + ka) $e^{ax}$. a]

$= r^k e^{ax}$ [a sin(bx + c + ka) +b cos (bx + c + ka)]

$= r^k e^{ax}$ [r cos a sin (bx + c + ka) + r sin a cos (bx + c + ka)]

$= r^{k+1} e^{ax}$ [sin(bx + c + ka) cos a + co (bx + c + ka) sin a]

$= r^{k+1} e^{x}$ sin (bx + c + ka + a)

$r^{k+1} e^{ax}$ sin [bx + c + (k + 1)a]

Putting the values of r and a, we have,

$$y_{k+1} = (a^2 + b^2)^{\frac{k+1}{2}} e^{ax} \sin (bx + c + (k+1) \tan^{-1} \frac{b}{a})$$

$\therefore$ this proves the result $f$ or n = k + 1 true.

Hence, by Induction, it is true for all positive integral values of n.

i.e., $y_n = (a^2 + b^2)^{\frac{n}{2}} e^{ax} \sin \left( bx + c + n \tan^{-1} \frac{b}{a} \right)$, $\forall$ n $\in$ N.

**Example 22:** let y = $\frac{1}{ax+b}$, x $\neq$ 0, a, b are arbitrary constants.

Then, we know that

$$\frac{dy}{dx} = \frac{a}{(ax+b)^2}, \ x \neq -\frac{b}{a}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} (-\frac{a}{(ax+b)^2} = -a \frac{d}{dx} (ax + b)^{-2}$$

$= (-1)(-2)(-3)\, a^3 (ax + b)^{-3}, \ x \neq -\frac{b}{a}$

$\frac{d^3y}{dx^3} = (-1)(-2)(-3)\, a^3 (ax + b)^{-4}, \ x \neq -\frac{b}{a}$

By Induction $\frac{d^ny}{dx^n} = (-1)(-2)(-3) \dots\dots\dots\dots (-n)\, an\, (ax + b)^{-n-1}, \ x \neq \frac{b}{a}$

Or $\frac{d^ny}{dx^n} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}, \ x \neq -\frac{b}{a}$

Or $\frac{d^x}{dx^x} \left( \frac{1}{(ax+b)} \right) = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}, \ x \neq -\frac{b}{a}$

**Example 23:** Let y = log (ax + b), a $\neq$ 0, x > $-\frac{b}{a}$

Then we know that,

$$\frac{dy}{dx} = \frac{1}{ax+b}.a, \ x > -\frac{b}{a}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{ax+b}\right) = a. \frac{d}{dx}(ax+b)^{-1}$$

$= a (-1) (ax + b)^{-2} -a$

$= (-1)a^2 (ax + b)^{-2}, \ x > -\frac{b}{a}$

$$\frac{d^3y}{dx^3} = (-1)(-2)a^3 (ax + b)^{-3}, \ x > -\frac{b}{a}$$

By Induction $\frac{d^n y}{dx^n} = (-1)(-2)............(-n + 1\ a^n(ax + b)^{-n}, \ x > -\frac{b}{a}$

$$= \frac{(-1)^{n-1}[n-1a^n}{(ax+b)^n}, \ x > -\frac{b}{a}$$

Or $\frac{d^n}{dx^\infty}$ [log(ax + b) $= \frac{(-1)^{n-1}[n-1.a^n}{(ax+b)^n}, \ x > -\frac{b}{a}$

**Note:** If a function $f$ has a derivative of order n at a point, it is not necessary that the (n+1)th order derivative at c must exist.

For example, Let $f(x) = f(x) = \begin{cases} x^2, \ if \ x \geq 0 \\ -x^2, \ if \ x < 0 \end{cases}$

then $f'(x) = \begin{cases} 2x \ if \ x > 0 \\ 0 \ if \ x = 0 \\ -2x \ if \ x < 0 \end{cases}$

Thus $f'(x)$ exists for all $x \in R$

Now $f''(x) = \begin{cases} 2, \ if \ x > 0 \\ -2 \ if \ x \neq 0 \end{cases}$

$\Rightarrow \quad f''(0_-) = -2$ and $f''(0_+) = 2$

$\Rightarrow \quad f''(0_-) \neq f''(0_+)$

$\Rightarrow \quad f''$ does not exist at $x = 0$

**Theorem 1:** Let $f$ and g be two real valued functions defined in a neighborhood of a point c. Let $f^n(c)$ and $g^n(c)$ exist.

Then (a) $\frac{d^n}{dx^n}[(\alpha f)(x)]_{x=c} = \alpha f^n(c)$, $\alpha$ is a constant

(b) $\frac{d^n}{dx^n}[(f \pm g)(x)]_{x=c} = f^n(c) \pm g^n(c)$

(exercise for you, use induction method)

**Example 24:** Let y = a sin x + b cos x, a, b are constants.

Prove that (1) $\frac{d^2y}{dx^2}$ + y = 0  (2) $\frac{d^2y}{dx^2}$ = y

**Solution:** y = a sin x + b cos x, a , b being constants

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(\text{a sin x}) + \frac{d}{dx}(\text{b cos x}) = \text{a cos x - b sin x}$$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}$(a cos x) - $\frac{d}{dx}$(b sin x) = -a cos x - b sin x

= -(a cos x + b sin x)

= - y

$\therefore$ $\frac{d^2y}{dx^2}$ + y = 0, this proves the required result (a).

Similarly $\frac{d^2y}{dx^2}$ = -(-a sin x + b cos x) = a sin x - b cos x

And $\frac{d^4y}{dx^4}$ = a cos x + b sin x

= y, this prove the required result (b).

**Example 25:** find the derivative of $\frac{1}{(x+2)(x+3)}$

**Solution:** y = $\frac{1}{(x+2)(x+3)}$

y = $\frac{1}{x+2} - \frac{1}{x+3}$

$$\therefore \frac{d^ny}{dx^n} = \frac{d^n}{dx^n}\left(\frac{1}{x+2}\right) - \frac{d^n}{dx^n}\left(\frac{1}{x+3}\right)$$

$$= \frac{(-1)^{n-1} n!}{(x+2)^{n+1}} - \frac{(-1)^{n-1} n!}{(x+3)^{n+1}}$$

$$= (-1)^{\infty} n! \left[ \frac{1}{(x+1)^{n+1}} - \frac{1}{(x+3)^{n+1}} \right]$$

**Example 26:** find the derivative of $\sin^2 x \cos^3 x$.

**Solution:** $y = \sin^2 x \cos^3 x = \frac{1}{4}(4 \sin^2 x \cos^2 x) \cos x$

$$= \frac{1}{8} \cos x - \frac{1}{16}(2 \cos 4x \cos x)$$

$$= \frac{1}{8} \cos x = - \frac{1}{16} (\cos 5x + \cos 3x)$$

$$= \frac{1}{8} \cos x - \frac{1}{16} \cos 3x - \frac{1}{16} \cos 5x$$

$$\therefore \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} \left( \frac{1}{8} \cos x (- \frac{d^n}{dx^n} \right) \frac{1}{16} \cos 3x \left( - \frac{d^n}{dx^n} \right) \frac{1}{16} \cos 5x)$$

$$= \frac{1}{8} \frac{d^n}{dx^n} (\cos x) - \frac{1}{16} \frac{d^n}{dx^n} (\cos 3x) - \frac{1}{16} \frac{d^n}{dx^n} (\cos 5x)$$

$$= \frac{1}{8} \cos \left( x + \frac{n\pi}{2} \right) - \frac{1}{16} 3^{\pi} \cos \left( 3x + \frac{n\pi}{3} \right) - \frac{1}{16} 5^n \cos \left( 5x + \frac{n\pi}{2} \right)$$

## 4.7 Leibnitz's Theorem

Leibnitz's Theorem for the nth derivative of product of two functions.

Statement. Let $u(x)$ and $\nu(x)$ be two real valued functions, each of which possesses nth order derivatives in any interval.

Then $(u\nu)_n = n_{c^0} u\nu_1 + nc_1 u_1 \nu n_{-1} + n_{c^2} u_2 \nu_{n-2} + \ldots\ldots\ldots$

$+ n_{cr} u_r \nu_{n-r} + \ldots\ldots. + n_{cn} u_n \nu$

Proof: We shall use method of induction to prove the above theorem.

$\therefore$ For $n = 1$

$(u\nu)_1 = u_1 \nu + u\nu_1 = c_a u_1 \nu + c_1 u\nu_1$

$\Rightarrow$ theorem is true for $n = 1$

Let us assume that it is true for $n = k$, $k$ being positive integer

$\therefore\ (u\nu)_k = k_{c_0}\, u_k\nu + k_{c_1}\, u_{k-1}u_1 + k_{c_2}\, u_{k-2}u_2 + ....$

$$+ k_{c_t}\, u_{k-r}u_r + ....... + k_{c_l}\, u\nu_k$$

Differentiating both sides w.r.t. + x. we get

$(u\nu)_{k+1} = k_{c_0}\, u_{k+1}\nu + k_{c_0}\, u\nu_1$

$+ k_{c_l}\, u_k\nu_1 + k_{c_l}\, u_{k-1}\nu_2$

+ ............................

+ ...........................

$+ k_{c_r}\, u_{k-r+1}u_r + k_{c_r}\, u_{k-r}u_{r+1}$

$+ k_{c_1}\, u_1\nu_k + k_{c_1}\, u\nu_{k+1}$

$= k_{c_0}\, u_{k+1}\nu + (k_{c_0} + k_{c_1})\, u_k\nu_1 + (k_{c_1} + k_{c_2})u_{k-2}\nu_2 + ........$

$+ (k_{c_{r-1}} + k_{c_r})uk\text{-}r\text{+}1\ \nu r + ......... + (k_{c_{k-1}} + k_{c_k})u_1\nu_k + k_{c_k}\, n\nu_{k+1}$

Hence $y_{k+1} = (u\nu)_{k+1} = k + 1_{c_0}\, u_{k+1}\nu + k + I_{c_1}\, 1_{c_1}\, u_k\nu_1 k + 1_{c_2}\, u_{k-1}\nu_2 + .... +$

$k + 1_{c_r}\, u_{k+1+r}\nu_r + ........ + k + 1_{c_k}\, u_1\nu_k + k + 1_{c_{k+1}}\, u\nu_{n+1}$

This proves the theorem for n = k + 1 too.

Hence, by mathematical induction, it is true for all $n \in N$.

**Note.** How to choose the first or second function for applying Leibnitz's theorem. If in the product of two functions one is a polynomial in x, choose it as the second function except in cases when the other function is inverse trigonometric, inverse hyperbolic or logarithmic functions in that case take that function as the second function.

**Note.** The nth derivative of $u\nu$ can also be written as :

$(u\nu)_n = n_{c_0}\, u\nu_n + n_{c_1}\, u_1\nu_{n-1} + n_{c_2}\, u_2\nu_{n-2} + ....... + n_{c_1}\, u_2\nu_{n-2} + ....... + n_{c_n}\, u_n\nu.$

**Example 27:** Find the nth derivative of (1) $e^x \sin x$

**Solution:** $y = e^x \sin x$

Take $u = e^x$ and $\nu = \sin x$

$\therefore\ u_1 = e^x,\ \nu_1 = \sin\left(x + \frac{\pi}{2}\right)$

$u_2 = e^x,\ \nu_2 = \sin\left(x + 2.\frac{\pi}{2}\right)$

..........................................

........................................

$u_n = e^x$, $\nu_n = \sin\left(x + n\dfrac{\pi}{2}\right)$

By Leibnitz's Theorem, we have,

$y_n = (u\nu)_n = n_{c_0}\, u\nu_n + n_{c_1}\, u_1\nu_{n+1} + n_{c_2}\, u_2\nu_{n-2} + \ldots.. + n_{c_1}\, u_2\nu_{n-2} + \ldots\ldots.. + n_{c_n}\, u_n\nu$

$$= \left[1.e^x \sin x + n.e^x \sin\left(x+\frac{\pi}{2}\right) + \frac{n(n-1)}{2.1}\sin\left(x+\frac{\pi}{2}\right) + \ldots\ldots + 1.e^x \sin\left(x+n\frac{\pi}{2}\right)\right]$$

$$= e^x\left[1.\sin x + n.\sin\left(x+\frac{\pi}{2}\right) + \frac{n(n-1)}{2.1}\sin\left(x+\frac{\pi}{2}\right) + \ldots\ldots + 1.\sin\left(x+n\frac{\pi}{2}\right)\right]$$

**Example 28:** Find the nth derivative of $e^x \log x$

**Solution:** Put $y = e^x \log x$

Take $u = e^x$ and $\nu = \log x$

$u_{n-1} = e_x$, $\nu_1 = \dfrac{1}{x}$

$u_{n-1} = e_x$, $\nu_2 = \dfrac{1}{x^2}$

$u_{n-2} = e_x$, $\nu_3 = \dfrac{2}{x^3}$

..........................

..........................

$u_1 = e^x$ $\qquad$ $\nu_n = \dfrac{(-1)^{n-1}[n-1}{x^n}$

By Leibnitz's Theorem, we have,

$y_n = (u\nu)_n = n_{c_0}\, u\nu_n + n_{c_1}\, u_1\nu_{n+1} + n_{c_2}\, u_2\nu_{n-2} + \ldots.. + n_{c_1}\, u_2\nu_{n-2} + \ldots\ldots.. + n_{c_n}\, u_n\nu$

$= n_{c_0}$ ex log x + $n_{c_1}\dfrac{1}{x} + n_{c_2}\dfrac{1}{x^2} + n_{c_3}\dfrac{1}{x^3} + \ldots\ldots. + n_{c_n}\dfrac{(-1)^{n-1}[n-1}{x^n}$

$e^x[\log x + \dfrac{n}{x} - \dfrac{n(n-1)}{x^2} + \dfrac{n(n-1)(n-2)}{3} . \dfrac{1}{x^3} + \ldots. + \dfrac{(-1)^{n-1}[n-1}{x^n}$

**Example 29 :** Find the nth derivative of $y = x \sin^2 x$

**Solution :** Take $u = \sin^2 x = \frac{1-\cos 2x}{2}$ and $\nu = x$

Or $u = \frac{1}{2} - \frac{1}{2}\cos 2x$ $\qquad \nu_1 = 1$ and $\nu_2 = \nu_3 = ... = \nu_n = 0$

$$\therefore\ u_n = 0 - \frac{1}{2} . 2^n \cos(2x + \frac{n\pi}{2})$$

$$u_{n-1} = -\frac{1}{2} . 2^{n-2} \cos(2x + \frac{n-1}{2})$$

..............................................

..............................................

By Leibnitz's Theorem, we have,

$$y_n = (u\nu)_n = n_{c_0}\, u\nu_n + n_{c_1}\, u_1\, \nu_{n-1} + n_{c_2}\, u_2\nu_{n-2} + ...+ n_{c_1}\ u_2\nu_{n-2} + ...+ n_{c_n}\ u_n\nu$$

$$n_{c_0} [-\frac{1}{2} . 2^n \cos(2x + \frac{n\pi}{2})] + n_{c_1} [--\frac{1}{2} . 2^{n-1} \cos(2x + \frac{(n-1)\pi}{2}).1 +0+0+...+0$$

$$=-2^{n-2} [\cos(2x + \frac{n\pi}{2}) + n \cos(2x + \frac{(n-1)\pi}{2})]$$

**Example 30 :** If $y = (\sin^{-1} x)^2$, find yn(0).

**Solution :** Here $y = (\sin^{-1} x)^2$ ....(1)

Differentiating w.r.t.x, we have

$$y_n = (\sin^{-1} x). \frac{1}{\sqrt{1-x^2}} \qquad ....(2)$$

Squaring and cross multiplying,

$$(1 - x^2)\, y_1^2 = 4(\sin^{-1} x)^2 = 4y$$

$$\text{or } (1 - x^2)\, y_1^2 - 4y = 0$$

Differentiating w.r.t.x., we have,

$$(1 - x^2)\, 2y_1y_1 - 2xy_1^2 = 4y_1 = 0$$

Dividing by $2y_1$, we have

$$(1 - x^2)\, y_2 - xy_1 - 2 = 0 \qquad ...(3)$$

Differentiating n-times by Leibnitz's theorem, we have

$$\left[1.y_{n+2}(1-x^2)+\frac{n}{2}y_{n+1}(-2x)+\frac{n(n-1)}{2.1}y_n.(-2)\right] - \left[1.y_{n+1}x-\frac{n}{1}y_n.1\right] - 0 = 0$$

Or $(1 - x^2)\, y_{n+1} - (2n + 1)xy_{n+1}) - n^2\, y_n = 0$ ...(4)

Putting x = 0, $y_1(0) = 0$, $y_2(0) = 1$ ...(5)

$y_{n+2}\,(0) = n^2\, y_n(0) = 1$ ....(6)

Putting n = 1, 2, 3, ... ... ... ... ... ... ... in (6) and using (5), we have,

$y_3\,(0) = 1^2$ $y_1(0) = 0$

$y_4\,(0) = 2^2$ $y_2(0) = 2.2^1$

$y_5\,(0) = 3^2$ $y_3(0) = 0$

$y_6\,(0) = 4^2$ $y_4(0) = 2.2^2 4^2$

And so on.

Hence

$$y_n\,(0) = \begin{cases} 2.2^2 4^2 ...(n-2)^2 & \text{when n is even and } n \neq 2 \\ 0, & \text{when n is odd} \end{cases}$$

**Example 31 :** If $f$ (x) = tan x, then prove that

$$n_{c_0}\, f^{n}\,(0) + n_{c_2}\, f^{n-2}\,(0) + n_{c_4}\, f^{n-4}\,(0) - ...\ ... = \sin\frac{n\pi}{2}.$$

**Solution :** $f$ (x) = tan x

$$\therefore\ f(x) \cos x = \sin x$$

Differentiating n-times w.r.t.x by Leibnitz's theorem, we have,

$$[f\,(x).\cos x]_n = (\sin x)_n$$

or $n_{c_0}\, f^{n}\,(x) \cos x$

$+\ n_{c_1}\, f^{n-1}\,(x)\,(-\sin x) + n_{c_2}\, f^{n-2}\,(x)\,(-\cos x) + n_{c_3}\, f^{n-3}\,(x)\,(\sin x)$

$+\ n_{c_4}\, f^{n-4}\,(x) \cos x + n_{c_5}\, f^{n-5}\,(x)\,(-\sin x) + n_{c_6}\, f^{n-6}\,(x)\,(-\cos x) + ....$

$= \sin\left(x + \frac{n\pi}{2}\right)$

Putting x = 0 and using sin 0 = 0, cos 0 = 1, we have,

$n_{c_0} f^n(0) + n_{c_2} f^{n-2}(0) + n_{c_4} f^{n-4}(0) - \ldots \ldots = \sin \frac{n\pi}{2}$.

## 4.8 Self Check Exercise-2

Q. 1 Using Leibnitz's theorem, find the nth derivative of $e^x \log x$

Q. 2 Using Leibnitz's theorem, find the nth derivative of $x^2 a^x$.

## 4.9 Summary

In this unit we have studied the following :

(i) Differentiability of a function

(ii) Derivative of function at a point and its interpretation

(iii) Geometrical interpretation and physical interpretation of derivative at a point

(iv) Algebra of differentiable function

(v) successive differentiation

(vi) Leibnitz's theorem

## 4.10 Glossary

(i) $\frac{dy}{dx}$ is the differential coefficient of a function y $f$ (x) and is called first differential coefficient of y w.r.t.x. Likewise $\frac{d^2y}{dx^2}$ is the second differential coefficient of y w.r.t.x.

(ii) The symbol $\frac{d^n}{dx^n}$ written before a function of x indicates that the function is to be differentiated n times in succession.

## 4.11 Answers to Self Check Exercises

Self Check Exercise - 1

Ans.1 $f$ is not differentiable at x = 2

Self Check Exercise - 2

Ans. 1 $e^x \left[ \log x + n_{c_1} \frac{1}{x^2} + \ldots (-1)^n \frac{n-1_{c_n}}{x^n} \right]$

Ans. 2 $a^x \left[ x^2.(\log a)^n + 2nx(\log a)^{n-1} + n(n-1)(\log a)^{n-2}.2 \right]$

## 4.12 Reference/Suggested Reading


1. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002

2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007

## 4.13 Terminal Questions

1. Differentiate y = $\frac{1+2x}{1+x}$, x > 0 w.r.t.x.

2. Find the third order derivative of $e^{2n}$ cos .

3. If y = $(\log x)^2$, find $\frac{dy}{dx}$.

4. Find $\frac{d^2y}{dx^2}$ if x = at2, y = 2at

5. Find the nth order derivative of $\sqrt{ax+b}$

6. Find the nth order derivative of $e^{3x} \sin^2 2x$

7. If y = $\frac{1}{x^2+a^2}$, find $y_n$

8. Find nth derivative of

   $\frac{1}{(x+1)(x+2)}$

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## Unit - 5

# Indeterminate Form

**Structure**

5.1 Introduction

5.2 Learning Objectives

5.3 Indeterminate Form

5.4 L' Hospital Rule

5.5 Self Check Exercise-1

5.6 Summary

5.7 Glossary

5.8 Answers to Self Check Exercises

5.9 Reference/Suggested Readings

5.10 Terminal Questions

## 5.1 Introduction

Dear students, you must have noticed that no meaning is given to the expression like $\frac{0}{0}$, $\frac{\infty}{\infty}$ $\infty$ - $\infty$, $1\infty$, $0\infty$ etc.. While evaluating the limit of such expressions. Infact they are called indeterminate Form. Obviously we need to develop new techniques for dealing with such situation. In this unit we shall study to evaluate the limit of above said indeterminate forms. We not here that a famous rule called L' Hospital Rule is used to evaluate the form $\frac{0}{0}$.

## 5.2 Learning Objectives

The main objectives of this unit are

(i) to study indeterminate form

(ii) to know about L' Hospital Rule

(iii) to study indeterminate form $\frac{\infty}{\infty}$

(iv) to learn about indeterminate form $\infty$, -$\infty$

(v) to study indeterminate form $0^0$, $1^\infty$, $\infty^0$ etc.

## 5.3 Indeterminate Forms

**Theorem 1 :** If $f$(x) and g(x) are two differentiable functions at x = 1 and

(i) $f$ 9x) = 0 = g(a)

(ii) g' (a) $\neq$ 0 then

$$\underset{x\to a}{Lim} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$ , g'(a) $\neq$ 0.

**Proof :** $\frac{f(x)}{g(x)} = \frac{f(x)-0}{f(x)-0} = \frac{f(x)-f(a)}{g(x)-g(a)} = \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$ , x $\neq$ a

$$\therefore \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} = \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{f'(a)}{g'(a)}$$ , g(a) $\neq$ 0.

$$\left[\begin{array}{c} f.g \text{ are differentiable at } x=a \\ \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = f'(a) \\ \text{and } \lim_{x\to a}\frac{g(x)-g(a)}{x-a} g'(a) \quad g'(a)\neq 0 \end{array}\right]$$

This completes the proof.

## 5.4 Theorem 2. L' Hospital Rule for Indeterminate from

**Statement.** Let $f$ (x) and g (x) be differentiable in a neighbourhood N of a point a except perhaps at the point a.

**Suppose**

(1) $\lim_{x\to a} f$ (x) = 0 $\lim_{x\to a}$ g (x) = 0

(2) g' (x) $\neq$ 0 for x $\in$ N, x $\neq$ a

(3) $\frac{f'(x)}{g'(x)}$ = l exists.

Then $\lim\limits_{x \to a} = \dfrac{f(x)}{g(x)} \lim\limits_{x \to a} \dfrac{f'(x)}{g'(a)}$ = l.

**Proof :** Since in the definition of limit as x → a, the value of the function at the point x = a is immaterial, therefore, without the loss of generality we can suppose

$f$ (a) = 0 = $\lim\limits_{x \to a} f(x)$ and g (a) = 0 = $\lim\limits_{x \to a}$ g(x)

This makes the function $f$ and g continuous at x = a.

For x∈ N = (a - δ , a + δ ), x ≠ a, we can now apply Cauchy's Mean Value Theorem to the functions $f$(x) and g (x) in the closed interval [a, x], since all the hypotheses as satisfied. Therefore, there exists a real number c, a < c < x such that

$$\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c)}{g'(c)}, \ a < c < x$$

Since $f$ (a) = 0 = g (a) by our assumption

$$\therefore \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}, \ a < c < x$$

Obviously, as x → a, c → a,

$$\therefore \lim_{x \to a} \frac{f(x)}{g(x)} \quad \lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = l, \text{ exists}$$

Hence $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(x)}{g'(x)} = l$

**Example 1:** Evaluate $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{1-\sin x}{\cos x}$

**Solution:** Let $f$(x) = 1 - sin x and g(x) = cos x

Obviously both are defined for all real values of x

Now $\lim\limits_{x \to \frac{\pi}{2}} f(x) = \lim\limits_{x \to \frac{\pi}{2}} (1 - \sin x) = 1 - 1 = 0$

and $\lim\limits_{x \to \frac{\pi}{2}} g(x) = \lim\limits_{x \to \frac{\pi}{2}} \cos x = 0$

$\therefore \dfrac{f(x)}{g(x)}$ is of the form $\dfrac{0}{0}$

Further $f$'(x) = - cos x and g'(x) = - sin x ≠ 0 in the nhood of $\dfrac{\pi}{2}$

and $\lim\limits_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to\frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \lim\limits_{x\to\frac{\pi}{2}}$ cot x = 0, exists

$\therefore$ L' Hospital rule

$\lim\limits_{x\to\frac{\pi}{2}} \frac{f(x)}{g(x)} = \lim\limits_{x\to\frac{\pi}{2}} \frac{f'(x)}{g'(x)} = 0$

Hence $\lim\limits_{x\to\frac{\pi}{2}} \frac{1-\sin x}{\cos x} = 0$

**Example 2:** Evaluate $li\,\mathrm{m}_{x\to 1} \frac{\log x}{x-\sqrt{x}}$

**Solution:** Let $f$(x) = log x and g(x) = x - $\sqrt{x}$

Obviously both are defined for x > 0

Now $\lim\limits_{x\to 1}$ log(x) = 0

and $\lim\limits_{x\to 1}$ g(x) = $\lim\limits_{x\to 1}\left(x+\sqrt{x}\right)$ = 1 - 1 = 0

$\therefore \frac{f(x)}{g(x)}$ = is of the form $\frac{0}{0}$

Further $f'$(x) = $\frac{1}{x}$ and g'(x) = 1 1 - $\frac{1}{2\sqrt{x}} \neq 0$ in the nhood. Of 1

and $\lim\limits_{x\to 1} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 1} \frac{\frac{1}{x}}{1-\frac{1}{\sqrt{x}}} = \lim\limits_{x\to 1} \frac{2}{\sqrt{x}(2\sqrt{x}-1)}$ = 2, exists

$\therefore$ by L' Hospital's rule, $\lim\limits_{x\to 1} \frac{f(x)}{g(x)} = \lim\limits_{x\to 1} \frac{f'(x)}{g'(x)} = 2$

Hence $\lim\limits_{x\to 1} \frac{\log x}{x-\sqrt{x}} = 2.$

**Note.** L' Hospital's rule is applicable only if the conditions are satisfied, in particular, the existence of the limit $\frac{f'(x)}{g'(x)}$ as x → a whether finite or infinite, is a must.

If this limit i.e. $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ does not exist, we should not conclude that $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ also does not exist. It may not exist.

**Example 3:** Let us consider the limit, $\lim\limits_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\tan x}$

Let us take $f(x) = x^2 \sin \frac{1}{x}$, and $g(x) = \tan x$.

Obviously both are defined for $x \neq 0$

Now $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 \sin \frac{1}{x} = 0$

and $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} \tan x = 0$

$$\therefore \frac{f(x)}{g(x)} \text{ is of the form } \frac{0}{0}$$

Further $f'(x) = x^2 \cos \frac{1}{x}\left(-\frac{1}{x^2}\right) + 2x \sin \frac{1}{x}$

$$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

and $g'(x) = \sec^2 x$

Now $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to 0} \frac{2x\sin\frac{1}{x} - \cos\frac{1}{x}}{\sec^2 x}$ does not exist, because $\lim\limits_{x\to 0} \cos \frac{1}{x}$ does not exist. Therefore we cannot apply the L' Hospital's rule. But it does not mean that the given limit also does not exist.

In fact this limit does exist. We prove it as follows:

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\tan x} = \lim_{x\to 0} \frac{x\sin\frac{1}{x}}{\frac{\tan x}{x}} = \frac{\lim\limits_{x\to 0} x\sin\frac{1}{x}}{\lim\limits_{x\to 0} \frac{\tan x}{x}} = \frac{0}{1} = 0, \text{ exists.}$$

**Note.** Suppose $f(x)$ and $g(x)$ are twice differentiable in a deleted nbd. of a and $g'(x) = 0$, $g''(x) \neq 0$ in this nbd.

**Suppose**

(1) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

(2) $\lim_{x \to a} f'(x) = \lim_{x \to a} g'(x) = 0$

(3) g''(x) ≠ 0 in the deleted nhood. of a

(4) $\lim_{x \to a} \frac{f''(x)}{g''(x)} = l$, exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)} = l.$$

Now, we state the following more general theorem.

**Theorem 3:** If $f$(x) and g(x) are differentiable function of order n in a deleted neighborhood N of a point a such that

(1) $\lim_{x \to a} f^k(x) = 0 = \lim_{x \to a} g^k(x)$, $1 \leq k \leq n - 1$

(2) $g^n(x) \neq 0$ x - N, $x \neq \alpha$

(3) $\lim_{x \to a} \frac{f^n(x)}{g^n(x)} = l$, exists

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = l$

We again stress that if in the repeated application of L' Hospital's rule, we get that $\lim_{x \to a} \frac{f^n(x)}{g^n(x)}$ does not exist, it does not imply that $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist. It may or may not exist.

**Example.** Evaluate the following limits:

(a) $\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)}$; (b) $\lim_{x \to \frac{\pi}{2}} \frac{\log \sin x}{1 - \sin x}$ (c) $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x - \sin x}$

**Solution:** (a) $\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)} = \lim_{x \to 0} \frac{2\tan^{-1} x . \frac{1}{1+x^2}}{\frac{1}{1+x^2} 2x} = \lim_{x \to 0} \frac{\tan^{-1} x}{x}$

is again of the form $\frac{0}{0}$ at x = 0

Therefore again by L' Hospital's rule

$$\lim_{x\to 0} \frac{(\tan^{-1} x)^2}{\log(1+x^2)} = \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1$$

(b) $\lim\limits_{x\to\frac{\pi}{2}} \frac{\log \sin x}{1-\sin x}$ is form of $\frac{0}{0}$ at x = $\frac{\pi}{2}$

$\therefore$ by L' Hospital's rule

$$\lim_{x\to\frac{\pi}{2}} \frac{\log \sin x}{1-\sin x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{1}{\sin x}\cos x}{0-\cos x} = \lim_{x\to\frac{\pi}{2}} \frac{1}{\sin x} = -1$$

(c) $\lim_{x\to 0} \frac{x-\tan^{-1} x}{x-\sin x}$ is of the form of $\frac{0}{0}$

$\therefore$ L' Hospital's rule

$$\lim_{x\to 0} \frac{x-\tan^{-1} x}{x-\sin x} = \lim_{x\to 0} \frac{1-\frac{1}{1+x^2}}{1+\cos x}$$ is again of the form of $\frac{0}{0}$ at x = 0

$\therefore$ L' Hospital's rule,

$$\lim_{x\to 0} \frac{x-\tan^{-1} x}{x-\sin x} = \lim_{x\to 0} \frac{0+\frac{2x}{\left(1+x^2\right)^2}}{0+\sin x}$$

$$= \lim_{x\to 0} \frac{2x}{(1+x^2)^2 \sin x}$$ is again $\frac{0}{0}$ at x = 0

[$\therefore$ Applying L' Hospital's rule again]

$$\lim_{x\to 0} \frac{2}{(1+x^2)^2 \cos x + 2(1+x^2).2x \sin x} = 2$$

L' Hospital's Rule for the Indeterminate form $\frac{\infty}{\infty}$

**Theorem 4.** (Statement): let $f$(x) and g(x) be two functions defined and differentiable in a deleted neighborhood N of the point a and g'(x) $\neq$ 0 for x $\in$ N. Suppose

(1) $\lim\limits_{x\to a} f(x) = \infty$, $\lim\limits_{x\to a} g(x) = \infty$

(2) $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ = l, exists then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ = l.

[we accept it without proof].

**Example 4:** Evaluate $\lim\limits_{x\to\infty} \frac{\log x}{x}$

**Solution:** $\lim\limits_{x\to\infty} \frac{\log x}{x}$ is of the form $\frac{\infty}{\infty}$

$$\left[\lim_{x\to\infty} \log x = \infty, \lim_{x\to\infty} x = \infty\right]$$

$\therefore$ by L' Hospital's rule

$$\lim_{x\to\infty} \frac{\log x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

**Example 5:** Evaluate $\lim\limits_{x\to\infty} \frac{2x^3 + x^2 - 3x + 1}{3x^3 - 2x^2 + x - 1}$

**Solution:** $\lim\limits_{x\to\infty} \frac{2x^3 + x^2 - 3x + 1}{3x^3 - 2x^2 + x - 1}$ is of the form $\frac{\infty}{\infty}$. therefore, by L' Hospital's rule,

$\lim\limits_{x\to\infty} \frac{2x^3 + x^2 - 3x + 1}{3x^3 - 2x^2 + x - 1} = \lim\limits_{x\to\infty} \frac{6x^2 + 2x - 3}{9x^2 - 4x + 1}$ is again of the form of $\frac{\infty}{\infty}$

$\therefore$ by applying L' Hospital's rule,

$\operatorname{Lim}\limits_{x\to\infty} \frac{12x + 2}{18x - 4}$ is again of the form $\frac{\infty}{\infty}$

$\therefore$ by L' Hospital's rule

$$= \operatorname{Lim}_{x\to\infty} \frac{12}{18} = \frac{2}{3}$$

**Example 6:** Evaluate $\operatorname{Lim}\limits_{x\to\frac{\pi}{2}} \frac{\tan x}{\log \cos x}$

**Solution:** $\operatorname{Lim}\limits_{x\to\frac{\pi}{2}} \frac{\tan x}{\log \cos x}$ is of the form $\frac{\infty}{\infty}$

$$\left[\underset{x\to\frac{\pi}{2}}{Lim}\tan x=\infty, \underset{x\to\frac{\pi}{2}}{Lim}\log\cos x=-\infty\right]$$

$\therefore$ by L' Hospital's rule

$$\underset{x\to\frac{\pi}{2}}{Lim}\frac{\tan x}{\log\cos x} = \underset{x\to\frac{\pi}{2}}{Lim}\frac{\sec^2 x}{\frac{1}{\cos x}(-\sin x)}$$

$$= \underset{x\to\frac{\pi}{2}}{Lim}\frac{1}{\sin x\cos x} = \underset{x\to\frac{\pi}{2}}{Lim}\frac{\sec^2 x}{\frac{1}{\cos x}(-\sin x)}$$

$$= \underset{x\to\frac{\pi}{2}}{Lim}\frac{1}{\sin x\cos x} = - \underset{x\to\frac{\pi}{2}}{Lim}\frac{\sec^2 x}{\frac{1}{\cos x}(-\sin x)}$$

$$= - \underset{x\to\frac{\pi}{2}}{Lim}\frac{1}{\sin x\cos x} = - \underset{x\to\frac{\pi}{2}}{Lim}\frac{2}{\sin 2^x} = -\infty$$

Indeterminate form $\infty$ - $\infty$ and 0. $\infty$ Indeterminate forms of the type $\infty$ - $\infty$ and 0. $\infty$ can often be evaluated by transforming these into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form to which L' Hospital's rule is easily applicable.

**Example 7:** Evaluate the limit $\lim\limits_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right)$.

**Solution:** obviously $\cot^2 x = \frac{1}{x^2}$ is of the form $\infty$ - $\infty$ at x = 0

$\therefore$ we write $\cot^2 x - \frac{1}{x^2} = \frac{1}{\tan^2 x} - \frac{1}{x^2} = \frac{x^2-\tan^2 x}{x^2\tan^2 x}$

$$\lim_{x\to 0}\left(\cot^2 x-\frac{1}{x^2}\right) = \lim_{x\to 0}\frac{x^2-\tan^2 x}{x^2\tan^2 x} = \lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4}\left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x\to 0}\frac{x^2-\tan^2 x}{x^4} \qquad \left[\therefore \lim_{x\to 0}\frac{x}{\tan x}=1\right]$$

is of the form $\frac{0}{0}$, by L' Hospital's rule,

$$\lim_{x\to 0} \frac{2x - 2\tan x \sec^2 x}{4x^3} = \lim_{x\to 0} \frac{x - \tan x(1+\tan^2 x)}{2x^3}$$

$= \lim_{x\to 0} \frac{x - \tan x - \tan^3 x}{2x^3}$ is of the form of $\frac{0}{0}$

$= \lim_{x\to 0} \frac{1 - \sec^2 x - 3\tan^2 x \sec^2 x}{6x^2}$ [by L' Hospital's rule]

$$= \lim_{x\to 0} \frac{1-(1+\tan^2 x) - 3\tan^2 x(1+\tan^2 x)}{6x^2}$$

$$= \lim_{x\to 0} \frac{-4\tan^2 x - 3\tan^4 x}{6x^2}$$

$$= \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^2 \left(\frac{4+3\tan^2 x}{6}\right)$$

$= - (1)^2 \left(\frac{4+3.0}{6}\right)$

$= - \frac{4}{6}$

$= - \frac{2}{3}$

**Example 8:** Evaluate $\lim_{x\to 0} \infty$ x sin $\frac{1}{x}$.

**Solution:** x sin $\frac{1}{x}$ is of the form 0. ∞.

$\therefore$ we write x sin $\frac{1}{x} = \frac{\sin\frac{1}{x}}{\frac{1}{x}}$, is of the form $\frac{0}{0}$ for x > 0

$$\therefore \lim_{x\to\infty} x \sin\frac{1}{x} = \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \cos \frac{1}{x} = \cos 0 = 1$$

**Indeterminate form of the type $0^0$, $\infty^0$, $1^\infty$**

Limits involving exponential expressions of the above type are evaluated taking logarithms and reducing them to one of standard forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Let F(x) = $\left[f(x)\right]^{g(x)}$ .....(1)

Now three cases arise:

(1) $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, in this case it is of the form $0^0$.

(2) $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = 0$, in this case it is of the form $\infty^0$.

(3) $\lim_{x\to 1} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$, in this case it is of the form $1^\infty$.

In all these cases, we proceed like this:

From (1), log F(x) = g(x) log $f$(x)

$$\lim_{x\to a} \log F(x) = \lim_{x\to a} g(x) \log f(x)$$

The R.H.S is of the form $0.\infty$ and therefore can be easily converted into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form to which L' Hospital's rule is applicable.

Suppose $\lim_{x\to a} g(x) \log f(x) = l$

Then $\lim_{x\to a} \log F(x) = l \Rightarrow \lim_{x\to a} F(x) = el \Rightarrow \lim_{x\to a} \left(f(x)\right)^{g(x)} = el$.

**Example 9:** Evaluate $\lim_{x\to 0+} x^{1/\log x}$

**Solution:** $x^{1/\log x}$ $f$ or x > 0 is of the form $0^0$.

Let F(x) = $x^{1/\log x}$, taking logs on both sides, we have,

$$\log F(x) = \frac{1}{\log x} . \log x = 1$$

$$\therefore \lim_{x\to 0+} \log F(x) = \lim_{x\to 0+} 1 \Rightarrow \lim_{x\to 0+} F(x) = e^1 = e$$

hence $\lim_{x \to 0+} x^{1/\log x} = e$

**Example 10:** Evaluate $\lim_{x \to 0} (\sin x / x)^{\frac{1}{x^2}}$

**Solution:** $\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ is of the form $1^{\infty}$ $f$ or x $\neq$ 0

Let F(x) = $\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$,

Taking logs on both sides, we have,

$$\log F(x) = \frac{1}{x^2} \log\left(\frac{\sin x}{x}\right) = \frac{\log \frac{\sin x}{x}}{x^2}$$

$$\therefore \lim_{x \to 0} \log F(x) = \lim_{x \to 0} \frac{\log \frac{\sin x}{x}}{x^3} \text{ is of the form } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\frac{x}{\sin x} \cdot \frac{x\cos x - \sin x.1}{x^2}}{2} \qquad \text{[by L' Hospital's rule]}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3} \qquad \left[\lim_{x \to 0} \frac{x}{\sin x} = 1\right]$$

is of the form $\frac{0}{0}$, by L' Hospital's rule

$$= \lim_{x \to 0} \frac{-x\sin x + \cos x - \cos x}{6x^2} = \lim_{x \to 0} \frac{-x\sin x}{6x^2}$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = -\frac{1}{6}$$

$$\therefore \lim_{x \to 0} F(x) = e^{\frac{-1}{6}} \text{ hence } \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{-1}{6}}$$

**Example 11:** Evaluate $\lim_{x \to 0+} (\cot x)^x$.

**Solution:** $(\cot x)^x$ for x > 0, is of the form $\infty^0$.

Let F(x) = (cot x)$^x$.

Taking logs on both sides, we have,

log F(x) = x log cot x, $\therefore \lim_{x\to 0+}$ log F(x)

= $\lim_{x\to 0+}$ x log cot x is of the form 0. ∞

Or $\lim_{x\to 0+}$ log F(x) = $\lim_{x\to 0+} \dfrac{\log \cot x}{\frac{1}{x}}$ is of the form $\dfrac{\infty}{\infty}$

= $\lim_{x\to 0+} \dfrac{\frac{1}{\cot x}(-\operatorname{cosec}^2 x}{-\frac{1}{x^2}}$ [by L' Hospital's rule]

= $\lim_{x\to 0+} \dfrac{x^2}{\sin^2 x}$ . tan x = $\lim_{x\to 0+} \left(\dfrac{x}{\sin x}\right)^2$ . $\lim_{x\to 0+}$ tan x = 1.0 - 0

$\lim_{x\to 0+}$ F(x) = e$^0$ = 1

Hence $\lim_{x\to 0+}$(cot x)$^x$ = 1

## 5.5 Self Check Exercise

Q.1 Evaluate $\lim_{x\to a} \dfrac{\sin ax}{\sin bx}$

Q.2 Evaluate $\lim_{x\to a} \dfrac{\log(1+x^3)}{\sin^3 x}$

Q.3 Evaluate $\lim_{x\to a} \dfrac{x^n}{e^x}$, n ∈ N.

**Some Illustrated Examples**

**Example 12:** Evaluate the Limit

$$\lim_{x\to 0} \frac{1-\cos^2 x}{\sin^2 x}$$

**Solution:** $\lim_{x\to 0} \dfrac{1-\cos^2 x}{\sin^2 x}$ $\left(\dfrac{0}{0}\ form\right)$

$$= \lim_{x\to 0} \frac{2\sin x\cos x}{2x\cos x^2} \quad \text{(L' Hospital Rule)}$$

$$= \lim_{x\to 0} \frac{\sin x\cos x}{x\cos x^2}$$

$$= \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{\cos x}{\cos^2 x}\right)$$

$$= \perp . \frac{1}{1} = 1 \left(\text{Q} \lim_{x\to 0}\frac{\sin x}{x} = 1, \lim_{x\to 0}\cos x = 1\right)$$

**Example 13:** Find $\displaystyle Lim_{y\to 0} \frac{e^y - e^{-y}}{\sin y}$

**Solution:** We have

$$Lim_{y\to 0} \frac{e^y - e^{-y}}{\sin y} \qquad \left(\frac{0}{0}\ form\right)$$

$$= Lim_{y\to 0} \frac{e^y - e^{-y}}{\sin y}$$

$$= \frac{e^0 + e^0}{\cos 0} = \frac{1+1}{1}$$

$$= \frac{2}{1} = 2.$$

**Example 14:** Show that

$$Lim_{x\to 1} \frac{x^x - x}{x-1-\log x} = 2$$

**Solution:** We have $\displaystyle Lim_{x\to 1} \frac{x^x - x}{x-1-\log x} \qquad \left(\frac{0}{0}\ form\right)$

$$= Lim_{x\to 1} \frac{x^x(1+\log x)-1}{1-\frac{1}{x}}$$

$$\left\{\begin{aligned}&Put\ \ y = x^x \\ &\Rightarrow Log\ y = x\log x \\ &\Rightarrow \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x.1 \\ &\Rightarrow \frac{dy}{dx} = x^x(1+\log x)\end{aligned}\right\}$$

$$= \underset{x\to 1}{Lim}\ \frac{x^x \times \frac{1}{x} + (1+\log x).x^x(1+\log x)}{\frac{1}{x^2}}$$

$$= \frac{1\times 1 + (1+0)\times 1(1+0)}{1}$$

$$= \frac{1+1}{1} = \frac{2}{1} = 2$$

**Example 15:** Evaluate $\underset{x\to\infty}{Lim}\dfrac{\sin^{-1}\frac{1}{x}}{\tan\frac{1}{x}}$

**Solution:** $\underset{x\to\infty}{Lim}\dfrac{\sin^{-1}\frac{1}{x}}{\tan\frac{1}{x}}$

$$= \underset{h\to 0}{Lim}\frac{\sin^{-1}h}{\tan \mathrm{h}} \qquad \left(\because if\ x = \frac{1}{h}(h>0) \Rightarrow as\, x \to \infty\ h \to 0\right)$$

Now $\underset{h\to 0}{Lim}\dfrac{\sin^{-1}h}{\tan \mathrm{h}} \qquad \left(\dfrac{0}{0}\ form\right)$

$$= \underset{h\to 0}{Lim}\ \frac{\frac{1}{\sqrt{1-h^2}}}{\sec^2 h} = \frac{1}{\sqrt{1-0}} = \frac{1}{1} = 1$$

**Example 16:** Evaluate $\underset{x\to\infty}{Lim}\dfrac{x^n}{e^x}$, n $\in$ N.

**Solution:** $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} \qquad \left(\dfrac{\infty}{\infty}\ form\right)$

$= \lim\limits_{x\to\infty} \dfrac{nx^{n-1}}{e^x} \qquad \left(\dfrac{\infty}{\infty}\ form\right)$

$= \lim\limits_{x\to\infty} \dfrac{n(n-1)x^{n-2}}{e^x} \quad \left(\dfrac{\infty}{\infty}\ form\right)$

$= \lim\limits_{x\to\infty} \dfrac{n(n-1)(n-2).......2.1\ x^0}{e^x}$

$= \lim\limits_{x\to\infty} \dfrac{n!}{e^x}$

$= n!\ \lim\limits_{x\to\infty} \dfrac{1}{e^x}$

$= n!.0 = 0 \qquad \left(\because \lim\limits_{n\to\infty} \dfrac{1}{e^x} = \lim\limits_{n\to\infty} e^{-x} = 0\right)$

**Example 17:** Evaluate $\lim\limits_{y\to 0} \left(\dfrac{1}{e^y - 1} - \dfrac{1}{y}\right)$

**Solution:** $\lim\limits_{y\to 0} \left(\dfrac{1}{e^y - 1} - \dfrac{1}{y}\right) \qquad (\infty - \infty\ form)$

$= \lim\limits_{y\to 0} \left(\dfrac{y - e^y + 1}{y(e^y - 1)}\right) \quad \left(\dfrac{0}{0}\ form\right)$

$= \lim\limits_{y\to 0} \left(\dfrac{1 - e^y + 0}{y\,e^y + (e^y - 1).1}\right)$

$= \lim\limits_{y\to 0} \dfrac{1 - e^y}{y\,e^y + e^y - 1} \qquad \left(\dfrac{0}{0}\ form\right)$

$= \lim\limits_{y\to 0} \dfrac{-e^y}{y\,e^y + e^y + e^y}$

$= -\ \dfrac{1}{0+1+1} = \dfrac{1}{2}$

**Example 18:** Evaluate $\underset{x \to 0+}{Lim}$ $x^x$

**Solution:** Let $y = x^x$.

$\Rightarrow$ Log y = x Log x

$\therefore$ $\underset{x \to 0}{Lim}$ log y = $\underset{x \to 0+}{Lim}$ x log x $(0.\infty \; form)$

$$= \underset{x \to 0+}{Lim} \frac{\log x}{\frac{1}{x}} \qquad \left(\frac{\infty}{\infty} \; form\right)$$

$$= \underset{x \to 0+}{Lim} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \underset{x \to 0+}{Lim} (-x) = 0$$

$\therefore$ Log $\underset{x \to 0+}{Lim}$ y = 0

$\Rightarrow$ $\underset{x \to 0+}{Lim}$ y = $e^0$ = 1

$\Rightarrow$ $\underset{x \to 0+}{Lim}$ $x^x$ = L

## 5.6 Summary

In this unit we have learnt

(i) indeterminate form

(ii) L' Hospital Rule

(iii) to evaluate the limit of the form

$\frac{\infty}{\infty}$, $\infty - \infty$, $0^0$, $1^\infty$, $\infty^0$ etc

## 5.7 Glossary

Some standard expansions

(i) ex = 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ +................

(ii) sin x = x - $\frac{x^3}{3!}$ + $\frac{x^5}{5!}$ - ...................

(iii) cos x = 1 - $\frac{x^2}{2!}$ + $\frac{x^4}{4!}$ - ..................

(iv) tan x = x + $\frac{x^3}{3}$ + $\frac{2x^5}{15}$ +....................

(iv) log (1 + x) = x - $\frac{x^2}{2}$ + $\frac{x^3}{3}$ - $\frac{x^4}{4}$ +..........

(v) $(1+x)^{\frac{1}{x}}$ = e - $\frac{ex}{2}$ + 11 $\frac{ex^2}{24}$ +............. (near x = 0)

## 5.8 Answers to Self Check Exercises

Ans. 1 $\frac{a}{b}$

Ans. 2 1

Ans. 3 0

## 5.9 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002


## 5.10 Terminal Questions

1. Evaluate $\underset{x\to\frac{\pi}{2}}{Lim} \frac{\tan x}{\tan 3x}$
2. Find $\underset{x\to\infty}{Lim} \frac{Log\, x}{x^n}$, x > 1
3. Evaluate $\underset{x\to 0+}{Lim} \frac{Log \sin x}{\cot x}$
4. Find $\underset{x\to 0}{Lim} \left(\cos ec\, x - \frac{1}{x}\right)$
5. Find $\underset{x\to 0}{Lim}\, x^{2x}$

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## Unit - 6

# General Theorems Rolle's And Lagrange's Theorem

**Structure**



## 6.1 Introduction

Dear students, by now you must have become familiar to classify to classify between theorems applicable to a class of functions such as logarithmic functions, trigonometric functions etc. The theorems applicable to class of functions are known as General Theorems. The most fundamental theorem is Rolle's theorem which play an important role out of these theorem. As the name suggests this theorem was given by the French Mathematician Michal Rolle in the year 1691.

## 6.2 Learning Objectives

The main objectives of this unit are:

(i) to state and prove Rolle's theorem

(ii) to give geometrical interpretation of Rolle's theorem

(iii) to state and prove Lagrange's mean value theorem

(iv) to give geometrical interpretation of Lagrange's mean value theorem

## 6.3 Rolle's Theorem.

Statement. Let $f$ be a function defined on a closed interval [a, b] such that

$f$ is continuous in a closed interval [a, b]

$f$ is derivable in the open interval (a, b)

$f$ (a) = $f$ (b)

then there exists at least one real number c $\in$ (a, b) such that $f'$ (c) = 0

Proof : If f is a constant function in [a, b] then obviously $f'$(x) = 0 for all x $\in$ $\in$ (a, b) so that the result is trivially true in this case.

Let $f$(x) be a non constant function in [a, b]. Since $f$ is continuous is the closed interval [a, b], therefore, it is bounded there in and attains its maximum and minimum value in [a, b] i.e. if M and m respectively denote the maximum and the minimum values of $f$ (x), then there exists two real points c and d in [a, b] such that $f$ 9c) = M and $f$(d) = m.

Since $f$ is not constant on [a, b], therefore M $\neq$ m. This implies that at least one of M or m is different from the common value $f$ (a) = $f$ (b) at the end points.

Let M $\neq$ $f$(a) = $f$(b), then $f$ (c) $\neq$ $f$(a) = $f$(b) because M = $f$(c) or some c $\in$ [a, b], but $f$(c) $\neq$ $f$(a) = $f$(b) = > c $\neq$ a, b = > c $\in$ (a, b).

Now we claim that $f'$(c) = 0

$$f(c) = M = \text{max. of } f(x) \ \forall\ x \in [a, b]$$

$$\therefore \quad f(x) \leq f(c) \qquad \forall\ x \in [a, b]$$

$$\Rightarrow \quad f(x) - f(c) \leq \qquad \forall\ x \in [a, b]$$

$$\Rightarrow \quad \frac{f(x)-f(c)}{x-c} \geq 0 \text{ if } x < c$$

$$\Rightarrow \quad \frac{f(x)-f(c)}{x-c} \leq 0 \text{ if } x < c$$

$$\Rightarrow \quad \lim_{x \to c-} \frac{f(x)-f(c)}{x-c} \geq 0 \text{ and } \lim_{x \to c+} \frac{f(x)-f(c)}{x-c} \leq 0$$

$$\Rightarrow \quad f'(c-) \geq 0 \text{ and } f'(c+) \leq 0$$

Since $f$ is differentiable at x = c

$$\therefore\ f'(c) = f'(c-) = f'(c+)$$

Consequently $f'$ (c) = 0,

Similarly if m $\neq$ $f$(a) = $f$ (b), then $f$ or some d $\in$ (a, b), $f'$ (d) = 0.

This completes the proof.

Note 1. It immediately follows from the above theorem that between two consecutive zeroes a real polynomial $f$(x), there exists at least one zero of $f'$ (x).

Note 2. The condition of Rolle's Theorem are sufficient but not necessary, i.e. $f'(c) = 0$ for some $c \in (a, b)$ even if the conditions of Rolle's Theorem are not satisfied e.g. $f(x) = \frac{1}{x} + \frac{1}{1-x}$ does not satisfy the conditions of Rolle's Theorem in [0, 1] but $f'\left(\frac{1}{2}\right) = 0$.

Note 3. For the validity of Rolle's Theorem the existence of the derivative of $f(x)$ at the end point a, b is not required.

For example, The function $f(x) = \sqrt{1-x^2}$ , $x \in [-1, 1]$ not differentiable at the end points -1 and 1, but $f'(x) = 0$ at $x = 0$ and $0 \in (-1, 1)$.

Note 4. Though the existence of the derivative at the end points is not required, all the three conditions stated in the theorem are essential for the validity of Rolle's Theorem i.e. if any one of the three conditions is not satisfied, while the other two conditions are satisfied, Roll's Theorem need not be true. We give examples in support of this assertion.

**Example 1 :** Let $f(x) = 1\ x\ 1$, $x\ \varepsilon\ /\ [-1, 1]$

Obviously, $f$ is continuous in [-1, 1] and $f(-1)\ f(1) = 1$

But $f$ is not differentiable at $x = 0$ and $0 \in (-1, 1)$. Thus $f$ satisfies the (1) and (3) conditions of Rolle's Theorem while the (2) condition is not satisfied by $f$, therefore Rolle's Theorem is not valid, since $f'(x) \neq 0$ for any $x \in (-1, 1)$

**Example 2 :** Let $f(x) = \begin{cases} x, & \text{if } 0 \le x < 1 \\ 0, & \text{if } x = 1 \end{cases}$

be a function defined on [0, 1].

Obviously, $f(a) = f(1) = 0$ and $f$ is differentiable in (0, 1)

But $f$ is continuous in [0, 1] as $f$ is not continuous to the left at x = 1. $\{\lim_{x\to 1} - f(x) = \lim_{x\to 1-} x = 1 \neq 0 = f(1)\}$

Thus $f$ satisfies the (2) and (3) conditions of Rolle's Theorem, while the (1) condition is not satisfied by $f$. Hence Rolle's Theorem is not valid, since $f'(x) \neq 0$ for any $x \in (0, 1)$

$[f'(x) = 1, x \in (0,1)]$

**Example 3 :** Let $f(x) = x^2$, $x \in [1, 2\}$

Obviously $f$ is continuous in [1, 2]

and $f$ is differentiable in [1, 2]

But $f(1) = 1 \neq 4 + f(2)$

Thus $f$ satisfies the conditions (1) and (2) of Rolle's Theorem while the condition (3) is not satisfied by $f$. Hence Rolle's Theorem is not valid, Since $f'(x) \neq 0$ for any $x \in (1, 2)$

$[f'(x) = 2x, x \in (1, 2)]$

## 6.4 Geometrical Significance of Rolle's Theorem

(1) $f(x)$ is continuous in [a, b] implies that the graphs of $y = f(x)$ does not break anywhere from $x = a$ to $x = b$.

(2) $f$ is differentiable in (a, b) implies that the tangent to the curve $y = f(x)$ exists at each point of (a, b).

(3) $f(a) = f(b)$ implies that the chord joining the points A (a, $f(a)$ and B [b, $f(b)$] on the curve $y = f(x)$ is parallel to the x -axis.

Hence geometrically speaking if the curve $y = f(x)$ satisfies all the above first three conditions, then there exists at least one point $c \in (a, b)$ where the tangent to curve at the point (c, $f(c)$ is parallel to x - axis. The point c need not be unique. See figures given below. In figure (3) there are four points in (a, b) where the tangent to the curve $y = f(x)$ is parallel to the a - axis.

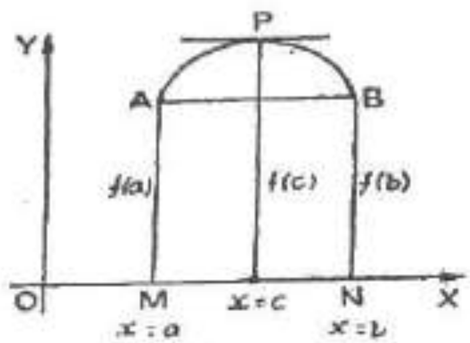


Fig. (i)

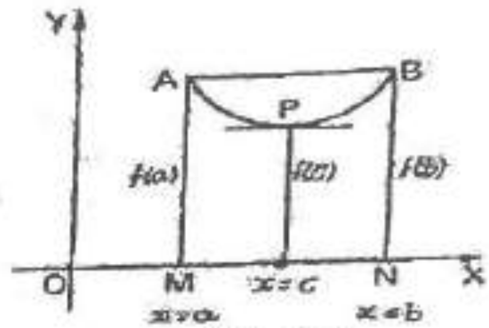
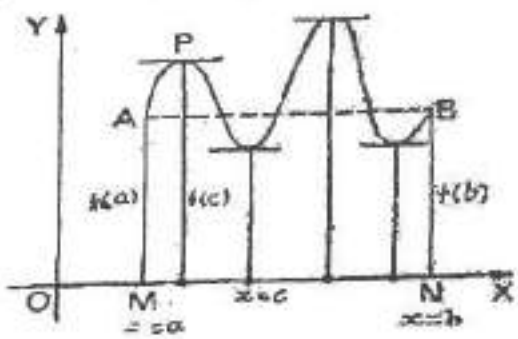


Fig. (ii)



Fig

**Example 4 :** Verify Rolle's Theorem for the function

$f(x) = (x - a)^m (x - b)^n$ in [a, b] ; m, n $\in$ N.

**Solution :** Here $f(x) = (x - a)^m (x - b)^n$ ....(1)

Obviously $f(x)$, being a polynomial function is continuous in [a, b] and derivable in (a, b).

Also $f(a) = f(b) = 0$

Thus $f$(x) satisfies all the three conditions of Rolle's Theorem in [a, b]

$\therefore$ there exists at least one c $\in$ (a, b) such that $f'$ (c) = 0, ...(2)

Differentiating (1) w.r.t. x, we have,

$$f'(x) = (x - a)^m n(x - b)^{n-1} + (x - b)^n m(n - a)^{m-1}$$
$$= (x - a)^{m-1} (x - b)^{n-1} [nx - na + mx - mb]$$
$$= (x - a)^{m-1} (x - b)^{n-1}[m + n) x - (mb + na)]$$
$$\therefore f'(c) = (c - a)^{m-1} (c - b)^{n-1} [(m + n)c - (mb + na)] = 0 \quad ....(3)$$

From (2) and (3), we have,

$$(m + n) c - (m b + na) = 0 \qquad [c \neq a, c \neq b]$$

Or $c = \dfrac{mb+na}{m+n}$

$\therefore$ c is a point in the interval (a, b) dividing the [a, b] internally in the ratio m : n.

Thus the Rolle's Theorem is verified.

**Example 5 :** Use Rolle's Theorem to find the position of real zeros of

$f'$ (x) where $f$(x) = x(x - 1) (x - 2) (x - 3).

**Solution :** $f$(x) = x(x - 1) (x - 2) (x - 3) [given] ....(1)

Since $f$(x) is a polynomial function, therefore, it is continuous and derivable for all real x. Also $f(0) = f(1) = f(2) = f(3) = 0$

Thus $f$(x) satisfies all the three conditions of Rolle's Theorem in each of the closed intervals [0, 1], [1, 2] and [2, 3] separately.

Hence $f'(x) = 0$ for some x $\in$ (0, 1), x $\in$ (1, 2), x $\in$ (2, 3) separately.

Since $f$(x) is a polynomial function of degree 4, therefore, $f'$(x) is a polynomial function of degree 3 and hence $f'$(x) can not have more than 3 zeros.

Combining the statements 2 and (3) together, we see, that $f'$(x) has exactly 3 distinct zeros, one in each of the open intervals (0, 1)] (1, 2) and (2, 3).

This completes the solution.

## 6.5 Lagrange's Mean Value Theorem

**Statement.** Let $f$ be a function defined on a closed interval [a, b] such that

(1) $f$ is continuous in the closed interval [a, b]

(2) $f$ is derivable in the open interval (a, b) then there exists at least one real number c $\in$ (a, b) such that

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

**Proof :** Let us consider a new function $\phi(x)$ define by

$$\phi(x) = f(x) + Ax$$

Where A is a real constant to be determined such that $\phi(a) = \phi(b)$

i.e. $f(a) + Aa = f(b) + Ab$

or $A(a - b) = f(b) - f(a)$

or $A = \dfrac{f(b) - f(a)}{b - a}$ ...(2)

Since $\phi(x)$ is the sum of two function $f(x)$ and Ax, both of which are continuous in {a, b} and derivable in (a, b), therefore $\phi(x)$ is,

(1) Continuous in [a, b]

(2) Derivable in (a, b)

(3) $\phi(a) = \phi(b)$.

$\therefore$ $\phi$ satisfies all the three condition of Rolle's Theorem.

$\therefore$ there exists at least one real number $c \in (a, b)$ such that

$\phi'(c) = 0$ ....(3)

Differentiating (1) w.r.t.x, we have

$$\phi'(x) = f'(x) + A$$

Putting x = c, $0 = f'(c) + A$

$\therefore$ $A = - f'(c)$ ...(4)

From (2) and (4), we have,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \text{ where } c \in (a, b).$$

This proves the theorem.

**Another from of Lagrange's Mean Value Theorem. Let $f$ be a**

function which is

(1) Continuous in the closed interval [a, a+h],

(2) Derivable in the open interval (a, a+h).

Then there exists a real number '$\theta$' between 0 and 1 such that $f(a + h) = f(a) + h f'(a + \theta h)$.

This can be proved from the above theorem by replacing b by a + h.

Obviously b - a = h and $c \in (a, b)$

$\Rightarrow c = a + \theta h, 0 < \theta < 1$

If we replacing b by x and a by $x_0$, we can express the Lagrange's Mean Value Theorem by the formula.

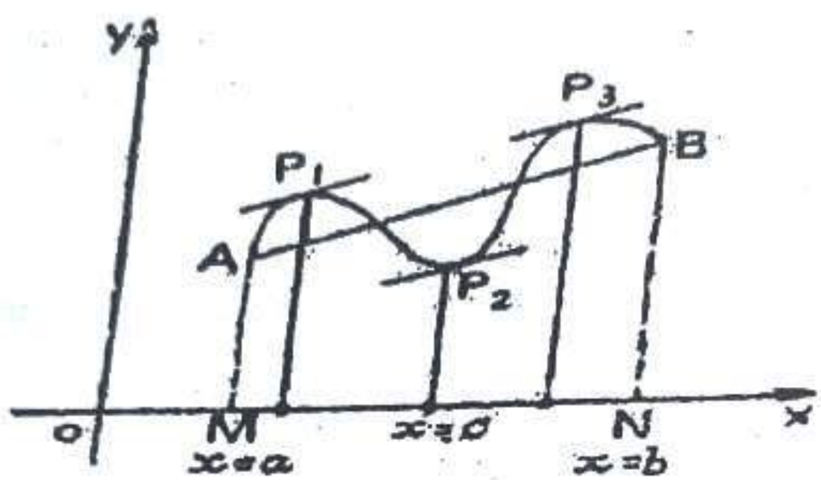
$$f(x) = f(x_0) + (x - x_0)\, f'[x_0 + \theta(x - x_0)],\ 0 < \theta < 1$$

## 6.6 Geometrical significance of Lagrange's Mean Value Theorem

If $y = f(x)$ be curve defined on the closed interval [a, b] such that

(1) The graph of the curve does not break any where from x = a to x = b and,

(2) The tangent to curve $y = f(x)$ exists at each point in (a, b), then, there exists a point c in (a, b) where he tangent at the point P(c, $f$(c)] is parallel to the chord joining the points A[a, $f$(a)] and B[b, $f$(b)] on the curve $y = f(x)$. The point c need not be unique. See figure below.




**Example 6 :** If $f(x)$ is a quadratic polynomial and a, b are any two numbers,

show that $\frac{a+b}{2}$ is the only value of c which satisfies the Mean value theorem in (a, b).

**Solution :** Let $f(x) = px^2 + qx + r$ ($p \neq 1$) be a quadratic polynomial in x.

Obviously $f(x)$ is continuous in [a, b] and derivable in (a, b). Therefore by Lagrange's Mean Value Theorem, there exists at least c $\in$ (a, b) such that.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\therefore\ 2pc + q = \frac{(pb^2 + qb + r) - (pa^2 + qa - r)}{b-a}$$

[$f$'(x) = 2pc + q] $\therefore$ $f$'(c) = 2pc + q]

or 2pc + q = $\frac{p(b^2 - a^2) + q(b+a)}{b-a}$

or 2pc + q = p(b + a) + q

or 2pc = p(b + a)

or c = $\frac{a+b}{2}$ ,

obviously, c $\in$ [a, b] [$\frac{a+b}{2}$ is the A.M. between a and b]

$\therefore$ c = $\frac{a+b}{2}$ is the only value of c which satisfies Mean Value Theorem in (a, b).

**Example 7:** If $f$(x) = $2x^{2/3}$ , a = -1, b = 1, show that there is no real number c which satisfies the Lagran ge's Mean Value Theorem. Explain.

Solution: $f$(x) = $2x^{2/3}$ , x $\in$ [-1, 1]

Obviously $f$(x) is continuous in [-1, 1]

But $f$'(x) = $\frac{4}{3} \cdot \frac{1}{x^{1/3}}$ does not exist at x = 0, 0 $\in$ (-1, 1)

$f$ is not differentiable in (-1, 1)

$\Rightarrow$ One of the two conditions of Lagrange's Mean Value Theorem is not satisfied.

$\Rightarrow$ That Lagrange's Mean Value Theorem is not valid.

$\therefore$ there exists no real number c $\in$ (-1, 1) for which Lagrange's Mean Value Theorem is satisfied.

i.e. $\frac{f(b) - f(a)}{b-a}$ = $f$'(c)

or $\frac{f(1) - f(-1)}{1+1} = \frac{4}{3} \; \frac{1}{e^{1/3}}$

or $\frac{1-1}{2} = \frac{4}{3e^{1/3}}$

or $\frac{1}{e^{1/3}}$ = 0,

Obviously, there is no such real c.

This completes the solution

**Some Illustrated Examples**

**Example 8:** Verify Rolle's theorem for

$f(x) = x^2 - 5x + 4$ in $1 \le x \le 4$

**Solution:** We here have

$$f(x) = x^2 - 5x + 4 \qquad \text{......(1)}$$

(i) Since (1) is a polynomial in x, therefore it is continuous function everywhere and in particular for $1 \le x \le 4$.

(ii) $f'(x) = 2x - 5$, which exists in $1 < x < 4$

$\Rightarrow$ $f(x)$ is differentiable in (1, 4)

$\therefore$ $f$ satisfies all the conditions of Rolle's theorem

$\therefore$ there must exist atleast one real $c \in (1, 4)$ such that $f'(c) = 0$

Now $f'(c) = 2c - 5 = 0$

$\Rightarrow$ $c = 5/2$ which lies in (1, 4)

Hence Rolle's theorem is verified.

