B.A.:3rd Year
MATHEMATICS

Course Code: MATH317TH
Course Credit: 04(SEC)

Transportation and Game Theory

UNITS: 1 to 19

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SYLLABUS

Course Code	MATH317TH
Credits= 4	L-4, T-0, P-0
Name of the Course	Transportation and Game Theory
Type of the Course	Skill Enhancement Course
Continuous Comprehensive Assessment: Based on Assignment	Max. Marks:30
End Semester Examination	Max Marks: 70 Maximum Times: 3 hrs.

Instructions

Instructions for paper setter: The question paper will consist of two Sections A& B of 70 marks. Section A will be Compulsory and will contain 8 questions of 16 marks (each of 2 marks) of short answer type having two questions from each Unit of the syllabus. Section B of the question paper shall have four Units I, II, III, and IV. Two questions will be set from each unit of the syllabus and the candidates are required to attempt one question from each of these units. Each question in Units I, II, III and IV shall be of 13.5 markseach.

Instructions for Candidates: Candidates are required to attempt five questions in all. Section A is Compulsory and from Section B they are required to attempt one question from each of the Units I, II, III and IV of the question paper.

SEC 3.11: Transportation and Game Theory

Unit-I

Transportation problem and its mathematical formulation. North west-corner method, least cost method.

Unit-II

Vogel approximation method for determination of starting basic solution, algorithm for solving transportation problem.

Unit-III

Assignment problem and its mathematical formulation, Hungarian method for solving assignment problem.

Unit-IV

Game theory: formulation of two person zero sum games, solving two person zero sum games. games with mixed strategies, graphical solution procedure.

Books Recommended

1. Mokhtar S. Bazaraa, John J. Jarvis and Hanif D. Sherali, Linear Programming and Network Flows, 2nd Ed., John Wiley and Sons, India, 2004.
2. F. S. Hillier and G. J. Lieberman, Introduction to Operations. Research, 9th Ed. Tata McGraw Hill, Singapore, 2009
3. HamdyA.Taha, Operations Research, An Introduction, 8th Ed., Prentice-Hall India, 2006.

CONTENTS

Unit - 1

The Transportation Problem

Structure

1.1 Introduction

Dear student, in this unit we will study about one of the application of operations research i.e. transportation problem operations research is a stats of art approach which is used for problem solving and decision making. With the help of operation research any organization can achieve its best performance under given contrarians or circumstances. Transportation Problem is one of the problems which is faced by many organization. It is the problem of transporting or shipping the commodities or items from the industry (sources) to the destinations with least possible cost, while satisfying the supply and demand limits. So, it is a special case of linear programming problem. So, in this unit we will study in detail about transportation problem, its mathematical representation and its types. Also, here we will study, how to solve a transportation problem by using North-West corner method.

1.2 Learning Objectives:

After studying this unit students will be able to

1. define transportation problem mathematically.
2. prepare cost matrix of given transportation problem
3. define the terminology related to transportation problem
4. solve transportation problem by using North-West corner method for initial feasible solution.

1.3 Transportation Problem

The transportation models were originated in 1941, when F.L. Hitch cock presented a study entitled "The Distribution of product from several sources to Numerous Localities" and this presentation is thought to be the first important controibution to the solution of Transportation problems. Further, in 1947 TC Koopman presented an independent study called "optimum utilization of the Transportation system." These two contribution assisted in the development of transportation models.

Transportation problem is a particular class of L.P.P. It deals with the situation in which a product is shipped from sources to destinations. The objective is to minimize the cost of transportation while meeting the requirement at the destinations. In a nutshell, transportation problem is used to optimise the distribution cost.

Suppose a company has three warehouses. W_1, W_2, W_3 having limited supply or capacity. From these ware house the product is transported to four stores S_1, S_2, S_3 and S_4 with specific or limited demand. Each warehouse can transport to each store but the cost of transportation very for all the combinations. The problem is to find out the quantity that each warehouse should transport to each store in order to minimize total transportation cost.

Mathematical Formulation of Transportation Problem

A transportation problem can be stated mathematically as a linear prrgramming problem as below.

Minimize $\quad Z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}\, c_{ij}$

Subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = a_i,\ i = 1, 2, ...m$$

$$\sum_{i=1}^{m} n_{ij} = b_j,\ j = 1, 2, ...n,\ x_{ij} \geq 0\ \forall\ u \text{ and } j$$

Where a_i = quantity of commodity available at origion i,

b_j = quantity of commodity needed at destination j,

c_{ij} = cost of transporting one unit of commodity from origin i to destination j.

and x_{ij} = quantity transport from origin i to destination j

Definitions:

1. **Feasible Solution:** Feasible solution is a set of non-negative values ($x_{ij} \geq 0$), which satisfies the row and column sum conditions (also called rim requirements) is called a feasible solution.
2. **Basic Feasible Solution:** If the total number of positive allocation (x_{ij}) is exactly equal to m+n-1, where m is number of rows and n is number of column, that solution is called basic feasible solution.
3. **Optimal Solution:** A feasible solution is said to be optimal if it minimize the total transportation problem.

4. **Balanced Transportation Problem:** A transportation Problem in which the total supply available equal to the total demand is called balanced transportation problem.

5. **Unbalanced Transportation Problem:** A transportation problem where the total availability at the origins is different from the total requirements at the destinations.

6. **Origin:** The multiple sites for the facilities location which are analyzed by the transportation method.

7. **Destination:** A location with a demand for material in the transportation method.

8. **Degenerate Basic Feasible Solution:** A basic feasible solution that contains loess than m+n-1 non negative allocations.

Non degenerate basic feasible solution: If the total member of positive allocations is exactly equal to (m+n-1) and these allocations are in independent positions that basic feasible solution is called non-degenerate basic feasible solutions.

Solution of a Transportation Problem: Solution of a transportation problem involve two

(1) Initial feasible solution

(2) Optimal solution

Here, in this unit we will study only about the initial feasible solution of a transportation problem. There are many methods to find initial feasible solutions like

(1) North-West comes method

(2) Least-cost entry method

(3) Vogel Approximation Method

Here we will study the first method i.e. North-West corner method

North West Corner Method

Algorithm for North West Corner Method

Step 1: Identify the North-West corner

- Start at the top-left (north west) corner of the cost matrix

Step 2: Allocate to the Extent Possible

- Allocate as much as possible to the cell that is being considered

Determine the maximum possible allocation based on supply and demand constraints

Step 3: Update supply and demand

- Adjust the supply and demand for the respective row and column after making the allocation
- Reduce the supply of the current row and decrease the demand of the current column

Step 4: Move to the next cell

- Move to the next cell to the right if possible. If not, move down to the next row
- Repeat steps 2 and 3 until all supply and demand constraints are satisfied

Step 5: Termination

- Continue the process until all supply and demand values are reduce to zero, indicating that all requirements have been met

Step 6: Calculation Total cost

- Once allocations are completed, calculate the total cost by multiplying allocated quantities with their respective costs and summing these values.

Let us try following examples to have better understanding of North-West corner Method.

Example 1: A company has production plants P, Q and R with weekly Production of 50, 60 and 40 units respectively. It has three warehouses U, V and W with weekly demands 40, 70 and 40 units respectively. Per unit shopping costs are given in the following table. Company wants to minimise its total transportation cost, How should it route its products?

Warehouses

		U	V	W
Plants	P	4	5	1
	Q	3	6	2
	R	5	4	5

Solution: Step 1: Start with the cell PU and allocate minimum of the demand (40) or supply (50) corresponding to this cell. So allocate 40 to this cell. Now the Demand at U having satisfied, move horizontally to the cell PV.

To / From	U	V	W	Supply
P	4 40	5 10	1	50
Q	3	6 60	2	60
R	5	4	5 40	40
Demand	40	70	40	150

Step 2: The column total is 70 while balance quantity left at P is 50-40 = 10. Thus assign 10 to the cell PV. Now the supply of the row P is exhausted.

Step 3: Now, we move vertically, to cell QV. The destination demand being 70 - 10 = 60 and source supply being 60, assign 60 to this cell and satisfy demand at V and exhaust the supply at Q.

Step 4: Next move diagonally to the cell RW. The destination demand (40) being equal to source supply (40), assign 40 to cell RW as shown in the table.

Total Cost:- $4 \times 40 + 5 \times 10 + 6 \times 60 + 5 \times 40 =$ Rs. 770.

Example 2:- Find out the Initial feasible solution with the help of North West corner method

To

From		D_1	D_2	D_3	D_4	Supply
	F_1	11	13	17	14	250
	F_2	16	18	14	10	300
	F_3	21	24	13	10	400
	Demand	200	225	275	250	

Solution: Step 1: Start with the cell F_1D_1 and allocate minimum of the demand (200) or supply (250) corresponding to this cell. So allocate 200 to this cell. Now the demand at D_1 having satisfied, move horizontally to the cell F_1D_2.

From \ To	D_1	D_2	D_3	D_4	Supply
F_1	11 200	13 50	17	14	250
F_2	16	18 175	14 125	10	300
F_3	21	24	13 150	10 250	400
Demand	200	225	275	250	950

Step 2: The column total is 225 while balance quantity left at F_1 is 250-200 = 50. Thus assign 50 to the cell F_1D_2. Now the supply of the row F_1 is over.

Step 3: We have move to the cell F_2D_2. The destination demand being 225-50 = 175 and source supply being 300, assign 175 to the cell F_2D_2. The requirement at D_2 is complete and move to the cell F_2D_3.

Step 4: The row total is 300, while remaining quantity at F_2 is 300 - 175 = 125. The requirement demand at D_3 is 275. So assign 125 to the cell F_2D_3. The supply of F_2 is complete. Now we move vertically to the cell F_3D_3.

Step 5: The destination demand being 275-125 = 150 and source supply being 400, assign the value 150 to the cell F_3D_3. The demand is complete on this cell. Now move on the cell F_3D_4. Assign the value 400-125 = 250 on the cell F_3D_3.

Total Cost:- 11 $\times$ 200 + 13 $\times$ 50 + 18 $\times$ 175 + 14 $\times$ 125 + 13 $\times$ 150 + + 10 $\times$ 250

Total Cost:-Rs. 12,200.

Example 3: Find out the Initial Feasible solution with the help of North West corner method.

Warehouse Factory	W_1	W_2	W_3	W_4	Factory Capacity
F_1	29	20	60	50	10
F_2	50	30	30	20	13
F_3	50	18	70	20	11
Requirement	6	7	11	10	34

Solution:

Factory ↓ / Warehouse →	W_1	W_2	W_3	W_4	Capacity
F_1	29 6	20 4	60	50	10
F_2	50	30 3	30 10	20	13
F_3	50	18	70 1	20 10	11
Requirement	6	7	11	10	34

Solution: Step 1: Start with the cell F_1W_1 and allocate minimum of the requirement (6) or capacity (10) corresponding to this cell. So allocate 6 to this cell. Now the requirement at W_1 having satisfied, move horizontally to the cell F_1W_2.

Step 2: The column total is 7 while balance quantity left at F_1 is 10-6 = 4. Thus assign 4 to the cell F_1W_2. Now the capacity of F is full-fill. Now, move vertically to the cell F_2W_2.

Step 3: The row total is 13 while remaining requirement at W_2 is 7-4 = 3. Thus assign 3 to the cell F_2W_2. Now the requirement of W_2 is over, move horizontally to the cell F_2W_3.

Step 4: The column total is 11 while balance quantity left at F_2 is 13-3 = 10. Thus assign 10 to the cell F_2W_3. Now the capacity of F_2 is complete. Now, move vertically to the cell F_3W_3.

Step 5: The row total is 11 while remaining requirement at W_3 is 11-10 = 1. Thus assign 1 to the cell F_3W_3. The requirement of W_3 is complete, the remaining capacity at F_3 is 11-1 = 10. assign 10 to the cell F_3W_4. So the capacity at F_3 and requirement at W_4 is fully satisfied respectively.

Total Cost: $6 \times 29 + 4 \times 20 + 3 \times 30 + 10 \times 30 + 1 \times 70 + 10 \times 20 =$ Rs. 914.

Example 4: Find the Initial feasible solution with the help of North West corner Method.

Market

		A	B	C	Capacity
Plant	X	7	24	17	20
	Y	15	9	13	20
	Z	14	20	21	36
	Requirement	23	25	28	

Solution:

Market / Plant	W_1	W_2	W_3	Capacity
X	7 20	24	17	20
Y	15 3	9 17	13	20
Z	14	20 8	21 28	36
Requirement	23	25	28	76

Step 1: Start with the cell × A and allocate minimum of the requirement (23) or capacity (20) corresponding to this cell. So allocate 20 to this cell. Now the capacity at X having satisfied, move vertically to the cell YA.

Step 2: The column total is 23, the remaining balance is 23-20 = 3. Thus assign 3 to the cell YA.

Step 3: We move horizontally, to cell YB. The remaining value at Y is 20-3 = 17. So assign the value 17 to this cell.

Step 4: When, we have vertically to the cell ZB. The remaining value is 25-17 = 8. Thus assign the value 8 to this cell. Since we left with the value 28. So assign this value to the cell ZC. So all the requirement and capacity are satisfied.

∴ Total Cost:- $20 \times 7 + 3 \times 15 + 17 \times 9 + 8 \times 20 + 28 \times 21$ = Rs. 1086.

Example 5: Solve the following transportation problem by North-West Corner Method:

Warehouse / Factory	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	34

Total (Balanced TP)

Solution:

19(5)	30(2)	50	10	7 — 2 — 0
70	30(6)	40(3)	60	9 — 3 — 0
40	8	70(4)	20(4)	18 — 14 — 0
5	8	7	14	
|	|	|	|	
0	6	4	0	
	|	|		
	0	0		

∴ Initial Feasible sol (Total Cost)

$= (19\times5) + (30\times2) + (30\times6) + (40\times3) + (70\times4) + (20\times4)$

Rs. 1015

Example 6: Find IFS of

Market

Plant		A	B	C	Capacity
	X	11	21	16	14
	Y	7	17	13	26
	Z	11	23	21	36
		18	28	25	

Solution: Here 18 + 28 + 25 = 61 (Demand)

14 + 26 + 36 = 76 (Supply)

Since here demand and capacity is different. So the given transportation problem is unbalanced transportation problem. So here we have to introduce the dummy market with zero cost in its column

Solution:

	A	B	C	D	
X	11(14)	21	16	0	14 — 0
Y	7(4)	17(22)	13	0	26 — 22 — 0
Z	11	23(6)	21(25)	0(5)	36 — 30 — 5 — 0
	18	28	25	5	
	|	|	|	|	
	4	6	0	0	
	|	|			
	0	0			

∴ Initial feasible sol (Total cost)

$= (11\times14) + (7\times4) + (17\times22) + (23\times6) + (21\times25) + (0\times5)$

= Rs. 1291

Example 7: A company has production plants located A, B and C with weekly production of 70, 50, 30 units resp. It has three warehouses at X, Y and Z with weekly demand of 20, 100, 30 units resp. Per unit shipping costs are given in the following table. Company wants to minimise its total transportation cost, how should it route its products?

	X	Y	Z
A	5	3	2
B	6	4	1
C	7	3	4

Solution:

5(20)	3(50)	2	70 — 50 — 0
6	4(50	1	50 — 0
7	3	4(30)	30 — 0
20	100	30	
|	|	|	
0	50	0	
	|		
	0		

∴ Initial feasible sol (Total cost) = $(5\times20) + (3\times50) + 4\ (50)$ = Rs. 570

Example 8: Find IFS by NWCM

				a_j
	10	9	8	8
	10	7	10	7
	11	9	7	9
	12	14	10	4
b_j	10	10	8	28

Solution:

10(8)		8	8 — 0
10(2)	7(5)	10	7 — 5 — 0
11	9(5)	7(4)	9 — 4 — 0
12	14	10(4)	4 — 0
10	10	8	
|	|	|	
2	5	4	
|	|	|	
0	0	0	

$\therefore$ Total Cost = $(10\times8) + (10\times2) + (7\times5) + (9\times5) + (7\times4) + (10\times4) = 248$

Self Check Exercise

Q.1: find out the IFS with the help of NWCM.

Warehouse

Factories	W_1	W_2	W_3	W_4	Capacity
F_1	10	5	10	20	40
F_2	15	25	5	15	30
F_3	15	30	10	5	10
Requirements	10	10	25	35	80

Q. 2: Find IFS by NWCM.

	A	B	C	Supply
I	10	9	8	8
II	10	7	10	7
III	11	9	7	9
IV	12	14	10	4
Demand	10	10	8	28

1.4 Summary:

The North West corner Method is a basic technique used in transportation and distribution problems to find an initial feasible solution. It starts allocating units from the top left corner of the transportation matrix and iteratively fills as much as possible in each cell, following the north-west direction, until all supplies and demands are met or allocated.

1.5 Glossary:

1. North-West corner rule:- This is a starting point for the northwest corner method. It involving beginning at the top-left corner of the transportation table and allocating shipments based on this starting point
2. Allocation:- The process of assigning quantities of goods from source to destinations in the transportation table.
3. Transportation Table:- A matrix or table used to represent the transportation problem, where rows represents sources, column represents destinations, and the cell represent the transportation costs or quantities.

1.6 Answer to Self Check Exercise

Q.1. Total Cost = $10\times10 + 5\times10 + 10\times20 + 2\times5 + 15\times25 + 5\times10$

Total Cost = Rs. 800

Q.2. Total Cost = $8\times10 + 2\times10 + 5\times7 + 5\times9 + 4\times7 + 4\times10$

Total Cost = Rs. 248

1.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.

4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

1.8 Terminal Questions

Q.1 Find the IFS with the help of NWCM.

Market

		A	B	C	Capacity
Plant	X	11	21	16	14
	Y	7	17	13	26
	Z	11	23	21	36
	Requirements	18	28	25	

Q.2. Find out the IFS with the help of North West corner method

Warehouse Factories	W_1	W_2	W_3	W_4	Factory Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirements	5	8	7	14	34

Unit - 2

Least Cost Entry Method

Structure

2.1 Introduction

Dear student in this unit we will study about more method to solve transportation problem. In last unit we studied the North-West corner method. In this unit we will study the least cost entry method, using this method we can improve our solution which is obtained by North-West corner method, means least cost entry method reduces the cost of transportation than the north-west corner method. In this unit we will learn the technique of least cost entry method and apply it can transportation problems. We will also try the problem with a tie having (Same least cost value) by using some examples.

2.2 Learning Objectives:

After studying this unit, students will be able to

1. define least-cost method for transportation problem
2. apply least-cost entry method transportation problem
3. solve T.P. by least cost entry method for initial feasible solution
4. resolve a lie (i.e. same least cost entry) using least-cost entry method.

2.3 Least cost entry Method

Algorithm of Least Cost Entry Method

Step 1: Select the cell with the lowest transportation cost in the whole matrix arbitrarily.

Step 2: Allocate an many units as possible to the cell determine in step 1 and eliminate that row/column in which either supply is exhausted or demand is satisfied.

Step 3: Re-select the cell with the lowest transportation cost from the remaining cost matrix or reduced table and repeated the step 2, till the rim-requirement is satisfied.

How to Resolve a Tie i.e. Two or More Cell Hauing Same Lest Cost

In case of tie in minimum cost, we check the entry where we can allocate the maximum.

Let us try following examples for finding basic feasible solution of a transportation problem by using least cost entry method.

Example 1: Find out the initial basic feasible solution by applying least cost method.

	1	2	3	4	Supply
1	9	2	15	11	20
2	11	9	14	22	20
3	6	12	16	16	10
Demand	6	19	10	15	

Solution: Step 1: Find out the smallest cost in the given table. The smallest cost element is 2 for the cell (1, 2) with corresponding supply and demand being 19 and 20 respectively. Allocate the minimum of 19 and 20, i.e. 19 to this cell, delete the column 2. The quantity of the supply at Row 1 is 20-19 = 1.

Sources \ Destination	1	2	3	4	Supply
1	9	2 19	15	11 1	20
2	11	9	14 10	22 10	20
3	6 6	12	16	16 4	10
Demand	6	19	10	15	50

Step 2: After deleting the column 2, the lowest cost cell is (3, 1). Allocate the minimum of 6 and 10 i.e. 6 to this cell and delete the column 1 and remaining quantity of supply at Row 3 is 10-6 = 4.

Step 3: The lowest cost cell is (1, 4) after deleting the column 1 and column 2. Allocate the minimum value 1 or 15 i.e. 1 to this cell. Now supply of the Row 1 is complete, so delete the Row 1.

Step 4: The lowest cost cell is (2, 3). Allocate the minimum of 10 and 20 to this cell. After allocate the value, remaining quantity at Row 2 is 20-10 = 10. Demand of column 3 is fully satisfied, delete the column 3.

Step 5: The lowest cost cell is (3, 4). Allocate the minimum value 4 or 14 to this cell. After deleting the Row 3, the lowest cost cell is (2, 4). Allocate the minimum value of 14-4 = 10 and 20-10 = 10. Now all the value are assigned.

Total Cost: $19\times2 + 1\times11 + 10\times14 + 6\times6 + 4\times16 + 10\times22$ = Rs. 509

Example 2: Find initial feasible solution using least cost entry method.

Ware house

		W_1	W_2	W_3	W_4	Capacity
Factories	F_1	12	7	8	16	30
	F_2	22	20	4	15	10
	F_3	10	25	15	5	40
	Requirements	35	10	25	10	80

Solution:

Factories \ Warehouse	W_1	W_2	W_3	W_4	Capacity
F_1	12 5	7 10	8 15	16	30
F_2	22	20	4 10	15	10
F_3	10 30	25	15	5 10	40
Requirements	35	10	25	10	80

Step 1: The smallest of the all cost element is 5 for the cell F_3W_4 with corresponding capacity and requirement being 40 and 10 respectively. Allocate 10 to this cell, delete the column 4, the remaining quantity of Row 3 is 40-10 = 30.

Step 2: In the reduce table the lowest cost cell is F_2W_3. Allocate the minimum value 10 or 25. After assign the value 10 to the cell F_2W_3, the capacity of F_2 is full-filled. Now delete the Row 2. Remaining Requirement of W_3 is 25-10 = 15

Step 3: At this stage, the lowest cost cell is F_1W_2. Assign the value (minimum of 10 and 30) 10 to this cell. The requirement of W_2 is complete, delete the column 2. The remaining capacity of F_1 is 30-10 = 20.

Step 4: The lowest cost cell is F_1W_3. Assign the minimum value 20 or 15. The requirement of W_3 is fully satisfied.

The remaining capacity of F_1 is 20-15 = 5

Total Cost = $5\times12 + 10\times7 + 15\times8 + 10\times4 + 30\times10 + 10\times5$ = Rs. 640

Example 3: Find the IFS using least cost method of given transportation problem

Destination

		D_1	D_2	D_3	D_4	Supply
Source	O_1	3	1	7	4	300
	O_2	2	6	5	9	400
	O_3	8	3	3	2	500
	Demand	250	350	400	200	1200

Solution:

Destination / Source	D_1	D_2	D_3	D_4	Supply
O_1	3	1 (300)	7	4	300
O_2	2 (250)	6	5 (150)	9	400
O_3	8	3 (50)	3 (250)	2 (200)	500
Demand	250	350	400	200	

Allocation:

Route	Cost
O1→D2	$1\times300 = 300$
O2→D1	$2\times250 = 500$
O2→D3	$5\times150 = 750$

O3→D2	3×50 = 150
O3→D3	3×250 = 750
O3→D4	2×200 = 400

Total Cost = Rs. 2850

Example 4: Find out IFS of given transportation problem using least cost method.

Source \ To	D	E	F	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

Solution:

Source \ To	D	E	F	Supply
A	5	8 50	4	50
B	6	6 5	3 35	40
C	3 20	9 40	6	60
Demand	20	95	35	150

Allocation:-

Route	Cost
A→E	8×50 = 400
B→E	5×6 = 30
B→F	3×35 = 105
C→D	3×20 = 60
C→E	9×40 = 360

Total Cost = Rs. 955

10	2(15)	20	11	15 — 0
12	7(1)	9(15)	20(9)	25 — 24 — 9 — 0
4(4)	14	16	18(6)	10 — 6 — 0
4	16	15	15	
|	|	|	|	
0	1	0	9	
	|		|	
	0		0	

∴ Initial feasible sol. (Total cost)

$= (2\times15) + (7\times1) + (9\times15) + (20\times9) + (4\times4) + 18\times6$

= Rs. 476

Note:- In case of tie in minimum cost, we check where we can allocate maximum.

Q.6 Find out Initial feasible sol. by applying least cost entry method.

8	7	3	60
3	8	9	70
11	3	5	80
50	82	82	82

Solution:

8	7	3(60)	60 — 0
3(50)	8	9(20)	70 — 20 — 0
11	3(80)	5	80 — 0
50	80	80	
|	|	|	
0	0	20	
		|	
		0	

∴ Initial feasible solution (Total cost)

= (3×60) + (3×50) + (9×20) + (3×80)

= Rs. 750

Q.7 Find the initial feasible sol. by least cost entry method

	W_1	W_2	W_3	W_4	Capacity
F_1	10	5	8	20	40
F_2	18	25	3	15	30
F_3	13	30	10	2	10
Requirement	10	10	25	35	

Solution:

10(10)	5(10)	8	20(20)	40 — 30 — 20 — 0
18	25	3(25)	15(5)	30 — 5 — 0
13	30	10	2(10)	10 — 0
10	10	25	35	
|	|	|	|	
0	0	0	25	
			|	
			20	
			|	
			0	

∴ Initial feasible solution (Total cost)

= (10×10) + (5×10) + (20×20) + (3×25) + (15×5) + (2×10)

= 720

Q.8 Solve the transportation problem with least cost entry method

2	7(1)	4(4)	5 — 1 — 0
3	3(8)	7	8 — 0
5	4	1(7)	7 — 0
1(7)	6	2(7)	14 — 7 — 0
7	9	18	
0	1	11	
	0	4	
		0	

∴ Min. Transportation Cost = (1×7) + (4×4) + (3×8) + (1×7) + (1×7) + (2×7)

= Rs. 75

Self Check Exercise

Q. 1 Find IFS by using LCM

	W_1	W_2	W_3	W_4	Supply
F_1	4	2	12	3	20
F_2	6	4	10	13	20
F_3	1	6	6	4	10
Demand	5	15	20	60	100

Q.2 Find initial feasible solution by using LCM.

To→ From↓	1	2	3	4	5	Supply
1	2	3	5	7	5	17
2	4	1	2	1	6	13
3	2	8	6	1	3	16
4	5	3	7	2	4	20
Demand	10	20	18	10	8	66

2.4 Summary:

The Least cost entry method is a strategy for solving transportation problems. It involves selecting the cell with the lowest transportation cost and assigning units based on the minimum of available supply and demand. This process is repeated until all supplies and demands are fulfilled, resulting in an optimal solution.

2.5 Glossary:

1. **Least cost method:-** A technique for finding an initial basic feasible solution in transportation problems by selecting the cell with the lowest transportation cost per unit.
2. **Feasible solution:-** A solution in which all supply and demand constraints are satisfied, and every cell in the transportation table has a non-negative value.
3. **Basic feasible solution:-** A solution where all supply and demand constraints are satisfied, and every cell in the transportation table has a non-negative value, achieved initially by methods such as the least cost entry method.

2.6 Answer to Self Check Exercise

Ans.1 Total Cost: $2\times15 + 3\times5 + 10\times20 + 1\times5 + 4\times5 + 20\times50$

Total Cost = Rs. 1270

Ans.2 Total Cost: $10\times2 + 7\times5 + 13\times1 + 10\times1 + 6\times3 + 7\times3 + 11\times7 + 2\times4$

Total Cost = Rs. 202

2.7 References/Suggested Readings

1. S.D. Sharma, Operations Reserach, KedarNath Ram Nath
2. R. Panneerselvam, Operations Research, Phi Learning Private Limited, Second Edition.

3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

2.8 Terminal Questions

Q.1 Solve the transportation problem LCEM

	1	2	3	Supply
I	2	7	4	5
II	3	3	7	8
III	5	4	1	7
IV	1	6	2	14
Demand	7	9	18	34

Q.2 Find the IFS of the following transportation problem using least cost method.

Warehouse

		W_1	W_2	W_3	W_4	Capacity
Factory	F_1	30	25	40	20	100
	F_2	29	26	35	40	250
	F_3	31	33	37	30	150
	Requirement	90	160	200	50	500

Unit - 3

Vogle's Approximation Method (VAM)

Structure

3.1 Introduction

Dear student, in this unit we will study about the most frequently method to find initial feasible solution of a transportation problem which is known as Vogel Approximation method (VAM). VAM gives the lowest transportation cost as compared with North-West corner method and least cost entry method. This method is also known as Penalty method, as for each row and column of the transportation problem penalties are calculated. Since VAM gives us the solution which is very close to optimal solution hence number of steps required to reach optimal solution reduces. In this unit we will study about Vogel approximation method and learn how it apply to find initial feasible solution of a transportation problem.

3.2 Learning Objectives:

After studying this unit, students will be able to

1. defineVogel Approximation method.
2. compare VAM with North-West corner method and least cost entry method
3. apply VAM to find initial feasible solution of a transportation problem

3.3 Vogle's Approximation Method (VAM)

Algorithm of Vogle's Approximation Method

Step 1: Calculate Penalty costs

- For each row and column in the cost matrix, calculate the difference between the two lowest transportation cost available for that row or column.

- This difference represents the penalty or opportunity cost of not using the next best option.

Step 2: Determine the largest Penalty:

- Identity the row or column with the largest penalty cost. If there is a tie (i.e. multiple rows or columns with the same largest penalty), you can choose arbitrarily or based on additional criteria.

Step 3: Allocate to the minimum cost cell

- From the identified row or column with the largest penalty cost, select the cell with the minimum transportation cost. This cell will be used for allocation.

Step 4: Allocate Maximum Possible Quantity

- Allocate as much as possible to the selected cell, based on the supply and demand constraints.
- Determine the maximum possible allocation based on the minimum of the available supply in the row and the demand in the column associated with the chosen cell.

Step 5: Repeat the Process

- Repeat Steps 1 to 4 until all supply and demand requirements are satisfied.
- Each iteration involves recalculation penalty costs identify the row or column with the largest penalty, and allocating to the minimum cost cell from that row or column.

How to Break a Tie (Having Same Penalties)

To break having same penalty, we always choose the lowest cost cell and allocate the minimum value of supply or demand.

Let us try following examples to find initial side solution by using Vogel Approximation method.

Example 1: Calculate the associated cost by using VAM.

	W_1	W_2	W_3	Supply
F_1	3	12	7	3
F_2	7	8	2	11
F_3	9	18	10	6
F_4	3	10	8	14
Demand	7	9	18	

Solution:

Table 1

Factories \ Warehouse	W_1	W_2	W_3	Supply	Penalty
F_1	3	12	7	3	4
F_2	7	8	2 11	0	5
F_3	9	18	10	6	1
F_4	3	10	8	14	5
Demand	7	9	18 7	34	
Penalty	0	2	5		

Step 1: The highest Penalty being 5, but there is tie amongst rows R_2, F_4, column W_3 for the highest penalty. Now find the lowest cost cell in these rows and column in order to break the tie. The lowest cost cell is F_2 W_3. Assign the minimum value of Supply (11) or demand (18). Allocate the value 11 to the cell F_2W_3. Delete the row F_2 because supply of F_2 is fully satisfied. Remaining value at column W_3 is 18-11 = 5. Move to table II.

Factories \ Warehouse	W_1	W_2	W_3	Supply	Penalty
F_1	3	12	7	3	4
F_3	9	18	10	6	1
F_4	3 7	10	8	14 7	5
Demand	7 0	9	7	23	
Penalty	0	2	1		

Step 2: The largest penalty being 8-3 = 5 for row F_4. In the lowest cost cell of this row (F_1W_1) quantity 7 is allocated and column W_1 is deleted. Move to table III.

Table III

Warehouse / Factories	W_2	W_3	Supply	Penalty
F_1	12	7	3	5
F_3	18	10 6	6 0	8
F_4	10	8	7	2
Demand	9	7 1	16	
Penalty	2	1	3	

Step 3: The largest penalty is 8 corresponding to the row F_3 and the lowest cost cell is F_3W_3. Allocate the value 6 to this cell and delete the row F_3 and move to table IV.

Table IV

Warehouse / Factories	W_2	W_3	Supply	Penalty
F_1	12	7 1	~~3~~ 2	5
F_4	10	8	7	2
Demand	9	1 0	10	
Penalty	2	1		

Step 4: The largest penalty is 5 corresponding to the row F_1 and the lowest cost cell is F_1W_3. Allocate the minimum of supply (3) or demand (1) i.e. 1 to the cell F_1W_3 and remove the column W_3. Move to table V.

Table V

Warehouse / Factories	W_2	Supply
F_1	12 (2)	2
F_4	10 (7)	7
Demand	9	9

In the reduce table IV, assign 2 and 7 to the cell F_1W_2 and F_4W_2 respectively. The Initial basic feasible solution is:

$F_1W_3 = 1$, $F_1W_2 = 2$, $F_2W_3 = 11$, $F_3W_3 = 6$, $F_4W_1 = 7$, $F_4W_2 = 7$.

Final solution show in table VI

Table VI

Warehouse / Factories	W_1	W_2	W_3	Supply
F_1	3	12 (2)	7 (1)	3
F_2	7	8	2 (11)	11
F_3	9	18	10 (6)	6
F_4	3 (7)	10 (7)	8	14
Demand	7	9	18	34

Total Cost: $2\times12 + 1\times7 + 11\times2 + 6\times10 + 7\times3 + 7\times10 =$ Rs. 204

Example 2: A manufacture wants to ships 8 loads of his product as shown below. The matrix gives the mileage from origin to the destination.

	A	B	C	Availability
X	70	40	190	4
Y	100	65	140	1
Z	90	150	60	3
Required	2	3	3	

Shipping cost areRs. 10 per load per mile. What shipping schedule should be used

Solution:

Table 1

	A	B	C	Availability	Penalty
X	70	40	190	4	30
Y	100	65	140	1	35
Z	90	150	60 3	~~3~~ 0	30
Required	2	3	~~3~~ 0	8	
Penalty	20	25	80		

Step 1: The highest penalty is 80 for column C and minimum cost cell is ZC corresponding to the column C. Allocate the minimum of availability (3) or required (3). After assigning the value 3 to cell ZC, all condition are satisfied for row Z and column C. Delete the Row Z and column C. Move to table II.

Table II

	A	B	Availability	Penalty
X	70	40	4	30
Y	100	65 1	~~1~~ 0	35
Required	2	~~3~~ 2	5	
Penalty	30	25		

Step 2: The highest penalty is 35 corresponding to row Y. Assign the minimum of Availability (1) or Required (3) to the lowest cost cell YB. Remove the Row Y. Move to table III.

Step 3: In the reduce table III, assign 2 and 2 to the cell ×A and ×B respectively which satisfies the availability and required condition.

Table III

	A	B	Availability
X	70 (2)	40 (2)	4
Required	2	2	4

Final solution is given in the table IV

	A	B	C	Availability
X	70 (2)	40 (2)	190	4
Y	100	65 (1)	140	1
Z	90	150	60 (3)	3
Required	2	3	3	8

The minimum cost is $2 \times 70 + 2 \times 40 + 1 \times 65 + 3 \times 60 = 465 \times 10 =$ Rs. 4650

Example 3: Find solution using VAM:

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	

Solution:

	D_1	D_2	D_3	D_4	Supply	Row Penalty
S_1	19 5	30	50	10 2	7	9 9 40 40 - -
S_2	70	30	40 7	60 2	9	10 20 20 20 20 20
S_3	40	8 8	70	20 10	18	12 20 50 - - -
Demand	5	8	7	14	34	
Column Penalty	21 21 - - - -	22 - - - - -	10 10 10 10 40 40	10 10 10 50 60 -		

The minimum transportation total cost

$= 19\times5 + 10\times2 + 40\times7 + 60\times2 + 8\times8 + 20\times10 =$ Rs. 779

Example 4: Solve the given transportation problem using Voglel's approximation method

Factories	Destination Centers				Supply
	D_1	D_2	D_3	D_4	
F_1	3	2	7	6	50
F_2	7	5	2	3	60
F_3	2	5	4	5	25
Demand	60	40	20	15	

Solution: Given, transportation problem is balanced transportation problem.

	D_1	D_2	D_3	D_4	Supply	Row Penalty
F_1	3 10	2 40	7	6	50	1 3 - - -
F_2	7 25	5	2 20	3 15	60	1 1 1 1 1
F_3	2 25	5	4	5	25	2 2 - - -
Demand	60	40	20	15		
Column Penalty	1 1 5 7 -	3 - - - -	2 2 2 2 2	2 2 2 3 3		

Total cost = $(10\times3) + (40\times2) + (25\times7) + (25\times2) + (20\times2) + (20\times2) + (15\times3)$

= Rs. 420

The Transportation Vogel's approximation method is the least. Let us Check this by following example.

Example: Solve the Transportation Problem for IFS by

(i) North West corner method (ii) least cost entry method

(iii) Vogel's Approximation Method

	C_1	C_2	C_3	C_4	Supply
W_1	14	28	20	16	60
W_2	14	24	22	12	60
W_3	10	30	16	18	60
Demand	40	50	40	60	180 / 190

Solution: Total demand = 190 and Total Supply = 180.

Given problem is unbalanced transportation problem. Introducing dummy origin (W_4) with fictitious supply = 10 and cost dummy row is assumed to be zero.

I. IFS by NWCM

	C_1	C_2	C_3	C_4	Supply
W_1	14 40	28 20	20	16	60
W_2	14	24 30	22 30	12	60
W_3	10	30	16 10	18 50	60
W_4	0	0	0	0 10	10
Demand	40	50	40	60	190

Total Cost = 40×14 + 20×28 + 30×24 + 30×22 + 10×16 + 50×18 + 10×0 = Rs. 3560

II. IFS by L.C.E.M.

	C_1	C_2	C_3	C_4	Supply
W_1	14	28 50	20 10	16	60
W_2	14	24	22	12 60	60
W_3	10 30	30	16 30	18	60
W_4	0 10	0	0	0	10
Demand	40	50	40	60	190

Total Cost = 50×28 + 10×20 + 60×12 + 30×10 + 30×16 + 10×0 = Rs, 3100

III. I.F.S. by V.A.M.

	C_1	C_2	C_3	C_4	Supply	Row Penalty
W_1	14	28 40	20 20	16	60	2 2 4 8
W_2	14	24	22	12 60	60	2 2 10 -
W_3	10 40	30	16 20	18	60	6 6 2 14
W_4	0	0 10	0	0	10	0 - - -
Demand	40	50	40	60	190	
Column Penalty	10 4 - -	24 4 4 2	16 4 4 4	12 4 4 -		

Total Cost = $40 \times 28 + 20 \times 20 + 60 \times 12 + 40 \times 10 + 20 \times 16 + 10 \times 0$ = Rs, 2,960

Here we can check that the least transportation cost is obtained by is Rs. 2,960 by VAM, than Rs. 3100 and Rs. 3560 by least cost entry method and North West corner method respectively.

Q.1 Calculate the IFS by Vogel Approx Method

	W_1	W_2	W_3	
F_1	4	14	8	5
F_2	6	6	1	8
F_3	10	16	14	7
F_4	2	12	4	14
	7	9	18	

Solution:

					Penalty			
	4(5)	14	8	5—0	4	4	4	-
	6	6(8)	1	8—0	5	-	-	-
	10(2)	16(1)	14(4)	7—5—4—0	4	4	4	4
	2	12	4(14)	14—0	2	2	-	-
	7	9	18					
Penalty	|	|	|					
	2	1	4					
	|	|	|					
	0	0	0					
	2	6↑	3					
	2	2	4↑					
	6↑	2	6					

So IFS (Total Cost) = $(4\times5) + (6\times8) + (10\times2) + (16\times1) + (14\times4) + (4\times14) = 216$

Q.2 A manufactures wants to ship 8 loads of his products as shown below. The matrix gives the mileage from the origin to destination

	A	B	C	Availability
X	50	30	220	1
Y	90	45	170	3
Z	50	200	50	4
Required	3	3	2	

Shipping cost areRs. 10 per load per mile. What shipping schedule should be used?

Solution:

	A	B	C	Availability	Penalty		
X	50(1)	30	220	1 — 0	20	20	20
Y	90	45(3)	170	3 — 0	45	45	45 ←
Z	50(2)	200	50(2)	4 — 2 — 0	0	150 ←	–
Required	3	3	2				
Penalty	1	0	0				
	0	15	120↑				
	0	15	-				
	0	15	-				

By applying Vogel's approximation method, the minimum cost is = (1×50) + (3×45) + (50×2) + (50×2) = 3850

Q. 3 Find the initial basic feasible solution of the following TP by VAM

	1	2	3	4		Penalty			
A	7	3(20)	8	6(40)	60 —20—0	3	3	4	-
B	4(20)	2(30)	5(50)	10	100 —80—50—0	2	2	2	2
C	2	6	5	1(40)	40—0	1	-	-	-
	20	50	50	80					
Penalty	0	30	0	40					
	0	0		0					
	2	1	0	5↑					
	3	1	3	4↑					
	3	1	3	-					

∴ The initial basic feasible solution is.

$= (3\times20) + (6\times40) + (4\times20) + (2\times30) + (5\times50) + (1\times40)$

= Rs. 730

Self Check Exercise

Q.1 A manufacturing company has distribution centers X, Y and Z. These centers have 40, 20 and 40 units of product. It retail outlets at A, B, C, D and E requires 25, 10, 20, 30 and 15 units respectively. The transport cost between each centre and each outlet is given in the following table:

	A	B	C	D	E
X	55	30	40	50	40
Y	35	30	100	45	60
Z	40	60	95	35	30

Find out the initial distribution cost by VAM.

Q.2 Find the IBFS of the following transportation problem by VAM.

Stores

		1	2	3	4	Supply
	A	7	3	8	6	60
Plant	B	4	2	5	10	100
	C	2	6	5	1	40
	Demand	20	50	50	80	200

3.4 Summary:

The Vogel's Approximation Method (VAM) is an algorithm used to solve transportation problem efficiently. It calculates penalties for each row and column based on the difference between the two smallest transportation costs in that row or column. Then, it prioritizes assigning units to cells with the highest penalties until all supplies and demands are met, aiming to find an optimal solution quickly.

3.5 Glossary

1. **Vogel's Approximation Method (VAM):** A heuristic method for finding an initial feasible solution in transportation problem by considering the penalty costs or differences between the two lowest costs in each row and column.
2. **Penalty Cost:** The difference between the two lowest transportation costs in each row and column, used in VAM to determine which row or column to allocate next.
3. **Supply:** Refers to the amount of a commodity available at each source.

3.6 Answers to Self Check Exercise

Q.1 Total Cost: $55\times5 + 30\times10 + 40\times20 + 50\times5 + 35\times20 + 35\times25 + 30\times15$

Total Cost = Rs. 3650

Q.2 Total Cost: $20\times3 + 40\times6 + 20\times4 + 30\times2 + 50\times5 + 40\times1$

Total Cost = Rs. 730

3.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

3.8 Terminal Questions

Q.1 Calculate the associate cost by VAM.

	W_1	W_2	W_3	Supply
F_1	4	14	8	5
F_2	6	6	1	8
F_3	10	16	14	7
F_4	2	12	4	14
Demand	7	9	18	

Q.2 Solve the transportation problem by using VAM:

	A	B	C	Supply
X	50	30	220	1
Y	90	45	170	3
Z	50	200	50	4
Demand	3	3	2	

Unit - 4

Modified Distribution Method (Modi)

Structure

4.1 Introduction

Dear student, in unit 1, 2 and 3 we discussed about the methods used to find the initial feasible solution of a transportation problem, which are North-West corner method, least cost entry method and Vogel's Approximation method. Since to main objective of a transportation problem is to find the minimum cost of transportation. After obtaining initial feasible solution the next step is to check that the cost we obtain by using above methods is optimum (or minimum) or not: So, in this unit we will learn to find the optimal solution of a transportation problem. To find optimal solution, we first have to cheek the optimality conditions and then proceed further. There are two methods to find optimal solution. In this unit we will learn the Modified distribution (MODI) method to find optimal solution of transportation problem. This method is also known as UV method.

4.2 Learning Objectives:

After studying this unit, students will be able to

1. define optimal solution of a transportation problem
2. define condition of optometry of a transportation problem
3. define MODI method of transportation problem.
4. apply MODI method to find optimal solution of a transportation problem.

4.3 Modified Distribution Method

Determination of Optimum Solution:

Optimality Test - After finding IFS by any of the method discussed earlier, the next important step is to see, can we decrease the total cost further by making changes in the occupied cells. This is known as testing optimality.

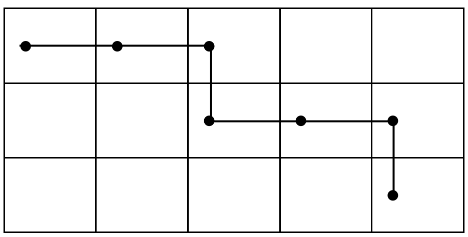
Two methods are uridely used for Testing the solution for optimality.

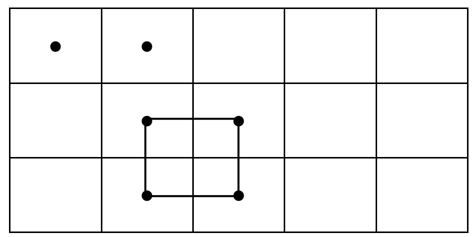
1. Stepping Stone Method

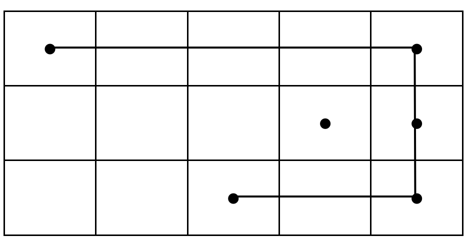
2. Modified Distribution (MODI) Method

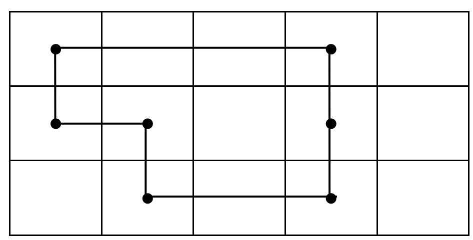
Note:- Optimality test is only applicable to a feasible solution consisting of (m+n-1) applications in independent position i.e. test is applicable only if the rim requirement is satisfied and allocations should be in independent positions.

Independent Position **Dependent Position**









* If we start from an allocation and making a loop, we did not return to the same pt. the positions is independent

* While making the loop the direction is changed from a allocated point only

Algorithm of Modified Distribution Method.

Step 1: Find initial basic feasible solution:

- Use any initial method to find an initial feasible solution

Step 2: Compute dual variables (u_i and v_j):

- Compute u_i and v_j for each row i and column j of the table using the following equations:

- $u_i - v_j = c_{ij}$ for non-empty cell (where c_i) is the cost in cell (i, j)).
- Choose any $u_i = 0$, where, most allocation in the matrix

Step 3: Calculate (Δ_{ij})

- For each cell (i, j), compute $\Delta_{ij} = c_{ij} - (u_i + v_j)$

Step 4: Identify Modified Costs (Cycle): Identify cycles in the Δ_{ij} matrix. A cycle is a closed path that alternate between occupied and unoccupied cells (starting and ending at the same point) in the Δ_{ij} matrix.

Step 5: Determine Feasibility and Improvement: Check the feasibility of the current solution. If any Δ_{ij} is negative within the cycle, determine the minimum value x among these negative values.

Step 6: Update Solution: Adjust the values of the basic variables along the cycle:

- Increase the value of x_{ij} for each (i, j) in the odd positions of the cycle by x.
- Decrease the value of x_{ij} for each (i, j) in the even positions of the cycle by x.

Step 7: Recomputeu_i and v_j: After updating the solution, recomputeu_i and v_j using the adjusted x_{ij} values.

Step 8: Iterate: Repeat steps 3 to 7 until no negative Δ_{ij} values are found in step 4.

Step 9: Optimal Solution: Once no negative C^*_{ij}value exist, the current solution is optimal.

Let us try following examples to find in optimal solution by MODI method.

Example 1: Solve the transportation problem

	A	B	C	Supply
X	3	1	20	2
Y	7	2	15	2
Z	1	18	3	4
Demand	3	3	2	

Solution: Initial feasible solution by apply VAM is:

Table I

	A	B	C	Supply	Penalty		
X	3 1	1 1	20	2	2	2	2
Y	7	2 2	15	2	5	5	5
Z	1 2	18	3 2	4	2	17 ←	-
Demand	3	3	2				
Penalty	2	1	↑12				
	2	1	-				
	4	1	-				

Total Cost = 1×3 + 1×1 + 2×2 + 2×1 + 2×3 = Rs. 16.

Conditions for Optimality:

1. Since number of allocation are 5 = m+n-1
2. All the allocations are at independent position i.e. no loop is formed by occupied cells.

For optimum solution we will use MODI.

We have added column u_i and row v_j to the transportation table.

Here u_i = Value of i^{th} row

v_j = Value of j^{th} column.

In case of occupied cells, the following equation exists:

$c_{ij} = u_i + v_j$

Table II

	A	B	C	Supply	Row Number (U_i)
X	3 1	1 1	20	2	U_1
Y	7	2 2	15	2	U_2
Z	1 2	18	3 2	4	U_3
Demand	3	3	2	8	
Column Number (V_j)	V_1	V_2	V_3		

In this problem, the unit transportation cost for the occupied cells can be described as:

$c_{11} = u_1 + v_1 = 3$, $c_{12} = u_1 + v_2 = 1$, $c_{22} = u_2 + v_2 = 2$,

$c_{31} = u_3 + v_1 = 1$, $c_{33} = u_3 + v_3 = 3$

In order to find the values of row and column numbers, one of the variables must be given an arbitrary value of zero. Here, we select u_1 and assign a zero value to it. Now, we can find out the values of the other variables as shown below:

Now $u_1 = 0$

$u_1 + v_1 = 3 \quad \Rightarrow \quad 0 + v_1 = 3 \quad \Rightarrow \quad v_1 = 3$

$u_1 + v_2 = 1 \quad \Rightarrow \quad 0 + v_2 = 1 \quad \Rightarrow \quad v_2 = 1$

$u_2 + v_2 = 2 \quad \Rightarrow \quad u_2 + 1 = 1 \quad \Rightarrow \quad u_2 = 0$

$u_3 + v_1 = 1 \quad \Rightarrow \quad u_3 + 3 = 1 \quad \Rightarrow \quad u_3 = -2$

$u_3 + v_3 \quad \Rightarrow \quad -2 + v_3 = 3 \quad \Rightarrow \quad v_3 = 5$

Now evaluate cost change for all the unoccupied cells by using following formula

(Δ_{ij}) cost change = (ij - ($u_i + v_j$))

The net cost charge (opportunity cost) for each of the empty cell is evaluated as below:

Unoccupied Cell	Δij=(i) - (Ui+Vj)	Net Cost Charge
XC	$\Delta_{13} = 20-0-5 = 15$	+15
YA	$\Delta_{21} = 7-0-3 = 4$	+ 4
YC	$\Delta_{23} = 15-0-5 = 10$	+ 10
ZB	$\Delta_{32} = 18+2-1 = 19$	+19

Since all the net cost charge are greater than or equal to zero ($\geq$ 0); so the transportation cost cannot be reduce further. So solution is optimum and total transportation cost = Rs. 16.

Example 2: Unit shipping costs per unit in rupees are as follows

Warehouse

		W_1	W_2	W_3	Supply
	F_1	9	18	20	1450
Factories	F_2	19	8	9	950
	F_3	30	22	13	800
	Demand	1400	1400	400	3200

Determine the optimal distribution for the company to minimize the shipping costs.

Firstly we will solve the above table by VAM to find initial feasible solution.

Warehouse / Factory	W_1	W_2	W_3	Supply	Penalty		
F_1	9 1400	18 50	20	1450	9	9	9
F_2	19	8 950	9	950	1	-	-
F_3	30	22 400	13 400	800	9	9	9
Demand	1400	1400	400				
Penalty	10	10↑	4				
	21↑	4	7				
	-	4	7				

Total Cost = 1400×9 + 50×18 + 950×8 + 400×22 + 400×13

= 12600 + 900 + 7600 + 8800 + 5200

= Rs. 35100

Test for Optimal:

Number of occupied cells = 5

Number of Rim Requirements = m+n-1 = 3+3-1 = 5

So number of occupied cells is equal to the number of Rim requirements. Hence solution is non-degenerate. Now, we will apply modified distribution method to find an optimum solution.

Table 1

	W_1	W_2	W_3	Supply	U_i
F_1	9 1400	18 50	20	1450	U_1
F_2	19	8 950	9	950	U_2
F_3	30	22 400	13 400	800	U_3
Demand	1400	1400	400		
V_j	V_1	V_2	V_3		

Next, we find the all values of u_i and v_j (i, j = 1, 2. 3). We have formula for occupied cell:

$c_{ij} = u_i + v_j$

for this problem, the unit transportation cost for the occupied cells can be described as:

$c_{11} = u_1 + v_1 = 9$, $c_{12} = u_1 + v_2 = 18$, $c_{22} = u_2 + v_2 = 8$,

$= c_{32} = u_3 + v_2$, $c_{33} = u_3 + v_3 = 13$

Now, we will assuming $U_1 = 0$

$0+v_1 = 9 \Rightarrow v_1 = 9$; $0+v_2 = 18 \Rightarrow v_2 = 18$; $u_2 + 18 = 8 \Rightarrow u_2 = -10$;

$u_3+18 = 22 \Rightarrow u_3 = 4$, $u_3+v_3 = 13 \Rightarrow u+v_3 = 13 \Rightarrow v_3 = 9$.

Now, we will evaluate cost change for all the unoccupied cells by using the following formula

Cost change $(\Delta_{ij}) = C_{ij} - (u_i + v_j)$

$\Delta_{13} = C_{13} - (u_1+v_3) = 20 - (0+9) = 11$

$\Delta_{21} = C_{21} - (u_2+v_1) = 19 - (-10+9) = 20$

$\Delta_{23} = C_{23} - (u_2+v_3) = 9 - (-10+9) = 10$

$\Delta_{31} = C_{31} - (u_3+v_1) = 30 - (4+9) = 17$

Since all the net cost charges are greater than or equal to zero i.e. ($\Delta ij \geq 0$); so the transportation cost cannot be reduced further. ∴ the solution is optimum and total transportation cost is Rs. 35700.

Example 3: A company has three factories at Solan, Shimla and Mandi; and four distribution centers at Patna, Mumbai, Bhopal and Chandigarh. With identical cost of Production at the three factories the only variable cost involved is transportation cost. The production at the three factories is 4,000 tones; 5,000 tones; 1500 tones respectively. The demand at four distribution centers is 5,000 tones; 3,000 tones; 1,000 tones and 500 tones respectively. The transportation costs per tone from different factories to different Centers are given below:

Distribution Centre				
Factory	Patna	Mumbai	Bhopal	Chandigarh
Solan	4	3	8	7
Shimla	8	6	3	4
Mandi	3	6	5	6

Suggest the optimum transportation schedule and find the minimum cost of transportation.

Solution: Firstly, we will find initial feasible solution using VAM.

	Patna	Mumbai	Bhopal	Chandigarh	Supply	Penalty
Solan	4 1000	3 3000	8	7	4000	1 3 - ←
Shimla	8 2500	6	3 1000	4 1500	5000	1 1 1
Mandi	3 1500	6	5	6	1500	2 2 2
Demand	5000	3000	1000	1500	10,500	
Penalty	1 1 ↑5	3↑ - -	2 2 2	2 2 2		

The transportation cost is 1000×4 + 3000×3 + 2500×8 + 1000×3 + 1500×4 + 1500×3 = Rs 46,500

The no of occupied cell is equal to the number of Rim requirement (i.e. m+n-1= 3+4-1=6). The solution is non-degenerate.

Hence, we shift to MODI test for optimality

We will find the values u_1, u_2, u_3 and v_1, v_2, v_3, v_4 Applying the formula for occupied cell

$c_{ij} = u_i + v_j$

Let $u_2 = 0$ (Q there is maximum allocation in this row)

$C_{21} = u_2+v_1 = 8 \Rightarrow v_1 = 8$,

$C_{23} = u_2+v_3 = 3 \Rightarrow v_3 = 3$,

$C_{24} = u_2+v_4 = 4 \Rightarrow v_4 = 8$,

$C_{11} = u_1+v_1 = 4 \Rightarrow u_1 + 8 = u_1 = -4$

$C_{12} = u_1+v_2 = 3 \Rightarrow -u_1 + v_2 = 3 \Rightarrow v_2 = 7$

$C_{31} = u_3+v_1 = 3 \Rightarrow u_3 + 8 = 3 \Rightarrow u_3 = -5$

Now, we will find net cost change (Δ_{ij}) by using the formula :$\Delta_{ij} = C_{ij} - (u_i+v_j)$ (for empty cell)

$\Delta_{13} = C_{13} - u_1 - v_3 = 8 + 4 - 3 = 9$

$\Delta_{14} = C_{14} - u_1 - v_4 = 7 + 4 - 4 = 7$

$\Delta_{22} = C_{22} - u_2 - v_2 = 6 - 0 - 7 = -1$

$\Delta_{32} = C_{32} - u_3 - v_2 = 6 + 5 - 7 = 4$

$\Delta_{33} = C_{33} - u_3 - v_3 = 5 + 5 - 3 = 7$

$\Delta_{34} = C_{34} - u_3 - v_4 = 6 + 5 - 4 = 7$

Not the all $\Delta_{ij} \geq 0$, so the solution is non-optimal.

Now we will choose the most negative cell, and give the value +n. We will make the closed path around this cell with the help of occupied cell shown in the table below.

	Patna	Mumbai	Bhopal	Chandigarh	Supply
Solan	4 1000 +x	3 3000 -x	8	7	4000
Shimla	8 2500 -x	6 +x	3 1000	4 1500	5000
Mandi	3 1500	6	5	6	1500
Demand	5000	3000	1000	1500	

Now, we have to choose x, for x we will choose that cell, where x is subtracting i.e.

x = min (25,00,3,000)

So x = 25,00

for the allocation for the cell (2, 2) = 25,00 for the cell (1, 2) is 3000-2500 = 500 and for the cell (1, 1) is 1000+2500 = 3500.

Now the new solution for above is given in table below.

	Patna	Mumbai	Bhopal	Chandigarh	Supply
Solan	4 3500	3 500	8	7	4000
Shimla	8	6 2500	3 1000	4 1500	5000
Mandi	3 1500	6	5	6	1500
Demand	5000	3000	1000	1500	

Now, again we will check the net cost change for all unoccupied cell. Firstly we will find u_1 and v_j for the above table using the formula: $C_{ij} = u_1 + v_j$.

Let $U_2 = 0$

$C_{22} = u_2 + v_2 = 6 \Rightarrow v_2 = 6$

$C_{23} = u_2 + v_3 = 3 \Rightarrow v_3 = 3$

$C_{24} = u_2 + v_4 = 4 \Rightarrow v_4 = 4$

$C_{11} = u_1 + v_1 = 4 \Rightarrow -3 + v_1 = 4 \Rightarrow v_1 = 7$

$C_{12} = u_1 + v_2 = 3 \Rightarrow u_1 + 6 = 3 \Rightarrow u_1 = -3$

$C_{31} = u_3 + v_1 = 3 \Rightarrow u_3 + 7 = 3 \Rightarrow u_3 = -4$

$\therefore$ We have $u_1 = -3$, $u_2 = 0$, $u_3 = -4$

$v_1 = 7$, $v_2 = 6$, $v_3 = 3$, $v_4 = 4$.

for the net cost (Δ_{ij}) = $C_{ij} - U_i - v_j$ (using formula) for unoccupied cell.

$\Delta_{13} = C_{13} - u_1 - v_3 = 8 + 3 - 3 = 8$

$\Delta_{14} = C_{14} - u_1 - v_4 = 7 + 3 - 4 = 6$

$\Delta_{21} = C_{21} - u_2 - v_1 = 8 - 0 - 7 = 1$

$\Delta_{32} = C_{32} - u_3 - v_2 = 6 + 3 - 6 = 3$

$\Delta_{33} = C_{33} - u_3 - v_3 = 5 + 4 - 3 = 6$

$\Delta_{34} = C_{34} - u_3 - v_4 = 6 + 4 - 4 = 6$

$\therefore$ All the $\Delta_{ij} \geq 0$ the solution not needs to improved further. So the solution in above table is optimum and the total cost is

$3500 \times 4 + 500 \times 3 + 2500 \times 6 + 1000 \times 3 + 1500 \times 4 + 1500 \times 3$

Total Cost = 44,000

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

If all Δ_{ij} are ≥ 0 then the corresponding solution of the transportation problem is optimum. If one or more of $\Delta_{ij} \leq 0$ then we select the cell with least Δ_{ij} and allocate as much as possible subject to the row and column constraints.

Step 4: A fresh set of dual variables is computed and entire procedure is repeated.

Q.4 Solve the transportation problem by MODI method

19	30	50	10	7
70	30	40	60	9
40	8	70	20	18
5	8	7	14	

Solution: First find initial basic feasible solution by VAM.

						Penalty				
	19(5)	30	50	10(2)	7—2—0	9	9	40	40	-
	70	30	40(7)	60(2)	9—2—0	10	20	20	20	-
	40	8(8)	70	20(10)	18—10—0	12	20	50 ←	-	-
	5—0	8—0	7—0	14—4—2—0						
Penalty	21	22↑	10	10						
	21↑	-	10	10						
	-	-	10	10						
	-	-	10	50↑						

∴ Initial basic feasible solution = $(19\times5) + (10\times5) + (40\times7) + (60\times2) + (8\times8) + (20\times10)$

= Rs. 779

Now to check the optimality

→ <u>To check the condition for non-degeneracy</u>

Initial basic feasible solution has allocation = 6

Rim condition = m+n-1

= 3+4-1

= 6 = no. of allocations

∴ $m+n-1 = 6$ (no. of allocations)

•			•
		•	•
	•		•

→Allocations are in independent positions

Hence optimality test is satisfied

→ Calculation of $\underline{u_i}$and $\underline{v_j \rightarrow}$

$C_{ij} = u_i + v_j$

u_i and v_j are calculated for occupied cells only

Rule - 1. Make any $u_i = 0$ for which there are maximum allocation

2. In case of tie, choose any one of them

19(5)			10(2)	u_1=0
		40(7)	60(2)	u_2=50
	8(8)		20(10)	u_3=10
v_i=19	$\frac{1}{2}$ =-2	$\frac{1}{3}$ =-10	$\frac{1}{4}$ =10	

$u_1 + v_1 = 19$

$u_1 + v_4 = 10$

$u_2 + v_3 = 40$

$u_2 + v_4 = 60$

$u_3 + v_2 = 8$

$u_3 + v_4 = 20$

Let $u_1 = 0$

$\Rightarrow \boxed{v_1 = 19}$

$\boxed{v_4 = 10}$

$u_2 + 10 = 60$

$\boxed{u_2 = 50}$

$u_3 + 10 = 20$

$\boxed{u_3 = 10}$

$10 + v_2 = 8$, $50 + v_3 = 40$

$\boxed{v_2 = -2}$ $\boxed{v_3 = -10}$

→ Calculation of cost difference for in occupied cells

$\Delta_{ij} = C_{ij} - (u_i + v_j)$

Cij

•	30	50	•
70	30	•	•
40	•	70	•

$u_i + v_j$

•	-2	-10	•	$u_1=0$
69	48	•	•	$u_2=50$
29	•	0	•	$u_3=10$
v_i (19)	v_2 (-2)	v_3 (-10)	v_4 (10)	

Δ_{ij}

•	32	60	•
1	-18	•	•
11	•	70	•

Optimality test - Check any Δ_{ij} is negative or not

1. if all $\Delta_{ij} > 0$ then solution is optimal
2. if any $\Delta_{ij} < 0$ so the IFS can be further optimise. Take the most negative value

Here Δ_{22} = -18 so x_{22} is entering the basis construction of loop starting from x_{22} and allocation of unknown quantity '0'

(5)			(2)
	(+Q)	(7)	(2-Q) (2)
	*(8) (8-Q)		*(10) (10+Q)

To find value of Q

min [8 - 0, 2-0] = 0

8 - Q = 0, 2 - Q = 0

Q = 8, Q = 2

min Q = 2

Im proved solution -

19(5)			10(2)
	30(2)	40(7)	
	8(6)		20(12)

∴ Total cost = (19×5) + (10×2) + (30×2) + (40×7) + (8×6) + (20×12)

= 743

Now to check the optimality of this improved solution -

No of allocations = 6

Now, m+n-1 = 3+4-1 = 6

∴ m+n-1 = no. of allocation

All allocations are at independent position

So non degeneracy

19			10	$u_1=0$
	30	40		$u_2=22$
	8		20	$u_3=0$
v_1	v_2	v_3	v_4	
(29)	(8)	(18)	(20)	

Let $u_1 = 0$ $\boxed{v_1 = 19}$

$u_1 + v_1 = 19$

$u_1 + v_4 = 10$

$u_2 + v_2 = 30$

$u_2 + v_3 = 40$

$u_3 + v_2 = 8$

$u_3 + v_4 = 20$

Now $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for unoccupied cells

	32	42	
19			18
11		52	

Since all Δ_{ij} are +ve, so solution is optimal solution

∴ Minimal or optimal cost = Rs 743

Q.5 Find the optimal sol. of the following transportation problem

	1	2	3	4	
A	7	3	8	6	60
B	4	2	5	10	100
C	2	6	5	1	40
	20	50	50	80	

Solution:

						Penalty	
7	3(20)	8	6(40	60—20—0	3	3	4
4(20)	2(30)	5(50)	10	100—70—50—0	2	2	2
2	6	5	1(40)	40—0	1	-	-

	20	50	50	82
	|	|	|	|
	0	30	0	40
		|		|
Penalty	2	1	0	5↑
	3	1	3	4↑
	3	1	3	-

$\therefore$ Initial feasible solution = $(3\times20) + (6\times40) + 4(20) + (2\times30) + (5\times50) + (1\times40)$

$= 60 + 240 + 80 + 60 + 250 + 40$

$=$ Rs. 730

Here no. of occupied cells = 6

And $m+n-1 = 3+4-1 = 6$

Also, Allocations are in independent position

$\therefore$ solution is non degenerate

Apply MODI method in order to test optimality

Calculation of u_i and $v_j \rightarrow$

$C_{ij} = u_i + v_j$

	3(20)		6(40)	$u_1=1$
4(20)	2(30)	5(50)		$u_2=0$
			1(40)	$u_3=-4$
$v_1 = 4$	$v_2 = 2$	$v_3 = 5$	$v_4 = 5$	

$u_1+v_2 = 3$

$u_1+v_4 = 6$

$u_2+v_1 = 4$

$u_2+v_2 = 2$

$u_2+v_3 = 5$

$u_3+v_4 = 1$

Let $u_2 = 0 \Rightarrow \boxed{v_1 = 4}$, $0 + v_2 = 2 \Rightarrow \boxed{v_2 = 2}$

$0 + v_3 = 5 \quad \Rightarrow \quad \boxed{v_3 = 5}$

$u_1 + 2 = 3 \quad \Rightarrow \quad \boxed{v_1 = 1}$

$1 + v_4 = 6 \quad \Rightarrow \quad \boxed{v_4 = 5}$

$u_3 + 5 = 1 \quad \Rightarrow \quad \boxed{v_3 = -4}$

Now $\Delta_{ij} = C_{ij} - (u_i + v_j)$ for unoccupied cell

$\Delta_{11} = C_{11} - (u_1 - v_1) = 7 - 5 = 2$

$\Delta_{13} = C_{13} - (u_1 + v_3) = 8 - 6 = 6$

$\Delta_{24} = C_{24} - (u_2 + v_4) = 10 - 5 = 5$

$\Delta_{31} = C_{31} - (u_3 + v_1) = 2 - 0 = 2$

$\Delta_{32} = C_{32} - (u_3 + v_2) = 6 -(-2) = 8$

$\Delta_{33} = C_{33} - (u_3 + v_3) = 5 -(1) - 4$

Since all Δ_{ij} for unoccupied cells have positive value so the solution is optimal

Stepping Stone Method -

Q.6 Find on optimal solution

	1	2	3	Supply
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	

Solution: 1. Find IBF solution by using VAM

Self Check Exercise

Q.1 Solve the transportation problem

	A	B	C	Supply
X	5	3	22	2
Y	9	4.5	17	2
Z	3	20	5	4
Demand	3	3	2	

Q.2 Solve the problem by MODI.

	1	2	3	Supply
1	2	7	4	5
2	3	3	1	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

4.4 Summary

The Modified Distribution method is a technique used to solve transportation problems by iteratively adjusting allocations to achieve an optimal solution. It involves evaluating each cell's cost and reallocating units based on the minimum cost increment until all supply and demand constraints are met, aiming for an efficient distribution of goods with minimized costs.

4.5 Glossary:

1. **U-V Method:** This is a technique within MODI (modified distribution) used to evaluate the opportunity cost of each non-basic variable in the transportation table.

2. **Loop or Closed Path:** In the U-V method, a closed path is a path that starts and ends at the same non-basic variable and alternates between row and column entries.

3. **Net Evaluation:** The net evaluation for a cell (i, j) in the table is the difference between the corresponding dual variable for row i and column j.

   Dual Variables : The represent the marginal cost of supplying an additional unit of supply or demand and are used to calculate the reduced cost.

4.6 Answers to Self Check Exercise

Q.1 Total Cost = $1\times5 + 1\times3 + 2\times4.5 + 2\times3 + 2\times5$

Total Cost = Rs, 33

Q.2 Total Cost = $2\times5 + 8\times1 + 7\times4 + 2\times1 + 2\times6 + 10\times2$

Total Cost = Rs. 80

4.7 References/Suggested Readings

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2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
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5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

4.8 Terminal Questions

Q.1 Solve the following transportation problem

		To			
		A	B	C	Available
From	I	50	30	220	1
	II	90	45	170	2
	III	250	200	50	5
	Requirement	4	2	2	

Q.2 Find the optimal solution of the given transportation problem.

		Warehouse				
		W_1	W_2	W_3	W_4	Capacity
Factory	F_1	19	30	50	10	7
	F_2	70	30	40	60	9
	F_3	40	8	70	20	18
	Requirement	5	8	7	14	

Unit - 5

Stepping-Stone Method

Structure

5.1 Introduction

Dear student, in this unit we will study the second method of find the optimal solution of a transportation problem. This method is known as steeping-stone method. Just like MODI method. This method also gives us the optimal solution of the transportation problem. In this method we move from an empty cell to another empty cell to improve the total cost of transportation, using loop or closed path. So in this unit we will study to find optimal solution using stepping stone method.

5.2 Learning Objectives:

After studying this unit, students will be able to

1. define stepping stone method
2. define optimal solution of a transportation problem.
3. cheek to condition of optimality
4. find optimal solution of a transportation problem by using stepping stone method

5.3 Stepping Stone Method

Algorithm of Stepping Stone Method

Step 1: Determine an initial basic feasible solution by using any one of the methods.

Step 2: Evaluate cost change of shipping goods via each unoccupied cell, which can be done as follows:

(a) Select an unoccupied cell to be evaluated.

(b) Starting from this unoccupied cell, trace a closed loop using the most direct route through at least three allocated cells used in the solution and then come back to the starting unoccupied cell. In the process of moving from one occupied cell to another, (i) move only vertically or horizontally but never diagonally, (ii) both empty or occupied cells may be skipped over, (iii) an even number of at least four cells must participate in a closed path. The cells at the turning points are known as stepping stones.

(c) After tracing the closed path, place plus (+) and minus (-) signs alternatively in the cells on each turn of the loop, beginning with a plus (+) sign in the empty cell.

(d) Find the 'net effect on the cost' along the closed loop. For this purpose add the unit cost figures found in each cell containing a plus sign and then subtracting the unit costs in each cell having the minus sign.

(e) Calculate the net change in cost for all the empty cells.

Step 3: Check the sign of each of the net cost changes. If all the net cost changes are greater than or equal to zero ($\rightarrow 0$); it indicates the solution is optimum.

Step 4: If at least one net cost change is negative. Then, choose the empty cell having the most negative net cost change and determine the maximum quantity that can be assigned to a cell marked with minus sign on the closed path. Add this quantity to the empty cell and to all other cells marked with a plus (+) sign. Subtract this quantity from the cells with a minus (-) sign.

Step 5: Go back to the step 2 and repeat the procedure until an optimum solution is obtained.

Let us try following examples to have more understanding of stepping stone method.

Example 1: Find the optimal solution to the following transportation problem using stepping stone method.

From \ To	A	B	C	Supply
X	140	170	100	12
Y	200	110	170	14
Z	120	260	60	10
Demand	8	13	15	36

Solution: Firstly we will find Initial feasible solution using Vogel's approximation method.

To From	A	B	C	Supply	UP_1	UP_2	UP_3
X	140 7	170	100 5	12	40	40	40
Y	200 1	110 13	170	14	60	30	30
Z	120	260	60 10	10	60	60	-
Demand	8	13	15	36			
UP_1	20	60	40				
UP_2	20	-	40				
UP_3	60		60				

Total Cost = 7×140 + 1×200 + 13×110 + 5×100 + 10×60 = Rs. 3710

The IFS has 5 occupied cells

$$m+n-1 = 3+3-1 = 5$$

Thus the IFS is non-degenerate and we can apply optimality test directly.

Step 1: To begin with, let us assess the unoccupied cell XB. Now, we will find a closed path, which contain at least three occupied cell. So the closed path is starting from the cell XB then it moves to YB, it moves to YA, it moves to XA and finally it wit comes to its original cell i.e. XB.

Step 2: Now we will find the net cost for this unoccupied cell. Firstly we give the +ve sign for the unoccupied cell and after that the sign will change alternatively.

Empty Cell	Closed Path	Net Cost Change	Effect on Total Cost
XB	XB→YB→YA→XA	170-110+200-140=+120	Cost increases

Step 3: Now we will find the same thing for all unoccupied cell. So all the unoccupied cells are evaluated as follows:

Empty Cell	Closed Path	Net Cost Change (Rs)	Effect on Total Cost
YC	YC→XC→XA→YA	170-100+140-200=+10	Cost increases
ZA	ZA→XA→XC→ZC	120-140+100-60=+20	Cost increase
ZB	ZB→YB→YA→XA→XC→ZC	260-110+200-140+100-60=+250	Cost increases

Step 4: All the unoccupied cells have positivevalues for the net cost change, the solution cannot be improved further. Hence the solution is the required optimum one.

Example 2: A cement factory manager is considering the least way to transport cement from his three manufacturing centers P, Q, R to depots A,B,C,D and E. The weekly production and demands along with transportation costs are given below:

	A	B	C	D	E	Supply (Tons)
P	6	3	5	6	6	50
Q	4	5	4	4	5	25
R	5	7	4	6	6	30
Demand (Tons)	20	40	20	10	15	105

What should be the distribution programme?

Solution: Initial solution by using Vogel Approximation method is presented below:

	A	B	C	D	E	Supply	Unit Penalty				
							I	II	III	IV	V
P	6	3 (40)	5	6	6	50	2	1	1	1	1
Q	4 (15)	5	4	4 (10)	5	25	0	0	0	-	-
R	5 (5)	7	4 (20)	6	6 (5)	30	1	1	1	1	1
Demand	20	40	20	10	15	105					
Unit Penalty I	1	2↑	0	2	1						
II	1	-	0	2↑	1						
III	1	-	0	-	1						
IV	1	-	0↑	-	0						
V	1	-	-	-	0						

After applying VAM, the initial solution is obtained as follows.

PB = 3×40 = 120, PE = 10×60, QA = 15×4 = 60, QP = 10×4 = 40.

RA = 5×5 = 25, RC = 20×4 = 80, RE = 5×6 = 30

Total Transportation Cost: 120+60+60+40+25+80+30 = Rs. 415

Optimal Test:-

Number of occupied cell = 7

Rim Requirements i.e. m+n-1 = 3+5-1 = 7

Since number of occupied cell and rim requirement are equal, hence the initial solution is non-degenerate.

Now we will evaluate the net cost for unoccupied cell.

Unoccupied Cell	**Closed Path**	**Net Cost**
PA	PA→PE→RE→RA	6-6+6-5 = +1
PC	PC→PE→RE→RC	5-6+6-4 = +1
PD	PD→PE→RE→RA→QA→QD	6-6+6-5+4-4 = +1
QB	QB→QA→RA→RE→PE→PB	5-4+5-6+6-6 = 13
QC	QC→QA→RA→RC	4-4+5-4 = +1
QE	QE→RE RA→QA	5-6+5-4 = 0
RB	RB→RE→PE→PB	7-6+6-3 = +4
RD	RD→RA→QA→QD	6-5+4-4 = +1

Since net cost of all unoccupied cell is greater than or equal to zero (Positive), which implies that initial solution cannot be improved furthers. Hence the initial solution is the optimum solution and optimum minimum transportation cost is Rs. 415.

Example 3: Using stepping stone method, solve the following transportation problem?

Factory	De Pot				Capacity
	D	E	F	G	
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Requirement	400	450	350	500	1700

Solution: Initial solution by Vogel Approximation method presented as below:

Factory \ Depot	D	E	F	G	Capacity	Unit Penalties			
A	4	6 400	8	6 300	700	2	2	0	0
B	3	5 50	2 350	5	400	1	2	0	0
C	3 400	9	6	5 200	600	2	2	4	-
Required	400	450	350	500	1700				
Unit Penalties	0	1	4	0					
	0	1	-	0					
	-	1	-	0					
	-	1	-	1					

Total Initial Cost = 6×400 + 6×300 + 5×50 + 2×350 + 3×400 + 5×200 = Rs. 7350

Optimal Test:-

Number of occupied cell = 6

Rim requirement = m+n-1

= 3+4-1 = 6

Since number of occupied cells is equal to rim requirement, hence initial solution is non-degenerate. We have to evaluate opportunity cost of unoccupied cell.

Un-Occupied Cell	Closed Path	Opportunity Cost
AD	AD→AG→CG→CD	4-6+5-3=0
AF	AF→BF→BE→AE	8-2+6-6=5
BD	BD→BE→AE→AG→CG→CD	3-5+6-6+5-3=0
BG	BG→AG→AE→BE	5-6+6-5=0

CE	CE→AE→AG→CG	9-6+6-5=4
CF	CF→BF→BE→AE→AG→CG	6-2+5-6+6-5=4

Since all the opportunity costs are positive which implies that solution cannot be improved further. Hence the initial solution is the optimum solution. Thus optimum solution is.

AE = 400×6 = 2400 BF = 350×2 = 700

AG = 300×6 = 1800 CD = 400×3 = 1200

BE = 50×5 = 250 CG = 200×5 = 1000

Example 4: Solve the transportation problem using stepping stone method.

	A	B	C	Supply
X	60	90	20	15
Y	20	30	90	10
Z	40	80	80	11
Demand	10	12	15	

We will find initial feasible solution using V.A.M.

To / From	A	B	C	Supply	Unit Penalties		
X	60	90 1	120 15	16	40	30	30
Y	20	30 10	90	10	10	10 ←	-
Z	40 10	80 1	80	11	40	40	40
Demand	10	12	15	37			
Unit Penalty	20	50	60↑				
	20	50	-				
	20	10	-				

Total Cost:- 1×90 + 15×20 + 10×30 + 10×40 + 1×80

= 90 + 300 + 300 + 400 + 80

= Rs. 1170

Test for Optimal:-

No. of occupied cell = 5

Rim requirement = m+n-1 = 5

Since number of occupied cell is equal to rim requirement

Hence initial solution is non-degenerate

Now we will check net cost change for all unoccupied cell, which is given in table below:

Un-Occupied Cell	Closed Path	Opportunity Cost
XA	XA→XB→ZB→ZA	60-90+80-40=+10
YA	YA→YB→ZB→ZA	20-30+80-40=+30
YC	YC→YB→XA→XC	90-20+90-30=+130
ZC	ZC→ZB→XB→C	80-20+90-80=+70

As all the unoccupied cells have positive values for the net cost change, the solution cannot be improved further.

Hence the solution obtained is the required optimum one.

XB = 1×90 = 90 ZA = 10×40 = 400

XC = 15×20 = 300 ZB = 1×80 = 80

YB = 10×30 = 300

Total Cost = Rs. 1170

					Penalty		
	2(5)	7	4	5—0	2	2	5 ←	-
	3	3	1(8)	8—0	2	-	-	-
	5	4(7)	7	7—0	1	1	1	1
	1(2)	6(2)	2(10)	14—4—2—0	1	1	5	5
	7—2—0	9—2—0	18—10—0					
Penalty	1	1	2↑					
	1	2	2↑					
	1	2	-					

∴ Initial feasible solution: $(2\times5) + (1\times8) + (4\times7) + (1\times2) + (6\times2) + (2\times10)$

$= 10 + 8 + 25 + 2 + 12 + 20$

$= 80$

2. Check for non-degeneracy

Here number of allocations = 6

$m+n-1 = 4+3-1 = 6$

∴ No of allocations = $m+n-1$

Also all allocations are in independent positions

To find cell evaluation/net cost change for unoccupied cell

C (1, 2) = 7-6+1-2 = 8-8 = 0

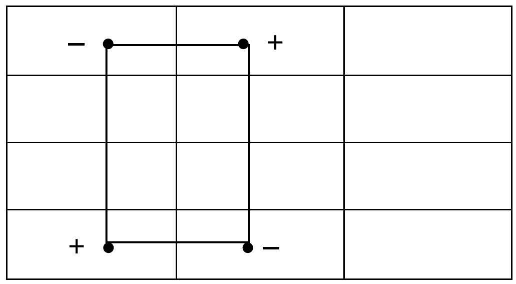
C (1, 3) = 4-2+1-2 = 5-4 = 1

C (2, 1) = 3-1+2-1 = 5-2 = 3

C (2, 2) = 3-1+2-6 = 5-7 = -2

C (3, 1) = 5-4+6-1 = 11-5 = 6

C (3, 3) = 7-2+6-4 = 13-6 = 7



→ (1, 2)

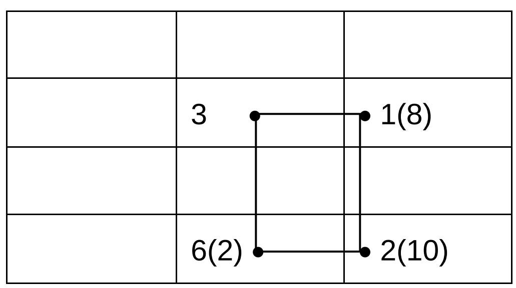
We find a loop starting from unoccupied cell and change direction from occupied cell and come back to the same cell from where are did starting and assign alternate sign to the cost values.

If all the values are positive or zero then solution is optimal. If not then solution can be improved.