**Example 9:** Verify Rolle's theorem for

$f(x) = \sqrt{1-x^2}$ in [-1, 1]

**Solution:** We have

$f(x) = \sqrt{1-x^2}$, $x \in [-1, 1]$

(i) $f$ is continuous for those x for which $\sqrt{1-x^2} \ge 0$

$\Rightarrow$ $1 - x^2 \ge 0$

or $x^2 \le 1$

$\Rightarrow$ $|x| \le 1$

i.e. $-1 \le x \le 1$

Thus $f$ is continuous in [-1, 1]

(ii) $f'(x) = \dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{-x}{\sqrt{1-x^2}}$, which exists in (-1, 1)

$\therefore$ $f$ is derivable in (-1, 1)

(iii) $f$ (-1) = $\sqrt{1-1}$ = 0

and $f$(1) = $\sqrt{1-1}$ = 0

$\Rightarrow$ $f$(-1) = $f$(1)

$\therefore$ all the conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist atleast one real c $\in$ (-1, 1) such that $f'(c) = \dfrac{-c}{\sqrt{1-c^2}} = 0$

$\Rightarrow$ c = 0

clearly c = 0 lies in (-1, 1)

Hence Rolle's theorem is verified.

**Example 10:** If $f$(x) = |x|, x $\in$ [-1, 1], discuss the applicability of Rolle's theorem.

**Solution:** We have

$f$(x) = |x|, x $\in$ [-1, 1]

Here we shall show that $f$ is not derivable at x = 0.

L. H. Derivative = $\displaystyle Lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$$= Lim_{x\to 0^-} \frac{|x|-|0|}{x-0}$$

$$= Lim_{x\to 0^-} \frac{|x|}{x} = Lim_{h\to 0} \frac{|0-h|}{0-h}$$

$$= Lim_{h\to 0} \frac{|-h|}{h} = Lim_{h\to 0} \frac{h}{-h} = -1 \qquad (\because x = 0 - h,\ h > 0 \Rightarrow h \to 0 \text{ as } x \to 0^-)$$

and similarly

R.H.D. = $\displaystyle Lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$$= Lim_{x\to 0^+} \frac{|x|}{x} = Lim_{x\to 0^+} \frac{|0+h|}{h} = 1$$

$\therefore$ Left hand Derivative $\neq$ Right hand derivative

$\Rightarrow$ $f$(x) = |x| is not differentiable at x = 0 $\in$ (-1, 1)

Hence Rolle's theorem cannot be applied to $f(x) = |x|$ in [-1, 1].

**Example 11:** Verify Lagrange's mean value theorem for $f(x) = x^3 + x^2 - 6x$ in [-1, 4]

**Solution:** Here $f(x) = x^3 + x^2 - 6x$ .....(1)

(i) clearly (1) is a polynomial in x therefore continuous for everyvalue of x, in particular in [-1, 4]

(ii) $f(x) = 3x^2 + 2x - 6$, which exists in (-1, 4)

$\Rightarrow$ $f(x)$ is derivable in (-1, 4)

Thus $f(x)$ satisfies both the conditions of Lagrange's mean value theorem.

$\therefore$ there must exists alteast one $c \in (-1, 4)$ s.t.

$$f'(c) = \frac{f(4) - f(-1)}{4-(-1)}$$

or $$3c2 + 2c - 6 = \frac{(64+16-24)-(-1+1+6)}{5}$$

or $$3c^2 + 2c - 6 = \frac{56-6}{5}$$

or $$3c^2 + 2c - 16 = 0$$

$\therefore$ $$c = \frac{-2 \pm \sqrt{4+192}}{2.3} = \frac{-2 \pm 14}{6} = \frac{8}{3}, 2$$

$\therefore$ $c = 2 \in (-1, 4)$

Hence Lagrange's theorem is verified

**Example 12:** Verify Lagrange's mean value theorem for the function

$f(x) = \log x$ in [1, e]

**Solution:** We have

$f(x) = \log x$ ...... (1)

(i) (1) is continuous in [1, e]

(ii) $f'(x) = \frac{1}{x}$, exists in (1, e)

$\Rightarrow$ $f$ is derivable in (1, e)

$\therefore$ Both the conditions of LMV (Legrange Mean Value) theorem are satisfied.

$\therefore$ there must exist atleast one $c \in (1, e)$ s.t.

$f'$ (c) = $\frac{f(e)-f(1)}{e-1}$

$\Rightarrow$ $\frac{1}{c} = \frac{1-0}{e-1}$

$\therefore$ c = e - 1 = 2.73 - 1 = 1.73 $\in$ (1, e) (e = 2.73 (app.)

Hence LMV theorem is verified.

**Example 13:** Verify LMV theorem for the function

$f$(x) = sin x, x $\in \left[0, \pi/2\right]$

**Solution:** Here $f$(x) = sin x

(i) Since the curve of sin x is a continuous one

$\therefore$ $f$ is continuous in $\left[0, \pi/2\right]$

(ii) $f'$(x) = cos x, which exists in $\left[0, \pi/2\right]$

$\therefore$ $f$ is derivable in $\left[0, \pi/2\right]$

$\therefore$ both the condition of LMV theorem are satisfied.

$\Rightarrow$ there must exist atleast one c $\in \left[0, \pi/2\right]$ such that

$$f'(c) = \frac{f\left(\pi/2\right)-f(0)}{\pi/2-0}$$

or $\cos c = \frac{\sin \pi/2 - \sin 0}{\pi/2}$

or $\cos c = \frac{1-0}{\pi/2}$

or $\cos c = \frac{2}{\pi}$

or $c = \cos^{-1}\left(\frac{2}{\pi}\right) \in \left(0, \frac{\pi}{2}\right)$

Hence LMV theorem is verified.

## 6.7 Self Check Exercise

Q.1 Verify Rolle's theorem for

$f(x) = -4x + 3,\ x \in [1,3]$

Q.2 Verify Rolle's theorem for

$f(x) = x^2$ in $[-1, 1]$

Q.3 Verify LMV theorem for

$f(x) = 2x - x^2$ in $[0, 1]$

Q.4 Verify LMV theorem for

$f(x) = x^3 - 3x - 1$ in $[1, 3]$

Q.5 In LMV theorem $f(a+h) = f(a) + h\,f'(a + \theta)\,h$.

Find $\theta$ if $f(x) = Ax^2 + Bx + C,\ A \neq 0$.

## 6.8 Summary

In this unit we have learnt

(i) Rolle's theorem and its geometrical significance

(ii) Lagrange's mean value theorem and its geometrical interpretation.

## 6.9 Glossary

(i) Lagrange's mean value theorem is also known as first mean value theorem or mean value theorem or Law of mean.

(ii) Application to mean value theorem

(a) If $f$ is

(i) contiguous in [a, b]

(ii) derivable in (a, b) and

(iii) $f'(x) = 0\ \forall\ x \in (a, b)$

then $f$ is constant in [a, b]

(b) If $f(x)$, $g(x)$ are two functions s.t.

(i) $f$, g are continuous in [a, b]

(ii) $f$; g are derivable in (a, b)

(iii) $f'(x) = g'(x)\ \forall\ x \in (a, b)$

then $f(x)$ and $g(x)$ differ by a constant in [a, b]

## 6.10 Answers to Self Check Exercises

Ans. 1 - 4 (Q1-Q4) Rolle's Theorem is verified.

Ans. $\theta = \frac{1}{2}$

## 6.11 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002


## 6.12 Terminal Questions

1. Verify Rolle's theorem for the functions;

   (i) $f(x) = x^2 - 11x + 28$ in $4 < x < 7$

   (ii) $f(x) = |9 - x^2|$ in $[-3, 3]$

   (iii) $f(x) = \cos 2x$ in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

2. Verify Lagrange's Mean value theorem for the functions:

   (i) $f(x) = x + \frac{1}{x}$ in $\left[\frac{1}{2}, 3\right]$

   (ii) $f(x) = (x - 1)(x - 2)(x - 3)$ in $[0, 4]$

   (iii) $f(x) = x(x - 1)(x - 2)$ in $\left[0, \frac{1}{2}\right]$

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## Unit - 7

# General Theorems (Cont) Cauchy's Mean Value Theorem (CMVT)

**Structure**



## 7.1 Introduction

Dear students, in this unit we shall continue with our study of General theory Cauchy's Mean Value Theorem is a powerful generalization of the more familiar mean value theorem. It provides a relationship between the change of two functions over a fixed interval with their derivatives. Cauchy mean value theorem is a special case of Lagrange Mean Value Theorem. Cauchy Mean Value Theorem is also called the Extended Mean Value Theorem or the second mean value theorem or the Generalized Mean Value Theorem.

## 7.2 Learning Objectives

The main objectives of this unit are :

(i) to state the Cauchy Mean Value Theorem (CMVT)

(ii) to prove CMVT

(iii) to give geometrical significance of CMVT

## 7.3 Cauchy's Mean Value Theorem

Statement. If two functions $f$, g are defined on a closed interval [a, b] such that both $f$ and g are

(1) Continuous in the closed interval [a, b];

(2) Derivable in the open interval (a, b) and

(3) $g'(x)$ does not vanish for any $x \in (a, b)$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}, \text{ where } a < c < b.$$

**Proof:** Let us consider a new function $\phi(x)$ on [a, b] defined by

$\phi(x) = f(x) + Ag(x)$ .....(1)

Where A is a real constant to be determined such that $\phi(a) = \phi(b)$

i.e. $f(a) + Ag(a) = f(b) + Ag(b)$

or $A[g(b) - g(a)] = -[f(b) - f(a)]$ ......(2)

We claim that $g(b) - g(a) \neq 0$ [if $g(b) = g(a)$, then g being continuous in [a, b] and derivable in (a, b) will satisfy all the conditions of Rolle's Theorem and thus $g'(c) = 0$ for at least one $c \in (a, b)$. This is not true, because it is given that $g'(x) \neq 0$ for any $x \in (a, b)$]

Now form (2), $A = -\frac{f(b)-f(a)}{g(b)-g(a)}$

Since $\phi(x)$ is the sum of two functions $f(x)$ and $g(x)$, both of which are continuous in [a, b] and derivable in (a, b), therefore, $\phi(x)$ is

1. Continuous in [a, b]
2. Derivable in [a, b]
3. $\phi(a) = \phi(b)$.

Thus $\phi$ satisfies all the conditions of Rolle's Theorem. Therefore, there exists at least one $c \in (a, b)$ such that $\phi'(c) = 0$.

Differentiating (1) w.r.t. x, we have,

$\phi'(x) = f'(x) = f'(x) + Ag'(x)$

Now $\phi'(c) = 0 \Rightarrow f'(c) + Ag'(c) = 0$ or $A = -\frac{f'(c)}{g'(c)}$ .....(4)

From (3) and (4), we have,

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \text{ where } a < c < b \qquad ......(5)$$

This prove the theorem.

Note. Lagrange's Mean Value Theorem is a special (particular) case of Cauchy's Mean Value Theorem.

If we take $g(x) = x$ in the Cauchy's Mean Value Theorem then

$$\frac{f(b)-f(a)}{b-a} = f'(c),\ a < c < b \left[\begin{array}{c} g(x)=x \\ \Rightarrow g(b)=b, g(a)=a \\ \text{and } \ g'(c)=1 \end{array}\right]$$

## 7.4 Geometrical Significance of Cauchy's Mean Value Theorem

Cauchy's Mean Value Theorem has geometrical significance similar to that of Lagrange's Mean Value theorem.

Let a curve be defined parametrically on the closed interval [a, b] by x = g(t), y = $f$(t), t being parameter, $a \le t \le b$.

g(t), $f$(t) are both continuous in [a, b] and derivable in (a, b) the curve is continuous from A to B and has a tangent at each point between A and B, also $g'(t) \neq 0$ for any t in (a, b).

The slope of tangent at a point (g(t), $f$(t)) is given by

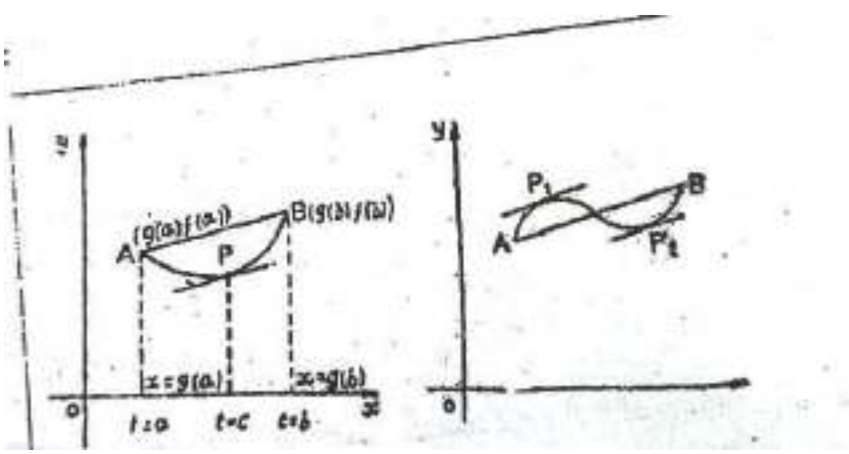
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)} \text{ and slope of the chord AB } = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Cauchy's Mean Value Theorem asserts that there is atleast one point $c \in (a, b)$ i.e. point P(g(c), $f$(c)) lying between A and B, the tangent at which is parallel to the chord AB.

i.e. $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$, $c \in (a, b)$

The point $c \in (a, b)$ need not be unique.

See figures given below:




**Example 1.** Verify Cauchy's Mean Value Theorem for the functions

$$f(x) = \cos x \text{ and } g(x) = \sin x \text{ in the interval } \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

**Solution:** Here $f$(x) = cos x and g(x) ....(1)

1. $f$ and g are both continuous in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

2. $f$ and g are both derivable in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and

3. g'(x) = cos x $\neq$ 0 for any x in $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$.

Hence $f$, g satisfy all the conditions of Cauchy's Mean Value Theorem.

$$\therefore \qquad \frac{f\left(\frac{\pi}{4}\right)-f\left(-\frac{\pi}{4}\right)}{g\left(\frac{\pi}{4}\right)-g\left(-\frac{\pi}{4}\right)} = \frac{f'(c)}{g'(c)}$$

[by Cauchy's Mean Value Theorem.]

Or $\dfrac{\cos\frac{\pi}{4}-\cos\left(-\frac{\pi}{4}\right)}{\sin\frac{\pi}{4}-\sin\left(-\frac{\pi}{4}\right)} = \dfrac{-\sin c}{\cos c}$ or $\dfrac{\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}$ = - tan c

or 0 = - tan c or tan c = 0

$$\therefore \ c = 0,\ 0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Hence the Cauchy's Mean Value Theorem is verified.

**Application of Mean Value Theorem.**

**Theorem 1.** If $f$(x) is a differentiable function in the open interval (a, b) and $f'$(x) = 0 for all x in (a, b) then $f$(x) is constant in (a, b).

**Proof:** Let $x_1$, $x_2$ be any two points in (a, b) such that

$$a < x_1 < x_2 < b$$

Now on the closed interval [$x_1$, $x_2$], the function $f$(x) satisfies both the conditions of Lagrange's Mean Value Theorem; since a function which is differentiable at point is also continuous there at. Therefore, there exists a real c $\in$ ($x_1$, $x_2$) and hence c $\in$ (a, b) such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \qquad [f \text{ or } a < x_1 < c < x_2 < b]$$

$$f'(x) = 0 \qquad \forall\ x \in (a, b)$$

$\therefore f'(c) = 0$

hence $f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$

$\Rightarrow f$ is constant in (a, b).

**Theorem 2.** If $f(x)$ is continuous in the closed interval [a, b] and $f'(x) = 0$ for all x in (a, b), then $f$ is constant in [a, b]

**Proof:** Let x be any point in [a, b] such that $a < x \leq b$. Now on the closed interval [a, x], the function $f(x)$ satisfies both the conditions of Lagrange's Mean Value Theorem; since $f(x)$ is continuous in [a, b] and $f'(x) = 0$ for all $x \in (a, b)$ implies that $f$ is differentiable in (a, b). Therefore, there exists a real $c \in (a, x)$ and hence $c \in (a, b)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(c) \qquad [f \text{ or } a < c < x \leq b]$$

$f'(x) = 0 \ \forall \ x \in (a, b)$ [given]

$\therefore f'(c) = 0$

hence $f(x) - f(a) = 0$ $[f \text{ or } a < x \leq b]$

$\Rightarrow f(x) = f(a)$

$\Rightarrow f$ is constant in [a, b].

**Theorem 3.** Let $f(x)$ and g(x) be differentiable in the open interval (a, b) and $f'(x) = g'(x)$ for all x in (a, b) then there exists a constant c such that $f(x) = g(x) + c$

For all x in (a, b)

**Proof:** Let $F(x) = f(x) - g(x)$

Since $f(x)$ and g(x) are differentiable in (a, b) .....(1)

$\therefore$ F(x) is also differentiable in (a, b)

Further $F'(x) = f'(x) - g'(x)$ .....(2)

$= 0 \ \forall \ x \in (a, b)$ $[f'(x) = g'(x) \ \forall \ x \in (a, b)$ is given]

Combining (1) and (2), we have,

F(x) is constant in (a, b)

$\Rightarrow \exists$ a constant c, that

$F(x) = c \qquad \forall \ x \in (a, b)$

i.e. $f(x - g(x) = c \ \forall \ x \in (a, b)$.

or $f(x) = g(x) + c \ \forall \ x \in (a, b)$

**Theorem 4:** Let $f(x)$ and g(x) be continuous in the closed interval [a, b] and $f'(x) = g'(x)$ for all x in (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all x in [a, b]

**Proof:** Let $F(x) = f(x) - g(x)$

Since $f(x)$ and g(x) are continuous in [a, b]

$\therefore$ F(x) is also continuous in [a, b]

Further $F'(x) = f'(x) - g'(x) = 0 \quad \forall x \in (a, b)$

Combining (1) and (2), we have,

F(x) is constant in [a, b]

$\Rightarrow$ $\exists$ a constant c, such that $F(x) = c, \quad \forall x \in [a, b]$

$\Rightarrow$ $f(x) - g(x) = c \quad \forall x \in [a, b]$

$\Rightarrow$ $f(x) = g(x) + c \quad \forall x \in [a, b]$.

**Theorem 5:** Let $f(x)$ be differentiable in the open interval (a, b).

(1) If $f'(x) > 0$ $f$or $a < x < b$, then $f$ is strictly monotonnically inveasing in (a, b)

(2) If $f'(x) \geq 0$ $f$or $a < x < b$, then $f$ is strictly monotonnically inveasing in (a, b)

(3) If $f'(x) < 0$ $f$or $a < x < b$, then $f$ is strictly monotonnically inveasing in (a, b)

(4) If $f'(x) \leq 0$ $f$or $a < x < b$, then $f$ is strictly monotonnically decreasing in (a, b)

**Proof:** Let $x_1$ and $x_2$ be any two arbitrary points in (a, b) with $a < x_1 < x_2 < b$.

$f(x)$ is differentiable in (a, b),

$\therefore$ $f(x)$ is continuous and differentiable in $[x_1, x_2]$

$\therefore$ $f(x)$ satisfies all the conditions of Lagrange's Mean Value Theorem in $[x_1, x_2]$ and, therefore, there exists at least one point $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \text{ for } a < x_1 < c < x_2 < b$$

or $f(x_2) - f(x_1) = (x_2 - x_1)\, f'(c)$ for $a < x_1 < c < x_2 < b$ .....(1)

(1) Since we are given that $f'(x) > 0 \ \forall x \in (a, b)$

$\therefore$ $f'(c) > 0$ and also $x_2 - x_1 > 0$

$\therefore$ R.H.S. of (1) is +ve $\Rightarrow$ L.H.S. of (1) is +ve

$\Rightarrow$ $f(x_1) < f(x_2)$ for $x_1 < x_2$ and $x_1, x_2 \in (a, b)$

$\Rightarrow$ $f(x_1)$ is m.I. in (a, b)

(2) Since we are given that $f''(x) \geq 0 \ \forall x \in (a, b)$

$\therefore$ $f'(c) \geq 0$

Also $x_2 - x_2 > 0$

$\therefore$ R.H.S. of (1) is $\geq 0 \Rightarrow$ L.H.S. of (1) $\geq 0$

$\Rightarrow$ $f(x_2) - f(x_1) > 0$

$\Rightarrow$ $f(x_2) > f(x_1)$ where $x_1 < x_2$

$\Rightarrow$ $f(x_1) \leq f(x_2)$ where $x_1 < x_2$ and $x_1, x_2 \in (a, b)$

$\Rightarrow$ $f(x)$ is m.I. in (a, b).

(3) Since we are given that $f'(x) < 0 \ \forall \ x \in (a, b)$

$\therefore$ $f'(c) < 0$

Also $x_2 - x_1 > 0$

$\therefore$ R.H.S. of (1) is $< 0 \Rightarrow$ L.H.S. of (1) $< 0$

$f(x_2) - f(x1) < 0$ where $x_1 < x_2$

$\Rightarrow$ $f(x_2) < f(x_1)$ where $x_1 < x_2$ and $x_1, x_2 \in (a, b)$

$\Rightarrow$ $f(x)$ is s.m.d in (a, b)

(4) Since we are given that $f'(x) \leq 0 \ \forall \ x \in (a, b)$

$\therefore$ $f'(c) \leq 0$

Also $x^2 - x1 > 0$

$\therefore$ R.H.S. of (1) is $\leq 0 \Rightarrow$ L.H.S. of (1) $\leq 0$

$\Rightarrow$ $f(x_2) - f(x_1) \leq 0$

$f(x_2) \leq f(x_1)$ where $x_1 < x_2$

$\Rightarrow$ $f(x_1) \geq f(x_2)$ where $x_1 < x_2$ and $x_1, x_2 \in (a, b)$

$\Rightarrow$ $f(x)$ is m.d. in (a, b)

**Theorem 6.** If a function $f$ is continuous in [a, b] derivable in (a, b) and

(1) If $f'(x) > 0 \ \forall \ x \in (a, b)$ then $f$ is strictly increasing in [a, b]

(2) if $f'(x) \geq 0 \ \forall \ x \in (a, b)$ then $f$ is increasing in [a, b]

(3) If $f'(x) < 0 \ \forall \ x \in (a, b)$ then $f$ is strictly decreasing in [a, b]

(4) if $f'(x) \leq 0 \ \forall \ x \in (a, b)$ then $f$ is decreasing in [a, b]

**Proof:** Let $x_1, x_2$ be any two arbitrary points in [a, b] such that $a \leq x_1 < x_2 \leq b$. obviously $[x_1, x_2]$ [a, b]

$f$ is continuous in $[x_1, x_2]$ and derivable in $(x_1, x_2)$

$\therefore$ $f$ satisfies both the conditions of Lagrange's mean Value Theorem in $[x_1, x_2]$

Hence there exists at least one real number $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \qquad [\text{for } a \leq < x_1 < c < x_2 \leq b]$$

Or $f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ [for $a \leq x_1 < c < x_2 \leq b$] ......(1)

(1) Since $f'(x) > 0 \ \forall \ x \in (a, b)$

$\Rightarrow$ $f'(c) > 0$

Also $x_2 - x_1 > 0$

$\therefore$ R.H.S. of (1) > 0 $\Rightarrow$ L.H.S. of (1) > 0

$\Rightarrow$ $f(x_2) - f(x_1) > 0$

$\Rightarrow$ $f(x_2) > f(x_1)$

$\Rightarrow$ $f(x_1) < f(x_2)$ where $x_1 < x_2$, $\forall \ x_1, x_2 \in [a, b]$

$\Rightarrow$ $f$ is strictly increasing in [a, b]

Note. The converse of the above theorems need not be true.

**Theorem 7:** If a function $f$ is derivable in (a, b) and

(1) If $f$ is increasing in (a, b), then $f'(x) \geq 0$ for all x in (a, b)

(2) If $f$ is decreasing in (a, b), then $f'(x) \leq 0$ for all x in (a, b)

**Proof:** let c be any real number in (a, b), choose a real number h such that a < c + h < b (such a choice of h is always possible).

(1) Since is increasing in (a, b), therefore we have,

$f$ or $h \geq 0$, $c + h \geq c \Rightarrow f(c + h) \geq f(c) \Rightarrow f(c + h) - f(c) \geq 0$

And $f$ or $h \leq 0$, $c + h \leq c \Rightarrow f(c + h) \leq f(c) \Rightarrow f(c + h) - f(c) \leq 0$

Thus, in both cases, we have,

$$\frac{f(c+h) - f(c)}{h} \geq 0, \ h \neq 0$$

$$\therefore \quad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \geq 0$$

$\Rightarrow$ $f'(c) \geq 0$ [$f$ is differentiable in (a, b) $\Rightarrow$ $f'(c)$ exists]

$\Rightarrow$ $f'(c) \geq 0 \ \forall \ x \in (a, b)$

(2) $f$ is decreasing in (a, b) therefore we have,

$f$ or $h \geq 0$, $c + h \geq c \Rightarrow f(c + h) \leq f(c) \Rightarrow f(c + h) - f(c) \leq 0$

And f or $h \leq 0$, $c + h \leq c \Rightarrow f(c + h) \geq f(c) \Rightarrow f(c) \Rightarrow f(c + h) - f(c) \geq 0$

Thus in both the cases, we have,

$$\frac{f(c+h) - f(c)}{h} \leq 0, \ h \neq 0$$

$\therefore \qquad \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \leq 0$

$\Rightarrow \qquad f'(c) \leq 0 \qquad$ [$f$ is differentiable in (a, b) $\Rightarrow f'(c)$ exists]

$\Rightarrow \qquad f'(c) \leq 0 \ \forall \ x \in (a, b)$

**Example 2:** If x > 0, prove that

$$x - \frac{x^2}{2(1+x)} > \log(1+x) > \left(x - \frac{x^2}{2}\right)$$

**Solution:** Let $f(x) = \left(x - \frac{x^2}{2(1+x)}\right) - \log(1+x)$

and $g(x) = \log(1+x) - \left(x - \frac{x^2}{2}\right)$

obviously both $f(x)$ and g(x) are derivable in $[0, \infty]$

Now $f'(x) = 1 - \frac{1}{2}\,\frac{(1+x)2x - x^2.1}{(1+x)^2} - \frac{1}{1+x}$

$$= 1 - \frac{x(2+x)}{2(1+x)^2} - \frac{1}{1+x} = \frac{2(1+x)^2 - x(2+x) - 2}{2(1+x)^2}$$

$$= \frac{2x^2 + 2 + 2x - 2x - x^2 - 2}{2(1+x)^2} = \frac{x^2}{2(1+x)^2} > 0$$

For all x > 0

$f(x)$ is s.m.i. in $[0, \infty]$ [see theorem 6]

$\Rightarrow \qquad f(x) > f(0)$ $f$ or x > 0

$\Rightarrow \qquad \left(x - \frac{x^2}{2(1+x)}\right) - \log(1+x) > 0,$ $f$ or x > 0 $\qquad [f(0) = 0]$

$\Rightarrow \qquad x - \frac{x^2}{2(1+x)} > \log(1+x)$ $f$ or x > 0 ....(1)

Also $g'(x) = \frac{1}{1+x} - 1 + x = \frac{1 - 1 - x + x + x^2}{1+x} = \frac{x^2}{1+x}$

> $f$ or all x > 0

$\Rightarrow \qquad f(x)$ is s.m. i. in $[0, \infty]$

$\Rightarrow$ g(x) > g(0), $f$ or x > 0

$\Rightarrow$ $\log(1 + x) - \left(x - \frac{x^2}{2}\right) > 0$ $f$ or x > 0 [g(0) = 0]

$\Rightarrow$ $\log(1 + x) > x = \frac{x^2}{2}$ $f$ or x > 0

Combining (1) and (2), we have,

$$x - \frac{x^2}{2(1+x)} > \log(1 + x) > x - \frac{x^2}{2} \ f \text{ or } x > 0$$

This completes the solution.

**Example 3:** Find the interval of increase and decrease of the following functions:

1. $f(x) = x \log x - x, x > 1$
2. $f(x) = x^4 - 4x$

**Solution :** 1. $f(x) = x \log x - x, x > 1$

$$\therefore f'(x) = x.\frac{1}{x} + \log x.1 - 1 = 1 + \log x - 1$$

$= \log x,$ x > 1

> 0 x > 1 $\Rightarrow$ log x > log 1 = 0

Also $f$ is continuous and derivable for x > 1

Hence $f$ (x) x.m.i. $f$ or x $\geq$ 1

i.e. $f(x)$ is s.m.i. in $[1, \infty)$.

2 $f(x) = x^4 - 4x$

$\therefore f'(x) = 4x^3 - 4$

$= 4(x^3 - 1)$

$= 4(x - 1)(x^2 + x + 1)$

Now $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + > 0 \ \forall \ x$

$\therefore f'(x) > 0$ if $f$ x > 1

Or $f'(x) > 0$ if x > 1

And $f'(x) < 0$ if x < 1

Also $f$(x) is s.m.i. in [1, ∞] and is s.m.d. in (-∞, 1].

**Example 4 :** Use Mean Value Theorem to prove the following inequalities :

(a) $1 + x < e^x \leq 1 + xe^x$ $f$ or all $x \geq 0$

(b) $\frac{x}{1+x} < \log(1+x) < x$ $f$ or all $x > -1$, $x \neq 0$

**Solution :** (a) $f$ or x = 0

$1 + x = e^x = 1 + xe^x = 1$ ....(1)

∴ the result is true with equality

Let x > 0

Consider the function $f$(x) = ex

Obviously $f$(x) is continuous in [0, x] and derivable in (0, x)

Therefore, by Lagrange's Mean Value Theorem,

$$\frac{f(x)-f(0)}{x-0} = f'(x),\ c \in (0, x)$$

$$\Rightarrow \quad \frac{e^x - e^0}{x-0} = e^c$$

$$\Rightarrow \quad \frac{e^x - 1}{x} = e^c \qquad ....(2)$$

Since 0 < c < x

∴ $e^0 < e^c < e^x$ [$e^x$ is a s.m. function]

Or $1 > e^c < e^x$ ...(3)

From (1) and (2), we have,

$$1 < \frac{e^x - 1}{x} < e^x \qquad [f \text{ or } x > 0]$$

Multiplying by x which is +ve, we have,

$x < e^x - 1 < xe^x$ [$f$ or x > 0]

Adding 1, we have, $1 + x < e^x < 1 + xe^x$, x > 0 ...(4)

Combining (1) and (4), we have,

$1 + x \leq e^x \leq 1 + xe^x$. $f$ or $x \geq 0$

b. case 1. When x > 0

We apply we Lagrange's Mean Value Theorem to the function log (1 + x) on the interval [0, x].

$\therefore$ $f$(x) = log (1 + x),

And $f'(x) = \frac{1}{1+x}$

Now $\frac{f(x)-f(0)}{x-0} = f'(c)$, 0 < c < x

$\Rightarrow \quad \frac{\log(1+x)-\log 1}{x} = \frac{1}{1+c}$

$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+c}$ ...(1)

Since 0 < c < x

$\therefore$ 1 < 1 + c < 1 + x

or $1 > \frac{1}{1+c} > \frac{1}{1+x}$

or $\frac{1}{1+x} < \frac{1}{1+c} < 1$ ...(2)

From (1) and (2), we have,

$$\frac{1}{1+x} = \frac{\log(1+x)}{x} < 1 \ f\text{or } x > 0$$

Case 2. When -1 < x < 0

Again applying the Lagrange's Mean Value Theorem to the function log (1 + x) on the interval [x, 0], we have,

Now $\frac{f(0)-f(x)}{0-x} = f'(c)$, 0 < c < x

$\Rightarrow \quad \frac{0-\log(1+x)}{-x} = \frac{1}{1+c}$ $f$ or x < c < 0

$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+c}$ $f$ or x < c < 0 ...(3)

Since x < c < 0

$\therefore$ 1 + x < 1 + c < 1

$\Rightarrow$ 1 + x < 1 + c < 1

or $1 > \frac{1}{1+c} > \frac{1}{1+x}$

or $\frac{1}{1+x} > \frac{1}{1+c} > 1$ ...(4)

From (3) and (4), we have,

$$\frac{1}{1+x} > \frac{\log(1+x)}{x} > 1 \quad f \text{ or } -1 < x < 0$$

Multiplying by x which is -ve, we have

$$\frac{x}{1+x} < \log(1+x)$$

$$< \in \text{ for } -1 < x < 0$$

Combining both the case, we have

$$\frac{x}{1+x} < \log(1+x) < x$$

for $x > -1, x \neq 0$

**Some Illustrated Examples**

**Example 5 :** Verify Cauchy's Mean Value Theorem for

$$f(x) = \cos x,\ g(x) = \sin x,\ \frac{-\pi}{4} < x < \frac{\pi}{4}$$

**Solution :** Here $f(x) = \cos x$, $g'(x) = \sin x$

$\Rightarrow f'(x) = -\sin x,\ g'(x) = \cos x$

We are that $f$, g are two functions s.t.

(i) $f$, g are continuous in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

(ii) $f$, g are differentiable in $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

(iii) $g'(x) \neq 0$ for any x in $\left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

$\therefore$ $f$, g satisfy all condition of CMVT in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

$\therefore$ by Cauchy Mean Value Theorem

$$\frac{f\left(\frac{\pi}{4}\right)-f\left(\frac{-\pi}{4}\right)}{g\left(\frac{\pi}{4}\right)-g\left(\frac{-\pi}{4}\right)} = \frac{f'(c)}{g'(c)} \text{ where } c \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$$

$\Rightarrow$ $$\frac{\cos\frac{\pi}{4}-\cos\left(\frac{-\pi}{4}\right)}{\sin\left(\frac{\pi}{4}\right)-\sin\left(\frac{-\pi}{4}\right)} = \frac{-\sin c}{\cos c}$$

$\Rightarrow$ $$\frac{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}} = -\tan c$$

$\Rightarrow$ tan c = 0

$\Rightarrow$ c = .... $-\pi$, 0, $\pi$, ....

But $c \in \left(\frac{-\pi}{4}, \frac{\pi}{4}\right)$

$\therefore$ CMVT is verified.

**Example 6 :** Verify CMVT for the function

$f(x) = e^x$, g(x) $e^{-x}$ in [a, b]

**Solution :** Here

$f(x) = e^x$, g(x) $e^{-x}$

$\Rightarrow$ $f'(x) = e^x$, g'(x) $-e^{-x}$

$\therefore$ $f(x)$, g(x) are two functions such that

(i) $f$, g are continuous in [a, b]

(ii) $f$, g are derivable in (a, b)

(iii) $g'(x) \neq 0$ for any $x \in (a, b)$

$f$, g satisfy all the condition of Cauchy Mean Value Theorem (CMVT)

$\therefore$ by CMVT

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

$\Rightarrow \qquad \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{e^c}{-e^{-c}}$

$\Rightarrow \qquad \dfrac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \dfrac{e^c}{-\frac{1}{e^c}}$

$\Rightarrow \qquad -e^{a+b} = -e^{2c}$

$\Rightarrow \qquad$ a + b = 2c

$\Rightarrow \qquad$ c = $\dfrac{a+b}{2} \in$ (a, b) as $\dfrac{a+b}{2}$ is arithmetic mean between a and b.

Hence CMVT is verified

**Example 7 :** Discuss the applicability of CMVT to the functions.

$$f(x) = \begin{cases} 2, & a \le x < b \\ 4, & x = b \end{cases}$$

and

g(x) = {x, $\quad$ x$\in$ [a, b], $\qquad\qquad$ x $\in$ [a, b]

**Solution :** We see that

$$\underset{x \to b^-}{Lim} f(x) = \underset{x \to b^-}{Lim} 2 = 2$$

and also $f$(b) = 4

$$\therefore \underset{x \to b^-}{Lim} f(x) \neq f(b)$$

$\Rightarrow \qquad f$(x) is not continuous at x = b

$\Rightarrow \qquad f$(x) is not continuous at [a, b]

Clearly, CMVT is not applicable to $f$(x) and g(x) in [a, b].

**Example 8 :** Use CMVT to Evaluate

$$\underset{x \to 1}{Lim} \frac{\cos \frac{\pi x}{2}}{\log \frac{1}{x}}$$

**Solution :** Let $f$(x) = cos x $\dfrac{\pi}{2}$, g(x) = log $\dfrac{1}{x}$, a = x (> 0) b = 1.

$\Rightarrow \qquad f$, g are two function such that

(i) $\qquad$ both $f$, g are continuous in [x, 1]

(ii) both $f$, g are differentiable in (x, 1)

(iii) g'(x) = $\frac{1}{x} \neq 0$ in (x, 1)

$\therefore$ all the conditions of CMVT are satisfied

$\therefore$ Using Cauchy Mean Value Theorem

$$\frac{f(1)-f(x)}{g(1)-g(x)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \quad \frac{\cos\frac{\pi}{2}-\cos\frac{\pi x}{2}}{\log 1-\log x} = \frac{-\frac{\pi}{2}\sin\frac{\pi c}{2}}{\frac{1}{c}}, \quad x < c < 1$$

$$\Rightarrow \quad \frac{0-\cos\frac{\pi x}{2}}{\log 1-\log x} = -\frac{c\pi}{2}\sin\frac{\pi c}{2}$$

$$\Rightarrow \quad \frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}} = +\frac{c\pi}{2}\sin\frac{\pi c}{2}$$

Taking limit x → 1, c → 1, we get

$$\lim_{x\to 1}\left(\frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}}\right) = \frac{\pi}{2}\sin\frac{\pi}{2}$$

$$\therefore \quad \lim_{x\to 1}\left(\frac{\cos\frac{\pi x}{2}}{\log\frac{1}{x}}\right) = \frac{\pi}{2}$$

## 7.5 Self Check Exercise

Q.1 Verify CMVT for

$f(x) = x^2$, $g(x) = \sqrt{x}$ in [1, 4]

Q.2 Discuss the applicability of CMVT.

for $f(x) = x^3$, g(x) = x4 in [-1, 1]

Q.3 Discuss the applicability of CMVT

For $f(x) = 2x$, $g(x) = x4$ in [-1, 2]

## 7.6 Summary

In this unit we have learnt about

(i) Cauchy mean value theorem

(ii) geometrical significance of CMVT

(iii) some application to CMVT.

## 7.7 Glossary

(1) Cauchy Mean Value Theorem cannot be deduced from Lagrange's mean value theorem because $\theta$ may be different for $f(x)$ and g(x) in [a, b]

(2) CMVT is slightly more general form of Lagrange's mean value theorem.

## 7.8 Answers to Self Check Exercises

Ans. 1 Verified

Ans. 2 CMT is not applicable

Ans. 3 CMVT is not applicable

## 7.9 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002


## 7.10 Terminal Questions

1. Verify CMVT for the functions

   (i) $f(x) = x^3$, $g(x) = 2-x$ in [0, 9]

   (ii) $f(x) = \sqrt{x}$, $g(x) = \dfrac{1}{\sqrt{x}}$ in [a, b], a > 0

2. Discuss the applicability of CMVT for

   $f(x) = x^2 - 1$, $g(x) = x^3$ in [-1, 2]

3. Discuss the applicability of CMVT for

   $$f(x) = \begin{cases} 1, & a \le x < b \\ 2, & x = b \end{cases}$$

   g(x) = 2x, in [a, b]

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## Unit - 8

# Taylor's Theorem With Lagrange's And Cauchy's Form Of Remainder

**Structure**



### 8.1 Introduction

Dear students, in this unit we shall study Taylor's Theorem and explore its Lagrange's and Cauchy form of remainders. Taylor Theorem provides a way to approximate a function using its derivatives at a specific point. It expresses a function as a polynomial centered around that point. The remainder term in Taylor's theorem can be expressed more conveniently using Lagrange form. One practical use of the Lagrange form of the remainder is to provide an upper bound on the error when approximating a function using a Taylor polynomial for instance, of we want to approximate a function $f(x)$ using its third degree Taylor polynomial centered at (x=2). The Lagrange's form of remainder would help us to estimate the error in this approximation.

### 8.2 Learning Objectives

The main objectives of this unit are

(i) to study Taylor's theorem with Lagrange's form of remainder

(ii) to learn Maclaurin's theorem with Lagrange's form of remainder

(iii) to study Taylor's theorem with Cauchy's form of remainder

(iv) to study Taylor polynomial

### 8.3 Taylor's Theorem With Lagrange's Form Of Remainder

**Statement:** If $f$ is a function defined in [a, b] such that

(1) $f, f', f''$.............$f^{n-1}(x)$ are continuous in [a, b]

(2) $f^n$ exists in (a, b), then there exists at least one real number

$c \in$ (a, b)

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(b-a)^n}{\lfloor n} f^n(c)$$

**Proof:** Consider a function $\phi$ defined on the closed interval [a, b] as

$$\phi(x) = f(x) + (b - x) f'(x) + \frac{(b-a)^2}{\lfloor 2} f''(x) + \ldots\ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(x) + \frac{(b-x)^n}{\lfloor n} \quad \ldots..(1)$$

Where A is a real constant to be determined such that

$\phi(a) = \phi(b)$

$$\text{i.e. } f(a) + (b - a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f'(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{(b-x)^n}{\lfloor n} A = f(b) \quad \ldots..(2)$$

Since $f(x)$, $f'(x)$, $f''(x)$,......,$f^{(n-1)}(x)$ and $(b - x)^n$, (n = 1,2,......(2), are continuous in [a, b] and derivable in (a, b), therefore

We see that.

1. $\phi(x)$ is continuous in [a, b]
2. $\phi(x)$ is derivable in (a, b)
3. $\phi(a) = \phi(b)$

Hence $\phi(x)$ satisfies all the three conditions of Rolle's Theorem in [a, b] and therefore there exists atleast one real number $c \in$ (a, b) such that $\phi'(c) = 0$

Differentiating (1) w.r.t x, we have,

$$\phi'(x) = f'(x) + [-f'(x) + (b - x) f''(x)] + \left[-(b-x) f''(x) + \frac{(b-x)^2}{\lfloor 2}'''(x)\right] + \ldots. +$$

$$\left[-\frac{(b-a)^{n-2}}{\lfloor n-2} f^{n-1}(x) + \frac{(b-a)^{n-1}}{\lfloor n-1} f^n(x)\right] + \left[-\frac{(b-x)^{n-1}}{\lfloor n-1} - A\right]$$

$$\Rightarrow \quad \phi'(x) = \left[-\frac{(b-x)^{n-1}}{\lfloor n-1} f^n(x)\right] - \left[-\frac{(b-x)^{n-1}}{\lfloor n-1} - A\right]$$

$$\text{Or } \phi'(x) = \frac{(b-x)^{n-1}}{\lfloor n-1} [f^n(x) - A]$$

Or $\phi'(c) = \dfrac{(b-x)^{n-1}}{\lfloor n-1}[f^n(x) - A]$ [by putting x = c]

Or $\dfrac{(b-c)^{n-1}}{\lfloor n-1}[f^n(c) - A]$ [$\phi'(c) = 0$]

Or $f^n(c) - A = 0$

Or $A = f^n(c)$

Putting this value of A in (2), we have,

$$f(a) + (b - a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{(b-a)^n}{\lfloor n} f^n(c)$$

This proves the Theorem.

Note. $\dfrac{(b-a)^n}{\lfloor n} f^n(c)$ is called Lagrange's remainder after n terms and is denoted by $R_n$.

Note. For n = 1 Taylor's Theorem reduces to Lagrange's Mean value Theorem.

**Another form of Taylor's Theorem with Lagrange's form of remainder.**

**If a function $f$ is defined in [a, a + h] such that**

1. $f, f', f'', \ldots. f^{n-1}$ are continuous in [a, a + h]
2. $f^n$ exists in {a, a + h),

Then there exists at least one real number $\theta$, $0 < \theta < 1$, such that

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n}{\lfloor n} f^n(a + \theta h)$$

(in the proof of above Theorem 6.7.1 take b = a + h and c = a + $\theta$h) $f$ or $0 < \theta < 1$, a < a < + $\theta$h < a + h

Here $R^n = \dfrac{h^n}{\lfloor n} f^n(a + \theta h)$, $0 < \theta < 1$, is the Lagrange's remainder after n terms.

Another form of Taylor's Theorem with Lagrange's form of remainder after n terms.

If a function $f$ is defined on $[x_0, x]$ such that

1. $f, f''$, ........... $f^{(n-1)}$ are continuous in $[x_0, x]$
2. $f^n$ exists in $(x_0, x)$

Then there exists at least one real number $\theta$, $0 < \theta < 1$ such that

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2} f''(x_0) + \ldots\ldots + \frac{(x-x_0)^{n-1}}{\lfloor n-1} f^{n-1}(x_0)$$

$$+ \frac{(x-x_0)^n}{\lfloor n} f^n [x_0 + \theta (x - x_0)]$$

[In the proof of above Theorem 6.7.1 take $a = x_0$ and $b = x$, $c = x_0 + \theta (x - x_0)$, f or $0 < \theta <$ 1, $x_0 < x_0 + \theta (x - x_0) < x$]

Hence $R_n = \frac{(x-x_0)^n}{\lfloor n} f^n [x_0 + \theta)$ is the Lagrange's form of remainder after n terms.

**8.4 Maclaurin's Theorem with Lagrange's form of remainder after n terms.**

Statement. If a function $f$ is defined in [0, x] and

1. $f, f'', f''' \ldots\ldots\ldots\ldots f^{(n-1)}$ are continuous in $[x_0, x]$
2. $f^n$ exists in $(x_0, x)$

Then there exists at least one real number $\theta$, $0 < \theta < 1$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{x^n}{\lfloor n} f^n(\theta x)$$

[In the proof of above Theorem 6.7.1 take a = 0 and b = x, $c = \theta (x)$, $f$ or $0 < \theta < 1$, $0 \le \theta (x) < x$]

Hence $R_n = \frac{(x)^n}{\lfloor n} f^n [\theta x]$ is the Lagrange's form of remainder after n terms in Maclaurin's expansion of $f(x)$.

Taylor's Theorem with Cauchy's' form of remainder.

**Statement:** If a function $f$ is defined in [a, b] and

1. $f, f', f'' \ldots\ldots\ldots\ldots f^{n-1}$ are continuous in [a, b]
2. $f^n$ exists in (a, b)

then there exists atleast one real number $\theta$, $0 < \theta < 1$ such that

$f(b)$

$$= f(a) + (b - a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{(n-1)}(a)$$

$$+\ \frac{(b-a)^n}{\underline{|n-1}}(1-\theta)^{n-1} f^n (a+\theta(b-a)])$$

**Proof:** Consider a function $\phi$ defined on the closed interval [a, b] as

$$\phi(x) = f(x) + (b-x) f'(x) + \frac{(b-x)^2}{\underline{|2}} f''(x) + \ldots\ldots.$$

$$+ \frac{(b-x)^{n-1}}{\underline{|n}-1} f^{(n-1)}(x) + (b-x)A \qquad \ldots.(1)$$

Where A is a real constant to be determined such that

$\phi(a) = \phi(b)$

i.e. $f(a) + (b-a) f'(a) + \frac{(b-a)^2}{\underline{|2}} f''(a) + \ldots\ldots.$

$$+ \frac{(b-a)^{n-1}}{\underline{|n}-1} f^{n-1}(a) + (b-a)A = f(b) \qquad \ldots\ldots(2)$$

Since $f(x)$, $f'(x)$, $f''(x)$, ......, $f^{(n-1)}(x)$ and $(b-x)^n$, (n = 1, 2, .....n-1), are continuous in [a, b] and derivable in (a, b), therefore

We see that,

1. $\phi(x)$ is continuous in [a, b]
2. $\phi(x)$ is derivable in (a, b)
3. $\phi(a) = \phi(b)$

Hence $\phi(x)$ satisfies all the three conditions of Rolle's Theorem in [a, b] and therefore there exists atleast one real number $\theta \in (0, 1)$ such that $\phi'[a - \theta(b-a)] = 0$

Differentiating (1) w.r.t. x, we have,

$$\phi'(x) = f'(x) + [-f'(x) + (b-x) f''(x)] + \left[-(b-x) f''(x) + \frac{(b-x)^2}{\underline{|2}} f'''(x)\right] + \ldots..$$

$$+ \left[-\frac{(b-x)^{n-2}}{\underline{|n}-2} f^{n-1}(x) + \frac{(b-x)^{n-1}}{\underline{|n}-1} f^n(x)\right] [-A]$$

$$\Rightarrow \qquad \phi'(x) = \left[\frac{(b-x)^{n-1}}{\underline{|n}-1} f^n(x)\right] - [A]$$

Or $\phi'[a + (b - a)\theta] = \frac{[b-a-(b-a)\theta]^{n-1}}{\lfloor n-1}$ $[f^n[a + (b - a)\theta] - A$

Or $\phi'(c) = \frac{(b-a)^{n-1}(1-\theta)^{n-1}}{\lfloor n-1}$ $[f^n[a + (b - a)\theta] - A$ [by putting $x = a + \theta(b - a)$]

Or $A = \frac{(b-a)^{n-1}(1-\theta)^{n-1}}{\lfloor n-1}$ $[f^n[a + (b - a)\theta]$

Putting this value of A in (2), we have,

$$f(a) + (b - a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{\lfloor n-1} f^{n-1}$$

$$(a) + \frac{(b-a)^n}{\lfloor n-1} (1 - \theta)^{n-1} f^n[a + \theta(b - a)]$$

**8.5 This proves the Theorem.**

Taylor's Theorem with Cauchy's form of remainder.

1. $f, f', f''$ ........$f^{n-1}$ are continuous in [a, a + h]
2. $f^n$ exists in (a, a + h).

then there exists atleast one real number $\theta$, between 0 and 1 such that

$$f(a + h) = f(a) + hf'(a) + \frac{(h)^2}{\lfloor 2} f''(a) + \ldots\ldots$$

$$+ \frac{(h)^{n-1}}{\lfloor n-1} f^{(n-1)}(a) + \frac{(h)^n}{\lfloor n-1} (1 - \theta)^{n-1} f^n (a + \theta h)$$

Here $R_n = \frac{(h)^n}{\lfloor n-1} (1 - \theta)^{n-1} f^n (a + \theta h)$ is called Cauchy's form of remainder after n terms.

[take b = a +h i.e. b - a = h in the above Theorem].

**Another form of Taylor's Theorem with Cauchy's form of remainder after terms.**

If a function $f$ is defined on $[x_0, x]$ such that

1. $f, f', f''$ ........... $f^{(n-1)}$ are all continuous in $[x_0, x]$
2. $f^n$ exists in $(x_0, x)$

Then there exists at least one real number $\theta$, $0 < \theta < 1$ such that

$$f(x) = f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2} f''(x_0) + \ldots\ldots. + \frac{(x-x_0)^{n-1}}{\lfloor n-1} f^{n-1}(x_0)$$

$$+ \frac{(x-x_0)^n}{\lfloor n} (1 - \theta)^{n-1} f^n [x_0 + \theta (x - x_0)]$$

Hence

$R_n = \dfrac{(x-x_0)^n}{\lfloor n} (1 - \theta)^{n-1} f^n [x_0 + \theta(x - x_0)$ is the Lagrange's form of remainder after n terms.

Is the Cauchy's remainder after n terms.

[take $a = x_0$, $b = x$ in the above Theorem 6.8]

**Maclaurin's Theorem with Cauchy's form of remainder after n terms**

**Statement.** If a function $f$ is defined on [0, x], such that

1. $f, f', f'' \ldots\ldots\ldots\ldots f^{(n-1)}$ are all continuous in [0, x]
2. $f^n$ exists in (0 x)

Then there exists a real number $\theta$, $0 < \theta < 1$, such that

$$f(x) = f(0) + f'(0) + \frac{x^2}{\lfloor 2} f''(0) \ldots\ldots.. + \frac{x^{n-1}}{\lfloor n-1} f^{n-1}(0) + \frac{x^n}{\lfloor n-1} (1 - \theta)^{n-1} f^n(\theta x),\ 0 < \theta < 1$$

Hence $R_n = \dfrac{x^n}{\lfloor n-1} (1 - \theta)^{n-1} f^n(\theta x)$, $0 < \theta < 1$

Is the Cauchy's form of remainder after n terms in the Maclaurin's expansion of $f(x)$.

[take a = 0, b, = x in the above Theorem]

**Taylor's Polynomial**

If a function $f(x)$ has derivatives of order n at $x_0$, then

$$f(x_0) + (x - x_0)\, f'(x_0) + \frac{(x-x_0)^2}{\lfloor 2} f''(x_0) + \ldots\ldots. + \frac{(x-x_0)^n}{\lfloor n} f^n(x_0)$$

is called nth Taylor's Polynomial of $f$ at $x_0$ and is denoted by $P_n(x)$.

Thus,

$$P_n(x) = f(x_0) + (x - x_0)\, f'(x_1) + \frac{(x-x_0)^2}{\lfloor 2} f''(x0) + \ldots.. + \frac{(x-x_0)^n}{\lfloor n} f^n(x)$$

is nth Taylor's Polynomial of $f$ at $x_0$.

**Some Illustrated Examples**

**Example 1:** Apply Taylor's Theorem with Lagrange's form of remainder to the function

$$f(x) = \cos x \text{ in } \left(\frac{\pi}{2}, x\right)$$

**Solution:** Let $f(x) = \cos x \quad x \in \left(\frac{\pi}{2}, x\right)$ ......(1)

By Taylor's Theorem with Lagrange's form of remainder we have from (1)

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} f''\left(\frac{\pi}{2}\right) + \ldots + \ldots\ldots + \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!} f^{n-1}\left(\frac{\pi}{2}\right)$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^n}{n!} f^n\left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$$

where $0 < \theta < 1$ .....(2)

Now

$f(x) = \cos x,$ $\qquad f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

$f'(x) = -\sin x,$ $\qquad f'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \cos \pi = -1$

$= \cos\left(x+\frac{\pi}{2}\right)$

$f''(x) = -\sin\left(x+\frac{\pi}{2}\right),$ $\qquad f''\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}+2.\frac{\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$

$= \cos\left(x+2.\frac{\pi}{2}\right)$

..............................., $\qquad$ ......................................$\frac{-\pi}{2}$]

$f^{n-1}(x) = \cos\left(x+(n-1)\frac{\pi}{2}\right),$ $\qquad f^{n-1}\left(\frac{\pi}{2}\right) = \cos\left[\frac{n}{2}+\frac{(n-1)\pi}{2}\right] = \cos\frac{n\pi}{2}$

$f$n(x) = cos $\left(x+\frac{n\pi}{2}\right)$, $\qquad f^{n}\left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$ = cos $\left[\frac{(n+1)\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$

$\therefore$ from (2)

$$\cos x = 0 + \left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^{2}}{2!} . 0 + \ldots\ldots$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!} \cos\left(\frac{n\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^{n}}{n!}$$

$$\times \cos\left[\frac{(n+1)\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$$

or

$$\cos x = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^{3}}{3!} \ldots\ldots\ldots\ldots\ldots + \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!} \cos\frac{n\pi}{2} + \frac{\left(x-\frac{\pi}{2}\right)^{n}}{n!}$$

$$\times \frac{\left(x-\frac{\pi}{2}\right)^{n}}{n!} \times \cos\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$$

where $0 < \theta < 1$

**Example 2:** Apply Taylor's Theorem with Lagrange's form of remainder to the function

$$f(x) = \sin x \qquad x \in \left(\frac{\pi}{2}, x\right)$$

**Solution:** Here $f$(x) = sin x, $\frac{\pi}{2} \leq x \leq x$ ......(1)

By Taylor's Theorem with Lagrange's form of remainder to (1), we have

$$f(x) = f\left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right) f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^{2}}{2!} f''\left(\frac{\pi}{2}\right)$$

$$+\ \frac{\left(x-\frac{\pi}{2}\right)^3}{3!}\ f'''\left(\frac{\pi}{2}\right)\ +.........$$

$$+\ \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!} f^{n-1}\left(\frac{\pi}{2}\right)+\ \frac{\left(x-\frac{\pi}{2}\right)^{n}}{n!} f\ \left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$$

Where $0 < \theta < 1$ .....(2)

Now

$f(x) = \sin x$, $\therefore\ f\left(\frac{\pi}{2}\right) = \sin\ \frac{\pi}{2}\ = 1$

$f(x) = \cos x$ $\therefore\ f'\left(\frac{\pi}{2}\right) = \sin\ \left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \sin \pi = 0 = \sin\ \left(x+\frac{\pi}{2}\right)$

$f''(x) = \cos\ \left(x+\frac{\pi}{2}\right)$ $\therefore\ f''\left(\frac{\pi}{2}\right) = \sin\ \left(\frac{\pi}{2}+2\frac{\pi}{2}\right) = \sin\ \frac{3\pi}{2}\ = -1 = \sin\ \left(x+2.\frac{\pi}{2}\right)$

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$f^{n-1}\ (x) = \sin\left[x+(n-1)\frac{\pi}{2}\right]$, $f^{n-1}\ \left(\frac{\pi}{2}\right) = \sin\left[\frac{\pi}{2}+(n-1)\frac{\pi}{2}\right] = \sin\ \frac{n\pi}{2}$

$f^{n}\ (x) = \sin\left[x+\frac{n\pi}{2}\right]$, $f^{n}\left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right] = \sin\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$ ....(3)

$\therefore$ from (2), we get

$$\sin x = 1 + \left(x-\frac{\pi}{2}\right).\ 0 + \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}\ (-1) + .........+\ \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!}\ \sin\ \left(\frac{n\pi}{2}\right)\ +$$

$$\frac{\left(x-\frac{\pi}{2}\right)^{n}}{n!}\ \sin\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$$

or

$$\sin x = 1 - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!} + \text{..........} + \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{(n-1)!} \sin \frac{n\pi}{2}$$

$$+ \frac{\left(x-\frac{\pi}{2}\right)^n}{n!} \sin \left[(n+1)\frac{\pi}{2} + \theta\left(x-\frac{\pi}{2}\right)\right]$$

**Example 3:** By Maclaurin's Theorem with Lagrange's form of remainder expand $f(x) = \sin x$ as for as nth terms of absconding power of x.

**Solution:** Here $f(x) = \sin x$ .....(1)

$\therefore$ By Maclarrin's Theorem with Lagrange's form of remainder to (1), we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \text{.......} + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!}$$

$f^n(\theta x)$, $0 < \theta < 1$ .....(2)

Now

$f(x) = \sin x$ $\therefore f(0) = \sin 0 = 0$

$f'(x) = \cos x = \sin\left(x+\frac{\pi}{2}\right)$ $\therefore f''(0) = \sin \pi = 0$

$f'''(x) = \cos\left(x+\frac{2\pi}{2}\right) = \sin\left(x+\frac{3\pi}{2}\right)$ $\therefore f'''(0) = \sin\frac{3\pi}{2} = -1$

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$f^{n-1}(x) = \sin\left[x+\frac{(n-1)\pi}{2}\right]$ $\therefore f^{n-1}(0) = \sin\frac{(n-1)\pi}{2}$

$f^n(x) = \sin\left(x+\frac{n\pi}{2}\right)$ $\therefore f^n(\theta x) = \sin\left[\theta x+\frac{n\pi}{2}\right]$

$\therefore$ from (2), we have

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \text{........} + \frac{x^{n-1}}{(n-1)!} \sin\frac{(n-1)^n}{2}$$

$$+ \frac{x^n}{n!} \sin\left[\theta x + \frac{n\pi}{2}\right]$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!} \sin \frac{(n-1)\pi}{2}$$

$$+ \frac{x^n}{n!} \sin\left[\theta x + \frac{n\pi}{2}\right],$$

$0 < \theta < 1.$

**Example 4:** Use Maclaurin's Theorem to prove that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta n},$$

$0 < \theta < 1.$

**Solution:** Here $f(x) = e^x$ ..... (1)

$\therefore$ $f(x) = e^x \Rightarrow f(0) = e^0 = L$

$f'(x) = e^x \quad \therefore \quad f'(0) = 1$

$f''(x) = e^x \quad \therefore \quad f''(0) = 1$

$f'''(x) = e^x \quad \therefore \quad f'''(0) = 1$

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$f^{n-1}(x) = e^x \quad \therefore \quad f^{n-1}(0) = 1$

$f^n(x) = e^x \quad \therefore \quad f^n(\theta x) = e^{\theta x}$

$\therefore$ by Maclaurin's formula, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots\ldots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n}{n!} f^n(\theta x)$$

$$\therefore \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x},$$

$0 < \theta < 1.$

Inequalities:

We shall make use of theorems to prove

Some important inequalities.

**Example 5:** Use mean value theorem to show that tan x > x > sin x, $x \in \left(0, \frac{\pi}{2}\right)$.

**Solution:** Let us consider the function

$$f(z) = \tan z - z$$

and

$$g(z) = z - \sin z, \qquad z \in [0, x],\ x \in R,\ x \in \left(0, \frac{\pi}{2}\right)$$

$$\therefore \qquad f'(z) = \sec^2 z - 1 = \tan^2 z > 0$$

and

$$g'(z) = 1 - \cos z > 0 \qquad \forall\ z \in (0, x)$$

clearly, $f$ and g are strictly increasing in [0, x]

$$\Rightarrow \qquad f(x) > f(0) \qquad \text{and} \qquad g(x) > g(0)$$

$$\Rightarrow \qquad \tan x - x > 0 \qquad \text{and} \qquad x - \sin x > 0 \qquad (\because\ f(0) = g(0) = 0)$$

$$\Rightarrow \qquad \tan x > x \qquad \text{and} \qquad x > \sin x$$

$$\Rightarrow \qquad \tan x > x > \sin x, \qquad x \in \left(0, \frac{\pi}{2}\right)$$

**Example 6:** By examining the sign of the derivatives of an appropriate function, prove that

$$\tan x > x \qquad \forall \left(0, \frac{\pi}{2}\right)$$

**Solution:** Let $f(y) = \tan y - y,\ y \in [0, x]$

$$\therefore \qquad f'(y) = \sec^2 y - 1 = \tan^2 y > 0$$

$\Rightarrow$ $f$ is strictly increasing function in [0, x]

$\Rightarrow$ $f(x) > f(0)$, i.e. $\tan x - x > 0$

or $\tan x > x \quad \forall\ x \in \left(0, \frac{\pi}{2}\right)$. Hence proved.

Now you can try the following exercises

## 8.6 Self Check Exercise

Q.1 Apply Taylor's Theorem to expand in the form

$$e^{\cos x} = 1 - \left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{2} - \frac{\left(x-\frac{\pi}{2}\right)^4}{8} + \ldots\ldots, \quad a = \frac{\pi}{2}$$

Q.2 If $f(x) = x^3 - 6x^2 + 7$, find the value of $f\left(\frac{21}{20}\right)$ by Taylor's Theorem

Q.3 Show that

$$\log (n+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots \frac{(-1)^{n-1}}{n} \quad \frac{x^n}{(1+\theta x)^n}$$

by using Maclaurin's Theorem

## 8.7 Summary

Dear students let us summarize what we have learnt from this unit:

(i) Taylor's Theorem with Lagrange's form of remainder

(ii) Maclaurin's Theorem with Lagrange's form of remainder

(iii) Taylor's Theorem with Cauchy form of remainder

(iv) Application of the above theorem to some important inequalities.

## 8.8 Glossary

(i) $R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$ is called Lagrange's remainder after n terms in Taylor's Theorem.

(ii) $R_n = \frac{x^n}{n!} f^n(\theta x)$ is called Lagrange's remainder after n terms in Maclaurin's Theorem.

(iii) $R_n = \frac{x^n}{(n-1)!} (1-\theta)^{n-1} f^n(\theta x)$, $0 < \theta < 1$, is called Cauchy form of remainder after n terms in Maclaurin's expansion of $f(x)$.

## 8.9 Answers to Self Check Exercises

Ans. 1 Apply Taylor's form of remainder

Ans. 2 1.542625

Ans. 3 Apply Maclaurin's theorem

## 8.10 Reference/Suggested Reading


1. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002
2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007


## 8.11 Terminal Questions

1. Expand tan x in power of $\left(x-\frac{\pi}{4}\right)$ up to four terms.
2. Using Taylor Theorem, express the polynomial $2x^3 + 7x^2 + x - 6$ in power of x - 2.
3. Show by Maclaurin's Theorem that sin ax = ax - $\frac{a^3x^3}{3!}$ + $\frac{a^5x^5}{5!}$ +.....+ $\frac{a^{n-1}}{(n-1)!}$

   $x^{n-1}$ sin $\frac{(n-1)\pi}{2}$ + $\frac{a^n x^n}{n!}$ sin $\left[\frac{n\pi}{2}+a\theta x\right]$, $0 < \theta < 1$.
4. Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$, $0 < x < \frac{\pi}{2}$, by examining the sign of the derivatives of an appropriate function.

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## Unit - 9

# Taylor's Series And
# Maclaurin's Series

**Structure**



## 9.1 Introduction

Dear students, in this unit we shall study two important series, namely, Taylor's series and Maclaurin's series. The Taylor's series or Taylor's expansion in an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common function, the function and the sum of its Taylor series are equal near this point. The Taylor series are named after Brook Taylor who introduced then in 1715. A Taylor's series is also called Maclaurin series when 0 is the point where the derivative are considered, after Colin Maclaurin, who made extensive use of this special case of Taylor series in 18th Century. The partial sum formed by the first (n+1) term of a Taylor series is a polynomial of degree n that is called nth Taylor polynomial of the function.

## 9.2 Learning Objectives

The main objectives of this unit are

(i) to define Taylor's series of real function

(ii) to deduce Maclaurin's series of real function

(iii) to find Taylor and Maclaurin's series of sin x, cos x, $e^x$, log (1+x), $(1+x)^m$.

## 9.3 Taylor's Series.

The Taylor's series of real function $f(x)$, that is infinitely differentiable at a real point a, is the power series.

$$f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots..$$

$$+ .. \quad = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x - a)^n.$$

Here n!, denotes the factorial n. The function $f^n(a)$ denotes the derivative of $f$ evaluated at the point a. The derivative of order o if $f$ is defined to be $f$ itself and $(x - a)^0$ and 0! are both defined to be 1.

We have seen that remainder $R_n$ after n terms in the Taylor's expansion of $f(x)$ at $x_0$ is given by

(a) $\quad R_n = \frac{(x-x_0)^n}{n!} f^n [x_0 + \phi (x - x_0), 0 < \phi < 1]$

(Called Legrange's Form of remainder)

(b) $\quad R_n = \frac{(x-x_0)^n}{n-1!}(1 - \theta)^{n-1} f^n [x_0 + \phi (x - x_0), 0 < \phi < 1]$

(Called Legrange's Form of remainder)

Now if $R_n \to 0$ as $n \to \infty$, then the Taylor's expansion of $f$ at $x_0$ is called Taylor's Series. In view of above, the Taylor's series can also be defined as :

Let a function $f$ has derivative of all orders in a neighbourhood (nhd). N of a real number $x_0$. Lt $R_n$ be the remainder after n terms in the Taylor's expansion of $f$ at $x_0$.

Further, Let $R_n \to 0$ as $n \to \infty$, then the infinite series.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x-x_0)^2}{2!} f''(x_0) +$$

$$+ \frac{(x-x_0)^3}{3!} f'''(x_0) + \ldots\ldots. + \ldots. \text{ in infinity}$$

is called Taylor's Series of $f$ around $x_0$ (or at $x_0$)

## 9.4 Machaurin's Series

If we take $x_0 = 0$, the Taylor series becomes Machaurin's Series of $f$ around 0 (or at $x_0$ = 0). Thus, Machaurin's Series is a special case of Taylor's Series.

Note : From the definition of Taylor Series it is quite obvious that it does not matter whether we take $R_n$, the reminder after n term in the Lagrange form of remainder or in the Cauchy Form of remainder (because $R_n \to 0$ as $n \to \infty$). But it is always convenient to take Rn in the Lagrange's form of remainder after n terms.