Here we choose most -ve net cell evaluation

Here negative values is for cell

C (2, 2) = -2



Add 2

so updated sol. is

2(5)	7	4
3	3(2)	1(6)
5	4(7)	7
1(2)	6	2(12)

Minimum Cost = $(2\times5) + (3\times2) + (1\times6) + (4\times7) + (1\times2) + (2\times12)$

$= 10 + 6 + 6 + 28 + 2 + 24$

= Rs. 76

Again to check the net cost change for unoccupied cell

C (1, 2) = 7-3+1-2+1-2 = 9-7 = 2

C (1, 3) = 4-2+1-2 = 5-4 = 1

C (2, 1) = 3-1+2+1 = 6-1 = 5

C (3, 1) = 5-4+3-1+2-1 = 10 -6 = 4

C (3, 3) = 7-4+3-1 = 10-5 = 5

C (4, 2) = 6-2+1-3 = 7-5 = 2

All net cost change are positive, so solution obtained is optimal

So minimum cost = Rs. 76

Self CheckExercise

Q.1 Using stepping stone method. Solve the given problem.

Factory	Depot				Capacity
	D	E	F	G	
A	4	6	8	6	700
B	3	5	2	5	400
C	3	9	6	5	600
Required	400	450	350	500	1700

Q.2 Find the optimum solution

	A	B	C	Supply
X	160	190	120	14
Y	220	130	190	60
Z	140	280	80	12
Demand	10	15	17	42

5.4 Summary:

The stepping stone method is an iterative procedure used to find an optimal solution in transportation problems. It involves moving units step-by-step chance the name "Stepping Stone" from occupied cells to empty cells along a path that increases the total transportation cost. This method helps to systematically test potential solution until the best one is found, ensuring efficiency and accuracy in optimizing transportation routes and costs.

5.5 Glossary:

1. **Basic Variable:-** Decision variables that are assigned values during the solution process of a transportation problem.
2. **Non-Basic Variable:-** Decision variables that are not assigned values during the solution process of a transportation problem.
3. **Net Evaluation Table:-** A table used to determine the optimal distribution of a product or service, considering various constraints and costs.

5.6 Answer to Self Check Exercise

Q.1 Total Cost = $6 \times 400 + 6 \times 300 + 5 \times 50 + 2 \times 350 + 3 \times 400 + 5 \times 200$

Total Cost = Rs. 7,350

Q.2 Total Cost = $9 \times 160 + 5 \times 120 + 1 \times 220 + 15 \times 130 + 12 \times 80$

Total Cost = Rs. 5,170

5.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

5.8 Terminal Questions

Q.1 Solve the given transportation problem

	A	B	C	D	E	Supply
P	4	1	3	4	4	60
Q	2	3	2	2	3	35
R	3	5	2	4	4	40
Demand	22	45	20	18	30	135

Q.2 Solve the given transportation problem

	W_1	W_2	W_3	Supply
F_1	11	20	22	1550
F_2	21	10	11	1050
F_3	32	24	15	900
Demand	1500	1500	500	3500

and determine the optimal distribution for the company to minimize the shipping costs.

Unit - 6

Degeneracy And Unbalanced Transportation Problem

Structure

6.1 Introduction

Dear student, in this unit we will study more about the transportation problem. Unbalanced transportation problem and degeneracy in transportation problem will be discussed in this unit. In a transportation problem when demand and supply are not equal the transportation problem becomes a unbalanced transportation problem. Here in this unit we will study about such problem and study the procedure to solve such problems. Also white finding initial feasible solution of a transportation problem it is found that solution did not complete the rim requirement member of allocation = m+n-1. If such situation comes then the case of degeneracy arises. In this unit we will also study and solve such transportation problems.

6.2 Learning Objectives:

After studying this unit, students will be able to

1. define unbalanced transportation problem
2. solve unbalanced transportation problem
3. define degeneracy in transportation problem
4. solve degeneracy in transportation problem

6.3 Degeneracy in Transportation problem

White finding the initial feasic solution of transportation problem using North West corner method, least cost entry method and VAM. If the number of allocation all less than m+n-1, where m is number of rows and n is number of column, then case of degeneracy arrases.

Whereas, if Initial basic feasible solution contains m+n-1 number of allocation and there allocation are at independent positions than the solution is known as non-degenerate solution.

Independent Position:- Allocations are said to be in independent position of it is impossible to draw a dosed loop/path using some or all occupied cells.

5		12	
	10		
20	18		6

To trace a path starting and end point are at occupied cells.

To Resolve Degeneracy:

To resolve the problem of degeneracy at stating stage, an artificial quantity and an infinitesimally small allocation called epision 'E' has been used in unoccupied cell, to full fill the condition of m+n-1=0 occupied cell. It should be noted that the value of E is very small or nearer to zero and does not affect the supply and demand constraints. It is nearer to zero hence it's value is also assumed to be zero. It is always placed to the unoccupied cell having least transportation cost and independent cell. After introducing Epision, we proceed in the usual manner. Some operations of Episilon are given below:

E+K = K

E-K = K

E+E = E

E-E = 0

E$\times$K = 0, where K is constant.

Let us understand more about degeneracy by following examples.

Example 1. Solve the transportation problem and obtain the optimal solution?

	D_1	D_2	D_3	Supply
S_1	2	2	3	10
S_2	4	1	2	15
S_3	1	3	1	40
Demand	20	15	30	

Solution: We will find IFS by using VAM.

	D_1	D_2	D_3	Supply	Penalty	
S_1	2 10	2	3	10	0	0
S_2	4	1 15	2	15	1	3 ←
S_3	1 10	3	1 30	40	0	2
Demand	20	15	30	65		
Penalty	1	1	1↑			
	1	1	-			

Total Cost = $2\times10 + 1\times15 + 1\times10 + 1\times30$

= Rs. 75

Test for optimal:-

No. of allocation cell = 4

No of Rim Requirement = m+n-1 = 3+3-1 = 5

Since no of allocation cell is not equal to the no. of Rim Requirement. so it is the case of degeneracy.

To resolve the case of degeneracy.

Among the empty cell, we choose an empty cell having the least cost, which is an independent position. In above table, there are two least empty cell in the position S_1D_2 and S_2D_3. Also both are the independent. We can choose anyone. Let, we select the empty cell S_1D_2.

Give the value E to this empty cell. (E > 0).

Now, we have the no of allocation cell is equal to the no. of requirement cell. Now we have to check this solution is optimal or not optimal.

We will find u_i and v_j for occupied cell using the formula $c_{ij} = u_i + v_j$.

	D_1	D_2	D_3	u_i
S_1	2 10	2 E	3	$u_1 = 0$
S_2	4	1 15	2	$u_2 = 1$
S_3	1 10	3	1 30	$u_3 = -1$
v_j	$v_1=2$	$v_2=2$	$v_3=2$	

Now, we will find net cost change (Δ_{ij}) for unoccupied cell.

We have $\Delta_{ij} = c_{ij} - u_i - v_j$

$\Delta_{13} = 1, \Delta_{21} = 3, \Delta_{22} = 1, \Delta_{32} = 2$

$\therefore$ all $\Delta_{ij} \geq 0$. Hence the solution is optimum and total cost is $10\times2 + E\times2 + 15\times1 + 10\times1 + 30\times1 = 75 + 2E$.

$= \text{Rs. } 75$ (as we will take 0). $E \times K = 0$

Example 2: Calculate the initial feasible solution using VAM method and apply MODI method to test its optimality.

	C_1	C_2	C_3	C_4	Supply
W_1	12	26	18	14	55
W_2	12	22	20	10	55
W_3	8	28	14	16	55
W_4	0	0	0	0	5
Demand	35	45	35	55	

Solution: We will use VAM for IFS.

	C_1	C_2	C_3	C_4	Supply	Penalty		
W_1	12	26 40	18 15	14	55	2	2	6
W_2	12	22	20	10 55	55	2	2	-
W_3	8 35	28	14 20	16	55	6	2	6
W_4	0	0 5	0	0	5	0	-	-
Demand	35	45	35	55	170			
Penalty	8	22↑	14	10				
	4	4	4	4↑				
	4	4	4	-				

Since no. of occupied cell are 6; whereas number of rim requirements is 4+4-1 = 7, the initial feasible solution is degenerate and optimality test cannot be applied directly.

The lowest cost cell in the above table in the position of W_4C_1, W_4C_3 and W_4C_4. We have to choose one cell among them and that cell must be independent.

The independent cell is W_4C_4. So we assign the value ε to that cell.

Now we will check the solution is optimal or not. Find u_i and v_j for occupied cell using $c_{ij} = u_i + v_j$.

	C_1	C_2	C_3	C_4	Supply	ui
W_1	12	26 40 - x	18 15	14 +n	55	$u_1 = 0$ (Let)
W_2	12	22	20	10 55	55	$u_2=-16$
W_3	8 35	28	14 20	16	55	$u_3=-4$
W_4	0	0 5 + x	0	0 ε - n	5	$u_4 = -26$
Demand	35	45	35	55	170	
v_j	$v_1=12$	$v_2=26$	$v_3=18$	$v_4=26$		

Net Cost ($\Delta_{ij} = c_{ij}- u_i - v_j$) for unoccupied cell is

Unoccupied Cell	Opportunity Cost
W_1C_1	$\Delta_{11} = 0$
W_1C_1	$\Delta_{14} = -12$
W_2C_1	$\Delta_{21} = 16$
W_2C_2	$\Delta_{22} = 12$
W_2C_3	$\Delta_{23} = 18$
W_3C_2	$\Delta_{32} = 6$
W_3C_4	$\Delta_{34} = -6$
W_4C_1	$\Delta_{41} = 14$
W_4C_2	$\Delta_{42} = 8$
W_4C_3	$\Delta_{43} = 0$

Not all the $\Delta_{ij} > 0$, So give the value x to the most negative cell and make closed path using the occupied cell.

Now choosing for n = min (40, ε)

So x = ε

So $40 - \varepsilon = 40; 5 + \varepsilon = 5$

after changing this value we will get

	C_1	C_2	C_3	C_4	Supply	ui
W_1	12	26 (40)	18 (15)	14 (ε)	55	Let u1=0
W_2	12	22	20	10 (55)	55	$u_2=-4$
W_3	8 (35)	28	14 (20)	16	55	$u_3=-4$
W_4	0	0 (5)	0	0	5	$u_4 = -26$
Demand	35	45	35	55		
v_j	$v_1=12$	$v_2=26$	$v_3=18$	$v_4=14$		

for u_i and v_j use ($c_{ij} = u_i + v_j$)

again we will check net cost effect.

We know $\Delta_{ij} = c_{ij} - u_i - v_j$.

$\Delta_{11} = 0$ $\Delta_{34} = 6$

$\Delta_{21} = 4$ $\Delta_{41} = 14$

$\Delta_{22} = 0$ $\Delta_{43} = 8$

$\Delta_{23} = 6$ $\Delta_{44} = 12$

$\Delta_{32} = 6$

Since all $\Delta_{ij} > 0$. Hence solution cannot be improved further. So the solution is optimum and the total cost = $40\times26 + 15\times18 + \varepsilon\times14 + 55\times10 + 35\times8 + 20\times14 + 5\times0$

Total cost = 2420 [as $\varepsilon\times14$ = as taken as zero.]

Self Check Exercise - 1

Q.1 Following the initial feasible solution calculated by Vogel's approximation method, applying MODI method to test the optimality.

Destination / Origin	C_1	C_2	C_3	C_4	Supply
W_1	14	28 40	20 20	16	60
W_2	14	24	22	12 60	60
W_3	10 40	30	16 20	18	60
W_4	0	0	0	0	10
Demand	40	50	40	60	190

Q.2 Initial solution using VAM is given below. Calculate optimal transportation cost by using MODI Test.

	N	X	Y	Z	Availability
A	17	13 5	16	0	5
B	14 5	16 3	18 2	0	10
C	15	18	14 5	0 2	7
D	18	14	13 3	2	3
Requirement	5	8	10	2	

6.4 Unbalanced Transportation Problem:-

Case 1: Supply Exceed demand: When aggregate supply exceeds aggregate demand, a dummy destination is introduced to absorb the excess capacity. The cost of transportation from each origin to the dummy destination is assumed to be zero.

Case 2: Demand exceeds supply: When aggregate demands at the destinations is greater than the aggregate supply at the origins, A dummy is introduced in the transportation table. The shipping cost from dummy origin to each destination is assumed to be zero.

	D_1	D_2	-	-	-	-	D_n	
S_1								a_1
S_2								a_2
'								'
'								'
'								'
'								'
'								'
S_n								a_n
	0	0	0	0	0	0	0	

Supply < Demand

Add dummy Source row with Cost Value zero

	D_1	D_2	-	-	D_n	
S_1						0
S_2						0
S_3						
'						0
'						0
S_n						0
	b_1	b_2	-	-	-	b_{n+1}

Supply > Demand

We add dummy Column with Cost Value zero

Example 1: Solve the following transportation problem to maximize profit and give criteria for optimality.

	1	2	3	4	Supply
A	35	20	17	28	200
B	39	30	25	25	60
C	33	33	23	25	140
Demand	80	40	120	60	

Solution:- Since the given transportation problem is in maximisation, we convert it into minimisation problem by selecting the largest value which is 39. Subtract all values of given problem from this largest value.

Further, total quantities of demand and supply are not equal,

Total Demand = 300

Total Supply = 400

So Demand < Supply, We introduce a dummy column with zero coefficient to make it a balanced problem. We will use VAM for initial Basic feasible solution.

	1	2	3	4	5 Dummy	Supply			
A	4 20	19	22 120	11 60	0	200	4	7	7
B	0 60	9	14	14	0	60	0	9 ←	-
C	6	6 40	16	14	0 100	140	6 ←	0	0
Demand	80	40	120	60	100	100			
Penalty	4	3	2	3	0				
	4	3	2	3	-				
	2	13	6	3	-				

Initial Total Profit:- $35\times20 + 17\times120 + 39\times60 + 33\times40 + 28\times60$

$= \text{Rs. } 8080$

Test For Optimal:-

Rim Requirement = m+n-1 = 5+3-1 = 7

Number of occupied cell = 6

Since number of occupied cell $\neq$ number of rim requirement, hence the initial basic feasible solution is degenerate.

For the test of optimality. We assign the value ε, which is least in the all unoccupied cell. The cell A5 is least and independent, Now we can apply MODI method.

We have to find u_i and v_j for occupied cell,

	1	2	3	4	5	ui
A	4 20	19	-x 22 120	11 60	0 ε+x	let u_1=0
B	0 60	9	14	14	0	u_2=-4
C	6	6 40	16 +x	14	0 100-x	u_3=0

$v_j = v_{1}$ ∴ $= 4$ $v_2 = 6$ $v_3 = 22$ $v_4 = 11$ $v_5 = 0$

So, net cost (Δ_{ij}) for all unoccupied cell is

$\Delta_{12} = 19 - 6 = 13$, $\Delta_{22} = 9 + 4 - 6 = 7$, $\Delta_{23} = 14 + 4 - 22 = -4$,

$\Delta_{24} = 14 + 4 - 11 = 7$, $\Delta_{25} = 0 + 4 - 0 = 4$, $\Delta_{31} = 6 - 4 = 2$,

$\Delta_{33} = 16 - 22 = -6$, $\Delta_{34} = 14 - 11 = 3$.

∴ all Δ_{ij} not greater than or equal to zero, so assign the value +x to most negative Δ_{ij} and make closed path by giving the alternative value of x.

So x = min (100, 120) = 100. after changing the value, we have

	1	2	3	4	5	ui
A	4 20 +n	19	-x 22 120 -n	11 60	0 100	let u_1=0
B	0 60 -n	9	14 +n	14	0	u_2=-4
C	6	6 40	16 100	14	0	u_3= -6

v_j $v_1 = 4$ $v_2 = 12$ $v_3 = 22$ $v_4 = 11$ $v_5 = 0$

again we will find ui and vj for occupied cell and net cost (Δij) for unoccupied cell.

So $\Delta_{12} = 19\text{-}12 = 7$, $\Delta_{22} = 9+4\text{-}12 = 1$, $\Delta_{23} = 18\text{-}22 = -4$

$\Delta_{31} = 6+6\text{-}4 = 8$, $\Delta_{34} = 14+6\text{-}11 = 9$, $\Delta_{35} = 6$

$\Delta_{24} = 14+4\text{-}11 = 7$, $\Delta_{25} = 0+4 = 4$

So most negative cell value is -4, assign the value x and make closed path with the help of occupied cell.

∴ x = min (20, 60) = 20, changing the value, we have

	1	2	3	4	5	ui
A	4 40	19	22	11 60	0 100	let u_1=0
B	0 40	9	14 20	14	0	u_2=-4
C	6	6 40	16 100	14	0	u_3= -2

v_j $v_1 = 4$ $v_2 = 8$ $v_3 = 18$ $v_4 = 11$ $v_5 = 0$

again, we will find u_i and v_j and Δ_{ij}.

$\therefore$ $\Delta_{12} = 19-8 = 11$, $\Delta_{13} = 22-18 = 4$, $\Delta_{22} = 9+4-8 = 5$

$\Delta_{24} = 14+4-11 = 7$, $\Delta_{25} = 0+4 = 4$, $\Delta_{31} = 6+2-4 = 4$

$\Delta_{34} = 14-11+2 = 5$, $\Delta_{35} = 0+2 = 2$.

$\therefore$ all$\Delta_{ij} \geq 0$, since no further improved in the solution. Hence this solution is optimum.

Origin	Destination	Quantities	Profit Per Unit	Total Profit Rs.
A	1	40	35	14,00
A	4	60	28	1680
A	5 (Dummy)	100	0	0
B	1	40	39	1560
B	3	20	25	500
C	2	40	33	1320
C	3	100	23	2300
Total Maximum profit				8,760

Example 2: National Oil Company has three refineries and four Depots. Transportation cost per ton and requirements are given below:

Refineries	D_1	D_2	D_3	D_4	Capacity
P_1	4	6	12	9	600
P_2	7	5	13	12	200
P_3	11	9	8	10	700
Requirement	200	500	600	300	

Determine optimal allocation of output.

Solution: Total capacity = 1500, Total Requirement = 1600

Since, Total capacity < Total requirement. Hence given problem

Unbalanced Transportation problem.make it a balanced problem by introducing a dummy refinery P_4 with zero coefficient. Further, initial feasible solution after applying VAM is presented below:

	D_1	D_2	D_3	D_4	Capacity	Penalty				
P_1	4 200	6 300	12	9 100	600	2	2	2	3	3
P_2	7	5 200	13	12	200	2	2	2	7 ←	-
P_3	11	9	8 600	10 100	700	1	1	1	1	1
P4	0	0	0	0 100	100	0	-	-	-	-
Requirement	200	500	600	300	16,00					
Penalty	4 3 3↑ - -	5 1 1 1 3	8 4↑ - - -	9↑ 1 1 1 1						

The initial feasible solution by applying VAM is

Total Cost = 200×4 + 300×6 + 100×9 + 200×5 + 600×8 + 100×10 + 100×0

= Rs. 10,300

Testing Optimality by MODI Method:-

Number of occupied cell = 7

Rim Requirement i.e. m+n-1 = 7

Since, number of occupied cell = rim requirement, hence initial feasible solution is non-degenerate. Now, we apply MODI method to get the optimum solution.

	D_1	D_2	D_3	D_4	Capacity	ui
P_1	4 200	6 300	12	9 100	600	Let u_1=0
P_2	7	5 200	13	12	300	u_2=-1
P_3	11	9	8 600	10 100	600	u_3=1
P4	0	0	0	0 100	100	u_4=-9
Requirement	200	500	600	300	16,00	
v_j	v_1=4	v_2=6	v_3=7	v_4=9		

Now, we have to find u_i, v_j and net cost (Δ_{ij}) for occupied cell and unoccupied cell respectively.

Δ_{13} = 12-7 = 5,Δ_{21} = 7+1-4 = 4, Δ_{23} = 13+1-7 = 7,

Δ_{24} = 12-9+1 = 4, Δ_{31} = 11-4-1 = 6, Δ_{32} = 9-6-1 = 2,

Δ_{41} = 9-4 = 5, Δ_{42} = 9-6 = 3, Δ_{43} = 9-7 = 2

Since the net cost or opportunity cost (Δ_{ij}) for all unoccupied cell is positive which implies that the initial solution cannot be improved further. Hence the initial solution is the optimum solution and optimum transportation cost is Rs. 10,300.

Self Check Exercise-2

Q.1 Solve the following transportation problem?

	C_1	C_2	C_3	C_4	Supply
W_1	14	28	20	16	60
W_2	14	24	22	12	60
W_3	10	30	16	18	60
W_4	0	0	0	0	10
Demand	40	50	40	60	

Q.1 Solve the given problem

	A	B	C	D	Supply
E	25	55	40	60	60
F	35	30	50	40	140
G	36	45	26	66	150
H	35	30	41	50	50
Demand	90	100	120	140	

6.5 Summary:

Degeneracy in the context of transportation problems occurs when the number of allocated cells (occupied by units) is less than the total of supply and demand points. It complicates the application of traditional methods and requires additional steps to resolve.

An unbalanced transportation problem arises when the total supply does not equal the total demand. To solve it, artificial supplies or demands are often introduced to balance the equation, ensuring all resources are allocated optimally.

6.6 Glossary:-

- **Degeneracy:-** Degeneracy refers to a situation in a transportation problem where the number of allocated cells equals m+n-1, rather than m+n, where m is the number of sources and n is the number of destinations.
- **Dummy source or Dummy Destination:-** Artificial sources or destinations added to balance the transportation problem dummy sources are added to mach excess supply, while dummy destinations are added to match excess demand.
- **Adjusted cost matrix:** The modified transportation cost matrix after introducing dummy sources or destinations to balance the supply and demand in an unbalanced transportation problem.

6.7 Answer to Self Check Exercise

Self Check Exercise-1

Q.1 Total Cost = Rs. 3160

Q.2 Total Cost = Rs. 320

Self Check Exercise-2

Q.1 Total Cost Rs. 2960

Q.2 Total Cost Rs. 12,300

6.8 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram nath.
2. R. Panneerselvam, Operations Research, Phi Learning Private Limited, Second Edition.
3. JK. Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

6.9 Terminal Questions

Q.1 Solve the transportation Problem

Destination

		1	2	3	4	5	Supply
Source	I	40	36	26	38	30	160
	II	38	28	34	34	198	280
	III	36	38	24	28	30	240
	Demand	160	160	200	120	240	

Q.2 Solve the Transportation Problem

	D_1	D_2	D_3	Supply
S_1	5	1	7	10
S_2	6	4	6	80
S_3	3	2	5	50
Demand	75	20	50	

Q.3 Solve the given Transportation Problem.

	D_1	D_2	D_3	D_4	D5	Supply
S_1	10	2	3	15	9	35
S_2	5	10	15	2	4	40
S_3	15	5	14	7	15	20
S_4	20	15	13	25	8	30
Demand	20	20	40	10	35	

Unit - 7

Assignment Problem-Hungarian Method

Structure

7.1 Introduction

Dear student, in this unit we will study a special type of transportation problem, known as assignment problem. Just like transportation problem, the main objective of assignment problem is to minimize the cost of allocating a numbers of jobs to a number of persons with a condition that one person can have only one jab. In this unit we will study assignment problem i.e. Hungarian method. We will also study the case when number of assignment with are not equal to cost matrix, in that case we will learn how to draw minimum number of lines. We also discuss special type of assignment problem i.e. minimization problem.

7.2 Learning Objectives:-

After studying this unit, students will be able to

1. define assignment problem
2. define method to solve assignment problem i.e. Hungarian method.
3. apply Hungarian method to solve assignment problem i.e. to find minimum cost
4. draw minimum number of lines, in the case when number of assignments is not equal to cost matrix and find the solution of assignment problem
5. solve the maximized assignment problem.

7.3 Assignment Problem

Assignment problem is a special type of linear programming problem in which the objective function is to minimize the cost or time of competing a number of jobs by a number of person. Here we assume that each person can perform each job but the extend of efficiency in each case is different. So the objective is to assign each person to a single job in such a way that total cost of assignment is minimum. The word assignment problem initialed from the historical problem where the objective is to assign a number of jobs or origins to the number of persons or destinations at minimum cost or maximum profit.

Balanced Assignment Problem:- If the cost matrix of assignment problem is a square matrix then the assignment problem is a balanced assignment problem.

Unbalanced Assignment Problem:- If the cost matrix of assignment problem is not a square matrix then the assignment problem is not a balanced assignment problem.

Mathematical Formulation of Assignment Problem

The assignment problem can be shown as mathematically:

Minimise = Z = $\sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij} X_{ij,}$ i = 1, 2, ..., N

j = 1, 2, ..., N.

Subject to restriction

$$X_{ij} = \begin{Bmatrix} 1 \; If\ the\ ith\ person\ is\ assigned\ \ jth\ job \\ 0 \; If\ the\ ith\ person\ is\ not\ assigned\ \ jth\ job \end{Bmatrix}$$

$\sum_{i=1}^{N} X_{ij} = 1$ (one job is done by the ith person)

and

$\sum_{i=1}^{N} X_{ij} = 1$(only one person should be assigned the j_{th} job).

Note: X_{ij} = J_{th} job is to be assigned to the i_{th} person

The cost matrix $[C_{ij}]$ for the assignment problem is cost matrix

Jobs

		1	2	3...	j...	....n
	1	C_{11}	C_{12}	C_{13}...	C_{ij}...	C_{1n}
	2	C_{21}	C_{22}	C_{23}...	C_{2j}...	C_{2n}
Persons	3	C_{31}	C_{32}	C_{33}...	C_{3j}...	C_{3n}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	i	C_{i1}	C_{i2}	C_{i3}...	C_{ij}...	C_{in}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n	C_{n1}	C_{n2}	C_{n3}...	C_{nj}...	C_{nn}

Hungarian Method

Step 1: Formulate the Cost Matrix

- Create an n×n matrix where n is the number of worker to be assigned. Each element Cij in the matrix represents the cost at assigning job j to worker i.

Step 2: Subtract Row Minima

- For each row in the cost matrix, Find the smallest element and subtract it from every element in that row. This step ensures that at least one zero is present in every row after adjustment.

Step 3: Subtract Column Minima

- Similarly, for each column in the matrix, find the smallest element and subtract it from every element in that column. This step ensures that at least one zero is present in every column after adjustment.

Step 4: Examine the rows successively until a row with exactly one unmarked zero is found. Enclose this zero in a box (W) as an assignment will be made there and cross (X) all other zeros in its column. Proceed this way until all the rows have been examined. If any row has more than one zero, then do not touch that row and pass on the next row.

Step 5: Examine the columns successively until a column with exactly one unmarked zero is found. Enclose the zero in a box (W) and cross out (X) all other zero in its row. Proceed in this way until all the columns have been examined.

Step 6: Repeat operations (1) and (2) until all the zeros are either assigned or crossed.

Step 7: If there is exactly one assignment in each row and each column, then the optimum assignment is made

Step 8: If, the total number of assigned zeros is less than the order of the matrix. Draw the minimum number of vertical and/or horizontal lines necessary to cover all the zeros as follows:

Step 9: Mark (√) all rows that do not have assignment.

Step 10: Mark (√) all column which have zeros in the marked rows.

Step 11: Mark (√) all rows (not already marked) that have assignment in marked columns.

Step 12: Repeat steps (10) and (11) until no more rows or columns can be marked.

Step 13: Draw straight lines through all unmarked rows and marked columns.

Step 14: Select the smallest uncovered element in the matrix. Subtract the minimum element from all uncovered elements, add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

Step 15: Repeat the process until we get a optimal solution.

To have more understanding of Hungarian method let us try following example.

Example 1: A charted Accountant has four articles for audit work and four audits have to be performed. Articles differ in efficiency and tasks differ in their intrinsic difficulty. Time each audit take is given in the effectiveness matrix. How the tasks should be allocated to each article so as to minimize the total man-hour?

Articles

		I	II	III	IV
Audit Type	A	14	50	32	20
	B	24	54	6	50
	C	74	36	34	28
	D	36	50	46	18

Solution:-

Step 1: Row Subtraction or row-reduce matrix:- Subtract the minimum element of each row from all the elements of that particular row and if there is zero present in th row, then there is no need for row subtraction for that particular row.

0	36	18	6
18	48	0	44
46	8	6	0
18	32	28	0

Step 2: Column Subtraction or Column Reduced Matrix:- Subtract the minimum element of each column from all elements of that column. There is no need for column subtraction if there is a zero present in the column.

0	28	18	6
18	40	0	44
46	0	6	0
18	24	28	0

Step 3: Draw minimum number of horizontal and vertical lines to cover all zeros and to draw minimum number of lines the following procedure may be followed:-

(a) Starting with row 1 of the matrix, check the rows one by one until a row having exactly single zero element is found, then a mark 'W' is made to that cell and cross all other zeros in the column in which the assignment has been made. This stapes the possibility of making future assignment in that column.

$\boxed{0}$	28	18	6
18	40	$\boxed{0}$	44
46	0	6	0
18	24	28	$\boxed{0}$

(b) After examining the rows, the same procedure be applied to columns. Strat with single zero element in the column. Then make an assignment mark in that position and cross other zeros in the row in which assignment has been done.

$\boxed{0}$	28	18	6
18	40	$\boxed{0}$	44
46	$\boxed{0}$	6	$\cancel{0}$
18	24	28	$\boxed{0}$

As in this case all four assignment have made and it is not possible to make additional assignment the given assignment is optimum.

Optimal	A-I	B-III	C-II	D-IV
Assignment	16	8	38	20

= 82 hours

Example 2: Five wagons are available at stations 1, 2, 3, 4 and 5. These are required at five stations I, II, III, IV and V. The distance in Km. between various stations are given in the table below. How should the wagons be transported to minimize the total distance travelled?

	I	II	III	IV	V
1	9	4	8	17	10
2	12	8	5	11	13
3	2	1	3	3	4
4	17	8	11	16	14
5	10	5	13	18	9

Solution: Subtract the smallest element of each row from all elements of that row, we get:

	I	II	III	IV	V
1	5	0	4	13	6
2	7	3	0	6	8
3	1	0	2	2	3
4	9	0	3	8	6
5	5	0	8	13	4

Subtracting the smallest element of each column from all the elements of that column and making assignments, we get

	√				
4	$\boxed{0}$	4	11	3	√
6	3	$\boxed{0}$	4	5	
$\boxed{0}$	$\cancel{0}$	2	0	$\cancel{0}$	
8	$\cancel{0}$	3	6	3	√
4	$\cancel{0}$	8	11	1	√

Now, we have to draw minimum number of verticle and 10r horizontal lines necessary to cover all the zeros as follows:

(a) Mark (√) all rows that do not have assignment.

(b) Mark (√) all columns which have zeros in the marked rows.

(c) Mark (√) all rows (not already marked) that have assignments in marked columns.

(d) Repeat steps (ii) and (iii) until no more rows or columns can be marked.

(e) Draw straight lines through all unmarked rows and marked columns.

Drawing lines over unmarked rows and marked columns since there is no assignment in Row 4 and Row 5. Hence subtract

$$\begin{array}{c c}
\begin{array}{ccccc} & \surd & & & \end{array} & \\
\left[\begin{array}{ccccc}
3 & \boxed{0} & 3 & 10 & 2 \\
6 & 4 & \boxed{0} & 4 & 5 \\
\boxed{0} & 1 & 2 & 0 & \cancel{0} \\
7 & \cancel{0} & 2 & 5 & 2 \\
3 & \cancel{0} & 7 & 10 & 0
\end{array}\right] & \begin{array}{c} \surd \\ \\ \\ \surd \\ \surd \end{array}
\end{array}$$

Least uncovered element which is 1 from all other uncovered elements and adding the same to those elements lying at the intersection of horizontal and vertical lines, we get the modified matrix as follows.

Again Row 4 has no assignment hence select the smallest element which is 2 not covered by horizontal and vertical lines. Subtract this element from all the uncovered elements and add it to those elements lying at the intersection of lines, we get the revised matrix as follows:

$$\begin{array}{c c}
\left[\begin{array}{ccccc}
1 & \boxed{0} & 1 & 8 & \cancel{0} \\
6 & 6 & \boxed{0} & 4 & 5 \\
\boxed{0} & 3 & 2 & \cancel{0} & \cancel{0} \\
5 & \cancel{0} & \cancel{0} & 3 & \cancel{0} \\
3 & 2 & 7 & 10 & \boxed{0}
\end{array}\right] & \begin{array}{c} \surd \\ \surd \\ \\ \surd \\ \surd \end{array} \\
\begin{array}{ccccc} & \surd & \surd & & \surd \end{array} &
\end{array}$$

Since, Row 4 has no assignment, hence subtract the smallest uncovered elements which is 1 from all the uncovered elements and add it to those elements lying at the intersection of lines and making fresh assignments, we get:-

$$\begin{array}{ccccc}
\boxed{0} & \cancel{0} & 1 & 7 & \cancel{0} \\
5 & 6 & \boxed{0} & 3 & 5 \\
\cancel{0} & 4 & 2 & \boxed{0} & 1 \\
4 & \boxed{0} & \cancel{0} & 2 & \cancel{0} \\
2 & 2 & 7 & 9 & \boxed{0}
\end{array}$$

Since, all assignments are made, the optimum assignment schedule is:

Transport Wagons from Stations	To Stations	Cost (Rs.)
1	I	10
2	III	6
3	IV	4
4	II	9
5	V	10
	Total Minimum Cost =	39

Example 3: A car hire company has one care at each 5 depots a, b, c, d and e. A customer requires a car in each town A, B, C, D and E. Distance (in Kms) between depots and towns are give in the following matrix.

	a	b	c	d	e
A	140	110	155	170	180
B	115	100	110	140	155
C	120	90	135	150	165
D	30	30	60	60	90
E	35	15	50	60	85

How should cars be assigned to customers so as to minimize the distance travelled?

Solution: Subtracting the smallest elements of each row from every elements of the corresponding row, we get the following reduced matrix.

30	0	45	60	70
15	0	10	30	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Subtracting the smallest element of each column from every element of the corresponding column, we get the following reduced matrix.

	√				
30	$\boxed{0}$	35	30	15	√
15	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
30	$\cancel{0}$	35	30	20	√
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	
20	$\cancel{0}$	25	15	15	√

Drawing the minimum no. of horizontal and vertical lines

Since Row 4 and Row 5 have no assignment. So select least element (15) in all uncovered elements and add to every element that lie at the Intersection of the lines, and subtract this value from all uncovered element. The following matrix is obtained:

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

again drawing the minimum number of horizontal and vertical lines.

Now, in the above table, every row and column has an assignment. The assignment is the required optimum solution.

Optimum assignment schedule

Route	Distance (Kms)
A - e	180
B - c	110
C - b	90
D - a	30
E - d	60
Total distance travelled	470 Kms.

Example 4: Solve the following assignment problem

		Man			
		1	2	3	4
Work	I	7	25	16	10
	II	13	28	4	26
	III	39	20	19	16
	IV	18	25	23	9

Solution: Subtract the smallest element of each row from other elements of that row.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Subtract the smallest element of each column from other elements of that column and making assignments.

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\cancel{0}$
9	12	14	$\boxed{0}$

To making assignments in every row and column. The all assignments is done.

The optimal solution schedule is

Work	Man	Cost
I	1	7
II	3	4
III	2	20
IV	4	9
	Minimum Cost	40

Difference between Transportation Problem and Assignment Problem

1. In transportation problem the no. of sources and no. of destinations need not be equal. Hence cost matrix is not necessarily a square matrix, while assignment is done on one basis, the no. of sources and the no. of destinations are equal. Hence, the cost matrix must be a square matrix.

2. The transportation problem is unbalanced if the total supply and total demand are not equal while in assignment, the problem is unbalanced if the cost matrix is not a square matrix.

3. In transportation problem, the capacity and the requirement value is equal to a_i and b_j for the i^{th} source and j^{th} destinations, while the capacity and the requirement value in assignment problem is exactly one i.e. for each source of each destination the capacity and the requirement value is exactly one.

4. The quantity to be transported from i^{th} origin to j^{th} destination can take any possible positive values and satisfies the rim requirements, while the j^{th} job is to be assigned to the i^{th} person and can take either the value 1 or 0.

Hungarian Method of Solving Assignment Problem -

Q.1 Solve the assignment problem

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	88	19	18	15
D	19	26	24	10

Solution: 1 Subtracting the smallest element in each row from every element of that row

Row reduced matrix

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

2. Subtracting the smallest element in each column from every element of that column

Column reduced matrix

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

3. (a) starting from Ist row

(b) single zero by W (in row) and in that particular column $\boxed{X}$ for all zero

If there is a single zero in any row then assign it by W and mark all zeros in that particular column by X

(c) Repeat the process till all zeros have either W or X

$\boxed{0}$	14	9	3
9	20	$\boxed{0}$	22
23	$\boxed{0}$	3	$\cancel{0}$
9	12	14	0

4. If no. of assignment = order of matrix then assignment is optimal

Here no. of assignment = 4

Order of matrix = 4

So the assignment is optimal or each row or column have assignment

Men Hours = 8 + 4 + 19 + 10

= 41 hours

Optimal assignment is

A - I	8
B - III	4
C - II	19
D - IV	10
	41 hours

Q.2 Solve the assignment problem

How should the task be allocated, one to a man, so as minimize the total man hours

	a	b	c	d
A	18	26	17	11
B	13	28	14	26
C	38	19	18	15
D	19	26	24	10

Solution: 1. Row reduced matrix

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

2. Column reduced matrix

7	11	5	$\boxed{0}$
$\boxed{0}$	11	$\cancel{0}$	3
23	$\boxed{0}$	2	$\cancel{0}$
9	12	13	$\cancel{0}$

Here no. of assignment (3) ≠order of matrix (4) so the solution is not optimal i.e. assignment is not optimal since row 4 and column 3 do not have any assignment. So this is not optimal assignment. Therefore we improve matrix

To draw minimum lines

(a) Tick the row having no assignment

(b) Tick all columns which have zeros in the marked row

7	11	5	$\boxed{0}$	√
$\boxed{0}$	11	$\cancel{0}$	13	
23	$\boxed{0}$	2	$\cancel{0}$	
9	12	13	$\cancel{0}$	√
			√	

(c) Tick all rows (not already marked) that have assignments in marked columns.

(d) Repeat step (a) and (b) until no more rows or column can be marked

(e) Draw straight lines through all unmarked rows and marked columns.

Here 2nd and 3rd rows are unticked and 4th column is ticked so draw lines on them and then we see all the zeros are covered under these lines.

From uncovered elements find the smallest one

Here smallest element from uncovered elements = 5

* From uncovered elements subtract the smallest element

* Add smallest element to the elements at intersection

	a	b	c	d
A	2	6	$\boxed{0}$	$\cancel{0}$
B	$\boxed{0}$	11	$\cancel{0}$	18
C	23	$\boxed{0}$	2	5
D	4	7	8	$\boxed{0}$

Here no. of assignment = 4

Order of matrix = 4

So solution is optimal

Optimal assignment = A - C, B - a, C - b, D - d

Men hours = 17 + 13 + 19 + 10

= 59 hrs.

Q.3 Consider the problem of assigning five jobs to five persons, the assignments are given as follows:

	I	II	III	IV	V
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

Solution: Row reduced Matrix

7	3	1	5	0
0	9	5	5	4
1	6	7	0	4
4	3	1	0	3
4	0	3	4	0

Column reduced matrix

$$\begin{array}{c|ccccc|}
 & I & II & III & IV & V \\
\hline
A & 7 & 3 & \cancel{0} & 5 & \boxed{0} \\
B & \boxed{0} & 9 & 4 & 5 & 4 \\
C & 1 & 6 & 6 & \boxed{0} & 4 \\
D & 4 & 3 & \boxed{0} & \cancel{0} & 3 \\
E & 4 & \boxed{0} & 2 & 4 & \cancel{0} \\
\hline
\end{array}$$

Here no. of assignment = 5

Order of matrix = 5

So solution is optimal

Optimal Assignment = A - V, B - I, C - N, D - III, E - II

Total men hours = 1 + 0 + 2 + 1 + 5 = 9 hrs.

Q.4 Solve the following assignment problem

$$\begin{array}{c|ccccc|}
 & I & II & III & IV & V \\
\hline
1 & 10 & 5 & 9 & 18 & 11 \\
2 & 13 & 9 & 6 & 12 & 14 \\
3 & 3 & 2 & 4 & 4 & 5 \\
4 & 18 & 9 & 12 & 17 & 15 \\
5 & 11 & 6 & 14 & 19 & 10 \\
\hline
\end{array}$$

Solution: Row reduced matrix

$$\begin{array}{|ccccc|}
\hline
5 & 0 & 4 & 13 & 6 \\
7 & 3 & 0 & 6 & 8 \\
1 & 0 & 2 & 2 & 3 \\
9 & 0 & 3 & 8 & 6 \\
5 & 0 & 8 & 13 & 4 \\
\hline
\end{array}$$

Column reduced matrix

4	$\boxed{0}$	4	11	3	√
6	3	$\boxed{0}$	4	5	
$\boxed{0}$	~~0~~	2	~~0~~	~~0~~	
8	~~0~~	3	6	3	√
4	~~0~~	8	11	1	√
	√				

Here no. of assignment = 3

Order of matrix = 5

∴ no. of assignment ≠ order of matrix

∴ Assignment is not optimal

Here smallest element from uncovered elements = 1

∴ Subtract '1' from uncovered elements and add one to the elements at intersection

Again, here no. of assignment = 4

Order of matrix = 5

∴ no. of assignment ≠ order of matrix

3	$\boxed{0}$	3	10	2	√
6	4	$\boxed{0}$	4	5	
$\boxed{0}$	1	2	~~0~~	~~0~~	
7	~~0~~	2	5	2	√
3	~~0~~	7	10	$\boxed{0}$	
	√				

Here smallest uncovered element = 2

So add '2' to intersection point and subtract '2' from uncovered element

1	$\boxed{0}$	1	8	$\cancel{0}$	√
6	6	$\boxed{0}$	4	5	√
$\boxed{0}$	3	2	$\cancel{0}$	$\cancel{0}$	
5	$\cancel{0}$	$\cancel{0}$	3	$\cancel{0}$	√
3	2	7	10	$\boxed{0}$	√
	√	√		√	

Here no. of assignment ≠ order of matrix

∴ Assignment is not optimal

Here smallest uncovered element = 1

So add '1' to intersection point and subtract '1' from uncovered element

	I	II	III	IV	V
1	$\boxed{0}$	$\cancel{0}$	1	7	$\cancel{0}$
2	5	6	$\boxed{0}$	3	5
3	$\cancel{0}$	4	2	$\boxed{0}$	$\cancel{0}$
4	4	$\boxed{0}$	$\cancel{0}$	2	$\cancel{0}$
5	2	2	7	9	$\boxed{0}$

Now, no. of assignment = 5

Order of matrix = 5

∴ no. of assignment = Order of matrix

So assignment is optimal optimal assignment 1 - I, 2 - III, 3 - IV, 4 - II, 5 - V

Men hours = 10 + 6 + 4 + 9 + 10

= 39

Q.5 Solve the assignment problem

	A	B	C	D
1	18	24	28	32
2	8	13	17	19
3	10	15	19	22

Here no. of rows $\neq$ no. of columns

So a dummy row is added

Solution:

18	24	28	32
8	13	17	19
10	15	1	22
0	0	0	0

Row reduced matrix

0	6	10	14
0	5	9	11
0	5	9	12
0	0	0	0

Column reduced matrix

$\boxed{0}$	6	10	14	√
$\cancel{0}$	5	9	11	√
$\cancel{0}$	5	9	12	√
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	
√				

Smallest uncovered element = 5

$\boxed{0}$	1	5	9	√
$\cancel{0}$	$\boxed{0}$	4	6	√
$\cancel{0}$	$\cancel{0}$	4	7	√
5	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	
√	√			

Smallest uncovered element = 4

$\boxed{0}$	1	1	5
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	3
9	4	$\cancel{0}$	$\boxed{0}$

Hence all allocation has been made, we get optimum assignment 1 - A, 2 - B, 3 - C, dummy - D

Q.6 Solve the assignment problem

	M_1	M_2	M_3
1	9	26	15
2	13	27	6
3	35	20	15
4	18	30	20

Solution:

9	26	15	0
13	27	6	0
35	20	15	0
18	30	20	0

Row reduced matrix

9	26	15	0
13	27	6	0
35	20	15	0
18	30	20	0

Column reduced matrix

9	26	15	0
13	27	6	0
35	20	15	0
18	30	20	0

Column reduced matrix

0	6	9	$\cancel{0}$
4	7	$\boxed{0}$	$\cancel{0}$
26	$\boxed{0}$	9	$\cancel{0}$
9	10	14	$\boxed{0}$

No. of assignment = order of matrix

∴ Optimal Assignment 1 - M_1, 2 - M_3, 3 - M_2, 4 - M_4

Total hours = 9 + 6 + 20 + 0

= 35

Q.7 Maximize the assignment problem (Assign five jobs to the five machine so as to maximize the total expected profit)

(Jobs)

		A	B	C	D	E
Machine	1	5	11	10	12	4
	2	2	4	6	3	5
	3	3	12	5	14	6
	4	6	14	4	11	7
	5	7	9	8	12	5

Solution: Convert it to minimization problem by subtracting all the elements from the largest element of given matrix Here largest element is 14

9	3	4	2	10
12	10	8	11	9
11	2	9	0	8
8	0	10	3	7
7	5	6	2	9

Row reduced Matrix

7	1	2	0	8
4	2	0	3	1
11	2	9	0	8
8	0	10	3	7
5	3	4	0	7

Column reduced matrix

3	1	2	$\boxed{0}$	7	√
$\boxed{0}$	2	$\cancel{0}$	3	$\cancel{0}$	
7	2	9	$\cancel{0}$	7	√
4	$\boxed{0}$	10	3	6	
1	3	4	$\cancel{0}$	6	√
			√		

Here no. of assignment = 3

Order of matrix = 5

So assignment is not optimal

Here smallest uncovered element is 1

2	$\cancel{0}$	1	$\cancel{0}$	6	√
$\cancel{0}$	2	$\boxed{0}$	4	$\cancel{0}$	
6	1	8	$\boxed{0}$	6	√
4	$\boxed{0}$	10	4	6	√
$\boxed{0}$	2	3	$\cancel{0}$	5	
	√		√		

Here no. of assignment = 4

Order of matrix = 5

So assignment is not optimal

Here smallest uncovered element is 1

	A	B	C	D	E
1	1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	5
2	$\cancel{0}$	3	$\cancel{0}$	5	$\boxed{0}$
3	5	1	7	$\boxed{0}$	5
4	3	$\boxed{0}$	9	4	5
5	$\boxed{0}$	3	3	1	5

No. of assignment = 5

Order = 5

So assignment is optimal

Optimal assignment = 1 - C, 2 - E, 3 - D, 4 - B, 5 - A

Max profit = 10 + 5 + 14 + 14 + 7 = 50

Self Check Exercise

Q.1 Consider the problem of assigning five jobs to five persons. The assignment costs are given as follows:

Jobs

		1	2	3	4	5
	A	8	4	2	6	1
Persons	B	0	9	5	5	4
	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimum assignment schedule?

Q.2 Solve the assignment problem

		Man 1	Man 2	Man 3	Man 4
Work	I	12	30	21	15
	II	18	33	9	31
	III	44	25	24	21
	IV	23	30	28	14

7.4 Summary

The Hungarian method is an efficient algorithm used to solve assignment problems, where task need to be assigned to workers based on cost or profit minimization. It involves constructing a cost matrix, iteratively finding a combination of assignments that minimize the total cost, and adjusting assignments until an optimal solution is reached, ensuring each task is assigned exactly once and each worker performs exactly one task.

7.5 Glossary:

1. **Assignment Problem :-** A type of optimization problem where the objective is to assign a set of agents to a set of tasks in away that minimizes or maximizes a given objective function.
2. **Cost Matrix :-** A matrix C = [C_{ij}} where Cij represents the cost of assigning agent i to task j.
3. **Hungarian Algorithm :-** An Algorithm used to solve assignment problems efficiently by finding a perfect matching that minimizes (or maximizes) the total cost.

7.6 Answer to Self Check Exercise

Q.1 A - 5, B - 1, C - 4, D - 3, E - 2,

Minimum Cost = 9

Q.2 I - 1, II - 3, III - 2, IV - 4

Minimum Cost = 60

7.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.

4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

7.8 Terminal Questions

Q.1 Solve the following Assignment

		Jobs			
		I	II	III	IV
Operators	A	10	12	9	11
	B	5	10	7	8
	C	12	14	13	11
	D	8	15	11	9

Q.2 Solve the assignment problem whose cost matrix is given below:

	1	2	3	4
I	5	6	7	8
II	7	8	9	10
III	10	11	12	11
IV	4	6	9	5

Unit - 8

Unbalanced And Maximized Assignment Problem

Structure

8.1 Introduction

Dear student, in this unit we will study about the unbalanced assignment problem. In the last unit we studied that in an assignment problem the cost matrix is a square matrix. But if the given cost matrix of assignment problem is not a square matrix then the assignment problem become unbalanced. In that case, we first have to make this unbalanced assignment to a balanced one by introducing dummy sows or columns as per the requirement. The dummy rows or columns has zero cost. Then the same method i.e. Hungarian method is applied to find the optimal solution of that problem. Unbalanced assignment problem and its method of solution will be discussed in this unit.

8.2 Learning Objectives:

After studying this unit, students will be able to

1. define unbalanced assignment problem
2. convert unbalanced assignment problem to balanced transportation problem
3. solve unbalanced transportation problem

8.3 Unbalanced And Maximized Assignment Problem

Unbalanced Assignment problem:-

If the cost matrix is not squared matrix, the problem is called unbalanced. In such case dummy rows or column depending upon need, shall be introduced with zero cost so as to form square matrix.

Maximization Assignment Problem:-

Step 1: Find cost matrix by subtracting all the elements from the largest element.

Step 2: Follow the Hungarian method to solve the minimization problem

Step 3: Take the solution from the original matrix i.e. maximized matrix.

Let us try following example to have understating of unbalanced assignment problem.

Example 1: A company has 4 machines with which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

		Machine			
		W	X	Y	Z
Job	A	14	20	24	28
	B	4	9	13	15
	C	6	11	15	18

What are the job assignment which will minimize the cost?

Solution: The cost matrix of an assignment problem is not a square matrix. Introduce one dummy column for a fictitious job say D in the matrix in order to get the problem balanced. The cost corresponding to dummy row are always taken as zero.

	W	X	Y	Z
A	14	20	24	28
B	4	9	13	15
C	6	11	15	18
D	0	0	0	0

Doing row subtraction, after that column subtraction. We get the following revised matrix.

√				
$\boxed{0}$	6	10	14	√
$\cancel{0}$	5	9	11	√
$\cancel{0}$	5	9	12	√
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	

giving the assignment and drawing the minimum number of horizontal and vertical lines. We get, Subtracting the least element (5), we get

$$
\begin{array}{ccccc}
\surd & \surd & & & \\
\boxed{0} & 1 & 5 & 9 & \surd \\
\cancel{0} & \boxed{0} & 4 & 6 & \surd \\
\cancel{0} & \cancel{0} & 4 & 7 & \surd \\
5 & \cancel{0} & \boxed{0} & \cancel{0} &
\end{array}
$$

again assign the value, and drawing minimum no. of horizontal and vertical lines. Row 3 is remaining with the assignment.

So Subtracting the least element (4) from the above matrix.

$$
\begin{array}{cccc}
\boxed{0} & 1 & 1 & 5 \\
\cancel{0} & \boxed{0} & \cancel{0} & 2 \\
\cancel{0} & \cancel{0} & \boxed{0} & 3 \\
9 & 4 & \cancel{0} & \boxed{0}
\end{array}
$$

after assigning the value, all assignment is done. We get the optimal assignment.

A $\rightarrow$ W,	B $\rightarrow$ X,	C $\rightarrow$ Y	Dummy $\rightarrow$ Z
14	9	15	

Total cost = Rs. 38

Example 2: A batch of four jobs can be assigned to five different machines. The set up time for each job on each machine is given in the following table. Find an optimal assignment of jobs to machines which will minimize the total set up time:

	Machine				
Jobs	1	2	3	4	5
1	15	16	9	7	13
2	12	16	15	19	17
3	10	11	14	17	19
4	18	20	16	15	12

Solution: Row Reduction matrix and introducing the dummy row

8	9	2	0	6
0	4	3	7	5
0	1	4	7	9
6	8	4	3	0
0	0	0	0	0

Now, giving the assignment and drawing the minimum number of horizontal and vertical line, we get.

√					
8	9	2	$\boxed{0}$	6	
$\boxed{0}$	4	3	7	5	√
$\cancel{0}$	1	4	7	9	√
6	8	4	3	$\boxed{0}$	
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	

Row 3 has no assignments, so delete the least element. We get

9	9	2	$\boxed{0}$	6
$\boxed{0}$	3	2	6	4
$\cancel{0}$	$\boxed{0}$	3	6	8
7	8	4	3	$\boxed{0}$
1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

The final solution as follows:

1	4	7
2	1	12
3	2	11
4	5	12
5	3	0
		42

Unbalanced Assignment with Maximization

Example 1: A production manager wants to assign one of the five new methods to each of the four operations. The following table summaries the weekly output in units:

Weekly Output

Operator	M_1	M_2	M_3	M_4	M_5
A	2	4	9	14	7
B	3	6	14	17	7
C	7	11	19	19	11
D	4	4	7	9	5

Cost per unit is Rs. 20, selling price per unit is 30. Find maximum Profit per month.

Solution: Since the problem is an unbalanced assignment problem, hence we make it balance problem by introducing a dummy row with zero cost coefficients:

Operator		M_1	M_2	M_3	M_4	M_5
	A	2	4	9	14	7
	B	3	6	14	17	7
	C	7	11	19	19	11
	D	4	4	7	9	5
	Dummy	0	0	0	0	0

Since the given problem is a maximisation problem, we convert it into minimisation problem by selecting the highest element which is 19 and subtracting all elements of maximisation problem matrix from this selected highest element, we get the modified minimisation assignment problem as:

	M_1	M_2	M_3	M_4	M_5
A	17	15	10	5	12
B	16	13	5	2	12
C	12	8	0	0	8
D	15	15	12	10	14
Dummy	19	19	19	19	19

Row Reduction :- Subtracting the smallest element of each row from all the element of that row, we get modified matrix as :

	M_1	M_2	M_3	M_4	M_5
A	12	10	5	0	7
B	14	11	3	0	10
C	12	8	0	0	8
D	5	5	2	0	4
Dummy	0	0	0	0	0

Drawing minimum number of horizontal and vertical lines to cover the zeros of above matrix, we get.

	M_1	M_2	M_3	M_4	M_5
A	12	10	5	0	7
B	14	11	3	0	10
C	12	8	0	0	8
D	5	5	2	0	4
Dummy	0	0	0	0	0

Since the number of lines is less than the order of matrix, hence subtract smallest element 4 in uncovered element. So the modified matrix is

	M_1	M_2	M_3	M_4	M_5
A	8	6	5	0	3
B	10	7	3	0	6
C	8	4	0	0	4
D	1	1	2	0	0
Dummy	0	0	4	4	0

Again the smallest element is 3, we have the modified matrix is

	M_1	M_2	M_3	M_4	M_5
A	5	3	2	0	0
B	7	4	0	0	3
C	8	4	0	3	4
D	1	1	2	3	0
Dummy	0	0	4	7	0

again the number of line are 4 as we need number of lines 5, smallest element is 1, subtracting the element from the all uncovered element and add in the intersection of horizontal and vertical lines, we get the modified matrix as:

	M_1	M_2	M_3	M_4	M_5
A	4	2	2	~~0~~	$\boxed{0}$
B	6	3	~~0~~	$\boxed{0}$	3
C	7	3	$\boxed{0}$	3	4
D	$\boxed{0}$	~~0~~	2	3	~~0~~
Dummy	~~0~~	$\boxed{0}$	5	8	1

Hence all assignments are done, the optimal solution is:

Operator	Method	Production Per Week
A	M_5	7
B	M_4	17
C	M_3	19
D	M_1	4
Dummy	M_2	0
Total Production Per Week		47

Production per month = 47×4 = 188 units

Profit per unit = Selling Price per unit - Cost Price Per unit

= 30 - 20 = Rs. 10

= Rs. 1880

Self Check Exercise:-

Q.1 Five different machine can do any of the five require jobs, with different profit resulting from each assignment.

Machines

		A	B	C	D	E
Jobs	1	20	27	30	18	30
	2	30	14	17	11	26
	3	30	22	23	20	25
	4	15	28	30	26	26
	5	19	52	31	24	29

Find the assignment which maximizes the profit.

Q.2 Solve the maximization assignment problem

Machine

Operations	A	B	C	D
1	11	6	8	9
2	12	5	10	11
3	9	5	10	8
4	8	6	7	5
5	9	10	8	6

8.4 Summary:

- An unbalanced assignment problem occurs when the number of workers does not equal the number of tasks, making direct assignment impossible without adjusting the problem by adding dummy workers or tasks to balance the equation.
- A maximization assignment problem aims to maximize the total profit or benefit of assignments rather than minimizing costs. It involves assigning tasks to worker in a way that maximizes the overall benefit, often using algorithms like the Hungarian method adopted for maximization criteria.

8.5 Glossary:

1. **Unbalanced assignment Problem:-** An assignment problem, where the number of agents (or tasks) does not equal the number of tasks (or agents), making it impossible to assign each agent to exactly one task without leaving some agents or tasks unassigned.
2. **Dummy Row/Column:-** In an unbalanced assignment problem, a dummy row or column is added to balance the matrix by either adding extra tasks or agents with zero cost.
3. **Maximization Assignment Problem:-** An assignment problem where the objective is to maximize the total profit or benefit instead of minimize cost.

8.6 Answer to Self Check Exercise

Q.1 1 → C, 2 → E, 3 → A, 4 → D, 5 → B

Maximum Profit = 164

Q.2 1 → A, 2 → D, 3 → C, 4 → Dummy, 5 → B

Maximum Production = 42

8.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

8.8 Terminal Questions

Q.1 A company has 4 machines on which to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table:

Machine

		W	X	Y	Z
Jobs	A	28	34	38	42
	B	18	23	27	29
	C	20	25	29	32

What are the job assignments which will minimize the cost?

Q.2 A company has a team of four salesmen and there are four districts where the company wants to start the business. After taking into accounts the capabilities of salesman and the nature of the districts the company estimates that the profit per day in rupees for each salesman in each district is ad below:

District

		1	2	3	4
Salesman	A	11	5	9	6
	B	9	6	10	10
	C	10	10	8	7
	D	8	7	9	10

Unit - 9

Special Assignment Problem-1

Air Crew Problem

Structure

9.1 Introduction

Dear student, in this unit we will study one special type of assignment problem known as Air Line crew problem (ACP). Airline crew problem. An airline company needs to determine the specific rosite flown by each airplane in the most cost effective way, from a given sets of flight to be flown. So, airline company should try to use as fewer airplanes as possible, but the same airplane can operate two subsequent flights only if the time interval between the arrival of the first flight and the departure of the next flight is longer than or equal to an hour. So the task of the airline operations team is to determine the minimum number of airplanes needed to operate the given list of flights. This is known as airline crew problem. In this unit we will study about this problem and use Hungarian method to solve airline crew problem.

9.2 Learning Objectives:

After studying this unit students will be able to

1. defineair line crew problem
2. make cost matrix for air crew problem
3. solve air crew problem using Hungarian method.

9.3 Air Crew Assignment Problem

Basic Terminology:-

1. **Tasks (Flights):** Each flight represents a task that needs to be assigned to a crew member. Flights have specific requirements such as departure times, duration, and destination.

2. **Resources (Air Crew Members):** The crew members are the resources available to perform the tasks (flights). Each crew member has constraints such as maximum flight hours allowed, legal rest periods between flights, and skill sets.

3. **Objective:** The objective is to assign flights to crew member is such a way that operational costs (like overtime pay or accommodation costs) are minimized, or operational efficiency (like minimizing delays or maximizing crew satisfaction) is maximized.

4. **Constraints:** Constraints may include crew availability, legal regulations (such as maximum flying hours per day or week) crew preferences, and logistical constraints (like aircraft type and availability).

Steps Involved In Aircrew Assignment Problem

1. From the given timetable of airline, find the layover time matrix for both the destinations.
2. Find the minimum layover time matrix.
3. Applying the Hungarian method as discussed earlier units.

Let us try following examples to have more understanding of Air crew problem.

Example 1: Prachi airlines that operates seven days a week has a time table shown below. Crews must have a minimum layover of 6 hours between flights. Obtain the pairing of flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the samller layover.

Flight No	ChannaiDeparture	Mumbai Airival	Flight No.	Mumbai Departure	ChannaiAirival
1	7 am	9 am	101	9.00 am	11 am
2	9 am	11 am	102	10 am	12 noon
3	1:30 pm	3:30 pm	103	3:30 pm	5:30pm
4	7:30 pm	9:30 pm	104	8 pm	10 pm

For each pair also mention the town where the crew should be based?

Solution: To determine the optimal assignment we first have to calculate layover times from the above timetable.

Calculating values i.e. layover time when crew is based at Channai:-

We start with arrival time at Mumbai and end with departure time at Mumbai. So

Ist Row Cell 1: Arrival time at Mumbai = 9:00 am

Departure time at Mumbai = 9:00 am

∴ Waiting time = difference between arrival and departure with layover time 6 hours = 24 hrs.

Cell 2: Arrival time at Mumbai = 9:00 am

Departure time at Mumbai = 10 am

Waiting time = 25 hrs

Cell 3: Arrival the at Mumbai = 9:00 am

Departure time at Mumbai = 3:30 pm

Waiting time = 6.5 hr.

Cell 4: Arrival time at Mumbai = 9:00 am

Departure time at Mumbai = 8 pm

Difference = 11 hr

Row 2. Cell 1: Arrival time at Mumbai = 11 am

Departure time at Mumbai = 9 am

Difference = 22 hr

Cell 2: Arrival time at Mumbai = 11 am

Departure time at Mumbai = 10 am

Difference = 23 hr.

Cell 3: Arrival time at Mumbai = 11 am

Departure time at Mumbai = 3:30 pm

Difference = 28.5hr.

Cell 4: Arrival time at Mumbai = 11 am

Departure time at Mumbai = 8 pm

Difference = 9 hr

Row 3.

Cell 1: Arrival time at Mumbai = 3:30 pm

Departure time at Mumbai = 9 am

Difference = 17.5 hr

Cell 2: Arrival time at Mumbai = 3:30 pm

Departure time at Mumbai = 10 am

Difference = 18.5 hr.

Cell 3: Arrival time at Mumbai = 3:30 pm

Departure time at Mumbai = 3:30 pm

Difference = 24 (Q layover time is 6 hours)

Cell 4: Arrival time at Mumbai = 3:30 pm

Departure time at Mumbai = 8 pm

Difference = 28.5 hr.

Row 4.

Cell 1: Arrival time at Mumbai = 9:30 pm

Departure time at Mumbai = 9 am

Difference = 11.5 hr.

Cell 2: Arrival time at Mumbai = 9:30 pm

Departure time at Mumbai = 10 am

Difference = 12.5 hr

Cell 3: Arrival time at Mumbai = 9:30 pm

Departure time at Mumbai = 3:30 pm

Difference = 18 hr.

Cell 4: Arrival time at Mumbai = 9:30 pm

Departure time at Mumbai = 8 pm

Difference = 22.5 hr.

Therefore. Crew based at Chennai has layover matrix as

	101	102	103	104
1	24	25	6.5	11
2	22	23	28.5	9
3	17.5	18.5	24	28.5
4	11.5	12.5	18	22.5

Now calculating layover time for crew based at Mumbai

We follow the same process as above. Now, we will start with arrival time at Chennai and end with departure time at Chennai. Like

Row 1

Cell 1: Arrival time at Channai = 11 am

Departure time at Chennai = 7 am

Difference = 20 hr.

Cell 2: Arrival time at Channai = 11 am

Departure time at Channai = 9 am

Difference = 22 hr

Similarly we will find other layover times, and hence find the Layover time matrix for crew based at Mumbai.

If crew based at Mumbai the Layover time matrix is

	101	102	103	104
1	20	19	13.5	9
2	22	21	15.5	11
3	26.5	25.5	20	15.5
4	8.5	7.5	26	21.5

Minimum Layover Time:-

We will select the entry from both the table and note the minimum one, and mark m or c or Mumbai and Channai For cell (iii) = min (24, 20)

= 20

20 is for Mumbai so we mentioned 20 m.

	101	102	103	104
1	20^{m}	19^{m}	6.5^{c}	9^{m}
2	$22^{c/m}$	21^{m}	15.5^{m}	9^{c}
3	17.5^{c}	18.5^{c}	20^{m}	15.5^{m}
4	8.5^{m}	7.5^{m}	18^{c}	21.5^{m}

M denotes Mumbai based crew, c denotes Chennai based crew.

Applying assignment technique No:-

Now will we apply same Hungarian method to the minimum layover time matrix.

Row Reduction:-

	101	102	103	104
1	13.5	12.5	0	2.5
2	13	12	6.5	0
3	2	3	4.5	0
4	1	0	10.5	14

Column Reduction:

	101	102	103	104
1	12.5	12.5	0	2.5
2	12	12	6.5	0
3	1	3	4.5	0
4	0	0	10.5	14

No. of Lines $\neq$ no of row or columns, so least element is L.

	101	102	103	104
1	12.5	12.5	$\boxed{0}$	3.5
2	12	11	5.5	$\boxed{0}$
3	$\boxed{0}$	2	3.5	$\cancel{0}$
4	$\cancel{0}$	$\boxed{0}$	10.5	13

So optimal solution is:-

1 $\rightarrow$ 103, 2 $\rightarrow$ 104, 3 $\rightarrow$ 101, 4 $\rightarrow$ 102.

$\therefore$ Minimum hours = 6.5 + 9 + 17.5 + 7.5 = 40.5 hours

Example 2: A trip from Madras to Bangalore takes six hours by bus. A table of the bus service in both directions is given as

Departure from Madras	Route Number	Arrival at Bangalore	Arrival at Madras	Route Number	Departure from Bangalore
6.00	a→	12.00	11.30	←1	5.30
7.30	b→	13.30	15.00	←2	09.00
11.30	c→	17.30	21.00	←3	15.00
19.00	d→	01.00	00.30	←4	18.30
00.30	e→	06.30	06.00	←5	00.00

The cost of providing this service by the transport depends upon the time spent by the bus crew away from their places in addition to the service time. There are 4 crews. There is a constraint that every crew should be provided with more than 4 hours of rest before return trip and should not wait. For more than 24 hours for the return trip. The company has residential facility for the crew at both stations. Suggest on optimal assignment of crew.

Solution: Crew Based at Madras

	1	2	3	4	5
a	17.5	21	-	6.5	12
b	16	19.5	-	5	10.5
c	12	15.5	21.5	-	6.5
d	4.5	8	14	17.5	23
e	23	-	8.5	12	17.5

Crew Based at Bangalore

	1	2	3	4	5
a	18.5	15	9	5.5	24
b	20	16.5	10.5	7	-
c	24	20.5	14.5	11	5.5
d	7.5	-	22	18.5	13
e	13	9.5	-	24	18.5

Minimum layover time

	1	2	3	4	5
a	17.5	15	9	5.5	12
b	16	16.5	10.5	5	10.5
c	13	15.5	14.5	11	5.5
d	4.5	8	14	17.5	13
e	13	9.5	8.5	12	17.5

Applying the Hungarian metho,

Raw Reduction :

	1	2	3	4	5
a	12	9.5	3.5	0	6.5
b	11	11.5	5.5	0	5.5
c	6.5	10.0	9.0	5.5	0
d	0	3.5	9.5	12	8.5
e	4.5	1	0	3.5	9

Column subtraction and Assignment

	1	2	3	4	5	
a	12.5	8.5	3.5	$\boxed{0}$	6.5	√
b	11	10.5	5.5	~~0~~	6.5	√
c	6.5	9	9	5.5	0	
d	0	2.5	9.5	13	8.5	
e	4.5	0	0	3.5	9	
				√		

Modified Matrix

	1	2	3	4	5
a	8.5	5.0	$\boxed{0}$	$\cancel{0}$	3
b	7.5	4.0	2.0	$\boxed{0}$	2
c	6.5	9.0	9.0	9.0	$\boxed{0}$
d	$\boxed{0}$	2.5	9.5	16.5	8.5
e	4.5	$\boxed{0}$	0	7	9

Since all assignment have been made, hence optimum solution is.

Route Paired	Residence at	Waiting Time
a-3	Bangalore	9
b-4	Madras	5
c-5	Bangalore	5.5
d-1	Madras	4.5
e-2	Bangalore	9.5
		33.5 hours

Example 3: XYZ air line operating 7 days a week has given the following time table. Crew must have a minimum layover of 5 hours between flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller lay over.

Hyderabad-Delhi			Delhi-Hyderabad		
Flight No.	Depart	Arrive	Flight Number	Depart	Arrive
A_1	5 AM	7AM	B_1	7 AM	9 AM
A_2	7 AM	9 AM	B_2	8 AM	10 AM
A_3	1 PM	3 PM	B_3	1 PM	3 PM
A_4	7 PM	9 PM	B_4	6 PM	8 PM

Solution: If the crew based at Hyderabad

	B_1	B_2	B_3	B_4
A_1	24	25	6	11
A_2	23	23	28	9
A_3	16	17	22	27
A_4	10	11	16	21

If crew is based at Delhi

	B_1	B_2	B_3	B_4
A_1	20	19	14	9
A_2	22	21	16	11
A_3	28	27	22	17
A_4	10	9	28	23

Minimum lay our table.

	B_1	B_2	B_3	B_4
A_1	20	19	6	9
A_2	22	21	16	9
A_3	16	17	22	17
A_4	10	9	16	21

Now, Hungarian method has been applied on this problem.

Row Reduction

	B_1	B_2	B_3	B_4
A_1	14	13	$\boxed{0}$	3
A_2	13	12	7	$\boxed{0}$
A_3	$\boxed{0}$	1	6	1
A_4	1	$\boxed{0}$	7	12

So, all the rows and column have zero. So after making assignments
The optimum assignment schedule

Fight Pair		City of which crew is based	Layover
A1	B3	Hyderabad	6
A2	B4	Hyderabad	9
A3	B1	Hyderabad	16
A4	B2	Delhi	9
			40 hrs.

Self Check Exercise:-

Q.1 Solve the given problem, minimum layover of hours between flight is 5 hours.

Crew assignment problem

Delhi - Mumbai			Mumbai - Delhi		
Fight No.	Departure	Arrival	Fight No.	Departure	Arrival
1	7.00	8.00	101	8.00	9.00
2	8.00	9.00	102	09.00	10.00
3	13.00	14.00	103	12.00	13.00
4	18.00	19.00	104	17.00	18.00

Q.2 Solve the given problem, minimum layover of hours between flights is 4 hours.

Shimla $\rightarrow$Patana			Patana$\rightarrow$ Shimla		
Fight No.	Departure	Arrival	Fight No.	Departure	Arrival
a	06.00	12.00	1	05.30	11.30
b	07.30	13.30	2	09.00	15.00
c	11.30	17.30	3	15.00	21.00
d	19.00	01.00	4	18.30	00.30
e	00.30	06.30	5	00.00	06.00

9.4 Summary:-

The Air crew Assignment problem involves assigning flight crews to flights in a way that minimizes costs or maximizes efficiency while meeting operational constraints. It typically considers factors like crew availability, qualifications flight schedules, and operational requirements to optimize crew assignments and ensure smooth airline operations.

9.5 Glossary:

1. **Air Crew Scheduling Problem:-** A type of optimization problem where the objective is to assign flight duties to air crew members while satisfying various constraints such s legality requirements, crew preferences, and operational rules.

2. **Crew Pairing Problem:-** The problem of generating optimal pairing or rotations for crew members to cover flight schedules efficiently while minimizing costs or maximizing utilization.

3. **Duty Time:-** The total time a crew member is on duty, including flight time, ground time and rest periods.

9.6 Answer to Self Check Exercise

Q.1 $1 \rightarrow 103$, $2 \rightarrow 104$, $3 \rightarrow 101$, $4 \rightarrow 102$

Total hours = 52

Q.2 $a \rightarrow 3$, $b \rightarrow 4$, $c \rightarrow 5$, $d \rightarrow 1$, $e \rightarrow 2$

Total hours = 33.5

9.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.

2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.

3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.

4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt., Ltd., Second Edition.

9.8 Terminal Questions

Q.1 An Aeroplane that operates 7 days a week follows time table as given. Crew must 6 hours layover time before starting a second flight. Solve the given problem.

Delhi - Sringar			Srinagar - Delhi		
Fight No.	Departure	Arrival	Fight No.	Departure	Arrival
1	7 AM	8 AM	5	8 AM	9 AM
2	8 AM	9 AM	6	9 AM	10 Am
3	2 PM	3 PM	7	12 noon	1 PM
4	7 PM	10 PM	8	5 PM	6 PM

Q.2 Solve the given problem. Given that a minimum layover of 6 hours between flights.

Delhi - Kolkata			Kolkata - Delhi		
Fight No.	Departure	Arrival	Fight No.	Departure	Arrival
1	7 AM	9 AM	101	9 AM	11 AM
2	9 AM	11 AM	102	10 AM	12 Noon
3	1.30 PM	3.30 PM	103	3.30 PM	5.30 PM
4	7.30 PM	9.30 PM	104	8 PM	10.00 PM

Unit - 10

Special Assignment Problem 2
Traveling Salesman Problem

Structure

10.1 Introduction

Dear student in this unit we will study about one another type of assignment problem i.e. travelling salesman problem. In travelling salesman problem we have to find the shortest possible route that a salesman have to visit to a set of cities and return back to the starting city. This can be understood easily as a news paper agent daily drops the newspapers to a assigned area in such a manner that he has to cover all the houses in that respected area with minimum travel cost. In this unit we will learn how to solve problems of travelling salesman. The route of salesman should be selected in such a manner that no city is visited twice and his total travelling time or distance is minimized. In travelling salesman problem the diagonal elements of cost matrix are represented by - or ∞, as the diagonal element represents the same city or destination i.e. c_{11} c_{22} c_{33} or c_{44} etc. We find the optimal solution of teavelling salesman problem by using Hungarian method.

10.2 Learning Objectives:

After studying this unit, students will be able to

1. define travelling salesman problem
2. find the optimal solution of travelling salesman problem

10.3 Travelling Salesman Problem

Statement: Given a set of cities and the distances between every pair of cities, the objective is to find the shortest possible route that visits each city exactly once and returns to the origin city (the starting point).

Key elements:

Cities: Represented as nodes in a graph, where each note (city) is connected to every other node by an edge whose weight represents the distance between the cities.

Routes: The problem seeks to find a Hamilton cycle (a closed loop) that visits every city exactly once and return to the starting city.

Objective: Minimize the total distance (or travel time) required to complete the tour.

Travelling Salesman Problem:

The travelling salesman problem can be solved liked an assignment problem with two additional conditions on the choice of an assignment i.e.

1. How should travelling salesmen travel starting from his home city. Assuming a salesman has to visit n cities. He wants to start from a particular city say A, visit each city once, and
2. returning to his home city (i.e. starting point A) without visiting single city twice.

His main purpose is to select the sequence in which the cites are visited in such a manner that his total travelling time or distance is minimized.

For example, given n cities and distance d_{ij} (cost of c_{ij} or time t_{ij}) from city i to city j, the salesman starts from city 1, then any permutation of 2, 3, ..., n represents the number of possible ways for his tour. To visit 2 cities (A and B) there is no choice. To visit 3 cities we have 2! possible routes. For 4 cities we have 3! possible routes. In general to visit n cities there are (n-1)! possible routes.

In travelling salesman problem, we cannot choose the elements along the diagonal and this can be avoided by filling the diagonal with infinitely large elements.

The travelling salesman problem is very similar to the assignment problem excepts that in the former case, there is an additional restriction, that x_{ij} is so chosen that no city is visited twice before the tour of all the cities is completed.

Example 1: Let us try following examples for travelling salesman problem solve the following travelling salesman problem so as to minimize the cost per cycle:

From/To

	A	B	C	D	D
A	-	5	8	4	5
B	5	-	7	4	5
C	8	7	-	8	6
D	4	4	8	-	8
E	5	5	6	8	-

Solution: Row Reduction:

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & D \\
\hline
A & \infty & 1 & 4 & 0 & 1 \\
B & 1 & \infty & 3 & 0 & 1 \\
C & 2 & 1 & \infty & 2 & 0 \\
D & 0 & 0 & 4 & \infty & 4 \\
E & 0 & 0 & 1 & 3 & \infty \\
\hline
\end{array}
$$

Colum Reduction

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & D \\
\hline
A & \infty & 1 & 3 & 0 & 1 \\
B & 1 & \infty & 2 & 0 & 1 \\
C & 2 & 1 & \infty & 2 & 0 \\
D & 0 & 0 & 3 & \infty & 4 \\
E & 0 & 0 & 0 & 3 & \infty \\
\hline
\end{array}
$$

Draw the minimum number of honzontal and vertical lines to cover all the zeros of above matrix after making rows and columns and drawing lines through unmarked rows and marked columns.

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & D \\
\hline
A & \infty & 0 & 3 & 0 & 1 \\
B & 1 & \infty & 2 & 0 & 0 \\
C & 2 & 1 & \infty & 2 & \boxed{0} \\
D & \boxed{0} & 0 & 3 & \infty & 4 \\
E & 0 & 0 & \boxed{0} & 3 & \infty \\
\hline
\end{array}
$$

$\therefore$ the smallest element is :

	A	B	C	D	D
A	∞	0	3	0	0
B	1	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	5
E	0	0	0	4	∞

The optimum assignment is A → B, B → D, D → A and not feasible (as the salesman is visiting home only A without visit C and E). Since the assignment schedule does not provide us the solution of travelling Salesman problem, hence we make a better solution by considering the next higher non-zero element by considering 1, we have

	A	B	C	D	D
A	∞	0	2	0	0
B	0	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	4
E	0	0	0	4	∞

The optimum assignment travelling schedule is

A → D →B → C → E → A and minimum cost = 4 + 4 + 7 + 6 + 5 = Rs. 26.

OR

	A	B	C	D	D
A	∞	0	2	0	0
B	0	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	4
E	0	0	0	4	∞

The alternate optimum assignment travelling schedule is:

$A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$

and minimum travelling cost = 5 + 6 + 7 + 4 + 4 = Rs 26.

Example 2: Solve the following Salesman problem given by the following data:

$C_{12} = 18$, $C_{13} = 2$, $C_{14} = 8$, $C_{23} = 3$, $C_{34} = 4$, $C_{25} = 8$, $C_{35} = 4$, $C_{45} = 8$.

Where $C_{ij} = C_{ji}$ and there is no route between cities i and j if a value of cij is not known.

Solution: The given problem can be expressed in the form of an assignment problem by putting the values given in the problem in the revert cell, we get:

	1	2	3	4	5
1	∞	10	2	8	∞
2	18	∞	3	∞	8
3	2	3	∞	4	4
4	8	∞	4	∞	18
5	∞	8	4	18	∞

Row Reduction:

	1	2	3	4	5
1	∞	16	0	6	∞
2	15	∞	0	∞	5
3	0	1	∞	2	2
4	4	∞	0	∞	14
5	∞	4	0	14	∞

Column Reduction:

	1	2	3	4	5
1	∞	15	0	4	∞
2	15	∞	0	∞	3
3	$\boxed{0}$	0	∞	0	0
4	4	∞	0	∞	12
5	∞	3	0	12	∞

Since the smallest element is 3. So modified matrix is.

	1	2	3	4	5
1	∞	12	0	1	∞
2	15	∞	0	∞	$\boxed{0}$
3	$\boxed{0}$	0	∞	0	0
4	1	∞	0	∞	9
5	∞	0	$\boxed{0}$	9	∞

Also Row 4 has no assignment. The smallest element is 1.

The modified matrix is

	1	2	3	4	5
1	∞	11	$\boxed{0}$	0	∞
2	12	∞	1	∞	$\boxed{0}$
3	$\boxed{0}$	0	∞	$\boxed{0}$	0
4	0	∞	0	∞	8
5	∞	$\boxed{0}$	1	9	∞

Since all assignment have bee made, hence optimum route is $1 \rightarrow 3 \rightarrow 4 \rightarrow 1$ which is not feasible because Salesman return to city 1 without visiting cities 2 and 5. Hence we select next higher number to zero which is 1 and make assignment on this selected number, we get:

	1	2	3	4	5
1	∞	11	0	$\boxed{0}$	∞
2	12	∞	$\boxed{1}$	∞	0
3	0	0	∞	0	0
4	$\boxed{0}$	∞	0	∞	8
5	∞	$\boxed{0}$	1	9	∞

The revised assignment schedule is $1 \rightarrow 4 \rightarrow 1$ which is not feasible. We select next higher number to 1 which is 8. Making assignment on the basis of this number, we get:

	1	2	3	4	5
1	∞	11	0	0	∞
2	12	∞	$\boxed{1}$	∞	0
3	$\boxed{0}$	0	∞	0	0
4	0	∞	0	∞	$\boxed{8}$
5	∞	$\boxed{0}$	1	9	∞

hence the optimum assignment schedule is

$1 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 1$

and minimum total cost = 8 + 8 + 8 + 3 + 2 = Rs. 39.

Q.8 Solve the following travelling salesman problem so as to minimize the cost per cycle

	A	B	C	D	E
A	-	3	6	2	3
B	3	-	5	2	3
C	6	5	-	6	4
D	2	2	6	-	6
E	3	3	4	6	-

Sol: how reduced matrix

∞	1	4	0	1
1	∞	3	0	1
2	1	∞	2	0
0	0	4	∞	4
0	0	1	3	∞

Row reduced matrix

∞	1	3	$\boxed{0}$	1
1	∞	2	0	1
2	1	∞	2	$\boxed{0}$
$\boxed{0}$	0	3	∞	4
0	0	$\boxed{0}$	3	∞

Column reduced matrix

Here smallest uncovered element = 1

	A	B	C	D	E
A	∞	$\boxed{0}$	2	0	0
B	0	∞	1	$\boxed{0}$	0
C	2	1	∞	3	$\boxed{0}$
D	$\boxed{0}$	0	3	∞	4
E	0	0	$\boxed{0}$	4	∞

Optimal assignment is

A → B, B → D, D → A

and not feasible (as the salesman is visiting home only A without visiting C and E). Since the assignment schedule does not provide us the solution of travelling salesman problem, hence, we make a better solution by considering the next higher non-zero element by considering 'I' we get

	A	B	C	D	E
A	∞	0	2	0	0
B	0	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	4
E	0	0	0	4	∞

The optimum assignment travelling schedule is A → D → B - C - E - A and minimum cost = 2 + 2 + 5 + 4 + 3 = Rs 16.

	A	B	C	D	E
A	∞	0	2	0	0
B	0	∞	1	0	0
C	2	1	∞	3	0
D	0	0	3	∞	4
E	0	0	0	4	∞

The alternate optimum assignment travelling schedule is A → E → C → B - D - A and minimum travelling cost = 3 + 4 + 5 + 2 + 2 + = Rs 16

Question 9: A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set up cost per change depends on the items presently on the machine and the set up to be made according to the following table

	A	B	C	D	E
A	∞	40	70	30	50
B	40	∞	60	30	40
C	70	60	∞	70	50
D	30	30	70	∞	70
E	40	40	50	70	∞

If he processes each type of item once and only once in each week, how should he sequence the items an his machine in order to minimise the total set up cost.