**Example 1 :** Use Taylor's Theorem (with Lagrange's form of remainder) to the function $f$ (x) = Cos x in $\left(\frac{\pi}{2}, x\right)$. Hence find the Taylor Series of Cos x around $\frac{\pi}{2}$

**Solution :** We have

$f$ (x) = Cos x $\quad\therefore f\left(\frac{\pi}{2}\right)$ = Cos $\frac{\pi}{2}$ = 0

$f'$ (x) = - Sin x $\quad\therefore f'\left(\frac{\pi}{2}\right)$ = - Sin $\frac{\pi}{2}$ = -1

$f''$(x) = - Cos x $\quad\therefore f''\left(\frac{\pi}{2}\right)$ = - Cos $\frac{\pi}{2}$ = 0

$f'''$ (x) = Sin x $\quad\therefore f'''\left(\frac{\pi}{2}\right)$ = Sin $\frac{\pi}{2}$ = 1

$f''''$(x) = Cos x $\quad\therefore f''''\left(\frac{\pi}{2}\right)$ = Cos $\frac{\pi}{2}$ = 0

$f'''''$ (x) = - Sin x $\quad\therefore f'''''\left(\frac{\pi}{2}\right)$ = - Sin $\frac{\pi}{2}$ = -1

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$f^{(n-1)}$ (x) = cos $\left(x+n-1\frac{\pi}{2}\right)$ and $f^{(n-1)}\left(\frac{\pi}{2}\right)$

$$= \cos\left(\frac{\pi}{2}+n-1\frac{\pi}{2}\right) = \cos\frac{n\pi}{2}$$

$$= \begin{cases} 0 & \text{if} \quad n \text{ is odd} \\ (-1)^{n/2} & \text{if} \quad n \text{ is even} \end{cases}$$

$f^{n}$(x) = cos (x + $\frac{n\pi}{2}$)

$$\Rightarrow \qquad f^{n} = \left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right] f^{n} = \left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)+\frac{n\pi}{2}\right]$$

$= f^{n}\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right]$

By Taylor's Theorem with Langrange's form of remainder after n terms in $\left[\frac{\pi}{2}, x\right]$, we have,

$$f(x) = \left(\frac{\pi}{2}\right) + \left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^2}{\lfloor 2}f''\left(\frac{\pi}{2}\right) + \ldots + \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{\lfloor n-1}f^{n-1}\left(\frac{\pi}{2}\right)$$

$$+\frac{\left(x-\frac{\pi}{2}\right)^n}{\lfloor n} f\left[\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right],\ 0 < \theta < 1$$

$$\cos x = -\left(x-\frac{\pi}{2}\right)$$

$$+\frac{\left(x-\frac{\pi}{2}\right)^3}{\lfloor 3} \ldots + \frac{\left(x-\frac{\pi}{2}\right)^{n-1}}{\lfloor n-1}\cos\frac{n\pi}{2}$$

$$+\frac{\left(x-\frac{\pi}{2}\right)^n}{\lfloor n}\cos\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right],\ 0 < \theta < 1$$

To find Taylor Series at $x_0 = \frac{\pi}{2}$

Here $R_n = \frac{\left(x-\frac{\pi}{2}\right)^n}{\lfloor n}\cos\left[(n+1)\frac{\pi}{2}+\theta\left(x-\frac{\pi}{2}\right)\right],\ 0 < \theta < 1$

Since $\left|\cos (n+1)\frac{\pi}{2} + \theta\left(x-\frac{\pi}{2}\right)\right| \leq 1$

And $x \in \left(\frac{\pi}{2}-\delta, \frac{\pi}{2}+\delta\right)$, $\delta > 0 \Rightarrow \left|x-\frac{\pi}{2}\right| < \delta < 1$

$$\therefore \lim_{n\to\infty}\frac{\left(x-\frac{\pi}{2}\right)^n}{\lfloor n} = 0$$

hence $R_n \to 0$ as $n \to \infty$

$\therefore$ Taylor Series of cos x around $\frac{\pi}{2}$ is

$$\cos x = -\left(x-\frac{\pi}{2}\right) + \frac{\left(x-\frac{\pi}{2}\right)^3}{\lfloor 3} - \frac{\left(x-\frac{\pi}{2}\right)^5}{\lfloor 5} \frac{\left(x-\frac{\pi}{2}\right)^7}{\lfloor 7} \text{ .... ....}\infty$$

$$[\lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0 \text{ provided } |x| < 1]$$

**Example 2 :** Use Maclaurin's Theorem (with Lagrange's form of remainder to expanded sin x and hence deduce the Macalurin Series.

**Solution :** $f(x) = \sin x$ $\quad\therefore f(0) = 0$

$f'(x) = \cos x \qquad \therefore f'(0) = 1$

$f''(x) = -\sin x \qquad \therefore f''(0) = 0$

$f'''(x) = -\cos x \qquad \therefore f'''(0) = -1$

$f''''(x) = \sin x \qquad \therefore f''''(0) = 0$

$f'''''(x) = \cos x \qquad \therefore f'''''(0) = 1$

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$$f^{n-1}(x) = \sin\left(x+(x-1)\frac{\pi}{2}\right) \qquad \therefore f^{n-1}(0) = \sin (n-1)\frac{\pi}{2}$$

$$= \begin{cases} 0 & \text{if } \quad n \text{ is even} \\ (-1)^{\frac{n-1}{2}} & \text{if } \quad n \text{ is odd} \end{cases}$$

$$f^n(x) = \sin\left(x+\frac{n\pi}{2}\right) \quad \therefore f^n(\theta x) = \sin(\theta x + \frac{\pi}{2})$$

By Maclaurin's Theorem with Lagrange's form remainder to the function $f(x) = \sin x$, we have,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor 2} f''(0) + \frac{x^3}{\lfloor 3} f'''(0) + \text{....}$$

$$+ \frac{x^{n-1}}{\lfloor n-1} f^{(n-1)}(0) + \frac{x^n}{\lfloor n} f^n(\theta x),\ 0 < \theta < 1 \qquad \text{.....(1)}$$

Hence sin x = 0 + x(1) + $\frac{x^2}{\lfloor 2}$ (0) + $\frac{x^3}{\lfloor 3}$ (-1) + .... + $\frac{x^{n-1}}{\lfloor n-1}$ sin $\frac{(n-1)\pi}{2}$ + $\frac{x^n}{\lfloor n}$ sin ($\theta$x + $\frac{n\pi}{2}$), 0 < $\theta$ < 1

Or sin x = x - $\frac{x^3}{\lfloor 3}$ + $\frac{x^5}{\lfloor 5}$ .... + $\frac{x^{n-1}}{\lfloor n-1}$ sin $\frac{(n-1)\pi}{2}$ + $\frac{x^n}{\lfloor n}$ sin ($\theta$x + $\frac{n\pi}{2}$), 0 < $\theta$ < 1

To find Taylor Series around $x_0 = 0$

Here $R_n = \frac{x^n}{\lfloor n}$ sin ($\theta$x + $\frac{n\pi}{2}$), 0 < $\theta$ < 1

Since sin ($\theta$x + $\frac{n\pi}{2}$) < 1 and x $\in$ (- δ , δ ), δ > 0

$$\Rightarrow |x| < \delta < 1$$

$$\therefore \lim_{n\to\infty} \frac{x^n}{\lfloor n} = 0$$

$$\Rightarrow R_n \to 0 \text{ as } n \to \infty$$

$\therefore$ the Maclaurin Series of sin x (Taylor Series of sin x around $x_0 = 0$

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + ....\infty.$$

**Example 3 :** Write the Taylor's formula $f$ or $f$(x) = log(1+x).$f$ or - 1 < x $\le$ 1 with $x_0 = 0$. Hence deduce the Taylor Series of log(1+x) around $x_0 = 0$

In particular, log 2 = 1 - $\frac{1}{2}$ + $\frac{1}{3}$ + $\frac{1}{4}$ + ....$\infty$

**Solution :** $f$(x) = log(1 + x) $\therefore f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\therefore f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\therefore f''(0) = -1$

$f'''(x) = \frac{1}{(1+x)^3}$ $\therefore f'''(0) = 2$

$f''''(x) = -\frac{6}{(1+x)^4}$ $\therefore f''''(0) = -6$

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.....................................................................................

$f^{n-1}(x) = \dfrac{(-1)^{n-2}\lfloor\underline{n-2}}{(1+x)^{n-1}}$ $\quad\therefore f^{n-1}(0) = (-1)^{n-2}\lfloor\underline{n} - 2$

$f^{n}(x) = \dfrac{(-1)^{n-1}\lfloor\underline{n-1}}{(1+x)^{n}}$ $\quad\therefore f^{n}(\theta x) = \dfrac{(-1)^{n-1}\lfloor\underline{n-1}}{(1+\theta x)^{n}}$

By Taylor's formula at $x_0 = 0$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{\lfloor\underline{2}} f''(0) + \frac{x^3}{\lfloor\underline{3}} f'''(0) + \ldots.$$

$$+ \frac{x^{n-1}}{\lfloor\underline{n-1}} f^{(n-1)}(0) + R_n$$

Where $R_n$ is the remainder after n terms.

$$\therefore \log(1+x) = 0 + x(1) + \frac{x^2}{\lfloor\underline{2}}(-1) + \frac{x^3}{\lfloor\underline{3}}(2) + \frac{x^4}{\lfloor\underline{4}}(-6) + \ldots. + \frac{x^{n-1}}{\lfloor\underline{n-1}} \cdot (-1)^{n-2}\lfloor\underline{n} - 2 + R_n$$

Or $\log(1 + x) = x - \dfrac{x^2}{\lfloor\underline{2}} + \dfrac{x^3}{\lfloor\underline{3}} + \dfrac{x^4}{\lfloor\underline{4}} + \ldots. + \dfrac{(-1)^{n-2}}{n-1} x^{n-1} + R_n$

Or $R_n = \dfrac{x^n(1-\theta)^{n-1}}{\lfloor\underline{n-1}} \cdot \dfrac{(-1)^{n-1}\lfloor\underline{n-1}}{(1+\theta x)^n}$, $0 < \theta < 1$

Or $R_n = \dfrac{(-1)^{n-1}x^n(1-\theta)^{n-1}}{(1+\theta x)^n}$, $0 < \theta < 1$

Since $-1 < x \leq 1$,

Therefore, for $x > -1$ and $0 < \theta < 1$

$-1 - \theta x < 1 - \theta < 1 + \theta x$

or $-(1 + \theta x) < 1 - \theta < 1 + \theta x$

$\Rightarrow \quad \left|\dfrac{1-\theta}{1+\theta x}\right| < 1$

$$|R_n| = \left|\frac{(-1)^{n-1}x^n(1-\theta)^{n-1}}{(1+\theta x)^n}\right| = \left|\frac{x_n}{1+\theta x}\right| \left|\left(\frac{1-\theta}{1+\theta x}\right)n-1\right|$$

$$= \left|\frac{x_n}{1+\theta x}\right| \left|\left(\frac{1-\theta}{1+\theta x}\right)n-1\right|$$

$$\leq \frac{|x_n|}{|1+\theta x|} \qquad \left[\left|\frac{1-\theta}{1+\theta x}\right|<1\right]$$

**Some Illustrated Examples**

**Example 4 :** Approximate the value of $\sqrt{10}$ to four decimal places by taking the first four terms of an appropriate Taylor's expansion.

**Solution :** Let

$f$(x + h) = $\sqrt{x+h}$ $\Rightarrow$ $f$(x) = $\sqrt{x}$ (Put h = 0)

$\therefore$ $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} x^{-\frac{1}{2}}$

$f''(x) = -\dfrac{1}{4} x^{-3/2}$

$f'''(x) = \dfrac{3}{8} x^{-5/2}$

$\therefore$ By Taylor's Theorem

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots$$

$$\therefore \quad \sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + \frac{h^3}{16x^2\sqrt{x}} + \ldots$$

Substitute x = 9 h = 1, we get

$$\therefore \quad \sqrt{10} = 3 + \frac{1}{2.3} - \frac{1}{8.9.3} + \frac{1}{16\times81\times3} + \ldots\ldots$$

$$= 3 + \frac{1}{6} - \frac{1}{216} + \frac{1}{3888} + \ldots..$$

= 3 + 0.16666 - 0.00463 + 0.00025 + ....

= 3.16228 nearly

= 3.1623 (nearly). up to four decimal places.

**Example 5 :** Find an approximation value of sin $31^0$ and estimate the error term.

**Solution :** Let $f$(x) = sin x

Take the interval $\left[\dfrac{\pi}{6}, \dfrac{\pi}{6}+\dfrac{\pi}{180}\right]$

Here $x_0 = \frac{\pi}{6} + \frac{\pi}{180} = 31^0$

Clearly $f(x)$ is derivable in N $\left(\frac{\pi}{6}\right)$

$\therefore$ $f'(x) = \cos x$ exists in N$\left(\frac{\pi}{6}\right)$

$\therefore$ By Mean Value Theorem

$$f(x) \cong f(x_0) + (x - x_0)\, f'(x_0)$$

$\Rightarrow$ $\sin\left[\frac{\pi}{6}+\frac{\pi}{180}\right] = \sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6}$

$\Rightarrow$ $\sin 310 \cong \frac{1}{2} + \frac{\pi}{180} \times \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\sin 310 \cong 0.5 + 0.015 = 0.515$

Now $|R(x)| \leq (x - x_0)^2 \,|\, f''(x)|,\ x_0 < c < x$

$(\therefore\ f''(x) = -\sin x)$

$\therefore |R(x)| \leq \left(\frac{\pi}{180}\right)^2 .|-\sin 4|\ ,\ \frac{\pi}{6} < c, < \frac{\pi}{180} + \frac{\pi}{6}$

$\leq \left(\frac{\pi}{180}\right)^2 .1$ $\qquad (\therefore\ 1 - \sin G| \leq 1)$

Which is the required error.

**Example 6 :** Approximate $\sqrt{17}$ up to four decimal term by taking first three terms of Taylor's expansion.

**Solution :** Let's take $f(x + h) = \sqrt{x} + h$

$\therefore$ $h = 0 \Rightarrow f(x) = \sqrt{x}$

$\therefore$ $f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}\ x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}\ x^{-3/2}$, ....

By Taylor's expansion

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!}\, f''(x) + \ ....$$

$\therefore \quad \sqrt{x+h} = \sqrt{x} + \frac{h}{2\sqrt{x}} - \frac{h^2}{8x\sqrt{x}} + ...$

Put x = 16, h = 1, we get

$\sqrt{17} = 4 + \frac{1}{8} - \frac{1}{512}$

= 4 + 0.125 - 0.00195 + ....

= 4.12305 (app)

= 4.1231 (nearly)

**Example 7 :** Show that $f$(ax) = $f$(x) + (a - 1) $f$'(x) + $\frac{(a-1)^2}{2!}$ $f$''(x) +....

**Solution :** $f$(ax) = $f$[x + (an -x)] = $f$ [x + (a -1) x]

$\therefore$ $f$(ax) = $f$ (a + h) , a = x, h = (a - 1) x

$\Rightarrow$ $f$(x) = $f$(a) + h $f$'(a) + $\frac{h^2}{2!}$ $f$''(x) + ....

$\Rightarrow$ $f$(ax) = $f$(x) + (a - 1) $f$'(x) + $\frac{(a-1)^2}{2!}$ $x^2$ $f$''(x)

Hence the result.

## 9.5 Self Check Exercise

Q.1 Evaluate the Taylor's Series for

$f$(x) $x^3 - 10x^2 + 6$ at x = 3

Q.2 Write the Taylor's series for $f$(x) = tan x

Q.3 Find the Machaurin's series expansion of $f$(x) = $e^x$

## 9.6 Summary

In this unit, we have learnt that

(i) Taylor's series.

(ii) Maclaurin's series

(iii) Application of Taylor's and Machaurim's series for sin x, cos x, $e^x$, lot (1+x) and $(1 + x)^m$

## 9.7 Glossary

(i) Maclaurin Polynomial : A Taylor polynomial centered at $\theta$, the nth degree polynomial for $f$ at 0 is the nth degree Maclaurin's polynomial for $f$.

(ii) Taylor Series : A power series at the point a that converges to a function on some open interval containing a.

(iii) Maclaurin's Series : A Taylor series for a function $f$ at x = 0 is known as Maclaurin series for $f$.

## 9.8 Answers to Self Check Exercises

Ans. 1 $[-57 - 33(x - 3) - (x - 3)^2 + (x - 3)^3]$

Ans. 2 $[x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots]$

Ans. 3 $[1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \ldots\ldots]$

## 9.9 Reference/Suggested Reading

1. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002
2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007

## 9.10 Terminal Questions

1. Find the Taylor Series for

   $f(x) = \frac{1}{x}$ at x = 1 and

   determine its interval of convergence.

2. Find the Taylor series for $f(x) = \frac{1}{2}$ at x = 2 and determine its interval of convergence.

3. Use the fourth Maclaurin polynomial for cos x to approximate $\cos\left(\frac{\pi}{12}\right)$

4. Find the first and second Taylor polynomial for $f(x) = \sqrt{x}$ at x = 4.

   Use these polynomial to estimate $\sqrt{6}$ . Use Taylor Theorem to bound the error.

5. Calculate the approximate value of $\sqrt{17}$ to four decimal places by taking first three terms of a Taylor expansion.

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**Unit - 10**

# Concavity And Convexity

**Structure**



## 10.1 Introduction

Dear students, in this unit we shall first define what we mean by concavity and convexity of a function (curve). To have a clear understanding of the concept concavity and convexity, w shall see that a curve is reported by two parts, one the convex part and the other is concave part and here comes the existence of point of inflexion.

## 10.2 Leaning Objectives

The main objectives of this unit are

(i) to know what we mean by concavity and convexity of a curve.

(ii) to learn about the point of inflexion

(iii) to learn the method to locate the point of inflexion

(iv) to know the working method for concavity and convexity

(v) to know the working method to find the point of inflexion

## 10.3 Concavity And Convexity of A curve

Let us consider a curve which is the graph of a single valued differentiable function $y = f(x)$

Definition

A curve is called concave downwards (or convex upward) on the interval (a, b) if all the points of the curve lie below any tangent to it on that interval (see fig. 1) It is said to be concave upward (convex downward) on the interval (a, b) if all the points of the curve lie above any tangent to it on that interval (see Fig. 2)

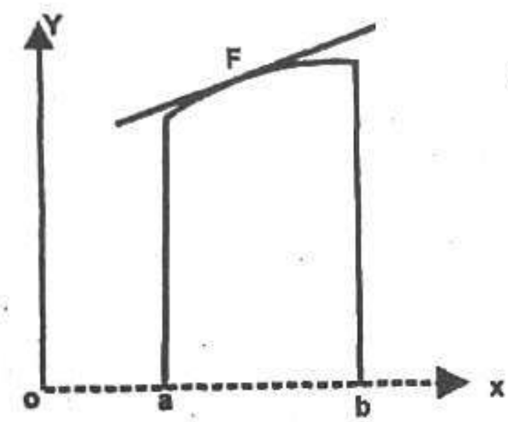


Fig. 1

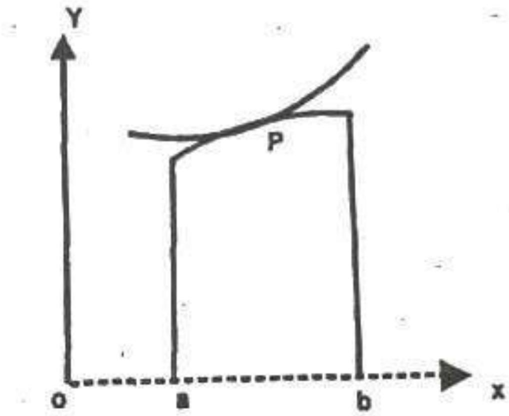


Fig. 2

A curve which is convex downward (or concave upward) is called a concave curve and a curve which is convex upward (or concave downward) is called a convex curve.

We now prove a problem

**Theorem:** The curve $y = f(x)$ on an interval (a, b) is convex upwards or concave downward, according as at all points of the interval (a, b) the second derivative is negative or positive respectively.

**Proof:** Take an arbitrary point $p(x=x_0)$ on the curve $y = f(x)$ in the interval (a, b) - (1)

In the tangent line line PT at P to the curve $y = f(x)$ is

$$Y - f(x_0) = f'(x_0)\,(x - x_0) \quad - (2)$$

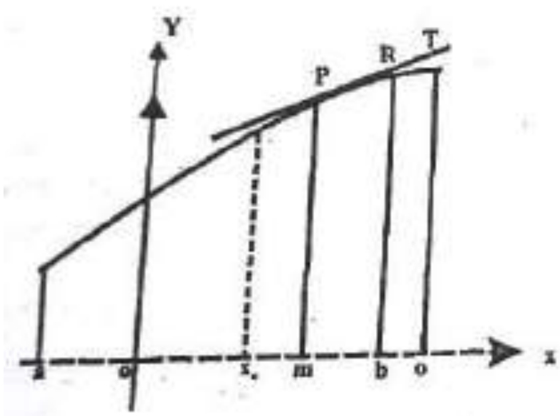


Fig. - 3

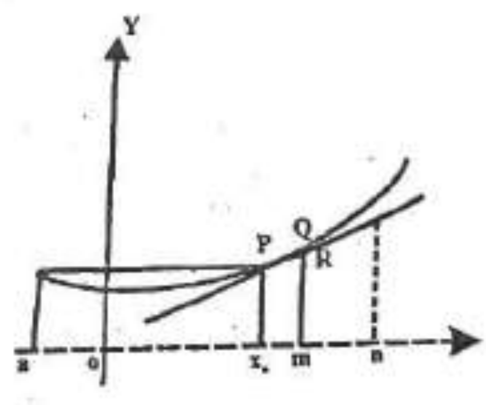


Fig. - 4

Let Q ($x_0$ + h, $f$($x_0$ + h) be a point on the curve in the neighborhood of P in (a, b)

Let the ordinate of QM of Q meet PT in R.

Since the abscissa of R = abscissa of Q = $X_0$ + h

MQ = F ($x_0$ + h) = $f(x_0) + hf\ (x_0) + \frac{h^2}{21} f''(x_0 + \theta h)\ 0 < \theta < 1$

MQ - MR = $\frac{h^2 21}{f^n}$ ($x_0$ + 0h) .....(3)

Let us assume $f'(x)$ to be continues at p and is non-zero at that point, so that the same sign as that of $f'(x_0 + \theta h)$ when |h| is very small hence from (3), MQ-MR has the same sign as that of $f'(x_0)$ for positive as well negative value of h, provided h is sufficiently small in magnitude.

**Now to cases arise**

Case when $f'(x_0)$ is negative

Then, MR - MR is negative

$\Rightarrow$ MQ < MR for Q on either side of P and in the nhd. Of P, so that portion of the curve on both side of a lies below the tangent at p(see Fig. 3)

Thus, the curve is convex upward (or concave downwards) in the ndd. Hence the curve y-$f$(x) is convex upwards (or concave downwards) when $\frac{d^2y}{dx^2}$ is negative at all points of (a, b).

**Case 2.** When $f''(x_0)$ is positive

Then, NQ-MR is positive

$\Rightarrow$ MQ > MR for Q an either side of p and in the nhd. Of p, so that the portion of the curve on both sides of P lie above the tangent at p (see Fig. 4)

Thus, curve is convex downwards (or concave upwards) in the nhd. Of p. Since, p is arbitrary, the curve is convex downwards (or concave upwards) When $\frac{d^2y}{dx^2}$ is positive at all points of (a, b). This completes the proof of the theorem.

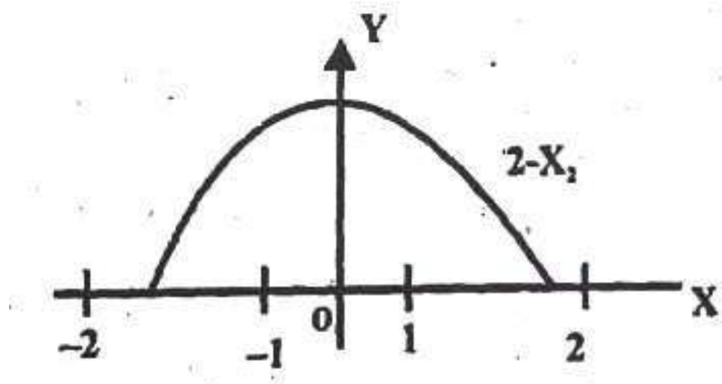
Let's now do now example to have better ides of the concept.

**Example 1.** Establish the intervals of convexity and concavity of the curve

Y = 2 - x2

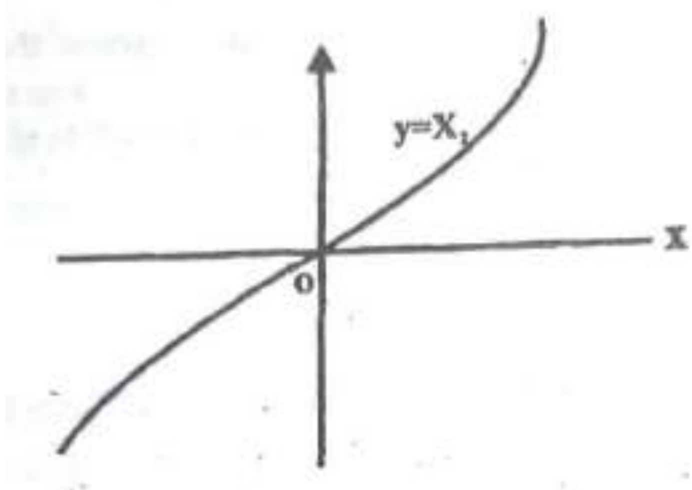
$\frac{dy}{dx}$ = - 2x and $\frac{d^2y}{dx^2}$ = - 2 < 0$\forall$ x $\in$ R

Hence curve is everywhere convex upwards. Let us draw this curve.




### 10.4 Point of Inflexion

A point that separates the concave part of the curve from the convex part of the curve is known as a point of inflexion. For example consider the curve $y = x^3$.




It is easy to se that O is point of inflexion for the curve $y = x^2$

You can see that at the point of inflexion, the tangent line, if it exists, cuts the curve, because on one side, the curve lies below the tangent and on the other side the curve lies above it.

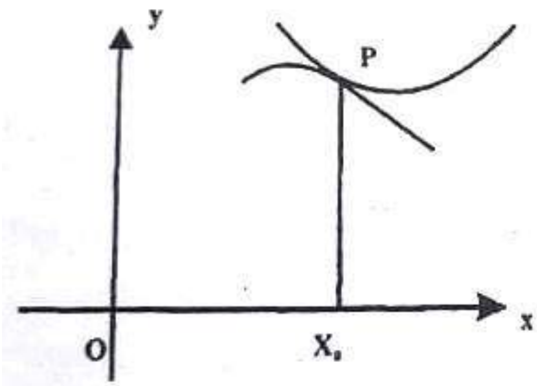
We now prove a theorem to find the conditions so that a point becomes a point of inflexion of the curve.

Theorem: A point of P of curve $y = f(x)$ is a point of inflexion if

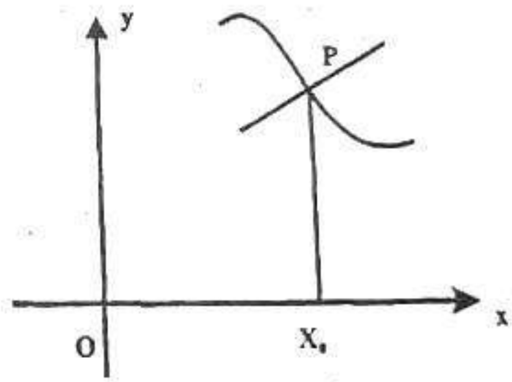
i. $\frac{d^2y}{dx^2}$ at P I zero (or does not exist)

ii. $\frac{d^2y}{dx^2}$ change sign white passing through P.

**Proof:** Let y = $f$(x) be a curve and $x_0$ be the abscissa o a point p on the curve. We have $\frac{d^2y}{dx^2}$ = $f''(x) = 0$ or it does not exists.



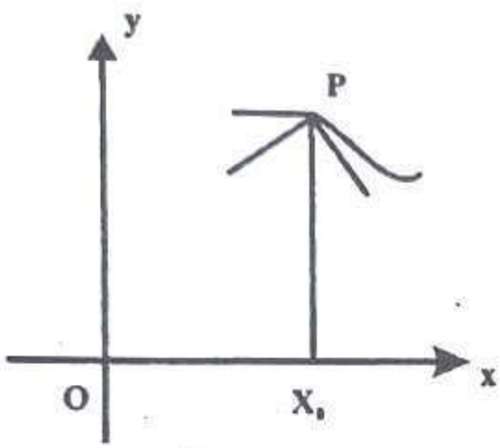
(Fig 5)



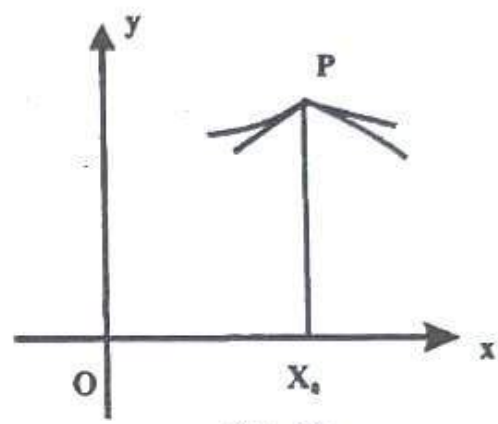
(Fig 6)

iii. Let $f''(x) < 0$ for $x < x_0$ and $f''(x) > 0$ for $x > x_0$.

We notice that the curve is convex upwards for $x < x_0$. Hence P (Se Fig. 5, 7) of the curve separates the convex upwards parts from concave upwards part, therefore P is a point of inflexion (see Fig. 5, 7)



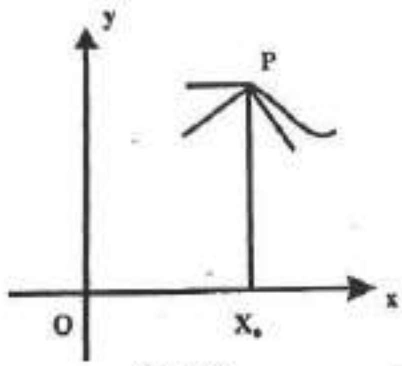
(Fig 7)



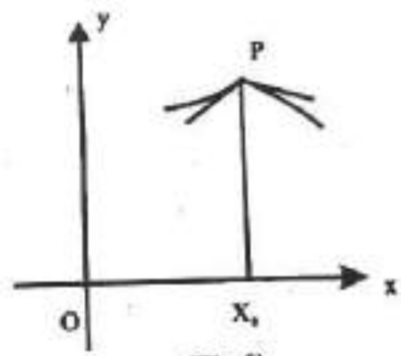
(Fig 8)

Let $f''(x) > 0$ or $x < x_0$ an $f''(x) < 0$ for $x > x_0$.

We notice that the curve is convex upwards for $x < x_0$ hence p (Se Fig 5, 7) for he curve separates the convex upwards part from concave upwards part, therefore p is a point of inflexion (see Fig 5, 7)



(Fig 7)



(Fig 8)

Let $f''(x) > 0$ or $x < x_0$ an $f''(x) < 0$ for $x > x_0$.

In this case we see that the curve is concave upwards for $x < x_0$ and convex upwards for $x > x_0$. Hence, the point p (see fig 6, 8) of the curve separates the concave upwards p part form convex upwards part. Therefore, p is a point of inflexion (see Fig. 8)

This completes the proof of the theorem.

Below we give method to find the point of inflexion of the curve $y = f(x)$.

1. Evaluate $\frac{d^2y}{dx^2}$, and find all possible values of x say $x_1, x_2, x_3$....... where $\frac{d^2y}{dx^2} = 0$

2. Also find all values of x (if any) $\frac{d^2y}{dx^2}$ does not exist. Let these values be $\alpha, \beta, \gamma$....X = $x_1, x_2, x_3$, ......$\alpha$, $\beta$, $\gamma$,......are possible points of inflexion.

3. X = $x_1$ will be a point of inflexion of the given curve if either $\frac{d^2y}{dx^2}$ = changes its sign at $x = x_1$. Or $\frac{d^2y}{dx^2}$ exists and is non is non zero at $x = x_1$.

Now, let us do one example.

**Example 2:** Find the points of inflexion and also determine the intervals of convexity and concavity of the curve y = $e^{-x^2}$

**Solution:** We have y = $e^{-x^2}$ as the eq = n of curve.

$\therefore \qquad \frac{dy}{dx}$ - $e^{-x^2}$ . 2x and

$$\frac{d^2y}{dx^2} = 2e^{-x^2} \text{ (2x2 - 1)} = 2e^{-x^2}\left(\sqrt{2x-1}\right)\left(\sqrt{2x-1}(2)\right)$$

Now $\frac{d^2y}{dx^2} = 0 \Rightarrow x_1 = \frac{1}{\sqrt{2}}$ and $x_2 = -\frac{1}{\sqrt{2}}$

For x < $\frac{1}{\sqrt{2}}$, $\frac{d^2y}{dx^2} < 0$ (From (2))

And for x > $\frac{1}{\sqrt{2}}$, $\frac{d^2y}{dx^2} : 0$ (From (2))

$\therefore$ For x1 = $\frac{1}{\sqrt{2}}$, there is point of inflexion on the curve and its coordinates are

$\frac{1}{\sqrt{2}}, e^{\frac{-1}{2}}$ Hence $\left(\frac{1}{\sqrt{2}}e^{\frac{-1}{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}e^{\frac{-1}{2}}\right)$ are two points inflexion for y = $e^{-x^2}$

Since $\frac{d^2y}{dn^2 > 0}$ for x > $\frac{1}{\sqrt{2}}$ or x < $\frac{1}{\sqrt{2}}$ of curve is convex downwards x <- $\frac{1}{\sqrt{2}}$, or x > $\frac{1}{\sqrt{2}}$

$\Rightarrow$ curve is concave for |x| > $\frac{1}{\sqrt{2}}$

Again $\frac{d^2y}{dx^2} < 0$ for - $\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ of course is convex upwards in $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ or curve is convex in $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

### 10.5 Working Method for Concavity and Convexity

Let y = $f$(x) be a curve in [a, b]

(i) Evaluate $\frac{d^2y}{dx^2}$

(ii) Find the interval (a, b) such that $\frac{d^2y}{dx^2} > 0$

Then (a, b) is the interval for y = $f$(x) of being concaves downwards (concave upwards)

(iii) Find the interval (a, b) for which $\frac{d^2y}{dx^2} < 0$.

Then (a, b) is the interval of being convex upward (concave downwards)

## 10.6 Working Method for Locating Points of Inflexion

(i) Evaluate $\frac{d^2y}{dx^2}$

(ii) Find the values of x for which $\frac{d^2y}{dx^2} = 0$, and those valves of x (if any) for which $\frac{d^2y}{dx^2}$ does not exist. x = a, b, c may be possible point of inflexion.

(iii) If x = a is a point of inflexion then either

(1) $\frac{d^2y}{dx^2}$ changes sign at x = a

(2) $\frac{d^3y}{dx^3}$ exists and is non-zero at x = a

**Note:** 1. $\frac{d^2y}{dx^2} = 0$ is not a sufficient condition for the graph of $f$ to have point of inflexion.

2. If $f^n(c) \neq 0$ at a point c, n even, then x = c is not a point f inflexion.

3. If $f^n(c) = 0$ at a point c, n even and $f$n+1(c) $\neq 0$ then the curve has a point of inflexion at x = c.

**Some Illustrated Examples**

**Example 3:** Examine for the function $y = 2 - x^2$, concavity upwards, concavity downwards and the points of inflexion.

**Solution:** We have $y = 2 - x^2$

$\Rightarrow$ $\frac{dy}{dx} = -2x$ and $\frac{d^2y}{dx^2} = -2$, $\frac{d^3y}{dx^3} = 0$.

Now $\frac{d^2y}{dx^2} = -2 < 0 \quad \forall x \in R$

$\Rightarrow$ the given curve is concave downwards $\forall x \in R$

Again $\frac{d^2y}{dx^2} > 0$ is not possible for any real x.

$\therefore$ the given curve is not concave upwards for any real x.

Also $\frac{d^2y}{dx^2} \neq 0$ for any $x \in R$

$\therefore$ the given curve has no point of inflexion.

**Example 4:** Examine the concavity and convexity of the curve $y = x + \frac{4}{x}$. Also find point of inflexion if any.

**Solution:** The given curve is

$$y = x + \frac{4}{x}$$

$\Rightarrow$ $\frac{dy}{dx} = 1 - \frac{4}{x^2}$ and $\frac{d^2y}{dx^2} = \frac{8}{x^3}$

Now $\frac{d^2y}{dx^2} > 0 \Rightarrow \frac{8}{x^3} > 0$ or $x^3 > 0 \Rightarrow x > 0$

$\therefore$ curve is concave upwards in $(0, \infty)$

Again $\frac{d^2y}{dx^2} < 0 \Rightarrow \frac{8}{x^3} < 0$ or $x^3 < 0$ or $x < 0$

$\therefore$ Curve is concave downwards in $(-\infty, 0)$

Now, domain of the given function is the set of all real values except zero.

$\therefore$ $\frac{d^2y}{dx^2} \neq 0$ for any real x,

also $\frac{d^2y}{dx^2}$ does not exist at x = 0 and 0 does not belong to domain of the function.

$\therefore$ given function has no point of inflexion.

**Example 5:** Examine the concavity and convexity of the curve y = sin x in $(0, 2\pi)$. Also locate point of inflexion.

**Solution:** Given function is

$$y = \sin x, x \in (0, 2\pi)$$

$\therefore$ $\frac{dy}{dx} = \cos x$, $\frac{d^2y}{dx^2} = -\sin x$, $\frac{d^3y}{dx^3} = -\cos x$

Now $\frac{d^2y}{dx^2}$ = - sin x > 0 when π < x < 2π

∴ given curve is concave upwards in (π, 2π)

Further $\frac{d^2y}{dx^2} < 0$ when 0 < x < π

∴ given curve is concave downwards in (0, π)

Now $\frac{d^2y}{dx^2} = 0$ ⇒ - sin x = 0 or sin x = 0

∴ x = n π, n ∈ z

Now $\left(\frac{d^3y}{dx^3}\right)_{x=n\pi}$ = - cos n π ≠ 0

∴ Points of inflexion are x = nπ, n ∈ z.

**Example 6:** Determine the point of inflexion on the curve

x = a(2θ - sin θ), y = a (2 - cos θ)

**Solution:** We here have

x = a(2θ - sin θ), y = a (2 - cos θ)

∴ $\frac{dy}{d\theta}$ = a sin θ

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(2-\cos\theta)} = \frac{\sin\theta}{(2-\cos\theta)}$$

$$\text{Also} \quad \frac{d^2y}{dx^2} = \frac{(2-\cos\theta)-\cos\theta-\sin\theta(\sin\theta}{(2-\cos\theta)^2} \cdot \frac{d\theta}{dx}$$

$$= \frac{2\cos\theta-1}{(2-\cos\theta)^2} \quad \frac{1}{a(2-\cos\theta)} = \frac{2\cos\theta-1}{a(2-\cos\theta)^3}$$ (Q sin - cos² θ = 1)

Now $\frac{d^2y}{dx^2}$ = 0 iff 2 cos θ - 1 = 0

∴ iff cos θ = $\frac{1}{2}$ = cos $\frac{\pi}{3}$

iff θ = 2 n π ± $\frac{\pi}{3}$, n ∈ z.

Now $\quad \dfrac{d^2y}{dx^2} < 0$ if $\cos\theta < \dfrac{1}{2}$ and $\dfrac{d^2y}{dx^2} > 0$ if $\cos\theta > \dfrac{1}{2}$

$\therefore \quad \dfrac{d^2y}{dx^2}$ changes sign at points where $\dfrac{d^2y}{dx^2} = 0$

$\Rightarrow \quad$ curve has points of inflexion where $\theta = 2n\pi \pm \dfrac{\pi}{3}$

Then $\quad x = a\left[4n\pi \pm \dfrac{2\pi}{3}\,\mathrm{m}\dfrac{\sqrt{3}}{2}\right]$

$$y = \frac{3a}{2}$$

$\therefore \quad$ points of inflexion are $\left[a\left\langle 4n\pi \pm \dfrac{2\pi}{3}\,\mathrm{m}\dfrac{\sqrt{3}}{2}\right\rangle, \dfrac{3a}{2}\right], n \in \mathrm{z}.$

**Example 7:** Prove that the origin is a point of inflexion on the curve $a^{m-1}\, y = x^m$, if m is odd and greater than 2.

**Solution:** The given curve is

$$a^{m-1} = x^m$$

$$\Rightarrow \quad y = \frac{x^m}{a^{m-1}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{m}{a^{m-1}}\, x^{m-1}, \quad \frac{d^2y}{dx^2} = \frac{m(m-1)}{a^{m-1}}\, x^{m-2}$$

$$\frac{d^3y}{dx^3} = \frac{m(m-1)(m-2)}{a^{m-1}}\, x^{m-3} \ldots\ldots\ldots\ldots\ldots\ldots \frac{d^my}{dx^m} = \frac{m!}{a^{m-1}}$$

Now $\quad \dfrac{d^2y}{dx^2} = 0 \quad \Rightarrow \quad \dfrac{m(m-1)}{a^{m-1}}\, x^{m-2} = 0$

$\therefore \quad x^{m-2} \quad \Rightarrow \quad x = 0$ if $m > 2$

and $\quad$ at $x = 0$, $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = \ldots\ldots\ldots\ldots \dfrac{d^{m-1}y}{dx^{m-1}} = 0$ and $\dfrac{d^my}{dx^m} \neq 0$

Hence origin is a point of inflexion of m is odd and greater than 2.

**Example 8:** Show that the curve

$x = \text{Log}\left(\frac{y}{x}\right)$,

has a point of inflexion at $(-2, -2e^{-2})$.

**Solution:** Here $x = \text{Log}\left(\frac{y}{x}\right)$

$\Rightarrow$ $e^x = \frac{y}{x}$ or $y = e^x$

$\therefore$ $\frac{d^2y}{dx^2} = x\,e^x + e^x.1 = e^x.\,(x + 1)$

and $\frac{d^2y}{dx^2} = e^x.1 + (x + 1)\,e^x = e^x\,(x + 2)$

and $\frac{d^3y}{dx^3} = e^x.1 + (x + 2)\,e^x = e^x\,(x + 3)$

Now $\frac{d^2y}{dx^2} = 0$ $\Rightarrow$ $e^x\,(x + 2) = 0$

$\Rightarrow$ $x + 2 = 0$ or $x = -2$

Now

When $x = -2$, $\frac{d^3y}{dx^3} = e^{-2}\,(-2 + 3) = e^{-2} \neq 0$

Also at $x = -2$ $y = -2\,e^{-2}$

Hence $(-2, -2\,e^{-2})$ is the point of inflex) on

## 10.7 Self Check Exercise

Q.1 Show that the curve $y = e^x$ is concave upwards for all real values of x

Q.2 Determine the point of inflexion on the graph of the function $y = x^4$

Q.3 Examine the concavity upwards and concavity down words for the curve $y = x^3 - 9x^2 + 10x + 5$

Q.4 Find the points of inflexion on the curve $xy = a^2 \log \frac{y}{x}$

## 10.8 Summary

Dear students, in this unit we have learnt

(i) concavity and convexity of a curve

(ii) concavity upwards and concavity downwards of a curve

(iii) the point of inflexion on the curve

(iv) working method for finding point of inflexion on a curve.

## 10.9 Glossary

(i) M a point of inflexion the curve changes from concave upwords to convex downwords and vice-versa, so at a point of inflexion $f''(x) = 0$

(ii) A point that separates the convex part of the curve from the concave part of the curve is called a point of inflexion.

## 10.10 Answers to Self Check Exercises

Ans. 1 find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and then proceed.

Ans. 2 0 is not a point of inflexion

Ans. 3 concave upward for (3, ∞) and convave downwords for (-∞, 3)

Ans. 4 $\left(\frac{3}{2}ae^{-3/2}, ae^{3/2}\right)$

## 10.11 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002


## 10.12 Terminal Questions

1. Prove that the curve y = log x is everywhere concave downwords for x > 0
2. Show that the origin is the point of inflexion of the curve y = $x^{1/3}$
3. Find the values of x for which the curve y = $\frac{x^2+1}{x^2-1}$, is concave upwords and concave downwords. Also find the points of inflexion.
4. For the curve y = ($x^2$ + 4x + 5) e-x, find the intervals of concave upwords or concave downwords. Also locate the points of inflexion.
5. Prove that the curve y = $\frac{a^2x}{a^2+x^2}$, has three points of inflexion. Also find the point.

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## Unit - 11

# Curvature

**Structure**



### 11.1 Introduction

Dear students, we have already introduced the concept of concavity, convexity and the point of inflexion of a curve in our previous unit. In this unit, we shall introduce the concept of curvature, radius of curvature and centre of curvature etc. The concept of curvature in geometry intuitively measure the amount by which a curve deviates from being a straight line or by which a surface deviates from being a plane. In 14th century philosopher and mathematician Nicole Oresme introduced the concept of curvature as a measure of departure from straightness. For circles he has the curvature as being inversely proportional to the radius and attempted to extend this idea to other curves as a continuously varying magnitude.

### 11.2 Learning Objectives

The main objectives of this unit are

(i) to define curvature

(ii) to study circle, centre and chord of curvature

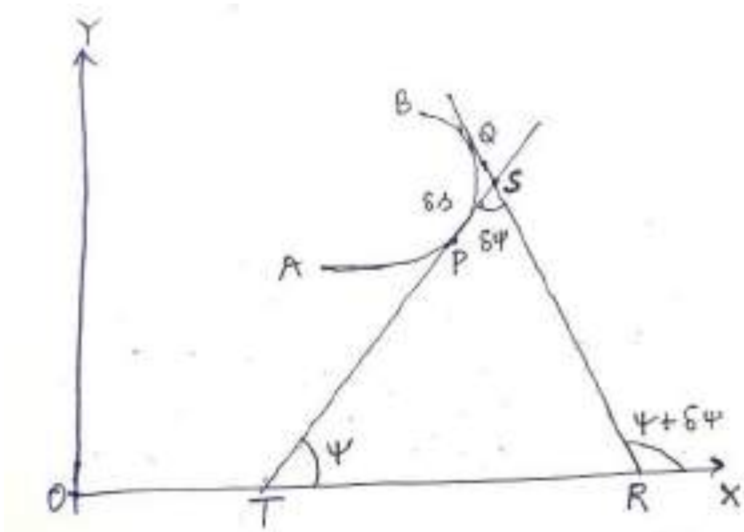
(iii) to study radius of curvature at the origin

(iv) to find radius of curvature at the origin by the method of expansion and Newton method.

(v) to find centre of curvature and rule to find centre of curvature etc.

## 11.3 Curvature

**Definition:** Let AB be a curve and P, Q be any two neighboring points on it such that are AP = s and are AQ = s+ δ s so that are PQ = δ s. (as shown in fig.)




Let us draw tangents to the curve AB at the points P and Q respectively. Let these tangents at P and Q make angles ψ and ψ + δ ψ with the x - axis so that <RST = δ ψ. Then,

(I) δ ψ (measured in radians) is called the total curvature or total bending of the arc PQ.

(II) the ratio $\frac{\delta\psi}{\delta s}$ is called the average curvature of the arc PQ.

(III) $\underset{\delta s \to 0}{Lim} \frac{\delta\psi}{\delta s}$, if exists, is called curvature of the curve at p and is denoted by K.

(IV) the reciprocal of curvature at any point P is called the Radius of curvature and is denoted by P.

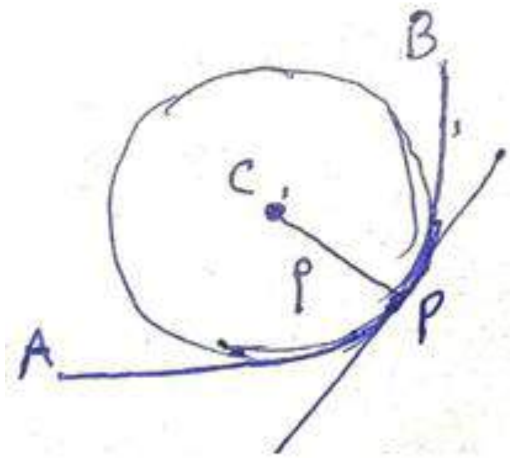
$$\therefore \quad \int = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}$$

**Note:** (1) The curvature at P is the limiting value, when it exists, of the average curvature when Q → P (from either side) along the curve as a limiting position.

(2) The curvature is the rate of change of direction of the curve with respect to the arc at that point

(3) The curvature of the curve is independent of the coordinate system.

(4) Circle, Centre and chord of curvature - The centre of curvature of a curve at a point P is the point C which lies on the positive direction of normal at P and which is at a distance P from it.



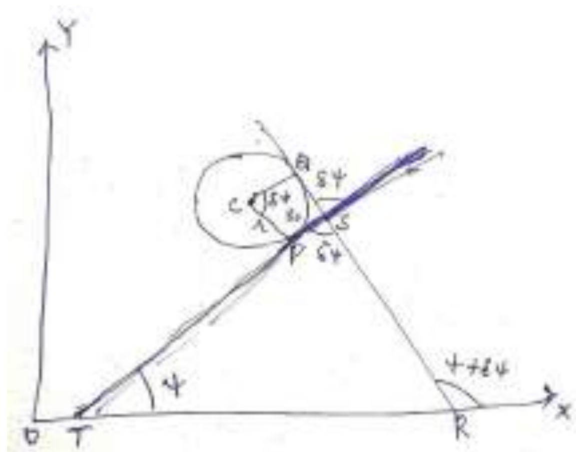

The circle with centre C and radius CP = P is called circle of curvature of the curve at P.

Any chord of circle of curvature at P passing though P is called chord of curvature through P.

**Art. 1.** Show that the curvature of a circle is constant and is equal to reciprocal of the radius

**Proof:** Let us consider a circle with centre C and radius r. Let P, Q be any two points on it such that are PQ = $\delta$ s. Let $\psi$ and $\psi + \delta\psi$ be the angels which the tangents PT and QR make with x-axis so the <RST = $\delta\psi$.

$\therefore$ <PCQ = $\delta\psi$




Now <PCQ = $\dfrac{arcPQ}{CP}$

$\therefore \qquad \delta\psi = \dfrac{\delta s}{r}$

$\Rightarrow \qquad \dfrac{\delta\psi}{\delta s} = \dfrac{1}{r}$

$\therefore \qquad \lim\limits_{\delta s \to 0} \dfrac{\delta\psi}{\delta s} = \dfrac{1}{r}$

$\Rightarrow \qquad \dfrac{d\psi}{ds} = \dfrac{1}{r}$

$\Rightarrow$ curvature at any point P of a circle is the reciprocal of its radius and so is constant

We note here that

Radius of curvature = radius of circle.

## 11.4 Radius of curvature for Cartesian curve

**Art.2.** Obtain the formula for radius of curvature of the curve y = $f$(x) (Cartesian explicit equation)

**Proof:** We have the equation of curve as

y = $f$(x)

From our knowledge of tangents and normal, we have

$$\frac{dy}{dx} = \tan\psi \qquad .....(2)$$

$\Rightarrow \qquad \dfrac{d^2y}{dx^2} = \dfrac{dx}{ds} = \sec^2\psi\,\dfrac{d\psi}{ds}$

or $\qquad \sec^2\psi\,\dfrac{d\psi}{ds} = \dfrac{d^2y}{dx^2}\cos\psi \qquad \left(\because\ \dfrac{dx}{ds} = \cos\psi\right)$

$\therefore \qquad \dfrac{ds}{d\psi} = \dfrac{\sec^3\psi}{d^2y/dx^2}$

$\Rightarrow \qquad \dfrac{ds}{d\psi} = \dfrac{\left(1+\tan^2\psi\right)^{3/2}}{d^2y/dx^2}$

$$\therefore \qquad \int = \frac{\left[1+\frac{dy^{2}}{dx}\right]^{3/2}}{d^{2}\,y/dx^{2}} \qquad \text{(Q of (2))}$$

$$\Rightarrow \qquad \int = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}, \text{ where } y_1 = \frac{dy}{dx},\ y_2 = \frac{d^2y}{dx^2}$$

**Remark:** We know that curvature of the curve at why point is independent of the coordinate system. Therefore interchanging x and y, we have

$$\int = \frac{\left(1+x_1^2\right)^{3/2}}{x_2},\ x_1 = \frac{dx}{dy},\ x_2 = \frac{d^2x}{dy^2}$$

**Art.3 Parametric Equation :** Find the radius of curvature at any point of the curve $x = f(t)$, $y = g(t)$

**Proof :** Let's Find the radius of curvature at any point of the curve

$$x = f(t),\ y = g(t)$$

$$\therefore \qquad \frac{dx}{dt} = f'(t),\ \frac{dy}{dt} = g'(t)$$

$$\text{Now} \qquad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = \frac{f'(t)g''(t)-g'(t)f''(t)}{[f'(t)]^3} \qquad \left(\text{Q } \frac{dt}{dx} = \frac{1}{f'(t)}\right)$$

$$\text{Also} \qquad \int \frac{\left[1+\left(dy/dx\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$\Rightarrow \qquad \int \frac{\left[1+\left(\frac{g'(t)}{f'(t)}\right)^2\right][f'(t)]^3}{f'(t)g''(t)-g'(t)f''(t)}$$

$$\Rightarrow \qquad \int \frac{\left[f'(t)\right]^2+\left[g'(t)\right]^2}{f'(t)g''(t)-g'(t)f''(t)}$$

**Art 4.** Obtain the formula for finding the radius of curvature when x and y are function of s.

**Proof:** Let equation of curve be x = f(s), y = g(s) we know that

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi$$

$$\Rightarrow \quad \left.\begin{aligned} \frac{d^2x}{ds^2} &= -\sin\psi.\frac{d\psi}{ds} \\ &= -\frac{\sin\psi}{\int} \end{aligned}\right\} \quad \frac{d^2y}{ds^2} = \cos\psi.\ \frac{d\psi}{ds}$$

$$= \frac{\cos\psi}{\int}$$

$$\therefore \quad \left(\frac{d^2x}{ds^2}\right) + \left(\frac{d^2y}{ds^2}\right) = \frac{1}{\int^2}(\sin^2\psi + \cos^2\psi)$$

$$\Rightarrow \quad \frac{1}{\int^2} = \left(\frac{d^2x}{ds^2}\right) + \left(\frac{d^2y}{ds^2}\right)$$

Cor. $$\int = \frac{\sin\psi}{\frac{d^2x}{ds^2}} = -\frac{dy/ds}{d^2x/ds^2}$$

also $$\int \frac{\cos\psi}{\frac{d^2y}{ds^2}} = \frac{dx/ds}{d^2y/ds^2}$$

**Art. 5. Radius of curvature for implicit Function**

**Proof:** Let $f$(x, y) = 0 be an implicit equation of the curve

$\therefore$ for implicit equation $f$(x y) = 0, we have

$$\int \qquad \frac{\left[(fx)^2 + (fy)^2\right]^{3/2}}{f_{xx}(fy)^2 - 2f_{xy}f_x f_y + f_{yy}(f_x)^2}$$

where $f_x = \frac{\partial f}{\partial x}$

$f_y = \frac{\partial f}{\partial y}$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

and $$f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$ respectively.

**Art.6. Radius of curvature for polar curves.**

Obtain the radius of curvature at any point p(r, $\theta$) of the curve r = $f(\theta)$.

**Proof:** Let p(r, $\theta$) be any point on the curve r = $f(\theta)$.

It $\psi$ is the angel which the tangent at p makes with the positive direction of 0x and let $\psi$ be the angle between tangent and radius vector, then

$$\psi = \theta + \phi$$

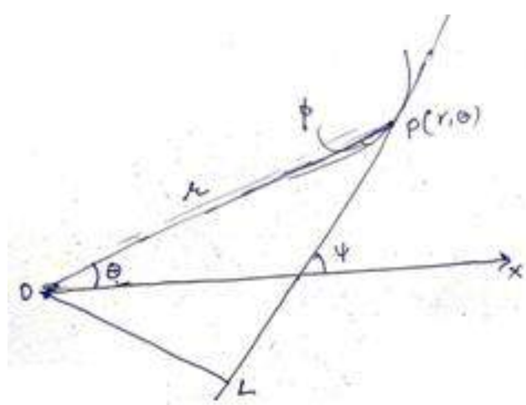
$$\Rightarrow \quad \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\phi}{ds}\left[1 + \frac{d\phi}{ds}, \frac{ds}{d\phi}\right]$$

$$= \frac{d\phi}{ds}\left[1 + \frac{d\phi}{d\theta}\right] \qquad ......(1)$$

From our knowledge of differential calculus, we have

$$\tan \phi = r\,\frac{d\phi}{dr}$$



$\Rightarrow \qquad \tan\phi = r.\dfrac{1}{dr/d\theta} = \dfrac{r}{r_1}$, r, = $\dfrac{dr}{d\theta}$

$\therefore \qquad \phi = \tan^{-1}\dfrac{r}{r_1}$

$$\Rightarrow \qquad \frac{d\phi}{d\theta} = \frac{1}{1+r^2/r_1^2}\cdot\frac{d}{d\theta}\left(\frac{r}{r_1}\right)$$

$$= \frac{r_1^2}{r_1^2+r_1^2}\cdot\frac{r_1.r_1-\gamma\gamma_2}{\gamma_1^2}$$

$$= \frac{r_1^2-\gamma\gamma_2}{r^2+r_1^2}$$

Also $\qquad \dfrac{ds}{d\theta} = \sqrt{r^2+r_1^2}$

$$\frac{1}{\int} = \frac{1}{\sqrt{r^2+r_1^2}}\left[1+\frac{r_1^2-r_2}{r^2+r_1^2}\right] \qquad \text{(using (1))}$$

Or $\qquad \int = \dfrac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-r_1r_2}$

**Some Illustrated Examples**

**Example 1:** find the radius of curvature at any point of the curve $xy = c^2$

**Solution:** We have

$$xy = c^2$$

or $\qquad y = \dfrac{c^2}{x}$

$\therefore \qquad y_1 = -\dfrac{c^2}{x^2}$, $y_2 = \dfrac{2c^2}{x^3}$

Now $\qquad \int = \dfrac{\left\{1+(y_1)^2\right\}^{3/2}}{y_2} = \dfrac{\left\{1+c^4/x^4\right\}^{3/2}}{\dfrac{2c^2}{x^3}}$

$$= \frac{\left(x^4 + c^4\right)^{3/2}}{2c^2x^3}$$

**Example 2:** Find the radius of curvature at the point (s, $\psi$) of the curve

$$s = a \sec \psi \tan \psi + a \log (\sec \psi + \tan \psi)$$

**Solution:** The given equation of curve is

$$s = a \sec \psi \tan \psi + a \log (\sec ++ \tan \psi)$$

$$\therefore \qquad \frac{ds}{d\psi} = a (\sec \psi \sec^2 \psi + \tan \psi \sec \psi \tan \psi)$$

$$+ \frac{a}{(\sec + \tan)} \sec \psi (\sec \psi + \tan \psi)$$

$$= a \sec \psi [(1 + \tan^2 \psi) + \sec^2 \psi)$$

$$= a \sec \psi (2 \sec^2 \psi)$$

$$= 2a \sec^3 \psi$$

$$\therefore \qquad \int = \frac{ds}{d\psi} = 2a \sec^3 \psi$$

**11.5 Self Check Exercise - 1**

Q.1 Find the radius of curvature at the point (x, y) on the curve $y = a \log \sec \left(\frac{x}{a}\right)$

Q.2 Find the radius of curvature at any point of the curve $ay^2 = x^3$

Q.3 Find the radius of curvature at any point (s, $\psi$) on the curve $s = c \tan \psi$.

**Art.7. Radius of curvature at the origin**

In this article we give two direct method to find the radius of curvature at the origin to the curve $y = f(x)$ i.e. at x = 0, y = 0 to the curve $y = f(x)$

**Method 1. (Method of Expansion by Maclaurin)**

We have

$$y = f(0) + x f(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots\ldots$$

Since the curve passes through origin

$$\therefore \qquad f(0) = 0$$

$\Rightarrow \qquad y = x\, f'(0) + \dfrac{x^2}{2!} f''(0) + \ldots\ldots\ldots\ldots$

$\Rightarrow \qquad y = px + q\, \dfrac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots,$

where $p = f'(0) = \left(\dfrac{dy}{dx}\right)_{(0,0)}$ and $q = f''(0) = \left(\dfrac{d^2y}{dx_2}\right)_{(0,0)}$

$\therefore \qquad \int$ (at the origin) $= \dfrac{\{1+p^2\}^{3/2}}{q}$

**Note:** We give rule to find radius of curvature at the origin

(i) Put $y = px + q\, \dfrac{x^2}{2!} + \ldots\ldots\ldots$ in equation of curve

(ii) Equate the coefficient of like powers of x on both side and find p, q.

(iii) $\int$ (at the origin) $= \dfrac{\{1+p^2\}^{3/2}}{q}$

**Method 2. (Newton's Method)**

If the curve passes through origin and the axis of x is a tangent to the curve at origin, then we have at the origin

$x = 0,\ y = 0,\ \dfrac{dy}{dx} = 0$ i.e. $p = 0$

$\therefore$ by Maclaurin's expansion, we get

$y = 0 + 0x + q\, \dfrac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots$

or $\dfrac{2y}{x^2} = q +$ term of x

$\therefore \qquad \lim\limits_{x\to 0} \dfrac{2y}{x^2} = q$ (Taking limit $x \to 0 \therefore y \to 0$)

Also $\int = \dfrac{\{1+p^2\}^{3/2}}{q} = \dfrac{\{1+0\}^{3/2}}{q} = \dfrac{1}{q}$

$\therefore \qquad \int = \underset{x \to 0}{Lim} \frac{x^2}{2y}$

Like wise, if the curve passes through the origin and the axis of y is the tangent, then the value of $\int$ at the origin is

$$\int = \underset{x \to 0}{Lim} \frac{y^2}{2x}$$

**Note:** We give rule to find radius of curvature at the origin

(i) obtain the equation of tangent (s) at the origin

(ii) then $\int$ (at origin) = $\underset{x \to 0}{Lim} \frac{x^2}{2y}$ if x - axis is tangent at the origin

and $\int$ (at origin) = $\underset{x \to 0}{Lim} \frac{y^2}{2x}$ if y-axis is the tangent at origin

(iii) if neither x-axis nor y-axis is tangent at the origin then put

$y = px + q \frac{x^2}{2!} + .........$

and proceed as above.

**Some Illustrated Examples**

**Example 3:** Find the radit of curvature at the origin to the curve

$a (y^2 - x^2) = x^3$

**Solution:** We have equation of curve as

$a (y^2 - x^2) = x^3$

Equating to zero the lowest degree terms,

$y^2 - x^2 = 0 \qquad \text{or} \qquad (y - x)(y + x) = 0$

$\Rightarrow \qquad y = -x, y = x$

Here neither x-axis nor y-axis is the tangent at the origin,

$\therefore$ putting $y = px + q \frac{x^2}{2!} + .............$ in (1) we get

$$a\left[\left(px + q\frac{x^2}{2} + ......\right)^2 - x^2\right] = x^3$$

or

a$\left[(p^2-1)x^2+pqx^3+........\right]$= x³ = ......(2)

Equating coefficient of $x^2$ in (2), we get

$a(p^2-1) = 0 \quad \Rightarrow \quad p = -1, 1$

and on equating coefficient of $x^3$ in (2), we get

apq = 1

when p = 1, $\quad q = \frac{1}{a}$

and when p = -1 $\quad q = \frac{-1}{a}$

$$\therefore \qquad \int = \frac{\{1+p^2\}^{3/2}}{q} = \frac{\{1+1\}^{3/2}}{\frac{-1}{a}} = 2\sqrt{2a} \quad \text{(in magnitude)}$$

when p = 1, $q = \frac{1}{a}$

$$\int = \frac{\{1+p^2\}^{3/2}}{q} = \frac{\{1+1\}^{3/2}}{\frac{1}{a}} = 2\sqrt{2a}$$

**Example 4:** Apply Newton's method to find radius of curvature at the origin for the cycloid

$x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

**Solution:** We have

$x = a(\theta + \sin\theta)$

$y = a(1 - \cos\theta)$

$$\therefore \qquad \frac{dx}{d\theta} = a(1+\cos\theta) = 2\cos^2\frac{\theta}{2}$$

$$\text{and} \qquad \frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \tan\frac{\theta}{2}$$

Now when $\theta = 0 \Rightarrow x = 0$ $y = 0$, $\frac{dy}{dx} = 0$

$\therefore$ curve posses though the origin and x-axis is the tangent at the origin

$\therefore$ $\int$ (at origin) = $\underset{\substack{x\to 0\\ y\to 0}}{Lim} \frac{x^2}{2y} = \underset{\theta\to 0}{Lim} \frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)}$

$= \frac{a}{2} \underset{\theta\to 0}{Lim} \frac{2(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta}$

$= a \frac{0+(1+1)(1+1)}{1}$

$= 4a$

**Self Check Exercise - 2**

Q.1 Find the radii of curvature at the origin of the curve

$y^2 - 3xy + 2x^2 - x^3 + y^4 = 0$

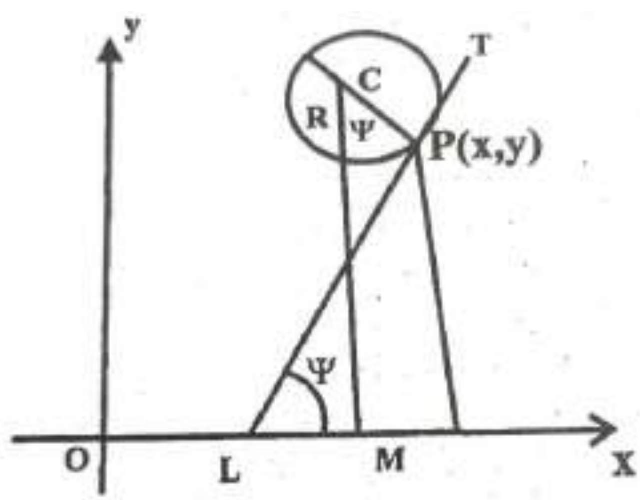
Q.2 Find the radius of curvature of the curve

$3x^2 + 4x^3 - 12y = 0$

by Newton's method

## 11.6 Centre of curvature

If a length PCI to p is measured from p along the positive direction of the normal, then the point c is called the centre of curvature at p, and the circle with centre cp (=p) I called the circle of curvature at p. Any chord of this circle through the point of contact is called chord of curvature.



The locus of the centres 0 curvature of a curve is called the envolute and the given curve is called involute the coordinate of centre of curvature is given by

$$= [x - \frac{y_1(1+y_1^2)}{y^2}, y + \frac{1+y_1^2}{y^2}]$$

and the equation of the circle of curvature is

$$(x - x)^2 + (y - y)^2 = p^2$$

Where $x = x - \frac{y_1(1+y_1^2)}{y^2}$

and $y = \frac{1+y_1^2}{y^2}$

Also the length of chord of curvature is

$\frac{2y1(1+y_1^2)}{y^2}$ (parallel to x - axis)

and $\frac{2(1+y_1^2)}{y^2}$ (parallel to y - axis)

Let us do some examples to understand the above concepts.

**Example 5:** Find the radius of curvature at origin for the curve

$$x^4 - x^4 - 4x^2 y + xy - x^2 + y = 0$$

**Solution:** Tangent at origin are obtained by equating to Zero the lowest term i.e. y = 0 or x-axis

Hence p = Lim

$x \to 0 \left(\frac{x^2}{2y}\right)$

$y \to 0$

Dividing equation by 2y, we get

$$X \frac{2x^2 2y}{2y} - \frac{y^2}{2} - 2x2 + \frac{x}{2} - \frac{x^2}{2y} + \frac{1}{2} = 0$$

$x \to 0$ $y \to 0$ $\therefore \frac{x^2}{2y} \to P$

Let $\therefore$ OP - 0 + 0 + 0 - P + $\frac{1}{2}$ = 0

Of $\Rightarrow$ P = $\frac{1}{2}$

**Example 6:** Apply Netwon method of find the radius of curvature at the origin for the curve

X = a (t + sin t)

Y = a (1 - cos t)

At t = 0: x = o. y = 0

$\therefore$ curve passes through origin.

Now $\frac{dx}{dt}$ = Q(1 + cos t and $\frac{dy}{dt}$ = a sin t

$$\therefore \quad \frac{dy}{dx} = \frac{dx/dt}{dn/dt} = \frac{a\sin t}{a(1+\cos t)} = \frac{\sin t}{1+\cos t}$$

At t = 0, $\frac{dy}{dx} = \frac{\sin 0}{1+\cos 0} = 0$

$\therefore$ x axis is tangent at the origin.

Hence, by Newton's method

p (at origin) = lim

x $\rightarrow$ 0 $\frac{x^2}{2y}$

$$= \underset{x\to 0}{Lim} \quad = \frac{a^2(t+\sin t)}{2a(1-\cos t)}$$

$$= \lim_{t\to 0} \quad = \frac{a(t+\sin t)}{2(1-\cos t)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{t\to 0} \quad = \frac{2a(t+\sin t)(1\cos t)}{2\sin t} \quad \left(\frac{0}{0} \text{ form L' Hospital Rule}\right)$$

$$= \lim_{t\to 0} \quad = \frac{a(t+\sin t)-(-\sin t)+(1+\cos t)(1+\cos t)}{\cos tt}$$

= 4a

**Example 7:** Find centre of curvature at the point (x, y) on the parabola $y^2 = 4ax$

Also find evolute of the Parabola.

**Solution:** We have $y^2 = 4x$

$$Y = 2\sqrt{a}\ \sqrt{x}$$

$\therefore$ $Y_1 = \dfrac{\sqrt{a}}{\sqrt{x}}$ and $y_2 = -\dfrac{1}{2}\ \dfrac{\sqrt{a}}{x3/2}$

Let ($\alpha$, $\beta$) be center of curvature

$\therefore$ $\alpha = x = \dfrac{y_1(1+y1^2)}{y^2}$

$$\left[\frac{\frac{\sqrt{a}}{\sqrt{x}}(1+\frac{a}{x})}{-\frac{1}{2}\frac{\sqrt{a}}{\frac{3}{x^2}}}\right]$$

Or $\propto$= x = 2 (x + a) = 3x + 2a (2)

Similarly

$$\beta\ y + \frac{1+y_1^2}{y^2}$$

$$= Y = \frac{1+\frac{a}{x}}{\frac{1\sqrt{a}}{2x^{\frac{3}{2}}}}$$

$$= Y = (2x)\left(\frac{\sqrt{x}}{a}\right)\left(1+\frac{a}{x}\right)$$

$$= 2\sqrt{a\sqrt{x}} - 2\ \frac{\sqrt{x}}{a}(x - a)$$

$$= \frac{2\sqrt{a\sqrt{x}}-2\sqrt{x}(x+a)}{\sqrt{a}}$$

$$= \frac{2\sqrt{x}^{\frac{3}{2}}}{\sqrt{a}} \qquad .....(3)$$

$(\alpha, \beta) = (3x + 2a, - \dfrac{2\sqrt{x}^{\frac{3}{2}}}{\sqrt{a}})$

Do find evolute

We have from (2)

$$X = \frac{\propto -2a}{3} \qquad ...... (4)$$

and from (3)

$$\beta^2 = \frac{4x^3}{a} = \frac{4}{a}\left(\frac{\alpha - 2\alpha}{3}\right)^3$$

$\Rightarrow$ $27\, a\beta^2 = 4\,(\alpha - 2a)^3$

$\therefore$ locus of $(\alpha, \beta)$ is

$27\, ay^2 = 4(x - 2a)^2$

which is required evolute.

## 11.7 Self Check Exercise

Q.1 Find the centre of curvature at any point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Ans. find its evolute

Q.2 Find the centre of curvature at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Also find its evolute.

## 11.8 Summary

In this unit we have learnt the followings

(i) curvature

(ii) circle, centre and chord of curvature

(iii) curvature of a circle

(iv) radius of curvature for Cartesian curve, parameter equation and for polar coordinates

(v) radius of curvature at the origin and different method to find it.

## 11.9 Glossary

(i) Evolute - The locus of the centre of curvature of a curve is called its evolute

(ii) Involute - The curve having evolute itself is called involute.

## 11.10 Answers to Self Check Exercises - 1

Ans. 1 a sec $\frac{x}{a}$

Ans. 2 $x^{\frac{1}{2}} \frac{\left(4a+9x\right)^{3/2}}{6a}$

Ans. 3 c $\sec^2 \psi$

**Self Check Exercise - 2**

Ans. 1 $\frac{5\sqrt{5}}{2}$

Ans. 2 $\frac{4}{5}$

**Self Check Exercise - 3**

Ans. 1 $\left\{\frac{\left(a^2-b^2\right)^{x^3}}{a^4}, \frac{y^3\left(b^2-a^2\right)^{x^3}}{b^4}\right\}, = evolute(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}$

Ans. 2 Centre $\left(\frac{a^2-b^2}{a}\sec^3\theta, -\frac{a^2+b^2}{a}\tan^3\theta\right)$

evolute is (a2 + b2) $(a^2+b^2)^{2/3}$ = $(ax)^{2/3}$ - $ax)^{2/3}$

## 11.11 Reference/Suggested Reading

1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Dovis, Calculus, John Wiley and Sons, Inc. 2002

## 11.12 Terminal Questions

1. Prove that the curvature of a straight line is zero
2. Find the least value of |p| for y = log x
3. Find radius of curvature at any point of the curve

   $x = a\cos^3\theta$, $y = a\sin^3\theta$
4. Find radii of curvature at the origin of the curve

   $x = 1 - t^2$, $y = t - t^3$
5. Prove that the evolute of the curve $x^{2/3} + y^{2/3} = a^3$

is 2 $(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$

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## Unit - 12

# Asymptotes

**Structure**



### 12.1 Introduction

Dear students,in this unit we shall study the concept of asymptor. In analytical geomerty, on asymptote of a curve is a line such that the distance between the curve and the line approaches zero as one or both of x or y coordinates tends to infinity. In projectile geometry and related content, an asymptote of a curve is a line which tangent to the curve at a point at infinity. The word asymptote is derived from greek work 'asumptotos', which means not falling together". This term was used by appallonius of Perga in his work on conic section, but in contrast to its modern meaning, he used it to mean any line that does not interest the given curve.

### 12.2 Learning Objectives

The main objectives of this unit are

(i) to define asymptote

(ii) to study rectangular and oblique asymptotes

(iii) to study asymptotes of general algebraic rational curve.

(iv) to study asymptotes by inspection.

(v) to study asymptotes by inspection

(vi) To learn about intersection of a curve and its asymptotes.

## 12.3 Asymptotes

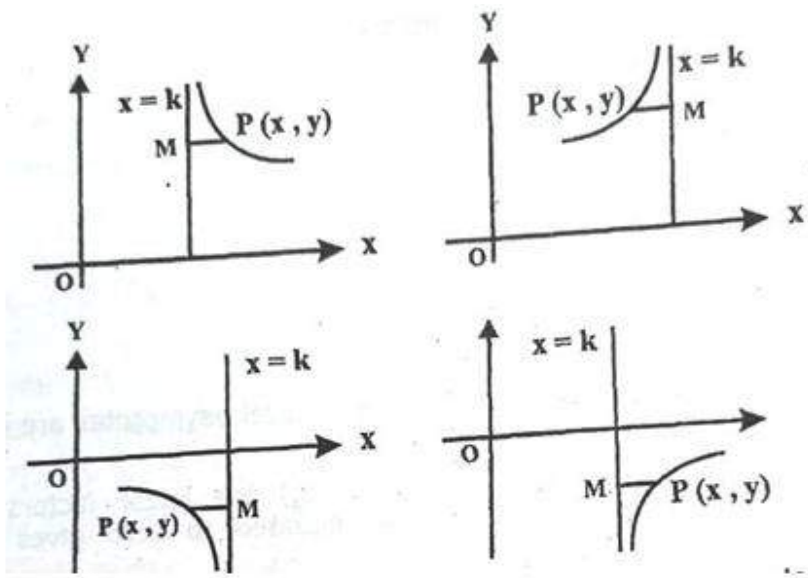
A straight line I is called an asymptote of an infinite branch of a curve iff the perpendicular distance of a point P on that branch from the straight line I tends to zero as P moves to infinity along the branch.

## 12.4 Rectangular Asymptote

If an asymptote to a curve is either parallel to x-axis or parallel to y-axis, then it is called a rectangular asymptote. An asymptote parallel to x-axis is usually called horizontal asymptote and an asymptote parallel to y-axis is called a vertical asymptote.

**Asymptotes Parallel to the Axes**

**(a) Asymptotes parallel to y-axis**




Let x = k b an asymptote of the curve y = f (x) parallel to the y-axis. We determine k. Let P(x, y) be any point on the curve. Draw PM perpendicular on x = k.

$|PM| = |x - k|$

Let P (x, y) move to infinity along the curve.

$\therefore$ y alone tends to + $\infty$ or - $\infty$

$\underset{x \to k}{L\text{im}} = \pm \infty$

equivalently k = $\underset{y \to \pm\infty}{Lim}\, x$

Thus k is determined.

**Case I.** Let the equation of the curve be of the form y = $\frac{N(x)}{D(x)}$ where N(x) and D(x) are polynomials in x without any common factor.

Let $x = k_1, k_2, \ldots\ldots\ldots k_r$ be the roots of D(x) = 0

Now → as $x \to k_1$, $D(x) \to 0$

$\Rightarrow y \to \infty$ as $x \to k_1$

$\therefore$ $x = k_1$ is a vertical asymptote of the curve $y = f(x)$

Similarly $x = k_2$, $x = k_3$, ..... are the vertical asymptotes of the curve y = $\frac{N(x)}{D(x)}$

By factor theorem, $(x - k_1)$, $(x - k_2)$, .... , $(x - k_n)$ are the linear factors of D(x).

Thus, linear factors of D(x) equated to zero determine the vertical asymptotes of the curve.

Case II. If the equation of the given curve cannot be put in the form y = $\frac{N(x)}{D(x)}$, then we arrange the equation of the curve in the descending powers of y so that it is

$$y^n \phi(x) + y^{n-1} \phi_1(x) + \ldots\ldots \phi_n(x) = 0 \qquad \ldots(1)$$

When $\phi(x)$, $\phi_1(x)$, $\phi_2(x)$, .... are polynomials in x.

Dividing (1) by $y^n$, we get,

$$\phi(x) + \frac{1}{y}\phi_1(x) + \frac{1}{y^2}\phi_2(x) + \ldots\ldots + \frac{1}{y^n}\phi_n(x) = 0 \qquad \ldots(2)$$

We already know, Lt $\underset{y \to \pm\infty}{Lim}\, x$ = k

proceeding to the limits, (2) gives $\phi(x) = 0$

which shows that k is a root of $\phi(x) = 0$

Let $k_1, k_2, \ldots\ldots k_n$, be the roots of $\phi(x) = 0$, then vertical asymptotes are given by $x = k_1$, $x = k_2$, ... $x = k_n$

By factor $(x - k_1)$, $(x - k_2)$, .... , $(x - k_n)$ the linear factors of the coefficient of the highest degree term in y, equated to zero gives vertical asymptotes.

(b) Asymptotes parallel to x-axis

As above, for asymptotes parallel to x-axis.... if equation of curve can be put in the form x = $\frac{N(y)}{D(y)}$ [Where N(y) and D(y) have no common factor] then the linear factors of D(y) equated to zero give the asymptotes parallel to x-axis.

In case, the equation cannot be put in the above form, then we arrange it in descending powers of x. Then, the linear factors of the coefficient of the highest power of x equated to zero give the asymptotes parallel to the x-axis.

**Note 1. Rule to find asymptotes parallel to x-axis.**

Equate to zero the final linear factors in the coefficient of highest power of x in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of x in the equation of the given curve is a constant or has not real linear factor, then the curve has no asymptote parallel to x-axis.

**Note 2. Rule to find asymptotes parallel to y-axis.**

Equate to zero the real linear factors in the coefficient of highest power of y in the equation of the given curve.

It should be noted properly that if the coefficient of highest power of y in the equation of the given curve is a constant or has no real linear factor, then the curve has no asymptote parallel to y-axis.

**Now, let us do some examples.**

**Example 1 :** Write down, by inspection or otherwise, the vertical and horizontal asymptotes of the curve $3xy + 5x - 4y - 3 = 0$

**Solution :** The equation of the given curve is $3xy + 5x - 4y - 3 = 0$ ....(1)

The coefficient of highest power of x In (1) is $3y + 5$

$3y + 5 = 0$ is the only asymptote parallel to x-axis

$\therefore$ horizontal asymptote of given curve is $3y + 5 = 0$

The coefficient of highest power of y in (1) $3x - 4$

$\therefore$ $3x - 4 = 0$ is the only asymptote parallel to y-axis

Vertical asymptote of the given curve is $3x - 4 = 0$

**Example 2 :** Find the asymptotes parallel to the axes of the curve $x^2y^2 + y^2 = 1$

**Solution :** The equation of the given curve is $x^2y^2 + y^2 = 1$. ...(1)

The coefficient of highest power of x in (1) is $y^2$

$y^2 = 0$ i.e., $y = 0$ is the only asymptote parallel to the x-axis

The coefficient of highest power of y in (1) is $x^2 + 1$. Now $x^2 + 1$ has no real linear factor.

$\therefore$ given curve has asymptote parallel to y-axis.

## 12.5 Oblique Asymptotes

An asymptote which is neither parallel to x-axis nor parallel to y-axis is called oblique asymptote.

Find the condition that $y = mx + c$ be an oblique asymptote of the curve $y = f(x)$.

Let y = mx + c be an oblique asymptote of the curve y = $f$(x) where m and c are finite.

Let us determine m and c.

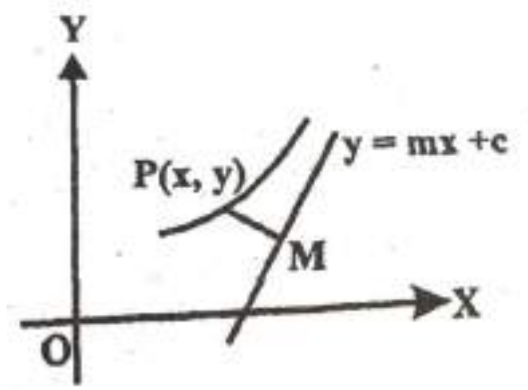
Take a point P(x, y) on the curve

Draw PM on the asymptote

y = mx + c (1)

The |PM| = $\dfrac{|y - mx - c|}{\sqrt{1+m^2}}$

As x → +∞ or x → -∞, $|\overset{1}{P}M| \to 0$




y-mx-c → 0 as x → ± ∞

$\lim\limits_{x\to\infty}$ (y - mx - c) = 0

$\lim\limits_{x\to\infty}$ (y - mx  = C

Again $\quad \underset{x \to \pm\infty}{\text{Lt}} \left[\dfrac{y}{x} - m\right] = \underset{x \to \pm\infty}{\text{Lt}} \dfrac{y - mx}{x} = \underset{x \to \pm\infty}{\text{Lt}} \dfrac{c}{x} = 0$

So if y = mx + c is an oblique asymptote of curve, then

$\underset{x \to \pm\infty}{\text{Lt}} \dfrac{y}{x} = m$ and $\quad \underset{x \to \pm\infty}{\text{Lt}} (y - mx) = c$

Note. Rule to find oblique asymptote

(i) Find $\lim\limits_{x\to\infty} \dfrac{y}{x}$ in the equation of the curve and denote it by m

(ii) Find $\lim\limits_{x\to\infty} \frac{y}{x}$ (y - mx) in the equation of the curve and denote it by c.

## 12.6 Asymptote of the General Rational Algebraic Curve

Let the equation of the curve be

$$x^n \phi_n \left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1}\left(\frac{y}{x}\right) + x^{n-2} \phi_{n-2}\left(\frac{y}{x}\right) + \ldots + x\phi_1\left(\frac{y}{x}\right) + \phi_0\left(\frac{y}{x}\right) = 0 \qquad \ldots(1)$$

Where $\phi n \left(\frac{y}{x}\right)$ represents a polynomial in $\frac{y}{x}$ of degree n.

Dividing, (1) by $x^n$, we get,

$$\phi_n \left(\frac{y}{x}\right) + \frac{1}{x} \phi_{n-1}\left(\frac{y}{x}\right) + \frac{1}{x^2} \phi_{n-2}\left(\frac{y}{x}\right) + \ldots + \frac{1}{x^{n-1}} \phi_1\left(\frac{y}{x}\right) + \frac{1}{x^n} \phi_0\left(\frac{y}{x}\right) = 0$$

Let $x \to \infty$ along the curve, we get,

$$\phi n\,(m) = 0 \qquad \ldots(2)$$

$$\left[ \mathop{\mathrm{Lt}}_{x \to \pm\infty} \frac{y}{x} = m, \text{ (slope of the asymptote)}\right]$$

Roots of (2) determine slopes of various asymptotes of the curve (1).

Let $m_1$ be one of the roots of (2)

Let $y = m_1x + c_1$ be the corresponding asymptote.

Suppose $y - m_1x = p_1$

$\therefore \quad \frac{y}{x} - m_1 = \frac{p_1}{x} \qquad \therefore\ p_1 \to c_1$ as $x \to \infty$

$\therefore \quad \frac{y}{x} - m_1 = \frac{p_1}{x}$ where $\to p_1 \quad c_1$ as $x \to \infty$

Substituting the value of $\frac{y}{x}$ in (1), we get.

$$x^n\phi_n \left[m_1 + \frac{p_1}{x}\right] + x^{n-1} \phi_{n-1} \left[m_1 + \frac{p_1}{x}\right] + x^{n-2} \phi_{n-2}\left[m_1 + \frac{p_1}{x}\right] + \ldots + x\phi_1\left[m_1 + \frac{p_1}{x}\right] + \phi_0\left[m_1 + \frac{p_1}{x}\right] = 0. (3)$$

Expanding each term by Taylor's Theorem, we get

$$x^n\left[\phi_n(m_1) + \frac{p_1}{x}\phi'_n(m_1) + \frac{p_2}{2x^2}\phi''_n(m_1) + \ldots\right]$$

$$+\ x^{n-1}\left[\phi_{n-1}(m_1)+\frac{p1}{x}\phi_{n-1}(m_1)+\frac{p1_2}{2x^2}\phi''_{n-1}(m_1)+.....\right]$$

$$x^{n-2}\left[\phi_{n-2}(m_1)+\frac{p_1}{x}\phi_{n-2}(m_1)+\frac{p1^2}{2x^2}\phi''_{n-2}(m_1)+.....\right] + .... = 0$$

$x^n\phi_n(m_1) + x^{n-1}$ $[p1\ \phi'_n(m_1) + \phi_{n-1}(m_1)]$ +

$$x^{n-2}\left[\frac{p1^2}{x}\phi''_n(m_1)+p_1\phi'_{n-2}(m_1)+\phi_{n-2}(m_1)\right]+.... = 0 \qquad ...(4)$$

Putting $\phi_n(m_1) = 0$ and dividing the remaining terms of (4) by $x^{n-1}$ we get,

$$[p1\ \phi'_n(m_1) + \phi_{n-1}(m_1)] + \frac{1}{x}\left[\frac{p1^2}{x}\phi''_n(m_1)+p_1\phi'_{n-1}(m_1)+\phi_{n-2}(m_1)\right] + .... = 0 \qquad ...(5)$$

Letting $x \to \infty$ and using $P_1 \to C_1$ (5) gives

$$c_1\ \phi'_n(m_1) + \phi_{n-1}(m_1) = 0$$

$$c_1 = \frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)} \qquad \text{(provided } \phi'_n(m_1) \neq 0)$$

Corresponding asymptote is

$$y = m_1x - \frac{\phi_{n-1}(m_1)}{\phi'_n(m_1)}$$

Similarly if $m_2, m_3$ ..... are the roots of (2), we have corresponding asymptotes

$$y = m_2x = \frac{\phi_{n-1}(m_2)}{\phi'_n(m_2)},\ y = m_2x = \frac{\phi_{n-1}(m_3)}{\phi'_n(m_3)}, ....$$

Provided $\phi'_n(m_2) \neq 0$, $\phi'_n(m_3) \neq 0$, ....

**Exceptional Case.** When $\phi'_n(m_1) \neq 0$

Suppose $\phi_{n-1}(m_1) \neq 0$

$c_1\phi'_n(m_1) + \phi_{n-1}(m_1) = 0$ does not determine any finite value of $c_1$. Hence there is no asymptote corresponding to the root $m_1$ of (2).

Now suppose $\phi'_n(m_1) = 0$ and $\phi_{n-1}(m_1) = 0$ becomes an identity.

[using $p_1 \to c_1$]

Letting $x \to \infty$ (4) gives

$$\frac{c1^2}{2}\phi''_n(m_1) + c_1\ \phi'_{n-1}(m_1)+\phi_{n-2}(m_1) = 0$$

Which is quadratic in $c_1$ and determines two values $\overset{1}{C_1}$, $C_2^n$ (say), provided $\phi''_n (m_1) \neq 0$

Then $y = m_1X + c'_1$ and $y = m_1x+c_2^n$ are the two parallel asymptotes corresponding to the slope $m_1$.

**Note 1. Rule to find oblique asymptotes of a rational algebraic curve :**

**Step I.** Find $\phi_n(m)$, $\phi_{n-1}(m)$ by putting x = 1 and y = m in nth degree terms and in the (n-1) th degree terms respectively of the given curve $f(x, y) = 0$.

**Step II.** Find all the real roots of $\phi_n (m) = 0$

**Step III.** If $m_1$ is non-repeated root of $\phi_n(m) = 0$, then the corresponding value of c is given by $c\phi'_n(m) + \phi_{n-1} (m) = 0$, provided $\phi'_n (m_1) \neq 0$.

If $\phi'_n (m_1) = 0$, then there is no asymptote to the curve corresponding to the value $m_1$ of m.

Step IV. If m1 is a repeated root occurring twice, then the corresponding value of c are given by $\frac{c1^2}{2}\phi''_n (m_1) + c_1 \phi'_{n-1}(m_1)+\phi_{n-2} (m_1) = 0$, provided $\phi''_n(m_1) \neq 0$.

In this case there are two parallel asymptotes to the curve.

Similarly we can proceed when $m_1$ is repeated three or more times.

**Note 2.** Rule explained above does not give us vertical asymptotes.

**Note 3.** A rational algebraic curve of degree n cannot have more than n asymptotes.

**Let us do some examples.**

**Example 3 :** Find the asymptotes of the curve y = $\frac{x^2+2x+1}{x}$

**Solution :** Here equation of Curve is y = $\frac{x^2+2x+1}{x}$

or $xy = x^2 + 2x - 1$

or $x^2 - xy + 2x - 1 = 0$

The coefficient of highest power of x is 1, which is constant.

Given curve has no asymptote parallel to x-axis

The coefficient of highest power of y is -x

The asymptote of given curve parallel to y-axis is -x = 0 or x = 0

Let us now find oblique asymptote y = mx + c.

To determine m and c :

$$m = \underset{x\to\pm\infty}{Lt} \frac{y}{x} = \text{Lt} \frac{x^2+2x+1}{x^2} = \underset{x\to\pm\infty}{Lt} \left[1 + \frac{2}{x} - \frac{1}{x^2}\right] = 1$$

$$c = \underset{x\to\pm\infty}{Lt} (y - mx) = \underset{x\to\pm\infty}{Lt} \left[ \frac{x^2+2x+1}{x^2} -1.x\right]$$

$$= \underset{x\to\pm\infty}{Lt} \frac{x^2+2x-1-x^2}{x} - \underset{x\to\pm\infty}{lim} \frac{2x-1}{x} = \underset{x\to\pm\infty}{lim} \left[2 \frac{1}{x}\right] = 2-0 = 2.$$

Therefore oblique asymptote is given by $y = x + 2$

Hence the given curve has two asymptotes given by $x = 0$ and $y = x+2$

**Example 4 :** Find all the asymptotes of the curve

$$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0.$$

**Solution :** Given equation is

$$x^3 + 2x^2y - xy^2 - 2y^3 + 4y^2 + 2xy + y - 1 = 0. \qquad ....(1)$$

(1) is an equation of degree 3 in x and y

Since coefficient of $x^3$ is 1, which is constant

So there is no asymptotes parallel to x-axis

Similarly coefficient of $y^3$ is -2, which is constant

there is no asymptote parallel to y-axis.

For oblique asymptotes, put $y = mx + c$ in (1), we get,

$$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + 4(mx + c)^2 + 2x(mx + c) + (mx + c) - 1 = 0.$$

$$x^3 (1+2m - m^2 - 2m^3) + x^2 (2c - 2mc - 6 m^2c + 4m^2 + 2m) + \times (-c^2 - 6mc^2 - 8mc + 2c + m) + (-2c^3 + 4 c^2 + c - 1) = 0.$$

Equating the coefficient of $x^3$ and $x^2$ to zero, we get,

$$1+ 2m - m^2 - 2m^3 = 0 \qquad ....(2)$$

$$2c - 2mc - 6m^2c + 4m^2 + 2m = 0 \qquad ....(3)$$

From (2), $1(1+2m) - m^2(1+2m) = 0$

$(1+m^2) (1 + 2m) = 0$

$(1 - m) (1 + m) (1 + 2m) = 0$

When $m = 1$, from (3), we have

$2c - 2c - 6c + 4 + 2 = 0$

$6c = 6$ or $c = 1$.

Corresponding asymptote is $y = x + 1$

When $m = -1$, from (3), we have

$2c - 2c - 6c + 4 - 2 = 0$

$2c = 2$ or $c = 1$

Corresponding asymptote is $y = x + 1$

When $m = \frac{-1}{2}$, from (3), we have,

$2c + c - \frac{3}{2}c + 1 - 1 = 0$ or $c = 0$

Corresponding asymptote is $y = -\frac{1}{2}x$.

$\therefore$ given curve has three asymptotes given by $y = x + 1$, $y = -x + 1$ and $y = -\frac{1}{2}x$.

## 12.7 Special Methods for finding Oblique Asymptotes of a Rational Algebraic Curve

Now we discuss some special methods of finding asymptotes of $f(x, y) = 0$ when the equation $f(x, y) = 0$ is of some special types.

**Method I.** If the equation of the curve is of the form

$$(ax + by + c)\, f_{n-1}(x, y) + g_{n-1}(x, y) = 0$$

Then the asymptote parallel to $ax + by + c = 0$ is given by $(ax + by + c) + \underset{x \to \pm\infty}{Lt} \frac{g_{n-1}(x, y)}{f_{n-1}(x, y)} = 0$ provided the limit exists

$$\frac{y}{x} \to \frac{a}{b}$$

**Method II.** If the equation of the curve is of the form

$$(ax + by)2\, f_{n-2}(x, y) + g_{n-2}(x, y) = 0$$

then the two asymptotes paraliel to $ax + by = 0$ are given by

$$(ax + by)^2 + \underset{x \to \pm\infty}{Lt} \frac{g_{n-2}(x, y)}{f_{n-2}(x, y)} = 0 \text{ provided the limit exists}$$

$$x \to \frac{a}{b}$$

**Method III.** If the equation of the curve is of the form

$$(ax + by)^2 f_{n-2}(x, y) + (ax + by)^2 g_{n-2}(x, y) + h_{n-2}(x,y) = 0$$

Then the two asymptotes parallel to $ax + by = 0$ are given by

$$(ax + by)^2 + (ax + by) \underset{x \to \pm\infty}{Lt} \frac{g_{n-2}(x, y)}{f_{n-2}(x, y)} + \underset{x \to \pm\infty}{Lt} \frac{h_{n-2}(x, y)}{f_{n-2}(x, y)} + = 0$$

$$x \to \frac{a}{b} \qquad \frac{y}{x} \to \frac{a}{b}$$

provided the limit exist.

**Note. Working Method to find asymptotes**

We now give

(I) Factorize the highest degree terms

(II) Retain one linear factor and divide by the product of other factors

(III) Take limits when $x \to \infty$, $y \to \infty$ in the direction of the retained factor.

Note. If limits do not exists, then there is no asymptote parallel to ax + by + c = 0

## 12.8 Asymptotes by Inspection

If the equation of the curve can be written as

$F_n(x, y) + F_{n-2}(x, y) = 0$

where $F_n(x, y)$ is a rational Integral function in x and y of degree n and $F_{n-2}(x, y)$ of degree (n-2) at the most then every linear factor ax + by + c of $F_n(x, y)$ equated to zero determines the asymptote of the curve, provided no two asymptotes so obtained are either parallel or coincident.

**Let us do some examples.**

**Example 5:** Find all the asymptotes of the curve

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$

**Solution:** The equation of the curve is

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0 \qquad ....(1)$$

$$x^3 - 2x^3y - xy^2 - 2y^3 + y(x - y) + 1 = 0$$

or $(x-y(x+y)(x+2y) + y(x-y)$

[$\therefore$ of factorizing third degree terms]

$\therefore$ possible asymptotes are parallel to the lines

x - y = 0, x + y = 0 and x + 2y = 0.

(I) The equation (2) of the curve can be written as

$$x - y + \frac{y(x-y)+1}{(x+y)(x+2y)} = 0$$

asymptote (if it exists) parallel to x -y = 0 is given by x - y + $\underset{x\to\infty}{Lt}\ \dfrac{y(x-y)+1}{(x+y)(x+2y)}$ = 0

y = x

or x - y + $\underset{x\to\infty}{Lt}\ \dfrac{0+1}{(x+x)(x+2x)}$ = 0 or x - y + $\underset{x\to\infty}{Lt}\ \dfrac{1}{6x^2}$ = 0

or x - y = 0, which is one asymptote

(II) The equation (2) of the curve can be written as

x + y + $\dfrac{y(x-y)+1}{(x-y)(x+2y)}$ = 0

$\therefore$ asymptote (if it exists) parallel to x + y = 0 is given by

x + y + $\underset{x\to\infty}{Lt}\ \dfrac{y(x-y)+1}{(x+y)(x+2y)}$ = 0

y = x

or x + y + $\underset{x\to\infty}{Lt}\ \dfrac{(-x)(2x)+1}{(2x)(-x)}$ = 0

or x + y + $\underset{x\to\infty}{Lt}\ \dfrac{1-2x^2}{-2x^2}$ = 0 or x + y + $\underset{x\to\infty}{Lt}\ \dfrac{x^{\frac{1}{2}-2}}{-2}$ = 0

or x + y + $\dfrac{0-2}{-2}$ = 0

or x + y + 1 = 0, which is second asymptote.

(III) the equation (2) of the curve can be written as

x + 2 y + $\dfrac{y(x-y)+1}{(x-y)(x+y)}$ = 0

$\therefore$ asymptote (if any) parallel to x + 2 y = 0 is given by

x + 2 y + $\underset{x\to\infty}{Lt}\ \dfrac{y(x-y)+1}{(x-y)(x+y)}$ = 0

y = - $\dfrac{x}{2}$

or x + 2y + $\underset{x\to\infty}{Lt}\ \dfrac{\left(-\frac{x}{2}\right)\left(x+\frac{x}{2}\right)+1}{\left(x+\frac{x}{2}\right)\left(x-\frac{x}{2}\right)}$ = 0

or x + 2y + $\underset{x\to\infty}{Lt}\ \dfrac{\frac{3x^3}{4}+1}{\frac{3x^2}{4}}$ = 0 or x + 2y + $\underset{x\to\infty}{Lt}\ \dfrac{4-3x^3}{3x^2}$ = 0

or x + 2 y + Lt $\dfrac{x^{\frac{4}{2}-3}}{3}$ = 0 or x + 2y + $\dfrac{-3}{3}$ = 0

or x + 2y - 1 = 0, which is third asymptote.

$\therefore$ asymptotes of the given curve are x - y = 0, x + y + 1 = 0 and x + 2y + 1 = 0

**Example 6 :** Find all the asymptotes of the following curve:

$$x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$$

**Solution:** The given equation is $x^3 + x^2y - xy^2 - y^3 + 2xy + 2y^2 - 3x + y = 0$

Or $x^2 (x + y) - y^2 (x + y) + 2xy + 2y^2 - 3x + y = 0$

or $(x + y) (x^2 + y^2) + 2xy + 2y^2 - 3x + y = 0$

or $(x - y) (x + y)^2 + 2xy + 2y^2 - 3x + y = 0$ ....(1)

The equation (1) can be written as

$$x - y + \frac{2xy+2y^2+3x+y}{(x+y)^2} = 0$$

$\therefore$ asymptote (it if exists) parallel to x - y = 0 is given by

$$x - y + \underset{x\to\infty}{Lt}\ \frac{2xy+2y^2+3x+y}{(x+y)^2} = 0$$

$$x - y + \underset{x\to\infty}{Lt}\ \frac{2xy+2y^2+3x+y}{(x+y)^2} = 0$$

or $x = y + \underset{x\to\infty}{Lt}\ \dfrac{4x^2-2x}{4x^2} = 0$

or $x - y + \underset{x\to\infty}{Lt}\ \dfrac{4\frac{2}{x}}{4} = 0$ or $x - y + \dfrac{4-0}{4} = 0$

$\therefore$ x = y + 1 = 0 is one asymptote.

The equation (1) can be written as

$$(x + y)^2 + (x + y) \frac{2y}{x-y} - \frac{3x-y}{x-y} = 0$$

$\therefore$ asymptotes (If they exist) parallel to x + y = 0 are given by

$$(x + y)^2 + (x + y) \underset{x\to\infty}{Lt} \frac{2y}{x-y} - \frac{3x-y}{x-y} = 0$$

$$y = x$$

or $$(x + y)^2 + (x + y) \underset{x\to\infty}{Lt} \frac{-2x}{x+x} - \underset{x\to\infty}{Lt} \frac{3x+x}{x+x} = 0$$

$$x \to \infty \qquad x \to \infty$$

or $(x + y)^2 - (x + y) - 2 = 0$ or $(x + y - 2)(x + y ++1) = 0$

$\therefore$ x + y - 2 = 0, x + y + 1 = 0 are the other two asymptotes

**Example 7:** Find all the asymptotes of the following curve:

$$xy (x^2 - y^2) (x^2 - 4y^2) + 3xy (x^2 - y^2) + x^2 + y^2 - 7 = 0$$

**Solution:** The equation of given curve is

$$xy (x^2 - y^2) (x^2 - 4y^2) + 3xy (x^2 - y^2) + x^2 + y^2 - 7 = 0$$

The given equation is of the form

$$F_6 (x, y) + F_4 (x, y) = 0$$

where $F6(x, y) = xy (x_2 - y_2) (x_2 - 4y_2)$, $F_4(x, y) = 3xy (x_2 - y_2) + x_2 + y_2 - 7$

Also $F_6(x, y)$ is the product of 6 linear factors (non-repeated) and $F_4(x, y)$ is of degree 4.

$\therefore$ asymptotes are given by $F_6(x, y) = 0$

or $xy(x_2 - y_2) (x_2 - 4y_2) = 0$ or $xy(x - y) (x + y)(x - 2y) (x + 2y) = 0$

$\therefore$ asymptotes of the given curve are

$$x = 0, y = 0, x - y = 0, x + y = 0, x - 2y = 0, x + 2y = 0$$

## 12.9 Intersection of a Curve and its Asymptotes

Prove that an asymptote of a rational algebraic curve of the nth degree cuts the curve in atmost (n - 2) points.

**Proof:** Let $y = m_1x + c_1$ ........(1) be an asymptote of the curve

$$x^n\phi_n\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right) + ......... = 0 \qquad ......(2)$$

We are to find the points of Intersection of (1) and (2), From (1), $\frac{y}{x} = m_1 + \frac{c_1}{x}$

Substituting the value of $\frac{y}{x}$ in (2), we get,

$x^n\phi_n\left(m_1 \frac{c_1}{x}\right) + x^{n-1}\phi_{n-1}\left(m_1 \frac{c_1}{x}\right) + x^{n-1}\phi_{n-1}\left(m_1 \frac{c_1}{x}\right) + \text{.........} = 0$

Using Taylor's Theorem, we get

$x^n\phi_n(m_1)+x^{n-1}[c_1\phi'_n(m_1) + \phi_{n-1}(m_1)] + x^{n-2}\left[c\frac{2}{1}\phi_{n-1}(m_1)+\phi_{n-2}(m_1)\right] + \text{.....,} = 0$ ....(3)

Since $\phi_n(m_1) = 0$ and $c_1\phi_n(m_1) + \phi_{n-1}(m_1) = 0$, (3)

becomes $x_{n-2}\left[c\frac{2}{1}\phi_{n-1}(m_1)+\phi_{n-2}(m_1)\right] + \text{.......} = 0$

which is an equation of degree (n-2) and correspondingly (1) and (2) Intersect in (n-2) points.

$\therefore$ asymptote (1) cuts the curve (2) in at the most (n-2) points.

Hence the result.

**Cor. 1.** Prove that all asymptotes of a curve of nth degree cut the curve in almost n(n-2) points.

**Proof:** We know that a curve of nth degree has atmost n asymptotes and each asymptote cuts the curve in atmost (n-2) points.

$\therefore$ all the asymptotes of a curve of nth degree cut the curve in atmost n(n-2) points.

**Cor. 2.** If the equation of the curve of nth degree is of the form $F_n + F_{n-2} = 0$ and curve has no parallel asymptotes, then the points of intersection of the curve and its asymptote lie on the curve $F_{n-2} = 0$

**Proof:** The equation of curve is $F_n + F_{n-2} = 0$

The equation of asymptote is $F_n = 0$

$\therefore$ the points of intersection of the asymptote and the curve satisfy the equations $F_n + F_{n-2} = 0$ and $F_n = 0$ and therefore they will satisfy.

$(F_n + F_{n-2}) - F_n = 0$ i.e., $F_{n-2} = 0$

Hence the result.

Let us do some examples

**Example 8:** Show that the asymptotes of the cubic curve $x^3 + xy^2 - 2xy + 2x - y - 1 = 0$ cut the curve in atmost three points which lies on the line $3xy - y - 1 = 0$

**Solution:** The equation of given curve is $x^3 + xy^2 - 2xy + 2x - y - 1 = 0$

Coefficient of highest power of y is -x.

$\therefore$ asymptote parallel to y-axis is -x = 0 or x = 0

The equation (1) can be written as x(x - y) (x + y) - 2xy + 2x - y - 1 = 0

The asymptote (If it exists) parallel to x - y = 0 is given

by x - y + $\underset{x\to\infty}{Lt}\dfrac{-2xy+2x-y-1}{x(x+y)}$ = 0

y = x

or x - y + $\underset{x\to\infty}{Lt}\dfrac{2x^2+2x-x-1}{x(x+x)}$ = 0

or x - y + $\underset{x\to\infty}{Lt}\dfrac{2x^2+x-1}{2x^2}$ = 0 or x - y + $\underset{x\to\infty}{Lt}\dfrac{-2\frac{1}{x}-\frac{1}{x^2}}{2}$ = 0

or x - y + $\dfrac{-2+0-0}{2}$ = 0

$\therefore$ x - y - 1 = 0 is the second asymptote.

The asymptoe (If it exists) parallel to x + y = 0 is given by

x + y + $\underset{x\to\infty}{Lt}\dfrac{-2xy+2x-y-1}{x(x+y)}$ = 0

y = -x

or x + y + $\underset{x\to\infty}{Lt}\dfrac{2x^2+2x+x+1}{x(x+x)}$ = 0

or x + y + $\underset{x\to\infty}{Lt}\dfrac{2x^2+3x-1}{2x^2}$ = 0 or x + y + $\underset{x\to\infty}{Lt}\dfrac{2\frac{1}{x}-\frac{1}{x^2}}{2}$ = 0

or x + y + $\dfrac{2+0-0}{2}$ = 0

$\therefore$ x + y + 1 = 0 is the third asymptote.

The joint equation of the asymptotes is

x(x - y - 1) (x + y + 1) = 0 or $x^3$ - $xy^2$ - 2xy - x = 0 ......(2)

Subtracting (2) from (1), we get,

3x - y - 1 = 0, which is a starlight line

Now curve (1) is of third degree and has 3 asymptotes. Therefore the curve and the asymptotes intersect in atmost 3(3-2) = 3 points which lie on the straight line 3x - y - 1 = 0

**Example 9:** Show that the asymptotes of the curve

$$x^4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

cut the curve in atmost eight points which lie on a rectangular hyperbola.

**Solution:** The equation of given curve is

$$x4 - 5x^2y^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0 \qquad ...(1)$$

Coefficient of highest power of x is 1, which is constant

$\therefore$ there is no asymptote parallel to x-axis.

Similarly there is no asymptote parallel to y-axis

Putting x = 1, y = m in (1), we get

$$\phi_4(m) = 1 - 5m^2 + 4m^4 = 4m^4 - 5m^2 + 1$$

$$\phi'_4(m) = 16m^3 - 10m$$

$$\phi_3(m) = 0$$

Now $\phi_4(m) = 0 \Rightarrow 4m^4 - 5m^2 + 1 = 0 \Rightarrow (m^2 - 1)(4m^2 - 1) = 0$

$\therefore$ $m^2 = 1, \frac{1}{4} \Rightarrow m = -1, 1, -\frac{1}{2}, \frac{1}{2}$ $\qquad \left[c = \frac{\phi_2(m)}{\phi'_4(m)}\right]$

When m = -1, $c = -\frac{0}{16(-1)^3 - 10(-1)} = 0$

$\therefore$ corresponding asymptote is y = -x + 0 or x + y = 0

When m = 1, $c = \frac{0}{16(-1)^3 - 10(-1)} = 0$

$\therefore$ corresponding asymptote is y = x + 0 - y = 0

When $m = -\frac{1}{2}$, $c = -\frac{0}{16\left(\frac{1}{2}\right)^3 - 10\left(\frac{1}{2}\right)} = 0$

$\therefore$ corresponding asymptote is $y = \frac{1}{2}x + 0$ or x + 2y = 0

The joint equation of asymptote is

$$(x - y)(x + y)(x - 2y)(x + 2y) = 0$$

or $\qquad (x^2 - y^2)(x^2 - 4y^2) = 0$

or $\quad (x^2 - y^2)(x^2 - 4y^2) = 0$

or $\quad x_4 - 5x_2 y_2 + 4y_4 = 0 \qquad .....(2)$

Subtracting (2) from (1), we get

$x^2 - y^2 + x + y + 1 = 0$, which is a rectangular hyperbola.

Now curve (1) is of 4th degree and has 4 asymptotes. Therefore the curve and the asymptotes intersect in atmost 4(4-2) = 8 points which lie on the rectangular hyperbola $x^2 - y^2 + x + y + 1 = 0$.

## 12.10 Self Check Exercise

Q.1 Find the asymptotes of the curve $x^3 + y^3 - 3\,axy = 0$

Q. 2 Show that the parabola $y^2 = 4ax$ has no asymptotes.

Q. 3 Find all asymptotes of the curve

$y^3 + x^2y + 2xy^2 - y + 1 = 0$

Q. 4 Find the asymptotes of the curve

$3x^3 + 2x^2y + 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$

Show that the asymptotes meet the curve again at three points which lie on a line. Find its equation.

## 12.11 Summary

In this unit we have learnt the followings :

(i) Asymptote

(ii) Rectangular and oblique asymptote

(iii) Asymptote by inspection

(iv) Intersection of a curve and its asymptote

(v) Besides above, special method has been given to find rectangular asymptote and oblique asymptote of rational algebraic curve.

## 12.12 Glossary

(i) Horizontal Asymptote - As asymptote parallel to x-axis is called horizontal asymptote.

(ii) Vertical Asymptote - An asymptote parallel to y-axis is called vertical asymptote.

## 12.13 Answers to Self Check Exercises

Ans. 1 $x + y + a = 0$

Ans. 2 Prove it

Ans. 3 $y = -x + 1$, $y = -x - 1$ or $x + y - 1 = 0$, $x + y + 1 = 0$

Ans. 4 Asymptotes are : $6x - 6y - 7 = 0$, $6x - 2y - 3 = 0$, $3x + 6y + 5 = 0$

and equation of line is $381x - 106y - 105 = 0$

## 12.14 Reference/Suggested Reading


1. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002
2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007


## 12.15 Terminal Questions

1. Find the asymptotes to the curve

   $$y = \frac{x^2 + 2x - 1}{x}$$

2. Find all asymptotes to the curve

   $ay^2 = x^3 (a - x)$

3. Show that the asymptotes of the curve

   $x^2y^2 = a^2 (x^2 + y^2)$

   Form a square of side 2a.

4. Find all asymptotes of the curve

   $f(x, y) = x (y - x)^3 - x (y - x) + 2 = 0$

5. Find the asymptotes of the curve

   $x^2y + xy^2 + 2x^2 - 2xy - y^2 - 6x - 2y + 2 = 0.$

   Also show that they cut the curve in atmost three points which lie on a straight line $2 x - 3y - 4 = 0$

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## Unit - 13

# Singular Points and Double Points

**Structure**



## 13.1 Introduction

Dear students,in this unit we shall study what we mean by singular point and double point. In geometry, a singular point on a curve is one where the curve is not given by smooth embedding of a parameter. The precise definition of a singular point depends on the type of curve being studiesd.

## 13.2 Learning Objectives

The main objectives of this unit are to-

(i) study singular point, double points

(ii) study multipoint

(iii) learn about classification of double point

(iv) study tangents at the origin

(v) to know the working rule for finding the nature of origin which is double point.

## 13.3 Singular Points, Double Points

### Singular Point

A point on the curve at which the curve behaves in an extraordinary manner is called a singular point.

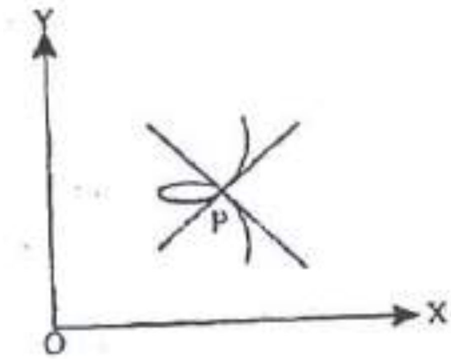
There are two types of singular points :

(i) Points of inflexion

(ii) Multiple points

We have already discussed point of inflexion. Now we discuss multiple points.

**Multiple Point :** A point on the curve through which more than one branches of the curve pass is called a multiple point.




In this chapter, we are mainly concerned with double points. Now we define double point.

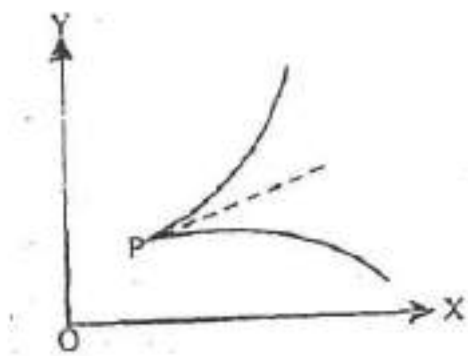
**Double Point :** A point on the curve through which two branches of the curve pass is called a double point.

**13.4 Classification of Double Points**

There are three kinds of double points.

**(i) Node :**

A node is a point on the curve through which two real branches of the curve pass and two tangents at which are real and distinct. Thus P is a node.

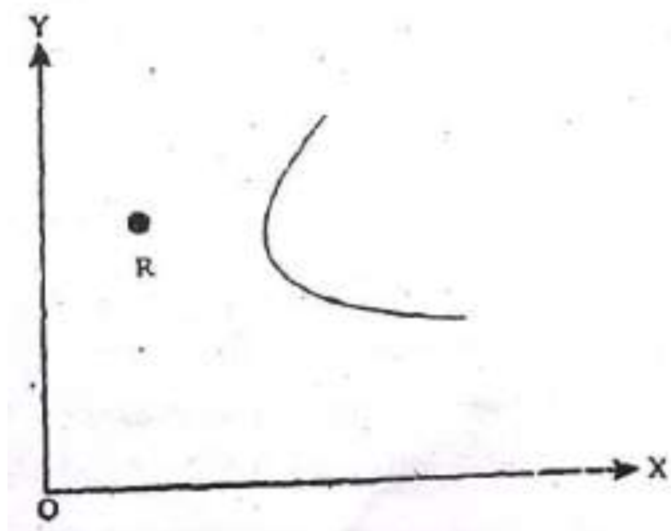



**(ii) Cusp :**

A double point on the curve through which two real branches of the curve pass and the tangents at which are real and coincident is called a cusp. Thus P is a cusp.

**(iii) Conjugate Point :**

A conjugate point on a curve is a point in the neighbourhood of which there are no other real points of the curve.




The two tangents at a conjugate point are in general imaginary but sometimes they may be real.

**Note 1. Conjugate point is also called isolated point.**

**Note 2.** the determination of the nature of double points depends basically on the nature of the two branches of the curve passing through it and not on the tangents to the curve at that point. Generally when the tangents at a double point are real, the branches are also real. But there are cases, when the tangents may be real, yet the branches may be imaginary.

### 13.5 Tangents at the Origin

If a rational algebraic curve $f(x, y) = 0$ passes through the origin, then the equation (or equations) of the tangent (or tangents) at the origin is (or are) obtained by equating to zero the lowest degree terms of $f(x, y)$.

**Proof :** Let the equation of curve be

$$f(x, y) = (a_1 x + a_2 y) + (b_1 x^2 + b_2 x y + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + c_3 xy^2 + c_4 y^3) + .... \quad ...(1)$$

[$\therefore$ the curve passes through origin, so its equation does not contain any constant term]

Let : $(\alpha, \beta)$ be any point in the nbh. of O (0, 0) on the curve (1).

$\therefore$ Slop of chord OP = $\dfrac{\beta - 0}{\alpha - 0} = \dfrac{\beta}{\alpha}$

Now chord OP becomes a tangent at O as $P \to O$ i.e. $\alpha \to 0$, $\beta \to 0$

$\therefore$ $m = \underset{P \to O}{Lt} \dfrac{\beta}{\alpha}$, where m is the slope of the tangent at O.

$\because$ P $(\alpha, \beta)$ lies on curve (1)

$\therefore$ $(a_1 \alpha + a_2 \beta) + (b_1 \alpha^2 + b_2 \alpha \beta + b_3 \beta^2) + ..... = 0$

Dividing throughout by $\alpha$ ($\neq 0$), we get

$$\left(a_1 + a_2 \frac{\beta}{\alpha}\right) + \left(b_1 \alpha + b_2 \beta + b_3 \beta . \frac{\beta}{\alpha}\right) + .... = 0 \quad ....(2)$$

Let $P \to O$ so that $\alpha \to 0$, $\beta \to 0$ and $\underset{P \to O}{Lt} \dfrac{\beta}{\alpha} = m$

$\therefore$ from (2), we get

$a_1 + a_2 m = 0$

If $\alpha \neq 0$, then $m = -\dfrac{a_1}{a_2}$ and hence the equation of the tangent at the origin is y = mx or y $= -\dfrac{a_1}{a_2} x$ or $a_1 x + a_2 y = 0$

If $a_2 = 0$ and $a_1 \neq 0$, then (1) reduces to

$$(b_1 x^2 + b_2 xy + b_3 y^2) + (c_1 x^3 + c_2 x^2 y + c_3 x y^2 + c_4 y^3) + .... = 0 \qquad ...(3)$$

Proceeding as above, we get

$b_1 + b_2 m + b_3 m^2 = 0$

Putting for m, the equation of the two tangents at origin is

$b_1 + x^2 + b_2 xy + b_3 y^2 = 0$

If $b_1 = b_2 = b_3 = 0$, we proceed as above and so on.

It follows that equation of tangent (or tangents) is obtained by equating to zero the lowest aegree terms in (x, y).

**Note.** Equating of tangents at any point

That the origin to the given point and find the equation of tangents at the new origin. The transform this equation to original axes.

## 13.6 Working rule for finding the Nature of Origin which is a Double Point.

Find the tangents at the origin by equating to zero the lowest degree terms in x and y of the equation of the curve. If the origin is a double point, then we shall get two tangents which may be real or imaginary.

(i) If two tangents are imaginary, then origin is a conjugate point.

(ii) If two tangents real and coincident, then origin is a cusp or a conjugate point.

(iii) If the two tangents are real and distinct, then origin is a node or a conjugate point.

To be sure, examine the nature of curve in the nbd. of origin. If the curve has real branches through the origin, then it is a node, otherwise a conjugate point.

To be sure, we test the nature of curve in the nbd. of the origin as above.

**Note. Test for nature of curve at origin.**

If the tangents at origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y containing powers above second. If the value of y, for small values of x are found to be real, the branches of the curve through the origin are real, otherwise imaginary.

If the tangents at origin $x^2 = 0$, solve the equation for x and proceed as above.

**Art-4.** Show that the necessary and sufficient conditions for any point.

(x, y) on $f$ (x, y) = 0 to be a multiple point are that $f_x$ (x, y) = 0, $f_y$ (x, y) = 0.

**Proof :** The equation of curve is $f$ (x, y) = 0 ....(1)

Differentiating (1) w.r.t. x, treating y as a function of x, we have

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0 \qquad ....(2)$$

where $\dfrac{dy}{dx}$ is the slope of the tangent at the point P (x, y)

If P (x, y) is a multiple point, there must be atleast two tangent which may be real, coincident or imaginary.

Thus must have atleast two value at (x, y). But (2) is first degree equation in $\dfrac{dy}{dx}$ and is satisfied by atleast two value of $\dfrac{dy}{dx}$, which is possible only when it is identity.

$$\therefore \qquad \frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

$\therefore$ necessary and sufficient conditions for any point (x, y) on the curve $f$ (x, y) = 0 to be a multiple point is

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0$$

i.e. $f_x$ (x, y) = 0 and $f_y$ (x, y) = 0

Classification of Double Points

Differentiating (2) w.r.t. x, we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx} + \left( \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dx} \right) \frac{dy}{dx} + \frac{\partial f}{\partial y} \cdot \frac{d^2 y}{dx^2} = 0$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \text{ and at a double point } \frac{\partial f}{\partial y} = 0$$

above equation becomes

$$\frac{\partial^2 f}{\partial y^2} \left( \frac{dy}{dx} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2} = 0 \qquad \text{....(4)}$$

It is a quadratic in $\dfrac{dy}{dx}$ and gives the two slopes of the tangents at the double point (x, y).

The tangents at (x, y) will be real and distinct; real and coincident or imaginary points as roots of (4) are real and different, real and equal or imaginary for which

$$4 \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 - 4 \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \overset{\geq}{<} 0 \qquad \left[ \text{Q disc, } b^2 - 4ac \overset{\geq}{<} 0 \right]$$

or $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} \gtrless 0$

Hence, in general, a double point will be a node, a cusp or a conjugate point according as

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \gtrless \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} \qquad ...(5)$$

at that point.

**Note.** The condition (5) is not a sure test for the node, cusp or a conjugate point. This in fact is the condition for the two tangents at the double points to be real and distinct, coincident or imaginary. This result lead us to wrong conclusion. We explain it by an example.

**Example 1 :** Find the position and nature of the double points on the curve.

$a^4 y^2 = x^4 (2 x^2 - 3 a^2)$.

**Solution :** The equation of given curve is

$$f(x, y) = 2 x^6 - 3 a^2 x^4 - a^4 y^2 = 0 \qquad ...(1)$$

$\therefore \quad \frac{\partial f}{\partial x} = 12x^5 - 12a^2 x^3, \ \frac{\partial f}{\partial y} = -2a^4 y$

Now for the double points, $\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 12x^5 - 12a^2 x^3 = 0 \Rightarrow x^3 (x^2 - a^2) = 0$$

$\Rightarrow \quad x = 0, a, - a$

and $\frac{\partial f}{\partial y} = 0 \quad \Rightarrow -2a^4 y = 0 \quad \Rightarrow y = 0$

$\therefore$ the possible double points are (0, 0), (a, 0), (-a, 0)

But (a, 0) (-a, 0) do not satisfy (1)

$\therefore$ (0, 0) is the only double point.

Now $\frac{\partial^2 f}{\partial x^2} = 60x^4 - 36a^2 x^2, \ \frac{\partial^2 f}{\partial y^2} = -2a^4, \ \frac{\partial^2 f}{\partial x \partial y} = 0$

At (0, 0), $\frac{\partial^2 f}{\partial x^2} = 0, \ \frac{\partial^2 f}{\partial y^2} = -2a^4, \ \frac{\partial^2 f}{\partial x \partial y} = 0$

$\therefore \quad \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}$ = $(0)^2$ $(-2a)^4$ = 0

$\therefore$ origin will be a cusp

Bu from (1), y = ± $\frac{x^2}{a^2}\sqrt{2x^2 - 3a^2}$

For small enough value of x (+ ve or - ve), 2 $x^2$ - 3 $a^2$ is -ve and so y is imaginary. Hence no portion of the curve lies in the neighbourhood of the origin.

$\therefore$ origin will be conjugate point.

$\therefore$ above method gives a wrong conclusion. So for greater accuracy, we must proceed as explained in working method given below.

**Working rule for finding the Position and Nature of Double Points of the Curve $f$ (x, y) = 0**

**Step I.** Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$

**Step II.** Solve the equations $\frac{\partial f}{\partial x}$ = 0 and $\frac{\partial f}{\partial y}$ = 0 to get possible double points.

Reject those points which do not satisfy the equation $f$ (x, y) = 0 of the curve. Remaining are the double points.

**Step III.** At each double point, calculate D = $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2}$

(a) If D is positive, double point is a node or conjugate point

(b) If D = 0, double point is a cusp or conjugate point.

In this case (a) and (b), find the nature by shifting the origin to the double points and then testing the nature of tangents and existence of the curve in the nbd. of new origin.

(c) If D is negative, double point is a conjugate point.

**Example 2 :** Examine the nature of origin for the curves :

(i) $x^4 - a x^2 y + a x y^2 + a^2 y^2 = 0, a > 0$

(ii) $x^4 + y^4 - 4 a x y = 0$

(iii) $y^3 = x^3 + a x^2, a > 0$

**Solution :** (i) The equation of curve is $x^4 - a x^2 y + a x y^2 + a^2 y^2 = 0$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$a^2 y^2 = 0$ i.e. $y^2 = 0$ i.e. y = 0 y = 0

$\therefore$ there are two real and coincident tangents at the origin

$\therefore$ origin is either a cusp or a conjugate point

To be sure, we study the nature of given curve near the origin.

From (1), $a(x + a)\, y^2 - a\, x^2 y + x^4 = 0$

$$\therefore \quad y = \frac{ax^2 \pm \sqrt{a^2x^4 - 4ax^4(x+a)}}{2a(x+a)} \Rightarrow y = \frac{ax^2 \pm x^2\sqrt{-4ax - 3a^2}}{2a(x+a)}$$

Now, for small values of $x \neq 0$, $-4ax - 3a^2$ is negative and so y is imaginary in the nbd. of origin.

$\therefore$ origin is a conjugate point

**Note.** Here (0, 0) is a conjugate point and tangents at (0, 0) are real.

$\therefore$ tangents at a conjugate point can be real

(ii) The equation of curve is $x^4 + y^4 - 4axy = 0$ ..(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$-4axy = 0$ or $xy = 0$ or $x = 0, y = 0$

Q the tangents are real and distinct

$\therefore$ origin is a node or a conjugate point

From (1), neglecting $y^4$, we get

$$x^4 - 4axy = 0 \quad \text{or } y = \frac{1}{4a} x^3$$

$\therefore$ y is real for values of x near origin

$\therefore$ origin is a node

(iii) The equation of curve is $y^3 = x^3 + a x^2$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$a x^2 = 0$ or $x^2 = 0$ or $x = 0, 0$

Q the two tangents are real and coincident

$\therefore$ origin is a cusp or a conjugate point

From (1), neglecting $x^3$, we get

$$a x^2 = y^2 \quad \text{or } x = \pm y \sqrt{\frac{y}{a}}$$

Since $a > 0$, therefore x is real for small positive values of y

$\therefore$ the two branches of the curve near the origin are real and so the origin is a cusp.

**Example 3 :** Examine the nature of origin for the curves :

(i) $y^2 = 2x^2y + x^4y - 2x^4$

(ii) $y^2(a^2 + x^2) = x^2(a^2 - x^2)$

(iii) $x^2(x - y) + y^2 = 0$

**Solution :** (i) The equation of curve is $y^2 = 2x^2y + x^4y - 2x^4$ ...(1)

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$y^2 = 0$ i.e. $y = 0, y = 0$

$\therefore$ there are two real and coincident tangents at the origin

$\therefore$ origin is either a cusp or a conjugate point

From (1), $y^2 - (2x^2 + x^4)y + 2x^4 = 0$

$$\therefore \quad y = \frac{(2x^2 + x^4) \pm \sqrt{(2x^2 + x^4) - 4(1)(2x^4)}}{2}$$

$$= \frac{(2x^2 + x^4) \pm \sqrt{x^8 + 4x^4 + 4x^6 - 8x^4}}{2}$$

$$\Rightarrow \quad y = \frac{(2x^2 + x^4) \pm x^2\sqrt{x^4 + 4x^2 - 4}}{2}$$

Now for small values of $x \neq 0$, $x^4 + 4x^2 - 4$ is negative and so y is imaginary in the nbd. of origin.

$\therefore$ origin is a conjugate point

(ii) The equation of curve is $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ .....(1)

or $a^2y^2 + x^2y^2 = a^2x^2 - x^4$ or $x^4 + x^2y^2 + a^2y^2 - a^2x^2 = 0$

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$a^2y^2 - a^2x^2 = 0$ i.e. $y^2 - x^2 = 0$ i.e. $y = \pm x$

$\therefore$ there are two real and distinct tangents at the origin

$\therefore$ origin is either a node or a conjugate point

From (1), $y = \pm x \dfrac{\sqrt{a^2 - x^2}}{a^2 + x^2}$

When x is small, $a^2 - x^2$ and $a^2 + x^2$ are both positive and so y is real.

$\therefore$ values of y near the origin are real and so the branches of the curve through the origin are real.

$\therefore$ origin is a node

(iii) The equation of curve is $x^2 (x - y) + y^2 = 0$ .....(1)

or $x^2 - x^2 y + y^2 = 0$

Equating to zero the lowest degree terms, the tangents at the origin are given by

$y^2 = 0$ i.e. $y = 0, y = 0$

$\therefore$ there are two real and coincident tangents at the origin

$\therefore$ origin is either a cusp or a conjugate point

From (1), $y^2 - x^2 y + x^3 = 0$

$\therefore$ $y = \dfrac{x^2 \pm \sqrt{x^4 - 4x^3}}{2} = \dfrac{x^2 \pm \sqrt{x^3(x-4)}}{2}$

When $x \neq 0$ is small and negative, then $x^3 (x - 4)$ is positive and so y is real in the nbd. of origin.

Hence there are two real branches of the curve near the origin

$\therefore$ origin is a cusp.

**Example 4:** Find the nature of origin of semi-cubical parabola $y^2 = x^3$

**Solution:** The equation of curve is $y^2 = x^3$ .....(1)

Equating to zero, the lowest degree terms in (1), the tangents at the origin are given by

$y^2 = 0$ or $y = 0, y = 0$

$\therefore$ there are two real and coincident tangents at the origin

$\therefore$ origin is either a cusp or a conjugate point

From (1), $y = \pm x^{\frac{3}{2}}$

$\therefore$ y is real for small positive values of x

$\therefore$ real branches of the curve pass through the origin.

$\therefore$ origin is a cusp

**Example 5:** Show that origin is a node, a cusp or a conjugate point on the curve

$y^2 = ax^2 + bx^3$, according as a is positive, zero or negative.

**Solution:** The equation of curve is $y^2 = ax^2 + bx^3$ .....(1)

Equating to zero, the lowest degree terms in (1), the tangents at the origin are given by

$y^2 = ax^2$ or $y = \pm \sqrt{ax}$ .......(2)

Since there are two tangents at the origin

$\therefore$ origin is a double point

**Case I.** When a is positive

$\therefore$ from (2), the tangents are real and distinct.

$\therefore$ origin is a node or a conjugate point

From (1), $y = \pm \sqrt{a+bx}$

For small positive values of x, y is real

$\therefore$ there are two real branches of the curve near the origin

$\therefore$ origin is a node.

**Case II.** When a = 0

From (2), tangents are y = 0, y = 0

$\therefore$ two tangents are real and coincident

$\Rightarrow$ origin is a cusp or a conjugate point

When a = 0, from (1),

$y^2 = bx^3$ or $y = +x\sqrt{bx}$, which gives real values of x when b and x have like signs.

$\therefore$ y is real for values of x near the origin

$\therefore$ origin is a cusp

**Case III.** When a is negative

From (2), two tangents at origin are imaginary

$\therefore$ origin is a conjugate point

**Example 6:** Show that the curve $y^2 = bx \sin \frac{x}{a}$ has a node or a conjugate point at the origin according as a and b have like or unlike signs.

**Solution:** The equation of given curve is $y^2 = bx \sin \frac{x}{a}$

or $$y^2 = bx\left[\frac{x}{a} - \frac{1}{\lfloor 3}\frac{x^3}{a^3} + \frac{1}{\lfloor 5}\frac{x^5}{a^5} - .....\right] \qquad \left[\because \sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - .....\right]$$

or $$y^2 = \frac{b}{a}\left[x^2 - \frac{x^4}{6a^2} + \frac{x^6}{120a^4} - .....\right] \qquad .....(1)$$

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$$y^2 = \frac{b}{a}x2 \text{ or } y = \pm x\sqrt{\frac{b}{a}}$$

$\therefore$ there are two distinct tangents at the origin

$\therefore$ origin is a double point

**Case I.** When a and b have like signs, then $\frac{b}{a}$ is positive and so $\sqrt{\frac{b}{a}}$ is real

$\therefore$ two tangents are real and distinct

$\therefore$ origin is either a node or a conjugate point

For small values of $x \neq 0$, the behaviors or R.H.S. of (1) is the same as that of its first term i.e. (1) behaves as $y^2 = \frac{b}{a}x^2$. Since $\frac{b}{a}$ is positive, so $y^2$ is positive for small values of x whether positive or negative.

$\therefore$ y is real in the nbd. of origin

$\therefore$ origin is a node

**Case II.** When a and b have unlike signs, then $\frac{b}{a}$ is negative and so $\sqrt{\frac{b}{a}}$ is imaginary.

$\therefore$ two tangents are imaginary

$\therefore$ origin is a conjugate point

**Example 7:** Show that the curve $y^2 = b\,x \tan\frac{x}{a}$ has a node or a conjugate point at the origin, according as a and b have like or unlike signs.

**Solution:** The equation of given curve is $y^2 = bx \tan\frac{x}{a}$

or $$y^2 = bx\left[\frac{x}{a}+\frac{1}{3}\frac{x^3}{a^3}+\frac{x}{a}+\frac{2}{15}\frac{x^5}{a^5}+.......\right]\left[\because \tan\theta = \theta+\frac{1}{3}\theta^3+\frac{2}{15}+\theta^5+.......\right]$$

or $$y^2 = \frac{b}{a}\left[x^2+\frac{x^4}{3a^2}+\frac{2x^6}{15a^4}+.......\right] \qquad .....(1)$$

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$$y^2 = \frac{b}{a}x^2 \text{ or } y = \pm x\sqrt{\frac{b}{a}}$$

$\therefore$ there are two distinct tangents at the origin

$\therefore$ origin a double point.

**Case I.** When a and b have like signs, then $\frac{b}{a}$ is positive and so $\sqrt{\frac{b}{a}}$ is real.

$\therefore$ two tangents are real and distinct

$\therefore$ origin is either a node or a conjugate point

For small values of $x \neq 0$, the behavior of R.H.S. of (1) is the same as that of its first term i.e. (1) behaves as $y^2 = \frac{b}{a}x^2$. Since $\frac{b}{a}$ is positive, so $y^2$ is positive for small values of x whether positive or negative.

$\therefore$ y is real in the nhd. of origin

$\therefore$ origin is a node.

**Case II.** When a and b have unlike signs, then $\frac{b}{a}$ is negative and so $\sqrt{\frac{b}{a}}$ is imaginary.

$\therefore$ two tangents are imaginary

$\therefore$ origin is a conjugate point

**Example 8:** Show that the curve $y^2 = 2x \sin 2x$ has a node at the origin.

**Solution:** The equation of curve is $y^2 = 2x \sin 2x$

or $$y^2 = 2x \left[2x - \frac{(2x)^3}{\lfloor 3} + \frac{(2x)^5}{\lfloor 5} - .......\right]$$

or $$y^2 = 4 \left[x^2 - \frac{4x^4}{6} + \frac{16x^6}{120} - .......\right] \qquad .....(1)$$

Equating to zero, the lowest degree terms, the tangents at the origin are given by

$y^2 = 4x^2$ or $y = \pm 2x$

$\therefore$ two tangents at origin are real and distinct

$\therefore$ origin is either a node or a conjugate point

For small values of $x \neq 0$, the behaviour or R.H.S. of (1) is the same as that of its first term i.e. (1) behaves as $y^2 = 4x^2$. Now $y^2$ is positive for small values of x whether positive or negative.

$\therefore$ y is real in the nhd. of origin

$\therefore$ origin is a node.

**Example 9:** Prove that the curve $y^2 = (x - a)^2 (x - b)$

has at x = a, a node if a > b, a cusp if a = b and a conjugate point if a < b.

**Solution:** The equation of curve is $y^2 = (x - a)^2 (x - b)$ ....(1)

When x = a, from (1), y = 0

$\therefore$ point under discussion in (a, 0)

Shifting origin to (a, 0) by transformations x = X + a, y = Y + 0 = Y

(1) becomes $Y^2 = X^2 (X + a - b)$ .....(2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$Y^2 = X^2 (a - b)$ or $Y = + X\sqrt{a-b}$ ....(3)

**Case I.** When a > b

From (3), two tangents at new origin are real and different

∴ new origin (a, 0) is a node or a conjugate point

From (2), $Y = + X \sqrt{X + a - b}$

For small non-zero value of X, Y is real as a - b > 0

∴ new origin (a, 0) is a node

**Case II.** When a = b

From (3), tangents are Y = 0, Y = 0

∴ two tangents are real and coincident

∴ origin is a cusp or a conjugate point

From (2), $Y^2 = X^3$ or $Y = + X \sqrt{X}$

For small positive values of X, Y is real

∴ new origin (a, 0) is a cusp.

**Case III.** When a < b

From (2), two tangents at new origin are imaginary

∴ (a, 0) is a conjugate point

**Example 10:** Determine the position and nature of the double point on the curve

$x^2 - y^2 - 7x^2 + 4y + 15x - 13 = 0$

**Solution:** The equation of curve is

$f(x, y) = x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$ .......(1)

$\frac{\partial f}{\partial x} = 3x^2 - 14x + 15$, $\frac{\partial f}{\partial y} = -2y + 4$

For the double points $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $f(x, y) = 0$

Now $\frac{\partial f}{\partial x} = 0$ $\Rightarrow$ $3x^2 - 14x + 15 = 0$

$\Rightarrow$ $(x - 3)(3x - 5) = 0 \Rightarrow x = 3, \frac{5}{3}$

and $\frac{\partial f}{\partial y} = 0 \Rightarrow -2y + 4 = 0 \Rightarrow y = 2$

$\therefore$ the possible double points are (3, 2), $\left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2)

Shifting the origin to the point (3, 2) by transformations $x = X + 3$, $y = Y + 2$

(I) become $(X + 3)^3 - (Y + 2)^2 - 7(X + 3)^2 + 4(Y + 2) + 15(X + 3) - 13 = 0$

or $X^2 + 9X^2 + 27 - Y^2 - 4Y - 4 - 7X^2 - 42X - 63 + 4Y + 8 + 15X + 45 - 13 = 0$

or $X^3 + 2X^2 - Y^2 = 0$ ....(2)

Equating to zero, the lowest degree terms, the tangents at the new origin are given by

$$2X^2 - Y^2 = 0 \quad \text{or} \quad Y = + \sqrt{2}\, X$$

which are real and distinct

$\therefore$ new origin is either a node or a conjugate point

From (2), $Y = + X\sqrt{X + 2}$

which gives real values of Y for small values of X, positive or negative

$\therefore$ real branches of the curve exist in the nbd. of the new origin (3, 2)

$\therefore$ (3, 2) is a node.