Q. 10: Solve

	A	B	C	D	E
A	∞	2	5	7	1
B	6	∞	3	8	2
C	8	7	∞	4	7
D	12	4	6	∞	5
E	1	3	2	8	∞

Solution: Row reduced matrix

∞	1	4	6	0
4	∞	1	6	0
4	3	∞	0	3
8	0	2	∞	1
0	2	1	7	∞

Column reduced matrix

∞	1	3	6	0	A-E
4	∞	0	6	0	B-C
4	3	∞	0	3	C-D
8	0	1	∞	1	D-B
0	2	0	7	∞	E-A

The optimal assignment travelling schedule is A $\rightarrow$ E $\rightarrow$ A and not feasible.

∞	1	3	6	0
4	∞	0	6	0
4	3	∞	0	3
8	0	1	∞	1
0	2	0	7	∞

Considering the next higher non-zero element by considering 1

The optimal assignment travelling schedule is A $\rightarrow$ B, B $\rightarrow$ C, C $\rightarrow$ D, D - E, E - A and minimum travelling cost is 2 + 3 + 4 + 5 + 1 = Rs 15.

Q.11 A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to the starting point. Cost of going from one city to another is shown below. You are required to find the best cost route.

	A	B	C	D	E
A	∞	4	10	14	2
B	12	∞	6	10	4
C	16	14	∞	8	14
D	24	8	12	∞	10
E	2	6	4	16	∞

Solution: Raw reduced Matrix

	A	B	C	D	E
A	∞	2	8	12	0
B	8	∞	2	6	0
C	8	6	∞	0	6
D	16	0	4	∞	2
E	0	4	2	12	∞

Column reduced matrix

	A	B	C	D	E
A	∞	2	6	12	0
B	8	∞	0	6	2
C	8	6	∞	0	6
D	16	0	2	∞	2
E	0	4	0	12	∞

The optimum assignment is A $\to$ E, B $\to$ C, C $\to$ D, D $\to$ B, E $\to$ A

However, sol. is not feasible we make a better sol. by considering the next higher non-zero element, by considering 2 and making assignment

	A	B	C	D	E
A	∞	2	6	12	0
B	8	∞	0	6	0
C	8	6	∞	0	6
D	16	0	2	∞	2
E	0	4	0	12	∞

Optimum assignment travelling schedule is A $\to$ B, B $\to$ C, C $\to$ D, D $\to$ E E$\to$ A

$\therefore$ Total minimum travelling cost = 4 + 6 + 8 + 10 + 2 = 30

Q.12 A salesman has to visit five cities A, B, C, D and E. The distance between the five cities are as follows.

$$
\begin{array}{c|ccccc|}
 & A & B & C & D & E \\
A & - & 7 & 6 & 8 & 4 \\
B & 7 & - & 8 & 5 & 6 \\
C & 6 & 8 & - & 9 & 7 \\
D & 8 & 5 & 9 & - & 8 \\
E & 4 & 6 & 7 & 8 & -
\end{array}
$$

If the salesman start from city A and has to come back to city A which route should he select so that the total distance travelled is minimum.

Solution: Row reduced matrix

∞	3	2	1	0
2	∞	3	0	1
0	2	∞	3	1
3	0	4	∞	3
0	2	3	4	∞

Column reduced matrix

∞	3	$\boxed{0}$	4	0
2	∞	1	$\boxed{0}$	1
$\boxed{0}$	2	∞	3	1
3	$\boxed{0}$	2	∞	3
0	2	1	4	∞

Modified matrix

	A	B	C	D	E
A	∞	3	0	4	0
B	3	∞	1	0	1
C	0	1	∞	2	0
D	4	0	2	∞	3
E	0	1	0	3	∞

A - E - C - A

Which is not feasible

	A	B	C	D	E
A	∞	3	0	4	0
B	3	∞	1	0	1
C	0	1	∞	2	0
D	4	0	2	∞	3
E	0	1	0	2	∞

Optimum assignment travelling schedule is A → C → D → B → E → A Min. Distance = 6 + 6 + 9 + 5 + 4 = 30

Self Check Exercise

Q.1: Solve the following travelling Salesman problem

	A	B	C	D	E
A	-	6	12	16	4
B	14	-	8	12	6
C	18	16	-	10	16
D	26	10	14	-	12
E	4	8	6	18	-

Traveling cost = Rs. 40

Q.2: Solve the following problem:

	A	B	C	D	E
A	-	17	16	18	14
B	17	-	18	15	16
C	16	18	-	19	17
D	18	15	19	-	18
E	14	16	17	18	-

10.4 Summary:

The traveling Salesman problem (TSP) is a classic optimization challenge where a salesman must visit a set of given cities exacity once and return to the starting city, aiming to minimize the total travel distance or time. It's crucial problem in computer science and operation research, with applications ranging from logistics to circuit design. Finding the most efficient route remains computationally intense due to its NP-hard nature, necessitating various algorithms for practice solutions.

10.5 Glossary:

- **Traveling Salesman Problem:** The classic optimization problem where a salesman seeks the shortest route that visit each city exactly once and returns to the origin city.
- **Cost/distances:** The measure of distance or cost between two cities in the traveling salesman problem, usually represented by a distance matrix.
- **Objective function:** The function that is to minimized in the traveling salesman problem, typically the total distance traveled.

10.6 Answer to Self Check Exercise

Q.1: A → B → C → D → E → A

Travelling cost = Rs 40.

Q.2: A → C → D → B → E → A

Travelling cost = Rs 80.

10.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.

3. JK. Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition

4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

10.8 Terminal Questions

Q. 1: Solve the travelling Salesman problem in the matrix shown below

	I	II	III	IV	V
A	∞	2	15	27	1
B	18	∞	3	15	2
C	16	15	∞	4	17
D	20	4	6	∞	5
E	1	3	2	18	∞

Q.2 A Salesman travels from one place to another; The distance between pairs of cities are given below:

	P	Q	R	S
P	-	15	25	20
Q	22	-	45	55
R	40	30	-	25
S	20	26	38	-

Find out the total distance covered by Lim. which is minimum.

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Unit - 11

Game Theory-Maximin+Minimax Principle

Structure

11.1 Introduction

Dear student in this unit we will study about game theory. In our day to day life we see there are competitive situations like sports, business completion, elections, marketing campaigns etc. Every person involved in such competitive situation wants to maximize his gain or minimize his less. Such competitive situations are known as games and the persons involved in game is known as prayers. In order to maximize gain or minimize loose a player use his technique which are known as strategies. So game theory deals with the competitive situations where one must win and other must lose. In this unit we will study about the basics of game theory and minimax or maximin principle to solve a game.

11.2 Learning Objectives:

After studying this unit, students will be able to

1. define game and game theory
2. define basic terminology related to game theory
3. defineSadde point of a game
4. definemaximin - minimax principle of game theory
5. solve game using maximin or minimax principle
6. define and find value of a game

11.3 Game And Game Theory

Basic Terminology

- **Game:** A competitive situation in which persons have conflict objectives is called competitive game.

- **Player:** Each participant in the game is called player.
- **Fair game:** If the value of game is zero, the game is sold to be fair game
- **Two person zero sum game:** The gain (loss) of one person is equal to the loss (gain) of another, such that the sum of gain and losses of the game is zero. In this, algebraic sum of gains and losses of both the player is equal to zero. Two person zero sum game is called rectangular game as it is usually represented by a pay-off matrix in rectangular form.
- **n-person game:** A game involving 'n' players is called a 'n-peron' game.
- **non-zero sum game:** A non-zero sum game is one in which the sum of the pay ofts from any player of the game may be +ve or -ve but not zero.
- **Strategy:** The term 'Strategy' refers to the total pattern of choice employed by any player. It is a rule for decision making in advance of all the plays by which he decides the activities he should adopt.
- **Pure Strategy:** It players use the same strategy throughout the play of game it is called pure strategy.
- **Mixed Strategy:** If a player uses the combinations of available strategies in some probability distribution it is called mixed strategy.
- **Optimum Strategy:** A play which puts a player in a most preferred position, irrespective of the strategy of his competitors is called optimum strategy.
- **Pay off:**Pay off is the outcome of playing the game.
- **Pay-off matrix:** A payoff matrix is a matrix showing the amount received by the player called maximizing player named at left hand side and the payment is made by the player called minimizing player name at the top of the table.

Mathematical Representation of Payoff Matrix

If a player m-courses of action and player B has n-courses of action then a payoff matrix may be constructed as

A's pay off matrix.

$$\text{Player A}\quad \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & \ldots & j & \ldots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{ij} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2j} & \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots & a_{3j} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix} \end{array}$$

B's Payoff Matrix

$$\text{Play of A}\quad \begin{array}{cc}
 & \begin{matrix} 1 & 2 & 3 & \cdots & j & \cdots & n \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ i \\ \vdots \\ m \end{matrix} &
\begin{bmatrix}
-a_{11} & -a_{12} & -a_{13} & \cdots & -a_{ij} & \cdots & -a_{1n} \\
-a_{21} & -a_{22} & -a_{23} & \cdots & -a_{2j} & \cdots & -a_{2n} \\
-a_{31} & -a_{32} & -a_{33} & \cdots & -a_{3j} & \cdots & -a_{3n} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
-a_{i1} & -a_{i2} & -a_{i3} & \cdots & -a_{ij} & \cdots & -a_{in} \\
\vdots & \vdots & \vdots & & \vdots & & \vdots \\
-a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mj} & \cdots & -a_{mn}
\end{bmatrix}
\end{array}$$

Value of Game: It is the expected payoff of play when all the players follows their optimal strategies. In other words it is the maximum guaranteed gain to the maximizing player if both player use their optimum strategy.

Maximin-Minimax Principle

Maximin Row

Step 1: Select the minimum element in each row and write it outside the pay-off matrix.

Step 2: Select the maximum element among them and encircle it i.e. '0'.

Minimax Column:

Step 3: Select the maximum element in each column and write it outside the pay off matrix.

Step 4: Select the minimum element among them and encircle it i.e. '0'.

Value of Game:

Step 5: The value at the intersection of these encircled elements is called value of game 0.

Saddle Point:

Steps to Find Saddle Point

Step 1: Find the minimum element of each row and encircle it.

Step 2: Find the greatest element of each column and mark by it 'W'.

Step 3: An element with both '0' 'W', indicates the presence of the saddle point.

Let us try to use minimax of maximin principle to find value of game

Example 1: Find value of game using maximum - minimax principle

	P	Q
L	-3	4
M	-2	3
N	2	3

Solution:

Step 1: Select the minimum element in each row and write it outside the pay-off matix and select the maximum element among them and encircle it i.e. '0'.

Step 2: Select the maximum element in each column and write it outside the pay-off matrix and select the minimum element among them and encircle it i.e. '0'.

	P	Q	Minimum Value
L	-3	4	-3
M	-2	3	-2
N	2	3	2
Maximum value	2	4	

Raw encircled element is equal to column encircled element. i.e. 2 = 2. Therefore, value of game is 2.

Example 2: Find the probabilities of strategies of player B for the following game. Also find the value of the game.

Player B

	B_1	B_2	B_3
A_1	12	-5	-2
A_2	7	8	5
A_3	0	7	5

Solution: Find minimum value from the each row and circle the maximum value among them. Also find the maximum value from the each column and circle the minimum value among them.

Player B

Player A

	B_1	B_2	B_3	Minimum value (i.e. maximin row)
A_1	12	-5	-2	-2
A_2	7	8	5	5
A_3	0	7	5	0
Maximum Value (i.e. Minimax Column)	12	8	5	

Raw encircled element is equal to column encircled element i.e. 5 = 5. Therefore, value of game is 5.

Example 3:

Player B

	B_1	B_2	B_3	B_4
A_1	170	40	190	250
A_2	70	20	90	110
A_3	250	80	120	100
A_4	300	40	150	90

Find the value of game using maximum-minimax principle.

Solution: Find minimum element from each row and circle the maximum element among them. Also find maximum element from each column and circle the minimum element among them.

	B_1	B_2	B_3	B_4	Minimum Value (i.e. maximin row)
A_1	170	40	190	250	40
A_2	70	20	90	110	20
A_3	250	80	120	100	80
A_4	300	40	150	90	40
Maximum Value (i.e. minimax row)	300	80	190	250	

Row encircled element is equal to column encircled element i.e. 80 = 80. Therefore, value of game is 80.

Example 4: for what value of λ, the game with following pay-off matrix is strictly determinable?

Player B

Player A		B_1	B_2	B_3
	A_1	λ	15	11
Player A	A_2	2	λ	0
	A_3	1	13	λ

Solution: Select the minimum element in each row and select the maximum element among them an encircle it. Also select the maximum element in each column and select the minimum element among them and encircle it.

Player B

		B_1	B_2	B_3	Minimum Value (i.e. maximin row)
	A_1	λ	15	11	11
Player A	A_2	2	λ	0	0
	A_3	1	13	λ	1
minimum value (i.e. minimax column		2	15	11	

Row encircled element is not equal to column encircled element i.e. $11 \neq 2$. It means that game is not strictly determinable and its value lies between 2 and 11.

However for the game to be strictly determinable value of λ, must be between minimax and maximin values. i.e. $2 \leq \lambda \leq 11$.

Example 5: Solve the game whose payoff matrix is

Player B

		B_1	B_2	B_3	B_4
	A_1	-6	4	-1	6
Player A	A_2	3	7	2	9
	A_3	-5	0	2	-3

Solution:

Player B

		B_1	B_2	B_3	B_4	Minimum Value
	A_1	-6	4	-1	6	-6
Player A	A_2	3	7	2	9	2
	A_3	-5	0	2	-3	-5
Minimax Column		3	7	2	9	

Value of game = 2, where maximin = minimax

Strategies for Player A = A_2

Player B = B_3.

Example 6: For the following payoff matrix for firm A, determine the optimal strategies for both the firms and the value of game.

Firm B

$$\begin{bmatrix} 4 & -1 & 6 & 8 & 9 \\ -1 & 10 & 4 & 6 & 14 \\ 16 & 10 & 8 & 9 & 15 \\ 2 & 9 & -4 & 3 & 2 \end{bmatrix}$$

Solution: Select the minimum value in each row and find out the maximum value among them and encircle it.

Select the maximum value in each column and find out the minimum value among them and encircle it.

Firm B

		I	II	III	IV	V	Maximin row
	I	4	-4	6	8	9	-1
Firm A	II	-1	10	4	6	14	-1
	III	16	10	8	9	15	8
	IV	2	9	-4	3	2	-4
	MaximinColumn	16	10	8	9	15	

Value of game = 6

Pure Strategies for Player A = III

Player B = III

Example 7: Two firm A and B manufactures of detergent powder are planning to make fund allocation for advertising their products. The matrix given below shows the percentage of market share of firm A for its various advertising policies.

Determine the optimal strategy for firm A

	No advertising	Medium Advertising	Large Advertising
No Advertising	80	70	60
Medium Advertising	50	35	25
Large Advertising	100	80	70

Solution:

	No Advertising	Medium Advertising	Large Advertising	Maximin ROW
No Advertising	80	70	60	60
Medium Advertising	50	35	25	25
Large Advertising	100	80	70	70
Minimax Column	100	80	70	

Value of game = 70

Pure Strategies fore

Player A = Large Advertising

Player B = Large Advertising

Following examples we will find saddle point of a game and value of a game.

Example 8: Solve the game whose pay off matrix is as

$$\begin{array}{cc} & \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{array}{c} B_1 \\ B_2 \\ B_3 \end{array} & \begin{pmatrix} 1 & 4 & 2 \\ 1 & -3 & -2 \\ 2 & 6 & -1 \end{pmatrix} \end{array}$$

Solution: Select all the minimum element in each row and encircle it. Select all the maximum element in each column and square around it.

Solution:

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 4 & 2 \\ 1 & -3 & -2 \\ 2 & 6 & -1 \end{bmatrix} \end{array}$$

The Saddle Point obtained is shown by having both circle and square around it. The game has two saddle points in positions (1, 1) and (1, 3)

Example 9: Player A can choose his strategies from A_1, A_2, A_3 only. While B can choose from B_1 and B_2 only. The rule of the game states that the payment should be made in accordance with the selection of strategies.

Strategy Pair Selected	Payment to be made	Strategy Pair Selected	Payment to be made
(A_1, B_1)	Player A PayesRs 3 to Player B	(A_2, B_2)	Player B Pays Rs 6 to Player A.
(A_1, B_2)	Player B PayesRs 8 to Player A	(A_3, B_1)	Player A Pays Rs 4 to Player B
(A_2, B_1)	Player B PayesRs 4 to Player A	(A_3, B_2)	Player A Pays to Rs 6 to Player B

What strategies should A and B Play in order to get the optimum benefit of the Play?

Solution: With the help of rules i.e. select row minima and encircle each. Select column maxima and put square around each.

Player B

Player A	A1	-3	8
	A2	4	6
	A3	-4	-6

Value of game :Rs 4

Saddle Point: At Position A_2 B_1

Strategies : Pure

(1) for Player A $\rightarrow A_2$

(2) for Player B $\rightarrow B_1$

Example 10: Solve the following game:

B

	1	-5	-4	4
A	2	4	0	4
	3	4	-4	-6

Solution: Select minimum element of each row and put '0' around it. Select maximum element of each column and put 'W' around it, we get,

B

	1	-5	-4	$\boxed{4}$
A	2	$\boxed{4}$	$\boxed{0}$	$\boxed{4}$
	3	$\boxed{4}$	-4	-6

Value of game: 0 i.e. game is FAIR where no one wins no one loses.

Strategies : Pure

(1) For Player A $\rightarrow A_2$

(2) For Player B → B_2

Example 11: Solve the game whose Payoff is given as:

	B_1	B_2	B_3
A_1	-5	13	-5
A_2	-8	-4	-5
A_3	-8	18	-9

Solution:

Saddle Point Saddle Point

	B_1	B_2	B_3
A_1	$\boxed{-5}$	13	$\boxed{-5}$
A_2	-8	-4	$\boxed{-5}$
A_3	-8	$\boxed{18}$	-9

In the above solution we have two saddle points.

Value of game = -5

Strategies:

For Player A = A_1

For Player B = B_1 or B_3.

Example 12: Solve the game whose Payoff matrix is given below:

$$\begin{bmatrix} -10 & 180 & -10 \\ -40 & -50 & -30 \\ -40 & 210 & -70 \end{bmatrix}$$

Solution:

$$\begin{array}{ccc} & & \text{B} \\ & & \begin{array}{ccc} \text{I} & \text{II} & \text{III} \end{array} \\ \text{A} & \begin{array}{c} \text{I} \\ \text{II} \\ \text{III} \end{array} & \begin{pmatrix} \boxed{-10} & 180 & \boxed{-10} \\ -40 & -50 & -30 \\ -40 & \boxed{210} & -70 \end{pmatrix} \end{array}$$

$\therefore$ Value of game = -20

Pure Strategies: For A $\rightarrow$ I

For B $\rightarrow$ I or III

Self Check Exercise

Questions 1: Solve the game whose Payoff is given as

	B_1	B_2	B_3
A_1	-2	15	-2
A_2	-5	-6	-4
A_3	-5	20	-8

Questions 2: Solve the following game

(a)

Player B

Player A

	B_1	B_2	B_3
A_1	8	14	10
A_2	16	16	14

(b)

Player B

Player B		B_1	B_2
	A_1	20	80
	A_2	40	30
	A_3	50	60

11.4 Summary:

In this unit we studied that

MaximinPrinciple : The maximin Principle focuses on minimizing the maximum possible loss a player could face.

Application: In decision making under uncertainly, it suggests choosing the option that guarantees the highest payoff in the worst case scenario.

Example: The maximin strategy is the one that ensures the least potential loss if the worst possible outcome occurs.

Minimax Principle: The minimax principle involves maximizing the minimum payoff a player can achieve.

Application: Used in zero sum games, it advices choosing strategies that maximize one's own minimum possible payoff.

Example: In competitive games, the minimax strategy seeks to maximize the smallest payoff one can guarantee, considering the opponent's strategies aimed at minimizing if.

11.5 Glossary:

- **Maximin Principle:** This principle suggests that a player should choose the strategy that maximizes their minimum possible payoff.
- **Minimax Principle:** This Principle suggests that minimizing the maximum possible loss. Players using this principle consider the maximum possible loss they could incur for each strategy and choose the one that minimizes this maximum loss.
- **Zero-Sum Game:** A type of game where one Player's gain is exactly equal to another player's loss. The total payoff across all players sums to zero.

11.6 Answer to Self Exercise

Q.1 Value of game = -2

Strategies For Player A = A_1

For Player B = B_1 or B_3

Q.3 (a) Value of game = 14;

Strategies = (A_2, B_3)

(b) Value of game = 50

Strategies = (A_3, B_1)

11.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

11.8 Terminal Questions

Q.1 Solve the following game

(a)

	B_1	B_2	B_3
A_1	-3	-2	6
A_2	2	0	2
A_3	5	-2	-4

(b)

Player B

Player A	10	6
	8	2

(c)

	Player B		
Player A	-1	2	-2
	6	4	-6

(d)

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 3 & 3 & 8 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \end{array}$$

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Unit - 12

Oddment Method

Traveling Salesman Problem

Structure

12.1 Introduction

Dear student in this unit we learn the method to solve a game when the saddle point does not exists. To solve such problem the methods of mixed strategies are used. In there are may methods to solve a game with mixed strategies. In unit we will learn first method known as oddment method odd's method to solve a game having no saddle point. Oddment method can be applied only to 2×2 matrix i.e. a game having two players each having two strategies. The concept of probability is used in this method i.e. the sum of all probability must be equal to one.

12.3 Learning Objectives:

After studying this unit, students will be able to

1. define a game with mixed strategy
2. define oddment method to solve a game
3. apply oddment method to solve a game
4. find value of game using oddment method
5. find the probabilities of selecting a strategy by the playes.

12.3 Odd's Method:

Oddment method is a method which is used to find to value of game, where saddle point does not exists. To apply this method. The pay off matrix must be square and it is only 2×2.

Steps Involved in Oddment Method

Step 1: Find out the difference between a_1 and a_2 of x_1.

Player B

			Y_1	Y_2
Player A	X_1		a_1	a_2
	X_2		b_1	b_2

and put it in the from t of X_2. Do the same thing with the element of X_2.

	Y_1	Y_2	
X_1	a_1	a_2	b_1-b_2
X_2	b_1	b_2	a_1-a_2

Step 2: Find out the difference between a_1 and b_1 and a_2, b_2 and place it below of Y_2 and Y_1 respectively.

	Y_1	Y_2
X_1	a_1	a_2
X_2	b_1	b_2
	a_2-b_2	a_1-b_1

Step 3: Find out the value of game with the help of the following equations

$$V = \frac{a_1(b_1 - b_2) + b_1(a_1 - a_2)}{(b_1 - b_2) + (a_1 - a_2)}$$

Probabilities for acceptance of player A

$$x_1 = \frac{b_1 - b_2}{(b_1 - b_2) + (a_1 - a_2)}$$

$$x_2 = \frac{a_1 - a_2}{(b_1 - b_2) + (a_1 - a_2)}$$

Probabilities for acceptance of player B

$$y_1 = \frac{a_2 - b_2}{(a_2 - b_2) + (a_1 - b_1)}$$

$$y_2 = \frac{a_1 - b_1}{(a_2 - b_2) + (a_1 - b_1)}$$

Note: Difference of two number is negative than ignore that negative sign and take the positive value.

To have more understanding of oddment method, Let us try following examples:

Example 1: Consider a modified form of 'matching biased coins! game problem. A wins Rs 5 when both the coins shows head and Rs 1 when both are tails. B coins Rs 2 when the coin do not matched. Give the choice of being matching player (A) or no matching player (B), which one would you choose and what would be your strategy? Solve the game by Odd's method.

Solution: Find out the Saddle Point: There is no saddle point is found, the optimal strategies will be mixed strategies.

Pay-off Matrix

		H	T
A	H	5	-2
	T	-2	1
		1-(-2)=3	-2-5=-7 (ignore-ve)

Find out the difference for first row i.e. -2-5 = -7 and write in the front of second row. Find out the difference for second row i.e. 1-(-2) = 3 and write this difference in front of the first row. Do the same thing for the both column. For modified pay off matrix, ignore the -ve sign.

Modified pay-off matrix

		B: H	T	
A	H	5	-2	3
	T	-2	1	7
		3	7	

∴ Value of game = $\frac{5 \times 3 + (-2 \times 7)}{3 + 7} = \frac{1}{10}$ (for A)

$$= \frac{-1}{10} \text{ (for B)}$$

Probabilities of

1) Selecting 1^{st} Strategy by A = $\frac{3}{10}$

2) Selecting II^{nd} Strategy by A = $\frac{7}{10}$

3) Selecting I^{st} Strategy by B = $\frac{3}{10}$

4) Selection II^{nd} Strategy by B = $\frac{7}{10}$

Example 2: In a game of matching coins with two players. Suppose A wins 2 unit of value when there are two heads, wins nothing when there are two tails and loses $1/2$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies and the value of game to A.

Solution:

Pay off Matrix

		H	T
A	H	2	$-1/2$
	T	$-1/2$	0

No Saddle Point exists. So we shift to odds method.

B

		H	T	odds
A	H	2	$-1/2$	$1/2$
	T	$-1/2$	0	$2\frac{1}{2}$
	odds	$1/2$	$2\frac{1}{2}$	

$$\text{Value of game} = \frac{2\times\frac{1}{2}+\left(\frac{-1}{2}\times\frac{5}{2}\right)}{\frac{1}{2}+\frac{5}{2}} = \frac{1-\frac{5}{4}}{\frac{6}{2}} = \frac{-1}{12} \text{ (for A)}$$

$$= \frac{1}{12} \text{ (for B)}$$

Strategies of A: I = $\frac{1}{6}$, II = $\frac{5}{6}$

Strategies of B. I = $\frac{1}{6}$, II = $\frac{5}{6}$

Example 3: Two players A and B match coins. If the coin match then A wins 20 unit of value. If the coins do not match then B wins 20 unit of value. Determine the optimum strategies for the players and the value of the game.

Solution:

Pay off Matrix

A \ B	H	T
H	20	-20
T	-20	20

Saddle does not exists, use odds

A \ B	H	T	odds
H	20	-20	40
T	-20	20	40
odds	40	40	

Value of game: $\frac{20\times 40+(-20\times 40)}{40+40} = \frac{0}{80} = 0$

Strategies for

Player A $\quad$ I = $\frac{1}{2}$

II = $\frac{1}{2}$

Player B $\quad$ I = $\frac{1}{2}$

II = $\frac{1}{2}$

Example 4: A and B each take out one or two matches and guess how many matches opponent has taken. If one of the players guesses correctly then the looser has to pay him as many rupees Rs the sum of the numbers held by both player otherwise, the payout is zero. Write down the payoff matrix and find the value of game.

Solution:

		B 1	B 2
A	1	2	0
	2	0	4

Saddle Point does not exist. Hence, we use odds.

		B 1	B 2	odds
A	1	2	0	4
	2	0	4	2
	odds	4	2	

Value of game = $\frac{2\times4+0\times2}{6}=\frac{8}{6} or \frac{4}{3}$

Strategies for

Player A : $\quad$ I = $\frac{4}{6}$ $\quad$ Player B : I = $\frac{4}{6}$

II = $\frac{2}{6}$ $\quad$ II = $\frac{2}{6}$

Example 5: Solve the game whose payoff matrix is as follows:

Player B

$$\text{Player A}\quad \begin{bmatrix} & B_1 & B_2 \\ A_1 & 8 & -4 \\ A_2 & -4 & 0 \end{bmatrix}$$

Solution:

Player B

		B_1	B_2	odds
	A_1	8	-4	4
Player A	A_2	-4	0	12
	odds	4	12	

Value of game = $\frac{(8\times 4)+(-4\times 12)}{4+12} = \frac{-16}{16}$ = -1

Strategies for:

Player A : I = $\frac{4}{6}$ Player B: I = $\frac{4}{6}$

II = $\frac{12}{16}$ II = $\frac{12}{16}$

Example 6: Solve the game whose payoff matrix is

Player B

$$\text{Player A}\quad \begin{array}{c} \\ A \\ B \end{array} \begin{array}{cc} X & Y \\ \left(\begin{matrix} 8 & -2 \\ 4 & 12 \end{matrix}\right. & \left.\right) \end{array}$$

Solution:

		B		
		X	Y	odds
Player A	A	8	-2	8
	B	4	12	10
	odds	14	4	

No Saddle point exist, we solve this by odd's

Value of game = $\dfrac{8\times 8+4\times 10}{10+8}=\dfrac{64+40}{18}=\dfrac{104}{18}=\dfrac{52}{9}$

Probability of Selecting Strategy A = $\dfrac{8}{18}=\dfrac{4}{9}$

Probability of Selecting Strategy B = $\dfrac{10}{18}=\dfrac{5}{9}$

Probability of Selecting Strategy X = $\dfrac{14}{18}=\dfrac{7}{9}$

Probability of Selecting Strategy B = $\dfrac{4}{18}=\dfrac{2}{9}$

Example 7: Solve the game whose payoff matrix and find the probability of selecting strategies of both players.

Player B

$$\text{Player A}\begin{bmatrix} & I & II \\ I & 10 & 0 \\ II & 0 & 12 \end{bmatrix}$$

Solution:

		Player B		
		I	II	odd's
Player A	I	10	0	12
	II	0	12	10
	odd's	12	10	

Value of game = $\frac{(10\times12)+(0\times10)}{10+12}=\frac{120}{22}=\frac{60}{11}$

Probabilities of selecting strategies player

A_1 = $\frac{12}{22}=\frac{6}{11}$ A_2 = $\frac{10}{22}=\frac{5}{11}$

B_1 = $\frac{12}{22}=\frac{6}{11}$ B_2 = $\frac{10}{22}=\frac{5}{11}$

Example 8: Solve the game whose payoff matrix is

Player B

$$\text{Player A}\begin{bmatrix} & B_1 & B_2 \\ A_1 & -4 & 6 \\ A_2 & 2 & -3 \end{bmatrix}$$

Solution:

Player B

		B_1	B_2	odds
	A_1	-4	6	5
Player A	A_2	2	-3	10
	odd's	9	6	

Value of game = $\frac{(-4\times5)+(2\times10)}{5+10}=\frac{-20+20}{15}=0$

Game is fair game.

Probabilities of selecting strategies

Player A_1 = $\frac{5}{15}=\frac{1}{3}$ Player A_2 = $\frac{10}{15}=\frac{2}{3}$

Player B_1 = $\frac{9}{15}=\frac{3}{5}$ Player B_2 = $\frac{6}{15}=\frac{2}{5}$

Example 9: Two players A and B match coins. If the coin match, then B wins 3 unit of value. Determine optimum strategies for the player and the value of the game.

Solution:

Player B

		H	T
Player A	H	3	-3
	T	-3	3

Since saddle point does not exist, so we will apply odder method to find the value of game.

Player B

		H	T	odd's
Player A	H	3	-3	6
	T	-3	3	6
	odd's	6	6	

Value of game = $\frac{(3\times6)+(-3\times6)}{12}=\frac{0}{12}=0$

∴ V = 0 (Fair game)

Probabilities of Selecting Strategies

	Head	Tail
Player A:	$\frac{1}{2}$	$\frac{1}{2}$
Player B:	$\frac{1}{2}$	$\frac{1}{2}$

Example 10: Solve the game whose payoff matrix is

Player B

$$\text{Player A} \begin{bmatrix} & B_1 & B_2 \\ A_1 & -6 & 4 \\ A_2 & 3 & -5 \end{bmatrix}$$

Solution:

		Player B		
		B_1	B_2	odd's
Player A	A_1	-6	4	8
	A_2	3	-5	10
	odd's	9	9	

Value of game : $\frac{(-6\times 8)+(3\times 10)}{8+10}=\frac{-48+30}{18}=\frac{18}{18}=1$

Probabilities of Selecting Strategies

Player A_1 = $\frac{8}{18}=\frac{4}{9}$ Player A_2 = $\frac{10}{18}=\frac{5}{9}$

Player B_1 = $\frac{9}{18}=\frac{1}{2}$ Player B_2 = $\frac{9}{18}=\frac{1}{2}$

Self Check Exercise

Q.1 Consider a modified form of 'matching biased coins' game problem. A win &Rs 8 when both coins show head and Rs 1 when both are tails. B wins Rs 3 when the coin do not match. Give the choice of being matching player (A) or non-matching player (B), which one would you choose and what would be your strategy.

Q.2 Solve the following game

		B	
		H	T
A	H	10	-10
	T	-10	10

Q.3 Two players A and B match coins. If the coin match, then A wins one unit value, if the coins do not match, then B wins one unit of value. Determine the optimum strategies for the players and the value of game.

12.4 Summary:

In this unit we studied that

(1) The odds method calculates the ratio of the probability of winning to the probability of losing in a game.

(2) It helps players determine the optimal amount to bet or invest based on the perceived likelihood of winning compared to losing

(3) In a betting scenario, if the odds are favorable (e.g. 2.1), a player might decide to bet higher because the potential payoff is twice the amount the risk losing.

12.5 Glossary:

(1) **Expected Value:** The weighted average of possible outcomes of a random variable, where the weights are probabilities.

(2) **Risk:** The potential for loss or negative outcome associated with a particular decision or strategy.

(3) **Saddle Point:** A saddle point is a pair of strategies such that:

(4) It is a nash equilibrium, meaning neither player can improve their payoff by unilaterally changing their strategy.

(5) The payoff to one player is the maximum value in its row, and the payoff to the other player is the minimum value in its column.

12.6 Answer to Self Check Exercise

Q.1. Value of game = $\frac{-1}{15}$ for A

= $\frac{1}{15}$ for B

Q.2. Value of game = 0

Strategies = A $\left(\frac{1}{2},\frac{1}{2}\right)$; B $\left(\frac{1}{2},\frac{1}{2}\right)$

Q.3. Value of game = 0

Strategies = A $\left(\frac{1}{2},\frac{1}{2}\right)$; B $\left(\frac{1}{2},\frac{1}{2}\right)$

12.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.

2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.

3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.

4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelfth Edition

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

12.8 Terminal Questions

Q. Solve the following game whose payoff mamix

(a)

$$\text{Player A}\quad \overset{\text{Player B}}{\begin{array}{c|cc} & B_2 & B_4 \\ \hline A_2 & 6 & -4 \\ A_3 & 2 & 10 \end{array}}$$

(b)

$$\text{Player A}\quad \overset{\text{Player B}}{\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -4 & 6 \\ A_2 & 2 & -3 \end{array}}$$

(c)

$$\text{Player A}\quad \overset{\text{Player B}}{\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 6 & -3 \\ A_2 & -3 & 0 \end{array}}$$

(d)

$$\text{Player A}\quad \overset{\text{Player B}}{\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 13 & -6 \\ A_2 & 6 & 8 \end{array}}$$

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Unit - 13

Arithmetic Method

Structure

13.1 Introduction

Dear students, in this unit we will study another method to solve a maxed game. Mixed game or a game with no Saddel point can be solve by using arithmetical method. Arithmetical method is applied on two person zero sum game means a game having two player such that gain of Ist is the loss of other. Again the probability of selecting a particular strategy by a player is calculated by the formula of the arithmetic method with the condition that sum of all probabilities of a particular player is equal to one.

13.2 Learning Objectives:

After studying this unit, students will be able to

1. define game with mixed strategy
2. define aesthetical method to solve two persons zero sum game.
3. find optimal strategies of players using arithmetical method.
4. find value of game by arithmetical method.

13.3 Arithmetical Method:

Just like oddment method arithmetical method is used to find value of game when saddle point does not exists and the strategy of the game is a mixed strategy.

Consider a 2×2 square matrix.

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} y & 1-y \end{array} \\ A & \begin{array}{c} x \\ 1-x \end{array} & \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array}$$

In this game, if x is the probability that A selects his first strategy and if y is the probability that B selects his first strategy, then from the following equation, we can find the probabilities for both player and value of game.

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

$$y = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$$

Value of game $(v) = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

Let us apply arithmetical method to find value of game with mixed strategy.

Example 1: The following is the payoff matrix of a game being played by A and B. Determine the optimal strategies for the players and the value of game.

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 11 & -4 \\ -3 & 7 \end{bmatrix} \end{array}$$

Solution: The given matrix has no saddle point and hence the players will use mixed strategies. We use arithmetical method to solve the game.

Let x and 1-x are the probabilities that A select his first and Second Strategies respectively.

Here $a_{11} = 11$, $a_{12} = -4$, $a_{21} = -3$, $a_{22} = 7$

Substituting the values in the formulae

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{7 - (-3)}{(11 + 7) - (-4 - 3)} = \frac{10}{18 + 7} = \frac{10}{25} = \frac{2}{5}$$

$1 - x = 1 - \frac{2}{5} = \frac{3}{5}$

Let y and 1-y are the probabilities that B selects his first and second strategies respectively.

$$y = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{7 - (-4)}{(11 + 7) - (-4 - 3)} = \frac{11}{25}$$

$1 - y = 1 - \frac{11}{25} = \frac{14}{25}$

Value of game $(v) = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \dfrac{11\times 7-(-4)\times(-3)}{(11+7)-(-4-3)} = \dfrac{77-12}{18+7} = \dfrac{65}{25} = \dfrac{13}{5}$

$$\left[A\left(\frac{2}{5},\frac{3}{5}\right); B\left(\frac{11}{25},\frac{14}{25}\right): v = \frac{13}{5}\right]$$

Example 2: In a game of matching coin with two players, suppose A wins two unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the pay-off matrix, the best strategies for each player and the value of game.

Solution: The payoff matrix for the given game problem as follows

		B	
		Head	Tail
A	Head	2	$\frac{-1}{2}$
	Tail	$\frac{-1}{2}$	0

The matrix has no saddle point and hence the players will use mixed strategies. We use arithmetical method to solve the game.

Here $a_{11} = 2$, a12 = $\frac{-1}{2}$, $a_{21} = \frac{-1}{2}$, $a_{22} = 0$.

Let x and 1-x are the probabilities that A select his first and second strategies respectively.

$$x = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{0-\left(\frac{-1}{2}\right)}{(2+0)-\left[\left(\frac{-1}{2}\right)+\left(\frac{-1}{2}\right)\right]} = \frac{\frac{1}{2}}{2+1} = \frac{1}{6}$$

1-x $= 1-\frac{1}{6} = \frac{5}{6}$

Let y and 1-y are the probabilities that B selects his first and Second strategies respectively.

$$y = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{0 - \left(\frac{-1}{2}\right)}{(2+0) - \left[\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)\right]} = \frac{\frac{1}{2}}{3} = \frac{1}{6}$$

1-y = $1 - \frac{1}{6} = \frac{5}{6}$

Value of the game $(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})}$

$$= \frac{(2 \times 0) - \left[\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)\right]}{(2+0) - \left[\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)\right]}$$

= $\frac{\frac{1}{4}}{3}$ = $\frac{-1}{12}$

$$\left[A\left(\frac{1}{6}, \frac{5}{6}\right); B\left(\frac{1}{6}, \frac{5}{6}\right); v = \frac{-1}{12}\right]$$

Example 3: Two players A and B match coins. If the coin match then A wins 10 unit of value. If the coins do not match then B wins Rs 10. Determine the optimum strategies for the players and the value of game.

Solution:

Pay-off matrix

Player B

		H	T
Player A	H	10	-10
	T	-10	10

The saddle point does not exist. We will use arithmetical method.

Here: a_{11} = 10, a_{12} = -10, a_{21} = -10, a_{22} = 10

Substituting these values in the formulae

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{10 - (-10)}{(10 + 10) - (10 + 10)} = \frac{20}{40} = \frac{1}{2}$$

1-y = $\frac{-1}{12} = \frac{1}{2}$

Value of game $(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{10 \times 10 - (-10) \times (-10)}{(10 + 10) - (-10 - 10)}$

= $\frac{100 - 100}{40} = 0$

$$\left[A\left(\frac{1}{2}, \frac{1}{2}\right); B\left(\frac{1}{2}, \frac{1}{2}\right); v = 0 \right]$$

Example 4: Solve the game whose payoff matrix is:

Player B

		B_1	B_2
Player A	A_1	-4	6
	A_2	2	-3

Solution: Saddle point does not exist. We will use arithmetical method.

Here, a_{11} = -4, a_{12} = 6, a_{21} = 2, a_{22} = -3.

Substituting these value in the formula

$$x = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{-3 - 2}{(-4 - 3) - (6 + 2)} = \frac{-5}{-7 - 8} = \frac{1}{3}$$

1-x $= 1 - \frac{1}{3} = \frac{2}{3}$

$$y = \frac{a_{22} - a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{-3 - 6}{(-4 - 3) - (6 + 2)} = \frac{-9}{-7 - 8} = \frac{3}{5}$$

1-y = $1 - \frac{3}{5} = \frac{2}{5}$

Value of game $(v) = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{22}) - (a_{12} + a_{21})} = \frac{(-4) \times (-3) - 6 \times 2}{(-4 - 3) - (6 + 2)}$ = 0

$$\left[A\left(\frac{1}{3},\frac{2}{3}\right);B\left(\frac{3}{5},\frac{2}{5}\right):v=0\right]$$

Example 5: Two players A and B match coins. If the coin match then wins one unit of value, if the coins do not match, then B wins one unit of value. Determine optimum strategies for the players and the value of the game.

Solution:

Player B

		H	T
Player A	H	1	-1
	T	-1	1

Since saddle point does not exist. So we will apply arithmetical method.

Here $a_{11} = 1$, $a_{12} = -1$, $a_{21} = -1$, $a_{22} = 1$.