Alter. The equation of the curve is

$f(x, y)\ x^3 - y^3 - 7x^2 + 4y + 15x - 13 = 0$

$\therefore$ $\frac{\partial f}{\partial x} = 3x^2 - 14x + 15, \ \frac{\partial f}{\partial y} = -2y + 4$ .....(1)

For the double points $\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0, \ f(x, y) = 0$

Now $\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad 3x^2 - 14x + 15 = 0$

$\Rightarrow$ (x - 3) (3x - 5) = 0 $\Rightarrow$ x = 3, $\frac{5}{3}$

and $\frac{\partial f}{\partial y}$ = 0 $\Rightarrow$ -2y + 4 = 0 $\Rightarrow$ y = 2

$\therefore$ the possible double points are (3, 2), $\left(\frac{5}{3}, 2\right)$

But $\left(\frac{5}{3}, 2\right)$ does not satisfy (1)

$\therefore$ (3, 2) is the only double point

Nature of the point (3, 2)

$$\frac{\partial^2 f}{\partial x^2} = 6x - 14,\ \frac{\partial^2 f}{\partial y^2} = -2,\ \frac{\partial^2 f}{\partial x \partial y} = 0$$

At (3, 2)

$$\frac{\partial^2 f}{\partial x^2} = 18 - 14 = 4,\ \frac{\partial^2 f}{\partial y^2} = -2,\ \frac{\partial^2 f}{\partial x \partial y} = 0$$

$\therefore$ $\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = (0)^2 - (4)(-2) = 8 > 0$

$\therefore$ (3, 2) is node.

**Example 11:** Find the position and nature of the double points on the curve

$$(x - 2)^2 = y(y - 1)^2$$

**Solution:** The equation of curve is

$$f(x, y) = (x - 2)^2 - y(y - 1)^2 = 0 \qquad ....(1)$$

$$\frac{\partial f}{\partial x} = 2(x - 2)$$

$$\frac{\partial f}{\partial y} = -2y(y - 1) - (y - 1)^2 = -(y - 1)(3y - 1)$$

For the double points $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$, $f(x, y) = 0$

Now $\frac{\partial f}{\partial x}$ = 0 $\Rightarrow$ 2(x - 2) = 0 $\Rightarrow$ x = 2 and $\frac{\partial f}{\partial y} = 0 \Rightarrow$ -(y - 1) (3y - 1) = 0 $\Rightarrow$ y = 1, $\frac{1}{3}$

$\therefore$ the possible double points are (2, 1) $\left(2, \frac{1}{3}\right)$

Now $\left(2, \frac{1}{3}\right)$ does not satisfy (1)

$\therefore$ (2, 1) is the only double point

Nature of the point (2, 1)

$$\frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial y^2} = -3(y - 1) - (3y - 1),\ \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial y^2} = -3(1 - 1) - (3 - 1) = -2\ \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\therefore \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} = (0)^2 - (2)(-2) = 4 > 0$$

$\therefore$ there is a node at the point (2, 1)

## 13.7 Self Check Exercise

Q.1 Find the position and nature of double points of the curve

y(y-6) = x2 $(x - 2)^3 - 9$

Q. 2 Find the position and nature of double points on the curve

$x^3 + y^3 = 3axy$

Q. 3 Show that the curve

$y^2 = 2x \sin 2x$

has a node at the origin

## 13.8 Summary

In this unit we have learnt

(i) What a singular and double point is

(ii) How to classify double points on a curve

(iii) to find the equation of tangent at the origin

(iv) the method for finding the nature of origin. Which is a double point.

## 13.9 Glossary

(i) **Node -** A node is a point on the curve through which two real branches of the curve pass and two tangents at which are real and distinct.

(ii) **Cusp -** A double point on the curve through which two real branches of the curve pass and the tangents at which are real and coincident

## 13.10 Answers to Self Check Exercises

Ans. 1 Only double points are (0, 3), (2, 3) (0, 3) is a conjugate point and (2, 3) is a cusp.

Ans. 2 (0, 0) is the only double point (0, 0) is a node.

Ans. 3 Prove it.

## 13.11 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002


## 13.12 Terminal Questions

1. Find the position and nature of the double points of the curve

   $x^2y^2 = (a + y)^2 (b^2 - y^2)$

2. Find the position and nature of double point for the curve

   $(x + y)^3 - \sqrt{2} (y - x + 2)^2 = 0$

3. Show that (0, 0) is a conjugate point on the curve $(x - y)^2 + x^4 = 0$
4. Find the position and nature of double points on the curve

   $y^2 = (x - 1) (x - 2)^2$

-----

## Unit - 14

# Curve Tracing

**Structure**



### 14.1 Introduction

Dear students, we have done curve tracing in our lower classes by giving different values to x and finding the corresponding values of y. Now, we will do curve tracing using idea of asymptotes, monotonicity, maxima and minima etc.

### 14.2 Learning Objectives

The main objectives of this unit are

(i) to trace the curve by using the ideas of multiple points and in particular double points.

(ii) to use the idea of tangents at the origin to trace the curve.

Before moving on to curve tracing, we assume that the students are aware of multiple points, classification of double points and tangent at the origin (unit-13).

### 14.3 Curve Tracing

Following points should be kept in mind for tracing the graph of the equation $f(x, y) = 0$.

**1. Symmetry-**

The curve $f(x, y) = 0$ is symmetric about

(i) x-axis if it remains unchanged on changing y to -y i.e. y

$f(x, -y) = f(x, y)$

(ii) y-axis if $f(-x, y) = f(x, y)$

(iii) the origin if $f(-x, -y) = f(x, y)$

(iv) the line $y = x$ if $f(x, y) = f(y, x)$

(v) the line $y = -x$ if $f(-y, -x) = f(x, y)$

**2. Domain and Range**

Find the domain and range of the function.

**3. Origin**

See whether origin lies on the curve. If so, then find the tangent at the origin and also determine whether origin in node, cusp or an isolated point (conjugate point)

**4. Asymptote**

Determine all the asymptotes of the curve and the position of the curve relative to its asymptotes.

**5. Points of Intersection**

Determine all the points of intersection of the curve with coordinate axes and find the equation of the tangent of these points. Find the nature of double points if any of these points is a double point.

Find also some other points on the curve by giving suitable values to x.

**6. Maxima and Minima**

Find the points where the function has maximum and minimum value. Also find minimum and maximum value at each point.

**7. Points of inflexion**

(a) Find the intervals of

(i) Increase and decrease of the curve

(ii) Concavity and convexity of the curve

(b) Find the points of inflexion, if any

**8. Discontinuities**

Find the points where the function is discontinuous. Examine the behaviour of the function near these points.

**Some Illustrative Examples**

**Example 1 :** Trace the curve

$$y = x^3 + 5x^2 + 3x - 4$$

**Solution :** The given equation of the curve is

$$y = x^3 + 5x^2 + 3x - 4 \quad ...(1)$$

(i) Symmetry

The curve is neither symmetrical about axes nor about the origin. Also it is not symmetrical about $y = x$ or $y = -x$

(ii) Origin

The curve does not pass through origin

(iii) Domain

D $f$ = $(-\infty, \infty)$

(iv) Points of intersection

The curve meets x-axis when $y = 0$

$\therefore$ From (1) $x^3 + 5x^2 + 3x - 4 = 0$ ...(2)

$\Rightarrow$ $x = -4$ satisfies (2)

$\therefore$ Other roots of the equation (2) are given by

$x^2 + x - 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+4}}{2.1} = \frac{-1 \pm \sqrt{5}}{2} = 0{,}6, -1.6$$

$\therefore$ Curve meets the x-axis at (-4, 0), (-1.6, 0) and (0.6, 0) resp.

Similarly, the curve meets the y-axis when $x = 0$

$\therefore$ From (1) $y = -4$

$\Rightarrow$ Curve (1) meets the y-axis at (0, -4).

(v) Asymptotes

The curve has no asymptote

(vi) Increasing and decreasing

$$\therefore \quad \frac{dy}{dx} = 3x^2 + 10x + 3, \quad \frac{d^2y}{dx^2} = 6x + 10$$

Now $\frac{dy}{dx} = 0 \Rightarrow 3x^2 + 10x + 3 = 0$

$\Rightarrow (3x + 1)(x + 3) = 0$

or $x = -1/3, -3$

$\therefore$ Tangents at $x = -3, -\frac{1}{3}$ are parallel to x-axis.

Also $\frac{dy}{dx} > 0$ if $(x + 3)(3x + 1) > 0$

i.e. if x does not lie between -3 and $-\frac{1}{3}$.

$\therefore$ function is increasing in $(-\infty, -3) \cup (-\frac{1}{3}, \infty)$

Also $\frac{dy}{dx} < 0$ if $(x + 3)(3x + 1) < 0$

i.e. if x lies between -3 and $-\frac{1}{3}$

$\therefore$ function is decreasing in $(-3, -\frac{1}{3})$

(vii) Points of inflexion

Now $\frac{d^2y}{dx^2} > 0$ when $6x + 10 >$ i.e. $x > -\frac{5}{3}$

$\therefore$ graph of function is concave upwards for $x > -\frac{5}{3}$

Also $\frac{d^2y}{dx^2} < 0$ when $6x + 10 <$ i.e. $x < -\frac{5}{3}$

$\therefore$ graph of function is concave downwards for $x < -\frac{5}{3}$

And $\frac{d^2y}{dx^2}$ changes sign as $x = -\frac{5}{3}$

$\therefore$ graph of function has a point of inflexion at

$x = -\frac{5}{3}$, $y = -\frac{125}{27} + \frac{125}{9} - 5 - 4 = \frac{7}{27}$

$\therefore$ $\left(\frac{5}{7}, \frac{7}{27}\right)$ is a point of inflexion.

(viii) Maxima and Minima

$\frac{dy}{dx} = 0 \Rightarrow x = -\frac{1}{3}, -3$

at $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = \frac{-6}{3} + 10 = 8 > 0$

$\therefore$ Function has local minima at $x = -\frac{1}{3}$

and the local minimum value = $\frac{-1}{27}$ + $\frac{5}{9}$ - 1 -4 = $\frac{-121}{27}$

At x = 3, $\frac{d^2y}{dx^2}$ = -18 + 10 = -8 < 0

$\therefore$ Function has local maxima at x = -3

and maximum value = -27 +45 - 9 - 4 = 5

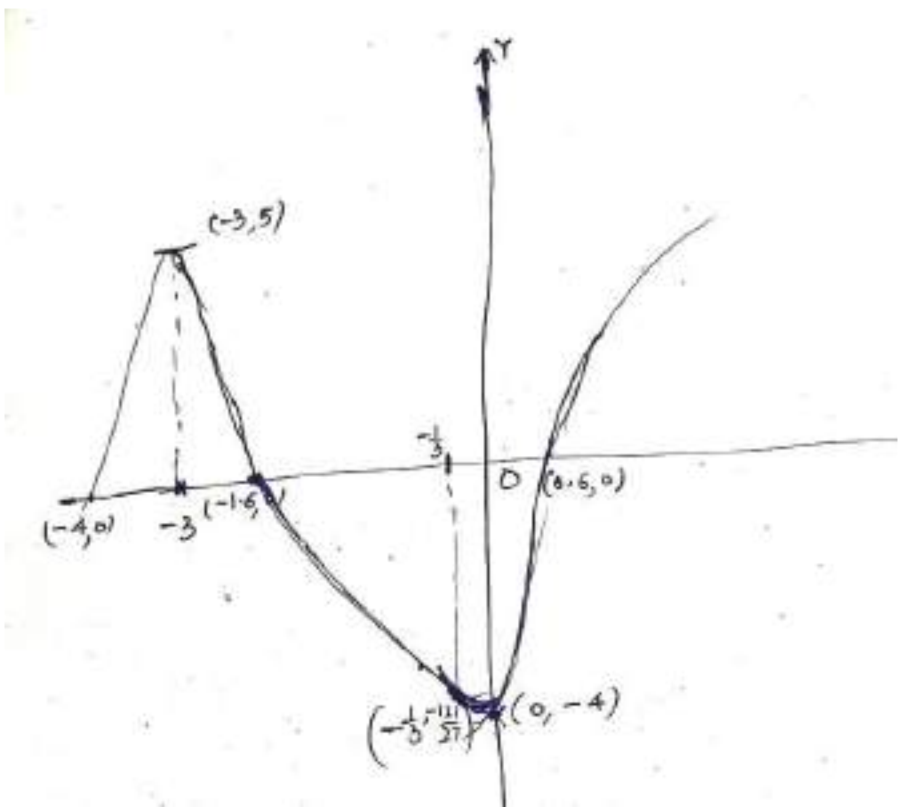
(ix) Additional Points

The curve passes through

(1, 5), (2, 30), (-1, -3), (-2, 2), )-3, 5).

Also y $\rightarrow \infty$ as x $\rightarrow \infty$ and y $\rightarrow - \infty$ as x $\rightarrow -\infty$

Now Let us draw a rough sketch of the curve




**Example 2 :** Trace the curve y = $x^3$ - $3x^2$ + 3

**Solution :** The given equation of the curve is

y = $f$ (x) = $x^3$ - $3x^2$ + 3 ...(1)

(i) The given curve is neither symmetrical about x-axis nor y-axis and not origin.

(ii) The curve does not pass through origin.

(iii) Domain of the function is $(-\infty, \infty)$

(iv) The curve meets y-axis at (0, 3).

The curve meets x-axis between -1 and 0, 1, and 2, 2 and 3, since $f$ (-1) = -ve, $f$ (0) = +ve, $f$ (1) = +ve, $f$ (2) = -ve, $f$ (3) = +ve.

(v) The curve has no asymptote.

(vi) $\frac{dy}{dx} = 3x^2 - 6x$, $\frac{d^2y}{dx^2} = 6x - 6$, $\frac{d^3y}{dx^3} = 6$

Now $\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

or x = 0, 2

$\Rightarrow$ tangents at x = 0, 2 are parallel to x-axis

At x = 0 , $\frac{d^2y}{dx^2} = -6 < 0$

$\therefore$ Curve has local maxima at x = 0, y = 3

Again at x = 2, $\frac{d^2y}{dx^2} = 6 > 0$

$\therefore$ Curve has local minima at x = 2, y = -1

Now $\frac{dy}{dx} = 3x^2 - 6x > 0$ if $3x(x - 2) > 0$

i.e. if x does not lie between 0 and 2

$\therefore$ Curve is increasing for x < 0 and x > 2

Likewise, curve is decreasing for 0 < x < 2.

$\frac{d^2y}{dx^2} = 6x - 6 > 0$ for x > 1

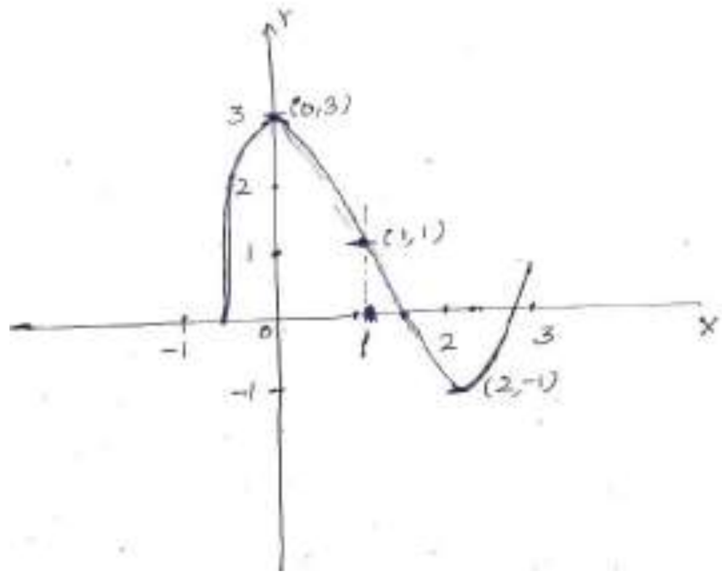
$\therefore$ Curve is concave upwards for x > 1 and concave downwards for x < 1.

Again $\frac{d^2y}{dx^2} = 0$ at x = 1 and $\frac{d^3y}{dx^3} \neq 0$ at x = 1, y = 1

$\therefore$ (1, 1) is a point of inflexion.

(vii) As $x \to \infty$, $y \to \infty$ and as $= -\infty$, $y \to -\infty$.

Now let us draw rough sketch of the curve.




**Example 3 :** Trace the curve

$$x = (y - 1)(y - 2)(y - 3)$$

**Solution :** The given curve is

$$x = (y - 1)(y - 2)(y - 3) \quad \text{...(1)}$$

(i) Symmetry

The curve is neither symmetrical about axes and not about origin. Also it can be seen that curve is neither symmetrical about line $y = x$ and about line $y = -x$.

(ii) The curve does not pass through origin

(iii) Points of intersection with axes -

the curve meets x-axis when $y = 0$

$\therefore$ From (1) curve meets x-axis at (-6, 0)

and it meets y-axis at (0, 1), (0, 2), (0, 3).

(iv) Asymptotes

The curve has not asymptotes.

(v) Tangents

$x = (y - 1)(y - 2)(y - 3) = y^3 - 6y^2 + 11y - 6$

$$\therefore \quad \frac{dy}{dx} \; 3y^2 - 12y + 11$$

$$\therefore \quad \frac{dy}{dx} = 0 \Rightarrow 3y^2 - 12y + 11 = 0$$

$$\Rightarrow \qquad y = \frac{+12 \pm \sqrt{144+132}}{243}$$

$$= \frac{12 \pm 2\sqrt{5}}{6}$$

$$= \frac{6 \pm 1.732}{3} = 2.6,\ 1.4 \text{ (nearly)}$$

Now when $y = 2.6 \Rightarrow x -0.384$,

and when $y = 1.4 \Rightarrow x = 0.384$, nearly

$\therefore$ tangents to the curve at (-0.384, 2.6) and (0.384, 1.4) are parallel to y-axis.

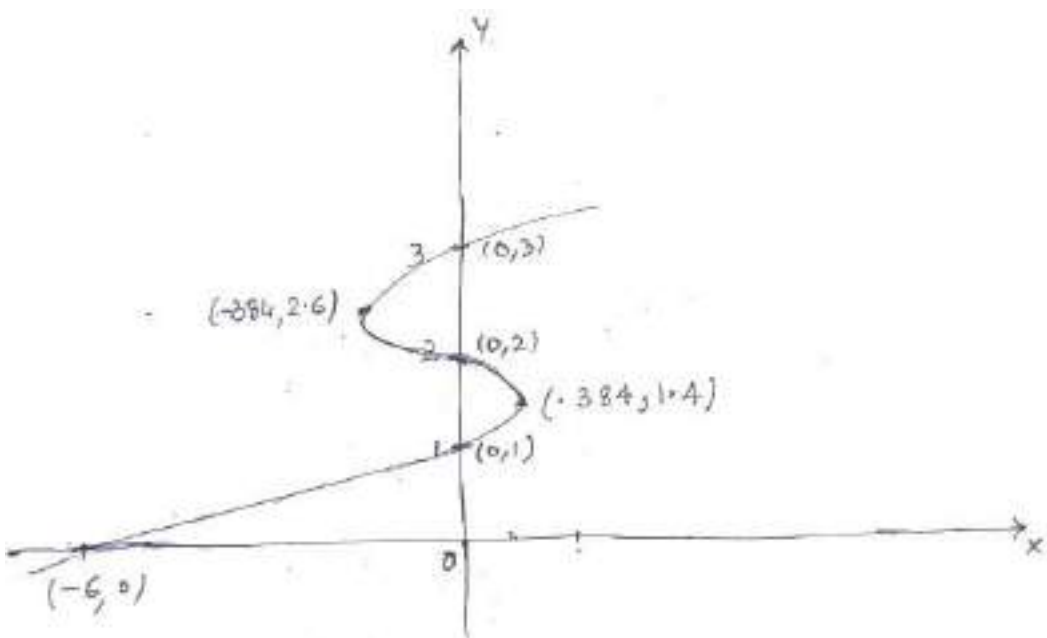
(vi) Additional points

$y < 0 \Rightarrow x < 0$

$\therefore$ No portion of the curve lies in the fourth quadrant.

| | | |
|---|---|---|
| $0 < y < 1$ | $\Rightarrow$ | $x < 0$ |
| $0 < y < 2$ | $\Rightarrow$ | $x > 0$ |
| $0 < y < 3$ | $\Rightarrow$ | $x < 0$ |
| $3 < y$ | $\Rightarrow$ | $x > 0$ |
| $x \to \infty$ | $\Rightarrow$ | $y \to \infty$ |

Below, we give a rough sketch of the curve.



**Example 4 :** Trace the curve

$$y = x + \frac{1}{x}$$

**Solution:** The given equation of curve is

$$y = x + \frac{1}{x} \qquad ....(1)$$

(i) Symmetry

The curve is symmetrical with respect to origin.

if $x < 0 \Rightarrow y < 0$

and $x > 0 \Rightarrow y > 0$

$\Rightarrow$ The graph of the curve lies in 1st and 3rd quadrant only

(ii) Origin

Origin does not lie on the graph

(iii) Asymptotes

The equation of curve is

$x^2 - xy + 1 = 0$

$\therefore$ asymptotes are $x (x - y) = 0$ (by inspection method)

i.e. $x = 0$, $y = x$ are the asymptotes

(iv) Points of intersection with coordinate axes the curve meets x-axis when $y = 0$

$\Rightarrow$ $x^2 + 1 = 0 \Rightarrow x = \pm 2$ which is imagining.

$\therefore$ Curve does not meet x-axis.

We note here that curve meets the y-axis if $x = 0$ which is not possible. Since domain if function is all reals except $x = 0$.

$\Rightarrow$ the curve does not meet y-axis, infact y-axis is the asymptote of the curve.

(v) Rising the falling-

$$y = x + \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = 1 - \frac{1}{x^2}, \quad \frac{d^2y}{dx^2} = \frac{2}{x^3}$$

$\therefore$ curve is rising where $\frac{dy}{dx} > 0$

$\Rightarrow$ 1- $\frac{1}{x^2}$ > 0 $\Rightarrow$ 1 < $x^2$ $\Rightarrow$ $x^2$ > 1 $\Rightarrow$ $|x|^2$ > 1

$\Rightarrow$ |x| > 1 $\Rightarrow$ x > 1 or x < -1

$\therefore$ the curve is rising in (-$\infty$, -1) U (1, $\infty$).

The curve is falling when

$\frac{dy}{dx}$ < 0 $\Rightarrow$ |x| < 1 $\Rightarrow$ -1 < x < 1

$\Rightarrow$ curve is falling in (-1, 1)

$\therefore$ $\frac{dy}{dx}$ = 0 when x = 1 or x = -1

Now when x = 1, $\frac{d^2y}{dx^2}$ > 0

$\therefore$ curve has a local maxima at (+1, +2)

when x = -1, $\frac{d^2y}{dx^2}$ < 0

$\therefore$ curve has a local minima at (-1, -2)

The curve is concave upwards if $\frac{d^2y}{dx^2}$ > 0

$\Rightarrow$ $\frac{2}{x^3}$ > 0 $\Rightarrow$ x > 0

$\therefore$ Curve is concave upward on (0, $\infty$)

Likewise, curve is concave downward, on (-$\infty$, 0)

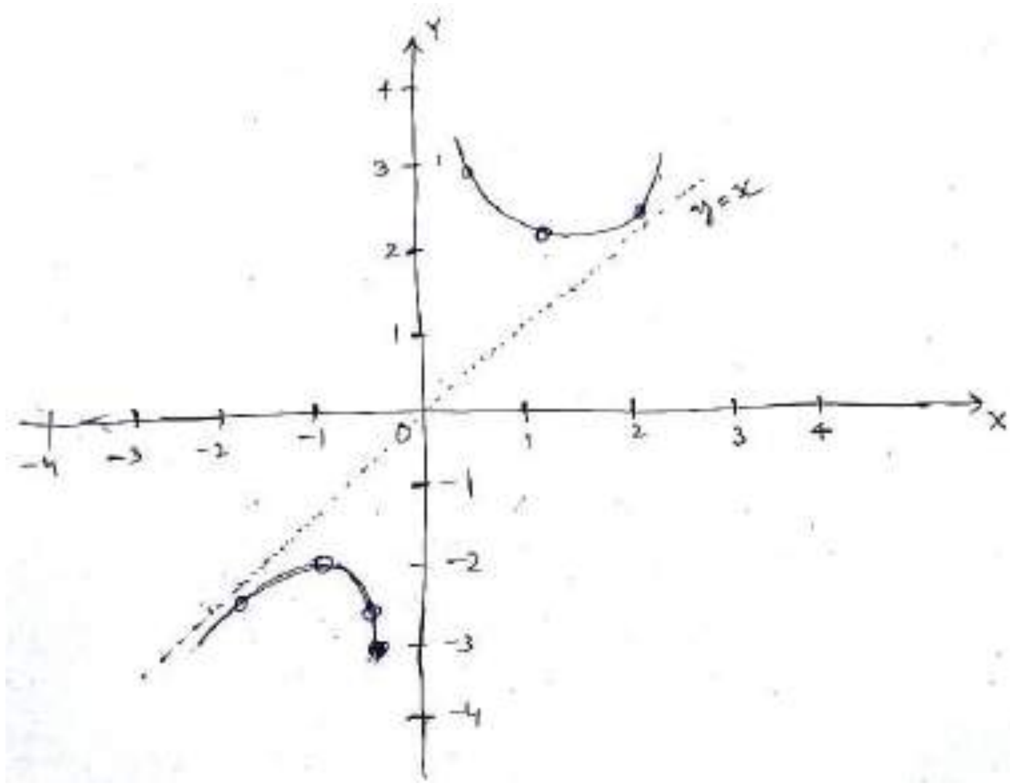
Now $\frac{d^2y}{dx^2} \neq 0 \ \forall$ x

$\Rightarrow$ there is no point of inflexion.

| x | y | $\frac{dy}{dx}$ | $\frac{d^2y}{dx^2}$ | |
|---|---|---|---|---|
| -2 | $\frac{-5}{2}$ | +ve | | rising |
| -$\frac{3}{2}$ | - $\frac{13}{6}$ | +ve | | rising |

| | | | | |
|---|---|---|---|---|
| -1 | -2 | 0 | -ve | maxima, split water, convex upwards |
| $-\frac{1}{2}$ | $\frac{-5}{2}$ | -ve | | falling |
| $\frac{1}{2}$ | $\frac{5}{2}$ | -ve | | falling |
| 1 | 2 | 0 | +ve | minima, holds water, concave upwards |
| $\frac{3}{2}$ | $\frac{13}{6}$ | +ve | | rising |
| 2 | $\frac{5}{2}$ | +ve | | rising |

The rough sketch of the curve is given below:




**Example 5 :** Trace the curve

$$y = \frac{x^2}{1+x^2}$$

**Solution :** The given equation is

$$\left.\begin{aligned} y &= \frac{x^2}{1+x^2} \\ \Rightarrow \quad (1+x^2)y &= x^2 \\ \text{or} \quad y(1+x^2)-x^2 &= 0 \end{aligned}\right\} \qquad \ldots(1)$$

(i) The curve is symmetrical about y-axis (since only even powers of x occur in the equation of given curve)

(ii) the curve passes through origin and the tangent at the origin is $y = 0$ i.e. x-axis.

(iii) Domain of the function is $(-\infty, \infty)$ and $0 \leq y < 1$

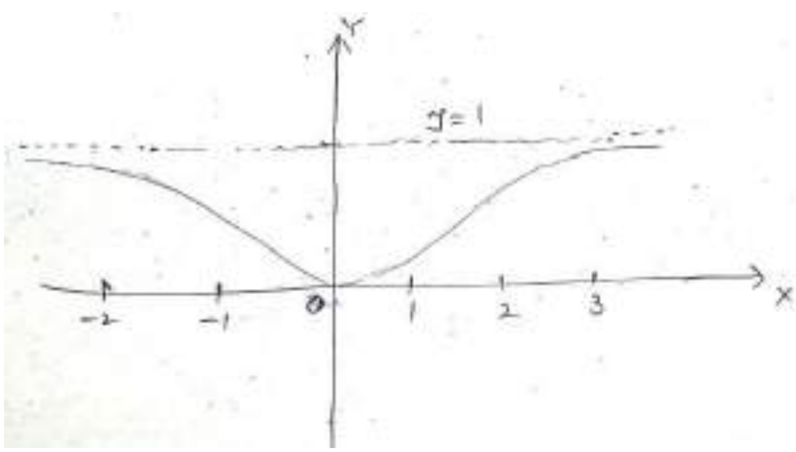
(iv) the curve meets both axis in (0, 0)

(v) the asymptote of the curve is $y - 1 = 0$. Since $y < 1$, the curve lies below the asymptote.

(vi) $\dfrac{dy}{dx} = \dfrac{2x}{\left(1+x^2\right)^2}$

Now $\dfrac{dy}{dx} > 0$ for $x > 0$ and $\dfrac{dy}{dx} < 0$ for $x < 0$.

The rough sketch of the curve is




### 14.4 Working Method for Tracing Parametric Curves

Two cases arises

**Case 1 -** If possible, eliminate the parameters and obtain the corresponding Cartesian equation of the curve. Now proceed as done earlier to trace the curve.

**Case 2 -** If case fails, then proceed as follows :

**(I) Symmetry**

(i) If $x = f(t)$ is an even function of t and y g (t) an odd function of t, then the curve is symmetrical about x-axis.

(ii) If $x = f(t)$ is an odd function of t and $y = g(t)$ is an even of t, then the cruve is symmetrical about y-axis.

(iii) If $x = f(t)$ and $y = g(t)$ are both odd function of t, then curve is symmetrical in opposite quadrants.

**(II) Origin**

If $x = 0 \Rightarrow t \in R$ which makes y equal to zero, then curve passes through origin.

**(III) Points of Intersection**

Find the points of intersection of the curve and the coordinate axes.

**(IV) Limitations**

Find the greatest and the least values, if possible, of x and y which give lines paraller to axes between which the curve lies of does not lie.

**(V) Points**

Find the points where $\frac{dy}{dx} = 0$, $\frac{dy}{dx} \to \infty$

**(VI) Region**

(i) obtain the regions in which curve does not lie

(ii) consider the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$

(iii) consider the values if x, y, $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dy}{dx}$

**(VII) Asymptotes**

Find asymptotes, if any

**Some Illustrated Examples**

**Example 6:** Trace the curve

$$x = a(\theta + \sin\theta)$$

$$y = a(1 + \cos\theta), \qquad -\pi \leq \theta \leq \pi$$

**Solution:** The given equation of the curve is

$$x = a(\theta + \sin\theta)$$

$$y = a(1 + \cos\theta), -\pi \leq \theta \leq \pi$$

which represents cycloid.

We note here that $\theta$ cannot be eliminated easily, so we proceed as follows:

(i) Symmetry -

The curve is symmetrical about the axis of y for ($\theta$ + sin $\theta$) is an odd function of $\theta$ and (1 + cos $\theta$) is an even function of $\theta$.

(ii) Origin -

Clearly the curve does not pass through origin

(iii) Intercepts -

The curve meets the x-axis when

$y = 0 \quad \Rightarrow \quad 1 + \cos\theta = 0 \quad \Rightarrow \quad \cos\theta = -1 \quad \Rightarrow \quad \theta = \pi, -\pi.$

$\therefore$ The point of intersection with x-axis are

A(a $\pi$, 0), B (-a $\pi$, $\theta$)

Again the curve meets the y-axis when x = 0

$\Rightarrow \quad \theta + \sin\theta = 0 \quad \Rightarrow \quad \sin\theta = -\theta$

or $\quad \theta = 0$

$\therefore$ the point of intersection with y-axis are

c (0, 2a)

(iv) Asymptotes -

There are no asymptotes of the curve

(v) Points -

$$\frac{dx}{d\theta} = a(1 + \cos\theta), \quad \frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)} = \frac{-2\sin\theta/2\ ¢ø\$\ \theta/2}{2\cos^2\theta/2} = -\tan\theta/2$$

$$\therefore \quad \frac{dy}{dx} = 0 \text{ when } \theta = 0$$

$\Rightarrow$ at (0, 2a) the tangent is parallel to x-axis

And $\frac{dy}{dx} \to \infty$ when $\theta = \pi, -\pi$

$\therefore$ at (a$\pi$, 0) and (-a$\pi$, 0), the tangent is perpendicular to x-axis

(vi) Region -

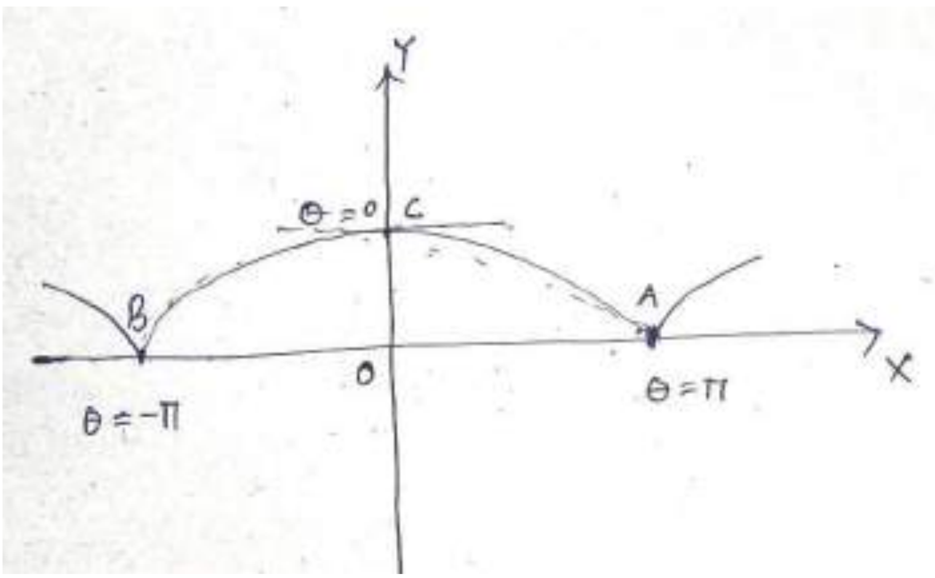
$\forall\ \theta,\ \dfrac{dx}{d\theta}$ is +ve

$\Rightarrow$ x always increases with $\theta$

also $\dfrac{dy}{dx}$ is +ve for $-\pi \leq \theta \leq \pi$

Thus y increases when $\theta$ increases from $-\pi$ to 0 and y decreases when $\theta$ increases from 0 to $\pi$.

The rough sketch of the graph of this curve is as below:




**Example 7:** Sketch the curve

$x = a \cos^3 \theta$

$y = a \sin^3 \theta$

**Solution:** We have equation of the curve

$x = a \cos^3 \theta,\ y = a \sin^3 \theta$ (Astroid)

Eliminating the parameter, we get the equation of the curve as

$$x^{2/3} + y^{2/3} = a^{2/3}$$

(i) Symmetry -

The curve is symmetrical about both the axes.

(ii) Origin -

The curve does not pass though origin

(iii) Intercepts -

The curve meets the x-axis at

$(\pm a, 0), (0, \pm a)$

$\therefore$ The equation of tangent at $(\pm a, 0)$ is the x-axis and the equation of tangent at $(0, \pm a)$ is the y-axis.

(iv) Asymptotes -

The curve has no asymptotes

(v) Points -

$\frac{dx}{d\theta}$ = a - 3a $\cos^2\theta \sin\theta$

and $\frac{dy}{d\theta}$ = + 3a $\sin^2\theta \cos\theta$

$\therefore$ $\frac{dy}{d\theta}$ = - $\tan\theta$ $\left(\frac{dy}{d\theta} = \frac{dy/d\theta}{dx/d\theta}\right)$

(a) As $\theta$ increases from 0 to $\frac{\pi}{2}$

(b) As $\theta$ increases from $\frac{\pi}{2}$ to $\pi$, x decreases from 0 to -a and y decreases from a to 0 because

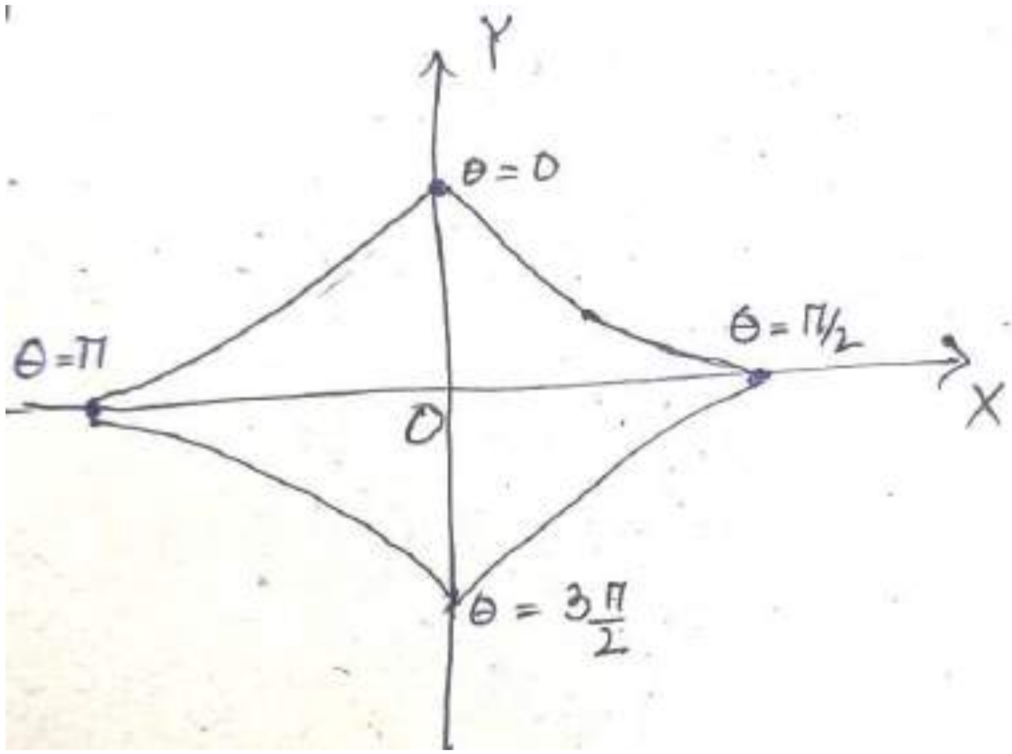
$\frac{dx}{d\theta} < 0$ and $\frac{dy}{d\theta} > 0$ for $\frac{\pi}{2} < \theta < \frac{\pi}{2}$

(c) As $\theta$ increases from $\pi$ to $\frac{3\pi}{2}$, x increases from -a to 0 and y decreases from 0 to -a because $\frac{dx}{d\theta} > 0$ and $\frac{dy}{d\theta} < 0$ for $\pi < \theta < \frac{3\pi}{2}$

(d) As $\theta$ increases from $\frac{3\pi}{2}$ to $2\pi$, x increases from 0 to a and y increase from -a to 0.

We don't get new points for other values of $\theta$ as x and y are periodic function of $\theta$ with period $2\pi$.

The rough sketch of the curve is given below:




## 14.5 Self Check Exercise

Q.1 Trace the curve y = x3

Q. 2 Trace the curve y = (x + 1)2 (x - 3)

Q. 3 Trace the curve $y = \frac{4}{x} + x$

Q. 4 Trace the curve

$x = a (t - \sin t)$

$y = a (1 - \cos t)$

## 14.6 Summary

Dear students, in this unit we have learnt.

(i) curve tracing

(ii) working method for tracing parametric curves

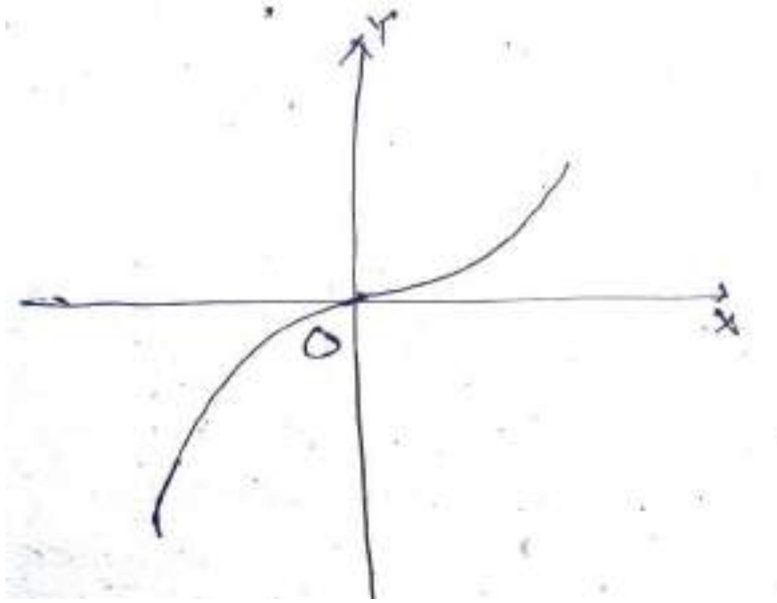
## 14.7 Glossary

(1) **Multiple Points -** A point though which two or more than two branches of a curve pass, is referred to as multiple point.

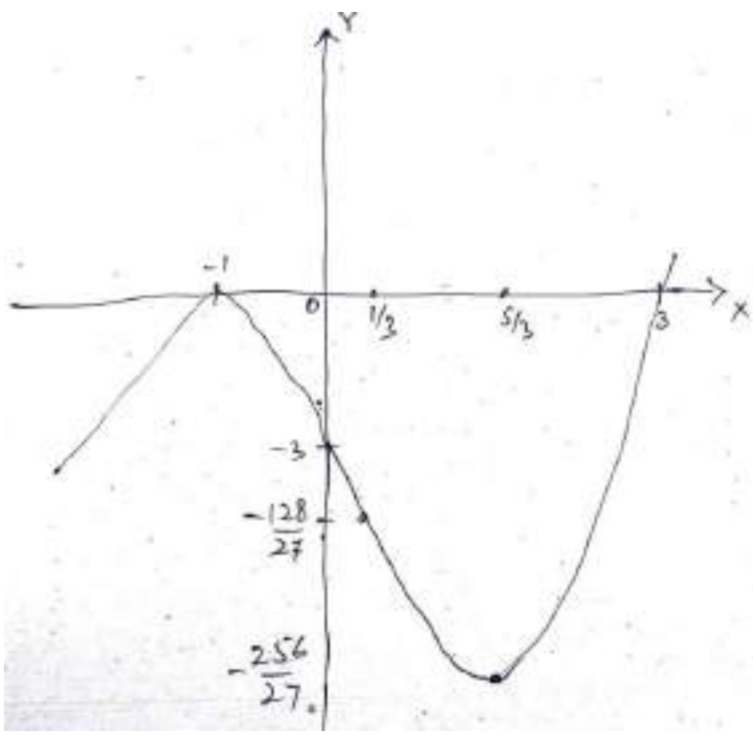
(2) **Isolated Point -** A double point is called on isolated point or conjugate point if two tangents at the double point are not real or there is no real point on the curve in the nhd. of double point.

## 14.8 Answers to Self Check Exercises

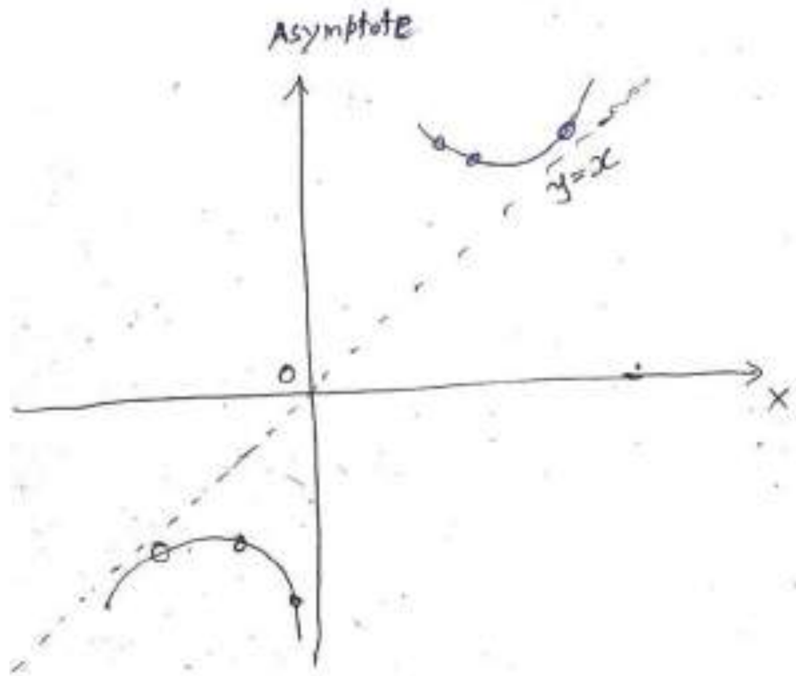
Ans. 1



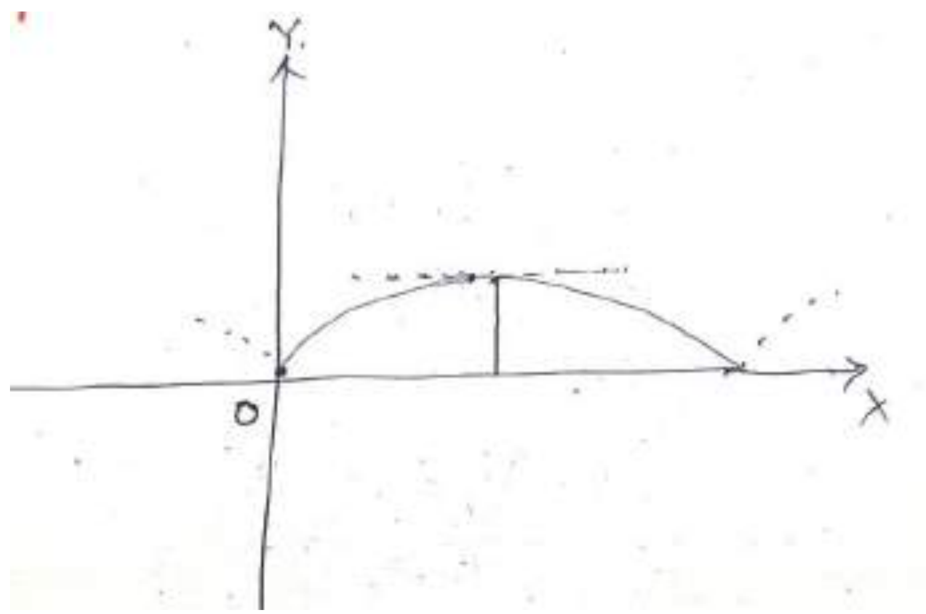

Ans. 2



Ans. 3




Ans. 4




## 14.9 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002

## 14.10 Terminal Questions

1. Trace the curve y = $\frac{-x^2}{1+x^2}$

2. Trace the curve y = $\frac{1+x^2}{x+1}$

3. Trace the curve y = $x^2$ (x - 3a), a > 0

4. Trace the curve

   x = a (cos t + $\frac{1}{2}$ log tan2 $t/2$)

   y = a sin t

   (Tractrix)

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## Unit - 15

# Polar Coordinates And Tracing of Curve In Polar Coordinates

**Structure**



### 15.1 Introduction

Dear students, we are generally introduced to the idea of sketching curves by relating x-values to y-values through a function $f$. That is, we set y = $f$(x) and plot lot of points pair (x, y) to get a good notion of how a curve looks. This method is useful but has limitations, not least of which is that curves that fall the vertical line test cannot be graphed without using multiple points. In previous unit introduced and studied a new way of plotting points in the x-y plane. Using parametric equations x and y values are computed indecently and thereafter plotted together. This method allows us to graph an extraordinary range of curves. This unit introduces another way to plot points in the plane; using polar coordinates.

### 15.2 Learning Objectives

The main objectives of this unit are :

(i) to define Polar coordinates

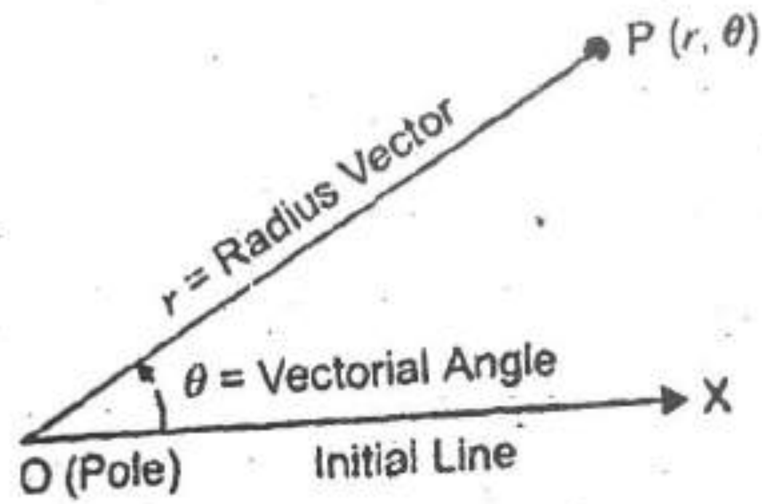
(ii) to study relation between rectangular (cartexin) and polar coordinates.

(iii) to trace the curve in polar coordinates.

### 15.3 Polar Co-ordinates

Let O be a fixed point and OX a fixed straight line through it, whose positive direction OX as shown by the arrow. O is called the pole, and OX is called the initial line.

Let P be a point in a plane through the initial line. Join OP. Then




(i) the length OP is called the radius vector of P, and is denoted by r,

(ii) the angle XOP is called the vectorial angle of P, and is denoted by $\theta$ and

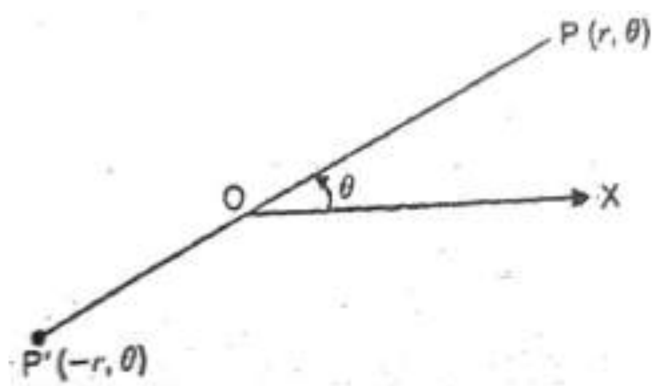
(iii) the two together, taken in this particular order, are called the polar co-ordinates of P, and are denoted by $(r, \theta)$.

Signs of the co-ordinates

(i) $\theta$ is regarded as positive, if it is traced in the counter-clockwise direction, and is negative, if it is traced in the clock-wise direction. Therefore, it follows that $(r, \theta)$ and $(r, \theta + 2n\pi)$, where $n \in Z$, represent the same point and consequently, the vectorial angle can have infinitely many values, with the same value of r.

On this account, we say that vectorial angle of a point is not unique.

(ii) 'r', the radius vector, is positive, if it is cut along the line bounding the vectorial angle and is negative, if it is cut along the opposite direction of the line bounding the vectorial angle.




Let P be the point having the co-ordinates $(r, \theta)$.

Produce PO to P', such that O is mid-point of PP; i.e., |OP| = |OP'| ; then P' will have polar co-ordinates $(-r, \theta)$

Note: OP' is a bounding line of vectorial angle $\theta + \pi$ or $\phi - \pi$

$\therefore$ polar co-ordinates of P' are also $(r, \theta \pm \pi)$

For vectorial angles $\theta + \pi$, or $\theta - \pi$, OP is negative.

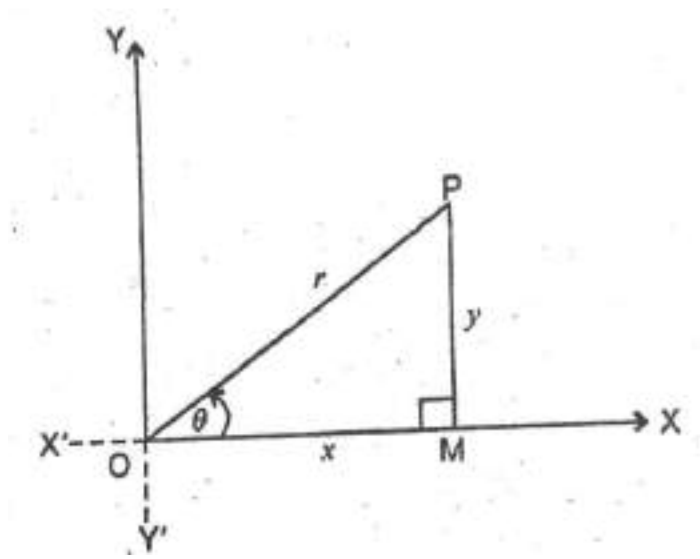
$\therefore$ co-ordinates of P are $(-r, \theta \pm \pi)$

$\therefore$ we conclude that, the polar co-ordinates of a point are not unique, whereas, the rectangular Cartesian co-ordinates of a point are unique.

Conclusion : Giving due consideration to the signs of r and $\theta$, we conclude that the general co-ordinates of a point $(r, \theta)$ are $((-1)^n r, \theta + n\pi)$; where n is any integer.

### 15.4 Relation between Rectangular (Cartesian) and Polar Co-ordinates

Given the point P (x, y) in Cartesian Co-ordinate system, express x and y in Polar form. Also, if P $(r, \theta)$ is a polar coordinate form, express $r, \theta$ in Cartesian form.




**Proof:** Let (x, y) be the Cartesian co-ordinates of P, and $(r, \theta)$ be its polar co-ordinates.

**(i) To express x and y in terms of r and θ**

From P draw PM $\perp$ X'OX

In rt. $\angle$d $\Delta$ OMP,

$$\frac{OM}{OP} = \cos\theta$$

$\therefore$ $OM = OP\cos\theta$

$\therefore$ $x = r\cos\theta$

and $\frac{OM}{OP}$ = sin θ

∴ MP = OP sin θ

∴ y = r sin θ

∴ we have x = r cos θ, y = r sin θ.

**(ii) To express r and θ in terms of x and y**

Now x = r cos θ .....(1)

y = r sin θ .....(2)

Squaring and adding (1) and (2), we get

$x^2 + y^2 = r^2, (\cos^2 \theta + \sin^2 \theta) = r^2 (1) = r^2$

∴ $r^2 = x^2 + y^2$, or $r = \sqrt{x^2+y^2}$

Dividing (2) by (1), we get

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

∴ $\tan\theta = \frac{y}{x}$, or $\theta = \tan^{-1}\frac{y}{x}$.

∴ we have $r = \sqrt{x^2+y^2}$, θ = tap$^{-1}\frac{y}{x}$

**Note 1. In practice we generally use the relations:**

$$r = \sqrt{x^2+y^2}, \cos\theta = \frac{x}{r}, \sin\theta = \frac{y}{r}$$

**Note 2. Rule to change a Cartesian equation to polar equation.**

In the given equation, put x = r cos θ, y - r sin θ, and simplify the result. The resulting equation is the required equation.

**Note 3. Rule to change a polar equation to Cartesian equation.**

(i) In the given equation, put $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and clear off fractions.

(ii) Put $r = \sqrt{x^2+y^2}$, and express the resulting equation in the rational form (i.e., free from fractional powers). The resulting equation is the required equation.

**Example 1:** Find the Cartesian coordinates of the points (2, 60º), (-2, 30º)

**Solution:** Let (x, y) be Cartesian coordinates of (2, 60º) i.e. $\left(2, \frac{\pi}{3}\right)$

$\therefore$ x = 2 cos $\frac{\pi}{3}$ = 2 × $\frac{1}{2}$ = 1

y = 2 sin $\frac{\pi}{3}$ = 2 × $\frac{\sqrt{3}}{2}$ = $\sqrt{3}$

$\therefore$ required Cartesian coordinates of (2, 60º) are $\left(1, \sqrt{3}\right)$

Again let (x, y) be Cartesian coordinates of (-2, 30º) i.e. $\left(-2, \frac{\pi}{6}\right)$

$\therefore$ x = -2 cos $\frac{\pi}{6}$ = -2 × $\frac{\sqrt{3}}{2}$ = -$\sqrt{3}$

y = -2 sin $\frac{\pi}{6}$ = -2 × $\frac{1}{2}$ = -1

$\therefore$ required Cartesian coordinates of $\left(-\sqrt{3}, -1\right)$

**Example 2:** Find the Cartesian coordinates for the point $\left(-\sqrt{3}, -\frac{5\pi}{6}\right)$

**Solution:** Let (x, y) be Cartesian coordinates of $\left(-\sqrt{3}, -\frac{5\pi}{6}\right)$

x = - 3 cos $\left(\frac{5\pi}{6}\right)$ = - 3 cos $\frac{5\pi}{6}$ = - 3 cos $\left(\pi - \frac{\pi}{6}\right)$

= 3 cos $\frac{\pi}{6}$ = 3 × $\frac{\sqrt{3}}{2}$ = $\frac{3\sqrt{3}}{2}$

y = -3 $\left(-\frac{5\pi}{6}\right)$ = 3 sin $\frac{5\pi}{6}$ = 3 sin $\left(\pi - \frac{\pi}{6}\right)$

= 3 sin $\frac{\pi}{6}$ = 3 × $\frac{1}{2}$ = $\frac{3}{2}$

$\therefore$ required Cartesian coordinates of $\left(-\sqrt{3}, -\frac{5\pi}{6}\right)$ are $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$

**Example 3:** (a) Plot the following points whose polar co-ordinates are

(i) $\left(3, \frac{\pi}{4}\right)$ (ii) $\left(-2, \frac{\pi}{3}\right)$ (iii) $\left(3, -\frac{\pi}{4}\right)$

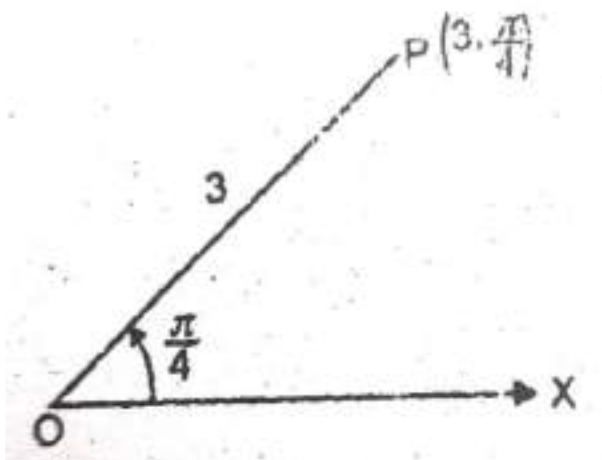
(iv) $\left(-3, -\frac{\pi}{4}\right)$ (v) $\left(2, -\frac{8\pi}{3}\right)$

Also determine their cooresponding Cartesian co-ordinates.

(b) Find the Cartesian coordinates of the points (2, 180o), $\left(3, \frac{\pi}{2}\right)$

(c) Find the Cartesian coordinates of the point P whose polar coordinates are $\left(6, \frac{\pi}{6}\right)$

**Solution:** (a) (i) P $\leftrightarrow \left(3, \frac{\pi}{4}\right)$




Let the +ve half ray of x-axis be taken as polar axis.

Let (x, y) correspond to P $\left(3, \frac{\pi}{4}\right)$

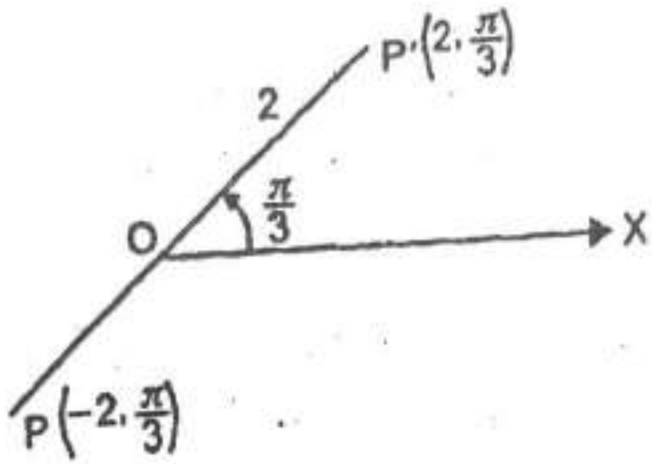
$\therefore$ x = 3 cos $\frac{\pi}{4}$, y = 3 cos $\frac{\pi}{4}$

$\Rightarrow$ x = $\frac{3}{\sqrt{2}}$, y = $\frac{3}{\sqrt{2}}$

Cartesian co-ordinates of P $\left(3, \frac{\pi}{4}\right)$ are $\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$

(ii) P $\leftrightarrow \left(-2, \frac{\pi}{3}\right)$

Here P is the reflection of P' $\left(2, \frac{\pi}{3}\right)$ in the pole.




Let (x, y) correspond to P $\left(-2, \frac{\pi}{3}\right)$

$\therefore$ x = -2 cos $\frac{\pi}{3}$ = -1

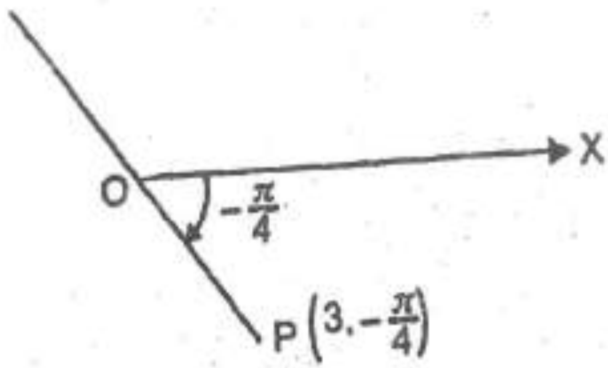
y = -2 sin $\frac{\pi}{3}$ = - $\sqrt{3}$

$\therefore$ $\left(-1, -\sqrt{3}\right)$ are the Cartesian co-ordinates of P $\left(-2, \frac{\pi}{3}\right)$

(iii) P $\leftrightarrow \left(3, -\frac{\pi}{4}\right)$

Let (x, y) correspond to P $\left(3, -\frac{\pi}{4}\right)$

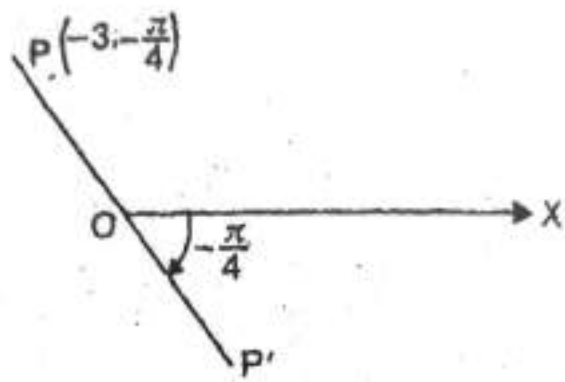
$\therefore$ x = 3 cos $\left(-\frac{\pi}{4}\right)$ = $\frac{3}{\sqrt{2}}$




y = 3 sin $\left(-\frac{\pi}{4}\right)$= - $\frac{3}{\sqrt{2}}$

$\therefore$ $\left(\frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right)$ are cartelist co-ordinates of P $\left(3, -\frac{\pi}{4}\right)$

(iv) P $\leftrightarrow$ $\left(-3, -\frac{\pi}{4}\right)$




Let (x, y) correspond to P $\left(-3, -\frac{\pi}{4}\right)$

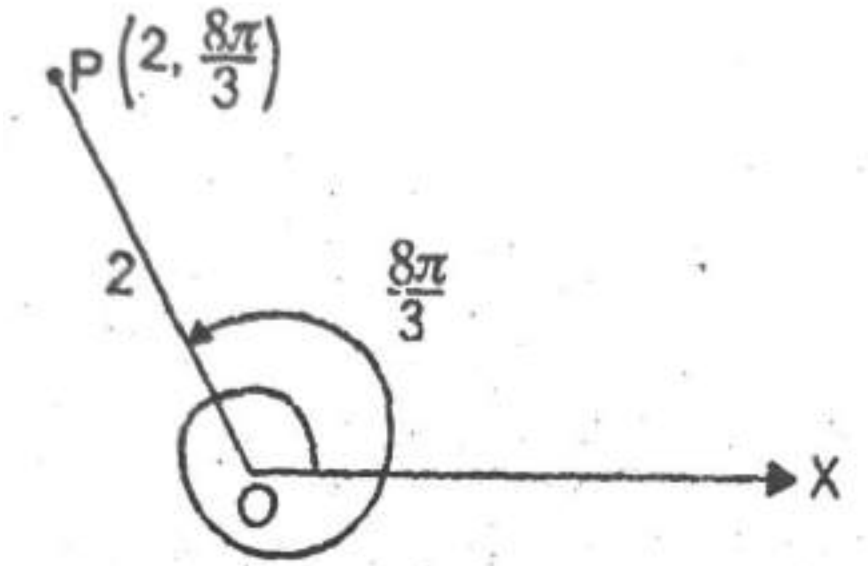
$\therefore$ x = -. cos $\left(-\frac{\pi}{4}\right)$ = - $\frac{3}{\sqrt{2}}$

y = -3 sin $\left(-\frac{\pi}{4}\right)$ = $\frac{3}{\sqrt{2}}$

$\therefore$ $\left(-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$ are the Cartesian co-ordinates of P$\left(-3, -\frac{\pi}{4}\right)$

(v) P $\leftrightarrow \left(2, -\frac{8\pi}{3}\right)$

Let (x, y) correspond to P $\left(2, -\frac{8\pi}{3}\right)$




$\therefore$ x = 2 cos $\frac{8\pi}{3}$ = 2 cos $\left(2\pi + \frac{2\pi}{3}\right)$

= 2 cos $\frac{2\pi}{3}$ = 2 $\left(-\frac{1}{2}\right)$ = -1

y = 2 sin $\frac{8\pi}{3}$ = 2 sin $\left(2\pi + \frac{2\pi}{3}\right)$ = 2 sin $\frac{2\pi}{3}$ = 2 $\left(\frac{\sqrt{3}}{2}\right)$ = $\sqrt{3}$

$\therefore$ $\left(-1, \sqrt{3}\right)$ are the Cartesian co-ordinates of P$\left(2, \frac{8\pi}{3}\right)$

(b) Let (x, y) be Cartesian coordinates of (2, 180o) i.e. (2, $\pi$)

y = 2 sin $\pi$ = 2 $\times$ 0 = 0

$\therefore$ required Cartesian coordinates of (2, $180^o$) are (-2, 0)

Again let (x, y) be Cartesian coordinates of $\left(3, \frac{\pi}{2}\right)$

$\therefore$ $x = 3\cos\frac{\pi}{2} = 3 \times 0 = 0$

$y = 3\sin\frac{\pi}{2} = 3 \times 1 = 3$

$\therefore$ required Cartesian coordinates of $\left(3, \frac{\pi}{2}\right)$ are (0, 3).

(c) Let (x, y) be Cartesian coordinates of $\left(6, \frac{\pi}{6}\right)$

$\therefore$ $x = 6\cos\frac{\pi}{6} = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3}$

$y = 6\sin\frac{\pi}{6} = 6 \times \frac{1}{2} = 3$

$\therefore$ required Cartesian coordinates of $\left(6, \frac{\pi}{6}\right)$ are $\left(3\sqrt{3}, 3\right)$

**Example 4:** Determine the polar co-ordinates of point $\left(2\sqrt{2}, 3\right)$

**Solution:** Let (r, $\theta$) be the point corresponding to $\left(2\sqrt{2}, 3\right)$

$\therefore$ $2 = r\cos\theta$ .....(1)

and $2\sqrt{3} = r\sin\theta$ ......(2)

Squaring (1) and (2) and adding, we get,

$4 + 12 = r^2(\cos^2\theta + \sin^2\theta)$

$\therefore$ $16 = r^2 \quad \Rightarrow \quad r = 4$

$\therefore$ from (1) and (2), we get,

$\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore$ $\theta = \frac{\pi}{3}$ as $\theta$ lies in Ist quadrant

**Example 5:** Find the polar coordinates for the points with the Cartesian coordinates $\left(-\sqrt{3}, 1\right)$

**Solution:** Let (r, $\theta$) be the point cooresponding to $\left(-\sqrt{3}, 1\right)$

$\therefore$ $-\sqrt{3}$ = r cos θ ....(1)

and 1 = r sin θ ......(2)

Squaring and adding (1) and (2), we get

3 + 1 = $r^2$ ($\cos^2$ θ + $\sin^2$ θ) or 4 = $r^2$ $\Rightarrow$ r = 2

$\therefore$ from (1) and (2), we get

cos θ = $-\dfrac{\sqrt{3}}{2}$, sin θ = $\dfrac{1}{2}$

$\therefore$ tan θ = - $\dfrac{1}{\sqrt{3}}$ $\Rightarrow$ θ = π = $\dfrac{\pi}{6}$

[Q θ lies in IInd quardrant]

$\Rightarrow$ θ = $\dfrac{5\pi}{6}$

$\therefore$ polar coordinates of $\left(-\sqrt{3},1\right)$ are $\left(2,\dfrac{5\pi}{6}\right)$

**Example 6:** Find the polar coordinates of the point whose Cartesian coordinates are (-2, 2).

**Solution:** Let (r, θ) be the point corresponding to (-2, 2)

$\therefore$ -2 = r cos θ ......(1)

and 2 = r sin θ ......(2)

Squaring and adding (1) and (2), we get

4 + 4 = $r^2$ ($\cos^2$ θ + $\sin^2$ θ) or 8 = $r^2$ $\Rightarrow$ r = $2\sqrt{2}$

$\therefore$ from (1) and (2), we get

cos θ = = $\dfrac{1}{\sqrt{2}}$, sin θ = $\dfrac{1}{\sqrt{2}}$

$\therefore$ tan θ = -1 $\Rightarrow$ θ = π - $\dfrac{\pi}{4}$ [Q θ lies in IInd quadrant]

$\Rightarrow$ θ = $\dfrac{3\pi}{4}$

$\therefore$ polar coordinates of (-2, 2) are $\left(2\sqrt{2}\cos\dfrac{3\pi}{4},2\sqrt{2}\sin\dfrac{3\pi}{4}\right)$

## 15.5 Polar Coordinates And Tracing Of Curves In Polar Coordinates

Procedure for Tracing Polar Curves

We shall keep in mind the following points for tracing the graphs of the equation $f(r, \theta) = 0$.

### I. Symmetry

(i) Symmetry about the initial line or x-axis : If the equation of the curve remains unchanged when $\theta$ is changed to $-\theta$, the curve is symmetrical about the initial line.

(ii) Symmetry about the line $\theta = \frac{\pi}{2}$ or y-axis : If the equation of the curve remains unchanged when $\theta$ is changed to $\pi - \theta$ or when $\theta$ changed to $-\theta$ and r to -r, the curve is symmetrical about the line $\theta = \frac{\pi}{2}$

(iii) Symmetry about the line $\theta = \frac{\pi}{4}$ or y = x : If the equation of the curve remains unchanged when $\theta$ is changed to $\frac{\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{\pi}{4}$

(iv) Symmetrical about the lie $\theta = \frac{3\pi}{4}$ or y = - x : if the equation of the curve remains unchanged when $\theta$ is changed to $\frac{3\pi}{2} - \theta$, the curve is said to be symmetrical about the line $\theta = \frac{3\pi}{4}$.

(v) Symmetry about the pole : If the equation of the curve remains unchanged when r is changed to -r, the curve is said to be symmetrical about the pole.

### II. Pole

(i) Find whether the curve passes through the pole or not. It can be done by putting r = 0 in the equation and then finding some real value of $\theta$. If it is not possible to find a real value of $\theta$ for which r = 0, then the curve does not pass through the pole.

(ii) Find the tangents at the pole. Putting r = 0, the real values of $\theta$ give the tangents at the pole.

(iii) Find the points where the curve meets the initial line and the line $\theta = \frac{\pi}{2}$.

**III. Value of $\phi$**

Find $\phi$ from the result $\tan \phi = r \dfrac{d\theta}{dr}$. Then find the points where $\phi = 0$ or $\dfrac{\pi}{2}$.

**IV. Asymptotes**

If $r \to \infty$ as $\theta \to \theta_1$ (any fixed number), then there is an asymptote. Find it by the method given below:

(i) Write down the given equation as $\dfrac{1}{r} = f(\theta)$, say.

(ii) Equate $f(\theta)$ to zero and solve for $\theta$. Let the roots be $\theta_1, \theta_2, \ldots\ldots$

(iii) Find $f'(\theta)$ and calculate it at $\theta = \theta_1, \theta_2, \ldots\ldots$

(iv) Asymptotes are $r \sin(\theta - \theta_1) = \dfrac{1}{f'(\theta)}$, $r \sin(\theta - \theta_2) = \dfrac{1}{f'(\theta_2)}$, ......

**V. Special Points**

Find some points on the curve for convenient values of $\theta$.

**VI. Region**

Solve the given equation for r or $\theta$. Find the region in which the curve does not lie. This can be done in the following manner.

(i) No part of the curve lies between $\theta = \alpha$ and $\theta = \beta$ if for $\alpha < \theta < \beta$, r is imaginary.

(ii) If the greatest numerical value of r be a, the curve lies entirely within the circle r = a. If the least numerical value of r be b, the curve lies outside the circle r = b.

**Example 7:** (a) Trace the curve $r = a(1 + \cos\theta)$, $a > 0$.

(b) Trace the curve $r = 5(1 + \cos\theta)$

When $\theta$ increases from 0 to $\pi$, r remains positive and decreases from 2 a to 0.

When $\theta$ increases from $\pi$ to $2\pi$, r remains positive and increases from 0 to 2 a.

The shape of the curve is an shown in the figure.

(b) Take a = 5.

**Example 8:** Trace the curve $r = a(1 - \cos\theta)$

Solution: The equation of the curve is $r = a(1 - \cos\theta)$ .....(1)

**I. Symmetry**

The equation of the curve remains unchanged when $\theta$ is changed to $-\theta$.

$\therefore$ curve is symmetrical about the initial line.

**II. Pole.**

Putting r = 0 in (1), we get,

a (1 - cos $\theta$) = 0 or cos $\theta$ = 1 $\Rightarrow$ $\theta = 0$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = 0$

The curve cuts initial $\theta = 0$ at (0, 0) and the lines $\theta = +\frac{\pi}{2}$ at $\left(a, \frac{\pi}{2}\right)$, $\left(a, -\frac{\pi}{2}\right)$

**III. Value of $\phi$**

$$\frac{dr}{d\theta} = a \sin \theta$$

$\therefore$ $\tan \phi = r \frac{d\theta}{dr} = a(1 - \cos \theta) \frac{1}{a \sin \theta} = \frac{2\sin^2 \frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

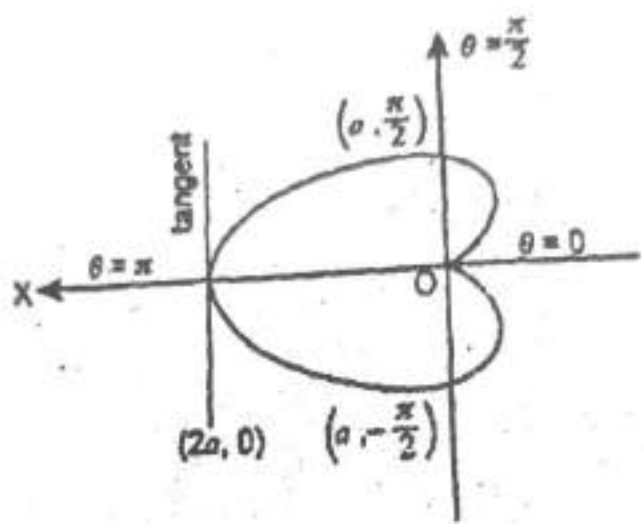
$\Rightarrow$ $\tan \phi = \tan \frac{\theta}{2}$ $\Rightarrow$ $\phi = \frac{\theta}{2}$

$\therefore$ $\phi = 0$ when $\theta$ when $\theta = 0$, r = 0

$\therefore$ at (0, 0), the tangent coincides with the initial line.

**IV. Asymptotes: Since r does not tend to infinity for any finite value of $\theta$.**

$\therefore$ curve has got no asymptote




**V. Special Points : We have**

$\theta$ : a $\frac{\pi}{4}$ $\frac{\pi}{2}$ $\pi$

r : 2a $\left(1-\frac{1}{\sqrt{2}}\right)$ a 2a

Solution: (a) The equation of the curve is r = a (1 + cos θ)

1. Symmetry : The equation of the curve remains unchanged when θ is changed to - θ.

∴ curve is symmetrical about the initial line.

**II. Pole :** Putting r = 0 in (1), we get

a (1 + cos θ) = 0 or cos θ = -1

∴ θ = π

∴ pole lies on the curve and tangent at the pole is θ = π,

The curve cuts the initial line θ = 0 at (2a, 0) and the lines θ = + $\frac{\pi}{2}$ at $\left(a,\frac{\pi}{2}\right)$, $\left(a,-\frac{\pi}{2}\right)$

**III. Value of φ**

$\frac{dr}{d\theta}$ = - a sin θ

∴ $\tan\phi = r\,\frac{d\theta}{dr} - a(1+\cos\theta)\times\frac{1}{-a\sin\theta} = -\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

∴ $\tan\phi = -\cot\frac{\theta}{2}$ ⇒ $\tan\phi = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$ ⇒ $\phi = \frac{\pi}{2}+\frac{\theta}{2}$

∴ $\phi = \frac{\pi}{2}$ when θ = 0, r = 2 a

∴ at (2 a, 0), the tangent is perpendicular to initial line.

**IV. Asymptotes:**

Since r does not tend to infinity for any finite value of θ.

∴ curve has got no asymptote

**V. Special points.**

We have

θ : 0 $\frac{\pi}{4}$ $\frac{\pi}{2}$ π

r : 2a $a\left(1+\frac{1}{\sqrt{2}}\right)$ a 0

**VI. Region.**

Since r = a (1 + cos $\theta$)

$\therefore$ max. value of r = 2 a

$\therefore$ curve lies entirely within the circle r = 2a

**VI. Region:**

Since r = a (1 - cos $\theta$)

$\therefore$ max. value of r = 2 a

$\therefore$ curve lies entirely within the circle r = 2 a.

When $\theta$ increases from 0 to $\pi$, r remains positive and increases from 0 to 2 a.

When $\theta$ increases from $\pi$ to 2 $\pi$, r remain positive and decreases from 2 a to 0.

The shape of the curve is given in the figure.

**Example 9:** Trace the curve r = a (1 + sin $\theta$)

**Solution:** The equation of given curve is

r = a (1 + sin $\theta$) .....(1)

**I. Symmetry.** The equation of the remains unchanged when $\theta$ is changed to $\pi$ - $\theta$.

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{2}$

**II. Pole.** Putting r = 0 in (1), we get

a (1 + sin $\theta$) = 0 or sin $\theta$ = -1 $\Rightarrow$ $\theta = \frac{3\pi}{2}$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = \frac{3\pi}{2}$

The curve cuts the initial line $\theta$ = 0 at (a, 0) and the line $\theta = \frac{\pi}{2}$ at $\left(2a, \frac{\pi}{2}\right)$

**III. Value of $\phi$**

$\frac{dr}{d\theta}$ = a cos $\theta$

$\therefore \quad \tan\phi = r\,\dfrac{d\theta}{dr} = a\,(1 + \sin\theta).\ \dfrac{1}{a\cos\theta} = \dfrac{\sin^2\dfrac{\theta}{2} + \cos^2\dfrac{\theta}{2} + 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}}{\cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2}}$

$$= \frac{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)} = \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}$$

$$= \frac{1 + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

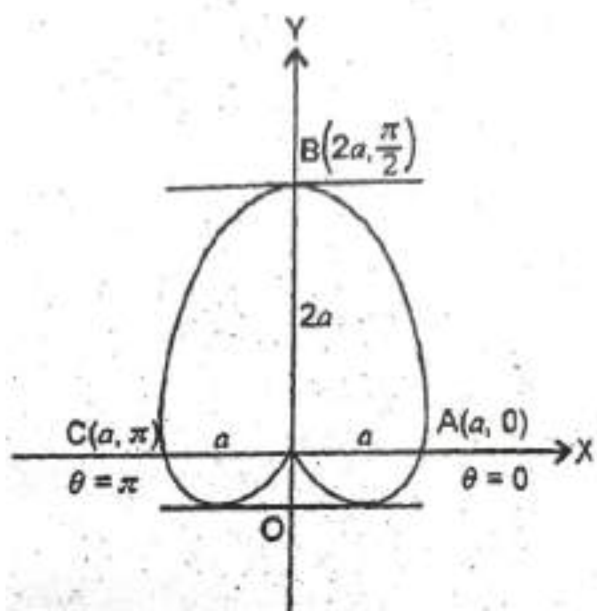
$\therefore \quad \phi = \dfrac{\pi}{4} + \dfrac{\theta}{2}$

When $\theta = 0$, $\phi = \dfrac{\pi}{4}$

When $\theta = \dfrac{\pi}{2}$, $\phi = \dfrac{\pi}{2}$ $\Rightarrow$ at $\theta = \dfrac{\pi}{2}$, the tangent is perpendicular to the line $\theta = \dfrac{\pi}{2}$.

**IV. Asymptotes.** Since r does not tend to infinity for any finite value of $\theta$.

$\therefore$ curve has got no asymptote



V **Special Points.** We have

| | | | | |
|---|---|---|---|---|
| $\theta$ : | $-\frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ | $\pi$ |
| r : | 0 | a | 2 a | a |

**VI Region.** We know that

$$|\sin \theta| \leq 1$$

$\therefore$ from (1), $r \leq 2a$

$\therefore$ curve lies entirely within the circle r = 2 a

When $\theta$ increases from 0 to $\frac{\pi}{2}$, the value of r remains positive and increases from a to 2a. As $\theta$ increases from $\frac{\pi}{2}$ to $\pi$, r remains positive but decreases from 2 a to a.

The shape of the curve is given in the figure.

**Example 10:** Trace the curve $r = a(1 - \sin\theta)$

**Solution:** The equation of given curve is

$$r = a(1 - \sin\theta) \qquad .....(1)$$

**I. Symmetry:** The equation of the curve remains unchanged when $\theta$ is changed to $\pi - \theta$

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{2}$

**II. Pole.** Putting r = 0 in (1), we get,

$$a(1 - \sin\theta) = 0 \quad \Rightarrow \quad \sin\theta = 1 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$$

$\therefore$ pole lies on the curve and tangent at the pole is $\theta = \frac{\pi}{2}$.

The curve cuts the initial line $\theta = 0$ at (a, 0) and the line $\theta = \frac{3\pi}{2}$ at $\left(2a, \frac{3\pi}{2}\right)$

**III. Value of $\phi$**

$$\frac{dr}{d\theta} = -a\cos\theta$$

$\therefore$ $\tan\phi = r\frac{d\theta}{dr} = a(1 - \sin\theta).\frac{1}{-a\cos\theta}$

$$= -\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}$$

$$= -\frac{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)^2} = \frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}$$

$$= -\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}} = -\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right) = \tan\left(\frac{3\pi}{4}+\frac{\theta}{2}\right)$$

$\therefore$ $\phi = \frac{3\pi}{4}+\frac{\theta}{2}$

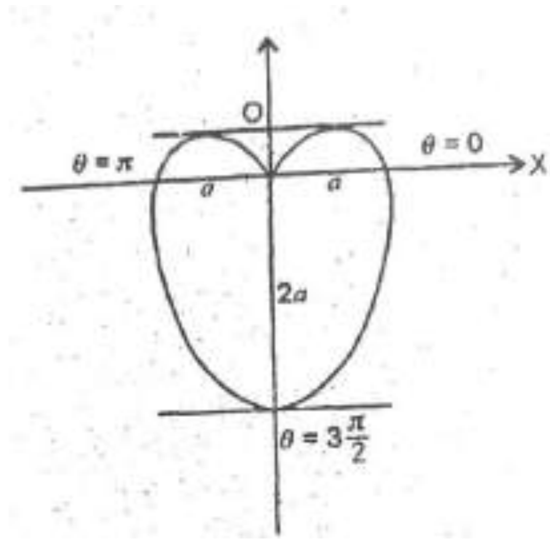
When $\theta = 0$, $\phi = \frac{3\pi}{4}$

When $\theta = \frac{3\pi}{2}$, $\phi = \frac{3\pi}{2}$

$\therefore$ at $\theta = \frac{3\pi}{2}$, tangent is perpendicular to the line $\theta = \frac{3\pi}{2}$

**IV. Asymptotes:** Since r does not tend to infinity for any finite value of $\theta$.

$\therefore$ curve has got no asymptote.




**V.** **Special Points.** We have

| | | | | |
|---|---|---|---|---|
| $\theta$ : | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
| r : | a | 0 | a | 2 a |

**VI.** **Region.** We know that $|\sin\theta| \le 1$

$\therefore$ from (1), $r \le 2a$

$\therefore$ curve lies entirely within the circle $r = 2a$

When $\theta$ increases from 0 to $\frac{\pi}{2}$, the value of r remains positive and decreases from a to 0. As $\theta$ increases from $\pi$ to $\frac{3\pi}{2}$, the value of r remains positive and increases from a to 2a.

The shape of the curve is given in the figure.

**Example 11:** Trace the curve $r = a + b\cos\theta$, $a > b$

**Solution:** The equation of curve is $r = a + b\cos\theta$ .....(1)

**I. Symmetry:** The equation of the curve remains unchanged when $\theta$ is changed to $-\theta$

$\therefore$ curve is symmetrical about the initial line.

**II.** **Pole.** Putting $r = 0$ in (1), we get,

$a + b\cos\theta = 0$ or $\cos\theta = \frac{a}{b}$

$\therefore$ $|\cos\theta| = \frac{a}{b} > 1$ as $a > b$

This is not possible

$\therefore$ for no value of $\theta$, is equal to zero

$\therefore$ curve does not pass through the pole.

**III. Value of $\phi$**

$$\frac{dr}{d\theta} = - b \sin \theta$$

$\therefore$ $\tan \phi = r \dfrac{d\theta}{dr} = \dfrac{a + b\cos\theta}{-b\sin\theta} \neq 0$

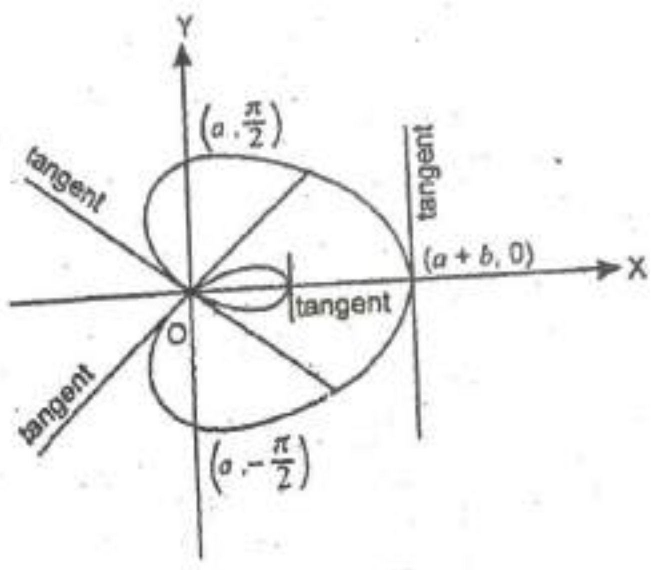
$\therefore$ $\phi \neq 0$ at any point

When $\theta = 0$, $r = a + b$, then $\phi = \dfrac{\pi}{2}$

$\therefore$ at $(a + b, 0)$, the tangent is perpendicular to initial line.

**IV. Asymptotes:** Since r does not tend to infinity for any finite value of $\theta$.

$\therefore$ curve has got no asymptote.




**V. Special Points.** We have

| | | | |
|---|---|---|---|
| $\theta$ : | 0 | $\frac{\pi}{2}$ | $\pi$ |
| r : | a + b | a | a - b |

**VI. Region.** Since $r = a + b \cos \theta$ and $|\cos \theta| \leq 1$

$\therefore$ $|r| \leq a + b$

$\therefore$ curve lies entirely within the circle r = a + b.

When $\theta$ increases from 0 to $\frac{\pi}{2}$, r remains positive but decreases from a + b to a.

When $\theta$ increases from $\frac{\pi}{2}$ to $\pi$, r decreases from a to a - b.

The shape of the curve is given in the figure.

**Example 12:** Trace the curve $r = 2 + 3 \cos \theta$

**Solution:** The equation of curve is

$$r = 2 + 3 \cos \theta \qquad .....(1)$$

**I. Symmetry:** The equation of the curve remains unchanged when $\theta$ is changed to $-\theta$.

$\therefore$ curve is symmetrical about the initial line.

**II. Pole.** Putting r = 0 in (1), we get,

$$2 + 3 \cos \theta = 0 \text{ or } \cos \theta = -\frac{2}{3} = \cos(\pi - \alpha), \text{ say}$$

$\therefore$ $\theta = \pi - \alpha$ is tangent to the curve at pole where $\alpha$ is given by

$$\cos(\pi - \alpha) = -\frac{2}{3}$$

**III. Value of $\phi$**

$$\frac{dr}{d\theta} = -3 \sin \theta$$

$\therefore$ $\tan \phi = r \frac{d\theta}{dr} = \frac{2 + 3\cos\theta}{-3\sin\theta}$

Now $\tan \phi = 0$ when $2 + 3 \cos \theta = 0$ or $\cos \theta = \frac{2}{3}$ or $\theta = \pi - \alpha$

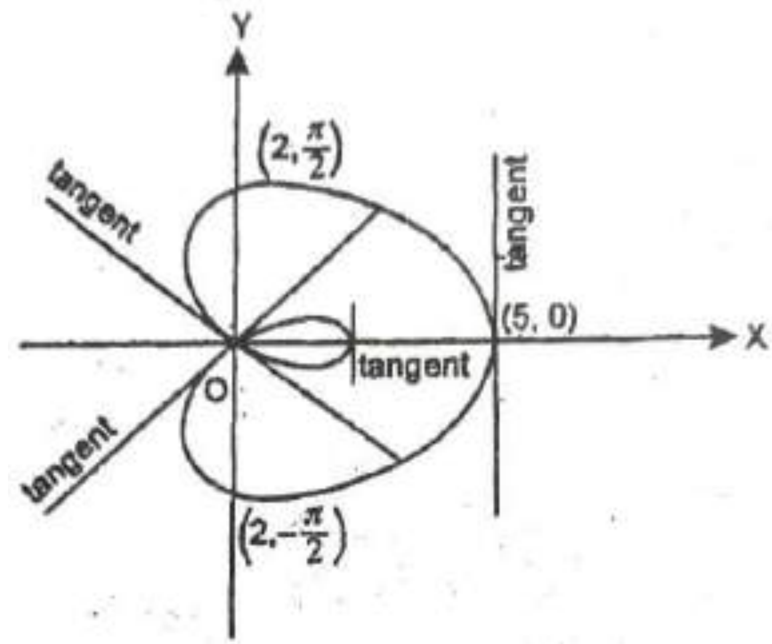
$\therefore$ at $(0, \pi - \alpha)$, the tangent to the curve is parallel to initial line.

Now $\tan \phi \to \infty$ when $\theta = 0$ or $\pi$

$\therefore$ at (5, 0) and $(-1, \pi)$, the tangent is perpendicular to the initial line.

**IV. Asymptotes:** Since r does not tend to infinity for any finite value of $\theta$.

$\therefore$ curve has got no asymptote.




**V.** **Special Points.** We have

| | | | |
|---|---|---|---|
| $\theta$ : | 0 | $\frac{\pi}{2}$ | $\pi$ |
| r : | 5 | 2 | -1 |

**VI.** **Region.** We know that $|\cos\theta| \le 1$

$\therefore$ from (1), $r \le 5$

$\therefore$ curve lies entirely within the circle r = 5.

As $\theta$ increases from 0 to $\frac{\pi}{2}$, r remains positive and decreases from 5 to 2. When $\theta$ increases from $\frac{\pi}{2}$ to $\pi$, r decreases from 2 to 0 and then from 0 to -1.

A rough sketch of the curve is given in the figure.

**Example 13:** Trace the curve $r = a\sin 3\theta$, $a > 0$.

**Solution:** The equation of curve is $r = a\sin 3\theta$ .....(1)

**I. Symmetry:** The equation of the curve remains unchanged when $\theta$ is changed to $\pi - \theta$.

$\therefore$ curve is symmetrical about the line $\theta = \frac{\pi}{2}$.

**II.** **Pole.** Putting r = 0 in (1), we get,

$a \sin 3\theta = 0$ or $\sin 3\theta = 0$

$\therefore$ $3\theta = 0, \pi, 2\pi, 3\pi, ....$

or $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, ....$

$\therefore$ the curve passes through the pole and the tangent at pole are

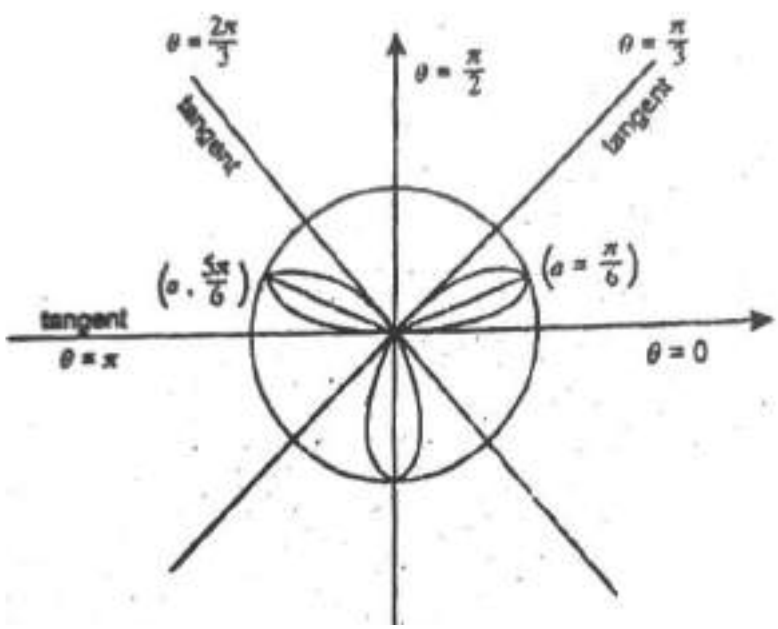
$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}$ as the other value of $\theta$ give the same tangents.

**III. Asymptotes:** Since r does not tend to infinity for any finite value of $\theta$.

$\therefore$ curve has got no asymptote.

**IV. Special Points.** We have

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\theta$ : | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{5\pi}{6}$ | $\pi$ |
| r : | 0 | a | 0 | a | 0 | a | 0 |




**V. Region.** From (1), $| r | = a |\sin \theta| \leq a$

$\therefore$ curve lies entirely within the circle r = a.

When $\theta$ increases from 0 to $\frac{\pi}{6}$, r positive and increases from 0 to a.

When $\theta$ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is positive and decreases from a to 0.

Thus we get a loop between the liens $\theta = 0$ and $\theta = \frac{\pi}{3}$.

As $\theta$ increase from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r is negative and numerically increases from 0 to a and when $\theta$ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r is negative and numerically decreases from a to 0.

$\therefore$ we get another loop between the lies $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$

When $\theta$ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r positive and increases from 0 to a and when $\theta$ increases from $\frac{5\pi}{6}$ to $\pi$, r is positive and decreases from a to 0.

$\therefore$ we get third loop between the lines $\theta = \frac{2\pi}{3}$ and $\theta = \pi$.

When $\theta$ varies from $\pi$ to $2\pi$, the same loops are repeated and we do not get any new loop as r is periodic.

The curve $r = a \sin n\theta$ or $r = a \cos n\theta$

consists of n or 2 n loops according as n is odd or even.

## 15.6 Self Check Exercise

Q.1 Determine the polar coordinates for the following point

(i) (-5, -12)

(ii) $(2, -2\sqrt{3})$

Q. 2 Transform the equation

$x^2 + y^2 - 2x + 2y = 0$

into polar coordinates

Q. 3 Trace the curve

$r = a \cos 3\theta$

## 15.7 Summary

In this unit we have learnt.

(i) polar coordinates

(ii) the relation between Cartesian (rectangular) and polar coordinates

(iii) to trace the curve in polar coordinates.

## 15.8 Glossary

(1) **Angle between radius vector and tangent -**

If $\phi$ = angle between radius vector and tangent then

$\tan \phi = r. \dfrac{d\theta}{dr}$, (r, $\theta$) is any pointer the curve.

(2) **Angle of intersection of two curves -**

If $\phi_1$ and $\phi_2$ are the angles which the tangents to the two curves make with radius vector then

Angle of intersection of two curves = $\phi_2 - \phi_1$

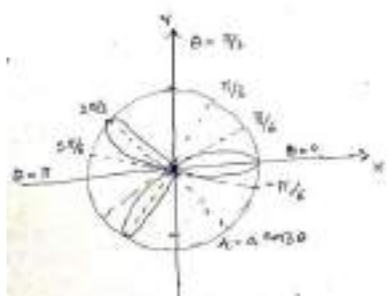
## 15.9 Answers to Self Check Exercises

Ans. 1 (i) $\tan \theta = 12/5$

(ii) $(4, -\dfrac{\pi}{3})$

Ans. 2 r = 1 (cos $\theta$ - sin $\phi$)

Ans. 3




## 15.10 Reference/Suggested Reading


1. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002

2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007


## 15.11 Terminal Questions

1. Find the polar coordinates of the points

(i) (-7, -12)

(ii) (-3, -4)

(iii) (-1, -1)

2. Transform the equation

$x^2 + y^2 - 2x + 2y = 0$

into polar coordinates

3. If $r = \frac{a}{\theta}$, prove that $\tan \phi = -\theta$

4. Find the angle between the curve

   $r = 2a \cos \theta$

   $r = 2a \sin \theta$

5. Trace the curve

   (i) $r\theta = a$

   (ii) $r2 = a2 \cos 2\theta$

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# Unit - 16

# Function of Several Variables
# (up to Three Variables)

**Structure**



## 16.1 Introduction

Dear students, we are already acquainted with the concepts of limit, continuity, differentiability and integrability of a real-valued function of a single real variable whose domain and range are the subsets of set R (set of real numbers). In this unit we shall study those functions whose domain and range may not be subsets of R, i.e. functions whose domain and range are subset of $R^n$ ($n \geq 2$). Such functions are called functions of several variables. However, our main concern is to study function which are dependent on up to three variables (two or three).

## 16.2 Learning Objectives

The main objectives of this unit are

(i) to study a function of two and three variables defined in a certain domain.

(ii) to define nhd. of a point in $R^2$ and in $R^3$.

(iii) to study limit and continuity of a function of two and three variables.

(iv) to study algebra of limit and continuity of function of two and three variables.

## 16.3 Function of Two and Three Variables

Let us recollect the definition of the Cartesian product of sets. Set A, B be any two sets, then the set

$A \times B = \{ (a, b) : a \in A, b \in B\}$, is called a Cartesian product of the sets and B and the element of the set (a, b) is called an ordered pair and (a, b) $\neq$ (a) unless a = b.

We define

$R^2 = R \times R$ = {x, y) : x, y $\in$ R}, which represents two dimensional plane.

Similarly

$R^3 = R \times R \times R$ = { (x, y, z) : x, y, z $\in$ R},

Which represents three dimensional plane.

Let x, y $\in$ R be two variables. One of these variables, say x, may have any value belonging to a certain given interval and corresponding to this value x, y may have any value belonging to any given interval or a set of intervals. In this way we obtain a system of order pairs of numbers (x, y). Now, if to each possible pair (x, y), we associate, in any manner whatsoever, a value of another variable, say x, then we say that z is a function of two variables x and y. We write it as z = $f$(x, y) where x, y, are independent variables and z is a dependent variable. The aggregate of the ordered pairs of numbers (x, y) is called the domain or region of definition of the functions and the set of corresponding value z $\in$ R is called range of the function.

**Similarly we can define a function of three variables.**

**Remark 1 :** In the theory of function of two and three variables, an ordered pair of numbers (x, y) is called point of $R^2$ and an ordered triad of numbers (x, y, z) is called point of $R^3$.

**Remark 2 :** the elements of R are scales whereas the elements of $R^2$ and $R^3$ are vectors.

Now we state a few definitions which will be used during the course of discussion.

**Definitions**

**Definition 1 :** Neighbourhood of a point in $R^2$.

Let (a, b) $\in R^2$ be a point of $R^3$. Let $\delta > 0$ (however small it may be) be a real number. Then the set of points lying within a circle having a centre at (a, b) and radius $\delta$, called a symmetric neighbourhood of the point (a, b) and is written as S((a, b), $\delta$ ). If P (x, y) is any point in this circular region, then

$$\sqrt{(x-a)^2+(y-b)^2} < \delta$$

**Remark 3 :** The circular region S((a, b), $\delta$ ) - [(a, b)] is called a deleted neighbourhood of (a, b).

**Definition 2 :** Neighbourhood of a point in $R^3$.

Let (a, b, c) $\in R^2$ be a point of $R^3$. Let $\delta > 0$ (however small) be a real number. Then the set of points lying within a circle having a centre at (a, b, c) and radius $\delta$, is called a symmetric neighbourhood of the point (a, b, c) and is written as S((a, b, c), $\delta$ ). If P (x, y, z) is any point in this spherical region, then

$$\sqrt{(x-a)^2+(y-b)^2+(z-c)^2<\delta}$$

**Remark 5 :** The spherical region S((a, b, c), $\delta$ ) - [(a, b, c)] is called a deleted neighbourhood of (a, b, c).

**Definition 3 : Open Set.**

An subset of $R^2$ is called an open set, if either A is an empty set or A is a neighbourhood of each of its points.

**Definition 4 :** Closed Set

A subset A $R^2$ is called a closed set if its complement $A^c$ is an open set.

**Definition 5 :** Limit point of a set in $R^2$.

**First Definition.** Let $A \subset R^2$. A point $a \in R^2$ is called a limit point of the set A if every nhood. Of a contains at least one point of A other than a.

**Second Definition.** Let $A \subset R^2$. A point $a \in R^2$ is called a limit point of the set A if every nhood, of a contains Infinitely many points of A.

**Remark 6.** It can be proved that the above two definitions are equivalent.

**Remark 7.** Limit point of a set, If it exists, may of may not belong to the set.

**Real Value Function of Two and Three Variables.**

**Definition 6 :**

Let $A \subset R^2$. The mapping $f : A \to R$ is called a real valued function of two variables.

e.g. $f$(x, y) = x2 + y2, $f(x, y) = \frac{x^3 - y^2}{x^2 + y^2}$ and $f$ (x, y) = sin (x + y) are functions of two variables x, y.

**Definition 7 :**

Let $A \subset R^3$. The mapping $f : A \to R$ is called a real valued function of three variables.

e.g. $f(x, y, z) = x^2 + y^2 + z^2$, $f(x, y, z) = xy + yz + zx$ and $f(x, y, z) = \log (x + y + z)$ are functions of three variables x, y, z.

## 16.4 Limit And Continuity of Function of Two And Three Variables.

**Simultaneous Limit of a Function of Two Variables**

**Definition 8 :** Let $A \subset R^2$. Let $f : A \to R$ be a real valued function of two variable x, y defined on a circular nhood with centre (a, b), except possibly at point (x, y)

Then a real number is called a limit of $f$ (x, y) at (a, b) If given > 0, there exists a real $\delta > 0$ ( $\delta$ depending upon and (a, b), such that

$|f(x, y) - l| < f$ or $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ and we write $\lim_{(x,y)\to(a,b)(x,y)\in A} f(x, y) = l$

**Remark 9 :** The region < $\sqrt{(x-a)^2+(y-b)^2}$ < δ can also be written as 0 < |(x, y) - (a, b)| < δ

**Cor.** The limit l is also called the double limit of $f$ when x, y → a, b simultaneously.

**Another definition of a function of two variables:**

Let $f$ be a real valued function of two variables x and y defined on a square neighbourhood with centre (a, b), except possibly at (a, b)

Then $f$ (x, y) is said to have a limit l ∈ R as (x, y) → (a, b) if for given > 0, however small, there exists a +ve real δ (depending upon) such that

|$f$(x, y) - l| < whenever

0 < |x - a| < δ , 0 < |y - b| < δ and we write it as

$$\lim_{(x,y)\to(a,b)} f\,(x, y) = l$$

**Definition 9 : Limit of a function of three variables.**

Let A ⊂ R3. let $f$ : A → R be a real valued function. Let (a, b, c) be a limit point of A then a real number l is called a limit of (x, y, z) at (a, b, c) if for a given > 0, there exists a real δ > 0) (depending upon and (a, b, c) such that |$f$(x,y,z) - l| < $f$ or 0 < $\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}$ < δ and we write

$$\lim_{(x,y,z)\to(a,b,c)} f\,(x, y, z) = l$$

**Remark 10 :** |$f$ (x, y, z) - l| < $f$ or 0 < |(x, y, z) - (a, b, c)| < δ

or |$f$(x, y, z) = l| < $f$ or 0 < |(x, y, z) - (a, b, c)| < δ .

**Algebra of Simultaneous Limits**

Theorems. Let $f$, g : A → $R^2$ where A ⊂ $R^2$, be two functions.

Let (a, b) be limit point of A, then

1. If $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} f(x, y)$ = l, exists then $f$ is bounded in some deleted neighborhood of (a, b) but the converse is not true.

2. If $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} f\,(x, y)$ = l, exists and $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} g(x, y)$ = m, exists then:

   (a) $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} (f + g)(x, y)$ = l + m, exists

   (b) $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} (fg)(x, y)$ = lm, exists

(c) $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} \left(\frac{1}{g}\right)$(x, y) = $\frac{1}{m}$, exists, provided m $\neq$ 0

(d) $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} \left(\frac{f}{g}\right)$(x, y) = $\frac{1}{m}$, exists, provided m $\neq$ 0

(e) $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}}$ (k$f$) (x, y) = kl, exists, k being a real number.

The converses of 4(a) to 4(e) are not ture.

3. If

$\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} f$(x, y) =

l, exists then $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} |f$(x, y)| = |l|, exists but its converse is not true.

All the above theorems are easy to prove. (Students are advised to prove themselves).

**Theorem 4:** $\lim_{(x,y)\to(a,b)} f$(x, y) exists then it is unique.

**Proof:** If possible, let $\lim_{(x,y)\to(a,b)} f$(x, y) = l and $\lim_{(x,y)\to(a,b)} f$(x, y) = l' where l' $\neq$ l

Let us assume that l' > l

let = $\frac{l'-l}{2}$ > 0

$\lim_{(x,y)\to(a,b)} f$(x, y) = l

$\therefore$ $f$ or given > 0, however small , $\exists$ a + ve number $\delta_1$ (depending upon)

|$f$(x,y) - l < $f$ or 0 < |(x, y) - (a, b)| < $\delta_1$ ....(1)

again $\lim_{(x,y)\to(a,b)} f$(x, y) = l'

$\therefore$ $f$ or given > 0, however small, $\exists$ a + ve number $\delta_2$ (depending upon)

|$f$(x, y) - l' < $f$ or 0 < |(x, y) - (a, b)| < $\delta_2$ .....(2)

let $\delta$ = min {$\delta_1$, $\delta_2$}

From (1) and (2), we have,

|$f$(x, y) - l| < $f$ or 0 < |(x, y) - (a, b)| < $\delta$ .....(3)

|$f$(x, y) - l'| < $f$ or 0 < |(x, y) - (a, b)| < $\delta$ .....(4)

Now l' - l = |l' - l| [l' > l]

$= |l' - f(x) + f(x) - l|$

$\leq |l' - f(x)| + |f(x) - l|$

$= |f(x) - l'| + |f(x) - l| < +$

$= 2$ [using (3) and (4)]

$\Rightarrow$ $l' - l < 2 = 2\left(\frac{l'-l}{2}\right) = l' = l' - l < l' - l$

Which is absurd. Our supposition is wrong.

$\therefore$ $l' = l$

Here $\lim_{(x,y)\to(a,b)} f(x, y)$ if exists is unique

**Theorem 5:** Let $f$: A $\rightarrow$ R, where A $\subset$ $R^2$ be an open set. Let $f(x, y)$ be defined in a neighbourhood of (a, b) not necessarily at (a, b)

$$\lim_{\substack{(x,y)\to(a,b)\\(x,y)A}} f(x, y) = l$$

Let g(x) be a function of a single variable, such that $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(x, g(x)) = l$

**Proof:** Since $\lim_{\substack{x,y\to(a,b)\\(x,y)\in A}} f(x, y) = l$ [given]

$\therefore$ given > 0, $\exists$ a real $\delta > 0$, (depending upon) s.t.

$|f(x, y) - l| < f$ or $0 <(x, y) - (a, b)| < \delta$

$\Rightarrow$ $|f(x, y) - l| < f$ or $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ ....(1)

Also $\lim_{x\to a} g(x) = b$ [given]

$\therefore$ Given $\varepsilon = \delta > 0, \exists$ a real number $\delta_1 > 0$ (take $\delta 1 < \delta$)

$|g(x) - b| < \frac{\delta}{2} f$ or $0 < |x - a| < \delta_1$ ......(2)

Now $0 < |(x, g(x) - (a, b)|$

$\Rightarrow$ $\sqrt{(x-a)^2 + (g(x)-b)^2} < \sqrt{\delta_1^2 + \frac{\delta^2}{4}}$ [ $\delta_1 < \delta$ ]

$< \sqrt{\frac{\delta^2}{2}} < \delta$ [ $\delta_1 < \delta$ ]

$\Rightarrow 0 < \sqrt{(x-a)^2 + \left(g(x)-b^2\right)} < \delta$

i.e. 0 < |x, g(x) - (a, b)| < δ ......(3)

∴ From (1) and (2), we see that

$|f$(x, g(x) - l| < ε whenever 0 < |x, g(x)) - (a, b) < δ

[Putting y = g(x) in (1)]

$\Rightarrow |f$(x, g(x) - l| < ε

$\lim_{x\to a} f(x_1, g(x) = l$

**Cor.** Let $f$: A → R, where A ⊂ $R_2$ is open and (a, b) is limit point of A. Let g1 and $g_2$ be the functions of a single variable x.

s.t. $\lim_{x\to a} g_1(x) = \lim_{x\to a} g_2(x) = b$

$\lim_{x\to a} f(x, g_1(x) \neq \lim_{x\to a} f(x, g_2(x))$

Then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

**Remark 11:** The corollary above is very useful to show the non existence of limits of unctions of two variables.

**Remark 12:** Roughly speaking. Theorem 5 shows that if $\lim_{(x,y)\to(a,b)} f(x, y)$ exists, then this limit is independent of the path along which we approach the point (a, b).

Thus if we can find two different paths of approach along which $f$(x, y) has diferent limits, then $\lim_{(x,y)\to(a,b)} f(x, y)$ doesn't exist.

**Remark 13:** If $\lim_{(x,y)\to(a,b)} f(x, y) = l$ and $\lim_{(y)\to(b)} g(y) = a$, then

$\lim_{(y)\to(b)} f(g1(y), y) \neq \lim_{(y)\to(b)} f(g_2(y), y)$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

For example. Consider $f(x, y) = \dfrac{x^3+y^3}{x^3-y^3}$

find out $\lim_{(x,y)\to(0,0)} f(x, y)$ exist or not.

First let us take (x, y) → (0, 0) along y-axis i.e. x = 0.

Then $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^3-y^3} = \lim_{(x,y)\to(0,0)} \dfrac{0+y^3}{0-y^3}$

$\lim_{(x,y)\to(0,0)} (-1) = -1$

Now let us take (x, y) → (0, 0) along x - along x - axis i.e. y = 0

Then $\lim_{(x,y)\to(0,0)} f$(x, y) = $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^3-y^3}$

$$= \frac{x^3+0}{x^3-0} = \lim_{(x,y)\to(0,0)} 1 = 1$$

**Here we get two different limits of the function when (x, y) approach along different parts.**

∴ The limit of this function as (z, y) → (0, 0) doesn't exist.

**Repeated Limits (or Iterated limits)**

$\lim_{x\to a}\left(\lim_{y\to a} f(x,y)\right)$ and $\lim_{y\to a}\left(\lim_{x\to a} f(x,y)\right)$ are called iterated limits or repeated limits. An iterated limit is a limit of a limit and can be found as in case of a function of a single variable. The two iterated limits, if they exist, need not be necessarily equal.

**Remarks 1:** The two repeated limits if they exist may not be equal.

2. If $\lim_{(x,y)\to(a,b)} f$(x, y) exists and the two iterated limits exist, then they must be equal.

3. However, if the tow iterated limits exist and are equal, there is no guarantee of the existence of $\lim_{(x,y)\to(a,b)} f$(x, y)

For the sake of distinction. $\lim_{(x,y)\to(a,b)} f$(x, y) is called simultaneous limit.

Let us look at some examples:-

**Example 1:** Let $f$(x, y) = $\frac{x^2-y^2}{x^2+y^2}$, (x, y) ≠ (0, 0)

Prove that $\lim_{(x,y)\to(a,b)} f$(x, y) does not exist

**Solution:** Let g(x) = mx, where m is any real number

Now $\lim_{x\to 0}$ g(x) = 0 for all real

and $\lim_{x\to 0} f$(x, g(x) = $\lim_{x\to 0} f$(x, mx)

$$= \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}$$

i.e. $\lim_{x \to 0} f(x, g(x))$ depends upon m, whereas $\lim_{x \to 0}$ g(x) = 0 is independent of m.

$\therefore \quad \lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. [See cor. Of theorem 5]

**Example 2:** Let $f(x, y) = x \sin \frac{1}{y} + y \sin \frac{1}{x}$, $x \neq 0$, $y \neq 0$. Use definition to prove that

$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$

**Solution:** Let $0 < |x - 0| < \delta_1 = \frac{\varepsilon}{2}$ and $0 < |y - 0) < \delta_2 = \frac{\varepsilon}{2}$

Now $|f(x, y) - 0| = \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right|$

$$\leq |x| \left|\sin \frac{1}{y}\right| + |y| \left|\sin \frac{1}{x}\right|$$

$$\leq |x| + |y| \qquad \left[\left|\sin \frac{1}{x}\right| \leq 1, \left|\sin \frac{1}{y}\right| \leq 1\right]$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \ f \text{ or } 0 < |x| < \delta_1 = \frac{\varepsilon}{2},\ 0 < |y| < \delta_2 = \frac{\varepsilon}{2}$$

$\therefore$ By definition, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$

**Example 3:** Let A = {(x, y): $0 < x \leq 1$, $0 \leq y < 1$,). Let : A $\rightarrow$ R be defined by $f(x, y) = x + y$.

Show that $\lim_{(x,y)\to(0,\frac{1}{2})} f(x, y) = \frac{1}{2}$

**Solution:** Let $0 < |x - 0| < \delta_1 = \frac{\varepsilon}{2}$ and $0 < \left|y - \frac{1}{2}\right| < \delta_2 = \frac{\varepsilon}{2}$

Then $\left|f(x,y) - \frac{1}{2}\right| = \left|x + y - \frac{1}{2}\right|$

$$= \left|(x - 0) + \left(y - \frac{1}{2}\right)\right|$$

$$< |x - 0| + \left|y - \frac{1}{2}\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \left[ 0 < |x - 0| < \frac{\varepsilon}{2} \, and \, 0 < \left| y - \frac{1}{2} \right| < \frac{\varepsilon}{2} \right]$$

$$\therefore \quad \lim_{(x,y)\to(0,\frac{1}{2})} f(x, y) = \frac{1}{2}$$

**Example 4:** Show that for the function $f$ defined by

$$f(x, y) = \frac{x^2y^2}{x^2y^2 + (x-y)^2}$$

The two repeated limits at (0, 0) exist and are equal, but the simultaneous limit does not exist.

**Solution:** $f(x, y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ [given]

Obviously, $D_f = R^2 - \{(x, y) \in R^2 : x^2y^2 + (x - y)^2 = 0\} = R^2 - \{(0, 0)\}$

Now $\lim_{y\to 0} [\lim_{x\to 0} f(x, y)] = \lim_{y\to 0} [\lim_{x\to 0} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}]$

$$= \lim_{y\to 0} \left[ \lim_{x\to 0} \frac{x^2}{x^2 + \left(\frac{x}{y} - 1\right)^2} \right]$$

$$= \lim_{y\to 0} (0)$$

$$= 0 \qquad .....(1)$$

And $\lim_{x\to 0} \left[ \lim_{y\to 0} f(x, y) \right] = \lim_{x\to 0} \left[ \lim_{y\to 0} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2} \right]$

$$= \lim_{x\to 0} \left[ \lim_{y\to 0} \frac{y^2}{y^2 + \left(1 + \frac{y}{x}\right)^2} \right]$$

$$= \lim_{x\to 0} (0)$$

$= 0$ .....(2)

From (1) and (2) we see that the two repeated limits at (0, 0) exist and are equal,

Now we shall prove that

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2+(x-y)^2}$$

Let g(x) - mx, where m is any real number

Now $\lim_{x\to 0}$ g(x) = $\lim_{x\to 0}$ (mx) = 0 $f$ or all real m

And $\lim_{x\to 0} f$(x, g(x)) = $\lim_{x\to 0} f$(x, mx)

$$= \lim_{x\to 0} \frac{m^2x^4}{m^2x^4+(1-m)^2}$$

$$= \lim_{x\to 0} \frac{m^2x^2}{m^2x^2+(1-m)^2}$$

$$= \begin{cases} 0, if \quad m \neq 1 \\ 1, if \quad m = 0 \end{cases}$$

This Implies that the simultaneous limit $\lim_{(x,y)\to(0,0)} f$(x, y) does not exist.

**Example 5:** Show that $\lim_{(x,y)\to(0,0)} f$(x, y) = $\dfrac{x^4y^4}{(x^2+y^4)^3}$ doesn't exist.

**Solution:** Let $f$(x, y) = $\dfrac{x^4y^4}{(x^2+y^4)^3}$

Let (x, y) → (0, 0) along the curve x = my$^2$

$$\therefore \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^4y^4}{(x^2+y^4)^3}$$

$$= \lim_{y\to 0} \frac{(my^2)^4y^4}{\left[(my^2)^2+y^4\right]^3} = \lim_{y\to 0} \frac{m^4y^{12}}{\left[m^2y^4+y^4\right]^3}$$

$$= \lim_{y\to 0} \frac{m^4y^{12}}{y^{12}\left[m^2+1\right]^3} = \lim_{y\to 0} \frac{m^4}{\left(m^2+1\right)^3}$$

$$= \frac{m^4}{\left(m^2+1\right)^3}$$

Which is not unique as it takes different values for different values of m.

$\therefore \lim_{(x,y)\to(0,0)} f(x, y)$ doesn't exist.

Continuity of a real valued function in $R^2$ and $R^3$.

**Definition 10.**

Let $f$ : A → R, where A ⊂ $R^3$. Let (a, b)

∈ A. The function $f$ is said to be continuous at the point (a, b) if

given ε > 0, ∃ a real δ > 0, s.t.

|$f$(x, y) - $f$(a, b) < ε $f$ or |(x, y) - (a, b) < δ (x, y) ∈ A

|$f$ (x, y) - $f$(a, b) < ε $f$ or |x - a| < δ , |y - b| < δ i.e. $f$(x, y) is continuous i.e. (a, b) if $\lim_{(x,y)\to(a,b)} f$(x, y) = $f$(a, b).

**Definition 11:**

Let $f$ : A → R, where A ⊂ $R^3$. Let (a, b, c)

∈ A the function $f$ is said to be continuous at the point (a, b, c) if given ε > 0, ∃ a real δ > 0 such that

|$f$ (x, y, z) - $f$(a, b, c)| < ε $f$ or |(x, y, z) - (a, b, c)| δ ,(x, y, z) ∈ A

|$f$(x, y, z) - $f$(a, b, c) < ε $f$ or |x - a| < δ , |y - b) < δ , |z - c| < δ .

**Algebra of Continuous Functions in $R^2$.**

**Theorem 8:** If $f$ is continuous at (a, 0), then $f$ is bounded in some neighbourhood of (a, b), converse in not true.

**Theorem 9:** If $f$, g: A → R, A ⊂ $R^2$ are continuous at (a, b), then

(a) $f$ + g, $f$ - g, are continuous at (a, b)

(b) $f$, g is continuous at (a, b)

(c) $\frac{L}{g}$ , g(a, b) ≠ 0 is continuous at (a, b)

(d) k$f$ where k is any real member, continuous at (a, b)

Converse of (a), (b), (c) and (d) are not true

To clarify what we have just said, consider the following examples:-

**Example 6:** Let $f$(x, y) = $f$(x) = $\begin{cases} xy\ \sin\dfrac{1}{x}, x \neq 0 \\ 0,\ x = 0 \end{cases}$

Discuss the continuity of (x, y) at the point (0, 0)

**Solution:** $|f$(x, y) - $f$(0, 0)| = $\left| xy\sin\dfrac{1}{x} - 0 \right|$

= |x| |y| $\left| \sin\dfrac{1}{x} \right|$

≤ |x| |y| $\left[ \left| \sin\dfrac{1}{x} \right| \leq 1 \right]$

< $\sqrt{\varepsilon}$ $\sqrt{\varepsilon}$ = ε [$f$ or |x| < $\delta_1 = \sqrt{\varepsilon}$ , |y| < $\delta_2 = \sqrt{\varepsilon}$ ]

∴ $f$(x, y) is continuous at the origin.

**Example 7:** Show that $f$ : $R^2 \rightarrow R$ defined by

$$f(x, y) = f(x) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), & (x, y) \neq (0,0) \\ 0, & (\&x, y) = (0,0) \end{cases}$$

is continuous at (0, 0)

**Solution:** $|f$(x, y) - $f$(0, 0) = $\left| xy\dfrac{x^2 - y^2}{x^2 + y^2} - 0 \right|$

= |x| |y| $\left| \dfrac{x^2 - y^2}{x^2 + y^2} \right|$

≤ |x| |y| $\left[ \left| \dfrac{x^2 - y^2}{x^2 + y^2} \leq 1 \right| \right]$

< $\sqrt{\varepsilon}$ $\sqrt{\varepsilon}$ = ε $f$ or |x| < $\sqrt{\varepsilon}$ , |y| < $\sqrt{\varepsilon}$

∴ $f$(x, y) is continuous at (0, 0)

**Example 8:** Prove that the function $f$ : $R^2 \rightarrow R$ defined by

$$f(x) = \begin{cases} 1, \ if \ \ y \neq 0 \\ 0, \ if \ \ y = 0 \end{cases}$$

is continuous nowhere on the x-axis,

**Solution:** $f$(x) will not be continuous at any point on the x-axis if we prove that $\lim_{(x,y)\to(a,0)} f$(x, y) does not exist where (a, 0) is any point on x - axis.

Let S(a, 0), δ ), δ > 0 be a nhood. of the point (a, 0). This nhood. of (a, 0) contains infinitely many points where y = 0 and infinitely many points where y ≠ 0.

Let $\left(a + \frac{1}{m}, 0\right)$ and $\left(a + \frac{1}{m}\right) \in$ S ((a, 0), δ ).

Obviously the sequences $\left\{\left(a + \frac{1}{m}, 0\right)\right\}$ and $\left\{\left(a + \frac{1}{m}\right)\right\}$ converge to (a, 0) but the sequences $\left\{f\left(a + \frac{1}{m}\right)\right\}$ converge to 0 and 1 respectively.

Hence $\lim\limits_{(x,y)\to(0,0)} f$(x, y) does not exist which implies that $f$(x, y) is not continuous at (0, 0)

**Example 9:** Let $f$(x, y) = $f$(x) = $\begin{cases} \left(\dfrac{2xy^2}{x^3 + y^3}\right), \ (x, y) \neq (0,0) \\ 0, \quad (x, y) = (0,0) \end{cases}$

Discuss the continuity of $f$ at the origin.

**Solution:** g(x) = mx, where m is any real number

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} mx = 0 \ f \text{ or all real m}$$

$$\lim_{x\to 0} f(x, g(x) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{2m^2x^3}{x^3 + m^3x^3}$$

$= \dfrac{2m^3}{1+m^3}$ depends upon m(m ≠ -1)

Where $\lim\limits_{x\to 0}$ g(x) = 0 is independent of m

$\therefore \lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Hence $f(x, y)$ is not continuous at (0, 0)

**Example 10:**

Let $f : R^2 \to R$ be a continuous function. Show that the function $g : R^2 \to R$ defined by g(x, y)

$$= \begin{cases} f(x,y), if \ (x,y) \neq (0,0) \\ f(x,y), if \ (\&x,y) = (0,0) \end{cases}$$

is not continuous at (0, 0)

**Solution:** $\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$

Also $g(0, 0) = f(0, 0) + 1$ [$f$ is continuous at (0, 0)]

Thus we see that $\lim_{(x,y)\to(0,0)} g(x, y) \neq g(0, 0)$

$\therefore$ g(x, y) is discontinuous at (0, 0).

## 16.5 Self Check Exercise

Q.1 Let $f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^4+y^4}, \ x^2+y^2 \neq 0 \\ 0, \quad x = 0 = y \end{cases}$

Prove that a straight line approach gives the limit (0, 0)

Q. 2 Let $f$ : R2 → R be defined by

$$f(x, y) = \begin{cases} 1, \ x \ rational \\ 0, \ x irrational \end{cases}$$

Prove that

$\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist at any point $(a, b) \in R^2$.

Q. 3 Examine the continuity of the function

$$f(x, y) = \begin{cases} xy\left(\dfrac{x^2-y^2}{x^2+y^2}\right), \ (x,y) \neq (0,0) \\ 0, \quad (x\,y) = (0\,0) \end{cases}$$

at the point (0, 0)

Q. 4 Prove that the function

$$f(x, y) = \begin{cases} \dfrac{x^2}{x^2 + y^2 - x}, & (x, y) \neq (0,0) \\ 0, & (x\, y) = (0\,0) \end{cases}$$

is discontinuous at (0, 0)

## 16.6 Summary

In this unit we have studied

(i) function of two and three variables

(ii) to find the limit and continuity of function of two and three variables

(iii) algebra of limit and continuity of these functions.

## 16.7 Glossary

(i) **Square nhd. of (a, b) -**

A square nhd of a point $(a, b) \in R^2$ is the set of points (x, y) that are inside on open square with centre (a, b) and side parallel to the coordinates axes such that

$|x - a| < \delta$ and $|y - b| < \delta$ for some $\delta > 0$

$\therefore$ square nhd. of (a b) = $\{(x, y) : |x - a| < \delta , |y - b| < \delta , \delta > 0\}$

(ii) **Circular nhd. of a point (a, b) -**

A circular nhd. of a point (a, b) in $R^2$ is the set of points (x, y) that are inside a circle with centre (a, b) such that

$(x - a)^2 + (y - b)^2 < \delta^2, \delta > 0$

$\therefore$ Circular nhd. of a point (a b)

$= \{(x, y) : (x - a)^2 + (y - b)^2 < \delta^2, \delta > 0\}$

## 16.8 Answers to Self Check Exercises

Ans. 1 $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{x\to 0} \dfrac{mx}{x^2 + m^2} = 0$

Ans. 2 Prove it.

Ans. 3 $f$ is continuous at (0, 0)

Ans. 4 $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. Therefore $f(x, y)$ is not continuous at (0, 0)

## 16.9 Reference/Suggested Reading


1. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002

2. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007

## 16.10 Terminal Questions

1. Prove that $\underset{(x,y)\to(0,0)}{Lim} \dfrac{xy^2}{x^2+y^4}$ does not exist.

2. Prove that $\underset{(x,y,z)\to(\frac{1}{2},0,0)}{Lim} f(x, y, z) = \dfrac{1}{2}$, $f : R^3 \to R$, $f(x, y, z) = x$

3. Let $f : R^3 \to R$ be defined by $f(x, y, z) = x^2 + 3y^2 + 5z^2$
   show that $f$ is continuous function.

4. Let $f(x, y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x, y) \neq 0 \\ 0, & (x\,y) = 0 \end{cases}$ is $f$ continuous at (0, 0)?

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## Unit - 17

# Partial Differentiation

**Structure**



## 17.1 Introduction

Dear students, in this unit we shall study the concept of partial derivatives. A partial derivative is defined as a derivative in which some variables are kept constant and the derivative of a function with respect to other variable can be determined the process of finding the partial derivatives of a function is called partial differentiation. The partial derivative of a function differentiation. The partial derivative of a function differentiation. The partial derivative of a function '$f$' with respect to 'x' is represented by $f_x$ or $\frac{\partial f}{\partial x}$.

## 17.2 Learning Objectives

The main objectives of this unit are

(i) to study directional derivatives

(ii) to study partial derivatives

(iii) to learn about first order partial derivatives of a function of three variables

(iv) to learn partial order of higher order

(v) to study partial derivative and continuity etc.

### 17.3 Directional Derivatives

**Definition :** Let $f : A \to R$, $A \subset R^n$ and A is an open set.

Let $a = (a_1, a_2, \ldots\ldots a_n) \in A$

$\nu = (\nu_1, \nu_2, \ldots\ldots \nu_n)$ be unit vector i.e.

$\sqrt{v_1^2 + v_2^2 + \ldots. + v_n^2} = 1$

Further Let $t \in R$ (t, scalar), s.t.

$a + t\nu = (a_1 + t\nu_1, a_2 + t\nu_2, \ldots.+ a_n + t\nu_n) \in A$

If $\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$ exists, then we say that $f$ has a directional derivative at a in the direction of $\nu$ and is denoted by $f\nu$ (a) or $\frac{\partial f}{\partial v}$ (a).

### 17.4 Partial Derivatives

If in the above definition, we take

$\nu = e_1 = (1, 0, 0, \ldots. 0)$ then

$$fe_1 (a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

$$= \lim_{t \to 0} \frac{f(a_1 + t, a_2, a_3 \ldots. a_n) - f(a_1, a_2, \ldots. a_n)}{t}$$

is called the partial derivative of $f$ at a in the direction of e1, provided the limit on the right hand side exists.

$$fe_2 (a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

$$= \lim_{t \to 0} \frac{f(a_1, a_2 + t, a_3 \ldots. a_n) - f(a_1, a_2, \ldots. a_n)}{t}$$

if it exists, is called the partial derivative of $f$ at a in the direction of $e_2$.

Proceeding in this way, we can define partial derivative of $f$ at a in the direction of

$e_i = (0, 0, \ldots. 1 < \text{ith place} >, 0, 0, \ldots.)$

$i = 1, 2, 3, \ldots. n$ as

$$fe_i (a) = \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

$$= \lim_{t \to 0} \frac{f(a_1 a_2, .....(ai+t), a_{i+1}, ....a_n) - f(a_1, a_2, ....a_n)}{t}$$

Provide the limit on the right hand side exists.

Remark.

1. $f e_1$(a), $f e_2$(a), $f e_3$(a)..... $f e_i$(a)..... $f e_n$(a) are also denoted by

$\frac{\partial f}{\partial x_1}$ (a), $\frac{\partial f}{\partial x_2}$ (a), $\frac{\partial f}{\partial x_3}$ (a)..... $\frac{\partial f}{\partial x_i}$ (a)..... $\frac{\partial f}{\partial x_n}$ (a) or by

$D_1$(a), $D_2$(a), $D_3$(a)..... $D_i$(a)..... $D_n$(a) and abe called first order partial derivatives of $f$ w.r.t. $x_1, x_2, x_3..... x_i..... x_n$, resp. at the point a.

2. If we take n=2 in the definition, the $f$ : A→R when A is an open sub set $R^2$. Since A is open,

∴ $f$(a, b) ∈ A we can find real number h, x st. (a + h, b + k) ∈ A, then

$$f_x \text{ (a, b)} = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

and

$$f_x \text{ (a, b)} \lim_{k \to 0} \frac{f(a, h+k) - f(a, b)}{k}$$

provided limits on the right hand side exist, are called the partial derivative of $f$ w.r.t. x and w.r.t. y resp. at the point (a, b)

It is customary to denote $f_x$ (a, b) by $f_1$ (a, b) or by $\frac{\partial f}{\partial x}$ (a, b) and $f$y (a, b) by $f_2$ (a, b) or by $\frac{\partial f}{\partial x}$ (a, b).

## 17.5 First Order Partial Derivatives of a Function of three Variables

If in the definition, we take n = 3 then

$f$ : A → R, where A is open subset of $R^3$. Since A is open, therefore, we can find real numbers h, k, l s.b (a, b, c) ∈ A ⇒ (a + h, b + k, c + l) ∈ A

$$\text{Then } f_x \text{ (a, b, c)} = \lim_{h \to 0} \frac{f(a+h, b.c) - f(a, b, c)}{h}$$

$$f_y \text{ (a, b, c)} = \lim_{k \to 0} \frac{f(a, b+k, c) - f(a, b, c)}{k}$$

$$\text{and } f_x \text{ (a, b, c)} = \frac{f(a, b, c+l) - f(a, b, c)}{l}$$

Provided limits on the right hand exist, are called the partial derivatives of $f$ w.r.t. x, and w.r.t. y and w.r.t. z respectively at the point (a, b, c).

$f_x$ (a, b, c) is also denoted by $f_1$ (a, b, c) or $\frac{\partial f}{\partial x}$ (a, b, c)

$f_y$ (a, b, c) is also denoted by $f_2$ (a, b, c) or $\frac{\partial f}{\partial x}$ (a, b, c)

And similarly

$f_x$ (a, b, c) is also denoted by $f_3$ (a, b, c) or $\frac{\partial f}{\partial x}$ (a, b, c).

Remark 1 : From above remarks it is obvious that the partial derivatives of $f$ w.r.t. x, w.r.t. y and w.r.t. z i.e. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ are the ordinary derivatives of $f$ w.r.t. x.

(taking all other variables as constants), w.r.t. y (taking all other variables as constants) and so on respectively.

2. Let $f : R^n \to R$, then $f$ has a partial derivative $f_1$ at a = $(a_1, a_2, a_3, \ldots\ldots a_n)$ if $f$ $f(a_1 + h, a_2, a_3, \ldots\ldots a_n) - f(a_1, a_2, a_3, \ldots\ldots a_n) =$

$hf_1(a_1, a_2, a_3, \ldots\ldots a_n) + h\in$ where $\in$ depends upon $a_1, a_2, \ldots\ldots a_n$ and h and $\in \to 0$ as $h \to 0$, $(a_1, a_2, a_3, \ldots\ldots a_n)$ being a fixed point.

**17.6 Geometrical Interpretation of Partial Derivatives of First Order.**

Let

$f : R^2 \to R$ be a real valued function of two variables and z =

$f$ (x, b) represents the curve which is the intersection of this

surface and the plane y = b. Now $f$ (x, b) can be regarded as a function of one variable

and we know that $\frac{d}{dx}$ (x, b) represent the slope of the tangents to the curve

z = $f$ (x, b) at the point (a, b, $f$(a, b) in the xz plane. But $\frac{\partial f}{\partial x}$ (a, b)

= $f_x$ (a, b), therefore the partial derivative $f_x$ at (a, b) represents the slope of the tangent to the curve z = $f$ (x, y), y = b. For example, if z = $f$(x, y)

$= x^2 + y^2$ then $\frac{\partial z}{\partial x}(1, 0) = 2$.

Which is obviously the slop of the tangent at (1, 0) to the curve $z = x^2$, which is the intersection of the surface $z = x^2 + y^2$ and the plane y = 0.

Let us see that the method is with the help of following examples :-

**Example 1 :** Let $f$(x, y) = $\begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}$

prove that $f_x$ (0, 0) and $f_y$ (0, 0) both exist but are not equal.

**Solution :** By definition,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h.0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{h^3 - 0}{h^2 + 0} - 0}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

and $f_y(0, 0) = \lim_{k \to 0} \dfrac{f(0.0 + k) - f(0,0)}{k}$

$$= \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\dfrac{0 - k^3}{0 + k^2} - 0}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

Thus, we see that both $f_x$ (0, 0) and $f_y$ (0, 0) ≠ $f_y$ (0, 0).

**Example 2 :** Let

$f : R^2 \to R$ be defined be $f$(x) = $\begin{cases} 2, & (x, y) \neq 0 \\ 0, & (x, y) = 0 \end{cases}$

prove that the partial derivatives of $f$ do not exist at (0, 0).

**Solution :** By definition,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{2 - 0}{h}$$

$$= 2\lim_{h\to 0} \frac{1}{h} \text{ does not exist,}$$

and

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{2-0}{k}$$

$$= 2\lim_{k\to 0} \frac{1}{k} \text{ does not exist.}$$

This complete the solution.

**Example 3 :** Let $f(x, y) = \begin{cases} \sin\left(\dfrac{xy}{x^2+y^2}\right) & \text{if } (x, y) \neq 0 \\ 0 & \text{if } (x, y) = 0 \end{cases}$

Evaluate $f_x(0, 0)$ and $f_y(0, 0)$. is $f$ continuous at (0, 0)?

**Solution :** By definition,

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \quad f_x(0, 0) = \lim_{h\to 0} \frac{\sin 0 - 0}{h} = 0$$

And $f_y(0, 0) = \lim_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} \quad f_y(0, 0) = \lim_{k\to 0} \dfrac{\sin 0 - 0}{k} = 0$

Thus both $f$x (0, 0) and $f$y (0, 0) exist and are equal.

To discuss the continuity of $f$ at (0, 0), we first discuss its limit at (0, 0).

Let g(x) = mx where m is any ream number and $\lim_{x\to 0}$ g(x) = $\lim_{x\to 0}$ mx = 0 for all real m.