Substituting these value in the formulae

$$x=\frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}=\frac{1-(-1)}{(1+1)-(-1-1)}=\frac{2}{4}=\frac{1}{2}$$

1-x = $\frac{-1}{12}=\frac{1}{2}$

$$y=\frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}=\frac{1-(-1)}{(1+1)-(-1-1)}=\frac{2}{4}=\frac{1}{2}$$

1-y = $\frac{-1}{12}=\frac{1}{2}$

Value of game $(v)=\frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}=\frac{1\times(-1)-(-1)(-1)}{(1+1)-(-1-1)}=0$

$$\left[A\left(\frac{1}{2},\frac{1}{2}\right);B\left(\frac{1}{2},\frac{1}{2}\right);v=0\right]$$

Self Check exercise

Q.1 The following is the pay-off matrix of a game being played by A and B. Determine the optimal strategegies for the players and the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 8 & -7 \\ -6 & 4 \end{bmatrix} \end{array}$$

Q.2 Solve the following game

B

		Head	Tail
A	Head	1	$\frac{-1}{2}$
	Tail	$\frac{-1}{2}$	0

13.4 Summary:

Dear students in this unit we studied that

1. The arithmetic method averages the payoffs of players in cooperative games to determine a fair distribution of gains.
2. It is used to find a compromise or fair allocation that balances the interests of all players based on their contributions or entitlements.
3. If players in a game contribute differently or have varying stake, the arithmetic method can help find a middle ground that respects each player's relative importance or investment.

13.5 Glossary

- **Game:** A formal model of strategic interaction between rational decision makers.
- **Strategy:** A complete plan of action specifying how a player will act in every possible situation with in the game
- **Probability:** The likelihood of a specific outcome or event happening, usually represented as a number between 0 (impossible) and 1 (certain)

13.6 Answer to Self Check Exercise

Q.1 $A\left(\frac{2}{5},\frac{3}{5}\right);B\left(\frac{11}{25},\frac{14}{25}\right)$

Value of game = $\frac{-2}{5}$

Q.2 $A\left(\frac{1}{4},\frac{3}{4}\right); B\left(\frac{1}{4},\frac{3}{4}\right)$

Value of game = $\frac{-1}{8}$

13.8 Terminal Questions

Q.1 Solve the following game problem, whose payoff matrix is

		B	
		H	T
A	H	11	-5
	T	-5	4

Q.2 Solve the game whose payoff matrix is as follows:

(a)

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 10 & -2 \\ -2 & 0 \end{bmatrix} \end{array}$$

(b)

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 40 & -40 \\ -40 & 40 \end{bmatrix} \end{array}$$

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Unit - 14

Dominance Property In Game Theory

Structure

14.1 Introduction

Dear students in this unit we will study about dominance property in game theory. Dominance property is applied to a game having two plays more than two strategies. Also dominance property can be applied to a pure game i.e. game having no saddle point and to a nixed game i.e. game having no saddle point. Dominance property works on the rule that inferior strategies of either player are deleted or ignored if dominated by other strategies. Since a player has no profits in using a inferior strategy. So doing this the cost size of pay off matrix is reduced. We apply dominance property again and again till the pay off matrix become 2×2 matrix. Then this 2×2 matrix is solved by any one of the rule discussed in units 12 and unit 13 i.e. oddment method or asthmatic method.

14.2 Learning Objectives:

After studing this unit students will be able to

1. define dominance property in a game.
2. apply dominance property to reduce the size of pay off matrix
3. find the value of game using dominance property

14.4 Dominance Property:

Sometimes, it is seen that one of the pure strategies of either player is inferior to atleast one of the remaining ones. The superior strategies dominates the inferior one. Obviously, a player would have no advent ange to use inferior strategies which are dominated by superior strategies.

Dominance principle states that inferior strategies are deleted or ignored if dominated by other strategies. For implementing the dominance principle the following rules should be followed.

Rule 1: If all the elements of a row (say i^{th} row) are less than or equal to the corresponding elements of another row (say j^{th} row) then i^{th} row is dominated by j^{th} row.

Rule 2: If all the element of a column (say i^{th} column) are greater them or equal to the corresponding elements of another column (say j^{th} column), then i^{th} column is dominated by j^{th} column Dominance principle may be applied in pure strategies and also to mixed strategies.

Rule 3: A pure strategy may be dominated if it is inferior to an average of two or more other pure strategies.

To have more understanding of dominance property let us try following examples:

Example 1: Reduce the following game by dominance property and solve it:

Player B

		1	2	3	4	5
	I	3	5	3	9	6
Player A	II	5	6	2	7	8
	III	8	7	9	8	7
	IV	4	0	8	5	3

Solution: In the above matrix, saddle point exist at cell (III, 2), but for getting the value of game, we are using the dominance principle.

Step 1: Since all the elements of ROW IV are less than corresponding elements of ROW III, hence ROW IV is deleted. We get:

Player B

		1	2	3	4	5
	I	3	5	3	9	6
Player A	II	5	6	2	7	8
	III	8	7	9	8	7

Step 2: All the elements of column 4 are greater than or equal to corresponding elements of column 1. Hence 4^{th} strategy of player A is dominated by 1^{st} strategy so deleting column 4, we get the reduced matrix as:

Player B

		1	2	3	5
	I	3	5	3	6
Player A	II	5	6	2	8
	III	8	7	9	7

Step 3: All the elements of I^{st} row is less than corresponding element of III^{rd} row which implies that row I is dominated by row III. Deleting ROW I, we get,

Player B

		1	2	3	5
Player A	II	5	6	2	8
	III	8	7	9	7

Step 4: All the elements of 5^{th} strategy of player B is more than that of 2^{nd} strategy, which implies that 5th of player B is dominated by 2^{nd} strategy, hence delete 5^{th} strategy of player B, we get

Player B

		1	2	3
Player A	II	5	6	2
	III	8	7	9

Step 5: Since all the element of column 1 is greater than or equal to average of 2^{nd} and 3^{rd} column which implies that 1^{st} strategy of player B is dominated by average of 2^{nd} and 3^{rd} strategy, hence delete column 1, we get:

Player B

		2	3
Player A	II	6	2
	III	7	9

Step 6: Now, applying saddle point method to above 2×2 problem, we get

Player B

		2	3
Player A	II	6	2
	III	$\boxed{7}$	$\boxed{9}$

Thus, Optimum strategy for player A = III

Optimum strategy for player B = 2

and Value of game = 5

Example 2: Using the dominance property obtain the optimal strategies for both the players and determine the value of game. The pay off matrix for player A is given:

Player B

		I	II	III	IV	V
	I	5	7	6	9	7
Player A	II	8	9	6	10	11
	III	9	10	12	11	10
	IV	7	5	11	6	6

Solution: Saddle point exist at cell (III, I), but for getting the value of game, we are using the dominance principle.

Step 1: From the above matrix we observe that ROW III dominate ROW I, hence ROW I is deleted. The reduce payoff matrix is as

Player B

		I	II	III	IV	V
Player A	II	8	9	6	10	11
	III	9	10	12	11	10
	IV	7	5	11	6	6

Step 2: All the element of column IV are greater than or equal to the elements of corresponding column II, So we delete column IV and get the reduced matrix.

Player B

		I	II	III	V
Player A	II	8	9	6	11
	III	9	10	12	10
	IV	7	5	11	6

Step 3: Since, all the elements of ROW IV are less than or equal to the corresponding elements of ROW III, we deleted the row IV and get the reduced matrix.

Player B

		I	II	III	IV
Player A	II	8	9	6	11
	III	9	10	12	10

Step 4: Since all the elements of column V are greater than or equal to the corresponding elements of column II, hence column V is deleted as it is inferior the reduce matrix is

Player B

		I	II	III
Player A	II	8	9	6
	III	9	10	12

Step 5: Since, all the elements of ROW II are less than the corresponding elements of ROW III. Hence ROW II is deleted.

Player B

		I	II	III
Player A	III	9	10	12

Step 6: Since, element of column Ii is 7 and element of column III is 9 is more than element of column I. column I dominates column II and column III hence both are deleted we get

Player B

		I
Player A	III	6

Hence Value of game is 6

Strategies : Pure

For A $\rightarrow$ III

For B $\rightarrow$ I

Example 3: Solve the game by using dominance principle

Player B

		I	II	III	IV	V
	I	3	4	2	9	0
Player A	II	7	6	5	7	8
	III	3	5	4	4	9
	IV	6	7	3	3	2

Solution: Since saddle point exist at cell (II, III), but to get this value, we use the dominance principle.

Step I: Since, elements of column II are dominates the elements of column III, hence delete column II, we get the reduce matrix.

Player B

		I	III	IV	V
	I	3	2	9	0
Player A	II	7	5	7	8
	III	3	4	4	9
	IV	6	3	3	2

Step 2: All element of column IV are greater than or equal to elements, of column III, hence delete column IV.

$$\text{Player A} \quad \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{array} \overset{\text{Player B}}{\begin{array}{ccc} \text{I} & \text{III} & \text{V} \\ \hline 3 & 2 & 0 \\ 7 & 5 & 8 \\ 3 & 4 & 9 \\ 6 & 3 & 2 \end{array}}$$

Step 3: All the elements of ROW II are greater than of ROW IV. Hence, we delete row II.

$$\text{Player A} \quad \begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \overset{\text{Player B}}{\begin{array}{ccc} \text{I} & \text{III} & \text{V} \\ \hline 3 & 2 & 0 \\ 7 & 5 & 8 \\ 3 & 4 & 9 \end{array}}$$

Step 4: All the elements of ROW II are greater than the row I. Hence, we delete row I.

$$\text{Player A} \quad \begin{array}{c} \\ \text{II} \\ \text{III} \end{array} \overset{\text{Player B}}{\begin{array}{ccc} \text{I} & \text{III} & \text{V} \\ \hline 7 & 5 & 8 \\ 3 & 4 & 9 \end{array}}$$

Step 5: All the elements of column V are greater than elements of column III and column I. As column V is dominate, hence delete it.

$$\text{Player A} \quad \begin{array}{c} \\ \text{II} \\ \text{III} \end{array} \overset{\text{Player B}}{\begin{array}{cc} \text{I} & \text{III} \\ \hline 7 & 5 \\ 3 & 4 \end{array}}$$

Step 6: All the elements of ROW III are less than the row II. Hence, row II dominates hence, deleting row II.

Player B

		I	III
Player A	II	7	5

Step 7: Column I is greater than Column III. Deleting column 7, we get

Player B

		III
Player A	II	5

Hence, value of game is 5

Strategies : Pure

For A : II

For B : III

Example 4: Solve the following game:

Player Q

		I	II	III	IV
	I	7	5	9	1
Player P	II	7	9	5	9
	III	9	5	9	1
	IV	1	9	1	17

Solution: First of all checking the solution by saddle point method:

Player Q

		I	II	III	IV
	I	7	5	9	1
Player P	II	7	9	5	9
	III	9	5	9	0
	IV	1	9	1	17

Since the above game does not have saddle point, hence the players will used mixed strategies. To use mixed strategy, the above game matrix will first of all be reduced to 2×2 matrix by using the principle of dominance.

Since the ROW I is dominated by ROW III, hence ROW I is eliminated.

$$\begin{array}{ccc} & & \text{Player Q} \\ & & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ & \begin{array}{c} \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{array}{|cccc|} \hline 7 & 9 & 5 & 9 \\ 9 & 5 & 9 & 1 \\ 1 & 9 & 1 & 17 \\ \hline \end{array} \end{array}$$

Player P (rows)

In the reduced matrix, column I is dominated by column III, hence column I is eliminated. The reduced matrix is:

$$\begin{array}{ccc} & & \text{Player Q} \\ & & \begin{array}{ccc} \text{II} & \text{III} & \text{IV} \end{array} \\ \text{Player P} & \begin{array}{c} \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{array}{|ccc|} \hline 9 & 5 & 1 \\ 5 & 9 & 1 \\ 9 & 1 & 17 \\ \hline \end{array} \end{array}$$

Since all elements of column II greater than or equal to corresponding elements of average of column III and IV. Hence column II is eliminated. The reduced matrix is:

$$\begin{array}{ccc} & & \text{Player Q} \\ & & \begin{array}{cc} \text{III} & \text{IV} \end{array} \\ \text{Player P} & \begin{array}{c} \text{II} \\ \text{III} \\ \text{IV} \end{array} & \begin{array}{|cc|} \hline 5 & 9 \\ 9 & 1 \\ 1 & 17 \\ \hline \end{array} \end{array}$$

Since all elements of ROW II equal to the average of each element of ROW III and ROW IV, which implies that ROW II is dominated by average of ROW III and IV, hence ROW II is eliminated.

Player Q

		III	IV	odd's
	III	9	1	16
Player P	IV	1	17	8
	odd's	16	8	

Player P Probabilities of Selecting Strategies	Player Q Probabilities of Selecting Strategies
I = 0 II = 0 III = $\frac{2}{3}$ IV = $\frac{1}{3}$	I = 0 II = 0 III = $\frac{2}{3}$ IV = $\frac{1}{3}$

Value of game i.e. $V = \frac{9\times16+1\times8}{16+8} = \frac{152}{24} = \frac{19}{3}$

Example 5: Solve the game by dominance method.

Player B

		I	II	III
	I	3	10	5
Player A	II	9	5	10
	III	8	5	9

Solution: It is clear that this game has no saddle point. Since row III is inferior to the row II, row III can be excluded from the pay off matrix. Thus, the reduced matrix is:

Player B

		I	II	III
Player A	I	3	10	5
	II	9	5	10

Again, column III is dominated by column I. Hence, column III can also be excluded from the payoff matrix. The reduced matrix is

Player B

		I	II
Player A	I	3	10
	II	9	5

This 2×2 game (without saddle point) can be solved by odds.

Player B

		I	II	odds
	I	3	10	4
Player A	II	9	5	7
	odds	5	6	

Value of game = $\frac{3\times 4+9\times 7}{4+7}=\frac{12+63}{11}=\frac{75}{11}$

Strategies for

1. Player A I = $\frac{4}{11}$

II = $\frac{7}{11}$

III = 0

2. Player B I = $\frac{5}{11}$

$II = \frac{6}{11}$

$III = 0$

Example 6: It is game between the two player where A is maximising player and B is minimising player. Player A wins B's coin of the two coins total are equal to odd number and losser his coin if total of two coin is even. It is game of 1, 2, 5, 10 and 50 rupees coins. Determine the pay off matrix, the best strategies for each player and the value of game to A.

Solution: Formulating the problem, we get the following game problem

Player B

		I	II	III	IV	V
	I	-1	2	-1	10	50
	II	1	-2	5	-2	-2
Player A	III	-5	2	-5	10	50
	IV	1	-10	5	-10	-10
	V	1	-50	5	-50	-50

Solution: Since the above game has no saddle point the player will use the mixed strategy by using concept of dominance.

Since each element of row IV and V are less than or equal to corresponding elements of row II. Hence row II dominants rows IV and V. Thus, Rows IV & V are eliminated.

The reduced matrix is given below:

Player B

		I	II	III	IV	V
	I	-1	2	-1	10	50
Player A	II	1	-2	5	-2	-2
	III	-5	2	-5	10	50

Since all elements of column IV and V are greater than or equal to corresponding elements of column II which implies column II dominates columns IV & V. Thus column IV & V are eliminated.

Player B

		I	II	III
	I	-1	2	-1
Player A	II	1	-2	5
	III	-5	2	-5

Since each element of column III are greater than or equal to corresponding element of column I. Hence column I dominates column III, Thus column III is eliminated, we get reduced matrix as:

Player B

		I	II
	I	-1	2
Player A	II	-5	2
	III	-5	2

Since each element of row III are less than or equal to corresponding element of row. 1. So row I dominates row III and hence row III is eliminated, we get reduced matrix as:

Player B

		I	II
Player A	I	-1	2
	II	1	-2

Since the above matrix does not possess saddle point hence we apply odds methods as follows:

Player B

		I	II	odds
	I	-1	2	3
Player A	II	1	-2	3
	odds	4	2	

Value of game = $\frac{-1\times 3+1\times 3}{3+3}=0$

Hence v = 0, it is a fair game

Probabilities of Selecting Strategy

Player	I	II	III	IV	V
A	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
B	$\frac{2}{3}$	$\frac{1}{3}$	0	0	0

Application of Dominance Principle in Pure Strategies -

It means that dominance principle may be used when the saddle point exists

Example 7: Reduce the following game by dominance property and solve it

Player B

		1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Solution: Since all the elements of Row IV are less than corresponding elements of Row III which implies that Row III dominates Row IV. Hence Row IV is deleted

Player B

		1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5

Since all elements of column 4 are greater than or equal to corresponding elements of column 1 hence 4th strategy of player B is dominated by 1st strategy so deleting column 4, we get reduced matrix.

$$\begin{array}{c} \\ \text{I} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \left|\begin{array}{cccc} 1 & 3 & 2 & 4 \\ 3 & 4 & 1 & 5 \\ 6 & 3 & 7 & 6 \end{array}\right| \end{array}$$

Since all elements of I row is less than corresponding element of III, which implies that row I is dominated by row III.

$$\begin{array}{c} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 5 \end{array} \\ \left|\begin{array}{cccc} 3 & 4 & 1 & 5 \\ 6 & 5 & 7 & 6 \end{array}\right| \end{array}$$

Since all elements of 5th strategy of player B is more than that of 2nd strategy, which implies that 5th of player B is dominated by 2nd strategy, hence delete 5th strategy of player B, we get

$$\begin{array}{c} \\ \text{II} \\ \text{III} \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \left|\begin{array}{ccc} 3 & 4 & 1 \\ 6 & 5 & 7 \end{array}\right| \end{array}$$

Since all the elements of column 1 is greater than or equal to overage of 2nd and 3rd column which implies that 1st strategy of player B is dominated by average of 2nd and 3rd strategy, hence delete column 1, we get

$$\begin{array}{cc} & \text{Player B} \\ & \begin{array}{cc} 2 & 3 \end{array} \\ \text{Player A} \quad \begin{array}{c} \text{II} \\ \text{III} \end{array} & \left|\begin{array}{cc} 4 & 1 \\ 5 & 7 \end{array}\right| \end{array}$$

Thus, Optimum strategy for player A = III

Optimum strategy for player B = 2

Value of Game = 5

Example 8: Using the dominance property obtain the optimal strategies for both the players and determine the value of the game. The payoff matrix for player A is

Player B

		I	II	III	IV	V
	I	2	4	3	6	4
Player A	II	5	6	3	7	8
	III	6	7	9	8	7
	IV	4	2	8	3	3

Solution: Row III dominate Row I, hence Row I is deleted Reduced Matrix -

		I	II	III	IV	V
	II	5	6	3	7	8
Player A	III	6	7	9	8	7
	IV	4	2	8	3	3

All element of column IV are greater than or equal to the elements of corresponding column II, so we delete column IV Reduced Matrix -

Player B

		I	II	III	V
	II	5	6	3	8
Player A	III	6	7	9	7
	IV	4	2	8	3

All the elements of Row IV are less than or equal to the corresponding elements of row III, we delete the row IV Reduced Matrix.

Player B

		I	II	III	V
Player A	II	5	6	3	8
	III	6	7	9	7

Column V elements are greater than or equal to the corresponding elements of column II, hence column V is deleted Reduced Matrix

	I	II	III
II	5	6	3
III	6	7	9

All the elements of row II are less than corresponding elements of row III, row II is deleted

Player B

		I	II	III
Player A	III	6	7	9

Since, element of column II i.e. 7 and element of column II i.e. 9 is more than element of column I. Column I dominates column II and column III, hence both are deleted

Player B

Player A	III	6

Hence value of game is 6

Strategies - Pure

For A $\rightarrow$ III

For B $\rightarrow$ I

Application of dominance principle in Mixed Strategies Game Matrix -

Example 9: Solve the following Game

Player Q

		I	II	III	IV
	I	6	4	8	0
Player P	III	6	8	4	8
	III	8	4	8	0
	IV	0	8	0	16

Solution: Checking the sol. by saddle point method

Player Q

		I	II	III	IV
	I	6	4	8	0
Player P	III	6	8	4	8
	III	8	4	8	0
	IV	0	8	0	16

Since the above game does not have a saddle point hence the players will use mixed strategy. To use mixed strategies, the above game matrix will first of all be reduced to 2×2 matrix by using the principle of dominance.

Row I is dominated by Row III

Player Q

		I	II	III	IV
	II	6	8	4	8
Player P	III	8	4	8	0
	IV	0	8	0	16

Since all elements of column II $\geq$ corresponding elements of average of column III and IV this implies that column II is dominated by average of column III and IV. Hence column II is eliminated

Player Q

		III	IV
	II	4	8
Player P	III	8	0
	IV	0	16

Since all the elements of column I $\geq$ corresponding elements of column III, column I is dominated by Col. III.

$$\begin{array}{llcccc} & & & & \text{Player Q} & \\ & & & \text{II} & \text{III} & \text{IV} \\ & & \text{II} & 8 & 4 & 8 \\ \text{Player P} & & \text{III} & 4 & 8 & 0 \\ & & \text{IV} & 8 & 0 & 16 \end{array}$$

Since all elements of Row II = average of each element of Row III and IV, Row II is dominated by averaged by average of Row III and IV, Row II is eliminated

$$\begin{array}{lcccc} & & \text{Player Q} & & \\ & & \text{III} & \text{IV} & \text{odds} \\ & \text{III} & 8 & 0 & 16 \\ \text{Player P} & \text{IV} & 0 & 16 & 8 \\ & \text{odds} & 16 & 8 & \end{array}$$

Value of Game $V = \frac{8(16)+0(8)}{16+8} = \frac{128}{24} = \frac{16}{3}$

Probabilities of selecting strategies for player P

Strategy I = 0

II = 0

III = $\frac{2}{3}$

IV = $\frac{1}{3}$

Probabilities of selecting strategies for player Q

Strategy I = 0

II = 0

III = $\frac{2}{3}$

IV = $\frac{1}{3}$

Example 10: Solve the following game by dominance method

		I	II	III
Player A	I	1	7	2
	II	6	2	7
	III	5	2	6

Solution: Game has no saddle point

Row II dominate row III

	I	II	III
I	1	7	2
II	6	2	7

Column III is dominated by Column I

	I	II
I	1	7
II	6	2

This is 2×2 game (with cut saddle point) can be solved by ODDS

	I	II	
I	1	7	4
II	6	2	6
	5	5	

Value of game = $\frac{(1\times4)+(6\times6)}{10}=\frac{40}{10}=4$

Strategies for

(1) Player A, I = $\frac{4}{10}$ or $\frac{2}{5}$

II = $\frac{6}{10}$ or $\frac{3}{5}$

III = 0

(2) Player B, I = $\frac{5}{10}$ or $\frac{1}{2}$

II = $\frac{5}{10}$ or $\frac{1}{2}$

III = 0

Example 11: Solve the game

Firm B

		B_1	B_2	B_3	B_4
	A_1	35	65	25	5
Firm A	A_2	30	20	15	00
	A_3	40	50	00	10
	A_4	55	60	10	15

Solution: Row A, dominates A_2, A_2 is deleted

Row A_4 dominates A_3, A_3 is deleted

	B_1	B_2	B_3	B_4
A_1	35	65	25	5
A_4	55	60	10	15

Column B_2 is dominated by B_1, B_2 is deleted

	B_1	B_3	B_4
A_1	35	25	5
A_4	55	10	15

Column B_1 is dominated by B_3, B_1 column is deleted

	B_3	B_4
A_1	25	5
A_4	10	15

the above 2×2 reduced game does not have a saddle point, hence we apply odds method

		B_3	B_4	odds
	A_1	25	5	5
Firm A	A_4	10	15	20
	odds	10	15	

Value of game = $\frac{25(5)+10(10)}{5+20} = 13$

Probability of selecting mixed strategy

Firm A : $A_1 = \frac{5}{25}$ or $\frac{1}{5}$ $A_2 = 0$, $A_3 = 0$, $A_4 = \frac{20}{25}$ or $\frac{4}{5}$

Firm B : $B_1 = 0$, $B_2 = 0$, $B_3 = \frac{10}{25}$ or $\frac{2}{5}$, By = $\frac{15}{25}$ or $\frac{3}{5}$

Example 12: Use the relation of dominance to solve the rectangular game where payoff matrix to A is given as

	I	II	III	IV	V	VI
I	0	0	0	0	0	0
II	4	2	0	2	1	1
III	4	3	1	3	2	2
IV	4	3	7	-5	1	2
V	4	3	4	-1	2	2
VI	4	3	3	-2	2	2

Solution: this game has no saddle point. Here row I and II are dominated by row III. Hence delete row I and II, we obtain the following matrix

	I	II	III	IV	V	VI
III	4	3	1	3	2	2
IV	4	3	7	-5	1	2
V	4	3	4	-1	2	2
VI	4	3	3	-2	2	2

Again column I, II and VI are dominated by column V So, delete column I, II and VI from the matrix

	III	IV	V
III	1	3	2
IV	7	-5	1
V	4	-1	2
VI	3	-2	2

Row VI is dominated by row V, Hence exclude strategy VI

	III	IV	V
III	1	3	2
IV	7	-5	1
V	4	-1	2

Average of player B's III and IV pure strategies is

$\left(\frac{1+3}{2}, \frac{7-5}{2}, \frac{4-1}{2}\right)$ i.e. (2, 1, 1.5)

Which is superior to player B's V^{th} strategy. Hence exclude the V^{th} strategy from the matrix

	III	IV
III	1	3
V	4	-1

The average of player A's III and IV pure strategies is

$$\left(\frac{1+7}{2},\frac{3-5}{2}\right) \text{i.e } (4, -1)$$

Which is same as player A's V^{th} strategy. Delete the V^{th} strategy. Finally (2×2) game has reached. The value of game is calculated by odds

	III	IV	
III	1	3	12
IV	7	-5	2
	8	6	

Value of game = $\frac{(1\times12)+(7\times2)}{14}=\frac{26}{14}$ or $\frac{13}{7}$

Strategies for

Player A	Player B
I $\rightarrow$ 0	I $\rightarrow$ 0
II $\rightarrow$ 0	II $\rightarrow$ 0
III $\rightarrow \frac{12}{14}$ or $\frac{6}{7}$	III $\rightarrow \frac{8}{14}$ or $\frac{4}{7}$
IV $\rightarrow \frac{2}{14}$ or $\frac{1}{7}$	IV $\rightarrow \frac{6}{14}$ or $\frac{3}{7}$
V $\rightarrow$ 0	V $\rightarrow$ 0
VI $\rightarrow$ 0	VI $\rightarrow$ 0

Example 13: Solve the following game by using principle of dominance (using average method)

	I	II	III	IV	V	VI
I	30	40	20	24	22	38
II	30	42	22	26	24	40
III	30	44	34	10	22	42
IV	30	46	28	18	24	44
V	30	48	26	16	24	46
VI	30	50	20	22	22	48

Solution: Column I Calumniates Column II

	I	III	IV	V	VI
I	30	20	24	22	38
II	30	22	26	24	40
III	30	34	10	22	42
IV	30	28	18	24	44
V	30	26	16	24	46
VI	30	20	22	22	48

Column IV dominates Column I, Column V dominates VI

	III	IV	V
I	20	24	22
II	22	26	24
III	34	10	22
IV	28	18	24
V	26	16	24
VI	20	22	22

Row II dominates row I

	III	IV	V
II	22	26	24
III	34	10	22
IV	28	18	24
V	26	16	24
VI	20	22	22

Row IV dominates row V

	III	IV	V
II	22	26	24
III	34	10	22
IV	28	18	24
VI	20	22	22

Row II dominates row VI

	III	IV	V
II	22	26	24
III	34	10	22
IV	28	18	24

Average of column III and IV dominates column V

	III	IV
II	22	26
III	34	10
IV	28	18

Average of rows II and III dominates row IV

	III	IV
II	22	26
III	34	10

Apply ODDS

	III	IV	Odds
II	22	26	24
III	34	10	4
odds	16	12	

Value of game = $\dfrac{(22\times 24)+(34\times 4)}{28}=\dfrac{332}{14}$

Probabilities for selecting strategies

A	I	II	III	IV	V	VI
	0	$24/28$	$4/28$	0	0	0
B	**I**	**II**	**III**	**IV**	**V**	**VI**
	0	0	$16/28$	$12/28$	0	0

Self Check Exercise

Q.1 Reduce the following game by dominance property and solve it

Player B

	1	2	3	4	5
I	1	3	2	7	4
II	3	4	1	5	6
III	6	5	7	6	5
IV	2	0	6	3	1

Q.2 Solve the game by using dominance principle

Player B

	I	II	III	IV	V
I	2	3	1	8	0
II	6	5	4	6	7
III	2	4	3	3	8
IV	5	6	2	2	1

14.4 Summary

In this unit we studied that

1. Dominance property occurs when one strategy for each player yields a betters outcome than any other strategy, regardless of the choices made by other players.
2. Players use dominance analysis to eliminate weak strategies and focus on those that offer better payoffs regardless of opponents decisions.
3. In a game where one strategy consistently provides higher payoff than another for each player, those dominated strategies can be eliminated from consideration, simplifying the decision-making process.

14.5 Glossary

- Dominant Strategy: A strategy that is optimal for a player regardless of the strategies chosen by other players. A dominant strategy always yields a higher payoff than any other strategy the player could choose.
- Mixed Strategy: A strategy where a player randomizes over possible pure strategies with specified probabilities
- Weak dominance: a strategy A weakly dominate strategy B if playing A gives a player at least as good an outcome as playing B, and strictly patter in some cases.

14.6 Answer to Self Check Exercise

Q.1 Value of game = 5;

A - II; B - 2

Q.2 Value of game = 4

A - II, B - III

14.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath
2. R. Panneerselvam, Operations Research, Phi. Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelfth Edition
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

14.8 Terminal Questions

Q.1 Solve the following game

Player Q

	I	II	III	IV
I	6	4	8	0
II	6	8	4	8
III	8	4	8	0
IV	0	8	0	16

Q.2 Solve the following game by dominance method

Player B

		I	II	III
Player A	I	1	7	2
	II	6	2	7
	III	5	2	6

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Unit - 15

Equal Gains Method

Structure

15.1 Introduction

Dear student in this unit we will learn to solve a game using equal gain method and equal gain method is applied only to game having two player each having two strategies (a game having 2×2 pay off matrix), without a saddle point. Then each prayer have to use mixed strategies. The selection of their combination of strategies will be alone in such a manner that the gain of one player is not influenced by the selection of any combination of strategy by the other player. To find the profanity of selecting a sectary we assumed that net gain to a player from his strategies will be equal, or in other words pay off under both the strategies is bound be equal whether other player adopt any strategy.

15.2 Learning Objectives:

After studying this unit students will be able to

1. define equal gain method
2. apply equal gain method to a game
3. find the value of game using equal gain method.

15.3 Equal Gains Method:

The method is applicable in case of game problem having 2×2 matrix or square matrix without saddle point. Let us take an example

$$
\begin{array}{ccc}
 & & B \\
 & & \begin{array}{cc} I & II \end{array} \\
A & \begin{array}{c} I \\ II \\ \\ \end{array} & \left|\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \downarrow & \downarrow \end{array}\right| \begin{array}{l} \rightarrow P \\ \rightarrow 1-P \\ \\ \end{array} \\
 & & \begin{array}{cc} q & 1-q \end{array}
\end{array}
$$

Player A has two choices I and II, If the probability of A to selecting Strategy I is P, then for strategy II is 1-P [as P+1-P=1]. We can find the expected gains of player A, if the player B uses his. I and II strategy, by putting both equal and hence probabilities of A's first and Second Strategies. Similarly, the probabilities of player B can be obtained. Let B uses I strategy with probability q and II with (1-q). We can find out the expected gain to B if A uses I and II by putting both equal and solving for q.

Let us try following examples where equal gain method is used to find value of a game.

Example 1: Solve the game by equal gains method

$$
\begin{array}{cc}
 & Y \\
X & \begin{bmatrix} & I & II \\ I & 30 & 10 \\ II & 15 & 20 \end{bmatrix}_{2\times 2}
\end{array}
$$

Solution: Since saddle point does not exists. The player may use equal gains method.

Let P_1 = Probability of X Selecting Strategy I

$(1-P_1)$ = Probability of X Selecting Strategy II

Similarly, q_1 = Probability of Y Selecting Strategy I

$(1-q_1)$ = Probability of Y Selecting Strategy II

If Player Y selects strategy I, the payoff X is

$30P_1 + 15\,(1-P_1)$

If Player Y selects strategy II, the payoff X is

$10P_1 + 20(1-P_1)$

As the payoff under the both situations is bound to be equal whether Y adopts first or second strategy. Equating the above pay-offs

$30P_1 + 15(1-A) = 10P_1 + 20\,(1-P_1)$

$15P_1 + 15 = -10P_1 + 20$

$25P_1 = 5$

$P_1 = \frac{5}{25} = \frac{1}{5}$

Or

$1-P_1 = 1-\frac{1}{5} = \frac{4}{5}$

Similarly, we determine the pay-offs to Y when X plays his first and second strategy respectively as follows:

$30q_1 + 10\ (1-q_1) = 15q_1 + 20\ (1-q_1)$

$30q_1 - 10q_1 + 10 = 15q_1 - 20q_1 + 20$

$20q_1 + 5q_1 = 10$

$q_1 = \frac{10}{25} = \frac{2}{5}$

$1-q_1 = 1-\frac{2}{5} = \frac{3}{5}$

Value of game = Expected Payoff of Player X

$$\overset{6}{\cancel{30}} \times \frac{1}{\cancel{5}} + \overset{3}{\cancel{15}} \times \frac{4}{\cancel{5}} = 16 + 12 = 8$$

Probability of A to selecting strategy I and II = $\frac{1}{5}$ and $\frac{4}{5}$ respectively.

Probability of B to selecting strategy I and II = $\frac{2}{5}$ and $\frac{3}{5}$ respectively.

Example 2: The two armies are at war. Army A has two air-bases, one of which is thrice as valuable as the other. Army B can destroy an undefended air base, but it can destroy only one of them. Army A can also defend only one of them. Find the best strategy for A to minimize its losses.

Answer:

	Army B	
	I	II
Army A	Attack the large base	Attack the smaller base
Defend the large base	0 (Both Survive)	-1 (Smaller one destroyed)

Defend the smaller base	-3 (Large one destroyed)	0 (Both Survive)

Since saddle point does not exist. Hence we may use equal gains method.

Let P = Probability of player A to select strategy I

(1-P) = Probability of player A to select strategy II

q = Probability of player B to select strategy I

(1-q) = Probability of player B to select strategy II

The Payoff for A, if B selects his first strategy is

$$OP - 3 (1-P)$$

The Pay off for A, if B selects his second strategy is

$$- P + 0 (1 - P):$$

Since the pay off under the both situation is bound to be equal

$$-3 (1-P) = -P$$

$$3P + P = 3$$

$$P = \frac{3}{4} \text{ and } 1-P = 1-\frac{3}{4} = \frac{1}{4}$$

Similarly, $0q - 1 (1-q) = -3q + 0 (1-q)$

$$-1 + q = -3q$$

$$4q = 1 \text{ or } q = \frac{1}{4} \text{ and } 1-q = \frac{3}{4}$$

Value of game = $0 \times \frac{3}{4} - 3 \times \frac{1}{4} = \frac{-3}{4}$

Probability of Selecting Strategies

Army	I	II
A	$\frac{3}{4}$	$\frac{1}{4}$
B	$\frac{1}{4}$	$\frac{3}{4}$

Example 3: A is paid Rs 10 if two coins turn both heads and Rs 3 if both coins turn tail. B is paid Rs 1 when the two coins do not match. Given the choice of being A or B, which one would you choose and what would be your strategy use equal gains method.

Solution:

		B	
		I (Head)	II (Tail)
A	I (Head)	10	-1
	II (Tail)	-1	3

Since saddle point does not exist. Use equal going method to solve.

Let P = Probability of Player A selecting strategy I

(1-P) = Probability of Player A selecting strategy II

q = Probability of Player B selecting strategy I

(1-q) = Probability of Player B selecting strategy II

If Player B selects strategy I, then pay off to

Player A = 10P (1-P)

If Player B selects strategy II, then payoff to

Player A = -P + 3(1-P)

Since Payoff in both situations is bound to be equal

10P - 1 + P = -P + 3 - 3P

11P + 4P = 4 or P = $\frac{4}{15}$ and 1-P = $\frac{11}{15}$

Similarly, we determine B's payoffs when A plays his first and second strategies respectively as follows

10q - 1 (1-q) = -q + 3 (1-q)

11q - 1 = -4q + 3

15q = 4 or q = $\frac{4}{15}$ or 1-q = $\frac{11}{15}$

V = (Expected payoff to B) = $10\times\frac{4}{5}-1\times\frac{11}{15}=\frac{29}{15}$

Example 4: In a game of matching coin. Player A wins Rs 20, if there are two heads, wins nothing if there are two tails and loses Rs. 1 when there are one head and one tail. Determine the payoff matrix and value of game.

Solution: The pay off matrix for A will be

		B	
		H	T
A	H	2	-1
	T	-1	0

Let P_1 = Probability of A selecting strategy I

$(1-P_1)$ = Probability of A selecting strategy II

q_1 = Probability of B selecting strategy I

$(1-q_1)$ = Probability of B selecting strategy II

Since the payoff is bound to be equal

Payoff of A $\Rightarrow 2P_1 - (1-P_1) = +\ 0\ (1)$

$= 3P_1 + P_1 = 1$ or $P_1 = \frac{1}{4}$ and $1-P_1 = \frac{3}{4}$

Similarly

Payoffs of B $2q_1 - (1-q_1) = -q_1 + 0\ (1-q_1)$

$4q_1 = 1$ or $q_1 = \frac{1}{4}$ and $1-q_1 = \frac{3}{4}$

Value of game = Expected Payoff of A = $2 \times \frac{1}{4} - 1 \times \frac{3}{4} = \frac{-1}{4}$

$$\therefore \qquad \left[A\left(\frac{1}{4},\frac{3}{4}\right), B\left(\frac{1}{4},\frac{3}{4}\right), V = \frac{-1}{4}\right]$$

Example 5: Two airlines operates the same air route, both trying to get as large market as possible. Based on certain market, daily gains and losses in rupees are shown in table below, in which +ve values favors airline B. Find the solution of the game.

			Air line B		
			Does nothing	Advertises special rates	Advertise special features
			1	2	3
Air line A	Advertises special rates	1	300	-25	100
	Advertise special features	2	150	155	175

Solution:

	1	2	3
1	300	-25	100
2	150	155	175

As all the elements of column 3 are more than all the element of column 2. It means column 2 dominates column 3. Hence column 3 deletes.

Reduced Matrix

	1	2
1	300	-25
2	150	155

Let P_1 = Probability for selecting I[st] strategy for player A

$(1-P_1)$ = Probability for selecting II[nd] strategy for player A.

Similarly, q_1 = Probability for selecting I[st] strategy for player B

$1-q_1$ = Probability for selecting II[nd]strategy for player B

If player B select strategy I, the payoff of A

$300P_1 + 150\ (1-P_1)$

If player B select strategy II, the payoff of A

$-25P_1 + 155\ (1-h)$

Since, the payoff under both the situations is bound to be equal

$300P_1 + 150 - 150P_1 = -25P_1 + 155 - 155P_1$

$150P_1 + 180P_1 = 5$

$$P1 = \frac{5}{330} = \frac{1}{66} \text{ and } 1\text{-}P_1 = \frac{65}{66}$$

Similarly, the pay off for B, if player A select his strategy I and II

$$300q1 - 25\,(1\text{-}q1) = 150q1 + 155\,(1\text{-}q1)$$

$$325q1 + 5q1 = 155 + 25 \text{ or } q1 = \frac{18\cancel{0}}{33\cancel{0}} = \frac{18}{33}$$

$$1\text{-}q1 = 1 - \frac{18}{33} = \frac{15}{33}$$

$$\text{Value of game} = 300 \times \frac{1}{66} + 150 \times \frac{65}{66} = 152.27$$

Self Check Exercise

Q.1 Solve the following game by equal gains method

(a)

$$\begin{array}{c} \\ X \end{array} \begin{array}{c} Y \\ \begin{bmatrix} & I & II \\ I & 25 & 5 \\ II & 10 & 15 \end{bmatrix} \end{array}$$

(b)

$$\begin{array}{c} \\ X \end{array} \begin{array}{c} Y \\ \begin{bmatrix} & I & II \\ I & 4 & 1 \\ II & 2 & 3 \end{bmatrix} \end{array}$$

(c)

$$\begin{array}{c} \\ \text{Player A} \end{array} \begin{array}{c} \text{Player B} \\ \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

15.4 Summary:

Dear students in this unit we studied that :

The equal game method is a cooperative solution concept where player aims to ensure that any change in strategy benefits all players equally. It focuses on distributing gains evenly among participants to promote fairness and cooperation.

Subgame method: The Subgame method is a strategic analysis technique in game theory used to solve multi stage games by breaking them down into smaller, more manageable sub games. It helps identify optimal strategies at each stage and predict outcomes by considering the strategic choices available to players at different points in the game.

15.5 Glossary

- Optimal Outcome: A solution where no player can be made better off without making another player worse off.
- Nash Equilibrium: A situation where no player has an incentive to change their strategy unilaterally.

15.6 Answer to Self Check Exercise

Q.1 (a) Value of game = 13 ; $X\left(\frac{1}{5},\frac{4}{5}\right);Y\left(\frac{2}{5},\frac{3}{5}\right)$

(b) Value of game = 2.5 ; $A\left(\frac{1}{4},\frac{3}{4}\right);B\left(\frac{1}{2},\frac{1}{2}\right)$

(c) Value of game = $\frac{7}{5}$

Player B

		$\frac{4}{5}$	$\frac{1}{5}$
Player A	$\frac{3}{5}$	3	5
	$\frac{2}{5}$	4	1

15.7 References/Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath.
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelfth Edition

5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

15.8 Terminal Questions

Q.1 Solve the game by equal gains method

(a)

		Y	
		B_1	B_2
X	A_1	1	5
	A_2	4	2

(b)

		B		
		1	2	3
A	1	275	-50	75
	2	125	130	150

(c)

		B	
		I	II
A	I	8	-3
	II	-3	1

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Unit - 16

Sub Game Method

Structure

16.1 Introduction

Dear student in this unit we will learn the method of sub games to solve the game problem. This method is applicable to the games having two players, one player has two strategies white the other has more than two strategies. On this basis the subgame method having two types (1) $2 \times n$ and $m \times 2$. If saddle point of such game does not exist and also the game is not solved by dominance property then method of subgames is applied. In this method given $2 \times n$ and $m \times 2$ game is divided into as many 2×2 sub games. Then using oddment method the value of all subgames is calculatesd. In case of $2 \times n$ game the subgame having lowest value is selected and in care of $m \times 2$ game, the subgame having highest value is selected.