Now $\lim_{x\to 0} f(x, g(x)) = \lim_{x\to 0} f(x, mx)$

$$= \lim_{x\to 0} \sin\left(\frac{mx^2}{x^2+m^2x^2}\right)$$

$$= \lim_{x\to 0} \sin \frac{mx^2}{(1+m^2)x^2}$$

$= \sin \dfrac{m^2}{1+m^2}$ depends upon m

$\therefore \quad \lim\limits_{(x,y)\to(0,0)} f$ (x, y) does not exist. This implies that $f$(x, y) is not continuous at (0 0).

**Example 4 :** Let $f : R^2 \to R$ be defined by $f(x, y) = x^3 + y^2$.

**Solution :** By definition,

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

$$= \lim_{h\to 0} \frac{(a+h)^3+b^2-(a^3+b^2)}{h}$$

$$= \lim_{h\to 0} \frac{h(a^3+3ah+h^2}{h}$$

$$= \lim_{h\to 0} (3a^2 + 3ah + h^2)$$

and $f_y(a, b) = \lim\limits_{k\to 0} \dfrac{f(a,b+k)-f(a,b)}{k}$

$$= \lim_{k\to 0} \frac{a^3+(b+k)^2-(a^3+b^2)}{k}$$

or $f_y(a, b) = \lim\limits_{k\to 0} = \dfrac{k(2b+k)}{k} \lim\limits_{k\to 0} (2b + k) = 2b$.

**Example 5 :** Let $f : R^3 \to R$ be defined by $f(x, y, z) = x^3y$.

Evaluate $f_x(1, 2, 3)$, $f_y(0, 3, 2)$, $f_z(2, 3, 4)$.

**Solution :** By definition,

$$f_x(1, 2, 3) = \lim_{h\to 0} \frac{f(1+h,2,3)-f(1,2,3)}{h}$$

$$= \lim_{h\to 0} \frac{2(1+h)^3-2}{h}$$

$$= \lim_{h\to 0} \frac{2(h^2+3h+3)h}{h} = \lim_{h\to 0} 2(h^2 + 3h + 3) = 6$$

and $f_y(0, 3, 2) = \lim\limits_{k\to 0} = \dfrac{f(0,3+k,2)-f(0,3,2)}{k} \lim\limits_{k\to 0} \dfrac{0-0}{k} = 0$

and $f_x(2, 3, 4) = \lim\limits_{l\to 0} \dfrac{f(2,3,4+l)-f(2,3,4)}{l} = \lim\limits_{l\to 0} \dfrac{24-24}{l} = 0$

**Example 6 :** Let $f(x, y) = \sqrt{x^4+y^4+l}$ Evaluate $f_x(1, 2)$ and $f_y(1, 2)$.

**Solution :** $f$ (x, y) = $\sqrt{x^4+y^4+l}$ = $(x^4 + y^4 + l)^{1/2}$. $4x^3$ = $\dfrac{2x^3}{\sqrt{x^4+y^4+l}}$

Putting x = 1, y = 2, we have,

$f^y$ (1, 2) = $\dfrac{16}{\sqrt{18}} = \dfrac{16}{3\sqrt{2}}$ = $\dfrac{8\sqrt{2}}{3}$

**Example 7 :** Let $f$ : $R^2 \to R$. If $f_x$ (x, y) = 0, then show that $f(x_1, y_0) = f(x_2, y_0)$ $f$ or all $x_1, x_2, y_0$.

**Solution :** Let us define F : R → R by F (x) = $f$ (x, $y_0$)

Now F' (x) = $\lim_{h\to 0} \dfrac{F(x+h)-F(x)}{h}$

[By definition or ordinary diff. coeff.]

$= \lim\limits_{h\to 0} \dfrac{f(x+h, y_0)-f(x, y_0)}{h}$

$= f_x (x, y_0)$ $\left[\text{by definition } \dfrac{\partial f}{\partial x}\right]$

= 0 $\forall$ x and $y_0$, $f_x$ (x, y) = 0

and F' (x) = 0 $\Rightarrow$ F is constant

$\Rightarrow$ F(x1) = F(x2) $\forall$ x1 and x2

$\Rightarrow f(x_1, y_0) = f(x_2, y_0)$ $\forall$ $x_1$, $x_2$ and $y_0$

This competes the proof.

## 17.7 Partial Derivative Higher Order

**Definition :** Let $f$ ; D → R, where D is an open subset of $R^2$. Let $f_x$, $f_y$ exist in the neighbour hood S of a point (a, b) $\in$ D.

Obviously S $\subset$ D and therefore X $\subset$ $R^2$ and is open.

Then $f_x$, $f_y$ : S → R are defined and are functions of two variables x and y.

Suppose $f_x$ and $f_y$ possess partial derivatives w.r.t. x and w.r.t. y.

Then $\dfrac{\partial}{\partial x}$ ($f_x$(x, y)) is denoted by $\dfrac{\partial^2 f}{\partial x^2}$ or $f_x$ or $f_{11}$

$$= \lim_{h\to 0} \frac{f_x(x+h, y)-f_x(x, y)}{h}$$

and $\dfrac{\partial}{\partial y}$ ($f_x$(x, y)) is denoted by $\dfrac{\partial^2 f}{\partial y \partial x}$ or $f_{yx}$ or $f_{21}$

$$= \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x, y)}{k}$$

Similarly $\frac{\partial}{\partial x}$ ($f_y$(x, y)) is denoted by $\frac{\partial^2 f}{\partial x \partial y}$ or $f_x$ or $f_{12}$

$$= \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}$$

and $\frac{\partial}{\partial y}$ ($f_y$(x, y)) is denoted by $\frac{\partial^2 f}{\partial y^2}$ or $f_{yx}$ or $f_{22}$

$$= \lim_{h \to 0} \frac{f_y(x, y+k) - f_y(x, y)}{k}$$

$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ are called the 2nd order partial derivatives of $f$, where $f$ is a function of two variables.

Remark, $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ need not necessarily be equal.

**Change of variables**

**Theorem 1 :** If z$f$(x, y) possess continuous partial derivatives and x$\phi$ (t), y$\psi$ (t) possesses continuous derivatives, then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} . \frac{\partial x}{dt} + \frac{\partial z}{\partial y} . \frac{\partial y}{\partial t}$$

**Solution :** Let $\delta$ t be a small change in the value of t.

Let $\delta$ x, $\delta$ y, $\delta$ z be the corresponding changes in x, y, z respectively. Obviously z is a composite function of a single variable t.

Then z + $\delta$ z $f$(x + $\delta$ x, y + $\delta$ y), where z $f$ (x, y) and $\delta$ x, $\delta$ y, $\delta$ z $\to$ 0 as $\delta$ t $\to$ 0.

$\therefore$ $\delta$ z = $f$(x + $\delta$ x, y + $\delta$ y) - $f$ (x, y)

($f$(x + $\delta$ x, y + $\delta$ y) - $f$(x + $\delta$ x, y) + ($f$(x + $\delta$ x, y) - $f$(x, y)

or $$\frac{\delta z}{\delta t} = \frac{(f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta t} + \frac{(f(x+\delta x, y) - f(x, y)}{\delta t}$$

or $$\frac{\delta z}{\delta t} = \frac{(f(x+\delta x, y+\delta y) - f(x+\delta x, y)}{\delta t} . \frac{\delta y}{\delta t}$$

$$+ \frac{(f(x+\delta x, y)-f(x,y)}{\delta t} \cdot \frac{\delta y}{\delta t}$$

or $\lim\limits_{\delta t \to 0} \frac{\delta z}{\delta t} = \lim\limits_{\delta t \to 0} \left[ \frac{(f(x+\delta x, y+\delta y)-f(x+\delta x, y)}{\delta t} \cdot \frac{\delta y}{\delta t} + \frac{(f(x+\delta x, y)-f(x,y)}{\delta t} \cdot \frac{\delta y}{\delta t} \right]$

$$\frac{\delta z}{\delta t} = \frac{\delta f}{\delta y} \cdot \frac{\delta y}{\delta t} + \frac{\delta f}{\delta x} \cdot \frac{dx}{dt}$$

or $\frac{\delta z}{\delta t} = \frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta t} + \frac{\delta z}{\delta y} \cdot \frac{dy}{dt}$

**Let us look at some examples :**

**Example 8 :** Compare the 2nd order partial derivatives for the function u = log$\sqrt{x^2+y^2}$ and verify the following :

(1) $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

(2) $u_{11} + u_{22} = 0$.

**Solution :** u = log$\sqrt{x^2+y^2}$ = $\frac{1}{2}$ log $(x^2 + y^2)$ ....(1)

$\therefore \frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2x = \frac{x}{x^2+y^2}$ ...(2)

and $\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot 2y = \frac{y}{x^2+y^2}$ ....(3)

Differentiating (2) and (3) partially w.r.t. x and w.r.t. y, we have

$$\frac{\partial^2 u}{\partial u^2} = \frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = -x(x^2+y^2)^{-2}. 2y = -\frac{2xy}{(x^2+y^2)^2}$$

and $\frac{\partial^2 u}{\partial x \partial y} = -y(x^2+y^2)^{-2}. 2x = -\frac{2xy}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Obviously $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -\frac{2xy}{(x^2+y^2)^2}$

and $u_{11} + u_{22} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$

or $u_{11} + u_{22} = 0$

**Example 9 :** Let $f(x, y) = \begin{cases} x^2 \tan^{-1}\frac{x}{y} - y^2 \tan^{-1}\frac{x}{y} & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Show that $f_{12}(0, 0) \neq f_{21}(0, 0)$.

**Solution :** $f_1(0, 0) \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} \lim_{h\to 0} \frac{0-0}{h} = 0$

and $f_2(0, 0) \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} \lim_{k\to 0} \frac{0-0}{k} = 0$

Let $(x, y) \neq (0, 0)$.

$$f_x(x, y) = x^2 . \frac{1}{1+\frac{y^2}{x^2}} . \left(-\frac{y}{x^2}\right) + 2x \tan^{-1}\frac{x}{y} - y^2 . \frac{1}{1+\frac{x^2}{y^2}} . \frac{1}{y}$$

$$= \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2} + 2x \tan^{-1}\frac{x}{y}$$

$$= \frac{y(x^2+y^2)}{x^2+y^2} + 2x \tan^{-1}\frac{x}{y}$$

$$= -y + 2x \tan^{-1}\frac{x}{y}$$

and $$f_y(x, y) = x^2 . \frac{1}{1+\frac{y^2}{x^2}} . \left(\frac{1}{x}\right) - y^2 . \frac{1}{1+\frac{x^2}{y^2}} . \left(-\frac{x}{y^2}\right) - 2y \tan^{-1}\frac{x}{y}$$

$$= \frac{x^3}{x^2+y^2} + \frac{xy^2}{x^2+y^2} - 2y \tan^{-1}\frac{x}{y}$$

$$= x - 2y \tan^{-1}\frac{x}{y}$$

$f_{yx}(0, 0)\ \lim_{k\to 0} \dfrac{f_x(0,k)-f_x(0,0)}{k}\ \lim_{k\to 0} \dfrac{-k-0}{k} = -1$

and $f_{xy}(0, 0) = \lim_{h\to 0} = \dfrac{f_y(h,0)-f_y(0,0)}{h}\ \lim_{h\to 0} \dfrac{h-0}{h} = 1$

Therefore $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ or $f_{12}(0, 0) \neq f_{21}(0, 0)$

**Example 10 :** Let $f(x, y) = \begin{cases} \dfrac{xy(x^2+y^2)}{(x^2+y^2)}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

Show that $f_{12}(0, 0) \neq f_{21}(0, 0)$.

**Solution :** $f_x(0, 0)\ \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}\ \lim_{h\to 0} \dfrac{0-0}{h} = 0$

and $f_y(0, 0)\ \lim_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k}\ \lim_{k\to 0} \dfrac{0-0}{k} = 0$

Let $(x, y) \neq (0, 0)$.

$$f_x(x, y) = \frac{(x^2+y^2)(3x^2y-y^3)-xy(x^2-y^2)2x}{(x^2+y^2)^2}$$

$$= \frac{3x^4-x^2y^3+3x^2y^3-y^5-2x^4.y+2x^2y^3}{(x^2+y^2)^2}$$

$$= \frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}$$

$$= y\left[\frac{(x^4-y^4)+4x^2y^2}{(x^2+y^2)^2}\right]$$

$$= y\left[\frac{x^2-y^2}{(x^2+y^2)}+\frac{4x^2y^2}{(x^2+y^2)^2}\right]$$

and $$f_y(x, y) = x\left[\frac{(x^2+y^2)(x^2-3y^2)-y(x^2-y^2).2y}{(x^2+y^2)^2}\right]$$

$$= x\left[\frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}\right]$$

$$= x \left[ \frac{x^2 - y^2}{(x^2+y^2)} - \frac{4x^2y^2}{(x^2+y^2)^2} \right]$$

Now

$$f_{yx}\ (0, 0)\ \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k}\ \ \lim_{k \to 0} \frac{-k-0}{k} = -1$$

$$\text{and } f_{xy}\ (0, 0) = \lim_{h \to 0} = \frac{f_y(h,0) - f_y(0,0)}{h}\ \ \lim_{h \to 0} \frac{h-0}{h} = 1$$

Therefore $f_{yx}$ (0, 0) $\neq$ $f_{xy}$ (0, 0) or $f_{12}$ (0, 0) $\neq$ $f_{21}$ (0, 0)

**Example 11 :** If u = $e^{ax}$ cos by, verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

**Solution :** $\frac{\partial u}{\partial x}$ = $ae^{ax}$ cos by

and $\frac{\partial u}{\partial y}$ = $-be^{ax}$ sin by

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\ (-be^{ax} \sin by) = -abe^{ax} \sin by$$

$$\text{and } \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y} (-ae^{ax} \cos by) = -abe^{ax} \sin by$$

Thus we see that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -abe^{ax}$ sin by

**Example 12 :** If $x^x y^y z^z = c_1$ show that $\frac{\partial^2 u}{\partial x \partial y}$ = -(x log ex)-1, where x = y = z.

**Solution :** We have $x^x y^y z^z = c$

Taking log of both sides, we get,

log $x^x$ + log $y^y$ + log $z^z$ = log c

$\Rightarrow$ x log x + y log y + z log z = log c

$\Rightarrow$ z log z = log c - x log x - y log y ...(1)

Differentiating (1) partially w.r.t. x, we get

$$z\ \frac{1}{z}\ \frac{\partial z}{\partial x} + \log z,\ \frac{\partial z}{\partial x} = 0 - \left( x.\frac{1}{x} + \log x \right)$$

$\Rightarrow$ (1 + log z) $\frac{\partial z}{\partial x}$ = -(1 + logx)

$$\Rightarrow \frac{\partial z}{\partial x} = \left(\frac{1+\log y}{1+\log x}\right) \qquad \text{...(2)}$$

Similarly, differentiating (1) partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right) \qquad \text{...(3)}$$

Again, differentiating (3) partially w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{1+\log y}{1+\log z}\right)$$

$$= -(1 + \log y)\ \frac{\partial}{\partial x}\ (1 + \log z)\text{-}1$$

$$= -(1 + \log y)\left[-(1+\log z)-2\frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$= \frac{(1+\log y)}{z(1+\log z)^2}\left[-\left(\frac{1+\log x}{1+\log z}\right)\right] \qquad \text{[using (2)]}$$

$$= \frac{(1+\log x)(1+\log y)}{z(1+\log z)^3}$$

Now $f$ or x = y = z, we get

$$\frac{\partial^2 y}{\partial x \partial y} = -\ \frac{(1+\log x)(1+\log x)}{x(1+\log x)^3}$$

$$= \frac{-1}{x(1+\log x)} = \frac{-1}{x(\log e+\log x)}$$

$$= \frac{-1}{x\log ex}$$

$$= -(x \log ex)^{-1}$$

Hence $\frac{\partial^2 x}{\partial x \partial y}$ = - (x log ex)$^{-1}$

**Example 13:** If u = $f$ (r), where r = $\sqrt{x^2+y^2}$ , prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r).$$

**Solution:** We are given r = $\sqrt{x^2 + y^2}$

$$\Rightarrow \qquad r^2 = x^2 + y^2 \qquad .....(1)$$

Differentiating (1) partially, w.r.t. x and y, we get

$$2r\,\frac{\partial r}{\partial x} = 2x \text{ and } 2r\,\frac{\partial r}{\partial y} = 2y$$

$$\Rightarrow \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad .....(2)$$

Now u = $f$ (r)

$$\therefore \qquad \frac{\partial u}{\partial x} = f'(r)\,\frac{\partial r}{\partial x} = \frac{x}{r} f'(r) \qquad \text{[using (2)]}$$

$$\therefore \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r} f'(r)\right)$$

$$= \frac{r\frac{\partial}{\partial x}(xf'(r)) = xf'(r)\frac{\partial r}{\partial x}}{r^2}$$

[r is a function of $f$ (x)]

$$= \frac{r\left[xf''(r)\frac{\partial r}{\partial x} + f'(r).1\right] - xf'(r)\frac{r}{x}}{r^2}$$

$$\frac{r\left[xf''(r)\frac{r}{x} + f'(r)\right] - \frac{x^2}{r}\, f'(r)}{r^2}$$

$$\text{i.e. } \frac{\partial^2 u}{\partial x^2} = \frac{x^2 f''(r) + rf'(r) - \frac{r^2}{r}\, f'(r)}{r^2} \qquad ......(3)$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2 f''(r) + rf'(r) - \frac{r^2}{r}\, f'(r)}{r^2} \qquad ......(4)$$

Adding (3) and (4), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = \frac{x^2 f''(r)+rf'(r)-\frac{x^2}{r}f'(r)}{r^2}+\frac{y^2 f''(r)+rf'(r)-\frac{y^2}{r}f'(r)}{r^2}$$

$$= \frac{(x^2+y^2)f''(r)+2rf'(r)-\frac{1}{r}(x^2+y^2)f'(r)}{r^2}$$

$$= \frac{r^2 f''(r)+2rf'(r)-\frac{r^2}{r}f'(r)}{r^2}$$

$$= \frac{r^2 f''(r)+2rf'(r)-rf'(r)}{r^2}$$

$$= \frac{r^2 f''(r)+rf'(r)}{r^2} = f''(r) + \frac{1}{r}f'(r)$$

Hence $\frac{\partial^2 u}{m^2}+\frac{\partial^2 u}{\partial y^2} = f'(r) + \frac{1}{r}f'(r)$

### 17.8 Partial Derivatives and Continuity

Students will recall that continuity of a function of a single real variable at a point does not necessarily imply the differentiability of the function there at. Likewise, in case of a function of several variables, the continuity does not necessarily imply the existence of partial derivatives.

Students may now be led to believe that the existence of partial derivatives will imply continuity, but unfortunately it is not true. (See example 3 above). However, the following theorem shows that an additional condition imposed on the partial derivatives will ensure the continuity.

**Theorem 1:** If a function $f(x, y)$ has partial derivatives $f_x$ and $f_y$ at every point of an open set $D(D \subset R^2)$ and if these partial derivatives are bounded in D, then $f(x, y)$ is continuous everywhere in D.

**Proof:** Since $f_x$ and $f_y$ are bounded in D therefore $\exists$ a real $M > 0$, s.t.

$$|fx(x, y) \leq M \text{ and } |f_y(x, y) \leq M \text{ } f\text{ or all } (x, y) \in D. \qquad ....(1)$$

Since D is open, therefore $\exists$ a real $\delta > 0$, s.t.

$$(a, b) \in D \Rightarrow (a + h, b + k) \in D \text{ } f\text{ or } |h| < \delta, |k| < \delta$$

We write

$$f(a + h, b + k) - f(a, b) = (f(a + h, b + k) - f(a, b + k)) + (f(a, b + k) - f(a, b)) \qquad ......(2)$$

Now we define two function $\phi(x)$ and $\psi(y)$ of a single real variable as:

$$\phi(x) = f(x, b + k),\ x \in [a, a + h] \qquad .....(3)$$

$$\text{and } \psi(y) = f(a, y),\ y \in [b, b + k] \qquad .......(4)$$

Now $\phi(a + h) - \phi(a) = f(a + h, b + k) - f(a, b + k)$

And $\psi(b + k) - \psi(b) = f(a, b + k) - f(a, b)$

And therefore (2) reduces to

$$f(a + h, b + k) - f(a, b) = (\phi(a + h) - \phi(a)) + (\psi(b + k) - \psi(b)) \quad ......(5)$$

From (3) and (4) it is obvious that

$$\phi'(x) = f_x(x, b + k), \text{ exists } \forall\ x \in [a, a + h] \qquad .....(6)$$

$$\text{and } \psi'(y) = f_y(a, y) \text{ exists } \forall\ y \in [b, b + k] \qquad ......(7)$$

Thus $\phi(x)$ and $\psi(y)$ satisfy the conditions of Lagrange Mean Value Theorem in the intervals [a, a + h] and [b, b + k] respectively and hence

$$\phi(a + h) - \phi(a) = h\phi'(a + \theta h),\ 0 < \theta < 1$$

$$= hf_x(a + \theta h, b + k) \qquad [\text{of } (6)]$$

$$\text{and } \psi(b + k) - \psi(b) = k\psi'(b + \phi k),\ 0 < \phi < 1$$

$$kf_y(a, b + \phi k) \qquad [\text{of } (7)]$$

and therefore (5) reduces to

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + kf_y(a, b + \phi k)$$

$$\text{or } |f(a + h, b + k) - f(a, b)| = |hf_x(a + \theta h, b + k) + kf_y(a, b + \phi k)|$$

$$\leq |h|f_x(a + \theta h, b + k) + |k|f_y(a, b + \phi k)|$$

$$\leq |h|\ M + |k||M|$$

$$\leq (|h| + |k|)M \quad [\text{of } (1)]$$

But the right hand side $\to 0$ as $(h, k) \to (0, 0)$ and this implies that

$$\lim_{(h,k)\to(0,0)} (f(a + h, b + k) - f(a, b) = 0$$

$$\text{i.e. } \lim_{(h,k)\to(0,0)} (f(a + h, b + k) - f(a, b)$$

Which implies that $f(x, y)$ is continuous at (a, b) where (a, b) is any point in D.

**Remark :** The above result is also true for functions defined in $R^n$.

**Remark :** We shall prove later on that if $f$x and $f$y are continuous in D, then $f(x, y)$ is also continuous in D.

## 17.9 Differentiability and Differentiable Functions

We have already seen (in lower classes) that if $f$ is a real valued function of a single real variable x, then $f$ is differentiable at $x_0$ only s.t.

$f(x_0 + h)\ f(x_0) - \alpha h = he$ where $\in \to 0$ as $h \to 0$ and the constant $\alpha = f(x_0)$

Now we want to extend this idea to functions defined in $R^n$.

Definition : $f : D \to R$, where $D \subset R^2$ and is open is said to be differentiable at the point $(a, b) \in D$ if $f\ \exists$ two constants $\alpha$ and $\beta$ [depending on $f$ and the point (a, b) only].

s.t. $f(a + h, b + k) - f(a, b) - ah - \beta k = \sqrt{h^2 + k^2}\ \phi(h, k)$

where $\phi(h, k)$ is a real valued function, s.t. $\phi(h, k) \to 0$ as

$(h, k) \to (0, 0)$.

**Definition :** $f : D \to R$, where D is an open subset of $R^3$ is said to be differentiable at the point (a, b, c)

$\in$ D, if $\exists$ three constants $\alpha, \beta, \gamma$ [depending on $f$ and the point (a, b, c) only] s.t.

$f(a + h, b + k, c + l) - f(a, b, c) - \alpha h - \beta k - \gamma l = \sqrt{h^2 + k^2 + l^2}\ \ \phi(h, k, l)$

Where $\phi(h, k, l)$ is real valued function, s.t.

$\phi(h, k, l) \to 0$ as $(h, k, l) \to (0, 0, 0)$.

**Remark :** It can easily proved that $f\ D \to R$, where $D \subset R^2$ and is open is differentiable at the point $(a, b) \in D$ if $f\ \exists$ constants $\alpha, \beta$ [depending on $f$ and the point (a, b) only], s.t.

$f(a, h, b + k) - f(a, b) - \alpha h - \beta k = \varepsilon h + \eta k$

Where $\varepsilon, \eta \to 0$ as $(h, k) \to (0, 0)$

Similarly,

$f : D \to R$, where $D \subset R^2$ and D is open is differentiable at $(a, b, c) \in$

D if $f\ \exists$ constants $\alpha, \beta, \gamma$ (depending on $f$ and (a, b, c) only], s.t.

$f(a + h, b + k, c + l) - f(a, b, c) - \alpha h - \beta k - \gamma l = h\varepsilon_1 + k\varepsilon_2 + l\varepsilon_3$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \to 0$ as $(h, k, l) \to (0, 0, 0)$ and so on.

Theorem : Let $f : D \to R$, where D is an open subset of $R^2$. If $f$ is differentiable at a point (a, b) of D, then $f$ has both the partial derivatives at (a, b). Moreover $f_x(a, b) = \alpha$ and $f_y(a, b) = \beta$, where $\alpha, \beta$ are the constants occurring in the definition of differentiability of $f$ at (a, b) given above.

Proof : Since $f$ is differentiable at (a, b), therefore, we have

$f(a + h, b + k) - f(a, b) - \alpha h - \beta k = h\varepsilon + k\eta$

Where $\varepsilon, \eta \to 0$ as $(h, k) \to (0, 0)$

Take k = 0, then

$f(a + h, b) - f(a, b) - \alpha h = h\varepsilon$

or $f(a + h, b) - f(a, b) = h (\alpha + \varepsilon)$

$$\lim_{h\to 0} = \frac{f(a+h,b)-f(a,b)}{h} = \lim_{h\to 0} (\alpha + \varepsilon)$$

$\Rightarrow \quad f_x (a, b) = \alpha \quad [\varepsilon \to 0 \text{ as } h \to 0]$

Similarly if we take h = 0, we have, $f_y (a, b) = \beta$.

**Remark :** If $f$ is differentiable at $(a, b) \in D$, where $D \subset R^2$ and is open, then $f(a + h, b+ k) - f (a, b) - h f_x (a, b) - k f_y (a, b) = h\varepsilon + k\eta$, where $\varepsilon, \eta \to 0$ as $(h, k) \to (0, 0)$.

**Theorem :** Let $f : D \to R$, where D is an open subset of $R^2$. If $f$ is differentiable at the point (a, b) of D, then $f$ is continuous at (a, b) given above.

Proof : Since $f$ is differentiable at (a, b), therefore,

$$f(a + h, b + k) - f(a, b) - hf_x (a, b) - kf_y (a, b) = h\varepsilon + k\eta$$

where $\varepsilon, \eta \to 0$ as $(h, k) \to (0, 0)$

i.e. $f(a + h, b + k) - f(a + b) - h[f_x (a, b) + \varepsilon] + k[f_y (a, b) + \eta]$

Taking limits on both sides as $(h, k) \to (0, 0)$, we have,

$$\lim_{(h,k)\to(0,0)} f(a + h, b + k) = \lim_{(h,k)\to(0,0)} [f(a, b) - h[f_x (a, b) + \varepsilon] + k[f_y (a, b) + \eta]$$

$$= f (a, b)$$

Which implies that $f (x, y)$ is continuous at (a, b).

**Remark :** The conditions mentioned in above theorems are necessary but not sufficient. In other words the converses of above theorem are not true.

i.e. (1) existence of the partial derivatives at a point does not imply the differentiability of $f$ at that point,

(2) Continuity of $f$ at a point need not imply the differentiability of $f$ at the point.

To clarify what we have just said, consider the following example :-

**Example 14 :** Let $f (x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & (x, y) \neq (0, 0) \\ 0, (x,y) = (0,0) & \end{cases}$

show that $f$ is continuous at (0, 0) and $f_x (0, 0)$, $f_y (0, 0)$ exist but

$f$ is not differentiable at (0, 0).

**Solution :** $|f (x, y) - f (0, 0)| = \left|\dfrac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right|$

$$= |x| \left|\sqrt{\frac{y^2}{x^2+y^2}}\right|$$

$$< |x| \left[\left|\sqrt{\frac{y^2}{x^2+y^2}}\right| \leq 1\right]$$

$< \varepsilon$

$f$ or $|x - 0| < \delta_1 = \varepsilon$

$f$ or $|y - 0| < \delta_2 = \varepsilon$

$\Rightarrow f$ (x, y) is continuous at (0, 0)

By definition, $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = 0$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0-0}{k} = 0$$

Thus we see both $f_x$ and $f_y$ exist at (0, 0)

Now $f(h, k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0) = \dfrac{hk}{\sqrt{h^2+k^2}}$

$$= \sqrt{h^2+k^2} \cdot \frac{hk}{\sqrt{h^2+k^2}}$$

where $\phi(h, k) = \dfrac{hk}{\sqrt{h^2+k^2}} \to 0$ as $(h, k) \to (0, 0)$

[ $\lim_{(h,k)\to(0,0)} \phi(h, k)$ does not exist. (Prove it)]

Hence by definition of differentiability of $f$, $f$ is not differentiable at (0, 0)

**Example 15:** Let $f(x, y) = |xy|^{1/2}$. Prove that $f$ is continuous at (0, 0) both the partial derivatives $f_x(0, 0)$, $f_y(0, 0)$ exist but $f$ is not differentiable at (0, 0)

**Solution:** $Lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} |xy|^{1/2} = 0 = f(0, 0)$

$\therefore$ $f$ is continuous at (0, 0)

Also $f_x(0, 0) = \lim_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = 0$

and $f$y (0, 0) $\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$ = $\lim_{k\to 0}\frac{0-0}{k}$ = 0

Thus we see that both $f_x$ and $f_y$ exist at (0, 0)

For differentiability at (0, 0)

$$f(h, k) - f(0, 0) - hf_x(0, 0) - kf_y(0, 0) = |hk|^{1/2}.$$

$$= \sqrt{h^2+k^2}\,.\,\frac{|hk|^{1/2}}{\sqrt{h^2+k^2}}$$

$$= \sqrt{h^2+k^2}\,\phi(h, k)$$

where $\phi(h, k) = \frac{|hk|^{1/2}}{\sqrt{h^2+k^2}} \to 0$ as $(h, k) \to (0, 0)$

[ $\lim_{(x,y)\to(0,0)} \phi(h, k)$ does not exist. (Prove (t)]

Hence by definition of differentiability $f$ is not differentiable at (0, 0)

**Theorem.** Let $f$ : D → R, where D is an open subset of $R^2$. If the partial derivatives $f$x and $f$y exist in a neighbourhood of a point (a, b) ∈ D and are continuous at (a, b), then $f$ is differentiable at (a, b)

**Proof:** Since D is open, therefore, ∃ a real δ > 0, s.t. (a, b) ∈ D

$$\Rightarrow \quad (a + h, b + k) \in D\ f \text{ or } |h| < \delta, |k| < \delta.$$

Now proceeding as in Theorem, we have,

$$f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + k) + kf_y(a, b + \phi k)$$

Where 0 < θ < 1, 0 < ϕ < 1.

Because $f_x$ and $f_y$ exist in the neighbourgood of (a, b) and are also continuous at (a, b), therefore,

$$\lim_{(x,y)\to(0,0)} f_x(a + \theta h, b + k) = f_x(a, b)$$

And $\lim_{(x,y)\to(0,0)} f_y(a, b + \phi k) = f_y(a, b)$

$$\therefore\ f\text{x}(a + \lim_{(x,y)\to(0,0)} h, b + k) = f\text{x}(a, b) + e$$

And $f$y(a, b + ϕk) = $f$y(a, b) + η

where ε, η are functions of h and k and ε, η → 0 as (h, k) → (0, 0)

Therefore $f(a + h, b + k) - f(a, b) = h[f_x(a, b) + \varepsilon] + k[f_y(a, b) + \eta]$

or $f(a + h, b + k) - f(a, b) - hf_x (a, b) - kf_y(a, b) = n\varepsilon + k\eta$

Where $\varepsilon.\eta \to 0$ as $(h, k) \to (0, 0)$

Hence $f$ is differentiable at (a, b)

Let us look at some examples:-

**Example 16:** Prove that $f(x, y) = e^{x+y}$ is differentiable at (1, 3)

**Solution:** Here, $f(x, y) = e^{x+y}$

Differentiating partially w.r.t. x and y, we have,

$f_x(x, y) = e^{x+y}$ and $f_y(x, y) = e^{x+y}$

Obviously both $f_x$ and $f_y$ exist in the neighbourhood of the point (1, 3) and are also continuous at (1, 3)

$\therefore$ $f(x, y)$ is differentiable at (1, 3).

**Example 17:** Let $f(x, y) = \cos (x + y)$. Prove that $f$ is differentiable at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

**Solution:** $f(x, y) = \cos (x + y)$

Obviously $f_x(x, y) = -\sin (x + y)$ and $f_y(x, y) = -\sin (x + y)$

Clearly, $f_x$ and $f_y$ exist in the neighbourhood of $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ and are

also continuous at $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$ therefore, $f(x, y)$ is differentiable at

$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

**Young's Theorem:**

Statement. Let $f : D \to R$, where D is an open subset of $R^2$

If $f_x$ and $f_y$ exist in the neighbourhood of a point (a, b)

$\in$ D and are differentiable at (a, b) then $f_{xy}(a, b) = f_{yx}(a, b)$.

**Proof:** Since $f_x$ and $f_y$ are differentiable at (a, b).

Therefore $f_{xy}$, $f_{yx}$, $f_{xy}$ and $f_{yy}$ exist at (a, b)

Since D is open, $\exists$ a real h > 0, then point $(a + h, b + h) \in D$

Now we write $F(x) = f(x, b + h) - f(x, b)$, $a \leq x \leq a + h$,

Then $F(a + h) - F(a) = f(a + h, b + h) - f(a + h, b) - f(a, b + h) + f(a, b)$ ......(1)

And $f'(x) = f_x(x, b + h) - f_x(x, b)$ .....(2)

Also by Lagrange's Mean Value Theorem in the interval [a, a + h]

$$F(a + h) - F(a) = h\, F'(a + 0h),\ 0 < \theta < 1$$

Using (2), we have,

$$F(a + h) - f(a) = h[f_x(a + \theta h, b + h) - f_x(a + \theta h, b)] \quad .....(3)$$

Since $f_x$ is differentiable at (a, b), therefore by definition of differentiability, we have,

$$f_x(a + \theta h, b + h) - f_x(a, b) - \theta h f_{xx}(a, b) - h f_{yx}(a,b) = 0hx + h\eta$$

Where $\varepsilon, \eta \to (0, 0)$, as $(h, h) \to (0, 0)$

$$f_x(a + \theta h, b + h) - f_x(a, b) - \theta h f_{xx}(a, b) - h f_{yx}(a, b) + h\varepsilon' \quad ......(4)$$

Where $\varepsilon' \to 0$ as $h \to 0$

Similarly $f_x(a + \theta h, b) - f_x(a, b) = \theta h f_{xx}(a, b) + h\varepsilon''$ .....(5)

Where $\varepsilon'' \to 0$ as $h \to 0$

Subtracting (5) from (4), we have,

$$f_x(a + \theta h, b + h) - f_x(a + \theta h, b) = h f_{yx}(a, b) + h(\varepsilon' + \varepsilon'') \quad .....(6)$$

Now putting (6) in (3), we have,

$F(a + h) - F(a) = h[f_{yx}(a, b) + h(\varepsilon' + \varepsilon'')] = h2[f_{yx}(a, b) + \varepsilon_1]$ where $\varepsilon_1 = \varepsilon' + \varepsilon'' \to$ as $h \to 0$

$$\text{or } \frac{F(a+h)-F(a)}{h^2} = f_{yx}(a, b) + \varepsilon'''$$

Using (1), we have,

$$\frac{f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)}{h^2} = f_{yx}(a, b) + \varepsilon_1 \quad .....(7)$$

Again writing $G(y) = f(a + h, y) - f(a, y),\ b \leq y \leq b + h$

And proceeding as above, we have,

$$\frac{f(a+h,b+h)-f(a+h,b)-f(a,b+h)+f(a,b)}{h^2} = f_{yx}(a, b) + \varepsilon_2 \quad .....(8)$$

Where $\varepsilon_2 \to 0$ as $h \to 0$

From (7) and (8) we see that

$$f_{yx}(a, b) + \varepsilon_1 = f_{xy}(a, b) + \varepsilon_2, \ \text{ where } \varepsilon_1, \varepsilon_2 \to 0$$

Taking limits on both sides as $h \to 0$, we at once get $f_{yx}(a, b) = f_{xy}(a, b)$.

This completes the proof:

**Schawrz's Theorem:**

Statement. Let $f : D \to R$, where D is an open subset of $R^2$.

If (1) $f_x f_y$ exist in the neighbourbood of a point (a, b) $\in$ D

And (2) $f_{xy}$ is continuous at (a, b) then, $f_{yx}$(a, b) exists and is equal to $f_{xy}$ (a, b).

Proof: (a, b) $\in$ D and D is open, therefore $\exists$ a real $\delta > 0$, s.t.

$(a + h, b + k) \in D\ f$ or $|h| < \delta$, $|k| < \delta$.

Consider $F(y) = f(a + h, y) - f(a, y)$, $b \leq y \leq b + k$ .....(1)

$F(b + k) - F(b) = f(a + h, b + k) - f(a, b + k) - f(a + h, b) + (a, b)$ ....(2)

Since $f_y$ exists, there F(y) is derivable in [b, b + k]

$\therefore$ by Lagrange's Mean Value Theorem,

$F(b + k) - F(b) = kF'(b + \phi k)$, $0 < \phi < 1$ .....(3)

Also from (1),

$F'(y) = f_y(a + h, y) - f_y(a, y)$

$F'(b + \phi k) = f_y(a + h, b + \phi k) - f_y(a, b + \phi k)$ .....(4)

From (3) and (4), we have,

$F(b + k) - F(b) = k[f_y(a + h, b + \phi k) - f_y(a, b + \phi k)]$ .....(5)

Let $G(x) = f_y(x, b + \phi k)$, $a < x < a + h$ ......(6)

Then $G(a + h) - G(a) = f_y(a + h, b + \phi k) - f_y(a, b + \phi k)$ ......(7)

Since $f$xy exists, therefore C(x) is derivable in (a, a + h) and $\therefore$ by Lagrange's Mean Value Theorem in [a, a + h],

From (6), $G'(x) = f_{xy}(x, b + \phi k)$

$\therefore G'(a + \theta h) = f_{xy}(a + \theta h, b + \phi k)$ ......(9)

From (8) and (9) we have,

$G(a + h) - G(a) = hf_{xy}(a + \theta h, b + \phi k)$ ......(10)

From (7) and (10), we have,

$f_y(a + h, b + \phi k) - f_y(a, b + \phi k) = hf_{xy}(a + \theta h, b + \phi k)$ ......(11)

From (5) and (11), we have

$F(b + k) - F(b) = hkf_{xy}(a + \theta h, b + \phi k)$ ......(12)

From (2) and (12), we have

$f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b) = hkf_{xy}(a + \theta h, b + k)$

or $\frac{1}{k}\left[\frac{f(a+h,b+k)-f(a+b,k)}{h}-\frac{f(a+h,b)-f(a,b)}{h}\right] = f_{xy}(a + \theta h, b + k)$

Taking limits on both sides as $h \to 0$

$$\frac{1}{k}\left[\lim_{h\to 0}\frac{f(a+h,b+k)-f(a+b,k)}{h}\lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}\right] = \lim_{h\to 0} f_{xy}(a + \theta h, b + k)$$

or $\frac{1}{k}$ [$f$x(a, b + k) - $f_x$(a, b)] = $f_{xy}$ (a + θh, b) [Since $f_{xy}$ is continuous]

$$\text{or } \frac{f_x(a,b+k)-f_x(a,b)}{k} = f_{xy}(a + \theta h, b)$$

Taking limits on both sides as k → 0

$$\lim_{k\to 0}\frac{f_x(a,b+k)-f_x(a,b)}{k} = \lim_{k\to 0} f_{xy}(a + \theta h, b)$$

or $f_{xy}$(a, b) = $f_{xy}$(a, b) [Since $f_{xy}$ is continuous]

This proves the theorem.

Remark. Schwarz's Theorem can also be stated as:

Let $f$:D → R, where D is an open subset of $R^2$.

If (1), $f_x f_y f_{yx}$ exist in the neighbourhood of a point (a, b)

2. $f_{yx}$ is continuous at (a, b).

Then $f_{yx}$(a, b) exists and is equal to $f_{yx}$(a, b).

**17.10 Self Check Exercise**

Q.1 Let $f$(x, y) = log ($x^2 + y^2$)

Find $f_x$ (1, 2), $f_y$ (0, 1)

Q. 2 Let $f$(x, y) = $x^3 + y^3$ - 3axu

find $f_x$ (a, a), $f_y$ (a, a)

Q. 3 Let $f$: $R^3$ → R be defined by $f$(x, y, z) = $x^2y^2z^3$, find

$f_x$(1, 1, 2) and $f_y$ (0, 2, 3)

Q. 4 Consider the function

u = log $\sqrt{x^2+y^2}$ , compute the second order partial derivative and

verify that $u_{11} + u_{22} = 0$

**17.11 Summary**

In this unit we have studied

(i) what do we mean by directional derivatives

(ii) definition of partial derivatives

(iii) first order partial derivatives of a function of three variables

(iv) the geometrical interpretation of partial derivatives of first order

(v) partial derivatives of higher order

(vi) partial derivatives and continuity

(vii) differentiability and differentiable functions

## 17.12 Glossary

(1) **Open set -**

Any subset A of $R^2$ is called on open set of either A is an empty set or A is a nhd. of each of its points.

(2) **Closed set -**

A subset A of $R^2$ is called a closed set of its compliment $A^c$ is an open set.

## 17.13 Answers to Self Check Exercises

Ans. 1 $\frac{2}{5}$, 2

Ans. 2 0, 0

Ans. 3 16, 0

Ans. 4 Find the derivatives and proceed.

## 17.14 Reference/Suggested Reading

1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, L. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002

## 17.15 Terminal Questions

1. Evaluate $f_x$ (0, 0, 0), $f_y$ (0, 1, 0) if $f$(x, y, z) = sin ($x^3 + 2xy + z^2$)

2. If $f(x, y) = |xy|^{1/2}$, prove that $f_x(0, 0) = f_y(0, 0) = 0$

3. It z = xy tan $\frac{y}{x}$, prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$.

$- r^2 4t$

4. If $\theta = t^n$ e, find the value of n s.t.

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$$

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## Unit - 18

# Partial Derivatives of Homogeneous Functions

**Structure**



### 18.1 Introduction

Dear students, in this unit we will study the concept of homogenous function and partial derivatives of these functions. The concept of homogeneous function was originally introduced for functions of several variables. With the definition of vector space at the end of 19th century, the concept has been naturally extended to functions between vector spaces, since a triple of variable valves can be considered as a coordinate vector.

### 18.2 Learning Objectives

The main objectives of this unit are

(i) to define homogeneous functions

(ii) to find the partial derivatives of homogeneous functions

(iii) to prove Euler's theorem on homogeneous functions of two or three variables

### 18.3 Homogeneous Functions

A function of two variables x and y of the form

$f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \ldots\ldots.. a_{n-1} xy^{n-1} + a_n y^n$

in which each term is of degree n is called.

homogeneous function

This function can be rewritten as

$$f(x, y) = x^n \left[ a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + .......+ a_n\left(\frac{y}{x}\right)^n \right]$$

$$= x^n\, g\left(\frac{y}{x}\right)$$

$$f(x, y) = y^n \left[ a_n + a_{n-1}\left(\frac{x}{y}\right) + a_{n-2}\left(\frac{x}{y}\right)^2 + .........+ a_0\left(\frac{x}{y}\right)^n \right] = y^n\, g\left(\frac{x}{y}\right)$$

Thus a function $f(x, y)$ if homogeneous of degree n in two variables x and y, can be expressed in the form.

$$x^n\, g\left(\frac{y}{x}\right) \text{ or } y^n\, g\left(\frac{x}{y}\right)$$

Another Definition 1. $f(x, y)$ is a homogeneous function of degree n if

$f(tx, ty) = t^n f(x, y)$ for all t independent of x and y

Definition 2. Let $f : R^n \to R$ be a function of n variable $x_1, x_2.......x_n$. Let $t \in R$.

if $f(tx_1, tx_2.......tx_n)$ can be expressed as $f(tx_1, tx_2,.......tx_n) = t^p f(x_1, x_2......x_n)$ for all real t, then $f(x_1, x_2.....x_n)$ is called a homogeneous function of order (or degree) p.

**For example:**

(i) $f(x, y) = \dfrac{x^2 + y^2}{x + y}$

$$\therefore f(tx, ty) = \frac{t^2x^2 + t^2y^2}{tx + ty} = \frac{t^2(x^2 + y^2)}{t(x + y)}$$

$$= t^1\left(\frac{x^2 + y^2}{x + y}\right)$$

$$= t^1 f(x, y)$$

$\therefore f(x, y)$ is a homogeneous function of order (degree) 1.

(ii) $f(x, y) = x^2 \tan^{-1}\left(\dfrac{y}{x}\right) = y^2 \tan^{-1} \dfrac{x}{y}$, $xy \neq 0$

$\therefore$ $f(tx, ty) = t^2x^2 \tan^{-1} \dfrac{ty}{tx} - t^2y^2 \tan^{-1} \dfrac{tx}{ty}$

$$= t^2 \left\{ x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y} \right\}$$

$$= t^2 f(x, y)$$

Hence $f(x, y)$ is a homogeneous function of order (degree) 2.

## 18.4 Euler's Theorem on Homogeneous function of Two Variables

**Theorem 1.** If $f : R^2 \rightarrow R$ is a homogeneous function of two variables x and y and is of order n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

**Proof:** Since $f(x, y)$ is a homogeneous function of two variables x, y of order n, therefore it can be written as $x^n F\left(\frac{y}{x}\right)$

$\therefore$ Let $f(x, y) = x\, F(\nu),\ \nu = \left(\frac{y}{x}\right)$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = x^n f'(\nu).\frac{\partial \nu}{\partial x} + n\,x^{n-1} F(\nu)$$

$$= x^n F'(\nu) \left\{\left(\frac{y}{x^2}\right)\right\} + nx^{n-1} F(\nu)$$

$$\Rightarrow \quad x\frac{\partial f}{\partial x} = -yx^{n-1} F'(\nu) + n\,x^n F(\nu) \qquad \text{......(1)}$$

and $$\frac{\partial f}{\partial y} = x^n \frac{\partial}{\partial y} F(\nu) = x^n F'(\nu).\frac{\partial \nu}{\partial y} = x^n F'(\nu).\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad y\frac{\partial f}{\partial y} = yx^{n-1} F'(\nu) \qquad \text{.....(2)}$$

Adding (1) and (2), we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\,x^n F(\nu)$$

or $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \qquad (Q\ f = x\ ;\ y) = x^n F(\nu))$$

Hence the theorem

**Cor.1.** If we take $z = f(x, y)$ then

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = nz.$$

**Cor.2.** From cor. 1., z = $f$ (x, y) then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$$

Differentiating partially w.r.t. x, we get

$$x\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}.1 + y\frac{\partial^2 z}{\partial x \partial y} = n\frac{\partial z}{\partial x}$$

$$\Rightarrow \qquad x\frac{\partial^2 z}{\partial x^2} + y\frac{\partial^2 z}{\partial x \partial y} = (n - 1)\,\frac{\partial z}{\partial x}$$

**Cor. 3.** Also

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = nz$$

Differentiating partially w.r.t. y, we get

$$x\,\frac{\partial^2 z}{\partial y \partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.1 = nz$$

Differentiating partially w.r.t. y, we get

$$x\,\frac{\partial^2 z}{\partial y \partial x} + y\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}.1 = n\,\frac{\partial z}{\partial y}$$

$$\Rightarrow \qquad x\frac{\partial^2 z}{\partial x \partial y} + y\frac{\partial^2 z}{\partial y^2} = (n\text{-}1)\,\frac{\partial z}{\partial y} \qquad \left(\begin{array}{c} essu\min g \\ \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x} \end{array}\right)$$

**Cor.4.** Multiplying the results of cor 2. by x and result of cor 3. by y on both sides and adding, we get

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = (n\text{-}1)\left\{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right]$$

$$\therefore \qquad x^2\frac{\partial^2 z}{\partial x^2} + 2xy\,\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n\text{-}1)z.$$

**Cor.5.** We have from theorem 1,

$$\frac{\partial z}{\partial x} = x^{n-1}\left\{\left(\frac{-y}{x}\right)g'\left(\frac{y}{x}\right)+ng\left(\frac{-y}{x}\right).\right\}$$

$$= x^{n-1} \,\&\left(\frac{y}{x}\right) \qquad \text{(Take } z = f(x, y))$$

$\Rightarrow$ $\frac{\partial z}{\partial x}$ is a homogeneous function of degree n-1 in x and y.

and similarly

$$\frac{\partial z}{\partial y} = x^{n-1}\, g'\left(\frac{y}{x}\right) = x^{n-1}\, \psi\left(\frac{y}{x}\right)$$

$\Rightarrow$ $\frac{\partial z}{\partial y}$ is a homogeneous functions of degree (n-1) in x and y.

### 18.5 Euler's Theorem on Homogeneous function of three Variables.

Theorem 2. If $f : R^3 \to R$ is a homogeneous function of three variables x, y and z and is of order n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$$

**Proof:** Since $f$ is a homogeneous function of three variables x, y, z of order n

$\therefore$ $f(tx, ty, tz) = t^n f(x, y, z)\ \forall$ Real t. ......(1)

Put $tx = u, ty = v, tz = w.$ ......(2)

$\Rightarrow$ $\frac{\partial u}{\partial t} = x,\ \frac{\partial v}{\partial t} = y,\ \frac{\partial w}{\partial t} = z.$ ......(3)

From (1), we have

$$f(u, v, w) = t^n f(x, y, z) \qquad \text{(using (2))}$$

Differentiating partially w.r.t. t, we have,

$$\frac{\partial f}{\partial u}.\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}.\frac{\partial v}{\partial t} + \frac{\partial f}{\partial w}.\frac{\partial w}{\partial t} = nt^{n-1} f(x, y, z)$$

$\Rightarrow$ $x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} + z\,\frac{\partial f}{\partial w} = n\,t^{n-1}\,(x, y, z)$ (using (3)) .....(4)

Taking t = 1, we have

$$x = u,\ y = v,\ z = w$$

$$\Rightarrow \qquad \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}, \frac{\partial f}{\partial v} = \frac{\partial f}{\partial y}, \frac{\partial f}{\partial w} = \frac{\partial f}{\partial z}$$

is from (4), we have

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)$$

Hence the proof of the theorem.

**Theorem 3**. Extension of Euler's Theorem in $R^2$.

If $f$ is a homogeneous function of two variables x and y and is order n, then

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = (n)(n-1) f.$$

**Proof:** By Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \qquad .....(1) \qquad (\because f \text{ is homogeneous})$$

Differentiating Partially w.r.t. x, we get

$$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}.1 + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x}$$

$$\Rightarrow \qquad x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = (n-1)\frac{\partial f}{\partial x}$$

Multiplying both sides by x, we get

$$x^2\frac{\partial^2 f}{\partial x^2} + xy\frac{\partial^2 f}{\partial x \partial y} = (n-1)\, x\frac{\partial f}{\partial x} \qquad .....(2)$$

Again differentiating (1) partially w.r.t. y, we get

$$x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial y} = n\frac{\partial f}{\partial y}$$

$$\text{or} \qquad x\frac{\partial^2 f}{\partial y \partial x} + y\frac{\partial^2 f}{\partial y^2} = (n-1)\frac{\partial f}{\partial y}$$

Multiplying both sides by y, we get

$$xy\frac{\partial^2 f}{\partial y \partial x} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)\, y\frac{\partial f}{\partial y} \qquad ......(3)$$

Adding (2) and (3), we get

$$x^2\frac{\partial^2 f}{\partial x^2} + xy\left\{x\frac{\partial^2 f}{\partial x\partial y}+y\frac{\partial^2 f}{\partial y\partial x}\right\} + y^2\frac{\partial^2 f}{\partial y^2} = (n-1)\left\{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right\}$$

Since $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = nf$, and $\frac{\partial^2 f}{\partial x\partial y}$, we have

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n\,(n-1)\,f.$$

Hence the proof of the theorem.

**Example 1:** Verify Euler's theorem for $z = x^4 \log\left(\frac{y}{x}\right)$

**Solution:** Since z is a homogeneous function of x, y and is of order 4, therefore by Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4z = 4x^4\log\left(\frac{y}{x}\right)$$

Verification

Since $z = 4x^4 \log\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial z}{\partial x} = x^4 \cdot \frac{1}{\left(y/x\right)}\left(-\frac{y}{x^2}\right) + 4x^3\log\frac{y}{x}$$

$$= -x^3 + 4x^3\log\left(\frac{y}{x}\right)$$

and

$$\frac{\partial z}{\partial y} = x4 \cdot \frac{1}{y/x}\cdot\frac{1}{x} = \frac{x^4}{y}$$

$$\therefore \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \left(-x^4+4x^4\log\left(\frac{y}{x}\right)\right) + x^4$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 4x^4\log\left(\frac{y}{x}\right) = 4z.$$

Hence the verification.

**Example 2 :** If z = $\tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \sin 2z$.

**Solution :** We have

$$z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$$

$$\Rightarrow \qquad \tan z = \frac{x^3+y^3}{x-y}$$

take tan z = u

$$\therefore \qquad u = \frac{x^3+y^3}{x-y}$$

Clearly, u is a homogeneous function of x and y and of order 2.

$$\therefore \qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u = 2\tan z \qquad ...(1)$$

Since u = tan z

$$\therefore \qquad \frac{\partial u}{\partial x} = \sec^2 z \,.\, \frac{\partial z}{\partial x} \text{ and } \frac{\partial u}{\partial y} = \sec^2 z\,\frac{\partial z}{\partial y}$$

$\therefore$ From (1), we have

$$x\sec^2 z\,\frac{\partial z}{\partial x} + y\,\sec^2 z\,\frac{\partial z}{\partial y} = 2\tan z$$

or

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{2\tan^2}{\sec x} = .\ \frac{2\tan z}{1/\cos^2 z} = \frac{2\tan z}{\cos \cancel{z}}.\cancel{\cos}^2 z$$

$$= 2\sin z\cos z$$

$$= \sin 2z.$$

**Example 3 :** If u = $\frac{x^2y^2}{x+y}$ then show that

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial y}$$

Solution : Clearly, u is a homogeneous function of x, y and is of order 3.

$\therefore$ by Euler's Theorem

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = 3u$$

Differentiating w.r.t. y , we get

$$x\,\frac{\partial^2 u}{\partial x \partial y} + y\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 3\frac{\partial u}{\partial y}$$

or $$x\,\frac{\partial^2 u}{\partial y \partial x} + y\,\frac{\partial^2 u}{\partial y^2} = 2\,\frac{\partial u}{\partial y}$$

**Example 4 :** Verify Euler's theorem for

$$u = \sqrt{x^2 + y^2}$$

**Solution :** Here u = $\sqrt{x^2+y^2}$ ...(1)

$$= \sqrt{x^2 \left\langle 1 + \left(\frac{y}{x}\right)^2 \right\rangle}$$

$$= \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

$$= x\, f\left(\frac{y}{x}\right)$$

$\therefore$ u is homogeneous function of x and y of degree 1.

We shall show now to verify Euler's Theorem, that

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = u \qquad ...(2)$$

From (1) we have

$$u = \sqrt{x^2 + y^2}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} \text{ and } \frac{\partial u}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x^2}{\sqrt{x^2+y^2}} \text{ and } y\,\frac{\partial u}{\partial y} = \frac{y^2}{\sqrt{x^2+y^2}}$$

$\therefore \quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{x^2}{\sqrt{x^2+y^2}} + \dfrac{y^2}{\sqrt{x^2+y^2}}$

$$= \frac{x^2+y^2}{\sqrt{x^2+y^2}} = \sqrt{x^2+y^2} = u$$

Hence

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Hence the verification of Euler's theorem.

## 18.6 Self Check Exercise

Q.1 Verify Euler theorem for

$u = xy + yz + zx$

Q. 2 If $u = e^{-x/y}$, verify Euler's theorem.

Q. 3 For u xn sin $\dfrac{y}{x}$, verify Euler's theorem.

Q. 4 If $u = \log\left(\dfrac{x^4-y^4}{x-y}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

## 18.7 Summary

In this unit we have learnt the following :

(i) definition of homogeneous function

(ii) Euler's theorem on homogeneous functions of two variables

(iii) Euler's theorem on homogeneous function of three variables.

## 18.8 Glossary

1. **Homogeneous Function of degree k**

   → a homogeneous polynomial of degree k defines a homogeneous function of degree k.

2. **Positive homogeneity -**

   $f$ (r, x) = r $f$ (x) ∀ x ∈ x . and all pontins real r > 0.

3. Absolute homogeneity -

   $f$ (s x) = | s | $f$ (x) ∀ x ∈ x. and all Scalars s ∈ F

## 18.9 Answers to Self Check Exercises

Ans. 1 u = xy + yz + zx = $x^2\left(\frac{y}{x}+\frac{y}{x}.\frac{z}{x}+\frac{z}{x}\right)$ = $x^2 f\left(\frac{y}{x},\frac{z}{x}\right)$ and then proceed.

Ans. 2 u = $e^{-x/y}$ = $x^0 e^{-\frac{1}{y/x}}$ = $x^0 f\left(\frac{y}{x}\right)$ and then proceed.

Ans. 3 Proceed to verify Euler's theorem.

Ans. 4 u = log $\frac{x^4-y^4}{x-y}$ $\Rightarrow e^u$ = $\frac{x^4-y^4}{x-y}$ = z (say), then proceed.

## 18.10 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002


## 18.11 Terminal Questions

1. Verify Euler's theorem for the functions

   (i) $f$ (x, y ) = $\frac{1}{x^2+xy+y^2}$

   (ii) $f$ (x y) an$^2$ + 2hxy + y$^2$b

   (iii) $f$ (x y) = $\frac{xy}{x+y}$

2. If $f$(x y) = sin$^{-1}$ + $\frac{x}{y}$ + tan$^{-1}$ $\frac{y}{x}$, then x$f$n + y$f$y = 0

3. It z = xy $f\left(\frac{x}{y}\right)$, then prove that

   x $\frac{\partial z}{\partial x}$ + y $\frac{\partial z}{\partial y}$ = 2z.

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## Unit - 19

# Maxima and Minima

**Structure**



### 19.1 Introduction

Dear students, we have already studied the concept of maxima and minima of a real valued function of a single variable, at our lower level in this unit we shall study maxima and minima of real valued function of two or more in depended variables. In mathematical analysis, the maxima and minima of a function are respectively the largest and smallest value taken by the function. Known generally as extermum they may be defined either within a given range or on the entire domain of a function. Pierre de Fermat was one of the first mathematician to propose a general technique, for binding maxima and minima of functions.

### 19.2 Learning Objectives

The main objectives of this unit are

(i) to define, definite, semi-definite and indefinite functions

(ii) to defined maximum and minimum value of a function

(iii) to find maximum and minimum value for functions of two variables.

(iv) to find maximum and minimum for functions of three variables.

(v) to study Lagrange's method of multiplier.

### 19.3 Definite, Semi-Definite and Indefinite Functions

**Def. 1. Definite Function.** A real valued function $f$ with domain $D_f \subset R^n$ is said to be positive definite if $f(x) > 0$ $f$ or all $x \in D_f$ and negative definite if $f(x) < 0$ for all $\in D_f$. A positive definite or a negative definite function is said to be definite function.

For example, the function

$f : R^2 \rightarrow R$ defined by $f(x, y) = x^2 + y^2 + 2 \ \forall \ (x, y) \in R^2$ is positive definite as $f(x, y) \geq 2 \forall \ (x, y) \in Df = R^2$ whereas the function $f : R^3 \rightarrow R$ defined by

$f(x, y, z) = -(x^2 + y^2 + z^2 + 2) \ \forall \ (x, y, z) \in R^3$ is negative definite as

$f(x, y, z) \leq -2 \forall \ (x, y, z) \in D_f = R^2$ .

**Def. 2. Semi-definite function :** A real value function $f$ with domain $D_f = R^n$ is said to be semi-definite if it vanishes at some points of $D_f$ and when it is not-zero, it is of the same sign throughout.

For example, the function $f : R^2 \rightarrow R$ defined by $f(x, y) = x^2 + y^2$ , $(x, y) \in R^2$ is semi definite as $f(0, 0) = 0$ and $f(x, y) > 0$ for all $(x, y) \in R^2$ , $(x, y) \neq (0, 0)$.

**Def. 3 Indefinite Function :** A real function $f$ with domain $D_f = R^n$ is said to be indefinite if it can assume values which have different signs i.e., $f$ is said to be indefinite if it is neither definite nor semi-definite.

For example, the function $f : R^3 \rightarrow R$ defined by $f(x, y, z) = 3x - 4y + 2z \ \forall \ (x, y, z) \in R^3$ is an indefinite function as $f(x, y, z)$ can be positive or zero or negative.

Sign of a Quadratic Expression

(a) Function of two real variables.

Let $f(x, y) = ax^2 + 2hxy + by^2$, $a \neq 0$

$$= \frac{1}{a}(a^2x^2 + 2ahxy + aby^2)$$

$$= \frac{1}{a}[(ax + hy)^2 + (ab - h)^2 y^2]$$

The following cases arise:

**Case 1.** When $ab - h^2 > 0$, then $f(x, y)$ has the same sign as that of a for all $(x, y) \in R^2$. This implies that $f$ is definite if $ab - h^2 > 0$ and is positive definite or negative definite according as a is positive or negative.

**Case 2.** When $ab - h^2 = 0$, then $f(x, y)$ has the same sign as that of 'a' for all $(x, y) \in R^2$. This implies that if $ab - h^2 = 0$ and $(ax + hy) \neq 0$ for any $(x, y) \in R^2$, then $f$ is definite. It is positive definite. It is positive definite or negative definite according as a is positive or negative.

**Case 3.** When $ab - h^2 = 0$ and $(ax + hy) = 0$ $f$ or some $(x, y) \in R^2$, then $f$ is semi definite.

**Case 4.** When $ab - h^2 < 0$, then we cannot be sure of the sign of $f(x, y)$ and hence $f$ is indefinite.

(b) Function of three realvariables.

Let $f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

(1) If $a \neq 0$, then $f(x, y, z) > 0$ if a, $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ are all positive

i.e. $f$ is positive definite.

(2) If $a \neq 0$, then $f < 0$ if the above three expressions are alternately negative and positive, i.e. $f$ is negative definite.

### 19.4 Maximum and Minimum for Functions of two Variables

**Def.** Let $D \subset R^2$ and (a, b) be an interior point of D where D is the domain of $f(x, y)$. Then $f(a, b)$ is said to be a local (or relative) maximum value of $f(x, y)$ if $f(x, y) < f(a, b)$ for all points (x, y) in a sufficiently small neighbourhood of (a, b), $[(x, y) \neq (a, b)]$.

**Alternative Definitions**

**Maximum Value**

Def. A function $f(x, y)$ is said to have a maximum value at a point (a, b) if $f(a, b) > f(a + h, b + k)$ for all small values of h and k, positive or negative of $f(a, b) - f(a + h, b + k) > 0 \; \forall$ h, k

**Minimum Value**

Def. A function $f(x, y)$ is said to have a minimum value at a point (a, b) if $f(a, b) < f(a + h, b + k)$ for all small value of h and k, positive or negative of $f(a, b) - f(a + h, b + k) < 0 \; \forall$ h, k.

**Extreme Values**

1. The local maximum and the local minimum value of $f(x, y)$ are also called the extreme values of $f(x, y)$ and the points where $f(x, y)$ has an extreme value are called extreme point.
2. A point $(a, b) \in D$ is called an interior point of D if every point of some neighbourhood of (a, b) is a point of D.

**Theorem 1.** The necessary conditions for $f(a, b)$ to be an extreme value of $f(x, y)$ are that $f$x(a, b) = $f$y(a, b) = 0, provided that $f$x(a, b) and $f$y(a, b) exist.

**Proof:** If $f(a, b)$ is an extreme value of $f(x, y)$ then, clearly it is also an extreme value of the value of $f(x, b)$ of one variable x for x = a and therefore its derivative w.r.t. x i.e. $f_x(a, b)$ for x = a, in case it exists, must necessarily be i.e. $f_x(a, b) = 0$. Similarly we have $f_y(a, b) = 0$.

**Remark 3.** The points where $f$x(x, y) = 0 = $f$y(x, y) are called critical or stationary points which not be extreme points. Hence every extreme point is a stationary point but not conversely.

**Remark 4.** Consider a function $f(x, y) = x^2 - y^2$.

Here $f$x(x, y) = 2x and $f$y(x, y) = -2y.

$$\therefore f_x(0, 0) = 0 \text{ and } f_y(0, 0) = 0$$

Now $f(0 + h, 0 + k) - f(0, 0) = f(h, k) - f(0, 0) = h^2 - k^2$.

which does not hold the same sign i.e. positive or negative for small values of h and k.

$\therefore$ (0, 0) is not an extreme point of $f(x, y)$.

**Remark 5.** $f(x, y)$ can have extreme value at (a, b) even though $f_x(a, b)$ do not exist.

For example, $f(x, y)$ $|x| + |y|$ has an extreme value at (0, 0) even though $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.

**Remark 6.** The greatest of all the extreme values is called absolute maximum or global maximum.

**Remark 7.** The smallest of all extreme values is called an absolute minimum or global minimum.

**Remark 8.** Saddle point. The point where $f(x, y)$ has neither maximum value nor minimum value is called a saddle point.

**Theorem 2.** Sufficient Conditions.

Let $f$ be a real valued function of two variables x and y and if $f_x(a, b) = 0 = f_y(a, b)$ and $f_{xy}(a, b) = A$. $f_{xy}(a, b) = B$ and $f_{yy}(a, b) = C$ then

1. $f(a, b)$ is a maximum value of $f(x, y)$ if $AC - B^2 > 0$ and $A < 0$
2. $f(a, b)$ is a minimum value of $f(x, y)$ if $AC - B^2 > 0$ and $A > 0$
3. $f(a, b)$ is not extreme value of $f(x, y)$ if $AC - B^2 < 0$
4. $f(a, b)$ may or may not be an extreme value of $f(x, y)$ if

   $AC - B^2 = 0$ [it is called a doubtful case]

**Proof:** We assume that $f$ has continuous partial derivatives of first, second and third order w.r.t. x and y both in a neighbourhood, say N of (a, b). By Taylor's theorem on two variables with remainder after three terms, we have,

$$f(a + h, b + k)$$

$$= f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f(a, b) + \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(a + th, b + tk)$$

Where $0 < t < 1$ and $(a + h, b + k) \in N$.

$$\Rightarrow \quad f(a + h, b + k) - f(a, b)$$

$$= hf_x(a, b) + kf_y(a, b) + \frac{1}{2}[h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)]$$

$$+\frac{1}{6}[h^3 f_{xxx}(u,v) + 3h^2 k f_{xxy}(u,v)\ 3hk^2 f_{xyy}(uv) + k^3 f_{yyy}(uv)]$$

Where $u = a + th$, $\nu = b + tk$, $0 < t < 1$.

Since $f$x(a, b) = 0 and $f$y(a, b) = 0, we have

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}(Ah^2 + 2Bkh + Ck^2) + g(h, k)$$

Where g (h, k) is a third degree expression in h, k.

We suppose for sufficiently small values of h and k, the sign of $\frac{1}{2}(Ah^2 + 2Bkh + Ck^2) + g(h, k)$ is the same as that of $Ah^2 + Bhk + Ck^2$.

The following cases arise:

Case 1. Let $AC - B^2 > 0$. Here nether A nor C can be zero, since, if A = 0 or C = 0 then $AC - B^2 = -B^2 < 0$, which is wrong.

Therefore, we write

$$Ak^2 + 2Bhk + Ck^2 = \frac{1}{A}[(Ah + Bk)^2 + (AC - B^2)k^2]$$

Now since $AC - B^2 > 0$, therefore, $(Ah + Bk)^2 + (AC - B^2)$ is always positive except when $Ah + Bk = 0$ and $k = 0$ i.e. except when $h = 0$ and $k = 0$ and in that case the value of this expression is zero.

Thus, we find that whenever $h \neq 0$ and $k \neq 0$, $Ak^2 + 2Bhk + Ck^2$ has the same sign as that of A.

Therefore, (1) If $AC - B^2 > 0$ and $A < 0$, then $f(a + h, b + k) - f(a, b) < 0$ $f$ or all $(a + h, b + k) \in N((a, b))$ i.e. (a, b) is a point of maxima of $f$ and therefore $f$ (a, b) is a maximum.

(2) If $AC - B^2 > 0$ and $A > 0$, then $f(a + h, b + k) - f(a, b) > 0$ $f$ or all $(a + h, b + k) \in N - \{(a, b)\}$ i.e. (a, b) is a point of minima of $f$ and $\therefore$ $f$(a, b) is minimum.