16.2 Learning Objectives:

After studying this unit students will be able to

1. definesubgame method
2. able to find the value of game
3. able to find the value of game for $m \times 2$ game.

5. Sub Games Method -

A game where one player has only two courses of action and the other has more than two courses of action is called a $2 \times n$ or $m \times 2$ game. If neither this game involve a saddle point not it is reducible to a 2×2 game the subgame method and the graphical method are very useful to solve the game. Under this where matrix is subdivided in 2×2 matrices.

Steps in Case of 2×n matrix

Step.1 Divide the game into as many 2×2 subgames as possible

Step.2 Take each game independently and find out the value of game

Step 3 Select the best sub game from the point of view of the player who has more than two courses of action.

Step. 4 Mix of strategies for this selected subgame will hold good for both the players and the value of the game of this subgame will also provide the solution of the whole game.

Let us try to solve the problems of game theory using subgame methods:

Example 1: Solve the following 2×3 game:

Player B

Player A		B_1	B_2	B_3
	A_1	3	5	13
	A_2	10	7	4

Solution: Since game has no saddle point hence the players will use mixed strategy. We using Sub-game method to solve it.

Sub game - I

Player B

Player A		I	II
	I	3	5
	II	10	7

∴ Value of game = 7

Sub game - II

Player B

Player A		I	III	
	I	3	13	6
	II	10	4	10
	odds	9	7	

Value of game = $\frac{3\times6+10\times10}{6+10}=\frac{18+100}{16}$

= $\frac{118}{16}=7.375$

Sub game - III

Player B

		I	III	odds
Player A		5	13	3
		7	4	8
	odds	9	2	

Value of game i.e. V = $\frac{5\times3+7\times8}{3+8}=\frac{15+56}{11}=\frac{71}{11}=6.454$

Since value of game i.e. V is the lowest in case of subgame - III, hence this sub-game is selected for the final solution

Probabilities of Selecting

	I	III	III
Player A	$\frac{3}{11}$	$\frac{8}{11}$	-
Player B	0	$\frac{9}{11}$	$\frac{2}{11}$

Value of game = 6.454

Example 2: Find the value of game by using subgame method.

B

		I	III	III
A	I	2	-4	-2
	II	1	4	2

Solution: Since, the game has no saddle point. We will use the sub-game method.

Sub game I

	I	III	odds
I	2	-4	3
II	1	4	6
odds	8	1	

V = $\dfrac{2\times3+1\times6}{3+6}=\dfrac{12}{9}=\dfrac{4}{3}$

Sub game II

	I	III	odds
I	2	-2	1
II	1	2	4
odds	4	1	

V = $\dfrac{2\times1+1\times4}{1+4}=\dfrac{6}{5}$

Sub game III

	I	II
I	-4	-2
II	4	2

V = 2, hence the lowest value of game is V = $\dfrac{6}{5}$

∴ Value of game = $\dfrac{6}{5}$

Strategies for Player A Player B

I = $\dfrac{1}{5}$ I = $\dfrac{4}{5}$

II = $\frac{4}{5}$ II = 0

III = $\frac{1}{5}$

Example 3: Solve the following game

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} \\ A & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \\ \mathrm{IV} \end{array} & \begin{bmatrix} 1 & 3 \\ 1 & 2 \\ 2 & 1 \\ -2 & 5 \end{bmatrix} \end{array}$$

Solution: Since, game has no saddle point. We apply sub games method to solve the above game.

Sub game I

$$\begin{array}{ccc} & & B \\ & & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} \\ A & \begin{array}{c} \mathrm{I} \\ \mathrm{II} \end{array} & \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \end{array}$$

V = 2

Sub game II

$$\begin{array}{cccc} & & B & \\ & & \begin{array}{cc} \mathrm{I} & \mathrm{II} \end{array} & \text{odds} \\ A & \begin{array}{c} \mathrm{I} \\ \mathrm{III} \end{array} & \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \\ & \text{odds} & \begin{array}{cc} 2 & 1 \end{array} & \end{array}$$

V = $\frac{1 \times 1 + 2 \times 2}{1 + 2} = \frac{5}{3}$

Sub game III

$$\begin{array}{cc} & B \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{array}{c} \text{I} \\ \text{IV} \end{array} & \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix} \end{array}$$

V = 1

Sub game IV

$$\begin{array}{cccc} & & B & \\ & & \begin{array}{cc} \text{I} & \text{II} \end{array} & \text{odds} \\ A & \begin{array}{c} \text{II} \\ \text{III} \end{array} & \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 1 \end{array} \\ & \text{odds} & \begin{array}{cc} 1 & 1 \end{array} & \end{array}$$

$$V = \frac{1 \times 1 + 2 \times 1}{2} = \frac{3}{2}$$

Sub game V

$$\begin{array}{cc} & B \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} \\ \begin{array}{c} \text{II} \\ \text{IV} \end{array} & \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} \end{array}$$

V = 1

Sub game VI

$$\begin{array}{ccc} & B & \\ & \begin{array}{cc} \text{I} & \text{II} \end{array} & \text{odd} \\ \begin{array}{c} \text{II} \\ \text{IV} \end{array} & \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix} & \begin{array}{c} 7 \\ 1 \end{array} \\ \text{odds} & \begin{array}{cc} 4 & 4 \end{array} & \end{array}$$

V = $\frac{1\times 7-2}{7+1}=\frac{12}{8}=\frac{3}{2}$

Player A will select SubgameII[nd] because value of game highest in this case

Thus the selected Sub game is II and value of game = $\frac{5}{3}$ Probability of player A selecting strategy

I	II	III	IV
$\frac{1}{3}$	0	$\frac{2}{3}$	0

Probability of player B selecting strategy

I	II
$\frac{2}{3}$	$\frac{1}{3}$

Case of m×2 matrix

Example 4: Solve the following game

		I	II
	I	2	4
A	II	2	3
	III	3	2
	IV	-1	6

Solution: Since given game does not have saddle point, hence the players will use mixed strategies. Further the above game is a rectangular game of 5×2 type, hence we apply subgames method to solve the above game.

Sub game I

Player B

		I	II
Player A	I	2	4
	II	2	3

$$V = 2$$

Sub game - II

		Player B		
		I	II	odds
Player A	I	2	4	1
	III	3	2	2
	odds	2	1	

$$V = \frac{(2\times1)+(3\times2)}{1+2} = \frac{8}{3}$$

Sub game III

		Player B	
		I	II
Player A	I	2	4
	IV	-1	6

$$V = 2$$

Sub game IV

		Player B		
		I	II	odds
Player A	II	2	3	1
	III	3	2	1
	odds	1	1	

$$V = \frac{(2\times1)+(3\times1)}{1+1} = \frac{5}{2}$$

Sub game V

		Player B	
		I	II
Player A	I	2	3
	IV	-1	6

V = 2

Sub game VI

		Player B		
		I	II	odds
Player A	III	3	2	7
	IV	-1	6	1
	odds	4	4	

$$V = \frac{(3\times 7)+(-1\times 1)}{7+1} = \frac{5}{2}$$

Player A will select subgame Ii because value of game is highest in this case

Thus the selected sub game is

		Player B		
		I	II	odds
Player A	I	2	4	1
	III	3	2	2
	odds	2	1	

Probability of Player A selecting strategy

I = $\frac{1}{3}$

II = 0

III = $\frac{2}{3}$

Probability of Player B selecting strategy

I = $\frac{2}{3}$

II = $\frac{1}{3}$

and value of game = $\frac{8}{3}$

Self Check Exercise

Q.1 Find the value of game by using sub game method

(a)

		B		
		I	II	III
A	I	3	-3	-1
	II	2	5	3

Ans. $\left[Value\, of\ game = \frac{11}{5}, A\left(\frac{1}{5}, \frac{4}{5}\right); B\left(\frac{4}{5}, 0, \frac{1}{5}\right) \right]$

(b)

		B			
		I	II	III	IV
A	I	2	2	3	-1
	II	4	3	2	6

Ans. Value of game = $\frac{5}{2}$

(c)

		B	
		I	II
	I	2	4
A	II	2	3
	III	3	2
	IV	-1	6

Ans. Value of game = $\frac{8}{3}$

16.4 Summary:

Dear students, in this unit we studied that

(1) a $2\times n$ game can be subdivided into a number of 2×2 games from which value of game is calculated. The subgame having lowest value of game is selected as the best strategy of the respective player.

(2) a $m\times 2$ game can be subdivided into a number of 2×2 game, which are further solved by any method (previously studied) and we select the subgame having highest value of game.

16.5 Glossary:

- **Subgame:** A smaller game that arises at any point in a larger game when players face a new decision mode.

16.6 Answers to Self Check Exercise

Q.1 (a) Value of game $\frac{11}{5}$

A $\left(\frac{1}{5}, \frac{4}{5}\right)$

B $\left(\frac{4}{5}, 0, \frac{1}{5}\right)$

(b) Value of game = $\frac{5}{2}$

(c) Value of game = $\frac{8}{3}$

15.7 References/Suggested Readings

1. S.D. Sharma, operations Research, KedarNathRath Math.
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16.8 Terminal Questions

Q.1 Solve the game using sub game method

(a)

		B		
		B_1	B_2	B_3
A	A_1	1	3	11
	A_2	8	5	2

(b)

	B	
A_1	1	-3
A_2	3	5
A_3	-1	6
A_4	4	1
A_5	2	2
A_6	-5	0

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Unit - 17

Graphical Method

Structure

17.1 Introduction

Dear student in this unit we will study about the graphical method of solving a game. The graphical method can be used to a game having $m \times 2$ or $2 \times n$ pay off matrix but no saddle point. In this unit we will study how graphical method is used to solve $m \times 2$ or $2 \times n$ game. The graphical method reduces the given into 2×2 game and then the game is solved by either oddment method or by automatic method. In graphical method we draw the feasible region for given game.

17.2 Learning Objectives:

Dear students, after studying this unit students will be able to

1. define graphical method of game theory
2. draw feasible region for a game for $m \times 2$ or $2 \times n$.
3. Able to reduce given game to 2×2 order
4. find value of game after using graphical method

17.3 Graphical Method

Conditions:

1. Pay-off matrix must be of $2 \times n$ or $m \times 2$ size
2. No saddle point exists
3. The matrix is not reduced to 2×2 size by using dominance principle

Steps Involved (For 2×n size)

1. Construct two vertical axis one unit a part and make a scale of each of them. These two lines represents the two strategies available to maximising player.
2. Draw lines to represent each of minimising player's strategies.
3. Determine the highest point from the lower boundary
4. Formulate (2×2) pay-off sub matrix corresponding to the strategies providing highest point.
5. Use odd's method to get the value of game.

Steps Involved (For m×2 size)

1. Construct two vertical axis one unit apart and make a scale on each of them. These two lines represents the two strategies available to minimising player.
2. Draw lines to represent each of maximising Player's strategies.
3. Determine the lowest point from the upper boundary
4. Formulate (2×2) payoff sub matrix corresponding to the strategies providing lowest point.
5. Use odd's method to get the value of game.

Let us try to find value of game using graphical methods.

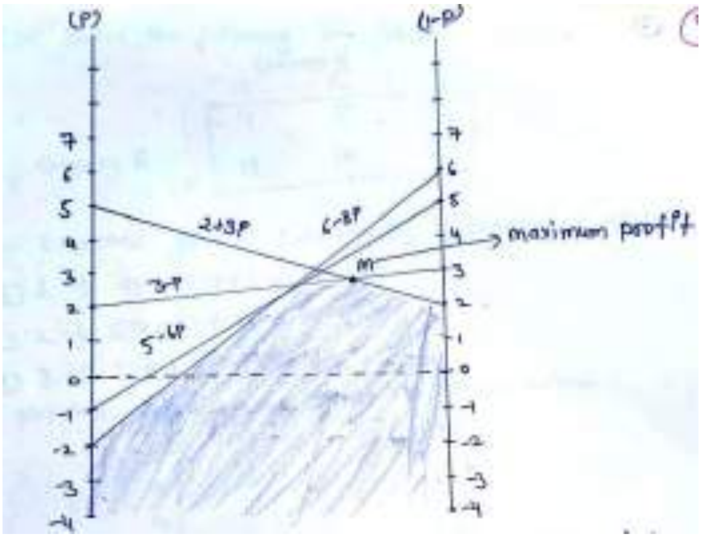
Example 1: Solve the game by using graphic method whose payoff matrix is given

$$
\begin{array}{cc|cccc|}
 & & & B & & \\
 & & I & II & III & IV \\
\hline
A & I & 2 & 5 & -1 & -2 \\
 & II & 3 & 2 & 5 & 6 \\
\hline
\end{array}
$$

Solution: Let player A choose his I strategy with probability p hence his II strategy will be with probability (1-P). The expected profit to A if B uses his I strategy $2 \times P + 3(1-P) = 2P + 3 - 3P$ or $3 - P$. If player B choose his II[nd] strategy, the expected profit to A will be $S \times P + 2(1-P) = 3P + 2$. If player B selects his III strategy the expected profit to A will be $-1 \times P + 5(1-P) = 5\text{-}6P$ and if player B select his last strategy, then expected profits to A will be $-2 \times P + (6(1-P) = 6 - 8P$.

Strategies of Player B	Expected Profits to A
I	3-P
II	3P+2
III	5-6P
IV	6-8P

We plot these values on the graph.



Now A choose a suitable value of P to maximize his gains. The highest point of bound is the required point. In the figure, M is the maximum point this bound there strategies will be selected where common difference intersects and the resultant matrix is produced as

$$
\begin{array}{cc}
 & B \\
A & \begin{pmatrix} & I & II \\ I & 2 & 5 \\ II & 3 & 2 \end{pmatrix}
\end{array}
$$

Now apply Odd's

B

		I	II	odd's
	I	2	5	1
A	II	3	2	3
	odd's	3	1	

Value of game = $\frac{2\times1+3\times3}{4}=\frac{2+9}{4}=\frac{11}{4}$

Probability of Selecting Strategy

Player A　　　　Player B

I = $\frac{1}{4}$　　　　I = $\frac{3}{4}$

II = $\frac{3}{4}$ II = $\frac{1}{4}$

III = 0

IV = 0

Example 2: Solve the following by Graphic Method

Company B

		1	2	3
Company A	1	4	5	13
	2	18	10	8

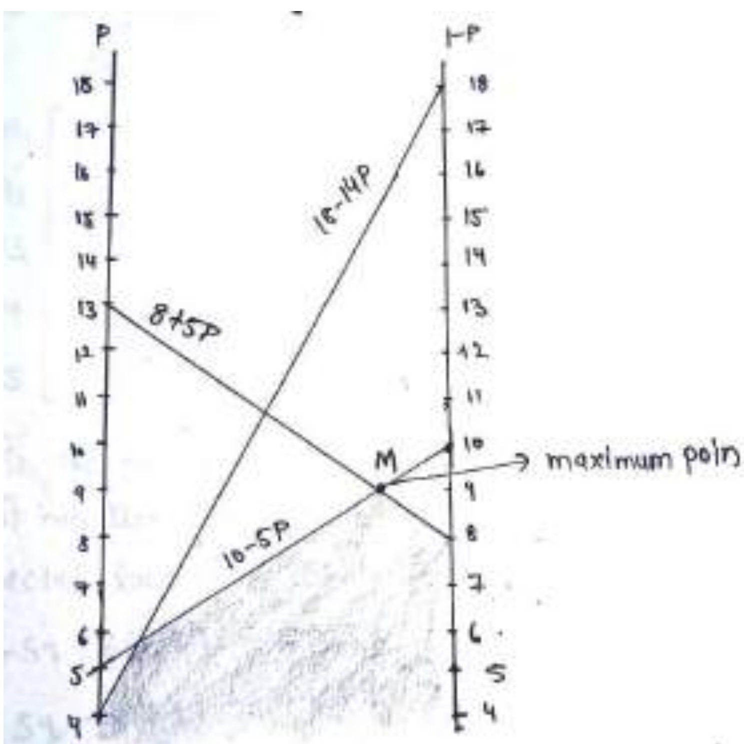
Solution: Expected gain of company A, if company B adopts.

Strategy 1 $\rightarrow$ 4P + 18(1-P) = 18 - 14P

Strategy 2 $\rightarrow$ 5P + (1-P)10 = 10 - 5P

Strategy 3 $\rightarrow$ 13P + 8(1-P) = 8 + 5P

Now, we plot these values on the graph as shown below.



The following matrix with point of Minimix M is a below

$$\begin{array}{cc|cc|c} & & \multicolumn{2}{c}{B} & \\ & & 2 & 3 & \text{odds} \\ \hline A & 1 & 5 & 13 & 2 \\ & 2 & 10 & 8 & 8 \\ \hline & \text{odds} & 5 & 5 & \end{array}$$

Value of game = $\frac{5 \times 2 + 10 \times 8}{2 + 8} = \frac{10 + 80}{10} = 9$

	Probability for A for selecting strategy		Probability for B for selecting strategy
1	$\frac{1}{5}$	1	0
2	$\frac{4}{5}$	2	$\frac{1}{2}$
3	0	3	$\frac{1}{2}$

Example 3: Solve the following game using graphic method

$$\begin{array}{cc|cc|} & & B_1 & B_2 \\ & A_1 & -5 & 8 \\ A & A_2 & 5 & -4 \\ & A_3 & 0 & -1 \\ & A_4 & -1 & 6 \\ & A_5 & 8 & -5 \end{array}$$

Solution: Let q is the probability of player B selecting strategy I^{st}

Hence probability of his II^{nd} strategy becomes 1-q.

Calculation of expected loss of B when A selects, his

I Strategy -5q + 8 (1-q) = 8 - 13q

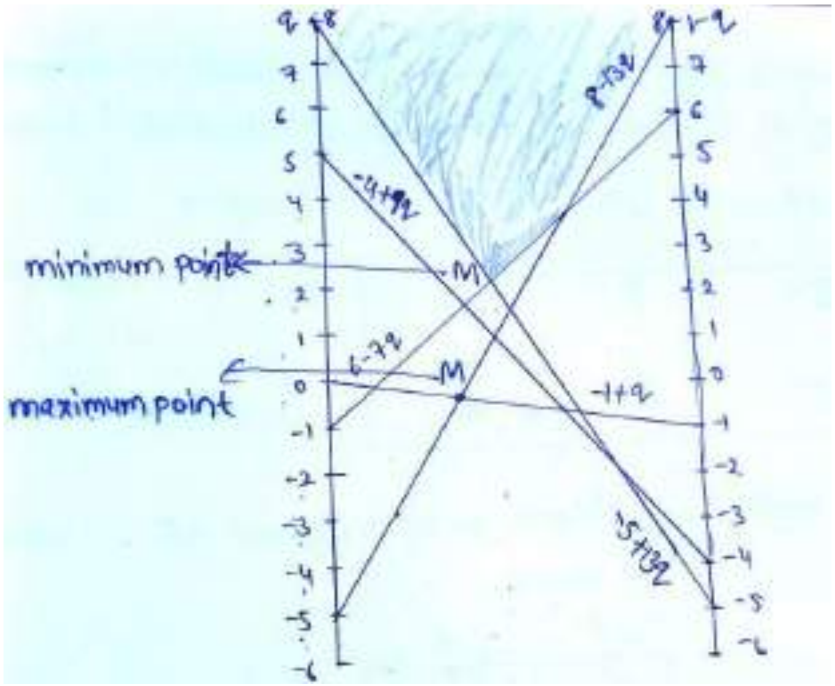
II Strategy 5q - 4 (1-q) = -4 + 9q

III Strategy $0 \times q - (1-q) = -1 + q$

IV Strategy $-1q + 6(1-q) = 6 - 7q$

V Strategy $8q - 5(1-q) = -5 + 13q$

We plot these figures on the graph



M is the minimax point. We have the following matrix from point of view of B.

		B		
		I	II	odd's
A	IV	-1	6	13
	V	8	-5	7
	odd's	11	9	

Value of game = $\frac{-1 \times 13 + 8 \times 7}{13 + 7} = \frac{-13 + 56}{20} = \frac{43}{20}$

Strategies	For Player A	Player B
I	0	$\frac{11}{20}$

II	0	$\frac{9}{20}$
III	0	-
IV	$\frac{13}{20}$	-
V	$\frac{7}{20}$	-

Example 4: Solve the following game for player A using graphical method. Pay-off matrix for player A is given below:

Player B

		a_1	a_2	a_3	a_4
Player A	b_1	-6	8	-3	9
	b_2	7	-3	-1	-5

Solution: The modified game matrix for player A is:

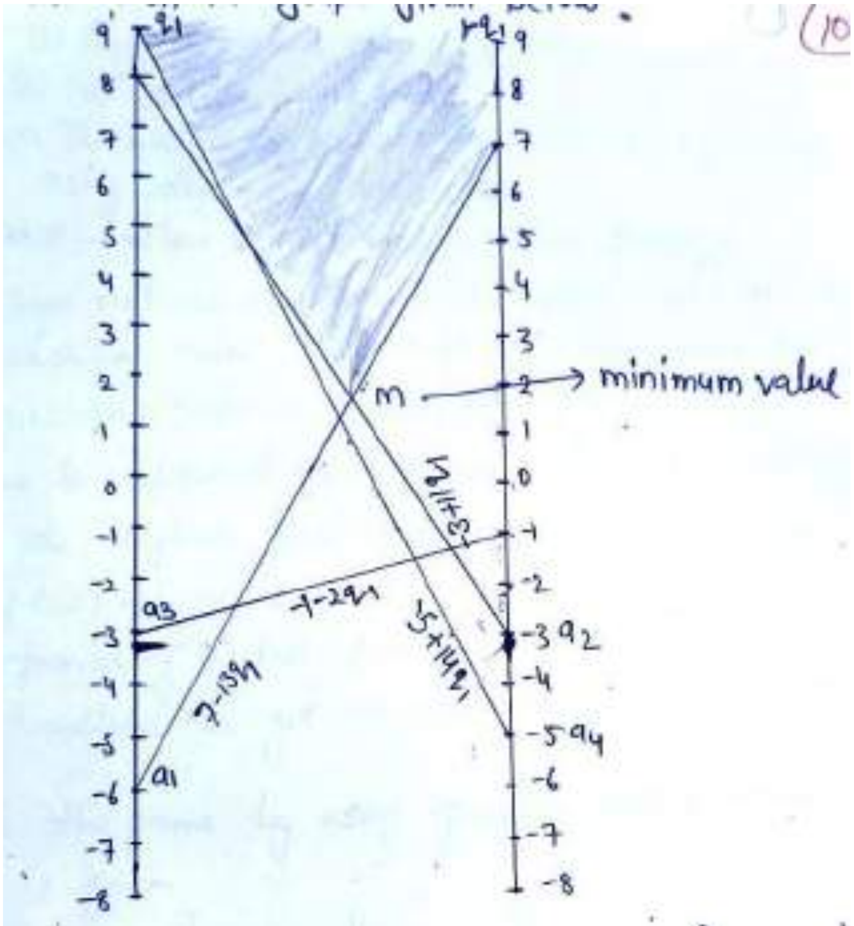
Player B

		b_1	b_2
	a_1	-6	7
Player A	a_2	8	-3
	a_3	-3	-1
	a_4	9	-5

Let q_1 be the probability of player B selecting strategy b1 and hence $(1-q_1)$ be the probability of player B selecting strategy b_2

If Player A Select Strategy	Expected Pay off of Player B
a_1	$-6q_1 + 7(1-q_1) = 7 - 13q_1$
a_2	$8q_1 - 3(1-q_1) = -3 + 11q_1$
a_3	$-3q_1 - (1-q_1) = -1 - 2q_1$
a_4	$9q_1 - (1-q_1)\ 5 = -5 + 14q_1$

Plotting these values on the graph given below:



Since M is the minimax point at the intersection of a_1 and a_2 line. The resultant matrix is

Player B

		b_1	b_2	odd's
Player A	a_1	-6	7	11
	a_2	8	-3	13
	odd's	10	14	

$$V = \frac{-6\times 11+8\times 13}{11+13} = \frac{-66+104}{24} = \frac{38}{24} = \frac{19}{12}$$

Probabilities for Selecting Strategy

Player A

$a_1 = \frac{11}{24}$

$a_2 = \frac{13}{24}$

$a_3 = 0$

$a_4 = 0$

Player B

$b_1 = \frac{11}{24} = \frac{5}{12}$

$b_2 = \frac{14}{24} = \frac{7}{12}$

Example 5: Solve the game by using graphic method whose pay off matrix is given

		B			
		I	II	III	IV
A	I	1	4	-2	-3
	II	2	1	4	5

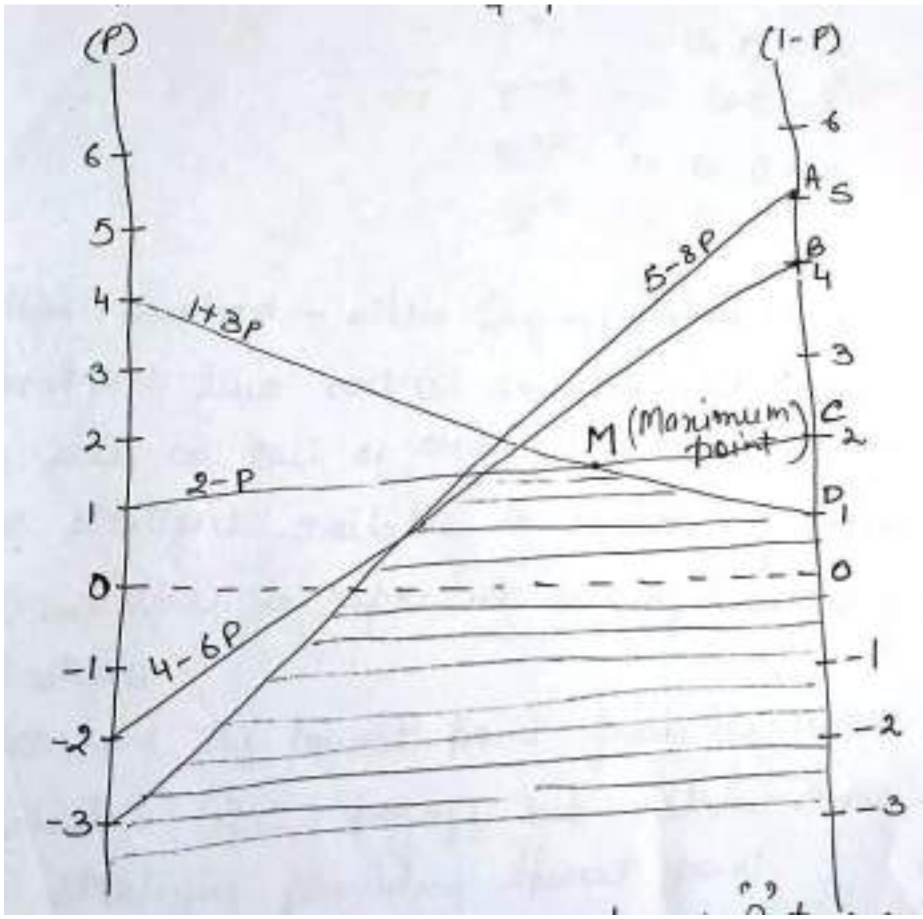
Solution: Let player A choose his I strategy with probability P hence his II strategy will be with probability (1-P). The expected profit to A if B uses his I strategy is $(1 \times P) + 2\ (1-P)$ i.e. P+2-2P or 2-P.

If player B choose his IInd strategy, the expected profit to A will be $(4 \times P) + 1(1-P)$ or 4P + 1 - p or 3P + 1.

If player B selects his IIIrd strategy the expected profit to A will be $(-2 \times P) + 4(1-P)$ or -2 + P+4 - 4 i.e. 4-6P and if player B selects his last strategy then expected profit of A will be -3P + 5(1-P) i.e. - 3P + 5 - 5 or 5 - 8P

Strategies of Player B	Expected profits to A
I	2-P
II	1+3P
III	4-6P
IV	5-8P

We plot these values on Graph



Now 'A' choose a suitable value of 'P' to manimize his gains. The highest point of bound is the required point. 'M' is the maximum point of this bound, those strategies will be selected where 'C' and 'D' intersects and the resultant matrix is produced as

$$A \begin{array}{cc} & \begin{array}{cc} I & II \end{array} \\ \begin{array}{c} I \\ II \end{array} & \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix} \end{array}$$ Now apply odd's

$$A \begin{array}{cc} & \begin{array}{ccc} I & II & odds \end{array} \\ \begin{array}{c} I \\ II \\ odds \end{array} & \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & \end{bmatrix} \end{array}$$

$$V = \frac{(1\times1)+(2\times3)}{4} = \frac{7}{4}$$

Probability of Selecting I[st] Strategy to A = $\frac{1}{4}$

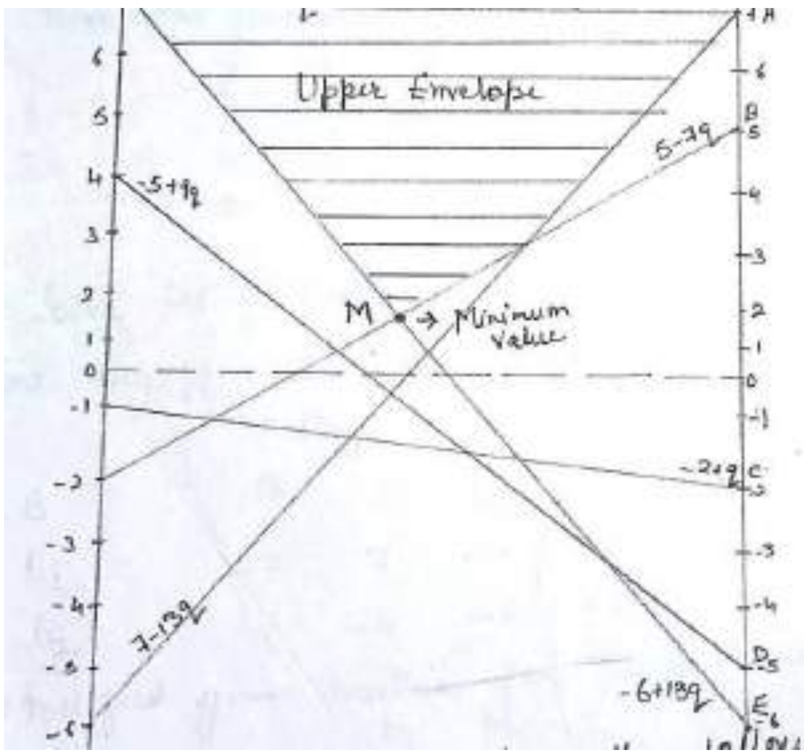
II[nd] Strategy to A = $\frac{3}{4}$

I[rd] Strategy to B = $\frac{3}{4}$

II[nd] Strategy to B = $\frac{1}{4}$

III[rd] Strategy to B = 0

IV[th] Strategy to B = 0



Example 6: Solve the following game using Graphical method

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{bmatrix} B_1 & B_2 \\ -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & -6 \end{bmatrix}$$

Solution: Let q is the probability of player B selecting strategy Ist Hence probability of his IInd strategy becomes (1-q) calculation of expected loss of 'B' when 'A' selects, his

I strategy	-6q + 7(1-q) = -6q + 7 - 7q = 7 - 13q
II strategy	4q - 5(1 - q) = 4q - 5 + 5q = -5 + 9q
III strategy	-q - 2)1 q) = -q - 2 + q = -2 + q
IV strategy	-2q + 5(1 - q) = -2q + 5 - 5q = 5 - 7q
V strategy	7q - 6 (1 - q) = 7q - 6 + 6q = - 6 + 13q

M is min max point we have the following matrix from point of view of B

$$A \begin{bmatrix} & I & I & Odds \\ Iv & -2 & -5 & 13 \\ V & 7 & -6 & 7 \\ odds & 11 & 9 & \end{bmatrix}$$

$$V = \frac{(-2\times13)+(7\times7)}{20} = \frac{23}{20}$$

Strategies	For Player A	Player B
I	0	11/20
II	0	9/20
III	0	-
IV	13/20	-
V	7/20	-

Self Check exercise

Question 1 Solve the following by Graphic method.

		B		
		1	2	3
A	1	6	7	15
	2	20	12	10

(b)

		I	II	III
A	I	8	4	-2
	II	-2	-1	3

Question 2: Solve the following game for player A using graphical method.

		Player A			
		a_1	a_2	a_3	a_4
Player B	b_1	1	4	-2	-3
	b_2	2	1	4	5

17.4 Summary:

In a game theory, the graphical method is used to analyze and visualize strategic interactions between players in a game. It typically involves representing players choices and payoff on a graph, such as a payoff matrix or a strategic form game, to determine optimal strategies and outcomes. This method helps illustrate how decisions made by each player can influence the overall outcome of the game.

17.5 Glossary

- **Feasible region:** The set of all possible combinations of strategies for the players that satisfy all constraints (pay off inequalities) in the game.
- **Optimal strategy:** A strategy or combination of strategies that maximizes a player's payoff given the strategies chosen by other players. This is typically found at the intersection of is profit lines where the highest payoff can be achieved.
- **Corner point solution:** A solution to the game where the optimal strategy combination lies at one of the vertices (corner points) of the feasible region.

17.6 Answers to Self check Exercise

Q. 1:

(a) Value of game = 11

Q.

(b) Value of game = 1

Q. 2:

Value of game = $\frac{7}{12}$

17.8 Terminal Questions

Q. 1 Solve the following game graphically, where payoff matrix for player A has been prepared

Player A

		1	2	3	4	5
Player B	1	1	5	-7	-4	2
	2	2	4	9	-3	1

Q.2 Solve the game graphically.

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{pmatrix} 1 & 3 & -3 & 7 \\ 2 & 5 & 4 & -6 \end{pmatrix} \end{array}$$

Q.3 Use graphical method to solve the game

$$\begin{array}{cc} & \text{Player A} \\ \text{Player B} & \begin{pmatrix} 2 & 2 & 3 & 2 \\ 4 & 3 & 2 & 6 \end{pmatrix} \end{array}$$

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Unit - 18

Algebraic And Approximation Method

Structure

18.1 Introduction

Dear student in this unit we will learn about two more methods related to game theory known as algebraic and approximation methods to solve game order 3×3. Algebraic method is normally used in the case when saddle point does not exist. Also this method is applied when the payoff matrix of a game is of order 3×3. Just like algebraic method approximation method is used to solve game having payoff matrix of order 3×3 and of higher order. The disadjuantage of this method is that of only gives approximate solutions not the exact solution.

18.2 Learning Objectives:

After studying this unit, students will be able to

1. define algebraic method of game theory
2. apply algebraic method to solve game having payoff matrix of order 3×3.
3. define approximation method of game theory
4. apply approximation method to solve game having payoff matrix of order 3×3 or higher.

18.3 Algebraic Method

The algebraic method is used to determine probability of using different strategies by player A and B.

Consider a game, where the payoff matrix is given by $\left[a_{ij}\right]_{m\times n}$

$$\begin{bmatrix} Player A & B_1 & B_2 & \dots & B_n \\ A_1 & a_{11} & a_{12} & \dots & a_{1n} \\ A_2 & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & & \\ A_m & a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Probability q_1 q_2 .... q_n

Since player A is the gainer player and expects at least v, therefore, we must have.

$a_{11}P_1 + a_{21}P_2 + \dots + a_{m1}P_m \geq V$

$a_{12}P_1 + a_{22}P_2 + \dots + a_{m2}P_n \geq V$

$\vdots \qquad \vdots$

$a_{1n}P_1 + a_{2n}P_2 + \dots + a_{mn}P_m \geq V$

Where $P_1 + P_2 + \dots + P_m = 1$, and $Pi \geq, 0 \ \forall$ i.

Similarly, the expected loss to player B when player A selects

Strategies A_1, A_2 ......A_m one by one also can be determined.

Since player B is the loser player, he must have

$a_{11}q_1 + a_{12}q_2 + \dots + a_{1n}q_n < V$

$a_{21}q_1 + a_{22}q_2 + \dots + a_{2n}q_n < V$

$\vdots \qquad \vdots$

$a_{m1}q_1 + a_{m2}q_2 + \dots + a_{mn}q_n < V$

Where $q_1 + q_2 + \dots + q_n = 1$, and $qj > 0 \ \forall$ V.

To get the value of p_i's and q_j's, above inequalities are considered as equations and are then solved for the given unknowns. However, if the system of equations so obtained is inconsistent, then at least one of the inequalities must hold as strict inequality. The solution can now be obtained only by applying trial and error method.

Note:- This method becomes quite lengthy. When number of strategies for both the players are more than two.

Let us try to understand algebraic method by following examples

Example 1: Solve the following game

		B		
		I	II	III
A	I	6	8	-4
	II	-6	0	2
	III	-2	-8	-4

Solution: As saddle point does not exist and dominance property is not applicable. Hence, we use algebraic method for calculating value of game.

Let P_1 = Probability of player A selecting strategy I

P_2 = Probability of player A selecting strategy II

P_3 = Probability of player A selecting strategy III

We know total or sum of probabilities = 1 i.e. $P_1 + P_2 + P_3 = 1$

Now expected payoff of Aare as under

If B selects first strategy the payoff of A is

$6P_1 - 3P_2 - 2P_3$ ...1

If B selects second strategy, the payoff of A is

$8P_1 + 0P_2 - 8P_3$ ...2

If B selects third strategy, the payoff of A is

$-4P_1 + 2P_2 + 4P_3$ ...3

Equate I and II, we get

$6P_1 - 6P_2 - 2P_3 = 8P_1 + 0P_2 - 8P_3$

$-2P_1 - 6P_2 + 6P_3 = 0$ ...4

Now equate 1 and 3, we get

$6P_1 - 3P_2 - \propto P_3 = -4P_1 + 2P_2 + 4P_3$

$10P_1 - 8P_2 - 6P_3 = 0$ ...5

Now take the value from 4 and 5 and cross multiply it by using formula "2312"

2	3	1	2
-6	6	-2	-6
-8	-6	10	-8

$$\frac{P_1}{(-6)\times(-6)-(-8\times 6)} : \frac{P_1}{(10\times 6)-(-6\times(-2))} : \frac{P_1}{(-8)\times(-2)-(-6\times 10)}$$

$$\frac{P_1}{36+48} : \frac{P_2}{60-12} : \frac{P_3}{16+60}$$

$$\frac{P_1}{84} : \frac{P_2}{48} : \frac{P_3}{76} = \frac{P_1+P_2+P_3}{84+48+76} = \frac{1}{208}$$

Now, $\frac{P_1}{48} = \frac{1}{208}$ or $P_1 = \frac{84}{208}$, $P_2 = \frac{48}{208}$ and $P_3 = \frac{76}{208}$

Now Consider payoff from B's Point of view

Let q_1 = Probability of player B selecting strategy I

q_2 = Probability of player B selecting strategy II

q_3 = Probability of player B selecting strategy III.

and we know $q_1 + q_2 + q_3 = 1$

Now expected payoff of B player is as follows:

If player A selects 1st strategy, payoff of B is

$6q_1 + 8q_2 - 4q_3$ ...6

If player A selects 2nd strategy, payoff of B is

$-6q_1 + 0q_2 + 2q_3$ 7

if player A selects 3rd strategy, payoff of B is

$-2q_1 - 8q_2 + 4q_3$ 8

equating 6 and 7, we get

$6q_1 + 8q_2 - 4q_3 = -6q_1 + 0q_2 + 2q_3$

$12q_1 + 8q_2 - 6q_3 = 0$ 9

Equating 6 and 8, we get

$6q_1 + 8q_2 - 4q_3 = 2q_1 - 8q_2 + 4q_3$

$8q_1 + 16q_2 - 8q_3 = 0$ ...10

take the values from 9 and 10 and cross multiply it by using formula 2321.

2	3	1	2
8	-6	12	8
16	-8	8	16

$$\frac{q_1}{-8\times 8-\left(-6\times 16\right)} : \frac{q_2}{-6\times 8-\left(-8\times 12\right)} : \frac{q_3}{12\times 16-8\times 8}$$

$$\frac{q_1}{32} : \frac{q_2}{48} : \frac{q_3}{128} = \frac{q_1+q_2+q_3}{32+48+128} = \frac{1}{208}$$

So $q_1 = \frac{32}{208}$, $q_2 = \frac{48}{208}$, $q_3 = \frac{128}{208}$

Value of game =

$$\left[\left(\frac{84}{208}\times 6\times\frac{32}{208}\right)+\left(\frac{84}{208}\times 8\times\frac{48}{208}\right)+\left(\frac{84}{208}\times(-4)\times\frac{128}{208}\right)+\right.$$

$$\left.\left(\frac{48}{208}\times 0\times\frac{48}{208}\right)+\left(\frac{84}{208}\times 2\times\frac{128}{208}\right)+\left(\frac{76}{208}\times(-2)\times\frac{32}{208}\right)+\left(\frac{76}{208}\times(-8)\times\frac{48}{208}\right)+\left(\frac{76}{208}\times 4\times\frac{128}{208}\right)\right]$$

value of game = $\frac{4}{13}$

Example 2: Solve the game by Algebraic Method:

B

		I	II	III
A	I	3	4	-2
	II	-3	0	1
	III	-1	-4	2

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Solution:

		B		
		I	II	III
A	I	3	4	-2
	II	-3	0	1
	III	-1	-4	2

Since, Saddle point does not exist, hence the player will used mixed strategies. Since dominance principle does not apply, the game will solved by Algebraic method.