**Case 2.** Let $AC - B^2 < 0$

If $A \neq 0$, we can write

$$Ah^2 + 2Bhk + Ck^2 = \frac{1}{A}[(Ah \therefore Bk)^2 - (B2 - AC)k^2]$$

Since $AC - B^2 < 0$ i.e. $B^2 - AC > 0$, we find that $Ah^2 + 2Bhk + Ck^2$ has the same sign as that of A when k = 0 and has the sign opposite to that of A when $Ah + Bk = 0$

Hence, $Ah^2 + 2Bhk + Ck^2$ has opposite sign when k = 0 and when $Ah + Bk = 0$ i.e. there exists points in the neighbourhood N of (a, b) where $Ah^2 + 2Bhk + Ck^2$ i.e. $f$(a, b) has opposite signs and hence (a, b) is not an extreme point of the function $f$.

Likewise, if $C \neq 0$, It can be shown that (a, b) is not an extreme point of $f$. A = 0 and C = 0, then

$$Ah^2 + 2Bhk + Ck^2 = 2Bhk$$

So that this expression takes values with opposite signs when hk > 0 and hk < 0. Hence, there exist points in the neighbourhood N of (a, b) where $f$(a + h, b + k) - $f$(a, b) has opposite signs, and hence (a, b) is not an extreme point of the function $f$.

**Case 3.** Let AC - $B^2$ = 0

If A $\neq$ 0, we write as,

$$Ah^2 + 2Bhk + Ck^2 = \frac{1}{A}(Ah + Bk)^2 + (Ac - B)^2k^2] = \frac{(Ah+Bk)^2}{A}$$

and the value of this expression becomes zero when h, k are so chosen that Ah + bk = 0. therefore, the sign of $f$(d + h, b + k) - $f$(a, b) depends upon that of g(h, k) and hence it needs further in vestigations.

If A = 0, then AC - $B^2$ = 0 or B = 0

If A = 0, then AC - $B^2$ = 0 or B = 0

$\therefore$ $Ah^2 + 2Bhk + Ch^2 = k^2$ and the value of this expression becomes zero when k = 0, whatever h may be. Again, in this situation, the sign of $f$(a + h, b + k) - $f$(a, b) depends upon that of g(h, k), and hence it needs further investigations.

Thus, if AC - $B^2$ = 0, then (a, b) may or may not be an extreme point. This is called a doubtful case. This completes the proof.

**Working Method to find Maxima and Minima.**

Let $f$(x, y) be a given function

Step 1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Step 2. Solve $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

Simultaneously to find x and y

Let $(x_1y_1)$, $(x_2y_2)$ ......be the solutions of these equations.

Step 3. Find A = $\frac{\partial^2 f}{\partial x^2}$, B = $\frac{\partial^2 f}{\partial x \partial y}$ and C = $\frac{\partial^2 f}{\partial y^2}$ and evaluate A, B, C for each point $(x_1y_1)$, $(x_2y_2)$ ......

Step 4. If for a point say $(x_1y_1)$ we have

(1) AC - $B^2$ > 0 and A < 0 then $(x_1y_1)$ is a maximum value and $f$(x, y) has a maxima for this pair.

(2) AC - $B^2$ > 0 and A < 0 then $(x_1y_1)$ is a maximum value and $f$(x, y) has a minima at $(x_1y_1)$

(3) If $AC - B^2 > 0$ and $A < 0$, then $f(x, y)$ has neither maxima nor minimum value at $(x_1y_1)$

(4) If $AC - B^2 = 0$, then it is a doubtful case and further investigation is required.

In this case, we consider:

(a) If $f(x_1 + hy_1 + k) - f(x_1y_1) < 0$, then $f(x, y)$ is maximum at $(x_1y_1)$

(b) If $f(x_1 + hy_1 + k) - f(x_1y_1) > 0$, then $f(x, y)$ is minimum at $(x_1y_1)$

(c) If $f(x_1 + hy_1 + k) - f(x_1y_1)$, does not keep the same sign, then $f(x, y)$ is neither maximum nor minimum at $(x_1y_1)$

**Note.** The points $(x_1y_1)$ $(x_2y_2)$ obtained are called critical points or stationary points or extreme points or turning points and the values of $f(x, y)$ at these points are called stationary values or extreme values.

To clarify what we have just said, consider the following examples:-

**Example 1:** Find the maximum and minimum value of

$$f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2$$

$$f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2$$

$$fx = 8x^3 - 4x$$

$$fy = 4y^3 - 4y$$

$$fxy = 0$$

$$fxx = 24x^2 - 4$$

$$fyy = 12y^2 - 4$$

For extreme points,

$$fx = 0,\ fy = 0$$

$$\therefore\ 8x^3 - 4x = 0 \quad \text{and} \quad 4y^3 - 4y = 0$$

i.e. $x = 0, \underline{+} \frac{1}{\sqrt{2}}$ and $y = 0, \underline{+} 1$.

$\therefore$ the possible extreme points are

$$(0, 0), (0, 1), (0, -), \left(\frac{1}{\sqrt{2}},0\right), \left(\frac{1}{\sqrt{2}},1\right), \left(\frac{1}{\sqrt{2}},-1\right), \left(-\frac{1}{\sqrt{2}},0\right), \left(-\frac{1}{\sqrt{2}},1\right), \left(-\frac{1}{\sqrt{2}},-1\right)$$

At (0, 0)

$$A = fxx(0, 0) = -4$$

$$B = fxy(0, 0) = 0$$

$$C = fyy(0, 0) = -4$$

$\therefore$ AC - B$^2$ = 16 > 0 and A = -4 < 0

$\therefore$ (0, 0) is a point of maximum and maximum value at (0, 0) is 0.

at (0, $\pm$ 1) [$f$(0, 0) = 0]

A = $f_{xx}(0, \pm 1) = -4$

B = $f_{xy}(0, \pm 1) = 0$

C = $f_{yy}(0, \pm 1) = 8$

$\therefore$ AC - B$^2$ = -32 < 0

$\therefore$ (0, $\pm$ 1) are not extreme points.

At $\left(\frac{1}{\sqrt{2}}, 0\right)$:

$$A = f_{xx}\left(\frac{1}{\sqrt{2}}, 0\right) = 8$$

$$B = f_{xy}\left(\frac{1}{\sqrt{2}}, 0\right) = 0$$

$$C = f_{yy}\left(\frac{1}{\sqrt{2}}, 0\right) = -4$$

$\therefore$ AC - B$^2$ = -32 < 0

$\left(\frac{1}{\sqrt{2}}, 0\right)$ is not an extreme point

At $\left(\frac{1}{\sqrt{2}}, \pm 1\right)$:

$$A = f_{xx}\left(\frac{1}{\sqrt{2}}, \pm 1\right) = 8$$

$$B = f_{xy}\left(\frac{1}{\sqrt{2}}, \pm 1\right) = 0$$

$$C = f_{yy}\left(\frac{1}{\sqrt{2}}, \pm 1\right) = 8$$

$\therefore$ AC - B$^2$ = 64 > 0 and A = 8 > 0

$\therefore \left(\frac{1}{\sqrt{2}}, \pm 1\right)$ are points of minimum and minimum value

$$= f\left(\frac{1}{\sqrt{2}}, \pm 1\right) = \frac{3}{2}$$

At $\left(-\frac{1}{\sqrt{2}}, 0\right)$:

$$A = f_{xx}\left(-\frac{1}{\sqrt{2}}, 0\right) = 8$$

$$B = f_{xy}\left(-\frac{1}{\sqrt{2}}, 0\right) = 0$$

$$C = f_{yy}\left(-\frac{1}{\sqrt{2}}, 0\right) = -4$$

$\therefore$ AC - B$^2$ = -32 < 0 and therefore $\left(-\frac{1}{\sqrt{2}}, 0\right)$ is not an extreme point.

At $\left(-\frac{1}{\sqrt{2}}, \pm 1\right)$:

$$A = f_{xx}\left(-\frac{1}{\sqrt{2}}, \pm 1\right) = 8$$

$$B = f_{xy}\left(-\frac{1}{\sqrt{2}}, \pm 1\right) = 0$$

$$C = f_{yy}\left(-\frac{1}{\sqrt{2}}, \pm 1\right) = 8$$

$$\therefore AC - B^2 = 64 > 0 \text{ and } A = 8 > 0$$

$\therefore \left(-\frac{1}{\sqrt{2}}, \pm 1\right)$ are points of minimum and minimum value = $f\left(-\frac{1}{\sqrt{2}}, \pm 1\right) = -\frac{3}{2}$

**Example 2:** Find all the maxima and minima of the function

$$f(x, y) = x^3 + y^3 - 63(x + y) + 12xy.$$

**Solution:** $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$

$$f_x = 3x^2 - 63 + 12y$$

$$f_y = 3y^2 - 63 + 12x$$

$$f_{xy} = 12, \qquad f_{xx} = 6x, \qquad f_{yy} = 6y$$

For extreme values $f_x = 0$ and $f_y = 0$

i.e. $3x^2 + 12y - 63 = 0$

and $3y^2 + 12x - 63 = 0$

$$\therefore x^2 + 4y - 21 = 0$$

$$\therefore y^2 + 4x - 21 = 0$$

Subtracting, we have

$$x^2 - y^2 - 4(x - y) = 0$$

$$\text{or } (x - y)(x + y - 4) = 0$$

$$\Rightarrow \quad \text{either } x - y = 0 \text{ or } \quad x + y - 4 = 0$$

$\therefore$ We get two sets of equations:

$$x^2 + 4y - 21 = 0$$

$$\Rightarrow x = 3, -7$$

$$\therefore y = 3, -7$$

Or $x^2 + 4y - 21 = 0$

and $x + y - 4 = 0$

Elimination y, we have,

$x^2 - 4x - 5 = 0$

$\Rightarrow x = 5, -1$

$\therefore y = -1, 5$

Thus the possible extreme points are (3, 3), (-7, -7), (5, -1), (-1, 5)

At (3, 3)

$A = f_{xx}(3, 3) = 63 = 18 > 0$

$B = f_{xy}(3, 3) = 12$

And $C = f_{yy}(3,3) = 63 = 18$

$$\therefore AC - B^2 = 18 \times 18 - 12 = 324 - 144 = 180 > 0, \text{ also } A = 18 > 0$$

$\Rightarrow$ (3, 3) is a point of local minima and local minimum value

$$= f(3, 3) = 27 + 27 - 63(3 + 3) + 12.3.3 = -261$$

At (-7, -7):

$A = -42 < 0$, $B = 12$ and $C = -42$

$$AC - B2 = (-42)(-42) - 12^2 = 1620 > 0$$

Also A = -42 < 0

$\Rightarrow$ (-7, -7) is a point of maxima (local) and (local) maximum value

$$= f(-7, -7) = (-7)3 + (-7)^3 - 63 \{(-7 -7) + 12(-7)(-7)] = 784$$

At (5, -1):

$$A = 30 > 0, B = 12 \text{ and } C = 6 (-1) = -6$$

$$\Rightarrow AC - B^2 = 30(-6) - (12)^2 = -180 - 144 = -324 < 0$$

$\therefore$ (5, -1) is not an extreme point.

At (-1, 5):

$$A = 6 (-1) = -6 < 0$$

$$B = 12$$

$$C = 6(5) = 30$$

$$\Rightarrow AC - B^2 = -6(30) - 12^2 = -324 < 0$$

$\therefore$ (-1, 5) is not an extreme point.

So $f$(x, y) has a local at (-7, -7) and maximum local value = 784 and has a (local) minima at (3, 3) and local minimum value = -216.

**Example 3:** Find the extreme values of the function

$$f(x, y) = (x - y)^4 + (y - 1)^4$$

**Solution:** We have

$$f(x, y) = (x - y)^4 + (y - 1)^4$$

$$\frac{\partial f}{\partial x} = 4(x - y)^3 \text{ and } \frac{\partial f}{\partial y} = -4(x - y)^3 + 4(y - 1)^3$$

Now for critical points $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\Rightarrow 4(x - y)^3 = 0$$

And $-4 (x - y)^3 + 4(y - 1)^3 = 0$

$$\Rightarrow y = x \text{ ......(1) and } -(x - y)^3 + (y - 1)^3 = 0 \qquad \text{......(2)}$$

Using y = x in (2), we get

$$0 = (y - 1)^3 \Rightarrow y = 1$$

Putting y = 1 in (1), we get

$$x = 1$$

$\therefore$ (1, 1) is the only critical point

Now A = $\frac{\partial^2 f}{\partial x^2}$ = $12(x - y)^2$

$$B = \frac{\partial^2 f}{\partial x \partial y} = 12(x - y)^2$$

and C = $\frac{\partial^2 f}{\partial y^2}$ = $12(x - y)^2 + 12(y - 1)^2$

At (1, 1),

$$A = 12(1 - 1) = 0$$

$$B = 12(1 - 1) = 0$$

and C = 12 (1 - 1) + 12(1 - 1) = 0

$\Rightarrow$ $AC - B^2 = 0$ is a doubtful case

$\therefore$ Further investigation is required.

Consider $f(1 + h, 1 + k) - f(1, 1)$

$$= (1 + h - 1 - k)^4 + (1 + k - 1)^4 - 0$$

$$= (h - k)^4 + k^4 > 0$$

For small +ve or negative values of h and k.

$\therefore$ $f(x, y)$ is maximum at (1, 1) and maximum value is

$$f(1, 1) = (1 - 1)^4 + (1 - 1)^4 = 0$$

**19.5 Maximum and Minimum for Functions of three variables**

The Local Maximum and Local Minimum values for functions of three variables are defined in the same way as for the functions of two variables.

Definition. A function $f(x, y, z)$ is said to have a maximum value at the point

$(x_1, y_1, z_1)$ if

$f(x_1, y_1, z_1) > f(x_1 + h, y_1 + k, z_1 + t)$

i.e. if $f(x_1 + h, y_1 + k, z_1 + l) - f(x_1, y_1, z_1) > 0$ for small values of h, k and l positive or negative.

Theorem 3. The necessary conditions for a real valued function $f(x, y, z)$ with domain $D \subset R^3$ (D is open) to have an extreme value at the point $(a, b, c) \in R$ are $(f_x(a, b, c) = f_y(a, b, c) = f_z(a, b, c) = 0$.

Proof: It $f(a, b, c)$ is an extreme value of the function $f(x, b, c)$ of one variable for x = a and, therefore, its derivative $f_x(a, b, c)$ for x = a in case it exists, must necessarily be 0 i.e. $f_x(a, b, c) = 0$.

Similarly, we have,

$$f_y(a, b, c) = 0,\ fz(a, b, c) = 0$$

Definition, Critical point (stationary point)

A point (a, b, c) is said to be a critical point or stationary point of function $f(x, y, z)$ if $f_x(a, b, c) = 0$, $f_y(a, b, c) = 0$ and $f_x(a, b, c) = 0$

Theorem. Let $f(x, y, z)$ be a function of three independent variables x, y, z. Then the sufficient conditions for a point (a, b, c) to be an extreme point are that

(1) $df(a, b, c) = f_x dx + f_x dy + f_x dz = 0$

or $f_x(a, b, c) = f_y(a, b, c) = f_z(a, b, c) = 0$

(2) $d^2f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{xx}(dz)^2 + 2f_{xy}(dxdy) + 2f_{zy}(dzdy) + 2f_{zx}(dzdx)$

keep same sign for arbitrary small values of dx, dy and dz. Further (a, b, c) is a point of maxima or minima according to as $d^2f$ is negative or positive. The point (a, b, c) is point (a, b, c) is not an extreme point if $d^2f$ does not keep the same sign and may or may not be an extreme point if $d^2f$ deeps the same sign but vanishes at some points in neighbourhood of (a, b, c).

(we assume it without proof)

Theorem. Let $f(x, y, z)$ be a function of three independent variables x, y, z. Then $d^2f$ will always be positive.

Iff $f_{xx} \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, \begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{xx} & f_{xy} & f_{zz} \end{vmatrix}$

are all +ve and $d^2f$ will always be-ve if their signs are alternatively negative and positive.

(we assume it without proof).

In view of the above results the sufficient condition for extreme values can be stated an follows:

Theorem (Sufficient conditions). Let $f(x, y, z)$ be a function (real valued) of three variables defined on an open det $D \subset R^3$ and $f_x(a, b, c) = f_z(a, b, c) = 0$, then (a, b, c) is a point of local maximum or local minimum according as, the three expressions:

$$f\text{xx} \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} \begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{xx} & f_{xy} & f_{zz} \end{vmatrix}$$

are alternately negative and positive or all are positive at (a, b, c). [we accept it without proof.]

**Working Method to find Extreme Values of a Function of Three variable**

Let $f(x, y, z)$ be a given function of three independent variables x, y and z.

Step 1. Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$

Step 2. Solve equations $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$ and $\frac{\partial f}{\partial z} = 0$ simultaneously for x, y, z.

Let (a, b, c) $(a_1, b_1, c_1)$.......be the solutions of these equation which are critical points.

Step 3. Find A = $\frac{\partial^2 f}{\partial x^2}$, B = $\frac{\partial^2 f}{\partial y^2}$, C = $\frac{\partial^2 f}{\partial z^2}$, F = $\frac{\partial^2 f}{\partial y \partial x}$, G = $\frac{\partial^2 f}{\partial z \partial x}$, H = $\frac{\partial^2 f}{\partial x \partial y}$ and evaluate the following expressions in order.

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

Step 4. Find the value of each of the three expressions at stationary point (a, b, c).

(1) If A < 0, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} < 0$$

Then $f$ (x, y, z) is maximum at (a, b, c) and maximum value is $f$(a, b, c).

(2) If A > 0 $\begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0$ $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} > 0$

Then $f$(x, y, z) is minimum at (a, b, c) and its minimum value is $f$(a, b, c).

(3) If we fall to get any idea of maximum value or minimum value from the expressions

A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ then we calculate $d^2 f$ at (a, b, c)

(a) If $d^2 f > 0$ for small values of dx, dy and dz then $f$(x, y, z) is maximum at (a, b, c)

(b) If $d^2 f > 0$ for small arbitrary values of dx, dy and dz then $f$(x, y, z) is minimum at (a, b, c)

(c) If $d^2 f$ does't keep the same sign for small arbitrary values of dx, dy and dz then $f$(x, y, z) is neither max. at (a, b, c) nor minimum.

Let us see what the method is with the following example:-

**Example 4:** Discuss the maximum and minimum values of the function $f(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$.

**Solution:** Here, $f(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$

$$f_x = 2x + 1 - y,\ f_y = 2y - x,\quad f_z = 2z - 2$$

$$f_{xx} = 2,\ f_{yy} = 2,\ f_{zz} = 2$$

$$f_{yx} = -1,\quad f_{xy} = -1,\quad f_{xx} = 0$$

$$f_{xx} = 0,\ f_{zy} = 0,\ f_{yz} = 0$$

For extreme values, $f_x = f_y = f_z = 0$

$$2x - y + 1 = 0 \qquad (1)$$

$$2y - x = 0 \qquad (2)$$

$$2z - 2 = 0 \qquad (3)$$

From (2), x = 2y, putting in (1) 3y + 1 = 0 or $y = -\frac{1}{3}$

$$\therefore x = -\frac{2}{3}$$

From (3), z = 1

Thus, we have $x = -\frac{2}{3}$, $y = -\frac{1}{3}$, z = 1

Now $f_{xx}\left(-\frac{2}{3}, -\frac{1}{3}, 1\right) = 2 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{xx} & f_{xy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 > 0$

Thus we see that all the above three quantities are all positive and therefore $f(x, y, z)$ has a minimum value at the point $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$

**Example 5:** Examine

$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$ for extreme values.

**Solution.** $f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

$$f_x = 2yz - 4z + 2x - 2$$

$$f_y = 2zx - 2z + 2y - 4$$

$$f_z = 2xy - 4x - 2y + 2z + 4$$

For extreme points, $f_x = 0$, $f_y = 0$, $f_z = 0$

i.e. $2yz - 4z + 2x - 2 = 0$

or $yz - 2z + x - 1 = 0$ .....(1)

$2zx - 2z + 2y - 4 = 0$

or $zx - z + y - 2 = 0$ .....(2)

and $2xy - 4x - 2y + 2z + 4 = 0$

or $xy - 2x - y + z + 2 = 0$ .....(3)

Adding (2) and (3), we have

$$zx + xy - 2x = 0$$

or $x(z + y - z) = 0$

$\therefore$ Either $x = 0$ or $x + y - 2 = 0$

Thus we have two sets of equations:

| | |
|---|---|
| $yz - 2z + x - 1 = 0$ | $yz + x - 2z - 1 = 0$ |
| $zx - z + y - 2 = 0$ | $zx - z + y - 2 = 0$ |
| $x = 0$ | $z + y - 2 = 0$ |

Solving these equations, we get solving these equ, we get

(0,3,1), (0,1, -1) (1,2,0), (2,1,1), (2,3, -1)

$\therefore$ the possible extreme points are (0,3,1), (0,0,-1), (1,2,0), (2,1,1) and (2,3,-1).

Again, we have

$$f_{xx} = 2,\ fxx = 2,\ f_{xy} = 2z.$$

$$f_{yz} = 2x - 2, \quad f_{yx} = 2z,$$

$$f_{xz} = 2y - 4m\ f_{xy} = 2x - 2,\ fxx = 2y - 4$$

For the point (0,3,1), we have,

$$f_{xx} = 2 > 0, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} = 4 - 4 = 0$$

and $$\begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{xy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 2 & 2 \\ 2 & 2 & -2 \\ 2 & -2 & 2 \end{vmatrix} = -32$$

Thus we see that the above three quantities are neither all positive nor alternately negative and positive.

$\therefore$ $f(x, y, z)$ is neither max. nor min. at the point (0,3,1)

It may similarly be shown that the function is neither a max, nor a min, at (0,1,-1), (2,1,1) and (2,3,-1).

At (1,2,0), we have

$$f_{xx} = 2 > 0, \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

and $$\begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{xy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0$$

since all the above 3 quantities are positive

$\therefore$ the function has a minimum at the point (1,2,0)

**Example 6:** Examine the following functions for extreme values:

$$f(x, y, z) = x^2 + y^2 + z^2 + 2xyz$$

**Solution.** $f(x, y, z) = x^2 + y^2 + z^2 + 2xyz$

$\therefore$ $f_x = 2x + 2yz$, $f_y = 2y + 2zx$, $f_x = 2z + xy$

$f_{xx} = 2, f_{yy} = 2, f_{xx} = 2; f_{xy} = f_{yx} = 2z,$

$f_{yz} = f_{zy} = 2x$ and $f_{xz} = f_{zx} = 2y$

For an extreme value

$$f_x = f_y = f_x = 0$$

$$\therefore\ 2x + 2yz = 2y + 2zx = 2z + 2xy = 0$$

$$\text{or } x + yz = 0,\ y + zx = 0,\ z + xy = 0$$

$$\therefore\ (x, y, z) = (0,0,0), (1,1,-1), (1,-1,1), (-1,1,1), (-1,-1,-1)$$

At (0,0,0): $f_{xx} = 2 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{xy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}$ = 8 > 0

this mean that $f$ has a minimum value at (0,0,0)

at (1,1,-1): $f_{xy} = 2 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ -2 & 2 \end{vmatrix} = 0$$

and $\begin{vmatrix} f_{xx} & f_{xy} & f_{xx} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{xy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & -2 & 2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix}$ = -32 < 0,

$\therefore$ we see that these values fall to give us an answer, and, therefore we calculate $d^2f$ at this point (1,1,-1)

Now $d^2f = f_{xx}(dx)^2 + f_{yy}(dx)^2 + f_{zx}(dx)^2 + 2f_{yz}dydz + 2f_{zx}dzdx + 2f_{xy}dxdy$

$= 2[(dx)^2 + (dy)^2 + (dz)^2] + 4[dydz + dzdx + dxdy]$

$= 2(dx + dy + dz)^2 - 8dxdy$

Obviously, $d^2f$ is indefinite as it can have positive as well as negative values.

(1,1,-1) is neither a point of maxima nor a point of minima.

Similarly, we can prove that (1,-1,1) and (-1,1,1) and (-1,-1,-1) are also not extreme points, though they are critical points.

## 19.6 Lagrange's Method of Multipliers

Stationary points under subsidiary conditions. To find the stationary points of the function $f(x_1, x_2,.....x_n)$ of n variables $x_1, x_2,........x_n$ which are further connected by m equations; $f_l(x_1, x_2,.......x_n) = 0$, l = 1,2,3,.....m. Lagrange has given very useful method known as Lagrange's method of undetermined multipliers i.e. this method is useful to find the maximum or minimum values of a function.

Under this method:

(1) We define a function called auxiliary function $F(x_1,x_2,.....x_n)$ by

$$F = f + \lambda_1 f_1 + \lambda_2 f_2 + ...... + \lambda_m f_m$$

Where $1_1,1_2,.......1_m$ are parameters independent of $x_1,x_2,......x_n$ and are called Lagrange's Undetermined Multipliers.

(2) The we find $\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2},......,\frac{\partial F}{\partial x_n}$ and equate each to zero

i.e. $\frac{\partial F}{\partial x_1}$, $\frac{\partial F}{\partial x_2}$ ,.....= $\frac{\partial F}{\partial x_n}$ = 0

(These are the necessary conditions for extreme values)

(3) Then solve these n equations with the help of given m constraints i.e. $f_1$ = 0, $f_2$ = 0 .....$f_m$ = 0 $f$ or $\lambda_1\lambda_2\lambda_3$.......$\lambda_3$ and stationary points.

(4) Find the values of $f$ at the stationary points if the variables in a function of three variable are not independent but are connected by some relations.

Working method for a function of three variables

Let $f$(x, y, z) be a function of x, y and z which are connected by relations $\phi_1$(x,y,z) = 0 and $\phi_2$(x, y, z) = 0

Step 1. Define the auxiliary function

$$F(x, y, z) = f(x, y, z) + \lambda_1\phi_1(x,y,z) + \lambda_2\phi_2(x,y,z)$$

Where $\lambda_1$ and $\lambda_2$ are parameters independent of x, y, z (Lagrange's Multipliers).

Step 2. Find $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ and equate each to zero which are necessary conditions for extreme points.

Step 3. Solve these equations along with constraints for the parameters $\lambda_1$, $\lambda_2$ and for critical points.

Then find the values of $f$(x, y, z) at the critical points which will give the maximum and minimum values of $f$(x, y, z)

Let us see what the method is with the following examples:-

**Example 7:** Find the maximum and minimum values of $x^2 + y^2$ subject to the condition $3x^2 + 4xy + 6y^2 = 140$

**Solution.** Let $f(x, y) = x^2 + y^2$ so that we have to find the extreme values of $f$ under the constraint

$$3x^2 + 4xy + 6y^2 = 140 = 0 \qquad ....(1)$$

Consider the function

$F(x, y) = x^2 + y^2 + \lambda(3x^2 + 4xy + 6y^2 = 140)$ where $\lambda$ is Lagrange's multiplier

$$\therefore \frac{\partial F}{\partial x} = 2x + \lambda(6x + 4y).$$

$$\frac{\partial F}{\partial y} = 2y + \lambda(4x + 12y)$$

For extreme points, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

$\therefore$ $(1 + 3\lambda)x + 2\lambda y = 0$ (2)

and $2\lambda x + (1 + 6\lambda)y = 0$ (3)

Since $(x, y) \neq (0, 0)$, form (2) and (3), we have

$$\begin{vmatrix} 1+3\lambda & 2\lambda \\ 2\lambda & 1+6\lambda \end{vmatrix} = 0$$

or $(1 + 3\lambda)(1 + 6\lambda) - 4\lambda^2 = 0$

or $14\lambda^2 + 9\lambda + 1 = 0$

or $\lambda = -\frac{1}{2}, -\frac{1}{7}$

Case 1. When $= -\frac{1}{2}$, from (2) we have, $x = -2y$ and putting in (1), we have

$12y^2 - 8y^2 + 6y^2 = 140$

$\therefore$ $y^2 = 14$ and then $x^2 = 4y^2 = 56$

$\therefore$ $x^2 + y^2 = 56 + 14 = 70$

Case 2. When $\lambda = -\frac{1}{7}$, from (2) we have $y = 2x$ and putting in (1), we have

$3x^2 + 8x^2 + 24x^2 = 140$ or $x^2 = 4$ and then $y^2 = 4x^2 = 16$

$\therefore$ $x^2 + y^2 = 16 + 4 = 20$

Therefore, the maximum and minimum values of $x^2 + y^2$ are 70 and 20 respectively.

**Example 8:** Determine the stationary values of the function

$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ subject to the condition $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$.

**Solution:** $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$ (1)

$f_1(x, y, z) = ax^2 + by^2 + cz^2 - 1 = 0$ (2)

and $f_2(x, y, z) = lx + my + nz = 0$ (3)

Let $F(x, y, z) = f + \lambda_1 f_1 + \lambda_2 f_2$

$= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \lambda_1(ax^2 + by^2 + cz^2 - 1) + \lambda_2(lx + my + nz)$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2} + 2a\lambda_1 x + l\lambda_2$$

$$\frac{\partial F}{\partial y} = \frac{2y}{b^2} + 2b\lambda_1 y + m\lambda_2$$

$$\frac{\partial F}{\partial z} = \frac{2x}{c^2} + 2c\lambda_1 z + n\lambda_2$$

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z}$$

$$\Rightarrow \frac{2y}{a^2} + 2a\lambda_1 x + 1\lambda_2 = 0 \qquad (5)$$

$$\frac{2y}{b^2} + 2b\lambda_1 y + m\lambda_2 = 0 \qquad (6)$$

$$\frac{2y}{c^2} + 2c\lambda_1 z + n\lambda_2 = 0 \qquad (7)$$

Multiplying (5) by x, (6) by y and (7) by z and adding,

$$2\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] + 2\lambda_1 (ax^2 + by^2 + cz^2) + \lambda_2 [lx + my + nz] = 0$$

Using (2) and (3), we have

$$2\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] + 2\lambda_1, 1 + \lambda_2 . 0 = 0$$

$$\therefore \lambda_1 = - \left(\frac{x^2}{a^2} \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = -f$$

Putting this value of $\lambda 1$ in (5), (6) and (7), we have

$$\frac{2x}{a^2} - 2ax f + l\lambda_2 = 0$$

$$\frac{2y}{b^2} - 2by f + m\lambda_2 = 0$$

$$\frac{2z}{c^2} - 2zc fz + n\lambda_2 = 0$$

$$\Rightarrow x = \frac{-a^2 l\lambda_2}{2(1-a^2 f)}, \quad y = \frac{-b^2 m\lambda_2}{2(1-b^3 f)}, \quad z = \frac{-c^2 n\lambda_2}{2(1-c^3 f)}$$

To find relation free of $\lambda 2$, lx + my + nz = 0

$$\Rightarrow \quad \frac{\lambda_2}{2}\left[\frac{a^2 l^2}{a^3 f-1}+\frac{b^2 m^2}{b^3 f-1}+\frac{c^2 n^2}{c^3 f-1}\right] = 0$$

or $\frac{a^2 l^2}{a^3 f-1} + \frac{b^2 m^2}{b^3 f-1} + \frac{c^2 n^2}{c^3 f-1} = 0$

Which is a quadratic in $f$ and gives two values of $f$ which are the extreme values of $f$.

**Example 9:** Find the minimum values of $f$ (x, y, z) = $x^2 + y^2 + z^2$ where x + y + z = 3a.

**Solution :** $f$ (x, y, z) = $x^2 + y^2 + z^2$ ...(1)

$f_1$ (x, y, z) = x +y + z - 3a ...(2)

Let F (x, y, z) = $(x^2 + y^2 + z^2) + \lambda(x + y + x - 3a)$ .... (3)

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda = 0$$

$$\therefore x = y = z = \frac{\lambda}{2}$$

Since x + y + z = 3a

$$-\frac{3\lambda}{2} = 3a$$

or $\lambda = 2a$

$\therefore$ the stationary value is given by (x, y, z) = (a, a, a)

Putting $\lambda = -2a$ in (3), we have

F (x, y, z) = $x^2 + y^2 + z^2$ 2a(x + y + z - 3a)

$F_x = 2x - 2a$, $F_y = 2y - 2a$, $F_z = 2z - 2a$,

$F_{xy} = 0$, $F_{zy} = 0$, $F_{xz} = 0$

$F_{xx} = 2$, $F_{yy} = 2$, $F_{zz} = 2$,

At (a, a, a)

$$F_{xx} = 2 > 0, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

And $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 > 0$

Thus all the above three quantities are positive.

$\therefore$ (a, a, a) is a point of minima of F (x, y, z) and therefore $f$(x, y, z) also has a minimum value at (a, a, a) and minimum value of

$f$ (x, y, z) = $f$(a, a, a) = $a^2 + a^2 + a^2 = 3a^2$

## 19.7 Self Check Exercise

Q.1 Find the local maxima, Local minima and saddle point, if any of the function.

$f(x, y) = zxy - 5x^2 - 2y^2 + 4x - 4$

Q. 2 Find the points of extreme volumes, if any of the function

$f(x, y) = x^3 + 3x + y^2 - y + 4$

Q. 3 Find the extreme values of the function

$f(x, y, z) = x^2 + y^2 + z^2 + 2xy + z$

Q. 4 Use Lagrange's methods of multiplier to find the point on the plane 2x - 3y + 6z = 49 nearest to the origin in $R^3$.

## 19.8 Summary

In this unit, we have learnt the following :

(i) definition, semi definite and indefinite functions

(ii) maximum and minimum for function of two and three variables

(iii) Lagrange's methods of multiplier

## 19.9 Glossary

1. **Maximum Value -**

$f$(x) is said to have a maximum value at x = a, y = b

if $f$ (a, b) > $f$ (a + h, b + k) for

small values of h and k, positive or negative.

2. **Minimum value -**

A function $f$ (x, y) is said to posses a minimum value at x = a, y = b if

$f$(a, b) < $f$(a + h, b + k) for small values of h, k positive or negative

3. Extreme Value -

A maximum or a minimum value of a function is called on extreme value.

## 19.10 Answers to Self Check Exercises

Ans. 1 Local maximum at $\left(\frac{4}{9}, \frac{2}{9}\right)$ and

local maximum value = $\frac{-28}{9}$.

Ans. 2 No point of extreme value of $f$.

Ans. 3 Local minimum at (0, 0, 0), and no other points have maximum or minimum points.

Ans. 4 (2, -3, 6) is the required points.

## 19.11 Reference/Suggested Reading


1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007
2. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002


## 19.12 Terminal Questions

1. Find the extreme value (if any) of the function

   $f(x, y) = 2x^4 - 3x^2y + y^2$

2. Find the extreme value, if any, of the function

   $f(x, y) = x^3y^2 (1 - x - y)$

3. Find the maximum and minimum value of the function

   $\sin x + \cos y + \cos (x + y)$

4. Using Lagrange's method find the points on the plane

   $2x + 3y - z = 5$,

   which is nearest the origin in $R^3$.

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# Unit - 20

# Jacobian

**Structure**



## 20.1 Introduction

Dear students, in this unit we shall study Jacobian of function. The term Jacobian of often interchangably used to refer to both the Jacobian matrix or its determinant. Both the matrix and the determinant have useful and important application. The Jacobian matrix aggregates the partial derivatives that are necessary for backpropagation, the determinant is useful in the process of changing between variables.

## 20.2 Learning Objectives

The main objectives of this unit are

(i) to define Jacobian of Function of several variables.

(ii) to study Jacobian of composite function

(iii) to find Jacobain of inverse of a function

(iv) to learn Jacobian of implicit functions.

## 20.3 Jacobian of Functions of Several Variables

**Definition.** Let $f = (f_1, f_2, .....f_n) : D \to R^n$, where D is an open subset of $R^n$. Let each of the function $f_1, f_2, .....f_n$ possess all the first order partial derivatives at a point $x = (x_1, x_2, .....x_n) \in D$, then the determinant.

$$\frac{\partial f_1}{\partial x_1} \frac{\partial f_1}{\partial x_2} \frac{\partial f_1}{\partial x_3} \cdots\cdots\cdots \frac{\partial f_1}{\partial x_n}$$

$$\frac{\partial f_2}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_2}{\partial x_3}\cdots\cdots\frac{\partial f_2}{\partial x_n}$$

$$\frac{\partial f_3}{\partial x_1}\frac{\partial f_3}{\partial x_2}\frac{\partial f_3}{\partial x_3}\cdots\cdots\frac{\partial f_3}{\partial x_n}$$

..............................

..............................

$$\frac{\partial f_n}{\partial x_1}\frac{\partial f_n}{\partial x_2}\frac{\partial f_n}{\partial x_3}\cdots\cdots\frac{\partial f_n}{\partial x_n}$$

is called the jacobian of $f_1, f_2, \ldots, f_n$ w.r.t. $x_1, x_2, \ldots, x_n$ and is denoted by $f_n$ (x) or $\dfrac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$

**Note.** In the above definition of Jacobian, the functions $f_1, f_2, \ldots, f_n$ are real valued functions defined on an open subset of $R^n$.

**Remark 1.** Let each $\dfrac{\partial}{\partial x_j}(f_i(x))$ be written as $D_j\,(f_j(x))$

$1 \leq i \leq n$, $\quad$ $1 \leq j \leq n$, then $j_j(x)$ can be written as

$$j_j(x) = \frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

$D_1f_1\ D_2f_1\ D_3f_1\ \ldots\ldots\ D_nf_1$

$D_1f_2\ D_2f_2\ D_3f_2\ \ldots\ldots\ D_nf_2$

$D_1f_3\ D_2f_3\ D_3f_3\ \ldots\ldots\ D_nf_3$

..........................................

..........................................

$= \det\,[D_if_i(x)]_{n\times n}$

**Theorem 1.** Let $f : R^n \to R^n$ be a differentiable function, then $f_j(x)$ exists at every point $x \in R^n$.

**Proof:** Since $f = (f_1, f_2, \ldots, f_n)$ is differentiable at each point $x = (x_1, x_2, \ldots, x_n) \in R^n$, therefore, each of the real valued function $f_1, f_2, \ldots, f_n$ is differentiable at x and further since each $f_1$, i = 1,2,3,........n is differentiable at $x = (x_1, x_2, \ldots, x_n)$, therefore each of $\dfrac{\partial f_i}{\partial x_1}\dfrac{\partial f_i}{\partial x_2}\dfrac{\partial f_i}{\partial x_3}\cdots\cdots\dfrac{\partial f_i}{\partial x_n}$, exists i.e. each $D_j(f_i(x))$ exists for $1 \leq i \leq n$, $1 \leq j \leq n$. Hence by definition $f_j(x)$ exists.

**Theorem 2.** Let $f = (f_1, f_2, \ldots, f_n)\ R^n \to R^n$.

If $f_1, f_2, \ldots, f_n$ are such that

$f_1 = f_1(x_1)$ i.e. $f_1$ is a function of $x_1$ only

$f_2 = f_2(x_1, x_2)$ i.e. $f_2$ is a function of $x_1$ and $x_2$ only

$f_3 = f_3(x_1, x_2, x_3)$ i.e. $f_3$ is a function of $x_1$, $x_2$ and $x_3$ only

$f_n = f_n(x_1,x_2,x_3,.....x_n)$ i.e. $f_n$ is a function of $x_1,x_2,x_3,.....x_n$ only then $\frac{\partial(f_1, f_2,.....f_n)}{\partial(x_1, x_2,.....x_n)} =$ $\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}......\frac{\partial f_n}{\partial x_n}$

**Proof:** Obviously $\frac{\partial}{\partial x_j} f_i(x) = 0$ for j > i, i = 1,2,3,......n - 1

$\therefore j_f(x) = \det[D_j(f_i(x)]_{n\times n}$ |where det $[D_j(f_i(x)]_{n\times n}$ is a lower triangular matrix and therefore $j_f(x)$ = product of the diagonal elements in det $[D_j(f_i(x)]_{n\times n}$

$$= \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}......\frac{\partial f_n}{\partial x_n}$$

This completes the proof.

Remark 2. Theorem 2 can also be stated as:

If the functions $f_1,f_2,.....f_n$ of n variables $x_1,x_2,x_3,......x_n$ be such that $f_i$ is independent of $x_j$ for j > i, i = 1,2,3,.....n - 1, then $\frac{\partial(f_1, f_2,.....f_n)}{\partial(x_1, x_2,.....x_n)}$, $\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3}......\frac{\partial f_n}{\partial x_n}$ = ....... $\frac{\partial f_1}{\partial x_1}$

Let us look at some examples:-

**Example 1:** Let $f(x, y) = (x\cos y, x\sin y)$ Evaluate $j_f(x, y)$

**Sol.** $f(x, y) = (f_1(x, y), f_2(x, y)) = (x \cos y, x \sin y)$

$\therefore$ $f_1(x, y) = x \cos y$ and $f_2(x, y) = x \sin y$

$\frac{\partial f_1}{\partial x} = \cos y$ and $\frac{\partial f_2}{\partial x} = \sin y$

$\frac{\partial f_1}{\partial y} = -x \sin y$ and $\frac{\partial f_2}{\partial y} = x \cos y$

By definition

$$j_f(x, y) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos y & -x\sin y \\ \sin y & x\cos y \end{vmatrix} = x\cos^2 y + x\sin^2 y = x(\cos^2 y + \sin^2 y) = x$$

**Example 2:** If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r \cos\theta$, find $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$

**Sol.** x = r sin θ cos ϕ

$$\therefore \frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi \text{ and } \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

Again y = r sin θ sin ϕ

$$\therefore \frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = -r\sin\theta\cos\phi$$

Also z = r cos θ, $\therefore \frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = r\sin\phi,\ \frac{\partial z}{\partial \phi} = 0$

$$\therefore \text{ by definition, } \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \cos\theta\,[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi] + r\sin\theta\,[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi]$

$= r^2\cos\theta\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) + r\sin\theta.\ r\sin^2\theta(\cos^2\phi + \sin^2\phi)$

$= r^2\sin\theta\cos^2\theta + r^2\sin^2\theta$

$= r^2\sin\theta(\cos^2\theta + \sin^2\theta)$.

**Example 3:** If $y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$, $y_3 = \frac{x_1x_2}{x_3}$, find $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$

**Sol.** Given $y_1 = \frac{x_2x_3}{x_1}$

$$\frac{\partial y_1}{\partial x_1} = -\frac{x_2x_3}{x_1^2},\ \frac{\partial y_1}{\partial x_2} = \frac{x_3}{x_1},\ \frac{\partial y_1}{x_3} = \frac{x_2}{x_1}$$

$$y_2 = \frac{x_3x_1}{x_2}$$

$$\therefore\ \frac{\partial y_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial y_2}{\partial x_2} = \frac{x_3 x_1}{x_1^2},\ \frac{\partial y_2}{\partial x_3} = \frac{x_1}{x_2}$$

and $y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial y_3}{\partial x_1} = \frac{x_2}{x_3},\ \frac{\partial y_3}{\partial x_2} = -\ \frac{x_1}{x_3},\ \frac{\partial y_3}{\partial x_3} = \frac{x_1 x_2}{x_3^2}$$

By definition, $$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{x_3 x_1}{x_1^2} & \frac{x_1}{x_3} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_1 x_2 \\ x_3 x_2 & -x_1 x_3 & x_2 x_1 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_2 x_3 . x_3 x_1 . x_1 x_2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

[operate $R_2 + R_1$, $R_3 + R_1$]

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} = \begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix} = 4$$

## 20.4 Jacobian of Composite Functions

**Theorem 3.** Let $f = (f_1, f_2, .....f_n)$: $R^n \rightarrow R^n$

$$\text{and } g = (g_1, g_2, \ldots\ldots. g_n): R^n \rightarrow R^n$$

Be two differentiable functions:

Then $j_{fog}(x) = j_f(g(x))\, j_g(x)$ where $x = (x_1, x_2, \ldots\ldots, x_n) \in R^n$

**Proof:** Here $f(x) = ((f_1, f_2, \ldots.., f_n)\,(x)$ and $(x) = (g_1, g_2, \ldots.. g_n)\,(x)$

Let $F(x) = (fog)\,(x)$

$$= (f_1, f_2, \ldots. f_n)og(x)$$

$$(f_1\, og,\ f_2 og,\ \ldots\ldots., f_n og)\,(x)$$

$$(F_1, F_2, \ldots\ldots., F_n)(x) \qquad \text{(say)}$$

Then $F_i(x) = (f_i og)(x)$, $i = 1,2,3,\ldots., n$

By definition, $j_F(x) = \det[D_j(F_i(x))]_{n\times n}$

Where $D_j(F_i(x)) = D_j g_i(x).\ D_1 f_i(g(x)) + D_j g_2(x) D_2 f_i(g(x)) + D_j g_3(x).\ D_3 f_1(g(x)) + \ldots\ldots + D_j g_n(x) = \det[D_j(F_i(x))]_{n\times n}$

By product of matrices, we have,

$$[D_j(F_i(x))]_{n\times n}.\ [D_j(F_i(x))]_{n\times n} = \det[D_i(F_i(x))]_{n\times n}$$

$$\therefore\ \det[D_j(F_i(x))]_{n\times n}.\ \det[D_j(F_i(x))]_{n\times n} = \det[D_1(F_i(x))]_{n\times n}$$

Hence $j_f(g(x)).j_g(x) = j_F(x) = j_{fog}(x)$.

This completes the proof.

Remark. The above theorem can also be stated as:

If $F_1$, $F_2$, $F_3$, .....$F_n$ are functions of $y_1$, $y_2$, $y_3$,......$y_n$ where $y_1$, $y_2$, $y_3$,.....yn are function of $x_1$, $x_2$, $x_3$, .....$x_n$ then

$$\frac{\partial(F_1, F_2, F_{3,\ldots\ldots.} F_n)}{\partial(x_1, x_2, x_{3\ldots\ldots\ldots} x_n)} = \frac{\partial(F_1, F_2, F_{3,\ldots\ldots.} F_n)}{\partial(y_1, y_2, y_{3\ldots\ldots\ldots} y_n)} \cdot \frac{\partial(y_1, y_2, y_{3\ldots\ldots\ldots} y_n)}{\partial(x_1, x_2, x_{3\ldots\ldots\ldots} x_n)}$$

And can be proved by taking $y_t = g_t(x)$, $1 \leq i \leq n$ in the above result.

## 20.5 Jacobian of inverse of a function

**Theorem 4.** Let D be an open subset of $R^n$ and let $f = (f_1,\ f_2,\ f_3, \ldots\ldots. f_n): D \rightarrow R^n$ be differentiable at every point of D. Suppose $f$ is invertible on D and let $f^{-1}$ be differentiable at every point of the range of $f$, then

$$j_{f^{-1}}(f(x)) = [j_f(x)]^{-1} = \frac{1}{j_f(x)}$$

**Proof:** Let $F = f^{-1}$ of then $F(x) = (f^{-1}\ of)(x) = 1\ (x)$ is the identity function By jacobian of composite functions.

$$j_F(x) = j_{f^{-1}}(f(x)).j_f(x) \qquad \ldots.(1)$$

Also $j_F(x) = j_1(x) = 1$ ....(2)

From (1) and (2),

$$j_{f^{-1}}(f(x)).\, j_f(x) = 1$$

or $$j_{f^{-1}}(f(x)) = \frac{1}{j_f(x)} = [j_f(x)]^{-1}$$

This completes the proof.

Remark. The above result can also be stated as:

$$\frac{\partial(x_1, x_2, x_{3.........}x_n)}{\partial(f_1, f_2, f_{3.........}f_n)} = \frac{1}{\dfrac{\partial(f_1, f_2, f_{3.........}f_n)}{\partial(x_1, x_2, x_{3.........}x_n)}}$$

or $$\frac{\partial(f_1, f_2, f_{3.........}f_n)}{\partial(x_1, x_2, x_{3.........}x_n)} \cdot \frac{\partial(x_1, x_2, x_{3.........}x_n)}{\partial(f_1, f_2, f_{3.........}f_n)} = 1$$

or $$\frac{\partial(y_1, y_2, y_{3,.......}y_n)}{\partial(x_1, x_2, x_{3.........}x_n)} \frac{\partial(x_1, x_2, x_{3,.......}x_n)}{\partial(y_1, y_2, y_{3.........}y_n)} = 1$$

Where $y_i = f_i(x_1, x_2, x_3, .....x_n)$ $i = 1,2,3,......n$

## 20.6 Jacobian of Implicit Functions

Theorem 5. If $u_1, u_2, ......\, u_n$ are functions of $x_1,x_2,.....x_n$ define emplicity by n equations

$$F_1(u_1,u_2,....u_n,x_1,x_2....x_n) = 0$$

$$F_2(u_1,u_2,....u_n,x_1,x_2....x_n) = 0$$

...........................................

$$F_n(u_1,u_2,....u_n,x_1,x_2....x_n) = 0$$

Then $$\frac{\partial(u_1, u_2, .......u_n)}{\partial(x_1, x_2, .......x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, .......F_n)}{\partial(x_1, x_2, .......x_n)}}{\dfrac{\partial(F_1, F_2, .......F_n)}{\partial(u_1, u_2, .......u_n)}}$$

Differentiating partially w.r.t. $x_1$ we have

$$\frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + ....... + \frac{\partial F_1}{\partial u_n} \cdot \frac{\partial u_n}{\partial x_1} + \frac{\partial F_1}{\partial x_1} = 0$$

$$\sum_{i=t}^{u} \frac{\partial F_t}{\partial u_i} \cdot \frac{\partial u_i}{\partial x_1} = \frac{\partial F_1}{\partial x_1} \quad .....................(1)$$

Similarly for all other partial derivatives, Now, $\dfrac{\partial(F_1,F_2,.......F_n)}{\partial(u_1,u_2,.......u_n)}\cdot\dfrac{\partial(u_1,u_2,.......u_n)}{\partial(x_1,x_2,.......x_n)}$

$$= \begin{vmatrix} \dfrac{\partial F_1}{\partial u_1} & \dfrac{\partial F_1}{\partial u_2} & \dfrac{\partial F_1}{\partial u_n} \\ \dfrac{\partial F_2}{\partial u_1} & \dfrac{\partial F_2}{\partial u_2} & \dfrac{\partial F_2}{\partial u_n} \\ \dfrac{\partial F_n}{\partial u_1} & \dfrac{\partial F_n}{\partial u_2} & \dfrac{\partial F_n}{\partial u_n} \end{vmatrix} \times \begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_n} \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum\dfrac{\partial F_1}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_1} & \sum\dfrac{\partial F_1}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_2} & \sum\dfrac{\partial F_1}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_n} \\ \sum\dfrac{\partial F_2}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_1} & \sum\dfrac{\partial F_2}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_2} & \sum\dfrac{\partial F_2}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_n} \\ \sum\dfrac{\partial F_n}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_1} & \sum\dfrac{\partial F_n}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_2} & \sum\dfrac{\partial F_n}{\partial u_i}\cdot\dfrac{\partial u_i}{\partial x_n} \end{vmatrix}$$

Using (1), we have

$$= \begin{vmatrix} -\dfrac{\partial F_1}{\partial x_1} & -\dfrac{\partial F_1}{\partial x_2} & -\dfrac{\partial F_1}{\partial x_n} \\ -\dfrac{\partial F_2}{\partial x_1} & -\dfrac{\partial F_2}{\partial x_2} & -\dfrac{\partial F_2}{\partial x_n} \\ -\dfrac{\partial F_n}{\partial x_1} & -\dfrac{\partial F_n}{\partial x_2} & -\dfrac{\partial F_n}{\partial x_n} \end{vmatrix} = (-1)^n \dfrac{\partial(F_1,F_2,.......F_n)}{\partial(x_1,x_2,.......x_n)}$$

Hence $\dfrac{\partial(u_1,u_2,.......u_n)}{\partial(x_1,x_2,.......x_n)} = (-1)^n \dfrac{\dfrac{\partial(F_1,F_2,.......F_n)}{\partial(x_1,x_2,.......x_n)}}{\dfrac{\partial(F_1,F_2,.......F_n)}{\partial(u_1,u_2,.......u_n)}}$

Remark 5. If the equations are of the form:

$F_1(x_1,x_2,.....x_n,u_1) = 0$

$F_2(x_1,x_2,.....x_n,u_1,u_2) = 0$

$F_3(x_1,x_2,.......x_n,u_1,u_2,u_3) = 0$

...............................................

...............................................

$F_n(x_1,x_2,........x_n,u_1,u_2,u_3.....u_n) = 0$

$$\frac{\partial(u_1, u_2, .......u_n)}{\partial(x_1, x_2, .......x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, .......F_n)}{\partial(x_1, x_2, .......x_n)}}{\dfrac{\partial(F_1, F_2, .......F_n)}{\partial(u_1, u_2, .......u_n)}}$$

**Example 4:** Let $f(x, y) = (\sin x, \cos y)$ and $g(x, y) = (x^2, y^2)$

If $F(x, y) = (f o g)(x, y) = (\sin x^2, \cos y^2)$, prove that $j_F(x, y) = -4xy \sin x^2 \cos y^2$ and verify the result by direct calculations.

**Sol.** $f(x, y) = (\sin x, \cos y)$ and $g(x, y) = (x^2, y^2)$

$$j_f(x, y) = \begin{vmatrix} \cos x & 0 \\ 0 & -\sin y \end{vmatrix} = -\cos x \sin y$$

$$\therefore\ j_f(g(x, y)) = j_f(x^2, y^2) = -\cos x^2 \sin y^2$$

$$\text{and } j_g(x, y) = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$$

$$j_F(x, y) = j_{fog}(x, y) = j_f(g(x, y)).\ j_g(x, y) = (-\cos x^2 \sin y^2)\, 4xy$$

$$= -4xy \cos x^2 \sin y^2$$

Direct method:

$$F(x, y) = (f o g)(x, y) = f[g(x, y)] = f(x^2, y^2)$$

$$= (\sin x^2, \cos y^2) = (F_1, F_2)$$

$$j_F(x, y) = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x\cos x^2 & 0 \\ 0 & -2y \sin y^2 \end{vmatrix}$$

**Example 5:** Prove that $j_{f^{-1}}(\zeta\ \eta) = \zeta$ for any $(\zeta\ \eta)$ belonging to the range of $f$, where $f(x, y) = \left(\sqrt{x^2+y^2}, \tan^{-1}\dfrac{y}{x}\right)$

**Sol.** Put $\zeta = \sqrt{x^2+y^2}$ and $\eta = \tan^{-1}\dfrac{y}{x}$ then $f(x, y) = (\zeta, n)$

Belongs to the range of $f$.

$$\therefore\ (x, y) = f^{-1}(\zeta, n)$$

Now $j_f(x, y) = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}}$

By Inverse theorem,

$$j_{f^{-1}}(f(x, y) = \frac{1}{j_f(x,y)}$$

$$j_{f^{-1}}(\zeta, \eta) = \sqrt{x^2+y^2} = \zeta$$

**Example 6:** If $f(x, y) = (x^2 + y^2, y^7 + e^x)$ and $g(x, y) = (\sin x, x^2)$ prove that $j_{fog}(x, y) = 0, \forall (x, y)$.

So. By the theorem on jacoblan of composite functions,

$$j_{fog}(x, y) = j_f(g(x, y)).j_g(x, y)$$

Now $jg(x, y) = \begin{vmatrix} \cos x & 0 \\ 2x & 0 \end{vmatrix} = 0 \;\forall\; (x, y)$

Hence $j_{fog}(x, y) = 0 \;\forall\; (x, y)$

**Example 7:** Let $f(x, y) = (x - y, x + y)$. Evaluate $j_{f^{-1}}(\zeta, \eta)$

**Sol.** $f(x, y) = (x - y, x + y) = (\zeta, \eta)$ where $\zeta = x - y, \eta = x + y$

$$j_f(x, y) = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

Since $j_{f^{-1}}(\zeta, \eta) = \frac{1}{j_f(x,y)}$

$$\therefore\; j_{f^{-1}}(\zeta, \eta) = \frac{1}{2}$$

**Example 8:** Let $f(x, y) = (e^x, \cos y)$ and $g(x, y) = (x^3, y^3)$.

Evaluate $j_F(x, y)$, where $F = fog$ and verify the result by direct calculation.

**Sol.** We have, $f(x, y) = (e^x, \cos y)$

$$= (f_1 f_2)$$

Where $f_1(x, y) = e^x$, $f_2(x, y) = \cos y$

And $g(x, y) = (x^3, y^3) = (g_1 g_2)$

Where $g_1(x, y) = x^3$, $g_2(x, y) = y^3$

Now $j_f(x, y) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x & 0 \\ 0 & -\sin y \end{vmatrix}$ = ie$^x$ sin y

$j_f(g(x, y) = j_f(x^3, y^3) = -e^{x^3} \sin y^3$

Also $j_g(x, y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x^2 & 0 \\ 0 & 3y^2 \end{vmatrix} = 9x^2y^2$

$\therefore J_f(x, y) = j_f(g(x, y)).\ j_g(x, y)\ j_f(g(x, y))$

$= (-e^{x^3} \sin y^3)9x^2y^2$

$= -9x^2y^2 e^{x^3} \sin y^3$ ......(1)

Verification by direct calculation

$F(x, y) = f(g(x, y)) = f(g(x, y)) = f x^3, y^3$

$= e^{x^3}, \cos y^3 = (F_1, F_2)$

Where $F1(x, y) = e^{x^3}$, $F_2(x, y) = \cos y^3$

$J_F(x, y) = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 3x^2 e^{x^3} & 0 \\ 0 & -3y^2 \sin y^3 \end{vmatrix}$

$= 9x^2y^2 e^{x^3} \sin y^3$

Which matches with the result (1).

**Example 9:** If $x_1 + x_2 + x_3 + x_4 = u_1$

$x_2 + x_3 + x_4 = u_1u_2$

$x_3 + x_4 = u_1u_2u_3$

$x_4 = u_1u_2u_3u_4$

Prove that $\frac{\partial(x_1, x_2, x_3, x_4)}{\partial(u_1, u_2, u_3, u_4)} = u_1^3u_2^2u_3$

**Sol.** Here $F_1(x_1,x_2,x_3,x_4,u_1) = x_1 + x_2 + x_3 + x_4 - u_1 = 0$

$F_2(x_1,x_2,x_3,x_4,u_1,u_2) = x_2 + x_3 + x_4 - u_1u_2 = 0$

$$F_3(x_1,x_2,x_3,x_4,u_1,u_2u_3) = x_3 + x_4 - u_1u_2u_3 = 0$$

$$F_4(x_1,x_2,x_3,x_4,u_1,u_2u_3u_4) = x_4 - u_1u_2u_3u_4 = 0$$

Obviously, $\frac{\partial F_1}{\partial x_1} = 1,\ \frac{\partial F_2}{\partial x_2} = 1,\ \frac{\partial F_3}{\partial x_3} = 1$

And $\frac{\partial F_1}{\partial u_1} = -1,\ \frac{\partial F_2}{\partial u_2} = -u_1,\ \frac{\partial F_3}{\partial u_3} = -u_1u_2,\ \frac{\partial F_4}{\partial u_4} = -u_1u_2u_3$

$$\frac{\partial(u_1,u_2,u_3,u_4)}{\partial(x_1,x_2,x_3,x_4)} = (-1)^4 \frac{\frac{\partial F_1}{\partial x_1}\cdot\frac{\partial F_2}{\partial x_2}\cdot\frac{\partial F_3}{\partial x_3}\cdot\frac{\partial F_4}{\partial x_4}}{\frac{\partial F_1}{\partial u_1}\cdot\frac{\partial F_2}{\partial u_2}\cdot\frac{\partial F_3}{\partial u_3}\cdot\frac{\partial F_4}{\partial u_4}}$$

$$= (-1)^4 \frac{1.1.1.1}{(-1)(-u_1)(-u_1u_2)(-u_1u_2u_3)} = \frac{1}{u_1^3u_2^2u_3}$$

Hence $\frac{\partial(x_1,x_2,x_3,x_4)}{\partial(u_1,u_2,u_3,u_4)} = u_1^3u_2^2u_3$.

**Example 10:** If u, v, w be the roots of equation

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

$$3\lambda^3 - 3(x + y + z)\lambda^2 + 3(x^2 + y^2 + z^2)\lambda - (x^3 + y^3 + z^3) = 0$$

u, $\nu$, w are its roots

$$\therefore\ u + \nu + w = x + y + z$$

$$\therefore\ u\nu + \nu w + wu = x^2 + y^2 + z^2$$

$$u\nu w = \frac{x^3 + y^3 + z^3}{3}$$

Let $F_1 \equiv u\ \nu + w - x - y - z = 0$

$$F_2 \equiv u\nu + \nu w + wu - x^2 - y^2 - z^2 = 0$$

$$F_3 \equiv u\nu w - \frac{1}{3}x^3 - \frac{1}{3}y^3 - \frac{1}{3}z^3 = 0$$

Now $\frac{\partial(F_1,F_2,.......F_n)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & z^2 \end{vmatrix} = -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$$= -2(x - y)\ (y - z)\ (z - x)$$

$$\frac{\partial(F_1, F_2, .......F_3)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ u+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= -(u - v)(v - w)(w - u)$$

Now $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \dfrac{\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\dfrac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}}$

$$= (-1)^3 \frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= -\frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

**Example 11:** If $u_3 + v_3 + w_3 = x + y + z$

$u_2 + v_2 + w_2 = x_3 + y_3 + z_3$

$u + v + w = x_2 + y_2 + z_2$ show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

**Sol.** $F_1 \equiv u^3 + v^3 + w^3 - x - y - z = 0$

$F_2 \equiv u^2 + v^2 + w^2 - x^3 - y^3 - z^3 = 0$

$F_3 \equiv u + v + w - x^2 - y^2 - z^2 = 0$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix} = -6 \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 6(x - y)(y - z)(z - x)$$

And

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix} = 6 \begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

$= -6(u - v)(v - w)(w - u)$

now $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \dfrac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}}$

$$= \frac{6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-z)}$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-z)}$$

**Example 12:** If u = $\dfrac{x}{\sqrt{1-r^2}}$ , v = $\dfrac{y}{\sqrt{1-r^2}}$ , w = $\dfrac{z}{\sqrt{1-r^2}}$

Where $r^2 = x^2 + y^2 + z^2$ prove that $\dfrac{\partial(u, v, w)}{\partial(x, y, z)} = (1 - r^2)^{-5/2}$

**Sol.** Given, u = $\dfrac{x}{\sqrt{1-r^2}}$ , v = $\dfrac{y}{\sqrt{1-r^2}}$ , w = $\dfrac{z}{\sqrt{1-r^2}}$ Where $r^2 = x^2 + y^2 + z^2$

$$\therefore u = \frac{x}{\sqrt{1-x^2-y^2-z^2}}, \ v = \frac{y}{\sqrt{1-x^2-y^2-z^2}}, \ w = \sqrt{1-x^2-y^2-z^2}$$

$$= \frac{\partial u}{\partial x} = \frac{1-y^2-z^2}{(-x^2-y^2-z^2)^{3/2}}, \ \frac{\partial u}{\partial y} = \frac{xy}{(1-x^2-y^2-z^2)^{3/2}}$$

$$\frac{\partial u}{\partial z} = \frac{xz}{(1-x^2-y^2-z^2)^{3/2}}$$

$$\frac{\partial v}{\partial x} = \frac{yx}{(1-x^2-y^2-z^2)^{3/2}}, \ \frac{\partial v}{\partial y} = \frac{1-z^2-x^2}{(1-x^2-y^2-z^2)^{3/2}}, \ \frac{\partial v}{\partial z} = \frac{zy}{(1-x^2-y^2-z^2)^{3/2}}$$

$$\frac{\partial w}{\partial x} = \frac{xz}{(1-x^2-y^2-z^2)^{3/2}}, \ \frac{\partial w}{\partial y} = \frac{yz}{(1-x^2-y^2-z^2)^{3/2}}, \ \frac{\partial w}{\partial z} = \frac{1-x^2-y^2}{(1-x^2-y^2-z^2)^{3/2}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1-y^2-z^2}{(1-x^2-y^2-z^2)^{3/2}} & \frac{xy}{(1-x^2-y^2-z^2)^{3/2}} & \frac{xz}{(1-x^2-y^2-z^2)^{3/2}} \\ \frac{yx}{(1-x^2-y^2-z^2)^{3/2}} & \frac{1-z^2-x^2}{(1-x^2-y^2-z^2)^{3/2}} & \frac{zy}{(1-x^2-y^2-z^2)^{3/2}} \\ \frac{xz}{(1-x^2-y^2-z^2)^{3/2}} & \frac{yz}{(1-x^2-y^2-z^2)^{3/2}} & \frac{1-x^2-y^2}{(1-x^2-y^2-z^2)^{3/2}} \end{vmatrix}$$

$$= \frac{1}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} 1-y^2-z^2 & xy & xz \\ yx & 1-z^2-x^2 & zy \\ xz & yz & 1-x^2-y^2 \end{vmatrix}$$

$$= \frac{1}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} x(1-y^2-z^2) & x^2y & x^2z \\ xy^2 & y(1-z^2-x^2) & zy^2 \\ xz^2 & yz^2 & z(1-x^2-y^2) \end{vmatrix}$$

$$= \frac{1}{(1-x^2-y^2-z^2)^{9/2}} .\mathrm{xyz} \begin{vmatrix} 1-y^2-z^2 & xy & xz \\ yx & 1-z^2-x^2 & zy \\ xz & yz & 1-x^2-y^2 \end{vmatrix}$$

Operating $c_1$ - $c_2$ - $c_3$

$$\frac{1}{(1-x^2-y^2-z^2)^{9/2}}$$

$$\begin{vmatrix} 1-x^2-y^2-z^2 & 0 & x^2 \\ -(1-x^2-y^2-z^2) & 1-x^2-y^2-z^2 & y^2 \\ 0 & -(1-x^2-y^2-z^2) & 1-x^2-y^2 \end{vmatrix}$$

$$= \frac{\frac{(1-x^2-y^2-z^2)^2}{(1-x^2-y^2-z^2)^{9/2}}}{(1-x^2-y^2-z^2)^{9/2}} \begin{vmatrix} 1 & 0 & x^2 \\ -1 & 1 & y^2 \\ 0 & -1 & 1-x^2-y^2 \end{vmatrix}$$

operating $R_1 + R_2 + R_3$

$$= \frac{1}{(1-x^2-y^2-z^2)^{5/2}} \begin{vmatrix} 1 & 0 & x^2 \\ -1 & 1 & y^2 \\ 0 & -1 & 1-x^2-y^2 \end{vmatrix}$$

Expanding w.r.t. $R_1$,

$$= \frac{1}{(1-x^2-y^2-z^2)^{5/2}} .1$$

$$= \frac{1}{(1-r^2)^{5/2}} = (1 - r^2)^{-5/2}$$

With this topic we come to end of this unit. This doesn't mean that we've exhausted all the methods, of even all the important ones. We have just exposed you to a few elementary one and some of their applications. As you study more mathematics you will come across these and several others.

New let us quickly go through what we covered in this unit.

## 20.7 Self Check Exercise

Q.1 Evaluate $I_f(x, y)$, for

$f(x, y, z) = (x^2 + y^2 + z^2, y, z^2)$

Q. 2 Let $f : R^n \to R^n$ be defined by $f(x) = x$ then find $J_f(x)\ \forall\ x \in R^n$.

Q. 3 If $f$ (r, θ) = (r cos θ, r sin θ) and (x, y) is an arbitrary point in the domain of $f^{-1}$ compute $j_{f^{-1}}(x, y)$

## 20.8 Summary

In this unit, we have learnt the following :

(i) what is Jacobian of function of several variables?

(ii) Jacobian of composite function

(iii) How to find Jacobian of inverse of a function

(iv) to find Jacobian of implicit functions.

## 20.9 Glossary

1. $j_{f^{-1}} f(x) = [j_f(x)]^{-1} = \dfrac{1}{j_f(x)}$

2. $J_{fog}(x) = J_f(g(x)).\ J_g(x)$

$x = (x_1, x_2 ......x_n) \in R_n$.

## 20.10 Answers to Self Check Exercises

Ans. 1 2x

Ans. 2 1

Ans. 3 $\dfrac{1}{\sqrt{x^2+y^2}}$

## 20.11 Reference/Suggested Reading

1. G.B. Thomas and R.L. Finney, Calculus, Pearson Education, 2007

2. H. Anton, I. Birens and S. Davis, Calculus, John Wiley and Sons, Inc. 2002

## 20.12 Terminal Questions

1. If $u_1 = \dfrac{x_2x_3}{x_1}$

   $u_2 = \dfrac{x_3x_1}{x_2}$

   $u_3 = \dfrac{x_1x_2}{x_3}$

   Prove that J $(u_1,u_2,u_3)$ = 4

2. Let $f_1(x, y) = \dfrac{x+y}{1-xy}$

   $f_2(x, y) = \tan^{-1} x + \tan^{-1} y$

   be two function. Are $f_1$, $f_2$ functionally related.

3. If u, $\nu$, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$.

   Prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{-2(y-z)((z-x)(x-y)}{(v-w)(w-u(u-v)}$

4. If $x = r\cos\theta$, $y = r\sin\theta$, verify that

   $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \dfrac{\partial(r.\theta)}{\partial(xy)} = 1$

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