Let P_1, P_2 and P_3 are the probability of Player A to selecting the strategy I, II and III respectively.

We know $P_1 + P_2 + P_3 = 1$

Similarly, let q_1 q_2 and q_3 are the probability of player B to selecting the strategy I, II and III respectively.

also $q_1 + q_2 + q_3 = 1$

Now expected payoff of A are as follows:

If B selects I strategy $3P_1 - 3P_2 - P_3$ ...1

If B selects II strategy $4P_1 + 0P_2 - 4P_3$ ...2

If B selects III strategy $-2P_1 + P_2 + 2P$...3

Now, equating 1 and 2, we get,

$$3P_1 - 3P_2 - P_3 = 4P_1 + 0P_2 - 4P_3$$

$$P_1 + 3P_2 - 3P_3 = 0 \qquad .....4$$

again equating 4 and 3, we get

$$3P_1 - 3P_2 - P_3 = -2P_1 + P_2 + 2P_3$$

$$5P_1 - 4P_2 - 3P_3 = 0 \qquad ...5$$

applying the formula "2312"

$$\begin{array}{cccc} 2 & 3 & 1 & 2 \\ \\ 3 & -3 & 1 & 3 \\ -4 & -3 & 5 & -4 \end{array}$$

$$\frac{P_1}{-9-12} : \frac{P_2}{-15+3} : \frac{P_3}{-4-15}$$

$$\frac{P_1}{21} : \frac{P_2}{-12} : \frac{P_3}{-19} = \frac{P_1+P_2+P_3}{-21-12-19} = \frac{-1}{52}$$

$\therefore$ $P_1 = \frac{21}{52}$, $P_2 = \frac{12}{52}$, $P_3 = \frac{19}{52}$

From the point of view of player B

If A selects I strategy $3q_1 + 4q_2 - 2q_3$ 6

If A selects II strategy $-3q_1 + 0q_2 + q_3$ 7

If A selects III strategy $-q_1 - 4q_2 + 2q_3$ 8

equating 6 and 7 we get

$3q_1 + 4q_2 - 2q_3 = -3q_1 + 0q_2 + q_3$

$6q_1 + 4q_2 - 3q_3 = 0$ 9

equating 6 and 8, we get

$3q_1 + 4q_2 - 2q_3 = -q_1 - 4q_2 + 2q_3$

$4q_1 + 8q_2 - 4q_3 = 0$ 10

from 9 and 10

$6q_1 + 4q_2 - 3q_3 = 0$

$4q_1 + 8q_2 - 4q_3 = 0$

$$\begin{array}{cccc} q_1 & q_2 & & q_3 \\ 4 & -3 & 6 & 4 \\ 8 & -4 & 4 & 8 \end{array}$$

$$\frac{q_1}{-16+24} : \frac{q_2}{-12+24} : \frac{q_3}{48-16}$$

$$\frac{q_1}{8}:\frac{q_2}{12}:\frac{q_3}{32}=\frac{q_1+q_2+q_3}{8+12+32}=\frac{1}{52}$$

$q_1 = \frac{8}{52}$, $q_2 = \frac{12}{52}$, $q_3 = \frac{32}{52}$

$\therefore$ value of the game = $\left(\frac{21}{52}\times 3\times\frac{2}{13}\right)$ + $\left(\frac{21}{52}\times 4\times\frac{3}{13}\right)$ + $\left(\frac{21}{52}\times(-2)\times\frac{8}{13}\right)$ + $\left(\frac{21}{52}\times(-3)\times\frac{2}{13}\right)$ + $\left(\frac{12}{52}\times 0\times\frac{3}{13}\right)$ + $\left(\frac{12}{52}\times 1\times\frac{8}{13}\right)$ + $\left(\frac{19}{52}\times(-1)\times\frac{2}{13}\right)$ + $\left(\frac{19}{52}\times(-4)\times\frac{3}{13}\right)$ + $\left(\frac{19}{52}\times 2\times\frac{8}{13}\right)$.

V = 0.154

Example 3: Solve the game by Algebraic Method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \end{array}$$

Solution: Since, the above game does not have saddle point. So, we will uqe algebraic method to solve this problem.

Let P_1, P_2, P_3 are the probability of A adopting strategy I, II and III respectively. So that $P_1 + P_2 + P_3 = 1$

Let q_1, q_2, q_3 are the probabilities of player B selecting I, II, III respectively. So that $q_1 + q_2 + q_3 = 1$

Now expected pay off of player A are as follows:

$0P_1 - P_2 + P_3$ 2

$P_1 + 0P_2 - P_3$ 3

Now, equating 1 and 2, we get

$-P_2 + P_3 = P_1 - P_3$

$P_1 + P_2 - 2P_3 = 0$ 4

Now, taking 4 and 3, we get

$-P_2 + P_3 = -P_1 + P_2$

$P_1 - 2P_2 + P_3 = 0$ 5

Now, taking equating 4 + 5, we get by cross multiplication

P_1 P_2 P_3

1	-2	1	1
-2	1	1	-2

$$\frac{P_1}{1-4} : \frac{P_2}{-2-1} : \frac{P_3}{-2-1}$$

$$\frac{P_1}{-3} : \frac{P_2}{-3} : \frac{P_3}{-3} = \frac{P_1 + P_2 + P_3}{-3-3-3} = \frac{1}{-9}$$

P_1 = $\frac{1}{3}$ and P_2= $\frac{1}{3}$, P_3 = $\frac{1}{3}$.

From the point of view of Player B

$0q_1 + q_2 - q_3$ 6

$-q_1 + 0q_2 + q_3$ ...7

$q_1 - q_2 + 0q_3$ 8

Equating 6 and 7, we get

$q_2 - q_3 = -q_1 + q_3$

$q_1 + q_2 - 2q_3 = 0$ ...9

Again, Equating 6 and 8, we get

$0q_1 + q_2 - q_3 = q_1 - q_2 + 0q_3$

$q_1 + 2q_2 - q_3 = 0$

from 9 and 10] we get

	q_1	q_2	q_3
1	-2	1	1
2	-1	-1	2

$$\frac{q_1}{1+4}:\frac{q_2}{2+1}:\frac{q_3}{2+1}$$

$$\frac{q_1}{3}:\frac{q_2}{3}:\frac{q_3}{3}=\frac{q_1+q_2+q_3}{3+3+3}=\frac{1}{9}$$

$q_1=\frac{1}{3}$, $q_2=\frac{1}{3}$ and $q_3=\frac{1}{3}$

$$V=\frac{1}{3}\times\frac{1}{3}\left[0+1-1-1+0+1+1-1+0\right]=0$$

Hence it is a fair game.

Self Check Exercise-1

Q.1 Solve the game by algebraic method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 6 & 8 & -4 \\ -6 & 0 & 2 \\ -2 & 8 & 4 \end{bmatrix} \end{array}$$

Q.2 Find the value of game by algebraic method whose payoff matrix is.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 0 \end{bmatrix} \end{array}$$

18.4 Approximation Method

Let us try following examples to have understanding of approximation method

Example 1: Solve the game by Iterative method?

B

	1	-1	-1
A	-1	-1	3
	-1	2	-1

Solution:

Step 1: Let Player A selects second row, place it below the matrix.

$$\begin{array}{c c} & B \\ A & \begin{array}{|ccc|} \hline 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \\ \hline \end{array} \\ & \begin{array}{ccc} -1 & -1 & 3 \end{array} \end{array}$$

Step 2 : Player B examines this row and choose a column corresponding to the smaller number of this row [there is tie of -1, let we consider -1 which is an extreme left]. This column is placed to the right of the matrix.

$$\begin{array}{c c} & B \\ A & \begin{array}{|ccc|} \hline 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \\ \hline \end{array} \\ & \begin{array}{ccc} -1 & -1 & 3 \end{array} \end{array}$$

Step 3: Player A examines this column and choose a row corresponding to the largest number in this column i.e. 1. This first row is then added to the last chosen row.

1	-1	-1	1
-1	-1	3	-1
-1	2	-1	-1

-1 -1 3

0 -2 2

Step 4: Player B, then chooses the column corresponding to the smallest number in the new row i.e. -2 and add this column to the last chosen column.

B

$$\begin{array}{c} \\ \\ A \\ \\ \\ \end{array} \begin{array}{|ccc|cc} \hline 1 & -1 & -1 & 1 & 0 \\ -1 & -1 & 3 & -1 & -2 \\ -1 & 2 & -1 & -1 & 1 \\ \hline \multicolumn{1}{c}{-1} & -1 & \multicolumn{1}{c}{3} & & \\ \multicolumn{1}{c}{0} & -2 & \multicolumn{1}{c}{2} & & \end{array}$$

Step 5: The Procedure is repeated for a finite number of steps. Here 10 iteration are presented are given below

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ -1 & -1 & 3 & -1 & -2 & 3 & -4 & -1 & -2 & 1 & 0 & 3 & 2 \\ -1 & 2 & -1 & -1 & 1 & 0 & 2 & 1 & 0 & -1 & 2 & -3 & -4 \\ -1 & -1 & 3 \\ 0 & -2 & 2 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 1 & -1 \\ -2 & 3 & -2 \\ -1 & 2 & -3 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

Strategies for player A is I : 4/10, II : 3/10, III : 3/10

Player B is I : 4/10, II : 2/10, III : 4/10

Value of game lis between -2/10 and $\frac{2}{10}$

Example 5: Solve the game approximately (Try 10 iterations)

$$\begin{array}{c} \\ A \end{array} \overset{B}{\begin{bmatrix} & I & II & III \\ I & 2 & 0 & 1 \\ II & 0 & 2 & 1 \\ III & 3 & 0 & 0 \end{bmatrix}}$$

Solution:

$$
\left[\begin{array}{cccc|cccccccccc}
 & I & II & III & & & & & & & & & & \\
I & 2 & 0 & 1 & 0 & 2 & 3 & 4 & 5 & 6 & 6 & 7 & 8 & 10 \\
II & 0 & 2 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 & 10 \\
III & 3 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 6 \\
\hline
2 & 0 & 1 & & & & & & & & & & & \\
2 & 2 & 2 & & & & & & & & & & & \\
5 & 2 & 2 & & & & & & & & & & & \\
5 & 4 & 3 & & & & & & & & & & & \\
7 & 4 & 4 & & & & & & & & & & & \\
7 & 6 & 5 & & & & & & & & & & & \\
9 & 6 & 6 & & & & & & & & & & & \\
9 & 8 & 7 & & & & & & & & & & & \\
9 & 10 & 8 & & & & & & & & & & & \\
9 & 12 & 9 & & & & & & & & & & &
\end{array}\right]
$$

Approx. Strategy for Player

A	I	3/10
	II	6/10
	III	1/10
B	I	2/10
	II	2/10
	III	6/10

Value of game - $\frac{9}{10} \leq V \leq \frac{10}{10}$

Example 6: In a well known children's game, each player says 'stone' or 'scissors' or 'papers'. If one says 'Stone' and another 'Scissors' then former win a rupee. Similarly, 'Scissors' beats 'Paper' and 'Paper' beats 'Stones' i.e. the player calling the former word wins a rupee. If the two player name the same item then there is a tie i.e. there is no payoff. Write down the payoff matrix. Find out the value of game and hence write down the optimal strategies of both players.

Solution: The payoff matrix is

B

$$
A \begin{bmatrix} & Sc & P & S \\ Sc & 0 & 1 & -1 \\ P & -1 & 0 & 1 \\ S & 1 & -1 & 0 \end{bmatrix}
$$

The iteration is

$$
\left[\begin{array}{ccc|ccc}
0 & 1 & -1 & -1 & -1 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 & 1 & 0 \\
\hline
0 & 1 & -1 & & & \\
-1 & 1 & 0 & & & \\
0 & 0 & 0 & & &
\end{array}\right]
$$

Value of game = Zero, it is a fair game

Strategies for player A is

I	1/3
II	1/3
III	1/3

Player B is

I	1/3
II	1/3
III	1/3

Example 7: Solve the game by Approximation method

$$
\begin{array}{c} \text{B} \\ A \begin{bmatrix} & I & II & III & IV \\ I & 2 & 3 & -1 & 0 \\ II & 5 & 4 & 2 & -2 \\ III & 1 & 3 & 8 & 2 \end{bmatrix} \end{array}
$$

The iteration is

	I	II	III											
I	2	0	1	0	2	3	4	5	6	6	7	8	10
II	0	2	1	2	2	3	4	5	6	8	9	10	10

III	3	0	0	0	3	3	3	3	3	3	3	3	6

2	0	1
2	2	2
5	2	2
5	4	3
7	4	4
7	6	5
9	6	6
9	8	7
9	10	8
9	12	9

Approx.	Strategy for Player	
A	I	3/10
	II	6/10
	III	1/10
B	I	2/10
	II	2/10
	III	6/10

Value of game: $\frac{9}{10} \leq V \leq \frac{10}{10}$

Example 6: In a well known Children's game, each player says 'Stone' or 'Scissors' or 'Papers'. If one says 'Stone' and another 'Scissors' then former win a rupee. Similarly, 'Scissors' beats 'Paper' and 'Paper' beats 'Stones' i.e. the player calling the former word wins a rupee. If the two players name the same item then there is a tie i.e. there is no payoff. Write down the payoff matrix. Find out the value of game and hence write down the optional strategies of both players.

Solution: The payoff matrix is

B

	Sc	P	S
Sc	0	1	-1
P	-1	0	1
S	1	-1	0

The integration is

0	1	-1	-1	-1	0
-1	0	1	1	0	0
1	-1	0	0	1	0
0	1	-1			
-1	1	0			
0	0	0			

Value of game = Zero, it is a fair game.

Strategies for player A is

I	1/3
II	1/3
III	1/3

Player B is

I	1/3
II	1/3
III	1/3

Example 7: Solve the game by Approximation Method

$$
\begin{array}{c c}
 & B \\
A & \begin{bmatrix} & I & II & III & IV \\ I & 2 & 3 & -1 & 0 \\ II & 5 & 4 & 2 & -2 \\ III & 1 & 3 & 8 & 2 \end{bmatrix}
\end{array}
$$

The iteration is

	I	II	III	IV										
I	2	3	-1	0	2	2	2	2	2	2	2	4	6	8
II	5	4	2	-2	5	3	1	-1	-3	-5	-7	-2	3	8
III	1	3	8	2	1	3	5	7	9	11	13	14	15	16
	1	3	8	2										
	6	7	10	0										

7	10	18	2
8	13	26	4
9	16	34	6
10	19	42	8
11	22	50	10
12	25	58	12
13	28	66	14
14	31	74	16

Value of game lies b/w $\frac{14}{10}$ and $\frac{16}{10}$

The approximate strategies for

Player A			Player B		
I	II	III	I	II	III
0	$\frac{1}{10}$	$\frac{9}{10}$	$\frac{4}{10}$	0	$\frac{6}{10}$

Self Check Exercise - 2

Q.1: solve the game approximately. (for 10 iteration)

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 5 & 4 & 2 & -2 \\ 1 & 3 & 8 & 2 \end{bmatrix}$$

Q.2 Find the approximate value of game and strategies by iterative method whose payoff matrix is given below

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 0 & 0 \end{bmatrix} \end{array}$$

18.5 Summary

Approximation Method: The approximation method refers to techniques that provide approximate solutions to complex games where exacte solutions are difficult to compute. This method often involve simplifying assumptions or numerical algorithms to estimate optimal strategies and outcomes.

Algebraic Method: The Algebraic method involves using mathematical equations and algebraic techniques to analyze strategic interactions and find optimal solutions. It typically involves setting up and solving system of equations derived from the game's structure to determine equilibrium strategies and payoffs.

18.6 Glossary

- **Best Response:** A strategy that yields the highest payoff for a player given the strategies chosen by the other players.
- **Normal form game:** A representation of a game showing all players, strategies, and payoffs in a matrix form.
- **Payoff matrix:** A matrix where each entry represents the payoff of a player given the combination of strategies. Chosen by all players.

18.7 Answers to Self Check Exercise

Self Check Exercise-1

Q.1 Value of game = $\frac{4}{13}$

Strategies for A $\left(\frac{84}{208}, \frac{48}{208}, \frac{76}{208}\right)$

B $\left(\frac{32}{208}, \frac{48}{208}, \frac{128}{208}\right)$

Q.2 Value of game = 0.67

Strategies for Player A $\left(\frac{6}{14}, \frac{4}{14}, \frac{4}{14}\right)$

Player B $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

Self Check Exercise-2

Q.1 Value of game = $\frac{4}{10} < V < \frac{16}{10}$

Approximate Strategies

A $\left(0, \frac{1}{10}, \frac{9}{10}\right)$ and B $\left(\frac{4}{10}, 0, \frac{6}{10}\right)$

Q.2 $\frac{9}{10} \leq$ Value of game $\leq \frac{11}{10}$

Approximate Stragies A $\left(\frac{3}{10},\frac{6}{10},\frac{1}{10}\right)$

B $\left(\frac{2}{10},\frac{3}{10},\frac{5}{10}\right)$

18.8 Suggested References/Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath
2. R. Panneerselvam, Operations Research, PHI Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., Third Edition.
4. K. Swarup, P.K. Gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

18.9 Terminal Questions

Q. 1 Solve the game by algebric Method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \end{array}$$

Q. 2 Solve the game by algebric Method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 8 & 10 & -2 \\ -4 & 2 & 4 \\ 0 & -6 & 6 \end{bmatrix} \end{array}$$

Q. 3 Solve the game by algebric Method

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 11 & 0 & 10 \\ 0 & 13 & 11 \\ 14 & 0 & 13 \end{bmatrix} \end{array}$$

Q.4 Find the value of game by iterative method whose payoff matrix is

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Q.5 Solve the game by method of approximation

$$\begin{array}{cc} & \text{B} \\ \text{A} & \begin{bmatrix} 3 & -3 & -3 \\ -6 & -3 & -6 \\ -3 & 6 & 3 \end{bmatrix} \end{array}$$

Unit - 19

Simplex Method

Structure

19.1 Introduction

Dear student, in this unit we will learn about the simplex method to solve a game. Simplex method is a basic method used to solve a linear programming problem. Since a two person zero sum game with mixed strategist is also a part of linear programming problem. All the methods which we learn yet reduce the game into 2×2 matrix then solved by oddment method but by using simplex method, we can solve a game of any size having mixed strategies.

19.2 Objectives:

After studying this unit, students will be able to

1. define simplex method in general
2. define simplex method in game theory
3. apply simplex method to find value of game of any order

19.3 Simplex Method

To apply simplex method we first have to connect the game problem with under programming problem. Let us do with this illustration.

Illustration: Consider a 3×3 payoff matrix given below:

$$\begin{array}{cc} & B \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \left[\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}\right] \end{array}$$

From A's point of view:

Suppose P_1, P_2 and P_3 be the probabilities that A plays the strategies A_1, A_2, and A_3 such that $P_1 + P_2 + P_3 = 1$. The objective of A is to maximize the minimum value of the game V.

Here, the problem is determine the variables P_1, P_2 and P_3 with the objective to maximize the value of game V subject to the following restrictions.

$a_{11}P_1 + a_{21}P_2 + a_{31}P_3 > V$

$a_{12}P_1 + a_{22}P_2 + a_{32}P_3 > V$

$a_{13}P_1 + a_{23}P_2 + a_{33}P_3 > V$

Divide all the constraints by V and try to minimize v rather than to maximize it.

$$\min \frac{1}{V} = \frac{P_1}{V} + \frac{P_2}{V} + \frac{P_3}{V}$$

$$a_{11}\left(\frac{P_1}{V}\right) + a_{21}\left(\frac{P_2}{V}\right) + a_{31}\left(\frac{P_3}{V}\right) \geq 1$$

$$a_{12}\left(\frac{P_1}{V}\right) + a_{22}\left(\frac{P_2}{V}\right) + a_{32}\left(\frac{P_3}{V}\right) \geq 1$$

$$a_{31}\left(\frac{P_1}{V}\right) + a_{23}\left(\frac{P_2}{V}\right) + a_{33}\left(\frac{P_3}{V}\right) \geq 1$$

For the propose of simplification, Put $\left(\frac{P_1}{V}\right) = x_1$; $\left(\frac{P_2}{V}\right) = x_2$; $\left(\frac{P_3}{V}\right) = x_3$ we have.

Minimize = $\frac{1}{V} = x_1 + x_2 + x_3$

Subject to constraints

$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \geq 1$

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \geq 1$

$a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$

From B_1 point of view:- Let q_1, q_2 and q_3 be the probabilities with B. Such that $q_1 + q_2 + q_3 = 1$. The objective of B is to minimize V. Problem is to determine the variable q_1, q_2 and q_3 with the objective to minimize the value of game V subject to the following restrictions.

$a_{11}q_1 + a_{12}q_2 + a_{13}q_3 \leq V$

$a_{21}q_1 + a_{22}q_2 + a_{23}q_3 \leq V$

$a_{31}q_1 + a_{32}q_2 + a_{33}q_3 \leq V$

Divide all constraints by V and try to maximize V rather than to minimize V.

$$\max : \frac{1}{V} = \frac{q_1}{V} + \frac{q_2}{V} + \frac{q_3}{V}$$

$$a_{11}\left(\frac{q_1}{V}\right) + a_{12}\left(\frac{q_2}{V}\right) + a_{13}\left(\frac{q_3}{V}\right) \leq 1$$

$$a_{21}\left(\frac{q_1}{V}\right) + a_{22}\left(\frac{q_2}{V}\right) + a_{23}\left(\frac{q_3}{V}\right) \leq 1$$

$$a_{31}\left(\frac{q_1}{V}\right) + a_{32}\left(\frac{q_2}{V}\right) + a_{33}\left(\frac{q_3}{V}\right) \leq 1$$

For the simplification:

Put $\left(\frac{q_1}{V}\right) = y_1$; $\left(\frac{q_2}{V}\right) = y_2$, $\left(\frac{q_3}{V}\right) = y_3$, so the given problem is

$$\text{maximise } Z = \frac{1}{V} = y_1 + y_2 + y_3$$

Subject to constraints

$a_{11}y_1 + a_{12}\,y_2 + a_{13}\,y_3 \leq 1$

$a_{21}y_1 + a_{22}\,y_2 + a_{23}\,y_3 \leq 1$

$a_{31}y_1 + a_{32}\,y_2 + a_{33}\,y_3 \leq 1$

$y_1, y_2, y_3 \geq 0$

Let us try following examples to find value of game on the basic of simplex method.

Example 1: Solve the following game by L.P. using simplex method. Also determine the best strategies for both the players.

Player B

$$\text{Player A} \begin{bmatrix} 6 & -1 & 5 \\ 4 & 0 & -4 \\ 1 & 7 & 10 \end{bmatrix}$$

Solution: Since saddle point does not exist, hence the players will use mixed strategy to solve the given game.

We are using LPP Simplex Method.

Let p_1 = Probability of Player A selecting I strategy

p_2 = probability of player A selecting II strategy

p_3 = Probability of player A selecting III strategy

So that $p_1 + p_2 + p_3 = 1$

From A Player's Point of view

$\text{Max}_v = \text{Min.}\ \frac{1}{V} = \text{Min}\ \frac{P_1 + P_2 + P_3}{V}$

Subject to

$6P_1 + 4P_2 + P_3 \geq V$

$- P_1 + 0.P_2 + 7P_3 \geq V$

$5P_1 - 4P_2 + 10P_3 \geq V]$ where as $P_1, P_2, P_3 \geq 0$

Divide the constraints by V

$\text{Min}_z = \frac{p_1}{V} + \frac{p_2}{V} + \frac{p_3}{V}$

Subject to

$\frac{6p_1}{V} + \frac{4p_2}{V} + \frac{p_3}{V} \geq 1$

$- \frac{p_1}{V} + \frac{0.p_2}{V} + \frac{2p_3}{V} \geq 1$

$\frac{5p_1}{V} - \frac{4p_2}{V} + \frac{10p_3}{V} > 1$, where $p_1, p_2, p_3 \geq 0$

Put $\frac{p_1}{V} = x_1$, $\frac{p_2}{V} = x_2$ and $\frac{p_3}{V} = x_3$.

So we get, Min $z = x_1 + x_2 + x_3$

Subject to $6x_1 + 4x_2 + x_3 \geq 1$

$-x_1 + 0.x_2 + 7x_3 \geq 1$

$5x_1 - 4x_2 + 10x_3 \geq 1$, where $x_1, x_2, x_3 \geq 0$

From B's Point of view

Let q1 = Probability of Player B Selecting I Strategy

q2 = Probability of Player B Selecting II Strategy

q3 = Probability of Player B Selecting III Strategy

So that $q_1 + q_2 + q_3 = 1$

$\text{Min}_v = \text{Max}\ \frac{1}{V} = \text{Max}\ \frac{q_1 + q_2 + q_3}{V}$

Subject to

$6q_1 - q_2 + 5q_3 \leq V$

$4q_1 - 0.q_2 - 4q_3 \leq V$

$q_1 + 7q_2 + 10q_3 \leq V$, where as $q_1, q_2, q_3 \geq 0$

Divide Constraints with V

$$Max_z = \frac{q_1}{V} + \frac{q_2}{V} + \frac{q_3}{V}$$

Subject to

$$\frac{6q_1}{V} - \frac{q_2}{V} + \frac{5q_3}{V} \leq 1$$

$$\frac{4q_1}{V} + \frac{0.q_2}{V} - \frac{4q_3}{V} \leq 1$$

$$\frac{q_1}{V} + \frac{7q_2}{V} + \frac{10q_3}{V} \leq 1, \text{ where } q_1, q_2, q_3 > 0$$

Let $\frac{q_1}{V} = Q$ (Q = 1, 2, 3)

$Maxz = Q_1 + Q_2 + Q_3$

Subject to

$6q_1 - q_2 + 5q_3 \leq 1$

$4q_1 - 0.q_2 - 4q_3 \leq V$

$q_1 + 7q_2 + 10q_3 \leq V$, where as $Q_1, Q_2, Q_3 \geq 0$

Now the objective function and constraints from each of the player point of view are reciprocal dual.

We introduce slack variable S_1, S_2, S_3 and assigning '0' co-efficient to the slack variables in the objective function we get.

$Max_z = Q_1 + Q_1 + Q_3 + 0S_1 + 0S_2 + 0S_3$

$6Q_1 - 1Q_2 + 5q_3 + S_1 + 0.S_2 + 0.S_3 = 1$

$4Q_1 + 0.Q_2 - 4Q_3 + 0.S_1 + S_2 + 0.S_3 = 1$

$1Q_1 + 7.Q_2 + 10.Q_3 + 0.S_1 + 0.S_2 + S_3 = 1$

Where $Q_1, Q_2, Q_3, S_1, S_2, S_3 \geq 0$

Simplex Table I

			1	1	1	0	0	0	R. Ratio
	Basic Variable	Q/y.	Q_1	Q_2	Q_3	S_1	S_2	S_3	

0	S_1	1	6	-1	5	1	0	0	1/6→Key row
0	S_2	1	4	0	-4	0	1	0	1/4
0	S_3	1	1	7	10	0	0	1	1
Total Contribution $Z_j = 0$			0	0	0	0	0	0	
Opportunity cost (C_j - Z_j)			1	1	1	0	0	0	
Here Key element is 6. So Incoming variable Q1 and outgoing variable S1. key column.									

Simplex Table II

			1	1	1	0	0	0	R. Ratio
	Basic Variable	Q/y	Q_1	Q_2	Q_3	S_1	S_2	S_3	
10	Q_1	1/6	1	-1/6	5/6	1/6	0	0	-1
0	S_2	1/3	0	2/3	-2/3	-2/3	1	0	1/2
0	S_3	5/6	0	43/6	55/6	-1/6	0	1	5/43 Key Row
Total Contribution Z_j=1/6			1	-1/6	5/6	1/6	0	0	
Opportunity cost (c_j-z_j)			0	7/6	1/6	$\frac{-1}{6}$	0	0	
Key element is 43/6. Incoming Variable Q_2 and outgoing variable is S_3.									

Simplex Table III

			1	1	1	0	0	0	

	Basic Variable	Q/y	Q_1	Q_2	Q_3	S_1	S_2	S_3	
1	Q_1	8/43	1	0	$\frac{45}{43}$	$\frac{7}{43}$	0	$\frac{1}{43}$	
0	S_2	$\frac{11}{43}$	0	0	$\frac{-352}{43}$	$\frac{-28}{43}$	0	$\frac{-4}{43}$	
1	Q_2	5/43	0	1	$\frac{55}{43}$	$\frac{-1}{43}$	0	$\frac{6}{43}$	
Total Contribution $Z_j = \frac{13}{43}$			1	1	$\frac{100}{43}$	$\frac{6}{43}$	0	$\frac{7}{43}$	
Opportunity Cost (C_j-Z_j)			0	0	$\frac{-57}{43}$	$\frac{-6}{43}$	0	$\frac{-7}{43}$	

Since all values in C_j-Z_j row are either negative or zero, we are with optimum solution.

$$\text{Max } \frac{1}{V} = \frac{13}{43} \Rightarrow \text{Max } V = \frac{43}{13}, \; Z_j = \frac{43}{13}$$

Now $\quad Q_1 = \frac{q_1}{V}$

$$q_1 = Q_1V = \frac{8}{43} \times \frac{43}{13} = \frac{8}{13}$$

$$q_2 = Q_2V = \frac{5}{43} \times \frac{43}{13} = \frac{5}{13}$$

$$q_3 = Q_3V = 0 \times \frac{43}{13} = 0$$

For player A's best strategies we read the dual values under slack variables ignoring signs.

$$x_1 = \frac{P_1}{V}$$

$$P_1 = x_1V = \frac{6}{43} \times \frac{43}{13} = \frac{6}{13}$$

P_2 - $x_2V = 0 \times \frac{43}{13} = 0$

$P_3 = x_3V = \frac{7}{43} \times \frac{43}{13} = \frac{7}{13}$

Best strategies for A $\left(\frac{6}{13}, 0, \frac{7}{13}\right)$

Best strategies for B $\left(\frac{8}{13}, \frac{5}{13}, 0\right)$

Example 2: Solve the following game by linear programming technique:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 2 & 0 & 4 \\ 4 & 6 & -2 \\ 7 & 3 & -1 \end{bmatrix} \end{array}$$

Solution: Since some of the entries in the payoff matrix are negative, we add a suitable constant to such of the entries to ensure them all positive. Thus adding a constant C=3 to each element, we get the following revised payoff matrix:

$$\begin{array}{cc} & \text{Player B} \\ \text{Player A} & \begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix} \end{array}$$

Let the strategies of the two players be

$$S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ p_1 & p_2 & p_3 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & q_3 \end{bmatrix}$$

Where $p_1 + p_2 + p_3 = 1$ and $q_1 + q_2 + q_3 = 1$

For Player A

Maximize V $\equiv$ Minimize $\frac{1}{V} = x_1 + x_2 + x_3$ Subject to the constraints:

$5x_1 + 7x_2 + 10x_3 > 1$, $3x_1 + 9x_2 + 6x_2 > 1$

$7x_1 + x_2 + 2x_3 > 1$; and $x_j > 0$ for j = 1, 2, 3.

For Player B

Minimize V = Maximize $\frac{1}{V}$ = $y_1 + y_2 + y_3$ subject to the constraints:

$5y_1 + 3y_2 + 7y_3 \leq 1$

$7y_1 + 9y_2 + y_3 \leq 1$

$10y_1 + 6y_2 + 2y_3 \leq 1$ and $y_1, y_2, y_3 \geq 0$

Let us now solve the problem for player B. By introducing slack

Variables the iterative simplex equations are:

$Max_z = y_1 + y_2 + y_3 + 0.S_1 + 0.S_2 + 0.S_3$

$5y_1 + 3y_2 + 7y_3 + S_1 = 1$

$7y_1 + 9y_2 + y_3 + S_2 = 1$

$10y_1 + 6y_2 + 2y_3 + S_3 = 1$

Where $y_1, y_2, y_3, S_1, S_2, S_3 > 0$

			1	1	1	0	0	0	R. Ratio
	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2	S_3	
0	S_1	1	3	0	7	1	0	0	$\frac{1}{7}$ →Key Row
0	S_2	1	9	0	1	0	1	0	
0	S_3	1	6	1	2	0	0	1	$\frac{1}{2}$
Total Contribution Z_j=0			0	1	0	0	0	0	
Opportunity Cost (C_j-Z_j)			1	0	1	0	0	0	
The key element is 7. Incoming Variable is y_3 and outgoing variable S_1.									

			1	1	1	0	0	0	R. Ratio

	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2	S_3	
1	q_3	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{3}{7}$	1	$\frac{1}{7}$	0	0	1/3
0	S_2	6/7	44/7	60/7	0	$\frac{-1}{7}$	1	0	$\frac{1}{6}\rightarrow$Key row
0	S_3	5/7	60/7	36/7	0	$\frac{-2}{7}$	0	1	
Total Contribution $Z_j = \frac{1}{7}$			$\frac{5}{7}$	3/7	1	$\frac{1}{7}$	0	0	
Opportunity Cost (C_j-Z_j)			$\frac{2}{7}$	4/7	0	$\frac{-1}{7}$	0	0	
The key element is 60/7. The incoming variable is y_2 and outgoing variable is S_2.									

			1	1	1	0	0	0	R. Ratio
	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2	S_3	
1	y_3	$\frac{1}{10}$	2/5	0	1	$\frac{3}{20}$	$\frac{-1}{20}$	0	
1	y_2	1/10	11/15	1	0	$\frac{-1}{60}$	$\frac{7}{60}$	0	
0	S_3	1/5	24/5	0	0	$\frac{-1}{5}$	$\frac{-3}{5}$	1	
Total Contribution $Z_j=\frac{1}{5}$			17/15	1	1	8/60	4/60	0	
Opportunity Cost (C_j-Z_j)			-2/15	0	0	-8/60	-4/60	0	

The expected value of game, is obtained as

$V = \frac{1}{V}$ i.e. $V = \frac{1}{1/5} = 5$, as 3 was added in the original payoff values. So game value is 5-3 = 2

Optimal strategies for player B

$$q_1 = 0 \times 5 = 0,\ q_2 = \frac{1}{10} \times 5 = \frac{1}{2},\ q_3 = \frac{1}{10} \times 5 = \frac{1}{2}$$

Making use of duality, the optimal strategies for player it.

$$P_1 = \frac{2}{15} \times 5 = \frac{2}{3},\ P_2 = \frac{1}{15} \times 5 = \frac{1}{3},\ P_3 = 0$$

Example 3: Solve the following game by linear programming technique:

Player B

$$\text{Player A} \begin{bmatrix} 2 & -2 & 3 \\ 5 & 6 & -3 \end{bmatrix}$$

Solution: Some of the elements is negative. We add a constant 4 to each element in above matrix. So modified matrix is

Player B

$$\text{Player A}\begin{bmatrix} 6 & 2 & 7 \\ 9 & 10 & 1 \end{bmatrix}$$

Now, let P_1 and P_2 represents the probabilities with which A choose strategies 1 and 2 respectively, while q_1, q_2 and q_3 be the probabilities in respect of B choosing strategies 1, 2 and 3 such that

$P_1 + P_2 = 1$ and $q_1 + q_2 + q_3 = 1$

If the value of the game is V, then for player A, we must have

$6p_1 + 9p_2 > \mu$, $2p_1 + 10p_2 > \mu$, $7p_1 + p_2 > \mu$

and for player B, we shall have

$6q1 + 2q2 + 7q3 < V$, $9q1 + 10q2 + q3 < V$

From A's point of view, the problem is

Minimise $\frac{1}{V} = x_1 + x_2$

Subject to the constraints

$6y_1 + 2y_2 + 7y_3 \leq 1$

$9y_1 + 10y_2 + y_3 \leq 1$

$y_1, y_2, y_3 \geq 0$, where $yj = \frac{q_j}{V}$ (j = 1, 2. 3)

The player B's problem is now solved by simplex method.

Introducing the slack variable $S_1 \geq 0$, $S_2 \geq 0$ and $S_3 \geq 0$ in the constraints, the above problem

Maximise $\frac{1}{V} = y_1 + y_2 + y_3 + 0.S_1 + 0.S_2$

Subject to the constraints

$6y_1 + 2y_2 + 7y + S_1 = 1$

$9y_1 + 10y_2 + y_3 + S_2 = 1$, where $y_1, y_2, y_3, S_1, S_2 \geq 0$

Simplex Table 1

										R. Ratio
	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2			
0	S_1	1	6	2	7	1	0			
0	S_2	1	9	10	1	0	1			
Total Contribution $Z_j=0$			0	0	0	0	0			
Opportunity cost (C_j-Z_j)			1	1	1	0	0			
Incoming variable = y_2, outgoing variable = S_2										

			1	1	1	0	0	R. Ratio
	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2	$\frac{2}{17}$ →key row
0	S_1	4/5	$\frac{21}{5}$	0	$\frac{34}{5}$	1	$\frac{-1}{5}$	1
1	y_2	$\frac{1}{10}$	$\frac{9}{10}$	1	$\frac{1}{10}$	0	$\frac{1}{10}$	
Total Contribution Zj = $\frac{1}{10}$			$\frac{9}{10}$	1	$\frac{1}{10}$	0	$\frac{1}{10}$	
Opportunity Cost (Cj-zj)			$\frac{1}{10}$	0	$\frac{9}{10}$	0	$\frac{-1}{10}$	
Incoming variable = y_3, outgoing variable = S_1								

Simplex Table III

			1	1	1	0	0	R. Ratio
	Basic Variable	Q/y	y_1	y_2	y_3	S_1	S_2	
1	y_3	2/17	21/34	0	1	5/34	$\frac{-1}{34}$	
1	y_2	3/34	$\frac{57}{68}$	1	0	$\frac{-1}{68}$	$\frac{7}{68}$	
Total Contribution Zj = $\frac{7}{34}$			$\frac{99}{68}$	1	1	$\frac{9}{68}$	5/68	
Opportunity Cost (Cj-zj)			$\frac{-29}{68}$	0	0	-9/68	-5/68	

The optimal values for y_1, y_2 and y_3 are 0, $\frac{3}{34}$ and $\frac{2}{17}$ respectively.

Also $\frac{1}{V}$ = 0 + $\frac{3}{34}$ + $\frac{2}{17}$ or V = $\frac{34}{7}$

Since a value 4 was added to the original payoff values, the game value is equal to V-3 = $\frac{34}{7}$ - 3 = $\frac{13}{7}$

Furthure, Since $q_1 = V \times y_i$ (i = 1, 2, 3)

$$q_1 = V \times y_1 = 0 \times \frac{34}{7} = 0$$

$$q_2 = V \times y_2 = \frac{3}{34} \times \frac{34}{7} = \frac{3}{7}$$

$$q_3 = V \times y_3 = \frac{34}{7} \times \frac{2}{17} = \frac{4}{7}$$

also $p_1 = \frac{9}{68} \times \frac{34}{7} = \frac{9}{14}$ and $p_2 = \frac{5}{68} \times \frac{34}{7} = \frac{5}{14}$

$$\therefore \quad \left[A\left(\frac{9}{14}, \frac{5}{14}\right); B\left(0, \frac{3}{7}, \frac{4}{7}\right), V = \frac{13}{7} \right]$$

Self Check Exercise:

Q.1 Solve the following game by linear programming technique:

Player B

$$\text{Player A} \begin{bmatrix} 8 & 9 & 3 \\ 2 & 5 & 6 \\ 4 & 1 & 7 \end{bmatrix}$$

Q.2 Solve the following game by L.P. Technique

Player B

$$\text{Player A} \begin{bmatrix} 3 & -1 & 7 \\ 6 & 7 & -2 \end{bmatrix}$$

19.4 Summary:

The Simplex method in a game theory is a mathematical technique used to find optimal strategies in zero-sum games, where one player's gain is another's loss. It involves iteratively adjusting probabilities or strategies to maximize or minimize payoffs, ensuring each player's strategy is optimal given the opponents strategy. This method simplifies complex decision-making by systematically improving strategies untill an equilibrium is reached.

19.5 Glossary:

- **Decision Variables:** Variables representing the choices or decisions available to players in a game. These variables are typically associated with strategies or action.
- **Objective function:** this function represents the goal or objective of players, often maximizing or minimizing their payoffs or utilities.
- **Constraints:** Limitations or restrictions on the strategies or action that players can choose. Constraints are formulated based on the structure of the game and the relationships between strategies.

19.6 Answers to Self Check Exercise

Q.1 V = $\frac{67}{13}$; A $\left(\frac{21}{52}, \frac{12}{52}, \frac{19}{52}\right)$; B $\left(\frac{2}{13}, \frac{3}{13}, \frac{8}{13}\right)$

Q.2 Value of game = $\frac{13}{7}$

A $\left(\frac{9}{14}, \frac{5}{14}\right)$; B $\left(0, \frac{3}{7}, \frac{4}{7}\right)$

19.7 Suggested Readings

1. S.D. Sharma, Operations Research, KedarNath Ram Nath
2. R. Panneerselvam, Operations Research, Phi Learning Private Limited, Second Edition.
3. JK Sharma, Operations Research Theory & Applications, Macmillan India Ltd., third Edition.
4. K. Swarup, P.K. gupta, M. Mohan, Operations Research, Sultan Chand & Sons, Twelth Edition.
5. S. Kalavathy, Operations Research, Vikas Publishing House Pvt. Ltd., Second Edition.

19.5 Terminal Questions

Q.1 Solve the following game by simplex method

(a)

Player B

$$\text{Player A} \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix}$$

Player B

(b)

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

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